Linear orbits of smooth plane curves

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§0. INTRODUCTION

There is a natural dominant rational map from the $\mathbf{P}^{14}$ of plane quartics to the moduli space $\mathcal{M}_3$ of smooth genus 3 curves. Restrict this map to a generically finite dominant rational map from a general (linear) $\mathbf{P}^6 \subset \mathbf{P}^{14}$ to $\mathcal{M}_3$. What is the degree of this map?

This natural question led us to studying the objects considered in this paper. It is readily understood that the fibers of the above map $\mathbf{P}^{14}$- - -> $\mathcal{M}_3$—or, more generally, of the natural map $\mathbf{P}^{\frac{d(d+3)}{2}}$ - - -> $\mathcal{M}_{\frac{(d-1)(d-2)}{2}}$—consist of the *orbits* of the action of PGL(3), the group of linear transformations of the plane, on the space $\mathbf{P}^N = \mathbf{P}^{\frac{d(d+3)}{2}}$ of plane curves of a given degree d. For a smooth plane curve C of degree $d \geq 3$, the set of all linear translations of C is an 8-dimensional quasi-projective subvariety of $\mathbf{P}^N$; the theme of this paper is that interesting information about plane curves gets naturally encoded in these subvarieties. For example, the answer to the question posed in the beginning of this introduction is the *degree* of the closure $\overline{O_C}$ of O_C in $\mathbf{P}^N$, for a general curve $C \in \mathbf{P}^N$.

Our purpose in this paper is to begin exploiting this connection, by studying the orbit closure $\overline{O_C}$ for *non-singular* curves C. The paper is focused on the computation of the degree of $\overline{O_C}$, on determining the 'boundary' of a given orbit—i.e., the complement of O_C in $\overline{O_C}$—and on studying the behavior of these orbits in one-parameter families of curves. We hope to complement this paper with similar results for *singular* curves, and with a study of the singular locus of the orbit closures, in future notes. Applications of the results will center on the study of the moduli spaces of *plane* curves: we include here (§5) results about the Chow groups (with $\mathbf{Q}$-coefficients) of the moduli spaces, in the spirit of [**Faber**]—this uses our result saying where the restriction of the above map to a general linear subspace of codimension 8 is proper. Also, the degree computation presented here has enumerative significance: if a curve C has no non-trivial automorphisms, then the degree of $\overline{O_C}$ is the number of linear translations of C that contain 8 given general points.

We attack the study of the orbit closure $\overline{O_C}$ of a curve C from two distinct sides. The degree computations (§§2,3) rest on dominating the orbit closure with a non-singular projective variety, and intersection theory computations using the blow-up formula of [**Aluffi1**]; the study of boundaries and families (§4) starts off with the Cartan-Iwahori decomposition in the (easy) $GL(3)$-case.

For the degree computations: given a non-singular plane curve C we are able to construct a non-singular compactification of the group PGL(3), with a dominant morphism to $\overline{O_C}$, by a suitable sequence of blow-ups of the space $\mathbf{P}^8$ of 3×3 matrices. The sequence depends on C, and in fact we find that it depends on local information about the *flexes* of C: revealing an intriguing connection between

$\overline{O_C}$ and the Hessian of C, which we believe will play a substantial role in future work on these objects. The variety is constructed by resolving the rational map $c : \mathbf{P}^8 \dashrightarrow \mathbf{P}^N$ extending the action of PGL(3) on O_C. The base locus of this map is supported on the set of matrices whose image is contained in C: if C is smooth this is a subset of $\mathbf{P}^8$ isomorphic to $\mathbf{P}^2 \times C$. Every point p of C contributes then to the base locus by the set of rank-1 matrices whose image is p; we will find (Theorem II) that to resolve the indeterminacies of the map at such points, one needs a number of blow-ups equal to the order of contact of the tangent line to C at p. For example, three suitable blow-ups suffice to resolve the map (thus constructing the variety) for a general curve. The construction leads to explicit formulas (Theorem III) for the degree of $\overline{O_C}$ in terms of the degree of C, the order of its group of automorphisms, and four numbers encoding the local information about the flexes of C. The degree of the orbit closure is maximal if and only if the curve has only simple flexes and no non-trivial automorphisms. The construction presented here should allow us to perform multiplicity computations on $\overline{O_C}$, to which we hope to devote a future note. Also, it should be possible to adapt the construction we present here to obtain compactifications of PGL(3) dominating the orbit closures of *singular* curves; we have some results in this direction (for mild singularities), which we do not present here.

For the study of boundaries: the 'boundary' of an orbit O_C (i.e., its complement in its closure $\overline{O_C}$) is the disjoint union of the orbits of different curves. These are necessarily singular, and have infinite automorphism group; we determine which curves arise in this way, depending on the flexes of the (smooth) curve C. Curves in the boundary arise either as images of rank-2 matrices by the above map c, or as limits of translations $c(\varphi)$ of C as φ approaches a point in the base locus of c. In the first case, the boundary curves consist of $d = \deg C$ lines through a point. Studying these curves amounts to studying the natural map from the g^2_d on C to the moduli space M_d of d-tuples of points on a line; we prove that this map is generically finite for $d \geq 5$. The other kind of boundary curves is found by shifting the point of view to the action of $GL(3)$ on $\mathbf{C}^{N+1}$, and studying all limits $\lim_{t\to 0} C \circ p(t)$, for $p : \mathbf{C}((t)) \to GL(3)$ a rational map. We use the Cartan-Iwahori decomposition and a case-by-case analysis to show that each k-flex of C (i.e., a point at which the tangent line intersects C with multiplicity k; in §§2,3 we also use the terminology 'flex of order $k-2$' for such a point) contributes to the boundary by the orbit of a curve consisting of the union of a k-th order cuspidal curve and the cuspidal tangent line, taken $d-k$ times (in coordinates, the orbit of the curve $x^{d-k}y^k + x^{d-1}z = 0$). A remarkable consequence (crucial in the application in §5) is that a general codimension-8 subspace of $\mathbf{P}^N$ will not contain any of the curves arising in this manner. This makes the natural map from this subspace to the moduli space of plane curves rather well-behaved.

Families are studied similarly. For a given C with a k-flex ($k \geq 4$), and with no non-trivial automorphisms for simplicity, consider a general 1-parameter family $C(u)$ of curves centered at $C(0) = C$. We observe that, as $u \to 0$, the orbit closure of $C(u)$ specializes to the union of the orbit closure of C and the (8-dimensional) orbit closure of another (singular) curve. This establishes a sort of liaison between C and a specific type of singular curve; the degree of $\overline{O_C}$ equals the degree of the orbit closure of the general curve minus the degree of the orbit closure of this specific singular curve. We determine what singular curves arise in this way, depending on the special flex of C, this time by computing limits $\lim_{t \to 0} C(t^e) \circ p(t)$, for p as above, and $e \in \mathbf{N}_+$.

The main body of the paper is preceded by a discussion of similar problems for the case of the action of the group PGL(2) on the spaces $\mathbf{P}^d$ parametrizing d-tuples of points on a line. This easy case makes for a good illustration of the techniques employed in the rest of the paper, and is of some interest in itself.

We conclude this introduction by answering the question posed in the beginning: the degree of the natural rational map from a general $\mathbf{P}^6 \subset \mathbf{P}^{14}$ to $\mathcal{M}_3$ is 14,280 ($d = 4$ in the corollary to Theorem III). Thus there are precisely 14280 quartics isomorphic to a given general one and containing 8 given general points.

As another illustration of the 'numerical' results in the paper, consider the d-uple Veronese embedding of $\mathbf{P}^2$: its trisecant variety can be identified with the closure of the set of degree-d curves that can be written as sum of 3 d-th powers of linear terms; i.e., with the orbit closure of the Fermat curve $x^d + y^d + z^d$. The Fermat curve has $6d^2$ distinct automorphisms and precisely $3d$ d-flexes, so (again as a consequence of Theorem III) the degree of the trisecant variety to the d-th Veronese embedding of $\mathbf{P}^2$ ($d \geq 3$) must be

$$\frac{1}{6}(d-2)(d^5 + 2d^4 - 26d^3 - 7d^2 + 192d - 192).$$

More examples of this kind may be found at the end of §3.

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§1. The PGL(2)-case.

We consider here the orbits of the action of PGL(2) on the space $\mathbf{P}^d$ of d-tuples of points of $\mathbf{P}^1$: this simpler context gives a good illustration of the techniques we will employ in the next sections to deal with PGL(3)-actions. (Some of these results appear also in [**Mukai-Umemura**]; in fact, the 'combinatorial' computation of the degree was done already in [**Enriques-Fano**].)

The first question we consider is the computation of the degree of the closure (in $\mathbf{P}^d$) of the orbit of a d-tuple: i.e., the intersection product of the orbit closure and *three* hyperplanes of $\mathbf{P}^d$.

It's worth observing that, in the PGL(2) case, this degree can be computed using simple combinatorics and a pinch of geometry. For the hyperplanes, take 3 distinct 'point-conditions', i.e., hyperplanes in $\mathbf{P}^d$ consisting of the d-tuples that contain a certain given point. One checks easily that the intersection multiplicity of the orbit closure and three point-conditions (determined by three distinct points p_1, p_2, p_3) at a d-tuple equals the product of the multiplicities of p_1, p_2 and p_3 in the d-tuple: so the intersection is automatically *transversal* if the d-tuple consists of d *distinct* points. Therefore, in this case the degree is just the number of points of intersection: the computation then comes down to counting the number of elements of PGL(2) that send a given d-tuple (consisting of d distinct points) to a d-tuple that contains 3 (distinct) given points. Using the fact that an element of PGL(2) is uniquely determined by prescribing the images of 3 distinct points, one immediately sees that the answer is

$$d(d-1)(d-2).$$

To get the degree of the orbit closure, we have to divide this number by the number of elements of PGL(2) sending a d-tuple to itself: i.e., the order of the stabilizer of the d-tuple.

EXAMPLES.

(1) The stabilizer of a 3-tuple consisting of 3 distinct points is S_3, so the degree of the orbit closure is 1 (the orbit closure is $\mathbf{P}^3$).
(2) A general 4-tuple has stabilizer $C_2 \times C_2$, so the degree of the orbit closure is 6. The 4-tuples with $j = 0$ (resp. 1728) have stabilizers A_4 (resp. D_4), so that the orbit closure has degree 2 (resp. 3).
(3) For $d \geq 5$, a general d-tuple has trivial stabilizer, so the degree of the orbit closure is $d(d-1)(d-2)$.

It would be easy to apply the same procedure to examine the case in which some points of the d-tuples appear with multiplicity. For example, suppose the d-tuple consists of an r-fold point and $d-r$ simple points, and (for simplicity) has trivial stabilizer. There are

$$(d-r)(d-r-1)(d-r-2)$$

ways to send the d-tuple to a d-tuple containing 3 given simple points, and

$$(d-r)(d-r-1)$$

ways to send it to a d-tuple with two assigned simple points and one given r-fold point. Arguing as above, the intersection is transversal at a d-tuple of the first kind, and has multiplicity r at a d-tuple of the second kind; and in the second case there are 3 possible choices for the r-fold point. So the degree of the orbit closure must be

$$(d-r)(d-r-1)(d-r-2)+3r(d-r)(d-r-1) = (d-r)(d-r-1)(d+2r-2).$$

However, this approach would not carry over to the PGL(3)-case. We will describe now the approach that does carry over to higher dimensions.

The idea is to resolve the indeterminacies of a rational map associated naturally to the given d-tuple. The easy combinatorics we have seen gets encoded in a geometric construction; as it happens, the geometry can be transferred to higher dimensions, while the combinatorics cannot.

