Running head: Spectral triples for Riemann surfaces 28th December 2007
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Zeta functions that hear the shape of a Riemann surface

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Abstract

To a compact hyperbolic Riemann surface, we associate a finitely summable spectral triple whose underlying topological space is the limit set of a corresponding Schottky group, and whose "Riemannian" aspect (Hilbert space and Dirac operator) encode the boundary action through its Patterson-Sullivan measure. We prove that the ergodic rigidity theorem for this boundary action implies that the zeta functions of the spectral triple suffice to characterize the (anti-)complex isomorphism class of the corresponding Riemann surface. Thus, you can hear the complex analytic shape of a Riemann surface, by listening to a suitable spectral triple.

Introduction

Let X denote a compact Riemann surface of genus $g \geq 2$. By the retrosection theorem of Koebe-Courant (e.g., [3], p. 317) X can be represented by a Schottky group Γ: we can write

$$X = \Gamma \backslash (\mathbf{P}^1(\mathbf{C}) - \Lambda_\Gamma),$$

where $\Lambda = \Lambda_\Gamma$ is the set of limit points of Γ and Γ is a free group of rank g, discrete in $\mathrm{PGL}(2, \mathbf{C})$.

Notice that, while the abstract group structure of Γ depends only on the genus g (i.e., the topology) of X, the way the Schottky group Γ is embedded in $\mathrm{PGL}(2, \mathbf{C})$ determines the complex structure on X through the Schottky uniformization. The limit set $\Lambda_\Gamma \subset \mathbf{P}^1(\mathbf{C})$ is in general not contained in $\mathbf{P}^1(\mathbf{R})$, except in the case of Fuchsian Schottky groups.

In complex analysis, it is well known that the dynamics of the action of Γ on the limit set endowed with Patterson-Sullivan measure encodes a lot about the structure of the

Riemann surface. Our purpose is to show that this action can be conveniently encoded by a notion from non-commutative geometry, namely a *spectral triple* ([11]) which provides the non-commutative analogue of a Riemannian manifold. As it will turn out, the spectral triple we will consider is commutative. But, as has been observed frequently, "even for classical spaces, which correspond to commutative algebras, the new point of view [of noncommutative geometry] will give new tools and results." (Connes, [10], p. 1). We show that the isometry class of the boundary action is encoded in the *zeta function* formalism of the spectral triple.

Construction of the spectral triple At the topological level, we consider the commutative unital C^*-algebra $A = C(\Lambda)$.

It might be natural to consider the boundary operator algebra $C(\Lambda) \rtimes \Gamma$ instead. However, by the non-amenability of the group Γ, the hyperfiniteness result of Connes implies that this algebra does not carry any finitely summable spectral triple ([9], Thms. 17 & 19, pp. 214–215). As we will indicate at the end of the proof of the main theorem, our construction can be extented to an AF-algebra that is Morita equivalent to a large subalgebra of the boundary operator algebra.

To retrieve the actual complex structure, we need to make the operator algebra act on a Hilbert space in a way compatible with a Dirac operator. The Hilbert space H is a particular GNS-representation of A. Its construction depends on chosing a state, and we make this choice in such a way that it encodes the metric action of Γ on Λ, expressed via the Patterson-Sullivan measure. More specifically, on certain elements of A related to words in a presentation of Γ, it gives the measure of the subset of Λ reached from that word in the representation of the limit set of Γ via word group completion (Floyd [18]). Finally, the Dirac operator D is composed from projection operators depending on the word length grading in a presentation for Γ. Let $\mathcal{S}_X$ denote the spectral triple so constructed (see Section 1 for details).

Proposition 1. *If X is a compact Riemann surface of genus at least 2, $\mathcal{S}_X$ is a 1-summable spectral triple.*

Zeta function rigidity The theory of finitely summable spectral triples comes with an elegant framework of zeta functions. Let $A_\infty = C(\Lambda, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C}$ denote the dense involutive subalgebra of $A = C(\Lambda)$ consisting of locally constant functions on Λ. For any $a \in A_\infty$, one has the spectral zeta function $\zeta_{X,a}(s) := \mathrm{tr}(a|D|^s)$.

Theorem 2. *Suppose X_1 and X_2 are compact Riemann surfaces of respective genera g_1 and g_2, assumed both to be at least 2. Let $\mathcal{S}_i = (A_i, H_i, D_i)\,(i = 1, 2)$ denote the corresponding spectral triples as constructed above.*

(i) *If*

$$\zeta_{X_1,1_{A_1}}(s) = \zeta_{X_2,1_{A_2}}(s)$$

for the respective units 1_{A_i} of A_i, then $g_1 = g_2$ and X_1 and X_2 are homeomorphic.

(ii) *If $g_1 = g_2$, there exists a C^*-algebra isomorphism $\iota : A_1 \cong A_2$ such that, if*

$$\zeta_{X_1,a_1}(s) = \zeta_{X_2,\iota(a_1)}(s)$$

for all $a_1 \in (A_1)_\infty$, then X_1 and X_2 are complex analytically equivalent or anti-equivalent (i.e., equivalent after complex conjugation) as Riemann surfaces.

In particular, the spectral triple $\mathcal{S}_X$ encodes the (anti-)complex analytic isomorphism type of X.

To ease notation, we will henceforth suppress any reference to the isomorphism ι and identify the algebras A_1 and A_2 when they are isomorphic.

About the proof: by the analogue of Fenchel-Nielsen theory (cf. Tukia [35]), the abstract isomorphism of Schottky groups of X_1 and X_2 induces a unique homeomorphism of limit sets, equivariant with respect to the group isomorphism (the *boundary map*). We show that equality of zeta functions implies that this boundary map is absolutely continuous. For this, one has to trace through the representation of the limit set in the sense of Floyd ([18]) to deduce fairly explicit expressions for various zeta functions. We do this by computing traces in an explicit orthonormal basis for H. The first part of the theorem follows from this explicit computation of $\zeta_{X,1}(s)$ as

$$\zeta_{X,1}(s) = 1 + \frac{2g-2}{2g-1} \cdot \frac{(2g)^{3s+1}}{1-(2g-1)^{3s+1}}, \tag{1}$$

cf. Lemma 2.6.

