ON FOURIER-MUKAI TRANSFORM ON THE COMPACT VARIETY OF RULED SURFACES

CRISTINA MARTÍNEZ

Abstract. Let C be a projective irreducible non-singular curve over an algebraic closed field k of characteristic 0. We consider the Jacobian $J(C)$ of C which is a projective abelian variety parametrizing topological trivial line bundles on C. We consider its Brill-Noether loci, which correspond to the varieties of special divisors. The Torelli theorem allows us to recover the curve from its Jacobian as a polarized abelian variety. We approach in the same way the analogous problem for the Quot scheme $Q_{d,r,n}(C)$ of degree d quotients of a trivial vector bundle on C, defining Brill-Noether loci and Abel-Jacobi maps. We define a polarization on the compactification $R_{C,d}$ of the variety of ruled surfaces considered as a Quot scheme and we prove an analogue of the Torelli theorem by applying a Fourier-Mukai transform.

1. Introduction

Let C be a complete nonsingular algebraic curve of genus g over an algebraic closed field k of characteristic 0, and let $D(X)$ denote the bounded derived category of coherent sheaves on a variety X. This is the category obtained by adding morphisms to the homotopic category of bounded complexes of coherent sheaves on X in such a way that any morphism of complexes which induces isomorphism in cohomology becomes an isomorphism.

For smooth projective curves, a derived equivalence always corresponds to an isomorphism. In particular this implies the classical Torelli theorem. If there is an equivalence between the derived categories of two smooth projective curves, then there is an isomorphism between the Jacobians of the curves that preserves the principal polarization, [Ber]. In general, for higher dimensional smooth projective varieties this is not true. The problem of the existence of smooth, projective non birational Calabi-Yau threefolds that have equivalent derived categories, has been studied by A. Caldararu in [Cal], where the author suggests a way to construct such examples of CY threefolds.

Due to a result of Orlov ([Orl]), an equivalence $F : D(X) \to D(X')$ between derived categories of coherent sheaves on smooth projective manifolds X, X' is always of Fourier-Mukai type, that is, there exists a unique (up to isomorphism) object $\mathcal{E} \in D(X \times X')$ such that the functor F is isomorphic to the functor:

$$\Xi_{\mathcal{E}}(-) := q_*(\mathcal{E} \otimes p^*(-)),$$

where p and q are the projections of $X \times X'$ onto X and X' respectively.

Date: November 2007.

2000 *Mathematics Subject Classification.* Primary 14F05; Secondary 14C40.

Key words and phrases. Quot schemes, Torelli Theorem, Fourier Mukai.

In particular, two smooth projective manifolds that have equivalent derived categories of coherent sheaves have isomorphic rational cohomology groups ([Huy]):

$$H^*(X, \mathbb{Q}) \cong H^*(X', \mathbb{Q}).$$

In general, it is of interest to know how much information about a space X can be recovered from Hodge data on X.

Let $M(r,d)$ be the projective non-singular variety of isomorphism classes of stable bundles on C of rank r and degree d. This space is closely related to the curve. Narasimhan and Ramanan proved in [NR] that the canonically polarised intermediate Jacobian of $M(r,d)$ corresponding to the third cohomology group, is naturally isomorphic to the canonically polarised Jacobian of C. It is believed that the theory of motives is an effective language for clearly and precisely expressing how the algebro-geometric properties of the curve influence those of the moduli space of stable vector bundles. S. del Baño studied in [Ba] the motive of the moduli space of rank two stable and odd determinant vector bundles over a curve. Here we show that a smooth projective curve C over k is determined by a certain Quot scheme compactification of the scheme of degree d morphisms from the curve to the Grassmannian $G(2,4)$ with a certain polarization on this Quot scheme. The proof uses the Fourier-Mukai transform along the lines of the Beilinson-Polishchuk proof of the classical Torelli theorem, [BP]. In the genus 0 case, these spaces are considered in [Mar1] as parameter spaces for rational ruled surfaces in order to solve a certain enumerative problem. However the Fourier-Mukai transform is defined on a general Quot scheme parametrizing quotient sheaves of a trivial bundle on C. This method was also applied to Prym varieties by J. C. Naranjo in [Nar].

2. Geometry of the Jacobian and Quot schemes

2.1. **Torelli problem for smooth projective curves.** Let C be a complete non-singular curve of genus g, and k an algebraically closed field. By $Pic\,(C)$ (resp. $Pic^d(C)$) we denote the Picard group of C (resp. the degree d subset within it). The Jacobian $J = J(C)$ of C is an abelian variety such that the group of $k-$points of J is isomorphic to $Pic^0(C)$, (resp. all topological trivial line bundles). Let $j : C \to J(C)$ be the canonical embedding of C into its Jacobian, normalized such that $j(P) = 0$ for some $P \in C$.