Choose coordinates in $\mathbf{P}^1$, and let C stand for a homogeneous form in 2 variables of degree $d \geq 3$, or for the d-tuple of points on $\mathbf{P}^1$ corresponding to it. The PGL(2)-orbit of C in $\mathbf{P}^d$ is the image of the map

$$c\colon \mathrm{PGL}(2) \to \mathbf{P}^d$$

sending $\alpha \in \mathrm{PGL}(2)$ to the form $C \circ \alpha$. Observe that this map is finite (if at least three points of the d-tuple are distinct), and its degree equals the order of the stabilizer of C. This map determines a rational map from the $\mathbf{P}^3$ of 2×2 matrices to $\mathbf{P}^d$, which we also denote by c.

Now we will resolve this rational map: i.e., we will construct a variety $\widetilde{V}$ filling a commutative diagram

$$\begin{tikzcd}
\mathrm{PGL}(2) \arrow[r, phantom, "\subset"] \arrow[d, equal] & \widetilde{V} \arrow[r, "\tilde{c}"] \arrow[d, "\pi"'] & \mathbf{P}^d \arrow[d, equal] \\
\mathrm{PGL}(2) \arrow[r, phantom, "\subset"] & \mathbf{P}^3 \arrow[r, dashed, "c"] & \mathbf{P}^d
\end{tikzcd}$$

The image of $\tilde{c}$ in $\mathbf{P}^d$ is precisely the orbit closure: therefore the degree of the orbit closure can be found by computing the third power of the pull-back of the hyperplane class of $\mathbf{P}^d$ to $\widetilde{V}$, and dividing by the order of the stabilizer of C. We call 'predegree' the 3-fold self-intersection of the pull-back of the hyperplane from $\mathbf{P}^d$.

The base locus of $c\colon \mathbf{P}^3 \dashrightarrow \mathbf{P}^d$ consists of the matrices α for which the form $C \circ \alpha$ is identically zero. This happens exactly when α is a rank-1 matrix with image a point of the d-tuple C. The base locus of c is therefore supported on a finite number of 'parallel' lines in the (non-singular) quadric of rank-1 matrices. There are as many distinct lines as there are distinct points in the d-tuple C.

CLAIM. *A variety $\widetilde{V}$ as above can be obtained by blowing up $\mathbf{P}^3$ along the support of the base locus of c.*

To see this, call 'point-conditions in $\mathbf{P}^3$' the inverse image of the point-conditions of $\mathbf{P}^d$ (defined above). The map c is then the map defined by the linear system generated by the point-conditions in $\mathbf{P}^3$, and therefore the base locus of c is actually cut out by the point-conditions. Now we argue that a point-condition in $\mathbf{P}^3$ is a degree-d hypersurface consisting of nothing but a collection of hyperplanes, one for each point in the d-tuple C, each appearing with the same multiplicity as the corresponding point appears in C. This is immediate: give coordinates

$$\begin{pmatrix} p_0 & p_1 \\ p_2 & p_3 \end{pmatrix}$$

to the $\mathbf{P}^3$ of matrices; and suppose C is given by the equation

$$F(x : y) = 0 \quad .$$

Then the point-condition corresponding to e.g. the point $(1:0)$ has equation

$$F(p_0 : p_2) = 0 \quad ,$$

so is indeed a union of hyperplanes as argued.

Now let $\widetilde{V}$ be the blow-up of $\mathbf{P}^3$ along the lines supporting the base locus of c. The (a priori rational) map $\tilde{c}$ making the above diagram commute is then defined by the linear system on $\widetilde{V}$ generated by the proper transforms of the point-conditions: so the base locus of $\tilde{c}$ is cut out by the proper transforms in $\widetilde{V}$ of the point-conditions. But since the point-conditions are supported on unions of *hyperplanes*, they necessarily intersect *transversally* in $\mathbf{P}^3$: therefore their intersection in $\widetilde{V}$ is empty, and we can conclude that the map $\tilde{c} : \widetilde{V} \longrightarrow \mathbf{P}^d$ is indeed a morphism.

This proves the Claim.

Now computing the 3-fold self-intersection of the class of the proper transform of a point-condition (i.e., the predegree of the orbit closure) is a straightforward intersection calculus exercise. As we will in later sections, we use a formula from [**Aluffi1**], which we will recall as Proposition 3.2. The self-intersection is computed as the self-intersection of the point-condition in $\mathbf{P}^3$ (i.e., d^3) *minus* contributions coming from each component of the base locus of c. The formula gives

$$\text{predegree} = d^3 - \sum_{i=1}^{s} \int_{L_i} \frac{(m_i + dh)^3}{1 + 2h} \quad ,$$

where the summation runs over the *distinct* points $p_1, \dots, p_s$ of the d-tuple, m_i is the multiplicity of p_i in the d-tuple, L_i is the line in the base locus corresponding to p_i, and h denotes the hyperplane class in L_i. The degree is computed by taking the coefficient of h in the expression under $\int$:

$$\begin{aligned} \text{predegree} &= d^3 - \sum_{i=1}^{s} m_i^2(3d - 2m_i) \\ &= d^3 - 3d(\sum_{i=1}^{s} m_i^2) + 2(\sum_{i=1}^{s} m_i^3) \quad . \end{aligned}$$

So the predegree of a d-tuple C can be written in terms of just *two* numbers, each of which is a sum of 'local contributions' given by each point of C. For example, if the d-tuple consists of $d - r$ simple points and one r-tuple point, then

$$\sum_{i=1}^{s} m_i^2 = r^2 + d - r, \qquad \sum_{i=1}^{s} m_i^3 = r^3 + d - r,$$

so

$$\begin{aligned} \text{predegree} &= d^3 - 3d(r^2 + d - r) + 2(r^3 + d - r) \\ &= (d - r)(d - r - 1)(d + 2r - 2) \quad . \end{aligned}$$

As we will see in section 3, this general feature of the predegree (being determined by a few numbers recording local data) is preserved in the PGL(3) case.

We turn now to the question of determining the 'boundary' of the orbit of a d-tuple C, by which we mean the complement of the orbit in its closure. Observe that the boundary of an orbit is necessarily itself the union of orbits, and has dimension ≤ 2. Since the orbit of a d-tuple has dimension 3 as soon as the d-tuple consists of at least 3 distinct points, we can conclude right away that the boundary of the orbit of a given d-tuple must consist of the union of the orbits of d-tuples concentrated in at most *two* points. We will show:

CLAIM. *The boundary of the (3-dimensional) orbit of C is the union of the 1-dimensional orbit of x^d and of those 2-dimensional orbits of $x^r y^{d-r}$ for which r is the multiplicity of a point of C.*

Again, we have two possible approaches. On the one hand, we can use the blow-up constructing the variety $\widetilde{V}$ above. The rank-1 matrices *not* in the base locus have image in the orbit of x^d; so we only have to determine the image in $\mathbf{P}^d$ of the components of the exceptional divisor in $\widetilde{V}$. Now, the blow up can be described easily in coordinates.

Give coordinates

$$\begin{pmatrix} p_0 & p_1 \\ p_2 & p_3 \end{pmatrix}$$

to the $\mathbf{P}^3$ of matrices. The locus of rank-1 matrices is given by $p_0p_3 - p_1p_2 = 0$. Suppose the d-tuple C has equation $a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d = 0$, corresponding to the point $(a_0 : a_1 : \cdots : a_d) \in \mathbf{P}^d$ (with obvious choice of coordinates there). Assume that $(1:0)$ is a point of multiplicity $r \geq 1$ in C, i.e., $a_0 = a_1 = \cdots = a_{r-1} = 0, a_r \neq 0$. Then $p_2 = p_3 = 0$ is a component of the base locus of c and we can study $\widetilde{V}$ locally by blowing up $\mathbf{P}^3$ along $p_2 = p_3 = 0$.

On the affine piece $p_0 = 1$ we have coordinates (p_1, p_2, p_3). On an affine piece of the blow-up, coordinates (q_1, q_2, q_3) are given by

$$\begin{cases} p_1 = q_1 \\ p_2 = q_2 \\ p_3 = q_2q_3 \end{cases}$$

The map induced by c is then given by

$$(q_1, q_2, q_3) \mapsto (b_0 : b_1 : \cdots : b_d)$$

with

$$b_0x^d + \cdots + b_dy^d \sim a_r(x + q_1y)^{d-r}(q_2x + q_2q_3y)^r + \cdots + a_d(q_2x + q_2q_3y)^d.$$

Note that we can factor out ${q_2}^r$ from the last expression, so that

$$\begin{aligned} b_0x^d + \cdots + b_dy^d \sim\ & a_r(x + q_1y)^{d-r}(x + q_3y)^r \\ & + a_{r+1}q_2(x + q_1y)^{d-r-1}(x + q_3y)^{r+1} + \cdots + a_dq_2^{d-r}(x + q_3y)^d. \end{aligned}$$

The exceptional divisor is given here by $q_2 = 0$. The restriction of the map $\tilde{c} : \widetilde{V} \longrightarrow \mathbf{P}^d$ to the component of the exceptional divisor of $\widetilde{V}$ corresponding to the r-fold point is then given by restricting the last expression to $q_2 = 0$: we get d-tuples corresponding to points

$$b_0 x^d + \cdots + b_d y^d \sim a_r (x + q_1 y)^{d-r} (x + q_3 y)^r :$$

we conclude that the image of the exceptional divisor corresponding to a point in C of multiplicity r is the closure of the PGL(2)-orbit of $x^{d-r} y^r$. (The boundary of *this* orbit is the orbit of x^d.) The claim follows.

Now for the alternative approach, which is the one we will use later on to study the question in the PGL(3) case. To determine the boundary, one has to write down all possible limits of the image $c(\alpha)$ in $\mathbf{P}^d$, as α moves in PGL(2). Now the question can be lifted to the same question in GL(2), where it can be reduced to finding all limits

$$\lim_{t \to 0} C \circ \lambda(t) \quad ,$$

where $\lambda(t)$ is a *1-parameter subgroup* of GL(2) (we'll say more about this reduction in §4, where we treat the PGL(3) case). Up to a choice of coordinates, and disregarding trivial cases, we are then reduced to studying the limits

$$\lim_{t \to 0} C \circ \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} \quad ,$$

with $a < b$. Now suppose as above that C has an r-fold point at $(1:0)$, i.e., it is given by an equation $a_r x^{d-r} y^r + a_{r+1} x^{d-r-1} y^{r+1} + \cdots + a_d y^d = 0$, with $a_r \neq 0$. Composing with the 1-parameter subgroup gives the equation

$$t^{a(d-r)+br} \left(a_r x^{d-r} y^r + a_{r+1} x^{d-r-1} y^{r+1} t^{(b-a)} + \cdots + a_d y^d t^{(d-r)(b-a)} \right) = 0,$$

and we see that the limit as $t \to 0$ must be $a_r x^{d-r} y^r$ if it exists at all. Again, we conclude that each point of multiplicity r on C contributes to the boundary with the orbit of $x^{d-r} y^r$, which gives the original claim.

It is this second approach that we will follow to determine the boundary of orbits of *smooth* curves in the PGL(3) case. Again we will find, as we have seen in the PGL(2) case, that the global features of the orbit reflect local information at the points of C.

Finally, we would like to analyze the behavior of the orbit closure in a family. Suppose $C(t) \subset \mathbf{P}^d$ is a 1-parameter family of d-tuples, such that $C(0)$ is a d-tuple with an r-fold point ($r > 1$) and $d-r$ simple points, while $C(t)$ consists of d simple points for each $t \neq 0$. The object is to compare the orbit (closure) of the central fiber with the limit of the orbits of the other fibers. The orbit of the central fiber is clearly a component of the limit of the orbits; to convince oneself that there are *other* components, it's enough to observe that, as seen above, the degree of the orbit closure of $C(t)$ for $t \neq 0$ is $d(d-1)(d-2)$, while the degree of the orbit closure of $C(0)$ drops down to $(d-r)(d-r-1)(d+2r-2)$: other components must account for the missing

$$(3d-2r-2)r(r-1) \quad .$$

Now this is the degree of the orbit closure of a d-tuple with r simple points and a $(d-r)$-fold one: it's not hard to see that this is precisely what makes up the missing component. This sets up a sort of 'duality' between different kinds of d-tuples, which will have a counterpart in the PGL(3) case, to be seen in §4.

