New results on the free boundary problem and generalization of Hartogs-theorem for CR functions

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<u>1.Statements.</u> The problem of description of functions that are given on some part of the boundary of a domain and can be extended holomorphically to the entire domain has been considered in the papers [2],[4],[6],[7],[5],[10], [11],[12],[13],[15],[16], [17],[19],[20]. (It is pressumed that the domain is not contained in the hull of holomorphy of the piece of the boundary.) A survey of almost all of these results can be found in [3]. In all papers mentioned above (with exception of Fock-Kuni theorem [10] and its generalization in [2]) the description is rather complicated. Recently, however, in [6], we have obtained a one-variable result similar to the spirit of the Fock-Kuni theorem but simpler both in its formulation and proof. Here the condition says that the moment integrals (i.e. integrals of the quotient of the function and the monomials z^{-k}) should not grow to fast for with k (Theorem A).

The easy results of this spirit for several-dimensional case are obtained in [7]. One can also apply one dimensional result from [6] (Theorem A) to the the sections in several-dimensional case (see Theorem C in [5])

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For the solution of this free boundary problem they used to work with some basis in a space of holomorphic or harmonic functions. We managed to find a very simple solution, using the complete system of functions (if we used the basis, the answer would not be so easy).

Let $\Omega = \{\zeta : \psi(\zeta) < 0\}$ be a $(p_1, \dots, p_n)$-circular domain in $\mathbf{C}^n$, where $p_1, \dots, p_n$ are natural numbers, i.e. $z \in \Omega$ implies $(z_1 e^{itp_1}, \dots, z_n e^{itp_n}) \in \Omega$ for $t \in \mathbf{R}$. In particular, for $p_1 = \dots = p_n = 1$ this circular domain is Cartan domain. In addition to this, let Ω be convex and bounded and $\partial\Omega \in C^2$. Furthermore let $D \subset \mathbf{C}^n$ be a domain bounded by a part of the $\partial\Omega$ and by a hypersurface $\Gamma \in C^2$ dividing the domain Ω into two parts, the complement of $\bar{D}$ containing the origin. Let us consider the Cauchy-Fantappié differential form

$$\omega(\zeta - z, w) = \frac{(n-1)!}{(2\pi i)^n} \frac{\sum_{k=1}^{n} (-1)^{k-1} w_k dw[k] \wedge d\zeta}{\langle w, \zeta - z \rangle^n},$$

where $dw[k] = dw_1 \wedge \dots \wedge dw_{k-1} \wedge dw_{k+1} \wedge \dots \wedge dw_n$, $d\zeta = d\zeta_1 \dots d\zeta_n$, $\langle a, b \rangle = a_1 b_1 + \dots a_n b_n$. Then $grad\psi = (\frac{\partial \psi}{\partial \zeta_1}, \dots, \frac{\partial \psi}{\partial \zeta_n})$. By Sard theorem $grad\psi \neq 0$ for almost all r on $\partial\Omega_r$, where $\Omega_r = r\Omega$ is a homothety of Ω with $0 < r < 1$. We will assume that $grad\psi \neq 0$ on Γ. We denote

$$c_q = \frac{(|q| + n - 1)!}{q!} \int_\Gamma f(\zeta) \left(\frac{grad\psi}{\langle grad\psi, \zeta \rangle} \right)^q \omega(\zeta, grad\psi),$$

where $q = (q_1, \dots, q_n)$, $q! = q_1! \dots q_n!$, $|q| = q_1 + \dots + q_n$, $w^q = w_1^{q_1} \dots w_n^{q_n}$,

$$a_k = \sum_{\substack{\langle p,q \rangle = k \\ \langle p,s \rangle = k}} b_{q,s} c_q \bar{c}_s,$$

where

$$b_{q,s} = \int_\Omega z^q \bar{z}^s dV,$$

dV is the volume element in Ω.

We emphasize that the integral moments c_q depend on f and Γ, but the moments $b_{q,s}$ depend only on Ω.

The question is the following: When can a continuous function f on Γ (we write $f \in C(\Gamma)$) be continued to a function F being holomorphic in D

(we write $f \in A(D)$) satisfying $F \in C(D \cup \Gamma)$ and $F|_\Gamma = f$. In order to simplify the formulation the extra condition $f \in L^1(\Gamma)$ was added and we will do so below as well. This produces no loss of generality since such a condition holds on any smaller hypersurface.