For the second part, we apply an ergodic rigidity theorem for Schottky uniformization that we deduce from a theorem of Yue (cf. [39] p. 79; from the long history we also mention the names Mostow, Kuusalo, Bowen, Sullivan, Tukia). This says that there are only two alternatives for the boundary map: either the Patterson-Sullivan measures are mutually singular with respect to the boundary map, or the map extends to a continuous automorphism of $\mathrm{PGL}(2, \mathbf{C})$. Absolute continuity excludes the first case.

We hope that the concrete examples considered in this paper will help in understanding the more general question of what type of geometric information is encoded in the zeta functions of a spectral triple, which are one of the main invariants associated to a (metric) noncommutative geometry. One of our sources of inspiration is the work of Consani and Marcolli ([13], [14]) on θ-summable spectral triples whose Ray-Singer regularized zeta functions reduce to classical zeta functions, and the fact that one can distinguish the isomorphism type of certain fake projective planes by associated spectral triples ([15], §5.1).

We will end the paper with a list of open questions.

Can you hear the shape of a Riemann surface? Theorem 2 fits into the framework of isospectrality questions, as coined by I.M. Gelfand for compact Riemannian manifolds. Vignéras ([37], [38]) and Sunada ([34]) constructed non-isometric surfaces with identical Laplace operator zeta function $\mathrm{tr}(\Delta^s)$ ('isospectral'). The work of Sunada, in particular, transports an idea from algebraic number theory due to Gassmann ([19]), where the same phenomenon is visible: there exist non-isomorphic algebraic number fields with identical Dedekind zeta function.

For the specific case of Riemann surfaces, Buser ([7], Thm. 13.1.1, p. 340) obtained a (finite) upper bound on the number of Riemann surfaces isospectral with a given Riemann surface, depending only on the genus of that surface, and work of Brooks, Gornet and Gustafson ([6]) shows this bound is of the correct order of magnitude.

As for Dirac operators instead of Laplace operators, Bär has constructed non-isometric space forms with the same Dirac spectrum ([1], Thm. 5).

For the case of planar domains, the problem of isospectrality was coined by Bochner, to quote Kac — quoting Lipman Bers — "can you hear the shape of a drum?" ([24], solved by Gordon, Webb and Wolpert [21]).

In this phrasing, Theorem 2 says you can hear the complex analytic type of a compact Riemann surface from listening to the noncommutative spectra of its associated spectral triple (that is, to the collection of the associated zeta functions). A main difference with respect to the classical isospectrality question is that in this case you do have to listen to $\mathrm{tr}(aD^s)$ for a dense subset of operators $a \in A$, and the eigenvalues of D themselves are not so interesting. For example, at the unit $1 \in A$, we find the innocent zeta function quoted above in (1).

We note that in the completely different construction of a "conformal" spectral triple by Bär, the eigenvalues themselves are uninteresting, too ([2]).

Thus, the main point of our discussion is that the "spectral object" $\mathcal{S}_X$ determines the "conformal object" X, up to complex conjugation.

1. A spectral triple associated to a Kleinian Schottky group

1.1. The aim of this section is to introduce a finitely summable spectral triple $\mathcal{S}_X := (A, H, D)$ associated to a Schottky group Γ that uniformizes a compact Riemann surface X of genus $g \geq 2$. Recall that a spectral triple is a noncommutative analogue of a Riemannian spin manifold, where A is a C^*-algebra, H is a Hilbert space on which A acts by bounded operators, and D is an unbounded self adjoint operator on H with compact resolvent $(D-z)^{-1}$ for $z \notin \mathbf{R}$, and such that the commutators $[D, a]$ are bounded operators for all a in a dense involutive subalgebra A_∞ of A. Connes has shown [10], p. 544 that if (A, H, D) arises from a Riemannian spin manifold, then the distance element is encoded by the inverse of the Dirac operator. A spectral triple as above is p-summable for some $p \in \mathbf{R}_+^*$ if $\mathrm{tr}(|D|^{-p}) < \infty$. It is finitely summable if it is p-summable for some p. On the other hand, it is called θ-summable if it is not finitely summable, but it satisfies the condition $\mathrm{tr}(e^{-tD^2}) < \infty$ for $t > 0$.

1.2. Let Γ denote a Schottky group of rank $g \geq 2$. As an abstract group, Γ is isomorphic to F_g, the free group on g generators. We think of Γ as being specified by an injective group homomorphism $\rho : F_g \hookrightarrow \mathrm{PGL}(2, \mathbf{C})$. Let $\Lambda = \Lambda_\Gamma$ denote the limit set of the action of Γ on $\mathbf{P}^1(\mathbf{C})$.

1.3 (Group completion and limit set). We recall what we need from Floyd's relation between the group completion of F_g and the limit set Λ of Γ ([18]). Let Y_g denote the Cayley graph (with unordered edges) of F_g for a presentation of F_g in a fixed alphabet on g letters, and let $\bar{Y}_g$ denote the completion of the Cayley graph as a metric space for the following metric. Let $|w|$ denote the reduced word length of a word in the generators of F_g. The edge between two words w_1 and w_2 is given length $\min\{|w_1|^{-2}, |w_2|^{-2}\}$ (with $|e|^{-2} := 1$ for the empty word e). The *group completion* of F_g is by definition the space $\bar{F}_g := \bar{Y}_g - Y_g$. It is a compact metric space. A different (finite) presentation for F_g leads to a Lipshitz equivalent group completion. Since F_g has no "parabolic ends" in the sense of Floyd, we have the following:

1.4 Lemma (Floyd, [18], p. 213–217). *Given a point $x_0 \in \mathbf{P}^1(\mathbf{C})$ and an embedding $\rho : F_g \hookrightarrow \mathrm{PGL}(2, \mathbf{C})$ as above, the following map is a continuous bijection:*

$$\begin{array}{cccc} \iota_\rho : & \bar{F}_g & \to & \Lambda \\ & \lim_i w_i & \mapsto & \lim_i \rho(w_i)(x_0). \end{array} \qquad \square$$

Recall that a word w in the generators of F_g is reduced if no two consecutive letters are each others inverses. The following definition of inclusion describes the simple operation of extending words to longer words, without introducing cancellations.

1.5 Definition. Given a reduced word w in the generators of F_g, let $i(w)$ respectively $t(w)$ denote the initial, respectively terminal letter of w. For two reduced words w and v (or v a limit of such), we write

$$w \subseteq v \text{ if } (\exists w_0)(v = w \cdot w_0) \text{ with } t(w) \neq i(w_0)^{-1}.$$

We write $w \subset v$ if $w \subseteq v$ and $w \neq v$. For example, in the alphabet $\{a, b\}$, $a \subset ab$ but $a \not\subset b$ although $a = b \cdot (b^{-1}a)$.