§2. A BLOW-UP CONSTRUCTION

In this section we construct a smooth projective variety surjecting onto the orbit closure $\overline{O_C}$ of a *smooth* plane curve $C \in \mathbf{P}^N = \mathbf{P}^{\frac{d(d+3)}{2}}$, where $d \geq 3$. As we will see, the construction depends essentially on the number and type of *flexes* of C.

Fix coordinates $(x_0 : x_1 : x_2)$ of $\mathbf{P}^2$, and assume the degree-d curve C has equation

$$F(x_0, x_1, x_2) = 0 \quad .$$

Consider the projective space $\mathbf{P}^8 = \mathbf{P}\mathrm{Hom}(\mathbf{C}^3, \mathbf{C}^3)$ of (homogeneous) 3×3 matrices $\alpha = (\alpha_{ij})_{i,j=0,1,2}$. So $\mathbf{P}^8$ is a compactification of $\mathrm{PGL}(3) = \{\alpha \in \mathbf{P}^8 : \det \alpha \neq 0\}$. To ease notations, in this section we will refer to a point in $\mathbf{P}^8$ and to any 3×3 matrix representing it by the same term; in the same vein, for $\alpha \in \mathbf{P}^8$ we will call '$\ker \alpha$' the linear subspace of $\mathbf{P}^2$ on which the map determined by α is not defined, '$\mathrm{im}\, \alpha$' will be the image of this map, and the rank '$\mathrm{rk}\, \alpha$' of α will be $1+\dim(\mathrm{im}\, \alpha)$. So:

$$\alpha \in \mathrm{PGL}(3) \iff \ker \alpha = \emptyset \iff \mathrm{im}\, \alpha = \mathbf{P}^2 \iff \mathrm{rk}\, \alpha = 3.$$

The curve C determines a rational map

$$c : \mathbf{P}^8 \dashrightarrow \mathbf{P}^N$$

as follows: for $\alpha \in \mathbf{P}^8$, let $c(\alpha)$ be the curve defined by the degree-d polynomial equation $F(\alpha(x_0, x_1, x_2)) = 0$. So $c(\alpha)$ is defined as long as $F(\alpha(x_0, x_1, x_2))$ doesn't vanish identically; i.e., precisely if $\mathrm{im}\, \alpha \not\subset C$.

If $\alpha \in \mathrm{PGL}(3)$, then $c(\alpha)$ is the translate of C by α; therefore, $c(\mathrm{PGL}(3))$ is just the orbit O_C of C in $\mathbf{P}^N$ for the natural action of PGL(3).

As an alternative description for the map c, consider for any point $p \in \mathbf{P}^2$ the equation

$$F(\alpha(p)) = 0 \quad .$$

As an equation 'in p', this defines the translate $c(\alpha)$; as an equation 'in α' this defines the hypersurface of $\mathbf{P}^8$ consisting of all α that map p to a point of C. We will call these hypersurfaces, that will play an important role in our discussion, 'point-conditions'. The rational map defined above is clearly the map defined by the linear system generated by the point-conditions on $\mathbf{P}^8$.

Our task here is to resolve the indeterminacies of the map $c : \mathbf{P}^8 \dashrightarrow \mathbf{P}^N$, by a sequence of blow-ups at smooth centers: we will get a smooth projective variety

$\widetilde{V}$ filling a commutative diagram

$$\begin{tikzcd} \mathrm{PGL}(3) \arrow[d, equal] & \subset & \widetilde{V} \arrow[r, "\widetilde{c}"] \arrow[d, "\pi"] & \mathbf{P}^N \arrow[d, equal] \\ \mathrm{PGL}(3) & \subset & \mathbf{P}^8 \arrow[r, dashed, "c"] & \mathbf{P}^N \end{tikzcd}$$

The image of $\widetilde{V}$ in $\mathbf{P}^N$ by $\widetilde{c}$ will then be the orbit closure $\overline{O_C}$. In §3 we will use $\widetilde{c}$ to pull-back questions about $\overline{O_C}$ to $\widetilde{V}$; the explicit description of $\widetilde{V}$ obtained in this section will enable us to answer these questions.

The plan is to blow-up the support of the base locus of c; we will get a variety V_1 and a rational map $c_1 : V_1 \dashrightarrow \mathbf{P}^N$. We will then blow-up the support of the base locus of c_1, getting a variety V_2 and a rational map $c_2 : V_2 \dashrightarrow \mathbf{P}^N$; in the case we are considering here (i.e., the curve C is smooth to start with), repeating this process yields eventually a variety $\widetilde{V}$ as above. The support of the first base locus is in fact a copy of $\mathbf{P}^2 \times C$ in $\mathbf{P}^8$ (see §2.1); if $(k, q) \in \mathbf{P}^2 \times C$, and c_i denotes the map obtained at the i-th stage, we will find that *c_i still has indeterminacies over (k, q) if and only if the tangent line to C at q intersects C at q with multiplicity $> i$.* So, for example, if C has only simple flexes then the map c_3 is regular (Proposition 2.9); and in general the number of blow-ups needed equals the highest possible multiplicity of intersection of a line with C.

We should point out that (even for smooth C) this is not the only way to construct a variety $\widetilde{V}$ as above: in fact, a different sequence of blow-ups is the one that seems to generalize naturally to approach the same problem for singular C.

§2.1. The first blow-up. The set of rank-1 matrices in $\mathbf{P}^8$ is the image of the Segre embedding

$$\check{\mathbf{P}}^2 \times \mathbf{P}^2 \hookrightarrow \mathbf{P}^8$$

given in coordinates by

$$((k_0 : k_1 : k_2), (q_0 : q_1 : q_2)) \mapsto \begin{pmatrix} k_0q_0 & k_1q_0 & k_2q_0 \\ k_0q_1 & k_1q_1 & k_2q_1 \\ k_0q_2 & k_1q_2 & k_2q_2 \end{pmatrix}$$

where $k_0x_0 + k_1x_1 + k_2x_2 = 0$ is the kernel of the matrix, and $(q_0 : q_1 : q_2)$ is its image. Intrinsically, this is just the map induced from the map

$$\begin{gathered} \check{\mathbf{C}}^3 \oplus \mathbf{C}^3 \to \check{\mathbf{C}}^3 \otimes \mathbf{C}^3 = \mathrm{Hom}(\mathbf{C}^3, \mathbf{C}^3) \\ (f, u) \mapsto f \otimes u \end{gathered}$$

We have already observed that the map $c : \mathbf{P}^8 \dashrightarrow \mathbf{P}^N$ is *not* defined at $\alpha \in \mathbf{P}^8$ precisely when $\operatorname{im}\alpha \subset C$; if C is smooth (therefore irreducible), this means that the image of α is a point of C. Therefore:
the support of the base locus of c is the image of $\check{\mathbf{P}}^2 \times C$ in $\mathbf{P}^8$ via the Segre embedding identifying $\check{\mathbf{P}}^2 \times \mathbf{P}^2$ with the set of rank-1 matrices.

In particular, the support of the base locus of c is *smooth*, since C is. We let then $B = \check{\mathbf{P}}^2 \times C$, and we let $V_1 \xrightarrow{\pi_1} \mathbf{P}^8$ be the blow-up of $\mathbf{P}^8$ along B. Since

$B \cap \mathrm{PGL}(3) = \emptyset$, V_1 contains a dense open set which we can identify with PGL(3). Also, the linear system generated by the proper transforms in V_1 of the point-conditions (which we will call 'point-conditions in V_1'), defines a rational map $c_1 : V_1 \dashrightarrow \mathbf{P}^N$ making the diagram

$$\begin{tikzcd} \mathrm{PGL}(3) \arrow[r, phantom, "\subset"] \arrow[d, equal] & V_1 \arrow[r, dashed, "c_1"] \arrow[d, "\pi_1"] & \mathbf{P}^N \arrow[d, equal] \\ \mathrm{PGL}(3) \arrow[r, phantom, "\subset"] & \mathbf{P}^8 \arrow[r, dashed, "c"] & \mathbf{P}^N \end{tikzcd}$$

commutative. The exceptional divisor E_1 in V_1 is the projectivized normal bundle of B in $\mathbf{P}^8$: $E_1 = \mathbf{P}(N_B\mathbf{P}^8)$. We will show now that *the base locus of c_1 is supported on a $\mathbf{P}^1$-subbundle of E_1 over B.*

Let (k, q) be a point of $B = \check{\mathbf{P}}^2 \times C$: i.e., a rank-1 $\alpha \in \mathbf{P}^8$ with $\ker\alpha = k$, $\operatorname{im}\alpha = q \in C$. Also, let ℓ be the line tangent to C at q, let p be a point of $\mathbf{P}^2$, and denote by P the point-condition in $\mathbf{P}^8$ corresponding to p.

LEMMA 2.1. *(i) The tangent space to B at (k,q) consists of all $\varphi \in \mathbf{P}^8$ such that $im\varphi \subset \ell$ and $\varphi(k) \subset q$.*

(ii) P is non-singular at (k,q), and the tangent space to P at (k,q) consists of all $\varphi \in \mathbf{P}^8$ such that $\varphi(p) \subset \ell$.

We are using our notations rather freely here. For example, in (i) $\alpha = (k,q)$ is in the tangent space since $\alpha(k) = \emptyset$ (as α is not defined along k).

PROOF: (i) The tangent space to B at (k,q) is spanned by the plane $\{(k',q) \in B : k' \in \check{\mathbf{P}}^2\} = \{\varphi \in \mathbf{P}^8 : \operatorname{im}\varphi = q\}$ and by the line $\{(k,q') \in B : q' \in \ell\} = \{\varphi \in \mathbf{P}^8 : \ker\varphi = k, \operatorname{im}\varphi \in \ell\}$. Both these subspaces of $\mathbf{P}^8$ are contained in $\{\varphi \in \mathbf{P}^8 : \operatorname{im}\varphi \subset \ell, \varphi(k) \subset q\}$; since this latter has clearly dimension 3, we are done.

(ii) For $\alpha = (k,q)$ and $\varphi \in \mathbf{P}^8$ consider the line $\alpha + \varphi t$. Restricting the equation for P to this line gives the polynomial equation in t

$$F((\alpha + \varphi t)(p)) = 0 \quad \text{, i.e.}$$
$$F(\alpha(p)) + \sum_i \left(\frac{\partial F}{\partial x_i}\right)_{\alpha(p)} \varphi_i(p)\, t + \cdots = 0$$

(where $\varphi_i(p)$ denotes the i-th coordinate of $\varphi(p)$).

$F(\alpha(p)) = 0$ since $\operatorname{im}\alpha = q \in C$; the line is tangent to P at α when the linear term also vanishes, i.e. if $\sum_i (\partial F/\partial x_i)_q \varphi_i(p) = 0$. This says precisely $\varphi(p) \subset \ell$, as claimed.

P is non-singular at α because any φ *not* satisfying the condition $\varphi(p) \subset \ell$ gives a line $\alpha + \varphi t$ intersecting P with multiplicity 1 at α, by the above computation. ∎

With the same notations, the tangent space to $\check{\mathbf{P}}^2 \times \mathbf{P}^2$ at α consists of all φ with $\varphi(\ker\alpha) \subset \operatorname{im}\alpha$ (intrinsically, all transformations φ inducing a map $\operatorname{coim}\alpha \longrightarrow \operatorname{coker}\alpha$).

The set of all φ such that $\operatorname{im}\varphi \subset \ell$ forms (for any α) a 5-dimensional space containing the tangent space to B at α, and therefore determines a 2-dimensional subspace of the fiber of $N_B\mathbf{P}^8$ over α. As α moves in B we get a rank-2 subbundle of $N_B\mathbf{P}^8$, and hence a $\mathbf{P}^1$-subbundle of $E_1 = \mathbf{P}(N_B\mathbf{P}^8)$, which we denote B_1. Notice that B_1 is non-singular, as a $\mathbf{P}^1$-bundle over the non-singular B.