Theorem 1 *For a function $f \in C(\Gamma) \cup L^1(\Gamma)$ to have a holomorphic continuation $F \in A(D) \cap C(D \cup \Gamma)$ with $F|_\Gamma = f$ it is necessary and sufficient that the following two conditions are fulfilled:*

1. *f is a CR function on Γ*
2. *$\overline{\lim}_{k\to\infty} \sqrt[k]{a_k} \leq 1$,*

The most easy results obtained as the solutions of free boundary problem are the consequences of this theorem:

Let $D_1 \in \mathbf{C}$ be a domain bounded by a part of the unit circle $\gamma_1 = \{z : |z| = 1\}$ and a simple , smooth and open arc Γ connecting two points of γ_1 and lying inside the circle. We assume that $0 \notin \bar{D}_1$.

Theorem A (Aizenberg) *For a function $f \in C(\Gamma) \cup L^1(\Gamma)$ to have a holomorphic continuation in D_1 it is necessary and sufficient that*

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|A_k|} \leq 1, \tag{1}$$

where

$$A_k = \int_\Gamma \frac{f(\zeta)d\zeta}{\zeta^{k+1}}, k = 0, 1, 2, \ldots$$

Let $\Omega \subset \mathbf{C}^n$ be a bounded convex n-circular domain. We assume $d_q(\Omega) = \max_\Omega |z^q|$.

Theorem B (Aizenberg-Kytmanov) *For a function $f \in C(\Gamma) \cup L^1(\Gamma)$ to have a holomorphic continuation in D as above it is necessary and sufficient that*

1. *f is a CR function on Γ*
2. $$\overline{\lim_{|q|\to\infty}} \sqrt[|q|]{c_q d_q(\Omega)} \leq 1. \tag{2}$$

Let D be a domain in $\mathbf{C}^n$, $\Gamma \subset \partial D$ a hypersurface of class C^1, transversal to the z_n-direction. Write $\alpha_{'z} = \{'z\} \times \mathbf{C}_{z_n}$, where $'z = (z_1, \ldots, z_{n-1})$. Assume that, for all $'z \in \mathbf{C}^{n-1}$ $\alpha_{'z} \cap D$ is connected and not empty. Then $\alpha_{'z} \cap \Gamma$ is not empty.

The following theorem, proved with the help of Theorem A, generalizes the well-known classical Hartogs theorem: if $f \in A(D_1)$ and can be holomorphically extended on z_n in the domain $D \supset D_1$, then $f \in A(D)$. In Theorem C we consider a function $f \in C(\Gamma) \cap CR(\Gamma)$.

Theorem C (Aizenberg-Rea) *Let $f : D \cup \Gamma \to \mathbf{C}$ be a function which is holomorphic with respect to the z_n-variable and, when restricted to $D \cap \alpha_{'z}$, is continuous up to $\Gamma \cap \alpha_{'z}$. Let $f \in C(\Gamma) \cap CR(\Gamma)$, then f is holomorphic in D with respect to all variables.*

This formulation can be improved using the methods of [14], [18]. Let $M = \{'z : D \cap \alpha_{'z} \neq \emptyset\}$, and $N \subset M$. We call $N \cap U$ for any U open in M, a portion of N.

Theorem 2 *Let f be a continuous CR function on Γ such for all $'z \in N$ its restriction on $\Gamma \cap \alpha_{'z}$ can be holomorphically extended with respect to z_n in the domain $D \cap \alpha_{'z}$ as a function continuous up to $\Gamma \cap \alpha_{'z}$. If any portion of N is not pluripolar in $\mathbf{C}^{n-1}_{'z}$, then f can be holomorphically extended in D as a function of all the variables: There exists $F \in A(D) \cap C(D \cup \Gamma)$ such that $F|_\Gamma = f$.*

In case when $f('z, z_n)$ for the fixed $'z \in N$ can be holomorphically extended on z_n for the whole $\mathbf{C}_{z_n}$ it is sufficient to require that N is not pluripolar (and not any of its portion).

Corollary 1 *Let $f \in C(\Gamma) \cap CR(\Gamma)$ and for $'z \in N$ the restriction of f on $\Gamma \cap \alpha_{'z}$ can be holomorphically extended in the $\mathbf{C}_{z_n}$. If N is not pluripolar in $\mathbf{C}^{n-1}_{'z}$, then f can be extended from Γ in $M \times \mathbf{C}_{z_n}$ as a holomorphic function, where $M = \{'z : \Gamma \cap \alpha_{'z} \neq \emptyset\} \subset \mathbf{C}^{n-1}_{'z}$.*

2. Proof of Theorem 1 and Corollaries. We call P_n weighted homogeneous polynomial of degree k if it is homogeneous of degree k with respect to $z_1^{1/p_1}, \ldots, z_n^{1/p_n}$.