Given $\rho : F_g \to \mathrm{PGL}(2, \mathbf{C})$ and a word $w \in F_g$, define the subset of Λ of *ends of w with respect to ρ* to be

$$\overrightarrow{w}_\rho := \{\iota_\rho(v) \ : \ v \in \bar{F}_g \text{ and } w \subseteq v\}.$$

1.6 Lemma. *$A = C(\Lambda)$ is the closure of the span of the characteristic functions $\chi_{\overrightarrow{w}_\rho}$ of the sets $\overrightarrow{w}_\rho$ for $w \in F_g$.*

Proof. This is immediate, since Λ is a totally disconnected compact Hausdorff space, and the sets $\overrightarrow{w}$ form a basis of clopen sets for its topology. $\square$

We denote by A_∞ the dense involutive subalgebra of A spanned by the characteristic functions $\chi_{\overrightarrow{w}_\rho}$.

1.7 Definition. Let μ_Λ denote the Patterson-Sullivan measure on Λ (cf. [29], [32]). Its main property is scaling by the Hausdorff dimension δ_H of Λ:

$$(\gamma^* d\mu)(x) = |\gamma'(x)|^{\delta_H}\, d\mu(x), \quad \forall \gamma \in \Gamma.$$

We define a state $\tau : A_\infty \to \mathbf{R}$ by

$$\tau(\chi_{\overrightarrow{w}_\rho}) := \int_\Lambda \chi_{\overrightarrow{w}_\rho} \mathrm{d}\mu_\Lambda = \mu_\Lambda(\overrightarrow{w}_\rho).$$

The above lemma shows that τ extends uniquely to a state on A. We define the Hilbert space H to be the GNS-representation of A arising from this state τ, that is, the completion of A/I with respect to the inner product $\langle a|b\rangle := \tau(b^* a)$, where I is the linear subspace of elements $a \in A$ with $\tau(a^* a) = 0$.

1.8 Definition. We now take our inspiration from the construction of Christensen and Ivan in [8]. The subalgebra A_∞ of $A = C(\Lambda)$ is a limit of finite dimensional subspaces $A_\infty = \varinjlim A_n$ with A_n the span of the characteristic functions of sets of ends of reduced words of length $\leq n$. This filtration is inherited by H. We denote by H_n the term of the filtration of H corresponding to A_n, that is, $H_n = \eta(A_n)$, where $\eta : A \to H$ is the linear map defined by the projection $A \to A/I$.

We let P_n denote the orthogonal projection operator onto H_n. We define the Dirac operator to be

$$D := \lambda_0 P_0 + \sum_{n\geq 1} \lambda_n (P_n - P_{n-1}),$$

where $\lambda_n = (\dim A_n)^3$. Note that $Q_n := P_n - P_{n-1}$ is the projection onto the graded pieces, identified with the orthogonal complements $H_n \ominus H_{n-1}$, which correspond to words of exact length n. The choice of λ_n arises from the fact that we then arrive at 1-summability (Proposition 1):

1.9 Proposition. *The triple $\mathcal{S}_X = (A, H, D)$ is a 1-summable spectral triple.*

Proof. The $*$-operation is complex conjugation, and since D is real, it is self-adjoint. For $a \in A_n$ and for any $m > n$, multiplication by a maps A_{m-1} and A_m into itself. Therefore, a commutes with the projections P_m and P_{m-1} and so $[Q_m, a] = 0$. Hence

$$[D, a] = \sum_{i=0}^{n} \lambda_i [Q_i, a]$$

is a finite sum (we set $P_{-1} = 0$ for convenience). Thus, the commutators of D with elements in the dense subalgebra A_∞ of A are bounded.

Moreover, one has $\dim A_n \geq n+1$, hence the 1-summability (and compact resolvent):

$$\mathrm{tr}((1+D^2)^{-1/2}) = 1 + \sum_{n=1}^{\infty}(1+\lambda_n^2)^{-1/2}(\dim H_n - \dim H_{n-1})$$

$$\leq 1 + \sum_{n=1}^{\infty}(1+\lambda_n^2)^{-1/2}\dim H_n \leq 1 + \sum_{n=1}^{\infty}(1+\lambda_n^2)^{-1/2}\dim A_n$$

$$\leq 1 + \sum_{n=1}^{\infty}(\dim A_n)^{-2} \leq 1 + \sum_{n=1}^{\infty}(n+1)^{-2} \leq 2,$$

where we used $\lambda_n = (\dim A_n)^3$ in the second-to-last inequality. This proves the proposition. □

1.10 Remark. A recent deep theorem of Rennie and Várilly ([30], Thm. 7.5) allows one to decide whether a given spectral triple is associated to an actual commutative Riemannian spin manifold. For the purpose of this paper, since we are mostly interested in the zeta functions, we do not consider any additional structure on the spectral triple. In particular, our Dirac operator is only considered up to sign, since the sign does not play a role in the zeta functions, while for [30], the sign provides the essential information on the K-homology fundamental class. It is possible that our construction may be refined to incorporate the further necessary properties of an abelian spectral triple to which the reconstruction theorem can be applied. In that case, it seems that the underlying metric geometry should probably relate to the existence of quasi-circles of limit sets of Schottky groups as in [5] — see also the next remark.

1.11 Remark. Notice that our construction provides a 1-summable spectral triple on the limit set, regardless of the actual value of its Hausdorff dimension (which can be greater than one). Thus, the metric dimension seen from this construction will be in general different from the actual metric dimension of the limit set embedded in $\mathbf{P}^1(\mathbf{C})$. It would be interesting to see if the 1-summable spectral triple on the limit set extends to a topologically 1-dimensional quasi-circle containing Λ (Bowen [5]). In the metric induced by the embedding in $\mathbf{P}^1(\mathbf{C})$, the quasi-circle need not be rectifiable (when the Hausdorff dimension of the limit set exceeds 1), but the existence of 1-summable spectral triples is compatible with the topological dimension being one in all cases.

2. Boundary isometry from the spectral zeta function

In this section we study the effect of equality of zeta functions on metric properties of the limit sets. Since we are dealing with two Riemann surfaces X_1, X_2, we will now sometimes index symbols $(H, D, \zeta, \lambda, \dots)$ by the index of the corresponding Riemann surface and will do so without further mention. If there is no index, we refer to any of the two Riemann surfaces.