For every $d > 0$ we denote by Sym^dC the dth symmetric power of a curve C. By definition, Sym^dC is the quotient of C^d by the action of the symmetric group S_d. We can identify the set of effective divisors of degree d on C with the set of $k-$rational points of the symmetric power Sym^dC, that is, Sym^dC represents the functor of families of effective divisors of degree d on C.

Theorem 2.1. *(Torelli) Let C_1 and C_2 be two smooth projective curves of genus $g > 1$ over k. If there is an isomorphism between the Jacobians $J(C_1)$ and $J(C_2)$ preserving the principal polarization then $C_1 \cong C_2$.*

The subset in $Pic^g(C)$ consisting of line bundles L with $h^0(L) = 1$ corresponds to the set of $k-$points of an open subset in Sym^gC. Translating this subset by various line bundles of degree $-g$ we obtain algebraic charts for $Pic^0(C)$. The Jacobian variety J is constructed by gluing together these open charts. It is a consequence of Torelli's theorem that if

$$Sym^{g-1}C_1 \cong Sym^{g-1}C_2, \quad then\ C_1 \cong C_2.$$

The next theorem states that the same result continues to hold for all $d \geq 1$ with one exception:

Theorem 2.2. [Fak] *Let C_1 and C_2 be two smooth projective curves of genus $g \geq 2$ over an algebraically closed field k. If $Sym^d C_1 \cong Sym^d C_2$ for some $d \geq 1$, then $C_1 \cong C_2$ unless $g = d = 2$.*

It is well known that there exist non-isomorphic curves of genus 2 over $\mathbb{C}$ with isomorphic Jacobians.

2.2. **Varieties of special divisors.** The closed subset $W_d^r \subset J^d = J(C)$ consists of all line bundles L of degree d such that $h^0(L) > r$. One has a canonical scheme structure on W_d^r, since it can be described as the degeneration locus of some morphism of vector bundles on J^d. The subscheme W_{g-1}^0 is exactly the theta divisor $\Theta \subset J^{g-1}$. All theta divisors in the Jacobian are translations of the natural divisor $\Theta \subset J^{g-1}$. We have a canonical involution corresponding to the map $\nu : \Theta \to \Theta$, $L \to K_C \otimes L^{-1}$, where K_C denotes the canonical line bundle over the curve C and L^{-1} is the dual line bundle of L. There is a canonical identification of $Pic^0(J^{g-1})$ with $Pic^0(J) = \widehat{J}$ induced by any standard isomorphism $J \to J^{g-1}$ given by some line bundle of degree $g-1$. The corresponding Fourier transform $\mathcal{F}$ on the derived categories of coherent sheaves on $\widehat{J}$ and J^{g-1} is an equivalence.

Denote by Θ^{ns} the open subset of smooth points of Θ. We can identify Θ^{ns} with an open subset of $Sym^{g-1}C$ consisting of effective divisors D of degree $g-1$, such that $h^0(D) = 1$.

For sufficiently large d, the morphism σ^d

$$\begin{array}{lcl} Sym^d C & \xrightarrow{\sigma} & J^d \\ D & \to & \text{isomorphism class of } \mathcal{O}_C(D) \end{array}$$

is a projective bundle. The fiber of σ^d over L is the variety of effective divisors D such that $\mathcal{O}_C(D) \cong L$. Further, $(\sigma^d)^{-1}(L) \cong \mathbb{P}H^0(C, L)$. Let us identify J^d with J using the line bundle $\mathcal{O}_C(dp)$, where $p \in C$ is a fixed point. Then we can consider σ^d as a morphism $Sym^d C \to J$ sending D to $\mathcal{O}_C(D - dp)$. In more invariant terms, let L be a line bundle of degree $d > 2g-2$ on C. Then the morphism $\sigma_L : Sym^d C \to J$ sending D to $\mathcal{O}_C(D) \otimes L^{-1}$ can be identified with the projective bundle associated with $\mathcal{F}(L)$, the Fourier transform of L.

2.3. **Higher rank divisors.** A divisor of rank r and degree d or (r, d)divisor, will be any coherent sub $\mathcal{O}_C$-module of $k^r = k^{\oplus r}$, having rank r and degree d.

This set can be identified with the set of rational points of an algebraic variety $Div_{C/k}^{r,d}$ which may be described as follows. For any effective ordinary divisor D, set:

$$Div_{C/k}^{r,d}(D) = \{E \in Div_{C,k}^{r,d} | E \subset \mathcal{O}_C(D)^r\},$$

where $\mathcal{O}_C(D)$ is considered as a submodule of k^r.