Any function $\phi \in A(\Omega)$ has a decomposition in Ω into a series of weighted polynomials (see [8], p.56)

$$\phi(z) = \sum_{k=1}^{\infty} P_k(z), \tag{3}$$

uniformly convergent on compact subsets of Ω, and the P_k are pairwise orthogonal in $L^2(\Omega)$. Hence, if $\phi \in h^2(\Omega) = A(\Omega) \cap L^2(\Omega)$, then

$$\|\phi\|^2_{L^2(\Omega)} = \sum_{k=0}^{\infty} \|P_k\|^2_{L^2(\Omega)} \tag{4}$$

and furthermore, a function $\phi \in A(\Omega)$ belongs to $L^2(\Omega)$ if and only if the series at the right hand side in (4) is convergent. Besides out the convergence of the series (3) in $L^2(\Omega)$ implies its uniform convergence on the compact subsets of Ω. Therefore, the sum $\phi(z)$ belongs to $h^2(\Omega)$ for any á priori given series (3), convergent in $L^2(\Omega)$.

We consider the function

$$\Phi(z) = \int_\Gamma f(\zeta)\omega(\zeta - z, grad\psi), \tag{5}$$

which is holomorphic in the origin. For z being sufficiently close to the origin $|\langle grad\psi, \zeta\rangle| > |\langle grad\psi, z\rangle|$, where $\phi \in \Gamma$. Therefore,

$$\frac{1}{\langle grad\psi, \zeta - z\rangle^n} = \sum_{q_1,\ldots,q_n \geq 0} \frac{(|q| + n - 1)!}{q!(n-1)!} \frac{z^q (grad\psi)^q}{\langle grad\psi, \zeta\rangle^{n+|q|}}$$

and it follows from (5), that in some neighbourhood of the origin

$$\Phi(z) = \frac{1}{(n-1)!} \sum_{q_1,\ldots,q_n \geq 0} c_q z^q = \frac{1}{(n-1)!} \sum_{k=0} P_{k,f}(z),$$

where

$$P_{k,f}(z) = \sum_{\langle p,q\rangle = k} c_q z^q$$

It follows from the results of [7] that the mentioned above in the formulation of Theorem 1 a holomorphic extension exists if and only if 1) is satisfied and the function $\Phi(z)$ extends holomorphically from a neighbourhood of the origin into Ω. Consider a weighted homothety $\Omega_r^p = \{w : w_j = r^{p_j} z_j, where \quad z \in \Omega\}, 0 < r < 1$. For $\Phi \in A(\Omega)$ it is necessary and sufficient that for any r the function $\Phi \in h^2(\Omega_r^p)$, i.e. the series

$$\sum_{k=0}^{\infty} \|P_{k,f}\|^2_{L^2(\Omega_r^p)} = \sum_{k=0}^{\infty} r^{2(k+|p|)} \|P_{k,f}\|^2_{L^2(\Omega)} = r^{2|p|} \sum_{k=0}^{\infty} (r^2)^k a_k$$

converges.

We have proved that the required holomorphic extension of Φ is equivalent to 2). □

One can prove that there exists a basis in $A(\Omega)$ consisting of weighted homogeneous polynomials, due to Gramm-Schmidt procedure (In [1] it was proved for $n = 2$ and was used to establish an isomorphism between $A(\Omega)$ for different Ω). Furthermore, one can use the decomposition of Φ in this basis, but in this case one can obtain the analogue of Theorem 1 which is not much more complicated both in the formulation and in the proof. Thus, in this case it is easier to use the complete system and not the basis.

If now Ω is an n-circular domain (Reinhardt domain), then it is also a $(1,\ldots,1)$-circular domain and Theorem 1 can be applied. In this case $b_{q,s} = 0$, if $q \neq s$, and, consequently,

$$a_k = \sum_{|q|=k} b_{q,q} |c_q|^2. \tag{6}$$

On the other side

$$\lim_{|q|\to\infty} \sqrt[|q|]{\frac{b_{q,q}}{d_q^2(\Omega)}} = 1$$

and the number of terms in the right hand side of (6) is equal to

$$\frac{(k+n-1)!}{k!(n-1)!},$$

therefore in this case condition 2) from Theorem 1 is equivalent to 2) from Theorem B, hence Theorem B is a consequence of Theorem 1. More precisely, both of this theorems are based on the results of [7], but Theorem

1 is true for a wider class of domains, and therefore it includes Theorem B. On the other hand for $n = 1$ Theorem B also implies Theorem A, since in this case

$$c_{q_1} = \frac{1}{2\pi i}\int_\Gamma f(\zeta)\left(\frac{\psi'}{\psi'\zeta}\right)^{q_1}\frac{\psi' d\zeta}{\psi'\zeta} = \frac{1}{2\pi i}A_{q_1},$$

but, of course, this is not the way to prove Theorem A, because the direct proof is easy and elementary. Probably, Theorem 1 (and Theorem B) are the most natural generalization of Theorem A for the several dimensional case.