As was already observed by Connes for the spectral triple associated to a usual spin manifold, only the action of A on H, or of D on H, doesn't capture interesting (metrical/conformal) information about the space, it is the interaction of the action of A and D that is important ([10], VI.1). For our purposes, this interaction will be encoded in the framework of zeta functions of spectral triples. The zeta functions are $\zeta_a(s) := \mathrm{tr}(a|D|^s)$, a priori defined for $\mathrm{Re}(s)$ sufficiently negative, but then meromorphically extended to the whole complex plane with poles at the dimension spectrum of the spectral triple (see [11], p. 219).

2.1 Theorem. *Let X_1 and X_2 be compact Riemann surfaces of genus at least 2. If $\zeta_{a,X_1}(s) = \zeta_{a,X_2}(s)$ for all $a \in A_\infty$, then $g_1 = g_2$ and*

$$\forall \eta \in F_g \ : \ \mu_1(\overrightarrow{\eta}_{\rho_1}) = \mu_2(\overrightarrow{\eta}_{\rho_2}).$$

2.2 Remark. As was indicated in the introduction, equality of zeta functions should be understood as follows: both the algebras A_1 and A_2 of X_1 and X_2, respectively, have a unit. If the zeta functions for this unit are equal, then we will conclude from this that the Riemann surfaces have the same genus. Therefore, the algebras A_1 and A_2 are isomorphic via the homeomorphism $\Phi : \Lambda_1 \to \Lambda_2$ induced from the Floyd maps in the triangle

$$\begin{tikzcd} & \Lambda_1 \arrow[dd, "\Phi"] \\ \bar{F}_g \arrow[ur, "\iota_{\rho_1}"] \arrow[dr, "\iota_{\rho_2}"'] & \\ & \Lambda_2 \end{tikzcd}$$

It then makes sense to interpret the expression $\zeta_{X_1,a}(s) = \zeta_{X_2,a}(s)$ for elements $a \neq 1$ in A_∞.

Proof. We make the convention that all words are reduced.

Let X be a Riemann surface of genus $g \geq 2$ and $\mathcal{S}_X$ its associated spectral triple. Suppose given an element $a = \chi_U$ in A_∞. We can assume $U = \overrightarrow{\eta}$ for a given word η of length $|\eta| = m$, since any a is a linear combination of such.

We now construct an orthogonal basis for H. First, we prove a lemma about ends of words.

2.3 Lemma. *Let w_1, w_2 denote two words. If $\overrightarrow{w_1} \cap \overrightarrow{w_2} \neq \emptyset$, then $\overrightarrow{w}_1 \subseteq \overrightarrow{w}_2$ or conversely $\overrightarrow{w}_2 \subseteq \overrightarrow{w}_1$. In particular, if we set $\max\{w, v\}$ to be the largest of the words w and v (if they* *are comparable in the order* $\subseteq$*) and* $\emptyset$ *otherwise, we find that*

$$\overrightarrow{w_1} \cap \overrightarrow{w_2} = \overrightarrow{\max\{w_1, w_2\}}$$

with the convention $\overrightarrow{\emptyset} = \emptyset$*, see Figure 1.*

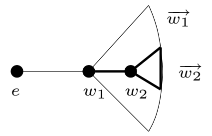

Figure 1: Illustration of Lemma 2.3

Proof. If this were not the case, then there is an end that lands in the nonempty intersection and starts from both segments w_1 and w_2, leading to a loop in the Cayley graph Y_g, but Y_g is a tree, so this is impossible, see Figure 2 (as usual, we identify the limit set Λ topologically with $\bar{F}_g$, by Floyd's Lemma).

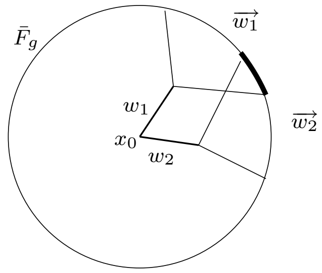

Figure 2: A forbidden situation.

$\square$

It is then easy to find a basis for the individual spaces H_n.

2.4 Lemma. *The functions* χ_w *for* $|w| = n$ *for a linear basis for* H_n*, and*

$$\langle \chi_w | \chi_v \rangle = \mu(\overrightarrow{\max\{v, w\}}).$$

Proof. The characteristic functions $\chi_{\overrightarrow{w}}$ for $|w| = n$ give a linear basis for H_n: they are linearly independent as their supports are disjoint, and they generate the space since for any word u of length $|u| < n$ one has

$$\chi_{\overrightarrow{u}} = \sum_{\substack{|w|=n \\ u \subset w}} \chi_{\overrightarrow{w}}.$$

Indeed, by the previous Lemma, all occuring $\chi_{\overrightarrow{w}}$ have disjoint support, and their union $\bigcup \overrightarrow{w}$ equals $\overrightarrow{u}$.

Now

$$\langle \chi_{\overrightarrow{w}_1} | \chi_{\overrightarrow{w}_2} \rangle = \tau(\chi^*_{\overrightarrow{w}_1} \chi_{\overrightarrow{w}_2}) = \mu_X(\overrightarrow{w}_1 \cap \overrightarrow{w}_2),$$

and the previous lemma applies. □

2.5 Lemma. *For all $n > 0$, we have*

$$\dim A_n = \dim H_n = 2g(2g-1)^{n-1}.$$

Proof. The space A_n is spanned by the linear basis $\chi_{\overrightarrow{w}_\rho}$ with w a word of exact length n, since as in the proof of Lemma 2.4, all functions corresponding to shorter words are dependent on these functions. An easy count gives the result: we pick the first letter from the alphabet on g letters or its inverses, and consecutive letters with the condition that they differ from the terminal letter of the word already constructed. □

We now construct a complete orthonormal basis for H inductively, by adding to a basis of H_n suitable elements of H_{n+1} in the style of a Gram-Schmidt process. Initially, we set $|\Psi_e\rangle = \chi_\Lambda$ and

$$|\Psi_w\rangle := \frac{1}{\sqrt{\mu_X(\overrightarrow{w})}} \chi_{\overrightarrow{w}} \quad (|w| = 1) \tag{2}$$

for w running through a set S of words of length one (viz., letters in the alphabet, and their inverses) not equal to one (arbitrarily chosen) letter. Set $I_1 := S \cup \{e\}$; then $\{|\Psi_w\rangle\}_{w \in I_1}$ is an orthonormal basis for H_1 by Lemma 2.4.

