Proper Actions of Lie Groups of Dimension n^2+1 on n-Dimensional Complex Manifolds[*†]

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In this paper we continue to study actions of high-dimensional Lie groups on complex manifolds. We consider connected complex manifolds M of dimension $n \geq 2$ on which connected Lie groups G of dimension n^2+1 act effectively and properly by holomorphic transformations. Such actions are transitive and are shown to split into three types according to the form of the linear isotropy subgroup. We give a complete explicit description of all pairs (M, G) for two of these types, and announce a classification for the third type. These results complement a classification obtained earlier by the author for $n^2+2 \leq \dim G < n^2+2n$ and a result due to W. Kaup for the maximal group dimension n^2+2n.

0 Introduction

Let M be a connected C^∞-smooth manifold and $\mathrm{Diff}(M)$ the group of C^∞-smooth diffeomorphisms of M endowed with the compact-open topology. A topological group G is said to act continuously on M by diffeomorphisms, if a continuous homomorphism $\Phi : G \to \mathrm{Diff}(M)$ is specified. The continuity of Φ is equivalent to the continuity of the action map

$$\hat{\Phi} : G \times M \to M, \quad (g, p) \mapsto \Phi(g)(p) =: gp.$$

We only consider effective actions, that is, assume that the kernel of Φ is trivial.

The action of G on M is called *proper*, if the map

$$\Psi : G \times M \to M \times M, \quad (g, p) \mapsto (gp, p),$$

is proper, i.e. for every compact subset $C \subset M \times M$ its inverse image $\Psi^{-1}(C) \subset G \times M$ is compact as well. For example, the action is proper if

[*]**Mathematics Subject Classification:** 32Q57, 32M10, 58D19
[†]**Keywords and Phrases:** complex manifolds, proper group actions.

G is compact. The properness of the action implies that: (i) G is locally compact, hence by [BM1], [BM2] (see also [MZ]) it carries the structure of a Lie group and the action map $\hat{\Phi}$ is smooth; (ii) Φ is a topological group isomorphism between G and $\Phi(G)$; (iii) $\Phi(G)$ is a closed subgroup of $\mathrm{Diff}(M)$ (see [Bi] for a brief survey on proper actions). Thus, one can assume that G is a Lie group acting smoothly and properly on the manifold M, and that it is realized as a closed subgroup of $\mathrm{Diff}(M)$.

Suppose now that M is equipped with a Riemannian metric $\mathcal{G}$, and let $\mathrm{Isom}(M,\mathcal{G})$ be the group of all isometries of M with respect to $\mathcal{G}$. It was shown in [MS] that $\mathrm{Isom}(M,\mathcal{G})$ acts properly on M (and so does its every closed subgroup). Conversely, by [Pal] (see also [A]), for any Lie group acting properly on M there exists a C^∞-smooth G-invariant metric $\mathcal{G}$ on M. It then follows that Lie groups acting properly and effectively on the manifold M by diffeomorphisms are precisely closed subgroups of $\mathrm{Isom}(M,\mathcal{G})$ for all possible smooth Riemannian metrics $\mathcal{G}$ on M.

If G acts properly on M, then for every $p \in M$ its isotropy subgroup

$$G_p := \{g \in G : gp = p\}$$

is compact in G. Then by [Bo] the isotropy representation

$$\alpha_p : G_p \to GL(\mathbb{R}, T_p(M)), \quad g \mapsto dg_p$$

is continuous and faithful, where $T_p(M)$ denotes the tangent space to M at p and dg_p is the differential of g at p. In particular, the linear isotropy subgroup

$$LG_p := \alpha_p(G_p)$$

is a compact subgroup of $GL(\mathbb{R}, T_p(M))$ isomorphic to G_p. In some coordinates in $T_p(M)$ the group LG_p becomes a subgroup of the orthogonal group $O_m(\mathbb{R})$, where $m := \dim M$. Hence $\dim G_p \le \dim O_m(\mathbb{R}) = m(m-1)/2$. Furthermore, for every $p \in M$ its orbit

$$Gp := \{gp : g \in G\}$$

is a closed submanifold of M, and $\dim Gp \le m$. Thus, setting $d_G := \dim G$, we obtain

$$d_G = \dim G_p + \dim Gp \le m(m+1)/2.$$

It is a classical result (see [F], [C], [Ei]) that if G acts properly on a smooth manifold M of dimension m and $d_G = m(m+1)/2$, then M is isometric (with respect to some G-invariant metric) either to one of the standard complete simply-connected spaces of constant sectional curvature $\mathbb{R}^m$, S^m, $\mathbb{H}^m$ (where $\mathbb{H}^m$ is the hyperbolic space), or to $\mathbb{RP}^m$. Next, it was shown in [Wa] (see also [Eg], [Y1]) that a group G with $m(m-1)/2+1 < d_G < m(m+1)/2$ cannot act properly on a smooth manifold M of dimension $m \neq 4$. The exceptional 4-dimensional case was considered in [Ish]; it turned out that a group of dimension 9 cannot act properly on a 4-dimensional manifold, and that if a 4-dimensional manifold admits a proper action of an 8-dimensional group G, then it has a G-invariant complex structure. Invariant complex structures will be discussed below in detail.

There exists also an explicit classification of pairs (M, G), where $m \geq 4$, G is connected, and $d_G = m(m-1)/2+1$ (see [Y1], [Ku], [O], [Ish]). Further, in [KN] a reasonably explicit classification of pairs (M, G), where $m \geq 6$, G is connected, and $(m-1)(m-2)/2+2 \leq d_G \leq m(m-1)/2$, was given. We also mention the classification of G-homogeneous manifolds for $m = 4$, $d_G = 6$ (see [Ish]) and the classifications of G-homogeneous simply-connected manifolds in the cases $m = 3$, $d_G = 3, 4$ and $m = 4$, $d_G = 5$ (see [C], [Pat]) obtained by E. Cartan's method of adapted frames introduced in [C]. There are many other results, especially for compact subgroups, but no complete classifications exist beyond dimension $(m-1)(m-2)/2+2$ (see [Ko2], [Y2] and references therein for further details).

We study proper group actions in the complex setting with the general aim to build a theory for group dimensions lower than $(m-1)(m-2)/2+2$, thus extending – in this setting – the classical results described above. In our setting real Lie groups act by holomorphic transformations on complex manifolds. Thus, from now on, M will denote a complex manifold of complex dimension n (hence $m = 2n$) and G will be a subgroup of $\mathrm{Aut}(M)$, the group of all holomorphic automorphisms of M. We will be classifying pairs (M, G), but we will not be concerned with determining G-invariant Riemannian metrics on M. Proper actions by holomorphic transformations are found in abundance. A fundamental result due to Kaup (see [Ka]) states that every closed subgroup of $\mathrm{Aut}(M)$ that preserves a continuous distance on M acts properly on M. Thus, Lie groups acting properly and effectively on M by holomorphic transformations are precisely closed subgroups of $\mathrm{Aut}(M)$ preserving a continuous distance on M. In particular, if M is a Kobayashi-hyperbolic manifold, then $\mathrm{Aut}(M)$ is a Lie group acting properly on M (see also [Ko1]).

In the complex setting, in some coordinates in $T_p(M)$ the group LG_p becomes a subgroup of the unitary group U_n. Hence $\dim G_p \leq \dim U_n = n^2$,

and therefore
$$d_G \leq n^2 + 2n.$$

We note that $n^2+2n<(m-1)(m-2)/2+2$ for $m=2n$ and $n\geq 5$. Thus, the group dimension range that arises in the complex case, for $n\geq 5$ lies strictly below the dimension range considered in the classical real case and therefore is not covered by the existing results. Furthermore, overlaps with these results for $n=3,4$ and $n=2$, $d_G=6$ occur only in relatively easy situations and do not lead to any significant simplifications in the complex case. The only interesting overlap with the real case occurs for $n=2$, $d_G=5$ (see [Pat]); we will briefly discuss it below. Note that in the situations when overlaps do occur, the existing classifications in the real case do not necessarily immediately lead to classifications in the complex case, since the determination of all G-invariant complex structures on the corresponding real manifolds may be a non-trivial matter.

It was shown in [Ka] that if $d_G=n^2+2n$, then M is holomorphically equivalent (in fact, holomorphically isometric with respect to some G-invariant metric) to one of $\mathbb{B}^n:=\{z\in\mathbb{C}^n:|z|<1\}$, $\mathbb{C}^n$, $\mathbb{CP}^n$, and an equivalence map F can be chosen so that the group $F\circ G\circ F^{-1}:=\{F\circ g\circ F^{-1}:g\in G\}$ is, respectively, one of the groups $\mathrm{Aut}(\mathbb{B}^n)$, $G(\mathbb{C}^n)$, $G(\mathbb{CP}^n)$. Here $\mathrm{Aut}(\mathbb{B}^n)\simeq PSU_{n,1}:=SU_{n,1}/(\text{center})$ is the group of all transformations
$$z\mapsto \frac{Az+b}{cz+d},$$
where
$$\begin{pmatrix} A & b\\ c & d\end{pmatrix}\in SU_{n,1};$$
$G(\mathbb{C}^n)\simeq U_n\ltimes\mathbb{C}^n$ is the group of all holomorphic automorphisms of $\mathbb{C}^n$ of the form
$$z\mapsto Uz+a, \tag{0.1}$$
where $U\in U_n$, $a\in\mathbb{C}^n$ (we usually write $G(\mathbb{C})$ instead of $G(\mathbb{C}^1)$); and $G(\mathbb{CP}^n)\simeq PSU_{n+1}:=SU_{n+1}/(\text{center})$ is the group of all holomorphic automorphisms of $\mathbb{CP}^n$ of the form
$$\zeta\mapsto U\zeta, \tag{0.2}$$
where ζ is a point in $\mathbb{CP}^n$ given in homogeneous coordinates, and $U\in SU_{n+1}$ (this group is a maximal compact subgroup of the complex Lie group

$\mathrm{Aut}(\mathbb{CP}^n) \simeq PSL_{n+1}(\mathbb{C}) := SL_{n+1}(\mathbb{C})/(\text{center})$). In the above situation we say for brevity that F *transforms* G into one of $\mathrm{Aut}(\mathbb{B}^n)$, $G(\mathbb{C}^n)$, $G(\mathbb{CP}^n)$, respectively, and, in general, if $F : M_1 \to M_2$ is a biholomorphic map, $G_j \subset \mathrm{Aut}(M_j)$, $j = 1, 2$, are subgroups and $F \circ G_1 \circ F^{-1} = G_2$, we say that F transforms G_1 into G_2.