PROPOSITION 2.2. *The base locus of the map* $c_1 : V_1 \dashrightarrow \mathbf{P}^N$ *is supported on* B_1.

PROOF: Since c_1 is defined by the linear system generated by all point-conditions in V_1, we simply need to show that the intersection of all point-conditions in V_1 is set-theoretically B_1. This assertion can be checked fiberwise over $\alpha = (k, q) \in B$; so all we need to observe is that the intersection of the tangent spaces to all point-conditions at α consists (by Lemma 2.1 (ii)) of the $\varphi \in \mathbf{P}^8$ such that $\varphi(p) \subset \ell$ for *all* p; i.e., the 5-dimensional space used above to define B_1. ∎

If $P_1^{(p)}$ denotes the point-condition in V_1 corresponding to $p \in \mathbf{P}^2$, we have just shown $\bigcap_{p \in \mathbf{P}^2} P_1^{(p)}$ is supported on B_1. The proof says a little more:

REMARK 2.3. $\bigcap_{p \in \mathbf{P}^2} P_1^{(p)} \cap E_1 = B_1$ *(scheme-theoretically).*

Indeed on each fiber of E_1 (say over $\alpha \in B$) the fiber of B_1, a linear subspace, is cut out by the fibers of the $P_1^{(p)} \cap E_1$, linear subspaces themselves; and the situation clearly globalizes as α moves in B.

§2.2. The second blow-up. Let $V_2 \xrightarrow{\pi_2} V_1$ be the blow-up of V_1 along B_1. The new exceptional divisor is $E_2 = \mathbf{P}(N_{B_1} V_1)$; call 'point-conditions in V_2' the proper transforms of the point-conditions of V_1. The linear system generated by the point-conditions defines a rational map $c_2 : V_2 \dashrightarrow \mathbf{P}^N$; again, we obtain a diagram

$$\begin{array}{ccccc} \mathrm{PGL}(3) & \subset & V_2 & \overset{c_2}{\dashrightarrow} & \mathbf{P}^N \\ \| & & \Big\downarrow{\scriptstyle \pi_2 \circ \pi_1} & & \| \\ \mathrm{PGL}(3) & \subset & \mathbf{P}^8 & \overset{c}{\dashrightarrow} & \mathbf{P}^N \end{array}$$

and we proceed to determine the support of the base locus of c_2.

Let $\widetilde{E}_1$ be the proper transform of E_1 in V_2. Then

LEMMA 2.4. *The base locus of* c_2 *is disjoint from* $\widetilde{E}_1$.

PROOF: This is basically a reformulation of Remark 2.3: $\widetilde{E}_1$ is the blow-up of E_1 along B_1, and B_1 is cut out scheme-theoretically by the intersections of E_1 with the point-conditions of V_1. So the intersection of the point-conditions in V_2 must be empty along $\widetilde{E}_1$, which is the claim. ∎

Lemma 2.4 reduces the determination of the support of the base locus of c_2 to a computation in $\mathbf{P}^8$. Denote by $\mathcal{B}$ the scheme-theoretic intersection of the point-conditions in $\mathbf{P}^8$, so the support of $\mathcal{B}$ is B. For $\alpha \in B$, let $th_\alpha(\mathcal{B})$ be the maximum length of the intersection with $\mathcal{B}$ of the germ of a smooth curve centered at α and transversal to B (the 'thickness' of $\mathcal{B}$ at α, in the terminology of [**Aluffi2**]).

LEMMA 2.5. *The base locus of* c_2 *is disjoint from* $(\pi_2 \circ \pi_1)^{-1}\alpha$ *if* $th_\alpha(\mathcal{B}) \leq 2$.

PROOF: The base locus of c_2 is the intersection of all point-conditions in V_2, i.e. the set of all directions normal to B_1 and tangent to all point-conditions in V_1. Let then $\gamma(t)$ be a smooth curve germ centered at a point of B_1 above α, transversal to B_1, and tangent to all point-conditions in V_1. By Lemma 2.4, γ is transversal to E_1; therefore $\pi_1(\gamma(t))$ is a *smooth* curve germ centered at α and *transversal* to B. Since $\gamma(t)$ intersects all point-conditions in V_1 with multiplicity 2 or more, $\pi_1(\gamma(t))$ must intersect all point-conditions in $\mathbf{P}^8$ with multiplicity 3 or more; B is the intersection of all point-conditions in $\mathbf{P}^8$, so this forces $th_\alpha(B) \geq 3$. ∎

Now the key computation is

LEMMA 2.6. *If $\alpha = (k, q) \in B$, and ℓ is the line tangent to C at q, then $th_\alpha(B)$ equals the intersection multiplicity of ℓ and C at q.*

PROOF: Let m be the intersection multiplicity of ℓ and C at q. To show $th_\alpha(B) \geq m$, we just have to produce a curve normal to B and intersecting all point-conditions with multiplicity at least m at α; such is the line $\alpha + \varphi t$, with $\varphi \in \mathbf{P}^8$ such that $\operatorname{im}\varphi = \ell$ and $\varphi(k) \neq q$. Indeed, the last condition guarantees normality (Lemma 2.1 (i)); and, for general p, $q = \alpha(p)$ and $\varphi(p)$ span ℓ: so $F((\alpha + \varphi t)(p))$ is just the restriction of F to a parametrization of ℓ, and it must vanish exactly m times at $t = 0$. Notice that these directions are precisely those defining B_1.

To show $th_\alpha(B) \leq m$, let $\gamma(t)$ be any smooth curve germ normal to B and centered at α; we have to show that γ intersects some point-condition with multiplicity $\leq m$ at α. In an affine open of $\mathbf{P}^8$ containing α, write

$$\gamma(t) = \alpha + \varphi t + \dots \quad .$$

The equation for the point-condition corresponding to p restricts on γ to

$$F((\alpha + \varphi t + \dots)(p)) = F(\alpha(p)) + \sum_i \left(\frac{\partial F}{\partial x_i}\right)_{\alpha(p)} \varphi_i(p)\, t + \dots = 0 \quad ,$$

where $\varphi_i(p)$ denotes the i-th coordinate of $\varphi(p)$. The coefficient of t^m in this expansion is

$$(*) \quad \frac{1}{m!} \sum_{i_1,\dots,i_m} \left(\frac{\partial^m F}{\partial x_{i_1} \cdots \partial x_{i_m}}\right)_{\alpha(p)} \varphi_{i_1}(p) \cdots \varphi_{i_m}(p)$$
$$+ \text{ terms involving derivatives of lower order,}$$

and to conclude the proof we have to show that this term is not identically 0.

To see this, observe that if ℓ and C intersect with multiplicity exactly m at q, then the form

$$\sum_{i_1,\dots,i_m} \left(\frac{\partial^m F}{\partial x_{i_1} \cdots \partial x_{i_m}}\right)_{\alpha(p)} x_{i_1} \cdots x_{i_m}$$

doesn't vanish identically on ℓ; since $\varphi(\ker\alpha) \not\subset q$ (γ is normal to B), this implies that the summand

$$\frac{1}{m!} \sum_{i_1,\dots,i_m} \left(\frac{\partial^m F}{\partial x_{i_1} \cdots \partial x_{i_m}}\right)_{\alpha(p)} \varphi_{i_1}(p) \cdots \varphi_{i_m}(p)$$

vanishes exactly $d - m$ times along the line $k = \ker\alpha$. But since all the other summands in (*) involve derivatives of order $< m$, they vanish with order $> d - m$ along k. Therefore (*) can't be identically 0, as we claimed. ∎

We adopt the following conventions:

DEFINITION. *A point q of C is a 'flex of order r' (or '$(r+2)$-flex') if the line tangent to C at q intersects C at q with multiplicity $r+2$. We will say that q is a 'flex' of C if $r \geq 1$, and that q is a 'simple flex' if $r = 1$.*

Now we observe that there is a section $s : B_1 \longrightarrow E_2$: for $\alpha_1 \in B_1$, let $\alpha = \pi_1(\alpha_1) \in B$, say $\alpha = (k, q)$, and let ℓ be the line tangent to C at q. By the construction of B_1, there is a matrix $\varphi \in \mathbf{P}^8$ with $\operatorname{im}\varphi \subset \ell$ such that α_1 is the intersection of E_1 and the proper transform of the line $\alpha + \varphi t$ in V_1; now let $s(\alpha_1)$ be the intersection of E_2 and the proper transform of the line $\alpha + \varphi t$ in V_2 (it is clear that $s(\alpha_1)$ does not depend on the specific φ chosen to represent α_1).

Let B_2 be the image via s of the set $\{\alpha_1 \in B_1 : q \text{ is a flex of } C\}$. Thus B_2 consists of a number of smooth three-dimensional components, one for each flex of C: each component maps isomorphically to a $\mathbf{P}^1$-bundle over one of the planes $\{(k, q) \in B : q \text{ is a flex of } C\}$.

PROPOSITION 2.7. *The base locus of the map $c_2 : V_2 \dashrightarrow \mathbf{P}^N$ is supported on B_2.*

PROOF: Let $\alpha_1 \in B_1$, and $\alpha = (k, q)$ the image of α_1 in B, as above. By Lemma 2.5 and 2.6, the intersection of the base locus of c_2 with the fiber $\pi_2^{-1}(\alpha_1)$ is *empty* if q is not a flex of C; it is at most *one point* even if q is a flex of C, because it misses a hyperplane in $\pi_2^{-1}(\alpha_1) \cong \mathbf{P}^3$ by Lemma 2.4. Thus all we have to show is that $s(\alpha_1)$ *is* in the base locus of c_2 if q is a flex of C (of order $r \geq 1$). But, as observed in the proof of Lemma 2.6, the line $\alpha + \varphi t$ determining α_1 intersects each point-condition in $\mathbf{P}^8$ with multiplicity at least $r + 2 \geq 3$; therefore the proper transform of $\alpha + \varphi t$ is *tangent* to all point-conditions in V_1, and it follows that $s(\alpha_1) \in$ all point-conditions in V_2, as needed. ∎

§2.3. **The third blow-up.** Let $V_3 \xrightarrow{\pi_3} V_2$ be the blow-up of V_2 along B_2. The new exceptional divisor is E_3; the 'point-conditions of V_3' are the proper transforms of the point-conditions of V_2. The linear system generated by the point-conditions defines a rational map $c_3 : V_3 \dashrightarrow \mathbf{P}^N$, making the diagram

$$\begin{tikzcd}
\mathrm{PGL}(3) \arrow[r, phantom, "\subset"] \arrow[d, equal] & V_3 \arrow[r, dashed, "c_3"] \arrow[d, "\pi_3 \circ \pi_2 \circ \pi_1"'] & \mathbf{P}^N \arrow[d, equal] \\
\mathrm{PGL}(3) \arrow[r, phantom, "\subset"] & \mathbf{P}^8 \arrow[r, dashed, "c"] & \mathbf{P}^N
\end{tikzcd}$$

commute. We will show now that c_3 is a *regular map* if all the flexes of C are simple, so that in this case V_3 is the variety we are looking for. For each flex of order > 1, we will find a four-dimensional component in the base locus of c_3, and more blow-ups will be needed.

Call $\mathcal{B}_2$ the scheme-theoretic intersection of the point-conditions in V_2, so $\mathcal{B}_2$ is supported on B_2. For $\alpha_2 \in B_2$, define the thickness $th_{\alpha_2}(\mathcal{B}_2)$ of $\mathcal{B}_2$ at α_2 as we did above for $th_\alpha(\mathcal{B})$. Also, let $\alpha = (k, q)$ be the image of α_2 in B. With these notations:

LEMMA 2.8. *If q is an flex of order r of C, then $th_{\alpha_2}(\mathcal{B}_2) = r$.*

PROOF: We have to show that if $\gamma(t)$ is a smooth curve germ in V_2, centered at α_2 and transversal to B_2, then the maximum length of the intersection of B_2 and γ at $t = 0$ is precisely r.