Now suppose

$$\{|\Psi_w\rangle \,:\, w \in I_n\}$$

is our inductively constructed basis for H_n, where I_n is an index set.

For every word w of length n, choose a set

$$V_w = \{wa \,:\, a \in S_w\}$$

where S_w consists of a choice of $2g-2$ letters among the $2g-1$ letters of the alphabet that are not equal to $t(w)^{-1}$, the inverse of the terminal letter of the fixed w. That is, we leave out one arbitrarily chosen letter from the possible extensions of w to an admissible word of length $n+1$. Let

$$I_{n+1} = I_n \cup \bigcup_{|w|=n} V_w.$$

Figure 2 has an example in the length ≤ 2 words in the Cayley graph Y_2 for $g = 2$.

We claim that $\{\chi_{\overrightarrow{w}}\}_{w \in I_{n+1} - I_n}$ is a basis for $H_{n+1} \ominus H_n$. The functions are linearly independent since their supports are disjoint. Hence it suffices to check dimensions. But

$$\dim(H_{n+1} \ominus H_n) = 2g(2g-1)^{n-1}(2g-2) = |I_{n+1} - I_n|.$$

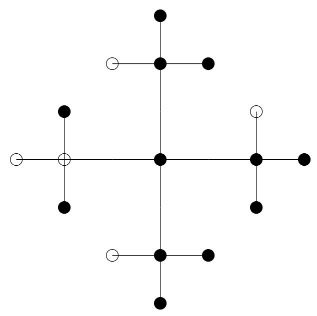

Figure 3: Black dots form a possible $I_2 \subseteq Y_2$

We define $\{|\Psi_w\rangle\}_{w\in I_{n+1}}$ as the Gram-Schmidt orthonormalisation of

$$\{|\Psi_w\rangle\}_{w\in I_n} \cup \{\chi_{\overrightarrow{w}}\}_{w\in I_{n+1}-I_n}.$$

We recall that this means we choose an enumeration of the words in $I_{n+1} - I_n$:

$$I_{n+1} - I_n = \{w_1, \ldots, w_r\}$$

and set inductively for $i = 1, \ldots, r$

$$|\Psi_{w_i}\rangle = \frac{|\phi_{w_i}\rangle}{\|\phi_{w_i}\|} \tag{3}$$

with

$$|\phi_{w_i}\rangle := \chi_{\overrightarrow{w_i}} - \sum_{w\in I_n\cup\{w_1,\ldots,w_{i-1}\}} \langle\Psi_w|\chi_{\overrightarrow{w_i}}\rangle|\Psi_w\rangle. \tag{4}$$

Then indeed, $\langle\Psi_v|\Psi_w\rangle = \delta_{v,w}$, for all $|v|, |w| \leq n + 1$.

Set

$$I_\infty = \bigcup_{n\geq 0} I_n.$$

We use the complete basis $\{|\Psi_w\rangle\}_{w\in I_\infty}$ of H to compute the trace of a trace-class operator T in the form $\mathrm{tr}(T) = \sum\langle\Psi_w|T\Psi_w\rangle$. With $T = aD^s$, we have

$$\mathrm{tr}(aD^s) = 1 + \sum_w \langle\Psi_w|a \sum_{n\geq 1} \lambda_n^s (P_n - P_{n-1})\Psi_w\rangle.$$

Now the projector $P_n - P_{n-1}$ onto $H_n \ominus H_{n-1}$ is $\sum\limits_{r\in I_n - I_{n-1}} |\Psi_r\rangle\langle\Psi_r|$, so we get

$$(P_n - P_{n-1})|\Psi_w\rangle = \sum_{r\in I_n - I_{n-1}} |\Psi_r\rangle\langle\Psi_r|\Psi_w\rangle = \delta_{|w|,n}\delta_{r,w}|\Psi_r\rangle = \delta_{|w|,n}|\Psi_w\rangle.$$

Thus, we rewrite the above as

$$\mathrm{tr}(aD^s) = 1 + \sum_{n\geq 1} \sum_{w\in I_n - I_{n-1}} \lambda_n^s \langle \Psi_w | a\Psi_w \rangle.$$

If we denote by

$$c_n(a) = \sum_{w\in I_n - I_{n-1}} \langle \Psi_w | a\Psi_w \rangle,$$

we can write

$$\zeta_{a,X}(s) = \mathrm{tr}(aD^s) = 1 + \sum_{n\geq 1} \lambda_n^s \, c_n(a), \quad \mathrm{Re}(s) \ll 0.$$

2.6 Lemma. *If $\zeta_{a,1}(s) = \zeta_{a,2}(s)$ for $a = 1 = \chi_\Lambda$ the identity of A_1, respectively A_2, then $g_1 = g_2$.*

Proof. We know that $\lambda_n = (\dim A_n)^3 = (2g)^3(2g-1)^{3n-3}$. By orthnormality, we find that

$$c_n(1) = \sum_{|w|\in I_n - I_{n-1}} \langle \Psi_w | \Psi_w \rangle = \sum_{|w|\in I_n - I_{n-1}} 1 \;=\; 2g(2g-1)^{n-2}(2g-2).$$

Hence we find for $a = 1$ that

$$\zeta_1(s) = 1 + \sum_{n\geq 1} \lambda_n^s c_n(1) = 1 + (2g)^{3s+1} \frac{2g-2}{2g-1} \sum_{n\geq 1} (2g-1)^{(3s+1)(n-1)},$$

and thus

$$\zeta_1(s) = 1 + \frac{2g-2}{2g-1} \cdot \frac{(2g)^{3s+1}}{1-(2g-1)^{3s+1}}. \tag{5}$$

For $a = 1$, the condition $\zeta_{a,1}(s) = \zeta_{a,2}(s)$ is thus equivalent to

$$\frac{2g_1-2}{2g_1-1} \cdot \frac{2g_2-1}{2g_2-2} \cdot \left(\frac{g_1}{g_2}\right)^{3s+1} = \frac{1-(2g_1-1)^{3s+1}}{1-(2g_2-1)^{3s+1}} \text{ for } \mathrm{Re}(s) \ll 0$$

If we let s tend to $-\infty$, the right hand side tends to 1. However, unless $g_1 = g_2$, the left hand side tends to zero for $g_1 > g_2$ or to infinity for $g_1 < g_2$. This finishes the proof that $g_1 = g_2$. ☐

As mentioned in the remark above, we conclude from this lemma that the algebras A_1 and A_2 are isomorphic via the induced Floyd homeomorphism $\Phi : \Lambda_1 \to \Lambda_2$. This makes the condition $\zeta_{a,1}(s) = \zeta_{a,2}(s)$ meaningful.