We are interested in characterizing pairs (M, G) for $d_G < n^2 + 2n$, where $G \subset \mathrm{Aut}(M)$ acts on M properly. In [IKra], [I1], [I2], [I3] we considered the special case where M is a Kobayashi-hyperbolic manifold and $G = \mathrm{Aut}(M)$, and explicitly determined all manifolds with $n^2 - 1 \le d_{\mathrm{Aut}(M)} < n^2 + 2n$, $n \ge 2$ (see [I4] for a comprehensive exposition of these results). The case $d_{\mathrm{Aut}(M)} = n^2 - 2$ represents the first obstruction to the existence of an explicit classification, namely, there is no good description of hyperbolic manifolds with $n = 2$, $d_{\mathrm{Aut}(M)} = 2$ (see [I1], [I4]); it is possible, however, that a reasonable classification exists in this case for $n \ge 3$. Our immediate goal is to generalize these results to arbitrary proper actions on not necessarily Kobayashi-hyperbolic manifolds by classifying all pairs (M, G) with $n^2 - 1 \le d_G < n^2 + 2n$, $n \ge 2$, where G is assumed to be connected.

This classification problem splits into two cases: that of G-homogeneous manifolds and that of non-G-homogeneous ones (note that due to [Ka] G-homogeneity always takes place for $d_G > n^2$). While the techniques that we developed for non-homogeneous Kobayashi-hyperbolic manifolds seem to work well for general non-transitive proper actions, there is a substantial difference in the homogeneous case. Indeed, due to [N] every homogeneous Kobayashi-hyperbolic manifold is holomorphically equivalent to a Siegel domain of the second kind, and therefore such manifolds can be studied by using techniques available for Siegel domains (see e.g. [S]). This is how homogeneous Kobayashi-hyperbolic manifolds with $n^2 - 1 \le d_{\mathrm{Aut}(M)} < n^2 + 2n$, $n \ge 2$, were determined in [IKra], [I1], [I2], [I4]. Clearly, this approach cannot be applied to general transitive proper actions, and one motivation for the present work is to re-obtain the classification of homogeneous Kobayashi-hyperbolic manifolds without using the non-trivial result of [N].

The first step towards a general classification for proper actions with $d_G < n^2 + 2n$ was made in [IKra] where we observed that if $d_G \ge n^2 + 3$, then, as in the case $d_G = n^2 + 2n$, the manifold must be holomorphically equivalent to one of $\mathbb{B}^n$, $\mathbb{C}^n$, $\mathbb{CP}^n$. However, in [IKra] we did not investigate the question what groups (if any) are possible for each of these three manifolds within the dimension range $n^2 + 3 \le d_G < n^2 + 2n$. We resolved this question in [I5]. In the same paper we gave a complete classification of all pairs (M, G) with $d_G = n^2 + 2$.

In the present paper we assume that $d_G = n^2 + 1$. Note that this is the lowest group dimension for which G-homogeneity always takes place; indeed, for $d_G = n^2$ both G-homogeneous and non G-homogeneous manifold occur (see [I4]). For $d_G = n^2 + 1$ we have $\dim G_p = (n-1)^2$, and we start by describing connected subgroups of the unitary group U_n of dimension $(n-1)^2$ in Proposition 1.1 (see Section 1), thus determining the connected identity components of all possible linear isotropy subgroups. According to this description, every action falls into one of three types. In Sections 2, 3 we deal with actions of type I and II, respectively, and obtain complete lists of the corresponding pairs (M, G) in Theorems 2.1 and 3.1. Actions of type III are more difficult to deal with. In Section 4 we give a large number of examples of such actions. Jointly with N. Kruzhilin, we are currently working on the problem of describing all actions of type III. This work is now essentially complete, and we use this opportunity to announce that the examples in Section 4 in fact give a complete description of such actions (see Theorem 4.1). Thus, Theorems 2.1, 3.1, 4.1 describe all pairs (M, G) with $d_G = n^2 + 1$.

Regarding Theorems 2.1, 3.1, 4.1 for $n = 2, 3$ some remarks are in order. Firstly, all connected 2- and 3-dimensional complex manifolds that admit transitive actions of Lie groups by holomorphic transformations were determined in [HL], [OR], [Wi]; however, it was not the aim of those articles to give a description of *all* possible transitive actions, and, indeed, most actions listed in Theorems 2.1, 3.1, 4.1 do not occur in [HL], [OR], [Wi]. Secondly, as we have already mentioned, a classification of all effective proper transitive actions of connected 5-dimensional Lie groups on simply-connected real 4-dimensional manifolds was given in [Pat] (see also [Ish]). Therefore, one can attempt to obtain Theorem 4.1 for $n = 2$ by determining all invariant complex structures and by passing to quotients to produce a list of non simply-connected manifolds from the list of simply-connected ones. Thirdly, we have been informed by G. Fels that he has recently obtained Theorem 4.1 for $n = 2$ by an alternative method.

Acknowledgements. Part of this work was done while the author was visiting the Ruhr-Universität Bochum in January-February 2007; the visit was partially supported by the Alexander von Humboldt-Stiftung. The classification of actions of type III (which is a joint work of the author with N. Kruzhilin) was completed during the author's visit to the Max-Plank Institut für Mathematik in Bonn in April-May 2007 and will appear in our future paper. We would like to thank G. Fels for making a large number of useful

comments and interest in our work.

1 Classification of Linear Isotropy Subgroups

In this section we prove the following proposition that extends Lemma 2.1 of [IKru].

Proposition 1.1 *Let H be a connected closed subgroup of U_n of dimension $(n-1)^2$, $n \geq 2$. Then H is conjugate in U_n to one of the following subgroups:*

I. $e^{i\mathbb{R}}SO_3(\mathbb{R})$ (here $n=3$);

II. $SU_{n-1} \times U_1$ realized as the subgroup of all matrices

$$\left(\begin{array}{cc} A & 0 \\ 0 & e^{i\theta} \end{array} \right), \tag{1.1}$$

where $A \in SU_{n-1}$ and $\theta \in \mathbb{R}$, for $n \geq 3$;

III. the subgroup $H^n_{k_1,k_2}$ of all matrices

$$\left(\begin{array}{cc} A & 0 \\ 0 & a \end{array} \right), \tag{1.2}$$

where k_1, k_2 are fixed integers such that $(k_1, k_2) = 1$, $k_1 > 0$, and $A \in U_{n-1}$, $a \in (\det A)^{\frac{k_2}{k_1}} := \exp(k_2/k_1\, Ln\,(\det A))$.[‡]

Proof: Since H is compact, it is completely reducible, i.e. $\mathbb{C}^n$ splits into a sum of H-invariant pairwise orthogonal complex subspaces, $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_m$, such that the restriction H_j of H to each V_j is irreducible. Let $n_j := \dim_{\mathbb{C}} V_j$ (hence $n_1 + \cdots + n_m = n$) and let U_{n_j} be the group of unitary transformations of V_j. Clearly, $H_j \subset U_{n_j}$, and therefore $\dim H \leq n_1^2 + \cdots + n_m^2$. On the other hand $\dim H = (n-1)^2$, which shows that $m \leq 2$.

Let $m = 2$. Then there exists a unitary change of coordinates in $\mathbb{C}^n$ such all elements of H take the form (1.2), where $A \in U_{n-1}$ and $a \in U_1$. We note that the matrices A and the scalars a corresponding to the elements of H form compact connected subgroups of U_{n-1} and U_1, respectively; we shall denote them by H_1 and H_2.

[‡]For $k_2 \neq 0$ the group $H^n_{k_1,k_2}$ is a k_1-sheeted cover of U_{n-1}.

If $\dim H_2 = 0$, then $H_2 = \{1\}$, and therefore $H_1 = U_{n-1}$. In this case we obtain the group $H^n_{1,0}$.

Assume that $\dim H_2 = 1$, i.e., $H_1 = U_1$. Then $(n-1)^2 - 1 \le \dim H_1 \le (n-1)^2$. Let $\dim H_1 = (n-1)^2 - 1$ first. The only connected subgroup of U_{n-1} of dimension $(n-1)^2 - 1$ is SU_{n-1}. Hence H is conjugate to the subgroup of matrices of the form (1.1) if $n \ge 3$ and to $H^2_{1,0}$ for $n = 2$. Now let $\dim H_1 = (n-1)^2$, i.e., $H_1 = U_{n-1}$. Consider the Lie algebra $\mathfrak{h}$ of H. It consists of matrices of the following form

$$\begin{pmatrix} \mathfrak{A} & 0 \\ 0 & l(\mathfrak{A}) \end{pmatrix}, \tag{1.3}$$

where $\mathfrak{A} \in \mathfrak{u}_{n-1}$ and $l(\mathfrak{A}) \not\equiv 0$ is a linear function of the matrix elements of $\mathfrak{A}$ ranging in $i\mathbb{R}$. Clearly, $l(\mathfrak{A})$ must vanish on the commutant of $\mathfrak{u}_{n-1}$, which is $\mathfrak{su}_{n-1}$. Hence matrices (1.3) form a Lie algebra if and only if $l(\mathfrak{A}) = c \cdot \mathrm{trace}\, \mathfrak{A}$, where $c \in \mathbb{R} \setminus \{0\}$. Such an algebra can be the Lie algebra of a closed subgroup of $U_{n-1} \times U_1$ only if $c \in \mathbb{Q} \setminus \{0\}$. Hence H is conjugate to $H^n_{k_1,k_2}$ for some $k_1, k_2 \in \mathbb{Z}$, where one can always assume that $k_1 > 0$ and $(k_1, k_2) = 1$.