Suppose first that γ is transversal to E_2: then, as argued in the proof of Lemma 2.5, the image of γ in $\mathbf{P}^8$ is a smooth curve germ centered at α and transversal to B: by Lemma 2.6, the length of the intersection of B and this curve is at most $r + 2$; it follows that the maximum length of the intersection of B_2 and such γ's is indeed r (attained for example by the proper transform of $\alpha + \varphi t$, with φ as in the proof of Lemma 2.6).

Thus we may assume that γ is tangent to E_2, and show that

CLAIM. *$B_2 \cap \gamma(t)$ vanishes at most r times at $t = 0$.*

This is a lengthy coordinate computation, and we encourage the hasty reader to skip it for the moment. The outcome is that the maximum length is r, and it is attained in the direction normal to B_2 in the section $s(B_1) \subset E_2$ defined in §2.2.

PROOF OF THE CLAIM: We express the blow-ups in coordinates. If $\alpha = (k, q)$ is the image of α_2 in B, we can assume that $q = (1 : 0 : 0)$ is a flex of order r of C, and that k is the line $x_0 = 0$. Also, we write the equation of C in a neighborhood of q (with affine coordinates x, y) as

$$y = f(x) \quad ,$$

where $f(x)$ is a convergent power series; since $q = (0, 0)$ is a (smooth) flex of order r of C, we can assume

$$f(x) = x^{r+2} + \text{ terms in higher powers of } x.$$

With this set-up, α is the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; give coordinates

$$\begin{pmatrix} 1 & p_1 & p_2 \\ p_3 & p_4 & p_5 \\ p_6 & p_7 & p_8 \end{pmatrix}$$

in a neighborhood of α in $\mathbf{P}^8$. The point-condition corresponding to $(\xi_0 : \xi_1 : \xi_2)$ has then equation

$$(*) \qquad \frac{p_6\xi_0 + p_7\xi_1 + p_8\xi_2}{\xi_0 + p_1\xi_1 + p_2\xi_2} = f\left(\frac{p_3\xi_0 + p_4\xi_1 + p_5\xi_2}{\xi_0 + p_1\xi_1 + p_2\xi_2}\right)$$

Equations for B in these coordinates are

$$\begin{cases} p_4 = p_1 p_3 \\ p_5 = p_2 p_3 \\ p_7 = p_1 p_6 \\ p_8 = p_2 p_6 \\ p_6 = f(p_3) \end{cases}$$

(the first four equations define rank-1 matrices, and the fifth one forces the image to land on C).

We can then choose coordinates $(q_1, \dots, q_8)$ in V_1 so that

$$\begin{array}{lll} p_1 = q_1 & p_4 = q_4 + q_1 q_3 & p_6 = q_4 q_6 + f(q_3) \\ p_2 = q_2 & p_5 = q_4 q_5 + q_2 q_3 & p_7 = q_4 q_7 + q_1 p_6 \\ p_3 = q_3 & & p_8 = q_4 q_8 + q_2 p_6 \end{array}$$

The exceptional divisor E_1 has equation $q_4 = 0$; the equation (*) above pulls-back to

$$\begin{aligned} f(q_3) + q_4 q_6 + q_4 \frac{q_7 \xi_1 + q_8 \xi_2}{\xi_0 + q_1 \xi_1 + q_2 \xi_2} &= f\left(q_3 + q_4 \frac{\xi_1 + q_5 \xi_2}{\xi_0 + q_1 \xi_1 + q_2 \xi_2}\right) \\ = f(q_3) + q_4 \frac{\xi_1 + q_5 \xi_2}{\xi_0 + q_1 \xi_1 + q_2 \xi_2} f'(q_3) &+ q_4^2 \left(\frac{\xi_1 + q_5 \xi_2}{\xi_0 + q_1 \xi_1 + q_2 \xi_2}\right)^2 \frac{f''(q_3)}{2} + q_4^3(\cdots) \end{aligned}$$

and therefore the point-conditions in V_1 have equation

$$\begin{aligned} (^{**}) \quad q_6 + \frac{q_7 \xi_1 + q_8 \xi_2}{\xi_0 + q_1 \xi_1 + q_2 \xi_2} &= \frac{\xi_1 + q_5 \xi_2}{\xi_0 + q_1 \xi_1 + q_2 \xi_2} f'(q_3) \\ &\quad + q_4 \left(\frac{\xi_1 + q_5 \xi_2}{\xi_0 + q_1 \xi_1 + q_2 \xi_2}\right)^2 \frac{f''(q_3)}{2} + q_4^2(\cdots) \end{aligned}$$

Equations for B_1 are

$$\begin{cases} q_4 = 0 \\ q_6 = 0 \\ q_7 = f'(q_3) \\ q_8 = q_5 f'(q_3) \end{cases}$$

(let $q_4 = 0$, and impose (**) to hold for all ξ_0, ξ_1, ξ_2) and we then choose coordinates $(r_1, \dots, r_8)$ in V_2 such that

$$\begin{array}{lll} q_1 = r_1 & q_4 = r_4 & q_6 = r_4 r_6 \\ q_2 = r_2 & q_5 = r_5 & q_7 = r_4 r_7 + f'(r_3) \\ q_3 = r_3 & & q_8 = r_4 r_8 + r_5 f'(r_3) \end{array}$$

In these coordinates the exceptional divisor E_2 has equation $r_4 = 0$, and (**) pulls-back to

$$\begin{aligned} r_4 r_6 + r_4 \frac{r_7 \xi_1 + r_8 \xi_2}{\xi_0 + r_1 \xi_1 + r_2 \xi_2} &+ \frac{\xi_1 + r_5 \xi_2}{\xi_0 + r_1 \xi_1 + r_2 \xi_2} f'(r_3) \\ &= \frac{\xi_1 + r_5 \xi_2}{\xi_0 + r_1 \xi_1 + r_2 \xi_2} f'(r_3) + r_4 \left(\frac{\xi_1 + r_5 \xi_2}{\xi_0 + r_1 \xi_1 + r_2 \xi_2}\right)^2 \frac{f''(r_3)}{2} + r_4^2(\cdots) \end{aligned}$$

so the point-conditions in V_2 have equation

$$(^{***}) \qquad r_6 + \frac{r_7 \xi_1 + r_8 \xi_2}{\xi_0 + r_1 \xi_1 + r_2 \xi_2} = \left(\frac{\xi_1 + r_5 \xi_2}{\xi_0 + r_1 \xi_1 + r_2 \xi_2}\right)^2 \frac{f''(r_3)}{2} + r_4(\cdots)$$

(also, the leading term in $(\cdots)$ is essentially $f'''(r_3)$). Now letting $r_4 = 0$ and imposing that (***) hold for all ξ_0, ξ_1, ξ_2 gives equations for B_2 at α_2:

$$\begin{cases} r_3 = 0 \\ r_4 = 0 \\ r_6 = 0 \\ r_7 = 0 \\ r_8 = 0 \end{cases} \qquad ;$$

from which it follows readily that the maximum length of the intersection with the general point-condition of a smooth curve germ $\gamma(t)$ transversal to B_2 and tangent to E_2 equals the order of vanishing of $f''(t)$ at $t = 0$, i.e. r, as claimed.

Tracing this coordinate description, equations for $s(B_1)$ in V_2 are $r_4 = r_6 = r_7 = r_8 = 0$[1]. Then $(r_1, \dots, r_8) = (*, *, t, 0, *, 0, 0, 0)$ is a direction normal to B_2 in $s(B_1)$ which intersects the general point-condition precisely r times at $t = 0$. ∎

The next results are now easy consequences.

PROPOSITION 2.9. *If all flexes of C are simple, then the map $c_3 : V_3 \dashrightarrow \mathbf{P}^N$ is regular.*

PROOF: We have to show that c_3 has no base locus, i.e. that the intersection of all point-conditions in V_3 is *empty*. But a point in the intersection of all point-conditions in V_3 would determine a direction normal to B_2 and *tangent* to all point-conditions in V_2; the thickness of B_2 would then be ≥ 2 at some point. By Lemma 2.8, if all flexes of C are simple (i.e., of order 1) the thickness of B_2 is precisely 1 everywhere on B_2, so this can't happen. ∎

By Proposition 2.9, we are done in the case when C has only simple flexes: V_3 is the variety $\widetilde{V}$ we meant to construct. We will show now that for each flex of C of order $r > 1$, the base locus of c_3 has a smooth four-dimensional connected component.

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[1] To see this, work over a neighborhood of $\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. B is parametrized by $(k_1, k_2, x) \mapsto \begin{pmatrix} 1 & k_1 & k_2 \\ x & k_1 x & k_2 x \\ f(x) & k_1 f(x) & k_2 f(x) \end{pmatrix}$. For each $q = (x, f(x)) \in C$ choose φ with ker on the line $x_1 + \ell x_2 = 0$, and image $\neq q$ on the tangent line to C at q: e.g. $(x \neq 1)$ at $(1, f(x) + f'(x)(1 - x))$. Then, in the given coordinates, the line $\alpha + \varphi t$ and its proper transforms in V_1 and V_2 are parametrized resp. by

$$t \mapsto \begin{pmatrix} 1 & k_1 + t & k_2 + \ell t \\ x & k_1 x + t & k_2 x + \ell t \\ f(x) & k_1 f(x) + t f(x) + t f'(x)(1-x) & k_2 f(x) + \ell t f(x) + \ell t f'(x)(1-x) \end{pmatrix} \quad ,$$

$$t \mapsto (k_1 + t, k_2 + \ell t, x, t(1 - x), \ell, 0, f'(x), \ell f'(x)) \quad ,$$

$$t \mapsto (k_1 + t, k_2 + \ell t, x, t(1 - x), \ell, 0, 0, 0) \quad .$$

As $t \to 0$, the last equation gives a parametrization of $s(B_1)$ in $(r_1, \dots, r_8)$:

$$(k_1, k_2, x, \ell) \mapsto (k_1, k_2, x, 0, \ell, 0, 0, 0) \quad ,$$

from which we can read the equations of $s(B_1)$.

Let $\alpha_2 \in B_2$, mapping to $\alpha = (k, q)$ in B, and assume q is a flex of C of order $r > 1$. B_2 is 3-dimensional, so the fiber $\pi_3^{-1}(\alpha_2)$ of $E_3 = \mathbf{P}(N_{B_2} V_2)$ over α_2 is a $\mathbf{P}^4$. We have two special points in this $\mathbf{P}^4$, namely the point determined by the proper transform of the line $\alpha + \varphi t$ used in §2.2 to define s, and the direction normal to B_2 in the section $s(B_1)$. We have seen in the proof of Lemma 2.8 that the length of the intersection of these directions with B_2 is exactly r; also, these points are *distinct* for all α_2 (since one of them corresponds to a direction contained in E_2, while the other comes from a direction transversal to E_2), so they determine a $\mathbf{P}^1$ in the fiber $\pi_3^{-1}(\alpha_2)$. As α_2 moves in the component of B_2 over q, this $\mathbf{P}^1$ traces a $\mathbf{P}^1$-bundle over that component, a smooth four-dimensional subvariety $B_3^{(q)}$ of E_3. Call B_3 the union of all these (disjoint) subvarieties of E_3, arising from non-simple flexes of C.