2.7 Lemma. *If $\zeta_{a,1}(s) = \zeta_{a,2}(s)$, for all $a \in A_\infty$, then $c_{n,1}(a) = c_{n,2}(a)$ for all $a \in A_\infty$.*

Proof. The equality $\zeta_{a,1}(s) = \zeta_{a,2}(s)$ is equivalent to

$$\sum_{n\geq 0} (c_{n,1}(a) - c_{n,2}(a))\, \lambda_n^s \equiv 0$$

for $\mathrm{Re}(s) \ll 0$. Here, λ_n is the same for the two Riemann surfaces, since it only depends on their genus and those have just been shown to be equal in Lemma 2.6. Now since all λ_n are *distinct* positive integers, we also have an identically zero Dirichlet series

$$\sum_{N\geq 0} \widetilde{c}_N N^s \equiv 0 \text{ for } \mathrm{Re}(s) \ll 0$$

with $\widetilde{c}_N = c_{n,1}(a) - c_{n,2}(a)$ if $N = \lambda_n$ for some n, and $\widetilde{c}_N = 0$ otherwise. Now clearly $\widetilde{c}_N = 0$ for all N, by the identity theorem for Dirichlet series (e.g., [23], 17.1). ☐

2.8 Lemma. *For $a = \chi_{\overrightarrow{\eta}}$ and w a word of length $n < |\eta|$, we have that*

$$\langle \Psi_w | a \Psi_w \rangle = \mu(\overrightarrow{\eta}) \cdot \kappa$$

where κ depends only on measures $\mu(\overrightarrow{v})$ of certain words v of length $|v| < |\eta|$.

Proof. This holds for w a word of length one, since by definition (2) and Lemma 2.4, we have

$$\langle \Psi_w | a \Psi_w \rangle = \frac{\mu(\overrightarrow{w} \cap \overrightarrow{\eta})}{\mu(\overrightarrow{w})} = \begin{cases} \frac{\mu(\overrightarrow{\eta})}{\mu(\overrightarrow{w})} & \text{if } w \subset \eta; \\ 0 & \text{otherwise.} \end{cases}$$

We then use induction on the word length of w. By construction of Ψ_w (looking at the definitions in Formulæ (4) and (3)) it suffices to prove that for w, u of length $\leq n$, we have that $\langle \chi_{\overrightarrow{w}} | a \chi_{\overrightarrow{u}} \rangle$ is of the required form $\mu(\overrightarrow{\eta}) \cdot \kappa$ where κ depends only on measures $\mu(\overrightarrow{v})$ of certain words v of length $|v| < |\eta|$: Ψ_w is a linear combination of such terms. Now

$$\langle \chi_{\overrightarrow{w}} | a \chi_{\overrightarrow{u}} \rangle = \mu(\overrightarrow{w} \cap \overrightarrow{\eta} \cap \overrightarrow{u}) = \begin{cases} \mu(\overrightarrow{\eta}) & \text{if } w, u \subset \eta \\ 0 & \text{otherwise,} \end{cases}$$

since η is longer than w and u. This proves the claim. ☐

Computing $c_{m-1}(\chi_{\overrightarrow{\eta}})$ as a linear combination of terms of the form $\langle \Psi_w | a \Psi_w \rangle$, we find from Lemma 2.8 (now indicating the representation ρ, since we will soon vary it) :

$$c_{m-1}(\chi_{\overrightarrow{\eta}_\rho}) = \mu(\overrightarrow{\eta}_\rho) \cdot \kappa, \tag{6}$$

where κ only depends on $\mu(\overrightarrow{v}_\rho)$ for $|v| < m$.

We need one more technical observation, namely that $\kappa \neq 0$ in (6) or, what is the same:

2.9 Lemma. $c_{m-1}(a) \neq 0$ *for $a = \chi_{\overrightarrow{\eta}}$ with $|\eta| = m$.*

Proof. Recall $c_{m-1}(a) = \sum_{w \in I_{m-1} - I_{m-2}} \langle \Psi_w | a \Psi_w \rangle$. The terms

$$\langle \Psi_w | a \Psi_w \rangle = \int |\Psi_w|^2 \cdot \chi_{\overrightarrow{\eta}} \ d\mu \geq 0$$

are all positive, but some might be zero. It therefore suffices to prove that at least one of them is non-zero, and for this, it suffices to find w such that the support of Ψ_w intersects $\overrightarrow{\eta}$. But if $x \in \overrightarrow{\eta}$, then there is a word v of length $m-1$ such that $x \in \overrightarrow{v}$, too, since

$$\bigcup_{|v|=m-1} \overrightarrow{v} = \Lambda.$$

Then $\chi_{\overrightarrow{v}}(x) \neq 0$, but, as $\{\Psi_w\}_{w \in I_{m-1} - I_{m-2}}$ is a basis for $H_{m-1} \ominus H_{m-2}$, we have

$$\chi_{\overrightarrow{v}}(x) = \sum_{w \in I_{m-1} - I_{m-2}} a_w \Psi_w(x)$$

for some coefficients a_w, hence there exists w of length $m-1$ such that $\Psi_w(x) \neq 0$. □

2.10 Proposition. *For all $\eta \in F_g$, $\mu_1(\overrightarrow{\eta}_{\rho_1}) = \mu_2(\overrightarrow{\eta}_{\rho_2})$.*

Proof. We prove this by induction on the word length of $\eta \in F_g$ If $|\eta| = 0$, we find $\overrightarrow{\eta}_{\rho_i} = \Lambda_i$ for $i = 1, 2$, so the identity holds. If it holds for all words of length $< m$, let η denote a word of length m.

Recall that the map $\Phi^* : A_1 \to A_2$ is such that $\Phi^*(\chi_{\overrightarrow{w}_{\rho_1}}) = \chi_{\overrightarrow{w}_{\rho_2}}$.