Now let $m = 1$. We shall proceed as in the proof of Lemma 2.1 in [IKra]. Let $\mathfrak{h}^{\mathbb{C}} := \mathfrak{h} + i\mathfrak{h} \subset \mathfrak{gl}_n$ be the complexification of $\mathfrak{h}$, where $\mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{C})$. The algebra $\mathfrak{h}^{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^n$ and by a theorem of E. Cartan (see, e.g., [GG]), $\mathfrak{h}^{\mathbb{C}}$ is either semisimple or the direct sum of the center $\mathfrak{c}$ of $\mathfrak{gl}_n$ and a semisimple ideal $\mathfrak{t}$. Clearly, the action of the ideal $\mathfrak{t}$ on $\mathbb{C}^n$ is irreducible.

Assume first that $\mathfrak{h}^{\mathbb{C}}$ is semisimple, and let $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h}_1 \oplus \dots \oplus \mathfrak{h}_k$ be its decomposition into the direct sum of simple ideals. Then the natural irreducible n-dimensional representation of $\mathfrak{h}^{\mathbb{C}}$ (given by the embedding of $\mathfrak{h}^{\mathbb{C}}$ in $\mathfrak{gl}_n$) is the tensor product of some irreducible faithful representations of the $\mathfrak{h}_j$ (see, e.g., [GG]). Let n_j be the dimension of the corresponding representation of $\mathfrak{h}_j$, $j = 1, \dots, k$. Then $n_j \ge 2$, $\dim_{\mathbb{C}} \mathfrak{h}_j \le n_j^2 - 1$, and $n = n_1 \cdot \ldots \cdot n_k$. The following observation is simple.

Claim: *If $n = n_1 \cdot \ldots \cdot n_k$, $k \ge 2$, $n_j \ge 2$ for $j = 1, \dots, k$, then $\sum_{j=1}^k n_j^2 \le n^2 - 2n$.*

Since $\dim_{\mathbb{C}} \mathfrak{h}^{\mathbb{C}} = (n-1)^2$, it follows from the above claim that $k = 1$, i.e. $\mathfrak{h}^{\mathbb{C}}$ is simple. The minimal dimensions of irreducible faithful representations of complex simple Lie algebras $\mathfrak{s}$ are well-known (see, e.g., [OV]). In the table below V denotes representations of minimal dimension.

$\mathfrak{s}$	$\dim V$	$\dim \mathfrak{s}$
$\mathfrak{sl}_k$ $k \geq 2$	k	$k^2 - 1$
$\mathfrak{o}_k$ $k \geq 7$	k	$k(k-1)/2$
$\mathfrak{sp}_{2k}$ $k \geq 2$	$2k$	$2k^2 + k$
$\mathfrak{e}_6$	27	78
$\mathfrak{e}_7$	56	133
$\mathfrak{e}_8$	248	248
$\mathfrak{f}_4$	26	52
$\mathfrak{g}_2$	7	14

Since $\dim_{\mathbb{C}} \mathfrak{h}^{\mathbb{C}} = (n-1)^2$, it follows that none of the above possibilities realize. Therefore, $\mathfrak{h}^{\mathbb{C}} = \mathfrak{c} \oplus \mathfrak{t}$, where the dimension of $\mathfrak{t}$ is equal to $n^2 - 2n$. Then, if $n = 2$, we obtain that H coincides with the center of U_2 which is impossible since its action on $\mathbb{C}^2$ is then not irreducible. Assuming that $n \geq 3$ and repeating the above argument for $\mathfrak{t}$ in place of $\mathfrak{h}^{\mathbb{C}}$, we see that $\mathfrak{t}$ can only be isomorphic to $\mathfrak{sl}_{n-1}$. But $\mathfrak{sl}_{n-1}$ does not have an irreducible n-dimensional representation unless $n = 3$.

Thus, $n = 3$ and $\mathfrak{h} \simeq \mathbb{C} \oplus \mathfrak{sl}_2 \simeq \mathbb{C} \oplus \mathfrak{so}_3$. Further, we observe that every irreducible 3-dimensional representation of $\mathfrak{so}_3$ is equivalent to its defining representation. This implies that H is conjugate in $GL_3(\mathbb{C})$ to $e^{i\mathbb{R}}SO_3(\mathbb{R})$. Since $H \subset U_3$ it is straightforward to show that the conjugating element can be chosen to belong to U_3.

The proof of the proposition is complete. $\square$

Let M be a connected complex manifold of dimension $n \geq 2$, and suppose that a connected Lie group $G \subset \text{Aut}(M)$ with $d_G = n^2 + 1$ acts properly on M. Fix $p \in M$, consider the linear isotropy subgroup LG_p, and choose coordinates in $T_p(M)$ so that $LG_p \subset U_n$. We say that the pair (M, G) (or the action of G on M) is of type I, II or III, if the connected identity component LG_p^0 of the group LG_p is conjugate in U_n to a subgroup listed in I, II or III of Proposition 1.1, respectively. Since M is G-homogeneous, this definition is independent of the choice of p.

We will now separately consider actions of each type.

2 Actions of Type I

In this section we prove the following theorem.

THEOREM 2.1 *Let M be a connected complex manifold of dimension 3 and $G \subset Aut(M)$ a connected Lie group with $d_G = 10$ that acts properly on M. If the pair (M, G) is of type I, then it is equivalent to one of the following:*

(i) $(\mathscr{S}, \mathrm{Aut}(\mathscr{S}))$, where $\mathscr{S}$ is the Siegel space

$$\mathscr{S} := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : Z\overline{Z} \ll id\right\},$$

with

$$Z := \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$$

(here $Aut(\mathscr{S})$ is isomorphic to $Sp_4(\mathbb{R})/\mathbb{Z}_2$);

(ii) $(\mathcal{Q}_3, SO_5(\mathbb{R}))$, where $\mathcal{Q}_3$ is the complex quadric in $\mathbb{CP}^4$, and $SO_5(\mathbb{R})$ is realized as a maximal compact subgroup of $Aut(\mathcal{Q}_3) \simeq SO_5(\mathbb{C})$;

(iii) $(\mathbb{C}^3, G_2(\mathbb{C}^3))$, where $G_2(\mathbb{C}^3)$ is the group that consists of all maps from $G(\mathbb{C}^3)$ with $U \in e^{i\mathbb{R}}SO_3(\mathbb{R})$ (see (0.1)).[§]

Proof: Fix $p \in M$. By Bochner's linearization theorem (see [Bo]) there exist an G_p-invariant neighborhood $\mathcal{V}$ of p in M, an LG_p-invariant neighborhood $\mathcal{U}$ of the origin in $T_p(M)$ and a biholomorphic map $F : \mathcal{V} \to \mathcal{U}$, with $F(p) = 0$, such that for every $g \in G_p$ the following holds in $\mathcal{V}$:

$$F \circ g = \alpha_p(g) \circ F,$$

where α_p is the isotropy representation at p. Let $\mathfrak{g}_M$ be the Lie algebra (isomorphic to the Lie algebra of G) of holomorphic vector fields on M arising from the action of G, that is, $\mathfrak{g}_M$ consists of all vector fields X on M for which there exists an element a of the Lie algebra of G such that for all $q \in M$ we have

$$X(q) = \frac{d}{dt}\Big[\exp(ta)(q)\Big]\Big|_{t=0}.$$

Next, let $\mathfrak{g}_{\mathcal{V}}$ be the Lie algebra of the restrictions of the elements of $\mathfrak{g}_M$ to $\mathcal{V}$ and $\mathfrak{g}$ the Lie algebra of vector fields on $\mathcal{U}$ obtained by pushing forward the elements of $\mathfrak{g}_{\mathcal{V}}$ by means of F. Observe that $\mathfrak{g}_M$, $\mathfrak{g}_{\mathcal{V}}$, $\mathfrak{g}$ are naturally isomorphic, and we denote by $\varphi : \mathfrak{g}_M \to \mathfrak{g}$ the isomorphism induced by F.

[§]In [I5] we introduced groups denoted by $G_1(\mathbb{C}^n)$, $G_2(\mathbb{C}^4)$ and $G_3(\mathbb{C}^4)$. Notation in the present paper is consistent with that in [I5].

Let (z_1, z_2, z_3) be coordinates in $T_p(M)$ in which $LG_p^0 = e^{i\mathbb{R}}SO_3(\mathbb{R})$. Since F transforms $G_p^0|_{\mathcal{V}}$ into $LG_p^0|_{\mathcal{U}}$ and since G acts transitively on M, the algebra $\mathfrak{g}$ is generated by $\langle Z_0\rangle \oplus \mathfrak{so}_3(\mathbb{R})$ and some vector fields

$$\begin{aligned} V_j &= \sum_{k=1}^{3} f_j^k\, \partial/\partial z_k, \\ W_j &= \sum_{k=1}^{3} g_j^k\, \partial/\partial z_k, \end{aligned}$$

for $j = 1, 2, 3$, where f_j^k, g_j^k are holomorphic functions on $\mathcal{U}$ such that

$$f_j^k(0) = \delta_j^k, \quad g_j^k(0) = i\delta_j^k.$$

Here

$$Z_0 := i\sum_{k=1}^{3} z_k\, \partial/\partial z_k,$$

and $\mathfrak{so}_3(\mathbb{R})$ is generated by the following vector fields on $\mathcal{U}$:

$$\begin{aligned} Z_1 &:= z_2\, \partial/\partial z_1 - z_1\, \partial/\partial z_2, \\ Z_2 &:= z_3\, \partial/\partial z_1 - z_1\, \partial/\partial z_3, \\ Z_3 &:= z_3\, \partial/\partial z_2 - z_2\, \partial/\partial z_3. \end{aligned}$$

Since $\mathfrak{g}$ contains the vector field Z_0, Hilfssatz 4.8 of [Ka] yields that every vector field in $\mathfrak{g}$ is polynomial and has degree at most 2. Considering $[Z_0, [V_j, Z_0]]$, $[Z_0, [W_j, Z_0]]$ instead of V_j, W_j if necessary, we can assume that V_j, W_j, $j = 1, 2, 3$, have no linear terms.

