ON THE MODULARITY OF THE GL_2-TWISTED SPINOR L-FUNCTION

BERNHARD HEIM

Abstract. In modern number theory there are famous theorems on the modularity of Dirichlet series attached to geometric or arithmetic objects. There is Hecke's converse theorem, Wiles proof of the Taniyama-Shimura conjecture, and Fermat's Last Theorem to name a few. In this article in the spirit of the Langlands philosophy we raise the question on the modularity of the GL_2-twisted spinor L-function $Z_{G\otimes h}(s)$ related to automorphic forms G, h on the symplectic group GSp_2 and GL_2. This leads to several promising results and finally culminates into a precise very general conjecture. This gives new insights into the Miyawaki conjecture on spinor L-functions of modular forms. We indicate how this topic is related to Ramakrishnan's work on the modularity of the Rankin-Selberg L-series

1. Introduction

This article is dedicated on the modularity of Dirichlet series. One of the fundamental questions in the field of number theory is how to decide whether an a priori defined Dirichlet series is modular. Sources of these series are Shimura varieties, convolutions of L-functions, and series of numbers with certain *global* properties. Since in general this question has a long history and influenced several areas in mathematics we briefly recapture some of its aspects and applications.

We begin with the prototype of a Dirichlet series which inherits the property to be modular. Hecke's converse theorem states the following: Given a sequence $a_0, a_1, a_2 \ldots$ of complex numbers with polynomial growth in n. Assume that $\Phi(s) := \sum_{n=1}^{\infty} a_n\, n^{-s}$ satisfy certain demanded analytic properties and fulfills a functional equation, then the function $f(\tau) := \sum_{n=0}^{\infty} a_n\, e^{2\pi\, i\, n\tau}$ ($\tau \in \mathfrak{H} := \{x + iy | y > 0\}$) is a modular form. This was generalized by Weil. As an application one has the Shimura correspondence between modular forms of half-integral weight and integer weight [17].

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Let $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$ with conductor N. Since for p with $(p,N)=1$ the curve has good reduction one can count the points on the reduced curve $\pmod p$. This leads to the numbers

$$\mathcal{A}_p(E) := p + 1 - |\overline{E}(\mathbb{F}_p)|,$$

and more general to $\mathcal{A}_n(E)$ for $(n,N)=1$. Then the Taniyama-Weil-Shimura conjecture states that the Dirichlet series $D(s) := \sum_{n=1,(n,N)=1}^{\infty} \mathcal{A}_n(E)\, n^{-s}$ is modular, i.e., there exists a primitive new form of weight $k=2$, such that the Fourier coefficients a_n, n coprime to N, coincide. This has been proven by Wiles [19] and has let him to the solution of Fermat's Last Theorem.

Let $F \in M_k^2$ be a Siegel modular form of degree 2 and weight k with respect to $\Gamma_2 = \mathrm{Sp}_2(\mathbb{Z})$. Let F be a Hecke eigenform with eigenvalues $(\lambda_n(F))_{n=1}^{\infty}$. Armed with a rich amount of numerical data Saito and Kurokawa [12] discovered, that for certain weights k and eigenforms F the Andrianov spinor L-function $L(s,F) := \prod_p L_p(p^{-s},F)$ degenerates. Let $\mu_{0,p}^F, \mu_{1,p}^F, \mu_{2,p}^F$ be the p-local Satake parameters of an arbitary Hecke eigenform F of degree 2. Then the local spinor factor $L_p(X,F)$ is given by

$$\Big\{(1-\mu_{0,p}^F X)(1-\mu_{0,p}^F\mu_{1,p}^F X)(1-\mu_{0,p}^F\mu_{2,p}^F X)(1-\mu_{0,p}^F\mu_{1,p}^F\mu_{2,p}^F X)\Big\}^{-1}. \tag{1.1}$$

The functions considered by Saito and Kurokawa turned out to be locally equal to

$$L_p(X,F) = \Big\{(1-p^{k+1}X)(1-p^{k+2}X)(1-\mathcal{B}_p(F)X + p^{2k-3}X^2)\Big\}^{-1}$$

for $p = 2,3,5$. Then they conjectured that

$$L(s,F) = \zeta(s-k+1)\zeta(s-k+2)\sum_{n=0}^{\infty} \mathcal{B}_n(F)\, n^{-s} \tag{1.2}$$

where $D_F(s) := \sum_{n=0}^{\infty} \mathcal{B}_n(F)\, n^{-s}$ is modular. Moreover it should belong to an element of M_{2k-2}, the space of elliptic modular forms of weight $2k-2$ with respect to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Finally it had been proven in the eighties by Andrianov, Maass, Saito, Kurokawa and Zagier [21] that $D_F(s)$ has this property and that the correspondence is $1-1$. These so-called Saito-Kurokawa lifts have been generalized by Ikeda [10] to Siegel modular forms of even degree.

The type of modularity we are most interested in is given by Ramakrishnan [16] related to a question first raised by Langlands on the modularity of the Rankin-Selberg product. Let g, h be two primitive new forms, not necessary holomorphic, of level N and M, respectively, with Hecke L-functions

$$L(s,g) = \sum_n a_n\, n^{-s} = \prod_p \Big\{(1-\alpha_p\, p^{-s})(1-\beta_p\, p^{-s})\Big\}^{-1} \tag{1.3}$$

$$L(s,h) = \sum_n b_n\, n^{-s} = \prod_p \Big\{(1-\alpha'_p\, p^{-s})(1-\beta'_p\, p^{-s})\Big\}^{-1}. \tag{1.4}$$

For p not dividing NM, we define $L_p(X, g\otimes h)$ by

$$\Big\{(1-\alpha_p\alpha'_p X)(1-\alpha_p\beta'_p X)(1-\beta_p\alpha'_p X)(1-\beta_p\beta'_p X)\Big\}^{-1}.$$

Let $L^*(s, g\otimes h)$ denote the product of $L_p(p^{-s}, g\otimes h)$ over all p not dividing NM. Then this L-function is closely related to the Rankin-Selberg Dirichlet series $\sum_n a_n b_n\, n^{-s}$. The precise form of the question is now: Does there exist an automorphic form $F := g\boxtimes h$ on $\mathrm{GL}_4/\mathbb{Q}$, whose L-function equals $L^*(s, g\otimes h)$ after removing the ramified factors? This was finally positive solved by Ramakrishnan [16].