The set of all $(r,d)-$divisors can be identified with the set of rational points of $Quot^m_{\mathcal{O}_C(D)^r/C/k}$ parametrizing torsion quotients of $\mathcal{O}_C(D)^r$ having degree $m = r\cdot deg\, D - d$. This is a smooth projective irreducible variety. As in the Jacobian case, tensoring by $\mathcal{O}_C(-D)$ defines an isomorphism between $\mathcal{Q}_{r,d}(D) = Quot^m_{\mathcal{O}_C(D)^r/C/k}$ and $Quot^m_{\mathcal{O}_C^r/C/k}$. Since the whole construction is algebraic, it can be performed over any complete normal field, for example, a $p-$adic field.

Let $Q_{d,r,n}(C)$ be the Quot scheme parametrizing rank r coherent sheaf quotients of $\mathcal{O}^n_C$ of degree d. There is a universal exact sequence over $Q_{d,r,n}(C)\times C$:

$$0 \to \mathcal{K} \to \mathcal{O}^n_{Q_{d,r,n}(C)\times C} \to \mathcal{E} \to 0$$

The universal quotient $\mathcal{E}$ is flat over the $Q_{d,r,n}(C)$ Quot scheme, that is, for each $q\in Q_{d,r,n}(C)$, $E_q := \mathcal{E}_{|\{q\}\times C}$ is a coherent sheaf over C and

$$h^0(E_q) - h^1(E_q) = d + 2\,(1-g),$$

is constant by Riemann-Roch, that is, does not depend on q.
By analogy with $Sym^d\, C$, it is natural to define maps

$$v : Quot^m_{\mathcal{O}_C^r/C/k} \to J(C),$$

of Abel-Jacobi type. The geometry of the curve C interacts with the geometry of $Q_{d,r,n}(C)$ and $J(C)$ via these maps.

Proposition 2.3. *For d sufficiently large and coprime with r, there is a morphism from the Quot scheme $Q_{d,r,n}(C)$ to the Jacobian of the curve J^d.*

Proof. Let $\mathcal{U}$ be a universal bundle over $C\times M\,(r,d)$. We consider the projective bundle $\rho : P_{d,r,n}(C) \to M(r,d)$ whose fiber over a stable bundle $[F]\in M(r,d)$ is $\mathbb{P}(H^0(C,F)^{\oplus n})$. We take the degree sufficiently large to ensure that the dimension of $\mathbb{P}(H^0(C,F)^{\oplus n})$ is constant. Globalizing,

$$P_{d,r,n}(C) = \mathbb{P}(\mathcal{U}^{\oplus n}).$$

Alternatively, $P_{d,r,n}(C)$ may be thought of as a fine moduli space for $n-$pairs $(F;\phi_1,\ldots,\phi_n)$ of a stable rank r, degree d bundle F together with a non-zero $n-$tuple of holomorphic sections $\phi = (\phi_1,\ldots,\phi_n) : \mathcal{O}^n \to F$ considered projectively. When ϕ is generically surjective, it defines a point of the Quot scheme $Q_{d,r,n}(C)$,

$$0 \to N \to \mathcal{O}^n \to E \to 0$$

where $N = F^\vee$. The induced map $\varphi : Q_{d,r,n}(C) \to P_{d,r,n}(C)$ is a birational morphism, so that $Q_{d,r,n}(C)$ and $P_{d,r,n}(C)$ coincide on an open subscheme and also the universal structures coincide.

From the universal quotient

$$\mathcal{O}^n_{Q_{d,r,n}(C)\times C} \to \mathcal{E}_{Q_{d,r,n}(C)\times C}$$

for all $q\in Q_{d,r,n}(C)$, we have a surjective morphism

$$\mathcal{O}^n_C \to E \to 0.$$

We now consider the canonical morphism to the Jacobian of the curve:

$$det : M(r,d) \to J^d.$$

Then the composition of the morphisms, $\rho = det \circ \rho \circ \varphi$ which gives a morphism from $Q_{d,r,n}$ to the Jacobian J^d. □

Remark 2.4. *For L a degree d line bundle on C, $\rho^{-1}([L]) \cong \mathbb{P}(H^0(C,L)^{\oplus n})$, where $L = \bigwedge^r F$ for some $[F] \in M(r,d)$. In particular, if $r = 0$ then $\rho^{-1}([L])$ corresponds to the variety of higher rank (n,d)-divisors $E \subseteq \mathcal{O}_C(D)^n$.*

3. The Torelli problem for the $R_{C,d}$–Quot scheme

Let K_C be the canonical bundle over C and π_1, π_2 be the projection maps of $Q_{d,r,n}(C) \times C$ over the first and second factors respectively. Tensorizing the sequence

$$0 \to \mathcal{K} \to \mathcal{O}^n_{Q_{d,r,n}(C)\times C} \to \mathcal{E} \to 0 \text{ over } Q_{d,r,n}(C) \times C$$

with the linear sheaf $\pi_2^*(K_C \otimes L^{-1})$, yields the exact sequence:

(1)
$$0 \to \mathcal{K}\otimes\pi_2^*(K_C\otimes L^{-1}) \to \mathcal{O}^n_{Q_{d,r,n}\times C}\otimes\pi_2^*(K_C\otimes L^{-1}) \to \mathcal{E}\otimes\pi_2^*(K_C\otimes L^{-1}) \to 0$$

Here L is a line bundle of fixed degree. The π_{1*} direct image of the above sequence yields the following long exact sequence on $Q_{d,r,n}(C)$:

$$0 \to \pi_{1*}(\mathcal{K} \otimes \pi_2^*(K_C \otimes L^{-1})) \to \pi_{1*}(\mathcal{O}^n_{Q_{d,r,n}} \otimes \pi_2^*(K_C \otimes L^{-1})) \to$$
$$\to \pi_{1*}(\mathcal{E} \otimes \pi_2^*(K_C \otimes L^{-1})) \to R^1\pi_{1*}(\mathcal{K} \otimes \pi_2^*(K_C \otimes L^{-1})) \to$$
$$\to R^1\pi_{1*}(\mathcal{O}^n_{Q_{d,r,n}} \otimes \pi_2^*(K_C \otimes L^{-1})) \to R^1\pi_{1*}(\mathcal{E} \otimes \pi_2^*(K_C \otimes L^{-1})) \to 0.$$

The universal element $\mathcal{E}$ considered as an object in the derived category $D^b(Q_{d,r,n}(C) \times C)$ of the product, defines an integral transform $\phi_{\mathcal{E}}(-)$ with kernel $\mathcal{E}$ to be the functor $D^b(Q_{d,r,n}(C)) \to D^b(C)$ between the bounded derived categories of coherent sheaves over $Q_{d,r,n}(C)$ and C respectively, given by the formula:

$$\phi_{\mathcal{E}}(-) = R^1\pi_{1*}(\mathcal{E} \otimes \pi_2^*(-)).$$

If such an integral transform is an equivalence, it is called a Fourier-Mukai equivalence.

The functor $\phi_{\mathcal{E}}$ has the left and right adjoint functors $\phi^*_{\mathcal{E}}$ and $\phi^!\mathcal{E}$ defined by the following formulas:

$$\phi^*_{\mathcal{E}}(-) = \pi_{1*}(\mathcal{E}^\vee \otimes \pi_2^*(K_C \otimes (-))),$$

$$\phi^!_{\mathcal{E}}(-) = K_{Q_{d,r,n}(C)}[dim\, Q_{d,r,n}(C)] \otimes \pi_{1*}(\mathcal{E}^\vee \otimes (-))),$$

where $K_{Q_{d,r,n}(C)}$ is the canonical sheaf on $Q_{d,r,n}(C)$, and
$\mathcal{E}^\vee := R^*\mathcal{H}om(\mathcal{E}, \mathcal{O}_{Q_{d,r,n}(C)\times C})$.
Note that the existence of the left adjoint functor immediately implies the existence of the right adjoint functor $\phi^!_{\mathcal{E}}$ by means of the formula:

$$\phi^!_{\mathcal{E}} = S_{R_{C,d}} \circ \phi^*_{\mathcal{E}} \circ S_C^{-1},$$

where $S_{Q_{d,r,n}(C)}$, S_C are the Serre functors on $D^b(Q_{d,r,n}(C))$ and $D^b(C)$.

Remark 3.1. Since $Quot_{d,r,n}(C)$ and C have different dimension, the functor $\phi_{\mathcal{E}}$ cannot be an equivalence between the corresponding derived categories of coherent sheaves.

Remark 3.2. The definition of the Fourier-Mukai transform uses the universal quotient sheaf $\mathcal{E}$ which is defined over the product $Quot_{d,r,n}(C) \times C$. However, since the Fourier-Mukai transform $\phi_{\mathcal{E}}$ pushes forward the corresponding coherent sheaf on the product to $Quot_{d,r,n}(C)$, it can be defined without reference to the curve C. In other words, we can take any $\mathcal{E} \otimes \pi_1^* N$, where N is a line bundle over $Quot_{d,r,n}(C)$, to define a Fourier-Mukai transform on $D^b(Quot_{d,r,n}(C))$.