Since $[V_1, Z_3]$ vanishes at the origin and has no linear terms, it is identically zero, which implies

$$\begin{aligned} V_1 = {} & (1 + \alpha_1 z_1^2 + \alpha_2 z_2^2 + \alpha_2 z_3^2 + \beta z_2 z_3)\, \partial/\partial z_1 + (\gamma_{12} z_1 z_2 + \gamma_{13} z_1 z_3)\, \partial/\partial z_2 + \\ & (-\gamma_{13} z_1 z_2 + \gamma_{12} z_1 z_3)\, \partial/\partial z_3, \end{aligned}$$

for some $\alpha_1, \alpha_2, \beta, \gamma_{12}, \gamma_{13} \in \mathbb{C}$. Similarly, considering $[W_1, Z_3]$ we obtain

$$\begin{aligned} W_1 = {} & (i + \alpha_1' z_1^2 + \alpha_2' z_2^2 + \alpha_2' z_3^2 + \beta' z_2 z_3)\, \partial/\partial z_1 + (\gamma_{12}' z_1 z_2 + \gamma_{13}' z_1 z_3)\, \partial/\partial z_2 + \\ & (-\gamma_{13}' z_1 z_2 + \gamma_{12}' z_1 z_3)\, \partial/\partial z_3, \end{aligned}$$

for some $\alpha_1', \alpha_2', \beta', \gamma_{12}', \gamma_{13}' \in \mathbb{C}$. Further, considering $[V_2, Z_2]$, $[W_2, Z_2]$, $[V_3, Z_1]$, $[W_3, Z_1]$ analogously implies:

$$
\begin{aligned}
V_2 &= (\delta_{12} z_1 z_2 + \delta_{23} z_2 z_3)\,\partial/\partial z_1 + (1 + \varepsilon_1 z_1^2 + \varepsilon_2 z_2^2 + \varepsilon_1 z_3^2 + \zeta z_1 z_3)\,\partial/\partial z_2 + \\
&\qquad (-\delta_{23} z_1 z_2 + \delta_{12} z_2 z_3)\,\partial/\partial z_3,\\
W_2 &= (\delta_{12}' z_1 z_2 + \delta_{23}' z_2 z_3)\,\partial/\partial z_1 + (i + \varepsilon_1' z_1^2 + \varepsilon_2' z_2^2 + \varepsilon_1' z_3^2 + \zeta' z_1 z_3)\,\partial/\partial z_2 + \\
&\qquad (-\delta_{23}' z_1 z_2 + \delta_{12}' z_2 z_3)\,\partial/\partial z_3,\\
V_3 &= (\eta_{13} z_1 z_3 + \eta_{23} z_2 z_3)\,\partial/\partial z_1 + (-\eta_{23} z_1 z_3 + \eta_{13} z_2 z_3)\,\partial/\partial z_2 + \\
&\qquad (1 + \mu_1 z_1^2 + \mu_1 z_2^2 + \mu_2 z_3^2 + \lambda z_1 z_2)\,\partial/\partial z_3,\\
W_3 &= (\eta_{13}' z_1 z_3 + \eta_{23}' z_2 z_3)\,\partial/\partial z_1 + (-\eta_{23}' z_1 z_3 + \eta_{13}' z_2 z_3)\,\partial/\partial z_2 + \\
&\qquad (i + \mu_1' z_1^2 + \mu_1' z_2^2 + \mu_2' z_3^2 + \lambda' z_1 z_2)\,\partial/\partial z_3,
\end{aligned}
$$

for some complex numbers δ_{12}, δ_{23}, ε_1, ε_2, ζ, δ_{12}', δ_{23}', ε_1', ε_2', ζ', η_{13}, η_{23}, μ_1, μ_2, λ, η_{13}', η_{23}', μ_1', μ_2', λ'.

Next, it is easy to see that $[V_1, Z_1] + V_2$ vanishes at the origin and has no linear terms. Therefore, $[V_1, Z_1] = -V_2$, and we obtain

$$
\begin{aligned}
&\beta = \gamma_{13} = \zeta = \delta_{23} = 0,\\
&\gamma_{12} = \delta_{12} = \alpha_1 - \alpha_2,\\
&\varepsilon_1 = \alpha_2, \varepsilon_2 = \alpha_1.
\end{aligned}
\tag{2.1}
$$

Similarly, we have $[V_1, Z_2] = -V_3$ which implies

$$
\begin{aligned}
&\lambda = \eta_{23} = 0,\\
&\eta_{13} = \alpha_1 - \alpha_2,\ \mu_1 = \alpha_2,\ \mu_2 = \alpha_1.
\end{aligned}
\tag{2.2}
$$

An analogous argument yields

$$
\begin{aligned}
&\beta' = \gamma_{13}' = \zeta' = \delta_{23}' = \lambda' = \eta_{23}' = 0,\\
&\gamma_{12}' = \delta_{12}' = \eta_{13}' = \alpha_1' - \alpha_2',\\
&\varepsilon_1' = \mu_1' = \alpha_2',\ \varepsilon_2' = \mu_2' = \alpha_1'.
\end{aligned}
\tag{2.3}
$$

Next, $[V_1, Z_0] - W_1$ vanishes at the origin, has no linear part and hence vanishes identically. This implies

$$
\alpha_1' = -i\alpha_1,\ \alpha_2' = -i\alpha_2. \tag{2.4}
$$

Further, since the commutator $[V_1, W_1]$ vanishes at the origin, its linear part $\mathcal{L}$ is an element of $\langle Z_0 \rangle \oplus \mathfrak{so}_3(\mathbb{R})$. It is straightforward to see that

$$
\begin{aligned}
\mathcal{L} &= 2(\alpha_1' - i\alpha_1) z_1\,\partial/\partial z_1 + \Big((\alpha_1' - i\alpha_1) - (\alpha_2' - i\alpha_2)\Big) z_2\,\partial/\partial z_2 + \\
&\qquad \Big((\alpha_1' - i\alpha_1) - (\alpha_2' - i\alpha_2)\Big) z_3\,\partial/\partial z_3.
\end{aligned}
$$

Clearly, $\mathcal{L}$ can lie in $\langle Z_0\rangle \oplus \mathfrak{so}_3(\mathbb{R})$ only if

$$\begin{aligned} &\alpha_1' - i\alpha_1 = -(\alpha_2' - i\alpha_2),\\ &\alpha_1' - i\alpha_1 \in i\mathbb{R}, \end{aligned}$$

which, together with (2.4), implies that $\alpha_2 = -\alpha_1$ and $\alpha_1 \in \mathbb{R}$. Using (2.1)–(2.3) and writing α_1 instead of α we obtain:

$$\begin{aligned}
V_1 &= (1+\alpha z_1^2 - \alpha z_2^2 - \alpha z_3^2)\,\partial/\partial z_1 + 2\alpha z_1 z_2\,\partial/\partial z_2 + 2\alpha z_1 z_3\,\partial/\partial z_3,\\
W_1 &= (i - i\alpha z_1^2 + i\alpha z_2^2 + i\alpha z_3^2)\,\partial/\partial z_1 - 2i\alpha z_1 z_2\,\partial/\partial z_2 - 2i\alpha z_1 z_3\,\partial/\partial z_3,\\
V_2 &= 2\alpha z_1 z_2\,\partial/\partial z_1 + (1 - \alpha z_1^2 + \alpha z_2^2 - \alpha z_3^2)\,\partial/\partial z_2 + 2\alpha z_2 z_3\,\partial/\partial z_3,\\
W_2 &= -2i\alpha z_1 z_2\,\partial/\partial z_1 + (i + i\alpha z_1^2 - i\alpha z_2^2 + i\alpha z_3^2)\,\partial/\partial z_2 - 2i\alpha z_2 z_3\,\partial/\partial z_3,\\
V_3 &= 2\alpha z_1 z_3\,\partial/\partial z_1 + 2\alpha z_2 z_3\,\partial/\partial z_2 + (1 - \alpha z_1^2 - \alpha z_2^2 + \alpha z_3^2)\,\partial/\partial z_3,\\
W_3 &= -2i\alpha z_1 z_3\,\partial/\partial z_1 - 2i\alpha z_2 z_3\,\partial/\partial z_2 + (i + i\alpha z_1^2 + i\alpha z_2^2 - i\alpha z_3^2)\,\partial/\partial z_3.
\end{aligned}$$

We will now show that the cases $\alpha < 0$, $\alpha > 0$ and $\alpha = 0$ lead to the manifolds and groups listed in (i), (ii) and (iii) of the theorem, respectively. For this purpose we refer to the general theory of Hermitian symmetric spaces (see [H] for details). By [Pal] one can find a G-invariant Hermitian metric on M. Since LG_q for every $q \in M$ contains the element $-\mathrm{id}$, the manifold M equipped with such a metric becomes a Hermitian symmetric space. The group LG_p^0 acts irreducibly on $T_p(M)$, and therefore M either is an irreducible Hermitian symmetric space, or is equivalent (holomorphically and isometrically) to $\mathbb{C}^3$ with the flat metric.

Suppose first that $\alpha < 0$. Then, changing coordinates as

$$z_j \mapsto \sqrt{-\alpha}\,z_j, \quad j = 1, 2, 3,$$

we can assume that $\alpha = -1$. In this case the algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{sp}_4(\mathbb{R})$. Then the group G is simple. Since a simple group cannot act by isometries on $\mathbb{C}^3$ and contain a symmetry at every point, we obtain that M is an irreducible Hermitian symmetric space. Furthermore, every group with Lie algebra isomorphic to $\mathfrak{sp}_4(\mathbb{R})$ is non-compact, and hence M is non-compact. It now follows from E. Cartan's classification of irreducible Hermitian symmetric spaces that M is equivalent to $\mathscr{S}$. Since $\mathrm{Aut}(\mathscr{S})$ is connected and 10-dimensional (in fact, isomorphic to $Sp_4(\mathbb{R})/\mathbb{Z}_2$), every equivalence map transforms G into $\mathrm{Aut}(\mathscr{S})$. Thus, we have obtained (i) of the theorem.