Since Hecke L-functions are spinor L-functions of elliptic modular forms, it seems to be natural to go one step further and ask about modularity if one exchanges one of the elliptic modular forms in the Ramakrishnan setting by a Siegel Hecke eigenform G of degree 2. We look at the convolution L-function $Z_{G\otimes h}(s)$ (GL_2-twisted spinor L-function) of the spinor L-functions attached to G and h. Hence we are searching for an *automorphic form* $F := G\boxtimes h$ such that the *L-series* of F is equal to $Z_{G\otimes h}(s)$. But at this point we have to confess that this question is somehow vague and has to be formulated in a more precise way.

It is not clear to which group the automorphic form F should belong and for which type of L-function one has to look for. Of course the general philosophy of Langlands predicts that we may should look for the standard L-function of an automorphic forms on $GL_8/\mathbb{Q}$, since the L-function $Z_{G\otimes h}(s)$ has locally degree 8.

On the other side we would like to have *holomorphic automorphic forms*, without any assumption on cohomology, to get arithmetic results on critical values in the sense of Deligne [5], [20]. Here one of the first main results has been to find the appropriate group and L-function to be able to sharpen the question on modularity. We believe that it is always a good advice to look at the properties of Eisenstein series. Let E_k^n be the Siegel type Eisenstein series of degree n and weight k. Then we have

$$L(s, E_k^3) = Z_{E_k^2 \otimes E_{k-2}}(s),$$

where the left side is the spinor L-function of E_k^3. So the question of modularity of $Z_{G\otimes h}(s)$ can be formulated in the final form: For which weights k_1 and k_2 and Hecke eigenform $G \in M_{k_1}^2, h \in M_{k_2}$ does there exist a Siegel modular form F of degree 3 of weight k_3, which is a Hecke eigenform, such that the spinor L-function of F is equal to the twisted spinor L-function $Z_{G\otimes h}(s)$. Does there exist a Siegel Hecke eigenform $F := G \boxtimes h$ of degree 3 such that

$$L(s, G \boxtimes h) = Z_{G\otimes h}(s). \tag{1.5}$$

We prove that the set of weight triples $(k_1, k_2, k_3) \subset \mathbb{N}^3$ which can have this property is distinguished. All this is also related to the Miyawaki conjectures. These are two types of conjectures on the spinor L-function of a Siegel modular form of degree 3. Miyawaki considered the two cuspidal Siegel Hecke eigenforms $F_{12} \in S_{12}^3$, $F_{14}^3 \in S_{14}^3$ of degree 3 and weight 12 and 14. His conjectures are generalizations of the expected degeneration of the spinor L-functions related to this two cusp forms.

Recently [9] we have proven the Miyawaki conjecture of the first type of F_{12}. Up to some non-vanishing condition our method covers mainly the full conjecture of the first type. Here we want to notice that our approach is related to the Miyawaki conjectures related to F_{14}, of the second type. We show in this paper that our modularity conjecture includes Miyawaki's second type conjecture completely.

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2. Statements of results

We first have to fix some notation and recall well-known properties of modular forms. Let n, k be positive integers and let $\mathcal{R}$ be a subring of $\mathbb{R}$. Then we put

$$\mathrm{G}^+\mathrm{Sp}_n(\mathcal{R}) :=$$
$$\left\{\gamma \in \mathrm{GL}_{2n}(\mathcal{R}) \,\middle|\, \gamma \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix} \gamma^t = \mu(\gamma) \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix},\ \mu(\gamma) > 0\right\}.$$

We define the symplectic group $\mathrm{Sp}_n(\mathcal{R}) := \mathrm{G}^+\mathrm{Sp}_n(\mathcal{R}) \cap \mathrm{SL}_{2n}(\mathcal{R})$. Then the Siegel modular group is given by $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$. The group $\mathrm{G}^+\mathrm{Sp}_n(\mathbb{R})$ acts on the Siegel upper half-space $\mathfrak{H}_n$ of degree n by

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z\right) \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix}(Z) := (AZ+B)\,(CZ+D)^{-1}.$$

Let $j(\gamma, Z) := \det(CZ+D)$ for $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $Z \in \mathfrak{H}_n$. Let M_k^n be the space of Siegel modular forms of weight k and degree n, i.e., complex valued holomorphic functions F on $\mathfrak{H}_n$ with the transformation law $F(\gamma(Z)) = j(\gamma, Z)^k\, F(Z)$ for all $\gamma \in \Gamma_n$ and certain growth conditions at the cusp ∞ in the case $n = 1$. We denote the subspace of cusp forms by S_k^n. Moreover let $\mathcal{H}^n$ be the Hecke algebra attached to the Hecke pair $\big(\Gamma_n, \mathrm{G}^+\mathrm{Sp}_n(\mathbb{Q})\big)$. For convenience we drop the index in the case $n = 1$ and use the symbols $\Gamma, \mathfrak{H}, \mathcal{H}, \ldots$. The Hecke algebra $\mathcal{H}^n$ acts on M_k^n and stabilizes the subspace of cusp forms. There exists a basis of *Hecke* eigenforms of S_k^n which can be extended to a basis of Hecke eigenforms of M_k^n for all Hecke operators.

Suppose $F \in M_k^n$ is a Hecke eigenform with p-Satake parameters $\mu_0, \mu_1, \ldots, \mu_n$ for every finite prime p. These are complex numbers, which are unique up to the action of the Weyl group of the symplectic group and normalized with $\mu_0^2 \mu_1 \cdot \ldots \cdot \mu_n = p^{nk - n(n+1)/2}$. Then we define the spinor L-function $L(s, F)$ by $\prod_p L_p(p^{-s}, F)$. Here

$$L_p(X, F) := (1 - \mu_0 X)^{-1} \prod_{r=1}^{n} \prod_{i_1 < \ldots < i_r} (1 - \mu_0 \mu_{i_1} \cdot \ldots \cdot \mu_{i_r} X)^{-1}. \tag{2.1}$$

The analytic properties of the spinor L-function are well understood in the cases $n = 1$ and $n = 2$ (meromorphic continuation to the complex plane, poles, functional equation). Recently, some progress has been made in the cases of Ikeda and Miyawaki lifts for $n \geq 3$ ([11], [9]). Let $h \in M_k$ be a Hecke eigenform with eigenvalues $\lambda_n(h)$ and Fourier expansion

$$h(\tau) = \sum_{n=0}^{\infty} a_n(h)\, q^n \quad (q = e^{2\pi i \tau}).$$

Let $a_1(h) = 1$ then $\lambda_n(h) = a_n(h)$. We introduce the local parameters α_p and β_p and the Hecke L-function $L(s,h)$ of the Hecke eigenform h:

$$L(s,h) = \sum_{n=1}^{\infty} \lambda_n(h)\, n^{-s} = \prod_p \Big\{(1-\alpha_p(h)\, p^{-s})(1-\beta_p(h)\, p^{-s})\Big\}^{-1}. \tag{2.2}$$

Here $\alpha_p(h) + \beta_p(h) = \lambda_p(h)$ and $\alpha_p(h)\beta_p(h) = p^{k-1}$. With this normalization the Hecke L-function of h is equal to the spinor L-function. We now recall the definition of the twisted spinor L-function [8], already mentioned in the introduction.