Recalling the notation of [Mar2], $R_{C,d}$ will be the Quot scheme compactifying the variety of morphisms $\mathrm{Mor}_d(C, G(2,4))$, so that we are fixing the integers r, n to be 2, 4 respectively. The image of a curve C by f is a geometric curve in $G(2,4)$ or equivalently a ruled surface in $\mathbb{P}^3$. For each $f : C \to G(2,4)$ there exists a unique corresponding quotient $\mathcal{O}_C^4 \to f^*\mathcal{Q} \to 0$ in $R_{C,d}$, where $\mathcal{Q}$ is the universal quotient over the Grassmannian. Let us denote by s the Segre invariant s of the bundle $f^*\mathcal{Q}$, which is defined as the maximal degree of $f^*\mathcal{Q}^\vee \otimes L$ having a non-zero section and which satisfies $s \equiv d \ (mod\, 2)$ and $2-2g \leq s \leq g$. Since the universal quotient $\mathcal{E}$ is flat over $R_{C,d}$, for each $q \in R_{C,d}$, $E_q := \mathcal{E}|_{\{q\}\times C}$ is a coherent sheaf over C and $\mathrm{Mor}_d(C, G(2,4))$ sits inside $R_{C,d}$ as the open subscheme of locally free quotients of $\mathcal{O}_C^4$.
Analogously to the case of the Jacobian, we can consider the following Brill-Noether loci associated with a line bundle L of degree $\frac{d+s}{2}$ on C for a fixed integer k.:

$$R^k_{C,d,s} = \{q \in R_{C,d} |\, h^0\,(C, E_q^\vee \otimes L) \geq k,\ deg\, L = \frac{d+s}{2}\} =$$

$$\{q \in R_{C,d} |\, h^1\,(C, E_q \otimes K_C \otimes L^{-1}) \geq k,\ deg\, L = \frac{d+s}{2}\} =$$

$$\{q \in R_{C,d} |\, h^0\,(C, E_q \otimes K_C \otimes L^{-1}) \geq d+3-2g+k,\ deg\, L = \frac{d+s}{2}\}.$$

The subset $R^k_{C,d,s}$ has a canonical scheme structure on $R_{C,d}$, since it can be described as the degeneration locus of some morphism of vector bundles on $R_{C,d}$. These sets are analogous to the varieties of special divisors in the Jacobian of a curve. Note that in the case $k=1$, this scheme corresponds exactly to the points $q \in R_{C,d}$ such that $f_q^*\mathcal{Q}$ has Segre invariant s, (see [Mar2]), and when s takes the value $2\,(g-1)$, this defines a codimension one locus inside $R_{C,d}$ which we will later take as a polarization for $R_{C,d}$.

3.1. **Tangent spaces.** Let $0 \to N_q \to \mathcal{O}_C^4 \to E_q \to 0$ be the quotient represented by a point $q \in Q_{d,r,n}(C)$. We consider the tangent space to that point,

$$\mathcal{T}_q Q_{d,r,n}(C) \cong Hom\,(N_q, E_q) \cong H^0(N_q^* \otimes E_q).$$

If $H^1(C, N_q^* \otimes E_q) \cong Ext^1(N_q, E_q)$ is trivial, then q is a smooth point in $Q_{d,r,n}(C)$. In that case, we compute using the Riemann-Roch theorem the dimension of $T_q Q_{d,r,n}(C)$ to be $deg\,(N_q^* \otimes E_q) + r \cdot (1-g)$.

Lemma 3.3. *The singular locus of $R^k_{C,d,s}$ is exactly equal to $R^{k+1}_{C,d,s}$.*

Proof. Since the schemes $R^k_{C,d,s}$ are determinantal varieties, locally there exists a morphism from $R^k_{C,d,s}$ to the variety of matrices $M_k(n,m)$ of rank less or equal than k, such that $R^k_{C,d,s}$ is the pull-back of the variety of matrices $M_k(n,m)$ of rank equal or less than k. Then the result follows from the analogous statement for the tangent space at $M_k(n,m)$. □

Lemma 3.4. *For d sufficiently large depending on s, the expected codimension of $R^k_{C,d,s}$ as a determinantal variety is $(2g-s-2+k)\cdot k$.*

Proof. $R^k_{C,d,s}$ is exactly the locus of degeneration of the morphism of vector bundles:

$$\phi : R^1\pi_{1*}(\mathcal{K} \otimes \pi_2^*(L^{-1} \otimes K_C)) \to R^1\pi_{1*}(\mathcal{O}^n_{R_{C,d}\times C} \otimes \pi_2^*(L^{-1} \otimes K_C)),$$

where L is a line bundle of degree $\frac{d+s}{2}$ and $d \equiv s \; mod\; 2$, (see Theorem 3.2 of [Mar2]). Then by the theory of determinantal varieties, a simple dimension computation gives the result. □

3.2. **A polarization and Torelli-type result for the variety $R_{C,d}$.** Let $\mathcal{F}$ be a flat family of coherent sheaves on a relative smooth projective curve $\pi : \mathcal{C} \to S$, such that for each member of the family $\chi(\mathcal{C}_s, \mathcal{F}_s) = 0$. We associate to $\mathcal{F}$ a line bundle $det^{-1}R\pi_*(\mathcal{F})$ (up to isomorphism) equipped with a section $\theta_{\mathcal{F}}$. We define the theta line bundle on J associated with a line bundle L of degree $g-1$ on C by applying this construction to the family $p_1^*L \otimes \mathcal{P}$ on $C \times J$, where $\mathcal{P}$ is the Poincare line bundle. Zeroes of the corresponding theta function θ_L constitute the theta divisor $\Theta_L = \{\xi \in J : h^0(L(\xi)) > 0\}$, which gives rise to a polarization for the Jacobian J. This means that isomorphism classes of theta line bundles have the form $det^{-1}\mathcal{S}(L)$, where $\mathcal{S}(L)$ is the Fourier-Mukai transform of a line bundle L of degree $g-1$ considered as a coherent sheaf on the dual Jacobian $\widehat{J}$ supported on C.