Suppose next that $\alpha > 0$. Then, changing coordinates as

$$z_j \mapsto \sqrt{\alpha} z_j, \quad j = 1, 2, 3,$$

we can assume that $\alpha = 1$. In this case the algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}_5(\mathbb{R})$. Again, G is simple, and, since every connected group with Lie algebra isomorphic to $\mathfrak{so}_5(\mathbb{R})$ is compact, M is a compact Hermitian symmetric space. It now follows from E. Cartan's classification of irreducible Hermitian symmetric spaces that M is equivalent to $\mathcal{Q}_3$ by means of a map that transforms G into $SO_5(\mathbb{R}) \subset \mathrm{Aut}(\mathcal{Q}_3)$. Thus, we have obtained (ii) of the theorem.

Suppose now that $\alpha = 0$. In this case $\mathfrak{g}$ is isomorphic to the semidirect sum of $\mathbb{R} \oplus \mathfrak{o}_3(\mathbb{R})$ and $\mathbb{C}^3$. A group with such Lie algebra cannot act by holomorphic isometries on a 3-dimensional irreducible Hermitian symmetric space and contain a symmetry at every point; therefore M is equivalent to $\mathbb{C}^3$ by means of a map $\mathcal{F}$ that transforms G into a subgroup of $G(\mathbb{C}^3)$ (recall that $G(\mathbb{C}^3)$ is the full group of holomorphic isometries of $\mathbb{C}^3$ with respect to the flat metric). Let $p_0 \in M$ be such that $\mathcal{F}(p_0) = 0$. Then $\mathcal{F}$ transforms $G^0_{p_0}$ into a subgroup H of $U_3 \subset G(\mathbb{C}^3)$ isomorphic to $e^{i\mathbb{R}}SO_3(\mathbb{R})$ and acting irreducibly on $T_0(\mathbb{C}^3)$. By Proposition 1.1, the subgroup H is conjugate in U_3 to the standard embedding of $e^{i\mathbb{R}}SO_3(\mathbb{R})$ in U_3, and hence there exists an equivalence map $\tilde{\mathcal{F}}$ between M and $\mathbb{C}^3$ that transforms $G^0_{p_0}$ into $e^{i\mathbb{R}}SO_3(\mathbb{R})$. We now argue as at the beginning of the proof of the theorem with $\tilde{\mathcal{F}}$ in place of F. For the corresponding vector fields V_j, W_j we then obtain

$$V_j = \partial/\partial z_j, \quad W_j = i\,\partial/\partial z_j, \quad j = 1, 2, 3.$$

This implies that $\tilde{\mathcal{F}}$ transforms G into $G_2(\mathbb{C}^3)$, and we have obtained (iii) of the theorem.

The proof is complete. $\square$

3 Actions of Type II

In this section we obtain the following result.

THEOREM 3.1 *Let M be a connected complex manifold of dimension $n \geq 3$ and $G \subset Aut(M)$ a connected Lie group with $d_G = n^2 + 1$ that acts properly on M. If the pair (M, G) is of type II, then it is equivalent to*

$(\mathbb{C}^{n-1} \times M', G_1(\mathbb{C}^{n-1}) \times G')$, where M' is one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, and G' is one of the groups $Aut(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively.[¶]

Proof: Fix $p \in M$, choose $\mathcal{V}$, $\mathcal{U}$, F and consider the Lie algebras $\mathfrak{g}_M$,$\mathfrak{g}_{\mathcal{V}}$, $\mathfrak{g}$ and the isomorphism $\varphi : \mathfrak{g}_M \to \mathfrak{g}$ as in the proof of Theorem 2.1. Next, we fix coordinates in $T_p(M)$ in which $LG_p^0 = SU_{n-1} \times U_1$. The algebra $\mathfrak{g}$ is generated by $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$ and some vector fields

$$\begin{aligned} V_j &= \sum_{k=1}^{n} f_j^k \, \partial/\partial z_k, \\ W_j &= \sum_{k=1}^{n} g_j^k \, \partial/\partial z_k, \end{aligned}$$

where the functions f_j^k, g_j^k, $j, k = 1, \dots, n$, are holomorphic on $\mathcal{U}$ and satisfy the conditions

$$f_j^k(0) = \delta_j^k, \quad g_j^k(0) = i\delta_j^k.$$

Here $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$ is realized as the algebra of vector fields on $\mathcal{U}$ of the form

$$\sum_{j=1}^{n-1} (a_{j\,1} z_1 + \cdots + a_{j\,n-1} z_{n-1}) \, \partial/\partial z_j + iaz_n \, \partial/\partial z_n,$$

with

$$\begin{pmatrix} a_{1\,1} & \dots & a_{1\,n-1} \\ \vdots & \vdots & \vdots \\ a_{n-1\,1} & \dots & a_{n-1\,n-1} \end{pmatrix} \in \mathfrak{su}_{n-1},$$

and $a \in \mathbb{R}$.

Let

$$Z_n := iz_n \, \partial/\partial z_n,$$

(observe that Z_n generates the $\mathfrak{u}_1$-component of $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$) and consider $[V_j, Z_n]$, $[W_j, Z_n]$ for $j = 1, \dots, n-1$. Since these commutators vanish at 0, they lie in $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$, which implies that the functions f_j^k, g_j^k are independent of z_n for $k = 1, \dots, n-1$ and that

$$\begin{aligned} f_j^n &= \tilde{f}_j^n(z_1, \dots, z_{n-1}) \, z_n, \\ g_j^n &= \tilde{g}_j^n(z_1, \dots, z_{n-1}) \, z_n, \end{aligned}$$

[¶]The group $G_1(\mathbb{C}^n)$ was introduced in [I5] and consists of all maps from $G(\mathbb{C}^n)$ with $U \in SU_n$ (we usually write $G_1(\mathbb{C})$ instead of $G_1(\mathbb{C}^1)$).

for some holomorphic functions $\tilde{f}_j^n$, $\tilde{g}_j^n$.

For every pair of indices $1 \le j, l \le n-1$, $j \ne l$, the vector fields

$$\begin{array}{rcl} X_{jl} & := & iz_j\,\partial/\partial z_j - iz_l\,\partial/\partial z_l, \\ Y_{jl} & := & z_l\,\partial/\partial z_j - z_j\,\partial/\partial z_l \end{array}$$

lie in the $\mathfrak{su}_{n-1}$-component of $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$. We now compute the commutators $[V_j, X_{j,l}]$, $[W_j, X_{jl}]$, $[V_j, Y_{jl}]$, $[V_l, Y_{jl}]$ and observe that $[V_j, X_{jl}] - W_j$, $[W_j, X_{jl}] + V_j$, $[V_j, Y_{jl}] + V_l$, $[V_l, Y_{jl}] - V_j$ vanish at the origin and hence lie in $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$. This yields

for $n \ge 4$, $j = 1, \dots, n-1$:

$$\begin{array}{rcl} \tilde{f}_j^n & = & i\rho_j + \lambda z_j, \\ \tilde{g}_j^n & = & i\sigma_j - i\lambda z_j, \end{array}$$

and

for $n = 3$:

$$\begin{array}{rcl} \tilde{f}_1^3 & = & i\rho_1 + \mu z_1 + \nu z_2, \\ \tilde{f}_2^3 & = & i\rho_2 - \nu z_1 + \mu z_2, \\ \tilde{g}_1^3 & = & i\sigma_1 - i\mu z_1 + i\nu z_2, \\ \tilde{g}_2^3 & = & i\sigma_2 - i\nu z_1 - i\mu z_2, \end{array}$$

where $\rho_j, \sigma_j \in \mathbb{R}$, $\lambda, \mu, \nu \in \mathbb{C}$. We now define: $V_j' := V_j - \rho_j Z_n$, $W_j' := W_j - \sigma_j Z_n$ for $j = 1, \dots, n-1$.

Further, consider the commutators $[V_n, X_{jl}]$, $[W_n, X_{jl}]$, $[V_n, Y_{jl}]$, $[W_n, Y_{jl}]$. Each of these commutators vanishes at the origin and hence lies in $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$. This gives that f_n^n, g_n^n are independent of $z_1, \dots, z_{n-1}$ and that for $k = 1, \dots, n-1$ the following holds:

$$\begin{array}{rcl} f_n^k & = & \alpha^k + \beta^k(z_n) z_k, \\ g_n^k & = & \gamma^k + \delta^k(z_n) z_k, \end{array}$$

where α^k and γ^k are linear functions independent of z_k, z_n.