Definition. *Let k_1, k_2 be positive even integers and let $G \in M_{k_1}^2$, $h \in M_{k_2}$ be two Hecke eigenforms. Then the twisted spinor L-function $Z_{G\otimes h}(s)$ attached to G and h is defined by*

$$Z_{G\otimes h}(s) := \prod_p \Big\{L_p\left(\alpha_p(h)p^{-s}, G\right)\ L_p\left(\beta_p(h)p^{-s}, G\right)\Big\}. \tag{2.3}$$

This series converges absolutely and locally uniformly for $\mathrm{Re}(s) \gg 0$. We will show in this paper the following

Theorem 2.1. *Let k_1, k_2 be positive even integers. Let $G \in M_{k_1}^2$ and $h \in M_{k_2}$ be two Hecke eigenforms. If G and h are cuspidal we have to assume that the first Fourier coefficient of G is not identical zero. Then the twisted spinor L-function $Z_{G\otimes h}(s)$ has a meromorphic continuation to the whole complex plane and posesses a functional equation.*

Now we come to the question of modularity. The twisted spinor L-function $Z_{G\otimes h}(s)$ is *modular* if there exists a Hecke eigenform $F \in M_{k_3}^3$ such that the spinor L-function of F is equal to $Z_{G\otimes h}(s)$. If this is the case we put $F := G \boxtimes h$. Here F is not unique. To be more precise, $G \boxtimes h$ should denote the vector space of all Hecke eigenforms F satisfying the modularity condition from above. There is also a local definition of modularity. We say G and h are p-modular for a prime p if there exists a Siegel Hecke eigenform F of degree 3 such that $Z_{G\otimes h,p}(X) = L_p(X,F)$. Then we put

$F = G \boxtimes_p h$. We would like to mention that it is an open questionwheter is follows from p-modularity for all finite primes p that G, h are globally modular.

Similar as in the approach of Maass towards the Saito-Kurokawa conjecture and the finding of the Maass Spezialschar [21], we look at properties of Eisenstein series. It turns out that a first result of global modularity, and hence p-modularity, is given by Siegel type Eisenstein series. Let $k > n+1$ be even. Then the Siegel type Eisenstein series is defined by

$$E_k^n(Z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_n} j(\gamma, Z)^{-k} \quad (Z \in \mathfrak{H}_n). \tag{2.4}$$

Here $\Gamma_\infty := \{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mid C = 0 \}$. Then it is well-known that this series is a Siegel modular form of weight k and degree n and an eigenform with respect to the Hecke algebra $\mathcal{H}^n$. In the case $n = 3$ it can been shown via Hecke summation that there exists a Eisenstein series of weight 4 (see also [18]).

Theorem 2.2. *Let k be an even positive integer. Then $Z_{E_k^2 \otimes E_{k-2}}(s)$ is modular.*

The next step is to open the door toward the modularity of cuspidal Hecke eigenforms. Again, as in the fundamental paper of Kurokawa [12] on the possible existence of a certain subspace of Siegel modular forms which servers as a counter example for the Ramanujan-Petersson conjecture, we have a result on p-modularity. Let

$$\Delta(\tau) := q \prod_p (1 - q^n)^{24}$$

be the Ramanujan function. The Fourier coefficients and eigenvalues are usually denoted by $\tau(n)$. It is the unique elliptic cusp forms of weight 12 with first Fourier coefficient equal to 1. The normalization of Siegel modular form is not canonical, since we do not know which Fourier coefficient is non-zero. Only in the case of Eisenstein series this can been done. Hence if we say that a Hecke eigenform $F \in S_k^n$ is unique, we mean unique up to a scalar. Employing results from Miyawaki [13] lead to the

Theorem 2.3. *Let G be the unique Siegel Hecke eigenform of weight 14 and degree 2. Then the pair (G, Δ) is p-modular for the prime 2.*

It would be desirable to have a clear picture, which pairs of weights (k_1, k_2) can be candidates for modularity. Theorem 2.3 provides us with the pair $(14, 12)$. But there are also counter examples. Let $\chi_{12} \in S_{12}^2$ be a Hecke eigenform of weight 12 Then it is easy to see by direct considerations that $Z_{\chi_{12}\otimes\Delta}(s)$ is not modular. Nevertheless using analytic properties of the standard L-function [3], [14], [18] attached to Siegel Hecke eigenforms of degree $1, 2$ and 3 leads to the following sharp result.

Theorem 2.4. *Let k_1, k_2 be positive even integer. Let $G \in S_{k_1}^2$ and $h \in S_{k_2}$ be Hecke eigenforms. If we assume that the twisted spinor L-function $Z_{G\otimes h}(s)$ is modular. Let $F := G \boxtimes h$ be cuspidal. Then $F \in S_{k_1}^3$ and $k_2 = k_1 - 2$.*

Let k be an even positive integer. From now on we consider pairs of the form $(k, k-2)$. Although the map $\boxtimes$ is not unique, it makes sense to define the space $\mathcal{L}_k$ obtained by the subspace generated by all $G \boxtimes h$, if they exist, where $G \in S_k^2$ and $h \in S_{k-2}$. Here we define $G \boxtimes h$ to be equal to zero if $Z_{G\otimes h}(s)$ is not modular to get a useful notation. Hence the map $\boxtimes : S_k^2 \times S_{k-2} \longrightarrow S_k^3$ is well defined if we fix a Hecke eigenbasis of S_k^2 and S_{k-2} and choose a suitable image.

Before presenting several illustrative examples, we first recall the two types of the conjectures of Miyawaki [13] that relate to investigations directly.

Miyawaki conjecture - Type I: There exists a map $S_k \times S_{2k-4} \longrightarrow S_k^3$ such that for pairs of Hecke eigenforms (f, g) the image F is a Hecke eigenform and the eigenvalues of F are expressed in terms of the eigenvalues of f and g.

Miyawaki conjecture - Type II: There exists a map $S_{k-2} \times S_{2k-2} \longrightarrow S_k^3$ such that for pairs of Hecke eigenforms (f, g) the image F is a Hecke eigenform and the eigenvalues of F are expressed in terms of the eigenvalues of f and g.