3. Proof of Theorem 2. Let us denote $D_- = K_1 \setminus \bar{D}_1$, where K_1 is the unit disc in $\mathbf{C}$, and D_1 the domain from Theorem A. We observe that condition 1) in Theorem A means, that the Cauchy type integral

$$F_-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)d\zeta}{\zeta - z}$$

extends holomorphically from the domain $D_- \ni 0$ into K_1 and Theorem A can be generalized as follows (cp. with Corollary 1' from [6])

Theorem A' *Let a simple smooth curve Γ belong to ∂D, $D \subset \mathbf{C}$, and D_- be a one side neighbourhood of Γ, lying out of D (i.e. an open arc $\Gamma \subset \partial D_-$ and $D \cap D_- = \emptyset$). For $f \in C(\Gamma) \cap L^1(\Gamma)$ to be holomorphically extendable in D to a function $F \in A(D) \cap C(D \cup \Gamma)$, $F|_\Gamma = f$ it is necessary and sufficient that the Cauchy-type integral $F_-(z)$ is holomorphically extendable from D_- over Γ in D.*

One can give an equivalent formulation of Theorem C, substituting the condition of holomorphy in z_n in $D \cap \alpha_{'z}$ by the condition of extendability of the Cauchy-type integral from every curve $\Gamma \cap \alpha_{'z}$.

Without loss of generality of the proof we assume that $f \in C(\bar{\Gamma})$, since we could provide the proof for a family of slightly smaller pieces of the hypersurface Γ_ϵ and, respectively, the smaller domains $D(\epsilon)$, for $\epsilon \to 0$. Therefore, we can consider that $f \in C(\bar{\gamma_{'z}})$ for $'z \in M$, where $\gamma_{'z} = \Gamma \cap \alpha_{'z}$. Consider the Cauchy-type integral

$$F_{\pm}(z) = \frac{1}{2\pi i}\int_{\gamma_{'z}} \frac{f('z,\zeta)d\zeta}{\zeta - z_n} \tag{7}$$

and a domain D_- being a one side neighbourhood of Γ, lying at the converse side of D with respect to Γ. Then (cp., e.g. with [9], Appendix 5.4; and [5]) $F_{\pm}$ is holomorphic with respect to all variables in $D_+ = D$ and in D_-, respectively. Now, by means of Theorem A', for fixed $'z \in N$ $F_-('z, z_n)$ extends holomorphically with respect to z_n from $D_- \cap \alpha_{'z}$ into the domain $D \cap \alpha_{'z}$. Let $\tilde{N}$ be a maximal subset of M with the described property of holomorphic extension, $N \subset \tilde{N} \subset M$. For $'z \in \tilde{N}$ denote by $W('z)$ the maximal subdomain of $D \cap \alpha_{'z}$ where the holomorphic extension $F_-('z, z_n)$ is possible, (or one of them, in case of multivalued extension). Let $W = \cup_{'z \in M} W('z)$, and W^0 - open kernel of W. Then, by Hartogs theorem F_- extends holomorphically into W^0. If $D \setminus W^0$ is not empty, then there exists an open set $U \subset M$ such that for all $'z \in U$ the domain $D \cap \alpha_{'z}$ is bigger than the domain $W^0 \cap \alpha_{'z}$. This is also true for a portion of the set N in U. Furthermore, taking [18] into consideration one can show that this portion is pluripolar in $\mathbb{C}^{n-1}_{'z}$, which contradicts with the promise of Theorem 2. Hence, $D = W^0$. Using the generalized formula of Sokhotskiĭ-Plemelj ([3], [6], [5]) we obtain that $F_+(z) \in C(D \cup \Gamma)$ and $F_+ - F_- |_\Gamma = f$, therefore $F_+ - F_-$ gives the required holomorphic extension of f from Γ into D.

One can give an equivalent formulation of Theorem 2, where instead of the condition of holomorphic extension of f from Γ with respect to z_n one can presume the extension of Cauchy-type integral $F_-('z, z_n)$ from $D_- \cap \alpha_{'z}$ into $D \cap \alpha_{'z}$. (See Theorems A and A'.)

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