PROPOSITION 2.10. *The base locus of the map $c_3 : V_3 \dashrightarrow \mathbf{P}^N$ is supported on B_3.*

PROOF: We have to show that in each fiber $\pi_3^{-1}(\alpha_2) \cong \mathbf{P}^4$ as above, the intersection of all point-conditions is supported on the specified $\mathbf{P}^1$. Observe that each point-condition determines a hyperplane in this $\mathbf{P}^4$, so that the intersection of the base locus of c_3 with $\pi_3^{-1}(\alpha_2)$ must be a linear subspace of this $\mathbf{P}^4$. Secondly, for the same reason, no directions tangent to the fiber of E_2 containing α_2 can be tangent to all point-conditions in V_2. The fibers of E_2 are three-dimensional and transversal to B_2, thus this shows that the base locus of c_3 must miss a $\mathbf{P}^2$ in the fiber $\pi_3^{-1}(\alpha_2)$. Thus, the intersection of the base locus of c_3 with $\pi_3^{-1}(\alpha_2)$ can consist of at most a $\mathbf{P}^1$.

Therefore, we just have to show that the two points of $\pi_3^{-1}(\alpha_2)$ used in the construction of B_3 are contained in all point-conditions of V_3; or, equivalently, the two directions in V_2 used to define these points are *tangent* to all point-conditions in V_2. But this is precisely the result of the computation in the proof of Lemma 2.8: the length of the intersection of these curves with all point-conditions is $r \geq 2$. ∎

§2.4 Further blow-ups. As we have seen in §2.3, each non-simple flex q of C gives rise to a smooth four-dimensional component of the support B_3 of the base locus of c_3; and B_3 is the union of all such components. The plan is still to blow-up the support of the base-locus; since the components are disjoint, we can concentrate on a specific one: say $B_3^{(q)}$, corresponding to a flex q of C of order $r \geq 2$.

Let $V_3^{(q)}$ be the complement of all components of B_3 *other than* $B_3^{(q)}$ in V_3. Let $V_4^{(q)} \to V_3^{(q)}$ be the blow-up of $V_3^{(q)}$ along $B_3^{(q)}$; again, the proper transforms in $V_4^{(q)}$ of the point-conditions define a map $c_4^{(q)} : V_4^{(q)} \dashrightarrow \mathbf{P}^N$. The base locus of $c_4^{(q)}$ might have components over $B_3^{(q)}$, whose union we denote $B_4^{(q)}$; in this case, we will let $V_5^{(q)}$ be the blow-up of $V_4^{(q)}$ along $B_4^{(q)}$. Iterating this process we get a tower of varieties and maps:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
B_{i+1}^{(q)} & \subset & V_{i+1}^{(q)} & \overset{c_{i+1}^{(q)}}{\dashrightarrow} & \mathbf{P}^N \\
\downarrow & & \downarrow & & \| \\
B_i^{(q)} & \subset & V_i^{(q)} & \overset{c_i^{(q)}}{\dashrightarrow} & \mathbf{P}^N \\
\downarrow & & \downarrow & & \| \\
\vdots & & \vdots & & \vdots \\
B_3^{(q)} & \subset & V_3^{(q)} & \overset{c_3}{\dashrightarrow} & \mathbf{P}^N
\end{array}$$

where, inductively for $i \geq 4$: $V_i^{(q)} \longrightarrow V_{i-1}^{(q)}$ is the blow-up of $V_{i-1}^{(q)}$ along $B_{i-1}^{(q)}$; $c_i^{(q)} : V_i^{(q)} \dashrightarrow \mathbf{P}^N$ is defined by the proper transforms in $V_i^{(q)}$ of the point-conditions (i.e., the 'point-conditions in $V_i^{(q)}$'); and (for $i \geq 3$) $B_i^{(q)}$ is the support of the intersection $\mathcal{B}_i^{(q)}$ of the point-conditions in $V_i^{(q)}$ (i.e., the base locus of $c_i^{(q)}$). Also, for $i \geq 3$ let $E_i^{(q)}$ be the exceptional divisor in $V_i^{(q)}$, and let $\widetilde{E}_i^{(q)}$ be the proper transform of $E_i^{(q)}$ in $V_{i+1}^{(q)}$.

LEMMA 2.11. *If q is a flex of order $r \geq 2$, then for $3 \leq i \leq r+1$:*

$(1)_i$: *$V_i^{(q)}$ is non-singular*

$(2)_i$: *the composition map $B_i^{(q)} \longrightarrow B_3^{(q)}$ is an isomorphism*

$(3)_i$: *the thickness of $\mathcal{B}_i^{(q)}$ is $r+2-i$ at each point of $B_i^{(q)}$*

$(4)_i$: *$B_{i+1}^{(q)} \cap \widetilde{E}_i^{(q)} = \emptyset$*

PROOF: We have $(1)_3$, $(2)_3$ trivially, and $(3)_2$ by Lemma 2.8. Also, since B_3 is cut out by linear spaces in each fiber of E_3, we have $(4)_3$. Now we will show that:

CLAIM. *For $4 \leq i \leq r+1$, $(1)_{i-1}$, $(2)_{i-1}$, $(3)_{i-2}$ and $(4)_{i-1}$ imply $(1)_i$, $(2)_i$, $(3)_{i-1}$, and $(4)_i$.*

Also, we will show that $(3)_r$, $(4)_{r+1}$ imply $(3)_{r+1}$: this will prove the statement.

PROOF OF THE CLAIM: In this proof we will drop the $^{(q)}$ notation, to ease the exposition. V_i is then the blow-up of V_{i-1} along B_{i-1}, and these are both non-singular by $(1)_{i-1}$, $(2)_{i-1}$: so V_i must also be non-singular, giving $(1)_i$.

Next, compute the thickness of $\mathcal{B}_{i-1}$: let $\gamma(t)$ be any smooth curve germ transversal to B_{i-1} and centered at any $\alpha_{i-1} \in B_{i-1}$. If γ is tangent to E_{i-1}, then by $(4)_{i-1}$ its proper transform will miss the general point-condition in V_i: i.e., the length of the intersection of $\gamma(t)$ with $\mathcal{B}_{i-1}$ at $t=0$ is 1. If γ is transversal to E_{i-1} (and B_{i-1}), then γ maps down to a *smooth* curve germ γ_* centered at a point of B_{i-2} and transversal to B_{i-2}. By $(3)_{i-2}$, the intersection of γ_* with the point-conditions in V_{i-2} has length at most $r-i+4$: it follows that the intersection of γ with the point-conditions in V_{i-1} has length at most $r-i+3 \geq 2$ (since $i \leq r+1$). Therefore the thickness of $\mathcal{B}_{i-1}$ at α_{i-1} is $r-i+3$, which gives $(3)_{i-1}$.

For $(2)_i$, look at the intersection of B_i with the fiber of E_i over an arbitrary $\alpha_{i-1} \in B_{i-1}$. First we argue this can't be empty: indeed, $th_{\alpha_{i-1}}(B_{i-1}) = r-i+3 \geq 2$, so through every α_{i-1} in B_{i-1} there are directions tangent to all point-conditions in V_{i-1}. To get $(2)_i$, we need to show that the fiber of B_i over α_{i-1} consists (scheme-theoretically) of a simple point. But this is the intersection of B_i with the fiber of E_i ($\cong \mathbf{P}^3$) over α_{i-1}, thus a nonempty intersection of linear subspaces in $\mathbf{P}^3$ missing a hyperplane (by $(4)_{i-1}$): precisely a point, as needed for $(2)_i$.

Finally, we need $(4)_i$. Once more observe that B_i intersects each fiber of E_i in an intersection of linear spaces: thus there are no directions in the fibers of E_i and tangent to all point-conditions in V_i. This says that B_{i+1} must avoid the proper transforms in V_{i+1} of all fibers of E_i, and therefore $\widetilde{E}_i$, giving $(4)_i$.

This proves the Claim. The only case not covered yet is $(3)_{r+1}$: to obtain this and conclude the proof of 2.11, apply the same argument as above to $(3)_r$, $(4)_{r+1}$. ∎

Lemma 2.11 describes the sequence of blow-ups over V_3 that takes care of a specific flex q on C of order $r \geq 2$. The case $i = r+1$ of the statement says that the variety $V_{r+1}^{(q)}$ is non-singular, and the base locus of the map $c_{r+1}^{(q)} : V_{r+1}^{(q)} \dashrightarrow \mathbf{P}^N$ is supported on a variety $B_{r+1}^{(q)}$ isomorphic to $B_3^{(q)}$; moreover, for all $\alpha_{r+1} \in B_{r+1}^{(q)}$, we got $th_{\alpha_{r+1}}(B_{r+1}) = 1$. Let then $V_{r+2}^{(q)} \to V_{r+1}^{(q)}$ be the blow-up of $V_{r+1}^{(q)}$ along $B_{r+1}^{(q)}$, and denote by $c_{r+2}^{(q)}$ the rational map $V_{r+2}^{(q)} \dashrightarrow \mathbf{P}^N$ defined by the point-conditions in $V_{r+2}^{(q)}$. Then $V_{r+2}^{(q)}$ is clearly non-singular, and

COROLLARY 2.12. *$c_{r+2}^{(q)}$ is a regular map.*

PROOF: Indeed, the point-conditions in $V_{r+2}^{(q)}$ cannot intersect anywhere along $E_{r+2}^{(q)}$: if they did, any intersection point would correspond to a direction normal to $B_{r+1}^{(q)}$ and tangent to all point-conditions in $V_{r+1}^{(q)}$, and the thickness of $B_{r+1}^{(q)}$ would be ≥ 2, in contradiction with Lemma 2.11. ∎

By this last result, the sequence of $r-1$ blow-ups over V_3 just described resolves the indeterminacies of $c_3 : V_3 \dashrightarrow \mathbf{P}^N$ over the component $B_3^{(q)}$ of B_3. To resolve *all* indeterminacies of c_3, we just have to apply the construction simultaneously to all components of B_3: build the sequence

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
B_{i+1} & \subset & V_{i+1} & \overset{c_{i+1}}{\dashrightarrow} & \mathbf{P}^N \\
\downarrow & & \downarrow & & \| \\
B_i & \subset & V_i & \overset{c_i}{\dashrightarrow} & \mathbf{P}^N \\
\downarrow & & \downarrow & & \| \\
\vdots & & \vdots & & \vdots \\
B_3 & \subset & V_3 & \overset{c_3}{\dashrightarrow} & \mathbf{P}^N
\end{array}$$

where, for $i \geq 4$, $V_i \longrightarrow V_{i-1}$ is the blow-up of V_{i-1} along B_{i-1}, $c_i : V_i \dashrightarrow \mathbf{P}^N$ is defined by the proper transforms in V_i of the point-conditions, and B_i is the support of the base locus of c_i. By Lemma 2.11 and Corollary 2.12 all V_i's are non-singular, and, for each flex q of C of order r, B_i has either exactly one component mapping isomorphically to $B_3^{(q)}$ if $i \leq r+1$, or *no* component over $B_3^{(q)}$ if $i \geq r+2$.

In particular, this construction will stop! If r is the maximum among the order of the flexes of C, let $\widetilde{V} = V_{r+2}$, $\widetilde{c} = c_{r+2}$, and let π be the composition of the $r+2$ blow-up maps; then we have shown

THEOREM II. *$\widetilde{c} : \widetilde{V} \longrightarrow \mathbf{P}^N$ is a regular map, and the diagram*

$$\begin{array}{ccccc} PGL(3) & \subset & \widetilde{V} & \xrightarrow{\widetilde{c}} & \mathbf{P}^N \\ \| & & \downarrow{\scriptstyle \pi} & & \| \\ PGL(3) & \subset & \mathbf{P}^8 & \overset{c}{\dashrightarrow} & \mathbf{P}^N \end{array}$$

commutes.

which was our objective.