Next, computing the commutators $[V_n, Z_n]$ and $[W_n, Z_n]$, we see that $[V_n, Z_n] - W_n$ and $[W_n, Z_n] + V_n$ vanish at 0 and hence are elements of $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$. This gives

$$\begin{array}{rcl} V_n & = & \displaystyle\sum_{k=1}^{n-1} \varepsilon^k z_k z_n\,\partial/\partial z_k + f_n^n\,\partial/\partial z_n \ (\mathrm{mod}\ \mathfrak{su}_{n-1}), \\ W_n & = & \displaystyle -i\sum_{k=1}^{n-1} \varepsilon^k z_k z_n\,\partial/\partial z_k + g_n^n\,\partial/\partial z_n \ (\mathrm{mod}\ \mathfrak{su}_{n-1}), \end{array}$$

for some $\varepsilon^k \in \mathbb{C}$, $k = 1, \ldots, n-1$, and we set

$$\begin{aligned} V_n' &:= \sum_{k=1}^{n-1} \varepsilon^k z_k z_n \, \partial/\partial z_k + f_n^n \, \partial/\partial z_n, \\ W_n' &:= -i \sum_{k=1}^{n-1} \varepsilon^k z_k z_n \, \partial/\partial z_k + g_n^n \, \partial/\partial z_n. \end{aligned}$$

Consider now for each $1 \le j \le n-1$ the commutator $[V_j', V_n']$. Its linear part $\mathcal{L}_j$ is easy to find:

for $n \ge 4$, $j = 1, \ldots, n-1$:

$$\mathcal{L}_j = \varepsilon^j z_n \, \partial/\partial z_j - \lambda z_j \, \partial/\partial z_n,$$

and

for $n = 3$:

$$\begin{aligned} \mathcal{L}_1 &= \varepsilon^1 z_3 \, \partial/\partial z_1 - (\mu z_1 + \nu z_2) \, \partial/\partial z_3, \\ \mathcal{L}_2 &= \varepsilon^2 z_3 \, \partial/\partial z_2 - (-\nu z_1 + \mu z_2) \, \partial/\partial z_3. \end{aligned}$$

Clearly, every commutator $[V_j', V_n']$ vanishes at 0. Hence it is an element of $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$ and thus coincides with $\mathcal{L}_j$. However, for $n \ge 4$ the vector field $\mathcal{L}_j$ can be an element of $\mathfrak{su}_{n-1} \oplus \mathfrak{u}_1$ only if $\varepsilon^j = \lambda = 0$. For $n = 3$ the vector fields $\mathcal{L}_1$ and $\mathcal{L}_2$ can be elements of $\mathfrak{su}_2 \oplus \mathfrak{u}_1$ only if $\varepsilon^1 = \varepsilon^2 = \mu = \nu = 0$. Therefore, V_j', W_j', for $j = 1, \ldots, n-1$, are independent of z_n and V_n', W_n' are independent of $z_1, \ldots, z_{n-1}$.

Thus, we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1$ is the ideal generated by $\mathfrak{su}_{n-1}$ and V_j', W_j', for $j = 1, \ldots, n-1$, and $\mathfrak{g}_2$ is the ideal generated by $\mathfrak{u}_1$ and V_n', W_n'.

Let G_j be the connected normal (possibly non-closed) subgroup of G with Lie algebra $\tilde{\mathfrak{g}}_j := \varphi^{-1}(\mathfrak{g}_j) \subset \mathfrak{g}_M$ for $j = 1, 2$. Clearly, for each j the subgroup G_j contains $\alpha_p^{-1}(L_{j\,p}) \subset G_p^0$, where $L_{1\,p} \simeq SU_{n-1}$ and $L_{2\,p} \simeq U_1$ are the subgroups of LG_p^0 given by $\alpha = 0$ and $A = \mathrm{id}$ in formula (1.1), respectively. Consider the orbit $G_j p$ and the isotropy subgroup $G_{j\,p}$ of the point p with respect to the G_j-action for $j = 1, 2$. Clearly, for each j we have $G_{j\,p}^0 = \alpha_p^{-1}(L_{j\,p})$. Furthermore, for each j there exists a neighborhood $\mathcal{W}_j$ of the identity in G_j such that

$$\begin{aligned} &\mathcal{W}_1 p = F^{-1}\left(\mathcal{U}' \cap \{z_n = 0\}\right), \\ &\mathcal{W}_2 p = F^{-1}\left(\mathcal{U}' \cap \{z_1 = \cdots = z_{n-1} = 0\}\right), \end{aligned}$$

for some neighborhood $\mathcal{U}' \subset \mathcal{U}$ of the origin in $T_p(M)$. Thus, each $G_j p$ is a complex (possibly non-closed) submanifold of M, and the ideal $\tilde{\mathfrak{g}}_j$ consists exactly of those vector fields from $\mathfrak{g}_M$ that are tangent to G_jp for $j = 1, 2$. Next, since $L_{j\,p}$ acts transitively on real directions in $T_p(G_jp)$ for $j = 1, 2$, by [GK], [BDK] we obtain that G_1p is holomorphically equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$ and G_2p is holomorphically equivalent to one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$.

We will now show that each G_j is closed in G. It is done as in the proof of Theorem 2.1 in [I5], but we repeat the argument here for the reader's convenience. We assume that $j = 1$; for $j = 2$ the proof is similar. Let $\mathfrak{U}$ be a neighborhood of 0 in $\mathfrak{g}_M$ where the exponential map into G is a diffeomorphism, and let $\mathfrak{V} := \exp(\mathfrak{U})$. To prove that G_1 is closed in G it is sufficient to show that for some neighborhood $\mathfrak{W}$ of $e \in G$, $\mathfrak{W} \subset \mathfrak{V}$, we have $G_1 \cap \mathfrak{W} = \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U}) \cap \mathfrak{W}$. Assuming the opposite we obtain a sequence $\{g_j\}$ of elements of G_1 converging to e in G such that for every j we have $g_j = \exp(a_j)$ with $a_j \in \mathfrak{U} \setminus \tilde{\mathfrak{g}}_1$. Observe now that there exists a neighborhood $\mathcal{V}'$ of p in M foliated by complex submanifolds holomorphically equivalent to $\mathbb{B}^{n-1}$ in such a way that the leaf passing through p lies in G_1p. Specifically, we take $\mathcal{V}' := F^{-1}(\mathcal{U}')$ for a suitable neighborhood $\mathcal{U}' \subset \mathcal{U}$ of the origin in $T_p(M)$, and the leaves of the foliation are then given as $F^{-1}(\mathcal{U}' \cap \{z_n = \text{const}\})$. For every $s \in \mathcal{V}'$ we denote by N_s the leaf of the foliation passing through s. Observe that for every $s \in \mathcal{V}'$ vector fields from $\tilde{\mathfrak{g}}_1$ are tangent to N_s at every point. Let $p_j := g_jp$. If j is sufficiently large, we have $p_j \in \mathcal{V}'$. We will now show that $N_{p_j} \neq N_p$ for large j.

Let $\mathfrak{U}'' \subset \mathfrak{U}' \subset \mathfrak{U}$ be neighborhoods of 0 in $\mathfrak{g}_M$ such that: (a) $\exp(\mathfrak{U}'') \cdot \exp(\mathfrak{U}'') \subset \exp(\mathfrak{U}')$; (b) $\exp(\mathfrak{U}'') \cdot \exp(\mathfrak{U}') \subset \exp(\mathfrak{U})$; (c) $\mathfrak{U}' = -\mathfrak{U}'$; (d) $G_{1\,p} \cap \exp(\mathfrak{U}') \subset \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U}')$. We also assume that $\mathcal{V}'$ is chosen so that $N_p \subset \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U}'')p$. Suppose that $p_j \in N_p$. Then $p_j = sp$ for some $s \in \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U}'')$ and hence $t := g_j^{-1}s$ is an element of $G_{1\,p}$. For large j we have $g_j^{-1} \in \exp(\mathfrak{U}'')$. Condition (a) now implies that $t \in \exp(\mathfrak{U}')$ and hence by (c), (d) we have $t^{-1} \in \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U}')$. Therefore, by (b) we obtain $g_j \in \exp(\tilde{\mathfrak{g}}_1 \cap \mathfrak{U})$ which contradicts our choice of g_j. Thus, for large j the leaves N_{p_j} are distinct from N_p. Furthermore, they accumulate to $N_p \subset G_1p$. At the same time, since vector fields from $\tilde{\mathfrak{g}}_1$ are tangent to every N_{p_j}, we have $N_{p_j} \subset G_1p$ for all j, and thus the orbit G_1p accumulates to itself (we will use this term in the future in analogous situations). Below we will show that this is in fact impossible thus obtaining a contradiction. Clearly, we only need to consider the case when G_1p is equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$.

By the result of [GK], the orbit G_1p is holomorphically equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$ by means of a map that maps p into the origin and transforms $G_{1\,p}^0$ into $SU_{n-1} \subset G(\mathbb{C}^{n-1})$. Consider the set $S := G_1p \cap G_2p$. The set

S contains a non-constant sequence of points converging to p, but does not contain any curve. Since G^0_{1p} preserves each of G_1p, G_2p, it preserves S. However, the G^0_{1p}-orbit of every point in G_1p other than p is a hypersurface in G_1p diffeomorphic to the sphere S^{2n-3}. This contradiction shows that G_1 is closed in G.

Thus, we have proved that G_j is closed in G for $j = 1, 2$. Hence G_j acts on M properly and G_jp is a closed submanifold of M for each j. Recall that G_1p is equivalent to one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$ and G_2p is equivalent to one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, and denote by F_1, F_2 the respective equivalence maps. Let $K_j \subset G_j$ be the ineffectivity kernel of the G_j-action on G_jp for $j = 1, 2$. Clearly, $K_j \subset G_{jp}$ and, since G^0_{jp} acts on G_jp effectively, K_j is a discrete normal subgroup of G_j for each j (in particular, K_j lies in the center of G_j for $j = 1, 2$). Since $d_{G_1} = n^2 - 2 = (n-1)^2 + 2(n-1) - 1$, Theorem 1.1 in [I5] yields that G_1p is in fact equivalent to $\mathbb{C}^{n-1}$ and that F_1 can be chosen to transform G_1/K_1 into $G_1(\mathbb{C}^{n-1})$. Further, since $d_{G_2} = 3$, the map F_2 can be chosen to transform G_2/K_2 into one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively. Here G_j/K_j is viewed as a subgroup of $\mathrm{Aut}(G_jp)$ for each j.

We will now show that the subgroup K_j is in fact trivial for each $j = 1, 2$. Let first $j = 1$. Clearly, $K_1 \setminus \{e\} \subset G_{1p} \setminus G^0_{1p}$, and if K_1 is non-trivial, the compact group G_{1p} is disconnected. Observe that any maximal compact subgroup of $G_1(\mathbb{C}^{n-1}) \simeq SU_{n-1} \ltimes \mathbb{C}^{n-1}$ is isomorphic to SU_{n-1} and therefore, if G_1/K_1 is isomorphic to $G_1(\mathbb{C}^{n-1})$, it follows that G_{1p} is a maximal compact subgroup of G_1. Since G_1 is connected, so is G_{1p}, and therefore K_1 is trivial. Let $j = 2$. If G_2/K_2 is isomorphic to either $\mathrm{Aut}(\mathbb{B}^1)$ or $G(\mathbb{C})$, the above argument can be applied. Suppose now that G_2/K_2 is isomorphic to $G(\mathbb{CP}^1) \simeq PSU_2$. If K_2 is non-trivial, then $G_2 \simeq SU_2$ and $K_2 \simeq \mathbb{Z}_2$. Then G^0_{2p} is conjugate in G_2 (upon the identification of G_2 with SU_2) to the subgroup of matrices of the form

$$\begin{pmatrix} 1/b & 0 \\ 0 & b \end{pmatrix},$$

where $|b| = 1$ (see e.g. Lemma 2.1 of [IKru]). Since this subgroup contains the center of SU_2, hence so does G^0_{2p}. In particular, $K_2 \subset G^0_{2p}$ which contradicts the non-triviality of K_2. Thus, G_1 is isomorphic to $G_1(\mathbb{C}^{n-1})$ and G_2 is isomorphic to one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$.