We apply the expression (6) to both Riemann surfaces, substituting $\rho = \rho_1$ and $\rho = \rho_2$, respectively. Since $\kappa \neq 0$, this is a genuine formula for $\mu(\overrightarrow{\eta}_\rho)$. The equality $c_{m-1,1}(\chi_{\overrightarrow{\eta}_{\rho_1}}) = c_{m-1,2}(\chi_{\overrightarrow{\eta}_{\rho_2}})$ is our assumption, and for the second factor on the right hand side we can inductively assume that the measures on the occuring words of length $< m$ agree in both representations. Hence we find indeed $\mu(\overrightarrow{\eta_{\rho_1}}) = \mu(\overrightarrow{\eta_{\rho_2}})$.

□

This proposition finishes the proof of Theorem 2.1. □

3. Rigidity from boundary isometry

We now prove Theorem 2 from the introduction:

3.1 Theorem. *If X_1 and X_2 are compact Riemann surfaces of respective genus $g_1, g_2 \geq 2$, such that $\zeta_{X_1,a}(s) = \zeta_{X_2,a}(s)$ for all $a \in A$, then X_1 and X_2 are complex analytically equivalent or anti-equivalent as Riemann surfaces.*

Proof. From Theorem 2.1, we find that X_1 and X_2 have the same genus $g \geq 2$. We consider the two Schottky groups Γ_1 and Γ_2 corresponding to X_1 and X_2, respectively. Let $\rho_i : F_g \hookrightarrow \mathrm{PGL}(2, \mathbf{C})$ denote the corresponding embeddings of the abstract group F_g (so $\rho_i(F_g) = \Gamma_i$), and let $\alpha := \rho_2 \circ \rho_1^{-1}$ denote the induced group isomorphism $\Gamma_1 \to \Gamma_2$. We consider the map $\Phi : \Lambda_1 \to \Lambda_2$ as in the diagram of the proof of 2.1: $x \in \Lambda_1$ can be written as $\iota_{\rho_1}(\lim w_i)$ for some Cauchy sequence $\lim w_i$ in Y_g. We then define $\Phi(x) := \iota_{\rho_2}(\lim w_i)$. As was remarked before, by Floyd's Lemma 1.4, Φ is a homeomorphism of Λ_1 onto Λ_2.

3.2 Lemma. *Φ is α-equivariant.*

Proof. Given $\gamma \in \Gamma_1$, we can find $g \in F_g$ such that $\gamma = \rho_1(g)$. For $x = \iota_{\rho_1}(\lim w_i)$, we find that $\gamma \cdot x = \iota_{\rho_1}(\lim g w_i)$. Hence

$$\Phi(\gamma \cdot x) = \iota_{\rho_2}(\lim g w_i) = \iota_{\rho_2}(g) \cdot \iota_{\rho_2}(\lim w_i) = \rho_2(\rho_1^{-1}(\gamma)) \cdot \Phi(x) = \alpha(\gamma)\Phi(x).$$

So we do find that Φ is α-equivariant. $\square$

This means that Φ is a boundary homeomorphism in the sense of Fenchel-Nielsen, see Tukia [35], 3C.

By Theorem 2.1, the equality of zeta functions implies that Φ is an isometry. Indeed,

$$\mu_2(\Phi^*(\chi_{\overrightarrow{w}_{\rho_1}})) = \mu_2(\chi_{\Phi(\overrightarrow{w}_{\rho_1})}) = \mu_2(\chi_{\overrightarrow{w}_{\rho_2}}) = \mu_1(\chi_{\overrightarrow{w}_{\rho_1}}),$$

where we use the definition of Φ in the second equality and the proposition in the third. Thus, since $\{\overrightarrow{w}_{\rho_i}\}$ is a basis for Λ_i, we find that $\mu_2 \circ \Phi^* = \mu_1$.

Now recall the following ergodic rigidity theorem:

3.3 Lemma (Chengbo Yue). *Let Γ_1 and Γ_2 be geometrically finite subgroups in two simple, connected and adjoint Lie groups G_1 and G_2 of real rank one, such that Γ_1 is Zariski dense in G_1. Let $\alpha : \Gamma_1 \to \Gamma_2$ be a type-preserving isomorphism. Then there exists a homeomorphism $\phi : \Lambda_{\Gamma_1} \to \Lambda_{\Gamma_2}$ which is equivariant with respect to α. If ϕ is absolutely continuous with respect to the Patterson–Sullivan measure, then α can be extended to a continuous homomorphism $G_1 \to G_2$.*

Proof. This is literally Corollary B from [39], apart from the fact that the extended homomorphism $G_1 \to G_2$ can be assumed *continuous*, but this follows by looking at the statement of Theorem A from which the corollary follows. $\square$

We want to apply this corollary with $G_1 = G_2 = \mathrm{PGL}(2, \mathbf{C})$ and Γ_i our Schottky groups, so let us check the conditions: Both Schottky groups are geometrically finite subgroups of $\mathrm{PGL}(2, \mathbf{C})$; $\mathrm{PGL}(2, \mathbf{C}) = \mathrm{PSL}(2, \mathbf{C})$ is a simple and connected adjoint real-rank-one Lie group; and finally:

3.4 Lemma. *A noncommutative Schottky group is Zariski dense in* $\mathrm{PGL}(2, \mathbf{C})$.

Proof. Since the group operations on such a Schottky group are induced from the algebraic operations on the algebraic group $\mathrm{PGL}(2)$, the Zariski closure $\hat{\Gamma}$ is itself an algebraic subgroup of $\mathrm{PGL}(2)$. Assume $\hat{\Gamma}$ is a strict algebraic subgroup of $\mathrm{PGL}(2)$. Let $\hat{\Gamma}_0$ denote its connected component of the identity. The group of connected components $\hat{\Gamma}/\hat{\Gamma}_0$ is finite, and $\Gamma \cap \hat{\Gamma}_0$ is a finite index subgroup of the free group Γ, hence free of the same rank g; and its Zariski closure is connected. It suffices that this group has full Zariski closure, hence we can assume without loss of generality that $\hat{\Gamma}$ is connected. However, if $\hat{\Gamma}$ is connected of dimension ≤ 2, then it is solvable (cf. [4], IV.11.6), and a solvable group cannot contain a free group of rank $g \geq 2$ (since the composition series of $\hat{\Gamma}$ would descend to one for Γ). On the other hand, if $\dim \hat{\Gamma} = 3$, then since $\mathrm{PGL}(2)$ is connected, we have $\hat{\Gamma} = \mathrm{PGL}(2)$. □

Since F_g has no parabolic points, we know that the equivariant boundary homeomorphism Φ is unique and type-preserving (Tukia, [36], p. 426), hence it coincides with the boundary homeomorphism ϕ in Yue's result. Since all conditions are satisfied, we can apply the result (replacing ϕ by Φ) to both the isometry $\Lambda_1 \to \Lambda_2$ and its inverse, we find that α extends to a continuous group automorphism $\mathrm{PGL}(2, \mathbf{C}) \to \mathrm{PGL}(2, \mathbf{C})$. Now recall that the automorphisms of $\mathrm{PGL}(2, k)$ over a field k have been classified by Schreier and van der Waerden (cf. [31], see also the supplement to [17]): the outer automorphisms are induced from field automorphisms of k. Now all continuous field automorphisms of $\mathbf{C}$ fix $\mathbf{R}$.