Let $F_{12} \in S_{12}^3$ and $F_{14} \in S_{14}^3$ be the two Hecke eigenforms analyzed by Miyawaki [13]. Then F_{12} belongs to Type I and F_{14} belongs to Type II. The Miyawaki conjectures are mainly based on these two examples and the consistency of the funcional equation of the standard L-function, which had been a highly non-trivial approach. Recently [9] we built on the work of Ikeda [11] to prove Miyawaki's Conjecture ([13], Conjecture 4.3). The conjecture of type II is still open. For further remarks and a different way to attack the conjectures as given in this paper, we refer to Ikeda's work [11]. Since $\dim S_k = 0$ for $k < 12$. The smallest possible k for modularity is given by

14.

EXAMPLE: k=14
Since $\dim S_{12} = \dim S^2_{14} = \dim S^3_{14} = 1$ every non-trivial cusp form in this case is a Hecke eigenform. Let a Hecke eigenform $G \in S^2_{14}$ be given, then the claim is that $Z_{G\otimes\Delta}(s) = L(s, F_{14})$, i.e., $F_{14} = G \boxtimes h$. This restates the conjecture suggested by Theorem 2.3 and gives a new interpretation of Miyawaki's conjecture of type II for $k = 14$.

Summary. We claim that the map $\boxtimes : S^2_{14} \times S_{12} \longrightarrow S^3_{14}$ is an isomorphism.

EXAMPLE: k=18
It is well known that $\dim S^2_{18} = 2$, $\dim S_{16} = 1$ and $\dim S^3_{18} = 4$. We expect that

$$\dim \left(S^2_{18} \boxtimes S_{16}\right) = 2.$$

Moreover the complement of $\mathcal{L}_{18}$ in S^2_{18} with respect to the Petersson scalar product is expected to be generated by lifts of Miyawaki Type I. Since S^2_{18} is generated by Saito-Kurokawa lifts, our modularity question is equivalent in this case to the Miyawaki conjecture of type II.

Summary. We expect that the space S^3_{18} decomposes into the direct sum of two Hecke invariant subspaces. These spaces are invariant with respect to $\mathrm{Aut}(\mathbb{C})$ and if one normalizes the Hecke eigenforms in an appropriate way and considers the corresponding vector spaces over $\overline{\mathbb{Q}}$, then these spaces should be preserved by the action of the absolute Galois group $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. The action is given on the Fourier coefficients. Hence the space S^3_{18} contains no Hecke eigenform which fulfills the generalized Ramanujan-Petersson conjecture.

Next, we arrive a very interesting case since it goes beyond the scope of Miyawaki's work and can be viewed as a touchstone of the new viewpoint.
EXAMPLE: k=20
It is well known that $\dim S^2_{20} = 3$, $\dim S_{18} = 1$, and $\dim S^3_{20} = 6$. We expect that

$$\dim \left(S^2_{20} \boxtimes S_{18}\right) = 3.$$

Miyawaki's conjectures of type I and II would give two Hecke invariant subspaces of S_{20}^3 of dimension 3 and 2. The space S_{20}^2 is generated by three Hecke eigenforms χ_a, χ_b, G, where χ_a, χ_b are Saito-Kurokawa lifts. Let $h_{18} \in S_{18}$ be a Hecke eigenform, then $\chi_a \boxtimes h_{18}$ and $\chi_b \boxtimes h_{18}$ correspond exactly to Miyawaki's conjecture of type II. Hence the modularity of $Z_{G\otimes h_{18}}(s)$ would provide a new lifting to S_{20}^3 with many applications. This would lead for example to a proof of Andrianov's conjecture on the functional equation of the spinor L-function of a Siegel Hecke eigenform of degree 3 for the full space of weight $k = 20$ (see [2], or [13] for further details).

Summary. We expect that the space S_{20}^3 decomposes into an orthogonal sum of Hecke invariant subspaces. We would have:

$$S_{20}^3 = \mathcal{L}_{20}^I \oplus \mathcal{L}_{20}^{II} \oplus \mathcal{L}_{20}^!, \tag{2.5}$$

where $\dim\mathcal{L}_{20}^I = 3$ and corresponds to Miyawaki lifts of type I, $\dim\mathcal{L}_{20}^{II} = 2$ and corresponds to Miyawaki lifts of type II, and finally $\dim\mathcal{L}_{20}^! = 1$. The space $\mathcal{L}_{20}^!$ is generated by $G \boxtimes h_{18}$, where $G \in S_{20}^2$ is a Hecke eigenform, which is not a Saito-Kurokawa lift. These spaces are invariant with respect to $\mathrm{Aut}(\mathbb{C})$, and invariant with respect to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, if the setting is suitable chosen.

All this observations lead us to the following

Main Conjecture

Let k be a positive even integer.

(1) Let $G \in M_k^2$ be a Hecke eigenform. Then there exists a $F \in M_k^3$ with $F = G \boxtimes E_{k-2}$.

(2) Let $G \in S_k^2$ and $h \in S_{k-2}$ Hecke eigenforms. Then $Z_{G\otimes h}(s)$ is modular and $F := G \boxtimes h \in S_k^3$. Hence we have

$$Z_{G\otimes h}(s) = L(s, F).$$

(3) The Hecke invariant subspace $\mathcal{L}_k$ of S_k^3 has the asymptotic dimension formula

$$\dim\Big(\mathcal{L}_k\Big) = \mathrm{O}\,(k^4).$$

Let $F \in S_k^r$ and $k > n + r + 1$. Then we denote by $E_k^{n,r}(F)$ the Klingen Eisenstein series attached to F. The Klingen Eisenstein series is an element of M_k^n. Let F be a Hecke eigenform then $E_k^{n,r}(F)$ is also a Hecke eigenform. We have the following explicit result.

Theorem 2.5. *Let k be a positive even integer. Let $G \in M_k^2$ be a Hecke eigenform. Then $F := G \boxtimes E_{k-2}$ is equal to E_k^3, $E_k^{3,1}(f)$ or $E_k^{3,2}(G)$ if $G = E_k^2$, $E_k^{2,1}(f)$ or a cusp form. Here $f \in S_k$ is a Hecke eigenform.*

Hence by applying the so called *Darstellungssatz* of Klingen for $k > 6$ and the procedure of Hecke summation in the case $k = 4$ we obtain:

Corollary 2.6. *Part* (1) *of the Main conjecture is true.*

Remark.
a) The result of the Corollary is maybe not expected, since it states that Klingen type Eisenstein series are in the image of the modularity map $\boxtimes$. Hence we have an interpretation of the Klingen lifts on the level of L-functions. This is different to the case of Saito-Kurokawa lifts. Since $L_{G\otimes E_{k-2}}(s)$ is modular for all Hecke eigenforms $G \in S_k^2$, we have another hint that part (2) of the Main conjecture could be true.

b) Let $k = 16$ then there exists no *modularity* subspace. But since $\dim S_{16}^3 = 3$ we expect that S_{16}^3 decomposes into a direct sum of two non-trivial Hecke invariant subspaces. The subspace with dimension 2 should correspond to the Miyawaki lifts of type I. We suggest that there is one eigenform of weight 16 which satisfies the Ramanujan-Petersson conjecture.