§3. THE DEGREE OF THE ORBIT CLOSURE

In this section we employ the blow-up construction of §2 to compute the degree of the orbit closure $\overline{O_C}$ of a smooth plane curve $C \in \mathbf{P}^N = \mathbf{P}^{\frac{d(d+3)}{2}}$ with at most finitely many automorphisms (if $d = 3$, we should specify 'induced from PGL(3)'. This will be understood in the following). The degree will depend on just *six* natural numbers: the order of the group of automorphisms of C, the degree d of C, and four numbers encoding information about the number and order of the flexes of C. In fact, the blow-up construction of §2 yields most naturally the 'predegree' of O_C, i.e. the product $\deg \overline{O_C} \cdot o_C$ of the degree of $\overline{O_C}$ by the order o_C of the group of automorphisms of C: this number depends only on d and on the flexes of C. Also, observe that for the general C of degree ≥ 4, the predegree of O_C equals the degree of the orbit closure.

Let $\widetilde{V}$ be the variety obtained in Theorem II: i.e., a smooth projective variety filling a commutative diagram

$$\begin{array}{ccccc} \mathrm{PGL}(3) & \subset & \widetilde{V} & \xrightarrow{\widetilde{c}} & \mathbf{P}^N \\ \| & & \downarrow{\scriptstyle \pi} & & \| \\ \mathrm{PGL}(3) & \subset & \mathbf{P}^8 & \overset{c}{\dashrightarrow} & \mathbf{P}^N \end{array}$$

where, for $\alpha \in \mathbf{P}^8$ a 3×3 matrix, $c(\alpha)$ is the translate of the curve C by α as defined in §2; and π is the sequence of blow-ups of §2. For any $p \in \mathbf{P}^2$, we have a 'point-condition in $\widetilde{V}$', i.e. the proper transform of the hypersurface of $\mathbf{P}^8$ determined by all $\varphi \in \mathbf{P}^8$ such that $\varphi(p) \subset C$.

DEFINITION. *The 'predegree' of O_C is the 8-fold self-intersection $\widetilde{P}^8$ of the class $\widetilde{P}$ of a point-condition in $\widetilde{V}$.*

LEMMA 3.1. *The predegree of O_C equals the product of the degree of the orbit closure of C by the order of the group of automorphisms of C induced from $PGL(3)$.*

PROOF: The map $\widetilde{c}$ is defined by the linear system generated by the point-conditions on $\widetilde{V}$, so $\widetilde{P}$ is the pull-back of the hyperplane class from $\mathbf{P}^N$. Therefore $\widetilde{P}^8$ computes the pull-back of the intersection of $\widetilde{c}(\widetilde{V}) = \overline{O_C}$ with 8 hyperplanes of $\mathbf{P}^N$: i.e., the product of $\deg(\overline{O_C})$ by the degree of the map $\widetilde{c}$. This latter equals o_C since, given a general $c(\alpha) \in O_C$ ($\alpha \in \mathbf{P}^8$), the fiber of $c(\alpha)$ consists of all products $\varphi\alpha$, where φ fixes C. ∎

Our aim here is to compute the predegree of O_C, by using the construction of $\widetilde{V}$ described in §2. We first collect most of the information we need from §2.

—The smooth projective variety $\widetilde{V}$ is obtained by a sequence of blow-ups at smooth centers over $\mathbf{P}^8$.

—The center of the first blow-up is the three-dimensional $B = \check{\mathbf{P}}^2 \times C$, embedded in $\mathbf{P}^8$ by Segre.

—Let V_1 be the blow-up of $\mathbf{P}^8$ along B, E_1 the exceptional divisor. The center of the second blow-up is a four-dimensional $\mathbf{P}^1$-bundle B_1 over B, a subbundle of $\mathbf{P}(N_B\mathbf{P}^8) = E_1$. Over each $\alpha = (k,q) \in \check{\mathbf{P}}^2 \times C$, the fiber of B_1 is $\mathbf{P}^1 = \mathbf{P}(T_\alpha Q_\alpha / T_\alpha B)$, where Q_α is the $\mathbf{P}^5$ of matrices $\varphi \in \mathbf{P}^8$ whose image is contained in the line tangent to C at q.

—Let V_2 be the blow-up of V_1 along B_1, E_2 the exceptional divisor. The center of the third blow-up is the three-dimensional union B_2 of disjoint components $B_2^{(q)}$, one for each flex q of C. Each $B_2^{(q)}$ maps isomorphically to the restriction of B_1 to $\check{\mathbf{P}}^2 \times q$, and is disjoint from the proper transform $\widetilde{E}_1$ of E_1.

—Let V_3 be the blow-up of V_2 along B_2, E_3 the exceptional divisor. The center of the fourth blow-up is the four-dimensional union B_3 of disjoint components $B_3^{(q)}$, one for each flex q of C of order ≥ 2. Each $B_3^{(q)}$ is a $\mathbf{P}^1$-bundle over the corresponding $B_2^{(q)}$, a subbundle of E_3; in the fiber of E_3 over q, a $\mathbf{P}^4$, the fiber of B_3 is a $\mathbf{P}^1$ spanned by points corresponding to a direction transversal to E_2 and a direction lying in E_2, transversal to the fiber of E_2.

—For $i \geq 4$, let V_i be the blow-up of V_{i-1} along B_{i-1}, E_i the exceptional divisor. The center of the $(i+1)$-st blow-up is the four-dimensional union B_i of disjoint components $B_i^{(q)}$, one for each flex q of C of order $\geq i-1$. Each $B_i^{(q)}$ maps isomorphically to the corresponding $B_{i-1}^{(q)}$, and is disjoint from the proper transform $\widetilde{E}_{i-1}$ of E_{i-1}.

—$\widetilde{V} = V_{r+2}$, where r is the maximum order of a flex of C.

Our tool will be a formula relating intersection degrees under blow-ups, from **[Aluffi1]**. In the form we will use, this can be stated as follows:

PROPOSITION 3.2. *Let $B \overset{i}{\hookrightarrow} V$ be non-singular projective varieties, and let $X \subset V$ be a codimension-1 subvariety, smooth along B. Let $\widetilde{V}$ be the blow-up of V*

along B, and let $\widetilde{X}$ be the proper transform of X. Then

$$\int_{\widetilde{V}} [\widetilde{X}]^{\dim V} = \int_V [X]^{\dim V} - \int_B \frac{([B] + i^*[X])^{\dim V}}{c(N_B V)}$$

where $\int_{\widetilde{V}}$, etc. denote the degree of a class in $\widetilde{V}$, etc., cf. **[Fulton]**, Def. 1.4. Note: we will omit the $\int$ sign and the class $[\cdot]$ brackets when this doesn't create ambiguities.

PROOF: This follows from **[Aluffi1]**, §2, Theorem II and Lemma (2), (3). ∎

We will compute the predegree of O_C (i.e. $\widetilde{P}^8$) by applying Proposition 3.2 to each blow-up in the sequence giving $\widetilde{V}$: the missing ingredients to be obtained at this point are the Chern classes of the normal bundles of the centers of the blow-ups, and calculations in their intersection rings.

In the following, $P, P_i, \widetilde{P}$ will denote resp. (the class of) point-conditions in $V, V_i, \widetilde{V}$. The embedding of B_j in V_j is denoted i_j, and p_{jk} will be used for the map $B_j \longrightarrow B_k$ (p_j will be p_{jj-1} for short). As a general convention, we will omit pull-back notations unless we fear ambiguity.

§3.1. The first blow-up. The center of the first blow-up is the variety $B = \check{\mathbf{P}}^2 \times C$; the embedding $i : B \hookrightarrow \mathbf{P}^8$ is given by composition with the Segre embedding:

$$B = \check{\mathbf{P}}^2 \times C \subset \check{\mathbf{P}}^2 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^8 \quad .$$

Call h, k resp. the hyperplane class in $\mathbf{P}^2, \check{\mathbf{P}}^2$. Our convention on pull-backs allows us to write k, h for the pull-backs of k, h from the factors to $\check{\mathbf{P}}^2 \times \mathbf{P}^2$, and to $B \subset \check{\mathbf{P}}^2 \times \mathbf{P}^2$. Also, since the Segre embedding is linear on each factor, the hyperplane class of $\mathbf{P}^8$ pulls-back to $k + h$ on B.

LEMMA 3.3. *If C has degree d:*

(i) In B: $k^3 = 0, k^2h = d, kh^2 = 0, h^3 = 0$

(ii) $c(N_B\mathbf{P}^8) = \dfrac{(1+k+h)^9(1+dh)}{(1+k)^3(1+h)^3}$

(iii) $P^8 = d^8$; and P pulls-back to $dk + dh$.

PROOF: (i) is immediate.

(ii) $c(N_B\mathbf{P}^8) = c(N_B\check{\mathbf{P}}^2 \times \mathbf{P}^2)c(N_{\check{\mathbf{P}}^2\times\mathbf{P}^2}\mathbf{P}^8)$ by the Whitney formula and the exact sequence of normal bundles. Now, since $B = \check{\mathbf{P}}^2 \times C$, $c(N_B\check{\mathbf{P}}^2 \times \mathbf{P}^2) = c(N_C\mathbf{P}^2) = 1 + dh$. The formula for $c(N_{\check{\mathbf{P}}^2\times\mathbf{P}^2}\mathbf{P}^8)$ is standard.

(iii) Recall from §2 that if $p \in \mathbf{P}^2$, P is the point-condition corresponding to p, and $F(x_0 : x_1 : x_2)$ is the (degree-d) polynomial defining C, then $\alpha \in P \iff F(\alpha(p)) = 0$: so P is defined by a degree-d equation in $\mathbf{P}^8$. ∎

We have already observed that the point-conditions are non-singular (Lemma 2.1 (ii)), so we are ready for the key computation needed to apply Proposition 3.2 to the first blow-up:

LEMMA 3.4.

$$\int_B \frac{(B + i^*P)^8}{c(N_B\mathbf{P}^8)} = d(10d - 9)(14d^2 - 33d + 21)$$

PROOF: By Lemma 3.3, this is

$$\int_{\mathbf{P}^2\times C} \frac{(1+dk+dh)^8(1+k)^3(1+h)^3}{(1+k+h)^9(1+dh)} \quad :$$

the statement follows by computing the coefficient of k^2h (the only term with non-zero degree, by Lemma 3.3(i)). ∎

§3.2. **The second blow-up.** The center of the second blow-up is a $\mathbf{P}^1$-bundle B_1 over B

$$\begin{array}{ccc} B_1 & \xrightarrow{i_1} & V_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ B & \xrightarrow{i} & \mathbf{P}^N \end{array}$$

so classes on B_1 are combinations of (the pull-backs of) k, h and $c_1(\mathcal{O}_{B_1}(-1))$; we call this latter e, and observe it is the pull-back from V_1 of the class of the exceptional divisor E_1.

LEMMA 3.5.

(i) $$p_{1*}e^i = \begin{cases} 0 & i=0 \\ -1 & i=1 \\ -3k+2dh-6h & i=2 \\ -6k^2+9dkh-27kh & i=3 \\ 24dk^2h-72k^2h & i=4 \end{cases}$$

(ii) $c(N_{B_1}V_1) = (1+e)(1+k+dh-e)^3$

(iii) $i_1^*P_1 = dk+dh-e$

PROOF: (iii) is immediate, as P is non-singular and pulls-back on B to $dk+dh$ (Lemma 3.3 (iii)).

For (i) and (ii) we need to produce $B_1 \subset E_1$ more explicitly as the projectivization of a rank-2 subbundle of $N_B\mathbf{P}^8$.