We remark here that since M is G-homogeneous and G_j is normal in G, the discussion above remains valid for any point $q \in M$ in place of p; in particular, all G_j-orbits are pairwise holomorphically equivalent, $j = 1, 2$.

Next, since $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, the group G is a locally direct product of G_1 and G_2. We claim that $\mathscr{T} := G_1 \cap G_2$ is trivial. Indeed, $\mathscr{T}$ is a discrete normal subgroup of each of G_1, G_2. However, every discrete normal subgroup of each of $G_1(\mathbb{C}^{n-1})$, $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$ is trivial, since the center of each of these groups is trivial. Hence $\mathscr{T}$ is trivial and therefore $G = G_1 \times G_2$.

We will now show that for every $q_1, q_2 \in M$ the orbits $G_1 q_1$ and $G_2 q_2$ intersect at exactly one point. Let $g \in G$ be an element such that $gq_2 = q_1$. It can be uniquely represented in the form $g = g_1 g_2$ with $g_j \in G_j$ for $j = 1, 2$, and therefore we have $g_2 q_2 = g_1^{-1} q_1$. Hence the intersection $G_1 q_1 \cap G_2 q_2$ is non-empty. We will now prove that $G_1 q \cap G_2 q = \{q\}$ for every $q \in M$. Suppose that for some q the intersection $G_1 q \cap G_2 q$ contains a point $q' \neq q$. Let $b_j \in G_j$ be elements such that $b_j q = q'$ for $j = 1, 2$. Then $b_1^{-1} b_2 \in G_q$. For $h \in G_q$ uniquely represented as $h = h_1 h_2$, with $h_j \in G_j$, we set $\Pi_1(h) := h_1$. Then $\Pi_1(G_q)$ is a compact subgroup of G_1 containing $G_{1\,q}$. Since G_1 is isomorphic to $G_1(\mathbb{C}^{n-1})$, the subgroup $G_{1\,q}$ is connected and is a maximal compact subgroup of G_1. Therefore, $\Pi_1(G_q) = G_{1\,q}$. It then follows that in this case $b_1 \in G_{1\,q}$, and hence $q' = q$. Thus, for every $p \in M$ the intersection $G_1 q \cap G_2 q$ consists of q alone.

Let, as before, F_1 be a biholomorphic map from $G_1 p$ onto $\mathbb{C}^{n-1}$ that transforms G_1 into $G_1(\mathbb{C}^{n-1})$, and by F_2 a biholomorphic map from $G_2 p$ onto M', where M' is one of $\mathbb{B}^1$, $\mathbb{C}$, $\mathbb{CP}^1$, that transforms G_2 into G', where G' is one of $\mathrm{Aut}(\mathbb{B}^1)$, $G(\mathbb{C})$, $G(\mathbb{CP}^1)$, respectively. We will now construct a biholomorphic map $\mathcal{F}$ from M onto $\mathbb{C}^{n-1} \times M'$. For $q \in M$ consider $G_2 q$ and let r be the unique point of intersection of $G_1 p$ and $G_2 q$. Let $g \in G_1$ be an element such that $r = gp$. Then we set $\mathcal{F}(q) := (F_1(r), F_2(g^{-1}q))$. Clearly, $\mathcal{F}$ is a well-defined diffeomorphism from M onto $\mathbb{C}^{n-1} \times M'$. Since the foliation of M by G_j-orbits is holomorphic for each j, the map $\mathcal{F}$ is in fact holomorphic. By construction, $\mathcal{F}$ transforms G into $G_1(\mathbb{C}^{n-1}) \times G'$.

The proof is complete. $\square$

4 Actions of Type III

In this section we give a large number of examples of actions of type III. Some of the examples can be naturally combined into classes and some of the actions form parametric families. In what follows $n \geq 2$.

(i). Here both the manifolds and the groups are represented as direct products.

(ia). $M = M' \times \mathbb{C}$, where M' is one of $\mathbb{B}^{n-1}$, $\mathbb{C}^{n-1}$, $\mathbb{CP}^{n-1}$, and $G = G' \times G_1(\mathbb{C})$, where G' is one of the groups $\mathrm{Aut}(\mathbb{B}^{n-1})$, $G(\mathbb{C}^{n-1})$, $G(\mathbb{CP}^{n-1})$, respectively.

(ib). $M = M' \times \mathbb{C}^*$, where M' is as in (ia), and $G = G' \times \mathrm{Aut}(\mathbb{C}^*)^0$, where G' is as in (ia).

(ic). $M = M' \times \mathbb{T}$, where M' is as in (ia) and $\mathbb{T}$ is an elliptic curve; $G = G' \times \mathrm{Aut}(\mathbb{T})^0$, where G' is as in (ia).

(id). $M = M' \times \mathcal{P}_>$, where M' is as in (ia) and $\mathcal{P}_> := \{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta > 0\}$; $G = G' \times G(\mathcal{P}_>)$, where G' as in (ia) and $G(\mathcal{P}_>)$ is the group of all maps of the form

$$\xi \mapsto \lambda\xi + ia,$$

with $a \in \mathbb{R}$, $\lambda > 0$.

(ii). Parts (iib) and (iic) of this example are obtained by passing to quotients in Part (iia).

(iia). $M = \mathbb{B}^{n-1} \times \mathbb{C}$, and G consists of all maps of the form

$$\begin{aligned} z' &\mapsto \frac{Az' + b}{cz' + d}, \\ z_n &\mapsto z_n + \ln(cz' + d) + a, \end{aligned}$$

where

$$\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in SU_{n-1,1},$$

$a \in \mathbb{C}$, and $z' := (z_1, \dots, z_{n-1})$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{B}^{n-1} \times \mathbb{C}$

$$\begin{aligned} z' &\mapsto \frac{Az' + b}{cz' + d}, \\ z_n &\mapsto z_n + T\ln(cz' + d) + a, \end{aligned} \tag{4.1}$$

where A, a, b, c, d are as above. Example (ia) for $M' = \mathbb{B}^{n-1}$ is included in this family for $T = 0$. If $T \neq 0$, then conjugating group (4.1) in $\mathrm{Aut}(\mathbb{B}^{n-1} \times \mathbb{C})$ by the automorphism

$$\begin{aligned} z' &\mapsto z' \\ z_n &\mapsto z_n/T, \end{aligned} \tag{4.2}$$

we can assume that $T=1$.

(iib). $M=\mathbb{B}^{n-1}\times\mathbb{C}^*$, and for a fixed $T\in\mathbb{C}^*$ the group G consists of all maps of the form

$$\begin{aligned} z' &\mapsto \frac{Az'+b}{cz'+d},\\ z_n &\mapsto \chi(cz'+d)^T z_n, \end{aligned} \tag{4.3}$$

where A,b,c,d are as in (iia) and $\chi\in\mathbb{C}^*$. Example (ib) for $M'=\mathbb{B}^{n-1}$ can be included in this family for $T=0$. This family is obtained from (4.1) by passing to a quotient in the last variable.

(iic). $M=\mathbb{B}^{n-1}\times\mathbb{T}$, where $\mathbb{T}$ is an elliptic curve, and for a fixed $T\in\mathbb{C}^*$ the group G consists of all maps of the form

$$\begin{aligned} z' &\mapsto \frac{Az'+b}{cz'+d},\\ [z_n] &\mapsto \left[\chi(cz'+d)^T z_n\right], \end{aligned}$$

where A,b,c,d,χ are as in (iib), $\mathbb{T}$ is obtained from $\mathbb{C}^*$ by factorizing by the equivalence relation $z_n\sim dz_n$, for some $d\in\mathbb{C}^*$, $|d|\neq 1$, and $[z_n]\in\mathbb{T}$ is the equivalence class of a point $z_n\in\mathbb{C}^*$. Example (ic) for $M'=\mathbb{B}^{n-1}$ can be included in this family for $T=0$. This family is obtained from (4.3) by passing to a quotient in the last variable.

(iii). Part (iiib) of this example is obtained by passing to a quotient in Part (iiia).

(iiia). $M=\mathbb{C}^n$, and G consists of all maps of the form

$$\begin{aligned} z' &\mapsto e^{\mathrm{Re}\, b}Uz'+a,\\ z_n &\mapsto z_n+b, \end{aligned}$$

where $U\in U_{n-1}$, $a\in\mathbb{C}^{n-1}$, $b\in\mathbb{C}$. In fact, for $T\in\mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^n$

$$\begin{aligned} z' &\mapsto e^{\mathrm{Re}\,(Tb)}Uz'+a,\\ z_n &\mapsto z_n+b, \end{aligned} \tag{4.4}$$

where U, a, b are as above. Example (ia) for $M'=\mathbb{C}^{n-1}$ is included in this family for $T=0$. If $T\neq 0$, then conjugating group (4.4) in $\mathrm{Aut}(\mathbb{C}^n)$ by the automorphism

$$\begin{array}{rcl} z' & \mapsto & z' \\ z_n & \mapsto & Tz_n, \end{array}$$

we can assume that $T = 1$.