Finally we have:

Theorem 2.7. *Assume that part (2) of the Main conjecture is satisfied then Miyawaki's conjecture of type* II *is true.*

3. Proofs

We begin by recalling briefly the definition of the so called Siegel Φ-operator introduced by Siegel. This operator has several remarkable properties. Let $F \in M_k^n$ be a Hecke eigenform and let F be not in the kernel of the Φ operator. Then the eigenvalues of F can be expressed by eigenvalues of Hecke eigenforms of smaller degrees.

This can be used to determine the spinor and standard L-function of non-cuspidal eigenforms in an interated way.

Let n, r, be integers and let $0 < r \leq n$. Consider the sequence $\tau^{(m)}$ $(m = 1, 2, \ldots)$ in $\mathfrak{H}_n$,

$$\tau^{(m)} := \begin{pmatrix} \tau & \tau_2^{(m)} \\ (\tau_2^{(m)})^t & \tau_4^{(m)} \end{pmatrix},$$

such that $\tau \in \mathfrak{H}_r$ is fixed, $\tau_2^{(m)}$ is bounded and all the eigenvalues of $\mathrm{Im}\tau_4^{(m)}$ tend to infinity. Then the limit

$$\left(\Phi^{n-r} F\right)(\tau) := \lim_{m \to \infty} F(\tau^{(m)})$$

exists and represents a modular form of weight k and degree r.

Moreover we have $\Phi^l = \Phi^{l-1} \circ \Phi$. This can be extended to $r = 0$ if we put $M_k^0 := S_k^0 := \mathbb{C}$. The map $\Phi : M_k^n \longrightarrow M_k^{n-1}$ is linear and surjective if $k > 2n$.

Let F be a Hecke eigenform with $\Phi F \neq 0$ then we have the following very useful result of Zarakovskaya [7]: The modular form ΦF is a Hecke eigenform and let $\mu_{0,p}^{\Phi F}, \mu_{1,p}^{\Phi F}, \ldots \mu_{n-1,p}^{\Phi F}$ be the p-Satake parameters of ΦF, then the p-Satake parameters of F are given by

$$\mu_{0,p}^{\Phi F}\, p^{k-n}, \mu_{1,p}^{\Phi F}, \ldots \mu_{n-1,p}^{\Phi F}, p^{n-k}. \tag{3.1}$$

Let F be any Hecke eigenform with $\Phi F \neq 0$, then the Zarakovskaya relation for the spinor L-function of F states that

$$L(s, F) = L(s, \Phi F)\, L(s - k + n, \Phi F). \tag{3.2}$$

Here for $c \in \mathbb{C}^x$ we put $L(s, c) := \zeta(s)$.

3.1. Liftings of type II and the proof of Theorem 2.7 and Theorem 2.3.

We first work out a precise formulation of Miyawaki's conjecture of type II (see also Ikeda [11]). We briefly recall Miyawaki's approach. His observation had been that the spinor L-function of certain Hecke eigenforms of degree 3 degenerate. This lead him to essentially to types of liftings (see [13], Proposition 7.7). Let $f \in S_{k_1}$ and $g \in S_{k_2}$ be two Hecke eigenforms. Then we restrict to $k_1 = k - 2$ and $k_2 = 2k - 2$ for $k > 4$ even, since the other possible case $k_1 = k$ and $k_2 = 2k - 4$ corresponds to type I. Let $F \in S_k^3$ be a Hecke eigenform with the property that the spinor L-function $L(s, F)$ of

F is equal to

$$L(s-k_2/2, f)\, L(s, k_2/2-1, f)\, L(s, f\otimes g). \tag{3.3}$$

Here $L(s, f\otimes g)$ is the Rankin-Selberg L-function of f and g, which we already have defined in the introduction. This leads to the conjecture of Type II: Let f, g be as above. Then there exists a Hecke eigenform $F \in S_k^3$ such that the spinor L-function of F is is equal to the expression (3.3). Conjecture 4.5 in Miyawaki's paper explicitly predicts that the spinor L-function of the Hecke eigenform $F_{14} \in S_{14}$ has the decomposition

$$L(s, F_{14}) = L(s-13, \Delta)\, L(s-12, \Delta)\, L(s, \Delta \otimes g_{26}),$$

where $g_{26} \in S_{26}$ is a primitive new form. Theorem 4.4 [13] states that this is true for $p = 2$. Let $g \in S_{2k-2}$ be a primitive Hecke eigenform. Then there exists a Siegel Hecke eigenform $G \in S_k^2$, such that for the spinor L-function of G and g we have have the relation

$$L(s, G) = \zeta(s-k+1)\ \zeta(s-k+2)\ L(s, g). \tag{3.4}$$

Moreover this correspondence is $1-1$, although we do not have multiplicity one for S_k^2 available. It is useful to describe this result on the level of p-Satake parameters. Here the p-Satake parameters of G are given by:

$$\mu_{0,p}^G = p^{k-1},\ \mu_{1,p}^G = \alpha_p(g)p^{1-k},\ \mu_{2,p}^G = \beta_p(g)p^{1-k}. \tag{3.5}$$

Up to the action of the Weyl group of the symplectic group, these parameters are unique. Here $\alpha_p(g), \beta_p(g)$ are the local parameters of g, with the normalization $\alpha_p(g)\beta_p(g) = p^{2k-3}$ and $\alpha_p(g) + \beta_p(g) = \lambda_p(g)$. This can be employed to get the following

Lemma 3.1. *Let $h \in S_{k-2}$ and $g \in S_{2k-2}$ be Hecke eigenforms. Let G be a Saito-Kurokawa lift of g. Assume that the Hecke eigenform $F := G \boxtimes h$ exists. Then the p-Satake parameters of F are:*

$$\mu_{0,p}^F = \alpha_p(h)p^{k-1} \qquad \mu_{1,p}^F = \alpha_p(g)p^{1-k}$$
$$\mu_{2,p}^F = \beta_p(g)p^{1-k} \qquad \mu_{3,p}^F = \frac{\beta_p(h)}{\alpha_p(h)}.$$