Beilison and Polishchuk give a proof of the Torelli theorem for the Jacobian J of a curve in [BP], based on the observation that the Fourier-Mukai transform of a line bundle of degree $g-1$ on C, is a coherent sheaf (up to shift), supported on the corresponding theta divisor in J. Here we present an analogue of the Torelli theorem for the variety $R_{C,d}$ of ruled surfaces, defining a polarization or theta divisor of $R_{C,d}$. Let C be an algebraic curve of genus $g \geq 2$ and define the map $\nu : Pic^{\frac{d+s}{2}}(C) \to Pic^{\frac{g-d-s-1}{2}}(C)$ by $\nu(L) = K_C \otimes L^{-1}$.

Theorem 3.5. *For $d > 2\,(g-1)$, and $L \in Pic^{\frac{d+s}{2}}(C)$, the Fourier-Mukai transform with kernel $\mathcal{E}$ of the line bundle $\nu(L)$ on C is a coherent sheaf F supported on the divisor $R^1_{C,d,2(g-1)}$ in $R_{C,d}$. Moreover, the restriction of F to the non-singular part of this divisor (understood as a polarization for $R_{C,d}$) is a line bundle and F can be recovered from this line bundle.*

Proof. For L a line bundle of degree $\frac{d+s}{2}$ on C ($d \equiv s \bmod 2$), let us consider the map $\nu : L \to K_C \otimes L^{-1}$ and the Fourier Mukai transform $\phi_{\mathcal{E}}(\nu(L))$, which is a coherent sheaf F supported on the divisor $R^1_{C,d,2(g-1)}$ in $R_{C,d}$, that is, on the locus of points $q \in R_{C,d}$ such that $h^0\left(C, E_q^{\vee} \otimes L\right) \geq 1$, or dually $h^1(C, E_q \otimes K_C \otimes L^{-1}) \geq 1$. Furthermore, F is the derived pushforward of $\mathcal{E} \otimes \pi_2^*(K_C \otimes L^{-1})$, so that we can represent it as the cone of the morphism of vector bundles on $R_{C,d}$, [Mar2]:

(2)
$$\phi : \mathcal{L}_0 := R^1\pi_{1*}(\mathcal{K} \otimes \pi_2^*(L^{-1} \otimes K_C)) \to \mathcal{L}_1 := R^1\pi_{1*}(\mathcal{O}^n_{R_{C,d}\times C} \otimes \pi_2^*(L^{-1} \otimes K_C)),$$

that is, by the complex $V_{\cdot} = [\mathcal{L}_0 \to \mathcal{L}_1]$ in $D^b(R_{C,d})$.
Since ϕ is an isomorphism outside of $R^1_{C,d,2(g-1)}$, it is injective and $F = coker\phi$. Moreover, when $s = 2\,(g-1)$, $R^1_{C,d,2(g-1)}$ is a divisor and it is a polarization for $R_{C,d}$. We see that

$$det\phi = det\, R^1\pi_{1*}(\mathcal{E} \otimes \pi_2^*(K_C \otimes L^{-1})) =$$

$$det\left(R^1\pi_{1*}(\mathcal{K} \otimes \pi_2^*(L^{-1} \otimes K_C))\right) \otimes det\left(R^1\pi_{1*}(\mathcal{O}^n_{R_{C,d}\times C} \otimes \pi_2^*(L^{-1} \otimes K_C)\right)^{-1}.$$

The line bundle $det(\phi_{\mathcal{E}}(\nu(L))^{-1}$ on $R_{C,d}$ has a canonical (up to non-zero scalar) global section θ_ϕ (canonically up to isomorphism). The zeroes of $\theta_\phi \in H^0(R_{C,d}, det(\phi_{\mathcal{E}}(L))^{-1})$ constitute a divisor which is supported on $R^1_{C,d,2(g-1)}$, that is, $det\, F$ is an equation of $R^1_{C,d,2(g-2)}$.
If we change the resolution $\mathcal{L}_0 \to \mathcal{L}_1$ for F, the pair $(det\, F, \theta_\phi)$ gets replaced by an isomorphic one.
There is a natural locally closed embedding $i : \left(R^1_{C,d,2(g-1)}\right)^{ns} \hookrightarrow R_{C,d}$.