(iiib). $M = \mathbb{C}^{n-1} \times \mathbb{C}^*$, and for a fixed $T \in \mathbb{R}^*$ the group G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & e^{T \,\mathrm{Re}\, b} U z' + a, \\ z_n & \mapsto & e^b z_n, \end{array}$$

where U, a, b are as in (iiia). Example (ib) for $M' = \mathbb{C}^{n-1}$ can be included in this family for $T = 0$. This family is obtained from (4.4) for $T \in \mathbb{R}^*$ by passing to a quotient in the last variable.

(iv). Parts (ivb) and (ivc) of this example are obtained by passing to quotients in Part (iva).

(iva). $M = \mathbb{C}^n$, and G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & Uz' + a, \\ z_n & \mapsto & z_n + \langle Uz', a \rangle + b, \end{array}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{C}$, and $\langle \cdot \, , \cdot \rangle$ is the inner product in $\mathbb{C}^{n-1}$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^n$

$$\begin{array}{rcl} z' & \mapsto & Uz' + a, \\ z_n & \mapsto & z_n + T\langle Uz', a \rangle + b, \end{array} \tag{4.5}$$

where U, a, b are as above. Example (ia) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. If $T \neq 0$, then conjugating group (4.5) in $\mathrm{Aut}(\mathbb{C}^n)$ by automorphism (4.2), we can assume that $T = 1$.

(ivb). $M = \mathbb{C}^{n-1} \times \mathbb{C}^*$, and for a fixed $0 \le \tau < 2\pi$ the group G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & Uz' + a, \\ z_n & \mapsto & \chi \exp\Big(e^{i\tau} \langle Uz', a \rangle\Big) z_n, \end{array} \tag{4.6}$$

where U, a are as in (iva) and $\chi \in \mathbb{C}^*$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^{n-1} \times \mathbb{C}^*$

$$\begin{array}{rcl} z' & \mapsto & Uz' + a, \\ z_n & \mapsto & \chi \exp\Big(T \langle Uz', a \rangle\Big) z_n, \end{array} \tag{4.7}$$

where U, a, χ are as above. Example (ib) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. For $T \neq 0$ this family is obtained from (4.5) by passing to a quotient in the last variable. Furthermore, conjugating group (4.7) for $T \neq 0$ in $\mathrm{Aut}(\mathbb{C}^{n-1} \times \mathbb{C}^*)$ by the automorphism

$$\begin{array}{rcl} z' & \mapsto & \sqrt{|T|}z' \\ z_n & \mapsto & z_n, \end{array}$$

we obtain the group defined in (4.6) for $\tau = \arg T$.

(ivc). $M = \mathbb{C}^{n-1} \times \mathbb{T}$, where $\mathbb{T}$ is an elliptic curve, and for a fixed $0 \leq \tau < 2\pi$ the group G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & Uz' + a, \\ [z_n] & \mapsto & \left[\chi \exp\left(e^{i\tau}\langle Uz', a\rangle\right) z_n\right], \end{array} \tag{4.8}$$

where U, a, χ are as in (ivb), $\mathbb{T}$ is obtained from $\mathbb{C}^*$ by factorizing by the equivalence relation $z_n \sim dz_n$, for some $d \in \mathbb{C}^*$, $|d| \neq 1$, and $[z_n] \in \mathbb{T}$ is the equivalence class of a point $z_n \in \mathbb{C}^*$. In fact, for $T \in \mathbb{C}$ one can consider the following family of groups acting on $\mathbb{C}^{n-1} \times \mathbb{T}$

$$\begin{array}{rcl} z' & \mapsto & Uz' + a, \\ [z_n] & \mapsto & \left[\chi \exp\left(T\langle Uz', a\rangle\right) z_n\right], \end{array} \tag{4.9}$$

where U, a, χ are as above. Example (ic) for $M' = \mathbb{C}^{n-1}$ is included in this family for $T = 0$. For $T \neq 0$ this family is obtained from (4.7) by passing to a quotient in the last variable. Furthermore, conjugating group (4.9) for $T \neq 0$ in $\mathrm{Aut}(\mathbb{C}^{n-1} \times \mathbb{T})$ by the automorphism

$$\begin{array}{rcl} z' & \mapsto & \sqrt{|T|}z' \\ \xi & \mapsto & \xi, \end{array}$$

where $\xi \in \mathbb{T}$, we obtain the group defined in (4.8) for $\tau = \arg T$.

(v). $M = \mathbb{C}^{n-1} \times \mathcal{P}_>$, and for a fixed $T \in \mathbb{R}^*$ the group G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & \lambda^T Uz' + a, \\ z_n & \mapsto & \lambda z_n + ib, \end{array}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{R}$, $\lambda > 0$. Example (id) for $M' = \mathbb{C}^{n-1}$ can be included in this family for $T = 0$.

(vi). $M = \mathbb{C}^n$, and for fixed $k_1, k_2 \in \mathbb{Z}$, $(k_1, k_2) = 1$, $k_1 > 0$, $k_2 \neq 0$, the group G consists of all maps of the form (0.1) with $U \in H^n_{k_1,k_2}$. Example (ia) for $M' = \mathbb{C}^{n-1}$ can be included in this family for $k_2 = 0$.

(vii). Part (viib) of this example is obtained by passing to a quotient in Part (viia).

(viia). $M = \mathbb{C}^{n*}/\mathbb{Z}_l$, where $\mathbb{C}^{n*} := \mathbb{C}^n \setminus \{0\}$, $l \in \mathbb{N}$, and the group G consists of all maps of the form

$$\{z\} \mapsto \{\lambda U z\},$$

where $U \in U_n$, $\lambda > 0$, and $\{z\} \in \mathbb{C}^{n*}/\mathbb{Z}_l$ is the equivalence class of a point $z \in \mathbb{C}^{n*}$.

(viib). $M = M_d/\mathbb{Z}_l$, where M_d is the Hopf manifold $\mathbb{C}^{n*}/\{z \sim dz\}$, for $d \in \mathbb{C}^*$, $|d| \neq 1$, and $l \in \mathbb{N}$; the group G consists of all maps of the form

$$\{[z]\} \mapsto \{[\lambda U z]\},$$

where U, λ are as in (viia), $[z] \in M_d$ denotes the equivalence class of a point $z \in \mathbb{C}^{n*}$, and $\{[z]\} \in M_d/\mathbb{Z}_l$ denotes the equivalence class of $[z] \in M_d$.

(viii). In this example the manifolds are the open orbits of the action of a group of affine transformations on $\mathbb{C}^n$. Let $G_{\mathcal{P}}$ be the group of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & \lambda U z' + a, \\ z_n & \mapsto & \lambda^2 z_n + 2\lambda \langle U z', a \rangle + |a|^2 + ib, \end{array}$$

where $U \in U_{n-1}$, $a \in \mathbb{C}^{n-1}$, $b \in \mathbb{R}$, $\lambda > 0$.

(viiia). $M = \mathcal{P}^n_>$, $G = G_{\mathcal{P}}$, where

$$\mathcal{P}^n_> := \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \operatorname{Re} z_n > |z'|^2 \right\}.$$

Observe that $\mathcal{P}^n_>$ is holomorphically equivalent to $\mathbb{B}^n$.

(viiib). $M = \mathcal{P}^n_<$, $G = G_{\mathcal{P}}$, where

$$\mathcal{P}^n_< := \left\{ (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \operatorname{Re} z_n < |z'|^2 \right\}.$$

Observe that $\mathcal{P}^n_<$ is holomorphically equivalent to $\mathbb{CP}^n \setminus (\overline{\mathbb{B}^n} \cup L)$, where L is a complex hyperplane tangent to $\partial \mathbb{B}^n$.

(ix). Here $n = 2$, $M = \mathbb{B}^1 \times \mathbb{C}$, and G consists of all maps of the form

$$\begin{array}{rcl} z_1 & \mapsto & \dfrac{az_1 + b}{\overline{b}z_1 + \overline{a}}, \\[2ex] z_2 & \mapsto & \dfrac{z_2 + cz_1 + \overline{c}}{\overline{b}z_1 + \overline{a}}, \end{array}$$

where $a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$, $c \in \mathbb{C}$.

(x). Here $n = 3$, $M = \mathbb{CP}^3$, and G consists of all maps of the form (0.2) for $n = 3$ with $U \in Sp_2$.

(xi). Here $n = 3$, M is obtained from $\mathbb{C}^2 \times \mathbb{C}^{2*}$ by factorizing by the equivalence relation $(z, \xi) \sim \nu(z, \xi)$, where $z \in \mathbb{C}^2$, $\xi \in \mathbb{C}^{2*}$, $\nu \in \mathbb{C}^*$, and G consists of all maps of the form

$$\begin{array}{rcl} z & \mapsto & Uz + C\xi, \\ \xi & \mapsto & V\xi, \end{array}$$

where $U, V \in SU_2$, and

$$C = \begin{pmatrix} c_1 & i\overline{c_2} \\ c_2 & -i\overline{c_1} \end{pmatrix},$$

with $c_1, c_2 \in \mathbb{C}$. Observe that $M = \mathbb{CP}^3 \setminus \mathfrak{C}$, where $\mathfrak{C}$ is the projective complex line given by $\xi = 0$ and (z, ξ) are considered as homogeneous coordinates in $\mathbb{CP}^3$.

(xii). Here $n = 3$, $M = \mathbb{C}^3$, and G consists of all maps of the form

$$\begin{array}{rcl} z' & \mapsto & Uz' + \overline{U}a, \\ z_3 & \mapsto & \det U\, z_3 + \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Uz' \right] \cdot \left[\overline{U}a \right] + b, \end{array}$$

where $z' := (z_1, z_2)$, $U \in U_2$, $a \in \mathbb{C}^2$, $b \in \mathbb{C}$, and $\cdot$ denotes the dot product in $\mathbb{C}^2$.

We announce the following theorem.

THEOREM 4.1 *Let M be a connected complex manifold of dimension $n \geq 2$ and $G \subset Aut(M)$ a connected Lie group with $d_G = n^2 + 1$ that acts properly on M. If the pair (M, G) is of type III, then it is equivalent to one of the pairs listed in (i)–(xii) above.*

A proof of Theorem 4.1 will appear in our future joint article with N. Kruzhilin.

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