Assume that the Main Conjecture (2) is true for the pair (G, h) of Hecke eigenforms, where $G \in S_k^2$ Saito-Kurokawa lift of $g \in S_{2k-2}$ and $h \in S_{k-2}$. Then $Z_{G\otimes h,p}(X) = L_p(X, F)$. Hence it follows that

$$\begin{aligned} L_p(X,F)^{-1} &= \left(1-\alpha_p(h)p^{k-1}X\right)\left(1-\alpha_p(h)p^{k-1}\alpha_p(g)p^{1-k}X\right) \\ &\quad \times \left(1-\alpha_p(h)p^{k-1}\beta_p(g)p^{1-k}X\right)\left(1-\alpha_p(h)p^{k-1}\frac{\beta_p(h)}{\alpha_p(h)}X\right) \\ &\quad \times \left(1-\alpha_p(h)p^{k-1}\alpha_p(g)p^{1-k}\beta_p(g)p^{1-k}X\right) \\ &\quad \times \left(1-\alpha_p(h)p^{k-1}\alpha_p(g)p^{1-k}\frac{\beta_p(h)}{\alpha_p(h)}X\right) \\ &\quad \times \left(1-\alpha_p(h)p^{k-1}\beta_p(g)p^{1-k}\frac{\beta_p(h)}{\alpha_p(h)}X\right) \\ &\quad \times \left(1-\alpha_p(h)p^{k-1}\alpha_p(g)p^{1-k}\beta_p(g)p^{1-k}\frac{\beta_p(h)}{\alpha_p(h)}X\right). \end{aligned}$$

This can be further simplified and leads to the expression

$$L_p(p^{k-1}X, h)\, L_p(p^{k-2}X, h)\; L_p(X, g\otimes h). \tag{3.6}$$

This proves Theorem 2.7.

Finally we prove Theorem 2.3. The Theorem says that there exists a Hecke eigenform F of degree 3 such that

$$L_2(X, F) = Z_{G\otimes h,2}(X).$$

Let the Hecke eigenforms $F_{14} \in S_{14}^3$ and $g_{26} \in S_{26}$ be given. Miyawaki ([13] [Theorem 4.4]) has proven that the spinor L-function the of the Hecke eigenform F_{14} satisfies locally

$$L_p(X, F_{14}) = L_p(Xp^{12}, \Delta)\, L_p(Xp^{13}, \Delta)\, L_p(X, \Delta \otimes g_{26})$$

for the Hecke eigenform $g_{26} \in S_{26}$ and the prime $p = 2$. With $h = \Delta$ and for G the Saito-Kurokawa lift of g_{26} we get the result.

3.2. **Standard L-functions.** In this subsection we prove Theorem 2.4. Although the focus of this paper is on properties of the twisted spinor L-function $Z_{G\otimes h}(s)$ and the spinor L-function of a Siegel modular forms of degree 3, it turns out to be very fruitful to use the related standard L-functions to obtain hidden properties.

Let $F \in M_k^n$ be a modular form of integer weight k with respect to $\Gamma_n = Sp_n(\mathbb{Z})$. Suppose F is a Hecke eigenform with p-Satake parameter $\mu_{0,p}^F, \mu_{1,p}^F, \dots \mu_{n,p}^F$. Langlands attached to F the standard L-function:

$$L(s,F,st) := \zeta(s) \prod_p \Big\{ \prod_{i=1}^n \left(1 - \mu_{i,p}^F \, p^{-s}\right) \left(1 - \left(\mu_{i,p}^F\right)^{-1} \, p^{-s}\right) \Big\}^{-1}. \tag{3.7}$$

The product converges absolutely and locally uniformly for $\mathrm{Re}(s) > n + k + 1$, and for $\mathrm{Re}(s) > n+1$ if F is a cusp form. Before we state the functional equation we complete the L-function at infinity. Let

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2) \text{ and } \Gamma_{\mathbb{C}}(s) = 2\,(2\pi)^{-s}\Gamma(s). \tag{3.8}$$

We also put $\varepsilon = 0$ if n is even and 0 otherwise. Then

$$\widehat{L}(s,F,st) := \Gamma_{\mathbb{R}}(s+\varepsilon) \left(\prod_{j=1}^n \Gamma_{\mathbb{C}}(s+k-j) \right) L(s,F,st). \tag{3.9}$$

This completed L-function has a meromorphic continuation to the whole complex s-plane and satisfies the functional equation

$$\widehat{L}(s,F,st) = \widehat{L}(1-s,F,st). \tag{3.10}$$

Proof of Theorem 2.4. Let $F \in S_k^n$ be a Hecke eigenform with degree at most three. Then $\widehat{L}(s,F,st)$ is entire [14]. In particular, let $F^{(i)} \in S_k^i$ be a Hecke eigenform for $i \leq 3$. Then

$$\begin{aligned}
\widehat{L}(s,F^{(1)},st) &= \Gamma_{\mathbb{R}}(s+1)\,\Gamma_{\mathbb{C}}(s+k-1)\,L(s,F^{(1)},st)\\
\widehat{L}(s,F^{(2)},st) &= \Gamma_{\mathbb{R}}(s)\,\Gamma_{\mathbb{C}}(s+k-1)\,\Gamma_{\mathbb{C}}(s+k-2)\,L(s,F^{(2)},st)\\
\widehat{L}(s,F^{(3)},st) &= \Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+k-1)\,\Gamma_{\mathbb{C}}(s+k-2)\Gamma_{\mathbb{C}}(s+k-3)L(s,F^{(3)},st).
\end{aligned}$$

Let $G \in S_{k_1}^2$ and $h \in S_{k_2}$ be Hecke eigenforms and $G \boxtimes h := F \in S_{k_3}^3$ be modular. To fix notation, let $\mu_{0,p}^G, \mu_{1,p}^G, \mu_{2,p}^G$ be the p-Satake parameter of G with

$$(\mu_{0,p}^G)^2 \mu_{1,p}^G \mu_{2,p}^G = p^{2k_1-3}$$

and $\alpha_p(h), \beta_p(h)$ the local parameter of h with $\alpha_p(h)\beta_p(h) = p^{k_2-1}$. Then we determine the standard L-function of F. We expect that $L(s,F,st)$ will be degenerated. Since $F = G \boxtimes h$, we can use the identity between the related spinor L-function to get an explicit form of the p-Satake parameters. This can been done in a straightforward manner (we omit the details). Finally we obtain

$$\mu_{0,p}^F = \mu_{0,p}^G \alpha_p(h) \qquad \mu_{1,p}^F = \mu_{1,p}^G$$
$$\mu_{2,p}^F = \mu_{2,p}^G \qquad \mu_{3,p}^F = \frac{\beta_p(h)}{\alpha_p(h)}.$$