Next we prove that F is the pushforward $F = i^{ns}_* M$ by i of a line bundle M on the non singular part of $R^1_{C,d,2(g-1)}$.
The singular locus of $R^1_{C,d,2\,(g-1)}$ is $R^2_{C,d,2\,(g-1)}$ and by Lemma 3.4 it has codimension in $R_{C,d}$ greater than 2 so that $M := i^{ns*}F$.
Let θ^{ns}_ϕ be the set $R_{C,d,2(g-1)}/R^1_{C,d,2(g-1)}$. This set corresponds to the non-singular part of θ_ϕ by Lemma 3.3. The restriction of F to θ^{ns}_ϕ is a line bundle from which F can be recovered by taking the push-forward with respect to the induced embedding $\theta^{ns}_\phi \hookrightarrow \theta_\phi$. By the base change theorem for a flat morphism:

$$Li^{ns*}F \cong R\pi_{2*}(\mathcal{E} \otimes \pi_2^*(K_C \otimes L^{-1}))|_{C\times\theta^{ns}_\phi}.$$

Since $h^0(C, Eq \otimes K_C \otimes L^{-1}) = 1$ for every $q \in \theta^{ns}_\phi$, by applying the base change theorem again, we deduce that

$$rk\, M|_q = 1 \text{ for every } q \in \theta^{ns}_\phi.$$

Since θ^{ns}_ϕ is reduced, M is a line bundle on θ^{ns}.

We can characterize the set $\mathcal{M}$ of all line bundles on θ^{ns}_ϕ in terms of $(R_{C,d}, \theta_\phi)$. The set $\mathcal{M}$ has two properties:

(1) For every $M \in \mathcal{M}$, $M \otimes \nu^* M \cong K_{\theta^{ns}}$, where ν is the map $L \to K_C \otimes L^{-1}$.

(2) The class of $\mathcal{M}$ generates the cokernel of the map

$$Pic(R_{C,d}) \to Pic(\theta_\phi^{ns}). \tag{3}$$

Since the Picard variety has the structure of a polarized abelian variety, the morphism (3) is a homomorphism of abelian groups induced by the inclusion $i : \theta_\phi^{ns} \hookrightarrow R_{C,d}$. Its cokernel $coker(Pic\,(R_{C,d}) \to Pic(\theta_\phi^{ns}))$ is a group. Moreover it is isomorphic to $\mathbb{Z}$ for d sufficiently large, (Prop. 2.3).

Thus to recover the curve C from $(R_{C,d}, \theta_\phi)$, by the involutive property of the Fourier-Mukai transform $\phi_{\mathcal{E}}$, we can recover the curve C by taking a line bundle M on $\mathcal{M}$, extending it to $\theta_\phi (= R^1_{C,d,2(g-1)})$ by taking the push-forward, and then apply Fourier-Mukai transform.

Now we show that $M \in \mathcal{M}$. First, we apply duality theory to the projection $\pi_1 : R_{C,d} \times C \to R_{C,d}$ to prove that

$$R\underline{Hom}(F, \mathcal{O}_{R_{C,d}}) \cong \nu^* F[-1], \tag{4}$$

where $\nu : Pic(C) \to Pic(C)$ corresponds to the involution $L \to L^{-1} \otimes K_C$. Applying the functor Li^{ns*} to the isomorphism 4, we obtain

$$R\underline{Hom}(Li^{ns*}F, \mathcal{O}_{\theta^{ns}}) \cong \nu^* Li^{ns*} F[-1]. \tag{5}$$

Since $Li^{ns*}F$ has locally free cohomology sheaves, this implies that $M^{-1} \cong \nu^* Li^{ns*} F[-1]$. But

$$L^{-1} i^{ns*} F \cong L^{-1} i^{ns*} i_*^{ns} M \cong M \otimes \mathcal{O}_{\theta^{ns}}(-\theta),$$

and that $\nu^* M^{-1} \cong M(-\theta)$, which proves condition (1) of $\mathcal{M}$.
In order to prove the second condition, we consider the universal quotient $\mathcal{E}|_{\{p\} \times R^1_{C,d,2\,(g-2)}}$ restricted to $\{p\} \times R^1_{C,d,2\,(g-2)}$. The line bundle M^{-1} on

$$\theta_\phi^{ns} = \{q \in R_{C,d} | h^0(C, E_q^\vee \otimes L) = 1,\ L \in Pic^{\frac{d+s}{2}}(C)\}$$

is isomorphic to $i_*^{ns} p_{2*}(\mathcal{O}(R_{C,d,2\,(g-2)}))(-R_p)$, where $p_2 : C \times R_{C,d} \to R_{C,d}$, i^{ns} is the embedding $i^{ns} : \theta_\phi^{ns} \hookrightarrow R_{C,d}$ and $R_p := R_{C,d,2\,(g-2)} \cap p \times R_{C,d}$. Therefore $M^{-1} \cong \alpha^*(\mathcal{O}_{R_{C,d}}(-R_p))$ which generates the cokernel of the map $Pic\,(R_{C,d}) \to Pic\,(\theta_\phi^{ns})$. ☐

Corollary 3.6. *Given two smooth projective curves C_1 and C_2, if there exist an isomorphism*

$$f : (R_{C_1,d}, \theta_1) \xrightarrow{\sim} (R_{C_2,d}, \theta_2)$$

of polarized Quot-schemes, then $C_1 \simeq C_2$.