We conclude that there is an isomorphism $\Gamma_1 \to \Gamma_2$ of the form

$$\gamma_1 \to g\gamma_1^\sigma g^{-1}$$

for $g \in \mathrm{PGL}(2, \mathbf{C})$ and $\sigma \in \mathrm{Aut}(\mathbf{C}/\mathbf{R})$, that is, Γ_1 and Γ_2^σ are conjugate in $\mathrm{PGL}(2, \mathbf{C})$. Now note that Γ_2^σ uniformizes the curve X_2^σ:

$$(\mathbf{P}^1(\mathbf{C}) - \Lambda_{\Gamma_2^\sigma})/\Gamma_2^\sigma = \big((\mathbf{P}^1(\mathbf{C}) - \Lambda_{\Gamma_2})/\Gamma_2\big)^\sigma = X_2^\sigma.$$

Hence X_1 and X_2^σ are isomorphic Riemann surfaces, so X_1 and X_2 are complex analytically equivalent or anti-equivalent, the former case arising when σ is trivial, and the latter case arising when σ is complex conjugation. □

3.5 Remark. The statement that two hyperbolic compact Riemann surfaces of the same genus are isomorphic if and only if the boundary map $S^1 \to S^1$ induced from the isomorphism of their fundamental groups, seen as Fuchsian groups of the first kind, is absolutely continuous in Lebesgue measure is originally part of Mostow's rigidity theorem (cf. Mostow [27] 22.14, p. 178). An easy proof for this case is in Kuusalo [25], and Bowen has given another proof using Gibbs measures in [5]. For more general Möbius groups (whose limit set is not necessarily the full boundary of the symmetric space, such as Schottky groups, and in higher dimensions), there is the work of Sullivan ([33]) and Tukia ([36], [35]). The typical ergodic rigidity theorem in this setting is that an absolutely continuous boundary map is identical to the restriction of a Möbius transformation *on the limit set*. What happens outside the limit set, however, depends on other considerations (cf. [35], Marden [26]). See, e.g., Tukia ([36]), Section 4D for a 3-dimensional example of an isomorphism of Schottky groups with absolutely continuous boundary map, that does *not* extend to a Möbius map outside the limit set.

However, in the case when Λ_1 is not contained in a geometric circle (i.e., in the image of S^1 under a Möbius transformation), Theorem 3.1 follows more easily from [36], Theorem 4B2, which shows that Γ_1 and Γ_2 are conjugate by a Möbius transformation of the real two-sphere.

3.6 Remark. The construction cannot be extended to the crossed product boundary operator algebra $C(\Lambda) \rtimes \Gamma$, since Connes by hyperfiniteness result [9] (in particular, Thms. 17 and 19 on pp. 214–215) the hyperbolic growth of the group Γ ($g \geq 2$) prevent this algebra from carrying finitely summable spectral triples. However, $C(\Lambda) \rtimes \Gamma$ can be identified with a Cuntz–Krieger algebra ([16]), which has a standard AF-subalgebra $\widetilde{A}$. This has a maximal abelian subalgebra that can be identified with our algebra A ([16], 2.5). One can construct a conditional expectation E from $\widetilde{A}$ to A. Thus, the construction of the spectral triple extends to this AF-algebra $\widetilde{A}$ by using an *unfaithful* state $\tau \circ E$ that is zero outside A and equals our state τ on A. Since this construction, however, is just a "factorisation" through the above commutative spectral triple, it doesn't give a lot of new information.

4. Remarks and Questions

4.1. An interesting question (e.g., in the light of Arakelov geometry) is how to generalize the result to the case of p-adic Mumford curves ([20], [28]). See [15] for some results on trees, inspired by earlier work on non-finitely summable spectral triples for tree actions by Consani and Marcolli [13].

4.2. Is there such a theory for Fuchsian uniformisation instead of Schottky uniformisation? In the noncompact case?

4.3. There are other ways of looking at Riemann surfaces from the point of view of non-commutative geometry. For example, as a commutative conformal manifold, it carries the non-commutative differential geometry of "quantized calculus", whose Fredholm module determines the conformal isomorphism type of the surface (Connes, Donaldson, Sullivan, N. Teleman [10], IV.4.α). Also, Consani and Marcolli ([14]) have constructed θ-summable, but non finitely summable spectral triples from the boundary action, where the Hilbert space is a symmetrized version of the L^2-space of the boundary that should have Hausdorff dimension < 1. There is the work of Bär ([2]) mentioned before. All of these constructions are rather different from the one in this paper. In particular, our spectral triple is finitely summable, so better suited to tools such as the local index formula of Connes and Moscovici ([12]). Nevertheless, the question arises whether the Consani-Marcolli spectral triple hears the conformal shape of the Riemann surface. Bär's spectral triple does enjoy this property.

4.4. Does our spectral triple $\mathcal{S}_X$ carry a real structure? Does it then arise from a commutative spin manifold ([22], [30])? Can one enhance $\mathcal{S}_X$ by additional classical structure, so this structure determines the complex analytic structure of X completely (not only up to complex conjugation)? Notice that in our construction we work only with the absolute value of the Dirac operator, and we use zeta functions that do not see the sign. In order to relate it to spectral triples coming from a Riemannian manifold one would need to first enrich it with a sign (which gives the fundamental class in K-homology) and then with a compatible real structure. It seems that the information on the sign will be needed to reconstruct completely the complex structure.

4.5. Since special values of the spectral zeta functions are just 0-Hochschild homology, it is interesting to compute higher characteristic classes of the spectral triple and relate them to the actual geometry of the original Riemann surface. This is especially tempting in arithmetically interesting cases, such as modular curves.

4.6. Can Theorem 2 be extended to an injective functor from the category of Riemann surfaces with complex analytic morphisms (up to complex conjugation) to a category of spectral triples?

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