This leads to the first relation between the even positive integers k_1, k_2, k_3:

$$\boxed{3k_3 - k_2 - 2k_1 = 2}.$$

Further we clarify some notation: Let $f \in S_l$ be a Hecke eigenform. The symmetric square L-function of f is usually defined by

$$L(s, \mathrm{Sym}^2(f)) := \prod_p \left\{(1 - \alpha_p(f)^2 p^{-s})(1 - \alpha_p(f)\beta_p(f)p^{-s})(1 - \beta_p(f)^2 p^{-s})\right\}^{-1}.$$

Then we have

$$L(s, f, st) = L(s + l - 1, \mathrm{Sym}^2(f)). \tag{3.11}$$

Armed with this data we determine the standard L-function directly.

$$\begin{aligned} L(s,F,st) &= L(s,G,st) \prod_p \left\{ \left(1 - \frac{\beta_p(h)}{\alpha_p(h)} p^{-s}\right) \left(1 - \frac{\alpha_p(h)}{\beta_p(h)} p^{-s}\right) \right\}^{-1} \\ &= \frac{L(s,G,st)}{\zeta(s)} \\ \times \prod_p & \left\{ \left(1 - \alpha_p(h)^2 p^{-s-k_2+1}\right) \left(1 - \beta_p(h)^2 p^{-s-k_2+1}\right) \left(1 - \alpha_p(h)\beta_p(h) p^{-s-k_2+1}\right) \right\}^{-1} \\ &= \frac{L(s,G,st)\, L(s + k_2 - 1, \mathrm{Sym}^2(h))}{\zeta(s)}. \end{aligned}$$

Here we also used the identities $\alpha_p(h)/\beta_p(h) = \alpha_p(h)^2 p^{1-k_2}$ and $\beta_p(h)/\alpha_p(h) = \beta_p(h)^2 p^{1-k_2}$. Let further

$$Z_{G\otimes h}^{st}(s) := \frac{L(s,G,st)\, L(s,h,st)}{\zeta(s)}. \tag{3.12}$$

We put $\widehat{\zeta}(s) := \Gamma_{\mathbb{R}}(s)\zeta(s)$ and

$$\begin{aligned}\gamma_{k_1,k_2}(s) &:= \Gamma_{\mathbb{R}}(s+1)\,\Gamma_{\mathbb{C}}(s+k_2-1)\,\Gamma_{\mathbb{C}}(s+k_1-1)\,\Gamma_{\mathbb{C}}(s+k_1-2)\\ \gamma_{k_3}(s) &:= \Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+k_3-1)\,\Gamma_{\mathbb{C}}(s+k_3-2)\Gamma_{\mathbb{C}}(s+k_3-3).\end{aligned}$$

With this notation we define

$$\widehat{Z}^{st}_{G\otimes h}(s) := \gamma_{k_1,k_2}(s)\, Z^{st}_{G\otimes h}(s). \tag{3.13}$$

This completed function satisfies the functional equation $s \mapsto 1-s$. It follows directly that

$$\mathcal{Q}_{k_1,k_2,k_3}(s) := \frac{\gamma_{k_3}(s)}{\gamma_{k_1,k_2}(s)} \tag{3.14}$$

also satisfies the functional equation $s \mapsto 1-s$. Since the k_j $(1 \leq j \leq 3)$ are positive integers larger than 8, the function $\mathcal{Q}_{k_1,k_2,k_3}(s)$ has to be constant. This leads to the equality

$$\left\{k_2-1, k_1-1, k_1-2\right\} = \left\{k_3-1, k_3-2, k_3-3\right\}. \tag{3.15}$$

But this forces $k_1 = k_3$. Hence we obtain the desired result. □

3.3. **On the Darstellungssatz and Eisenstein series.**

Generalized Eisenstein series have been introduced by Klingen to analyze the kernel of the Siegel Φ-operator and the metric structure of the space of Siegel modular forms M_k^n of even weight $k > 2n$ and degree n.

Let $F \in S_k^r$. For $k > n+r+1$ even and $n \geq r$, Klingen defined the following element of M_k^n:

$$E_k^{n,r}(F)(Z) := E_k^{n,r}(F,Z) := \sum_{\gamma \in P_{n,r}\backslash \Gamma_n} F\left(\gamma(Z)^{(r)}\right)\, j(\gamma, Z)^{-k}, \tag{3.16}$$

where $Z^{(r)} = \tau \in \mathfrak{H}_r$ with $Z = \left(\begin{smallmatrix}\tau & * \\ * & *\end{smallmatrix}\right)$. Moreover $P_{n,r}$ is the subgroup of Γ_n given by all γ, with the property that the lower left $(n-r, n+r)$ block is zero. We declare in the case $r=0$ the Klingen Eisenstein series to be equal to the Siegel Eisenstein series E_k^n and in the case $r=n$ the series equal to the cusp form. Assume that $k > n+r+1$ is even as above. Then we define

$$M_k^{n,r} := \left\{E_k^{n,r}(F) \,|\, F \in S_k^r\right\}. \tag{3.17}$$

We have for $n > r$:

$$\Phi\left(E_k^{n,r}(F)\right) = E_k^{n-1,r}(F). \tag{3.18}$$

For $r = n$ we have $\Phi(F) = 0$. Let F be a Hecke eigenform then $E_k^{n,r}(F)$ is also a Hecke eigenform. Now the Darstellungssatz of Klingen states:
Let k be even and $k > 2n$. Then

$$\begin{aligned} M_k^n &= \oplus_{r=0}^n M_k^{n,r} \\ &= \mathbb{C}\, E_k^n \oplus M_k^{n,1} \oplus \ldots \oplus M_k^{n,n-1} \oplus S_k^n \end{aligned}$$

and $\Phi(M_k^{n,r}) = M_k^{n-1,r}$. We know that this decomposition is preserved by the action of the Hecke algebra $\mathcal{H}^n$ (see Harris [7] for more details and references).

For $n = 1, 2, 3$ we have the following decomposition. Let k be even and $k \geq 4$ then E_k exists. For $k < 12$ we have $\dim S_k = 0$. Moreover we have $M_k = \mathbb{C}\, E_k \oplus S_k$. In the case $n = 2$ we have for $k = 4 : M_k^2 = \mathbb{C}\, E_k^2$ and for $k > 4$ even we have:

$$M_k^2 = \mathbb{C}\, E_k^2 \oplus \left\{ E_k^{2,1}(f) \,|\, f \in S_k \right\} \oplus S_k^2. \tag{3.19}$$

Here $\dim S_k^2 = 0$ for $k < 10$. In the case $n = 3$ the case $k = 4$ is exceptional ($\dim M_4^3 = 1$). The element comes from the real analytic Eisenstein series via Hecke summation. Further we have $M_6^3 = \mathbb{C}\, E_6^3$ and for $k > 6$ we have can apply the Darstellungssatz of Klingen. Here $\dim S_k^3 = 0$ for $k < 12$.