Proof. By Theorem 3.5, the restriction of $F = \phi_{\mathcal{E}}(\nu(L))$ to the non-singular part of θ_i ($L \in Pic^{\frac{d+s}{2}}$), is a line bundle $i_*^{ns} M$ and F can be recovered from this line bundle since $M := i^{ns*}F$. Therefore

$$f|_{supp(i_{1*}^{ns})} : supp(i_{1*}^{ns} M_1) \xrightarrow{\sim} supp(i_{2*}^{ns} M_2),$$

and $C_1 \cong C_2$, where $i_1 : \theta_1^{ns} \hookrightarrow R_{C,d}$, and $i_2 : \theta_2^{ns} \hookrightarrow R_{C,d}$. ☐

Acknowledgments. I would like to thank Professor Yuri Manin for introducing me to the world of the derived categories. I thank Oren Ben-Bassat for explanations that helped me clarify some points, and So Okada for comments. I thank Guillaume Thèret for discussions to try to link our different approaches to geometry. Finally I thank Prof. Joergen Andersen for suggesting me the reference [NR]. This work has been partially supported by a Spanish Postdoctoral Fellowship of the Ministry of Science and Culture.

References

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris, *Geometry of Algebraic Curves, Springer-Verlag, New York 1985.*

[At] M. F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc. (3) 7 (1957), 414-452.

[Ba] S. del Baño, *On the Motive of Moduli Spaces of Rank Two Vector Bundles over a Cruve*, Comp. Mathematica **131**:1-30, 2002.

[Ber] M. Bernardara *Fourier-Mukai transforms of curves and principal polarizations*, C. R., Math., Acd. Sci. Paris 245, No. 4, 203-208 (2007).

[BDW] A. Bertram, G. Daskalopoulos, R. Wentworth, *Gromov-Witten invariants for holomorphic maps from Riemann surfaces to Grassmannians*, J. Amer. Math. Soc. 9 (1996), n. 2, 529-571.

[BGN] L. Brambila, I. Grzegorczyk, P.E. Newstead, *Geography of Brill-Noether loci for small slopes*, J. Algebraic Geometry, 6 (1997), no. 4, 645-669.

[BGL] E. Bifet, F. Ghione and M. Leticia, *On the Abel-Jacobi map for divisors of higher rank on a curve*, Math. Ann. 299 (1994), no. 4, 641-672.

[BP] A. Beilinson, A. Polishchuk, *Torelli Theorem via Fourier-Mukai transform*, Moduli of abelian varieties (Texel Island), Progr. Math. 195, Birkhäuser, Basel (1999), 127-132.

[Cal] A. Caldararu, *Non-birational Calabi-Yau threefolds that are derived equivalent*, math.AG/0601232.

[Fak] N. Fakhruddin, *Torelli's theorem for high degree symmetric products of curves*, math.AG/0208180.

[GLO] V. Golyshev, V. Lunts, and D. Orlov, *Mirror Symmetry for abelian varieties*, math.AG/9812003.

[Huy] D. Huybrechts, *Fourier-Mukai transforms in Algebraic Geometry*, Oxford Math. Monographs (2006).

[Mar1] C. Martínez, *The degree of the variety of rational ruled surfaces and Gromov-Witten invariants*, Trans. of AMS. **358**, 11-24 (2006).

[Mar2] C. Martinez, *On the cohomology of Brill-Noether loci over Quot schemes*, MPIM preprint series 2006 (33).

[Nar] J. C. Naranjo, *Fourier transform and Prym varieties*, J. Reine Angew Math. 560 (2003), 221-230.

[NR] M. S. Narasimhan, S. Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve*, Ann. Math. (2) 101 (1975), 391-417.

[Orl] D. Orlov, *Equivalences of derived categories and K3 surfaces*, math.AG/9606006.

[Po] A. Polishchuk, *Abelian Varieties, Theta Functions and the Fourier Transform*, Cambridge University Press.

Institut for Matematike Fag (CTQM), Aarhus Universitet, Ny Munkegade, 8000 Aarhus, Denmark

E-mail address: cmartine@imf.au.dk

Max Planck Institute for Mathematics, Vivatgasse 7, Bonn, 53111.
E-mail address: cmartine@mpim-bonn.mpg.de