Let $F \in S_k^r$ be a Hecke eigenform with $k > n + r + 1$ and $n > r$. Then the Zarakovskaya relation leads to

$$L(s, E_k^{n,r}(F)) = L(s, E_k^{n-1,r}(F))\; L(s - k + n, E_k^{n-1,r}(F)). \tag{3.20}$$

Moreover such kind of relations also exist for the standard L-function. Let $F \in M_k^n$ with $\Phi(F) \neq 0$, then

$$L(s, F, st) = \zeta(s - k + n)\, \zeta(s + k - n)\, L(s, \Phi(F), st). \tag{3.21}$$

3.4. **Proof of Theorem 2.2 and Theorem 2.5.** We have normalized our Siegel type Eisenstein series such that $\Phi(E_k) = 1$. Since $L(s, E_k) = \zeta(s)\zeta(s - k + 1)$ the p-Satake parameters of E_k are given by

$$\mu_{0,p}^{E_k} = 1 \text{ and } \mu_{1,p}^{E_k} = p^{k-1}. \tag{3.22}$$

Here want to note that this parametrization is equivalent to $\mu_{0,p}^{E_k} = p^{k-1}$ and $\mu_{1,p}^{E_k} = p^{1-k}$.

Proof of Theorem 2.2. Let $k > 2n$. Then it follows from the the Zarkovskaya relations and the observations from above, that

$$\mu_{0,p}^{E_k^n} = 1,\ \mu_{1,p}^{E_k^n} = p^{k-1}, \dots,\ \mu_{n,p}^{E_k^n} = p^{k-n}. \tag{3.23}$$

The spinor L-function of E_k^3 is equal to

$$\zeta(s)\,\zeta(s-3k+6)\prod_{j=1}^{3}\zeta(s-k+j)\prod_{i=1}^{3}\zeta(s-2k+2+i). \tag{3.24}$$

On the other side we have in the case $n = 2$:

$$L(s, E_k^2) = \zeta(s)\,\zeta(s-k+1)\zeta(s-k+2)\,\zeta(s-2k+3).$$

Since $\alpha_p(E_k) = 1$ and $\beta_p(E_k) = p^{k-1}$, this leads to the identity

$$Z_{E_k^2 \otimes E_{k-2}}(s) = L(s, E_k^3).$$

This proves that $E_k^2 \boxtimes E_{k-2}$ is modular. □

Proof of Theorem 2.5. Let $f \in S_k$ be a Hecke eigenform. Then $E_k^{n,1}(f)$ is a Hecke eigenform for $n \in \mathbb{N}$ with $k > n+2$. We have $\alpha_p(E_{k-2}) = 1$ and $\beta_p(E_{k-2}) = p^{k-3}$. From the Zarkovskaya relations for the spinor L-function and the known interaction of the Siegel operator Φ with Klingen Eisenstein series we deduce that

$$L(s, E_k^{3,1}(f)) = L\left(s, E_k^{2,1}(f)\right)\ L\left(s-k+3, E_k^{2,1}(f)\right). \tag{3.25}$$

Twisting the spinor L-function of $E_k^{2,1}(f)$ locally with the parameters 1 and p^{k-3} leads to $E_k^{2,1}(f) \boxtimes E_{k-2} = E_k^{3,1}(f)$. Let $G \in S_k^2$ be a Hecke eigenform then

$$L(s, E_k^{3,2}(G)) = L\left(s, G\right)\ L\left(s-k+3, G\right). \tag{3.26}$$

We see that this is equal to $Z_{G\otimes E_{k-2}}(s)$ and to complete the proof we recall that we have already shown that $E_k^2 \boxtimes E_{k-2} = E_k^3$. □

Combining this result with the Darstellungssatz and the well-known fact that the space $M_k^{n,r}$ for $k > 2n$ has a Hecke eigenbasis for all $0 \le r \le n$ leads to the proof of Corollary 2.6, since the smaller weights can be examened case by case.

3.5. **Analytic properties of the modular twisted spinor L-function.** The modularity of $Z_{G\otimes h}(s)$ for suitable Hecke eigenforms $G \in M^2_{k_1}$ and $h \in M_{k_2}$ leads to new insides towards the spinor L-function $L(s,F)$ of $F := G \boxtimes h$.

Proof of Theorem 2.1. Let $G \in S^2_{k_1}$ and $h \in S_{k_2}$ with k_1, k_2 even be Hecke eigenforms. We assume that the first Fourier-Jacobi coefficient of G is not identically zero. Then the twisted spinor L-function $Z_{G\otimes h}(s)$ has a meromorphic continuation to the whole s-plane and satisfies a functional equation [4], [8]. Let $k_1 - k_2 \geq 0$ then the functional equation is given by

$$\widehat{Z}_{G\otimes h}(s) = \widehat{Z}_{G\otimes h}(2k_1 + k_2 - 3 - s). \tag{3.27}$$

Here $\widehat{Z}_{G\otimes h} := \widetilde{\gamma}_{k_1,k_2}(s)\, Z_{G\otimes h}(s)$ with

$$\widetilde{\gamma}_{k_1,k_2}(s) := (2\pi)^{-4s}\, \Gamma(s-k_1+2)\, \Gamma(s)\, \Gamma(s-k_2+1)\, \Gamma(s-k_1+1).$$

For $k_2 - k_1 > 0$ we have a similar functional equation (see [4] for more details). Let $h = E_{k_2}$ then $\alpha_p(E_{k_2}) = 1$ and $\beta_p(E_{k_2}) = p^{k_2-1}$. This leads to the degeneration

$$Z_{G\otimes E_{k_2}}(s) = L(s,G)\ L(s-k_2+1,G) \tag{3.28}$$

and to the meromorphic continuation of the twisted spinor L-function. The functional equation of $L(s,G)$ gives a functional equation of $Z_{G\otimes E_{k_2}}(s)$. The case $G = E^2_{k_1}$ and $h = E_{k_2}$ is obvious. Assume that $G = E^2_{k_1}$ or $G = E^{2,1}_{k_1}(f)$ with Hecke eigenforms $f \in S_{k_1}, h \in S_{k_2}$ then we have

$$\begin{aligned} Z_{E^2_{k_1}\otimes h}(s) &= L(s,h)\, L(s-k_1+1,h)\, L(s-k_1+2,h)\, L(s-2k_1+3,h) \\ Z_{E^{2,1}_{k_1}(f)\otimes h}(s) &= L(s,f\otimes h)\, L(s-k_1+2,f\otimes h). \end{aligned}$$

This finally proves the Theorem. □

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Max-Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: heim@mpim-bonn.mpg.de