First define for any $p \in \mathbf{P}^2$ a rank-8 subbundle H_p of the trivial bundle $B \times \mathbf{C}^9$ over B: if F is a polynomial defining C, and $(k,q) \in B$, $A \in \mathbf{C}^9 = \mathrm{Hom}(\mathbf{C}^3, \mathbf{C}^3)$, say

$$((k,q),A) \in H_p \iff \sum_{i=0}^{2} \left(\frac{\partial F}{\partial x_i}\right)_q A(p)_i = 0$$

where $A(p)_i$ is the i-th coordinate of $A(p)$. So the fiber of H_p over q is the hyperplane of matrices $A \in \mathbf{C}^9$ such that $A(p) \in$ line tangent to C at q. Notice that the above equation has degree $d-1$ in the coordinates of q: thus (denoting by $\mathbf{C}^9$ the trivial bundle $B \times \mathbf{C}^9$, for short)

$$c_1\left(\frac{\mathbf{C}^9}{H_p}\right) = (d-1)h \quad .$$

Now restrict the Euler sequence for $\mathbf{P}^8$ to B via $B \xrightarrow{i} \mathbf{P}^8$: $H_p \subset \mathbf{C}^9$ determines a subbundle $\mathcal{H}_p$ of $i^*T\mathbf{P}^8$ and we have the following diagram of bundles over B

(suppressing pull-back as usual)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, equal] & H_p \otimes \mathcal{O}_{\mathbf{P}^8}(1) \arrow[r] \arrow[d] & \mathcal{H}_p \arrow[r] \arrow[d] & 0 . \\
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d] & \mathbf{C}^9 \otimes \mathcal{O}_{\mathbf{P}^8}(1) \arrow[r] \arrow[d] & T\mathbf{P}^8 \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \dfrac{\mathbf{C}^9}{H_p} \otimes \mathcal{O}_{\mathbf{P}^8}(1) \arrow[r, equal] \arrow[d] & \dfrac{T\mathbf{P}^8}{\mathcal{H}_p} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

from which it follows

$$c\left(\frac{T\mathbf{P}^8}{\mathcal{H}_p}\right) = c\left(\frac{\mathbf{C}^9}{H_p} \otimes \mathcal{O}_{\mathbf{P}^8}(1)\right) = 1 + k + dh \quad .$$

Also, observe that each $\mathcal{H}_p$ contains TB.

Now let p_1, p_2, p_3 be non-collinear points. A matrix has image contained in a line if and only if it sends three non-collinear points to that line, thus the intersection $H_{p_1} \cap H_{p_2} \cap H_{p_3}$ is the rank-6 bundle over $B = \check{\mathbf{P}}^2 \times C$ whose fiber over $(k, q) \in B$ consists of all matrices whose image is contained in the line tangent to C at q. This is the space we used to define B_1: if we set $Q = \mathcal{H}_{p_1} \cap \mathcal{H}_{p_2} \cap \mathcal{H}_{p_3}$, then

$$B_1 = \mathbf{P}\left(\frac{Q}{TB}\right) \subset \mathbf{P}(N_B\mathbf{P}^8) = E_1, \quad \text{and} \quad c\left(\frac{T\mathbf{P}^8}{Q}\right) = (1 + k + dh)^3 \quad .$$

Finally, the Euler sequences for E_1 and B_1 give the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, equal] & \dfrac{Q}{TB} \otimes \mathcal{O}_{B_1}(1) \arrow[r] \arrow[d] & TB_1|B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d] & N_B\mathbf{P}^8 \otimes \mathcal{O}_{B_1}(1) \arrow[r] \arrow[d] & TE_1|B \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \dfrac{T\mathbf{P}^8}{Q} \otimes \mathcal{O}_{B_1}(1) \arrow[r, equal] \arrow[d] & N_{B_1}E_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

(here $TB_1|B$, $TE_1|B$ denote the relative tangent bundles of B_1, E_1 over B) from which

$$c(N_{B_1}E_1) \doteq c\left(\frac{T\mathbf{P}^8}{Q} \otimes \mathcal{O}_{B_1}(1)\right) = (1+k+dh-e)^3 \quad .$$

From this discussion, it's easy to obtain (i) and (ii):

$$\begin{aligned}
\text{(i) } p_{1*}\sum_i(-1)^i e^i &= c\left(\frac{Q}{TB}\right)^{-1} && \text{by [Fulton], Proposition 3.1 (a)}\\
&= c\left(\frac{T\mathbf{P}^8}{Q}\right)c(N_B\mathbf{P}^8)^{-1} && \text{by Whitney's formula}\\
&= \frac{(1+k+dh)^3(1+k)^3(1+h)^3}{(1+k+h)^9(1+dh)} && \text{by the above and Lemma 3.3 (ii)}\\
&= 1-3k+2dh-6h+6k^2-9dkh+27kh+24dk^2h-72k^2h \quad .
\end{aligned}$$

(ii) $c(N_{B_1}V_1) = c(N_{E_1}V_1)c(N_{B_1}E_1) = (1+e)(1+k+dh-e)^3$. ∎

Lemma 3.5 allows us to compute the term needed to apply Proposition 3.2 to the second blow-up:

LEMMA 3.6.

$$\int_{B_1}\frac{(B_1+i_1^*P_1)^8}{c(N_{B_1}V_1)} = d(2d-3)(322d^2-1257d+1233)$$

PROOF: This is

$$\int_{B_1}\frac{(1+dk+dh-e)^8}{(1+e)(1+k+dh-e)^3}$$

by Lemma 3.5 (ii) and (iii). Since the degree doesn't change after push-forwards, this is also

$$\int_B p_{1*}\frac{(1+dk+dh-e)^8}{(1+e)(1+k+dh-e)^3} \quad .$$

Computing the degree-4 term in the expansion of the fraction and applying Lemma 3.5 (i) and the projection formula, this is computed as a sum of degree-3 terms in k, h over B. Lemma 3.3 (i) is used then to obtain the stated expression. ∎

§3.3. The third blow-up. At this point we have to start taking flexes into account. For any $q \in C$, let $f\ell(q)$ be the order of q as a flex of C, in the sense of §2.2: so $f\ell(q) = 0$ if q is *not* a flex of C, $f\ell(q) = 1$ if q is a *simple* flex of C, and so on.

The center $B_2 \overset{i_2}{\hookrightarrow} V_2$ of the third blow-up is the disjoint union

$$B_2 = \bigcup_{f\ell(q)>0} B_2^{(q)} \quad ,$$

where each $B_2^{(q)}$ maps isomorphically to the restriction $B_1^{(q)}$ of the $\mathbf{P}^1$-bundle B_1 to $\check{\mathbf{P}}^2 \times \{q\} \subset B$. Moreover, $B_2 \cap \widetilde{E}_1 = \emptyset$ (Lemma 2.4). As h restricts to 0 on each $\check{\mathbf{P}}^2 \times \{q\}$, the intersection ring of $B_2^{(q)}$ is generated by k, e (defined as in §3.2). Also, we denote by e' the pull-back of E_2 to $B_2^{(q)}$, and by p_{20} the map $B_2^{(q)} \to \check{\mathbf{P}}^2 \times \{q\} \cong \mathbf{P}^2$.

LEMMA 3.7.

(i) $e' = e$

(ii) $p_{20*}e^i = \begin{cases} 0 & i=0 \\ -1 & i=1 \\ -3k & i=2 \\ -6k^2 & i=3 \end{cases}$

(iii) $c(N_{B_2^{(q)}}V_2) = (1+e)(1+k-2e)^3$

(iv) $i_2^* P_2 = dk - 2e$

PROOF: (ii) follows from Lemma 3.5 (i), since the restriction of h to $B_2^{(q)}$ is 0.

The key observation for the other points is that $B_2^{(q)} \cap \widetilde{E}_1 = \emptyset$. Realize $B_2^{(q)} \subset \mathbf{P}(N_{B_1}V_1)$ as $\mathbf{P}(\mathcal{L})$, where $\mathcal{L}$ is a sub-line bundle of $N_{B_1}V_1$. $\widetilde{E}_1 \cap E_2$ is the exceptional divisor of the blow-up of E_1 along B_1, i.e. the projectivization of $N_{B_1}E_1$ in $N_{B_1}V_1$. That $\mathbf{P}(\mathcal{L})$ and $\mathbf{P}(N_{B_1}E_1)$ are disjoint says that $\mathcal{L} \cap N_{B_1}E_1$ is the zero-section of $N_{B_1}V_1$, and therefore

$$\mathcal{L} \cong \frac{N_{B_1}V_1}{N_{B_1}E_1} = N_{E_1}V_1 \quad \text{as bundles on } B_1^{(q)}.$$

(i) With the same notations, $\mathcal{L}$ is tautologically the universal line bundle over $\mathbf{P}(\mathcal{L})$; it must then equal the restriction to $B_2^{(q)}$ of the universal line bundle $\mathcal{O}_{E_2}(-1) \cong N_{E_2}V_2$. In other words

$$\mathcal{L} \cong N_{E_2}V_2 \quad \text{as bundles on } B_2^{(q)}.$$

Since the projection from $B_2^{(q)}$ to $B_1^{(q)}$ is an isomorphism, it follows that

$$e = c_1(N_{E_1}V_1) = c_1(\mathcal{L}) = c_1(N_{E_2}V_2) = e' \quad .$$

(iii) Call $E_2^{(q)}$ the restriction of $E_2 = \mathbf{P}(N_{B_1}V_1)$ to $B_1^{(q)}$. We have Euler sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{L}\otimes\mathcal{O}(1) & \longrightarrow & TB_2^{(q)}|B_1^{(q)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & N_{B_1}V_1\otimes\mathcal{O}(1) & \longrightarrow & TE_2^{(q)}|B_1^{(q)} & \longrightarrow & 0 \end{array}$$

and we just argued $\mathcal{L} \cong \mathcal{O}(-1)$: so

$$\begin{aligned} c(N_{B_2^{(q)}}E_2^{(q)}) &= c\left(\frac{N_{B_1}V_1}{\mathcal{L}} \otimes \check{\mathcal{L}}\right) \qquad (\text{restricted to } B_2^{(q)}) \\ &= \frac{(1+e-e')(1+k-e-e')^3}{(1+e'-e')} \\ &= (1+k-2e)^3 \quad \text{by (i);} \end{aligned}$$

next, since $N_{B_1^{(q)}}B_1$ is clearly trivial, we have $c(N_{E_2^{(q)}}E_2) = 1$; so putting $N_{B_2^{(q)}}V_2$ together:

$$c(N_{B_2^{(q)}}V_2) = c(N_{E_2}V_2)c(N_{E_2^{(q)}}E_2)c(N_{B_2^{(q)}}E_2^{(q)}) = (1+e)(1+k-2e)^3 \quad ,$$

as claimed.

(iv) Since P_1 is non-singular along B_1, P_2 restricts to $dk - e - e' = dk - 2e$ by (i). ∎

We are ready for the term needed to apply Proposition 3.2 to the third blow-up:

LEMMA 3.8.

$$\int_{B_2} \frac{(B_2 + i_2^* P_2)^8}{c(N_{B_2} V_2)} = \sum_{f\ell(q)>0} (196d^2 - 960d + 1125)$$

PROOF: By Lemma 3.7 (iii) and (iv), this is

$$\sum_{f\ell(q)>0} \int_{B_2^{(q)}} \frac{(1+dk-2e)^8}{(1+e)(1+k-2e)^3} = \sum_{f\ell(q)>0} \int_{\mathbf{P}^2} p_{20*} \frac{(1+dk-2e)^8}{(1+e)(1+k-2e)^3}$$

(pushing forward doesn't change degrees) and one concludes with the projection formula and Lemma 3.7 (ii). ∎

§3.4. **Further blow-ups.** Further blow-ups are necessary if there are points q on C with $f\ell(q) > 1$. We first attack the initial step.

The center $B_3 \overset{i_3}{\hookrightarrow} V_3$ of the fourth blow-up is the union

$$B_3 = \bigcup_{f\ell(q)>1} B_3^{(q)} \quad ,$$

where each $B_3^{(q)}$ is a $\mathbf{P}^1$-bundle over $B_2^{(q)}$. The intersection ring of $B_3^{(q)}$ is generated by (the pull-back of) the classes k, e of $B_2^{(q)}$, and by the class of the universal line bundle, i.e. the pull back f of E_3 from V_3. Denote by p_3 the projection $B_3^{(q)} \longrightarrow B_2^{(q)}$.

LEMMA 3.9.

(i) $p_{3*} f^i = \begin{cases} 0 & i=0 \\ -1 & i=1 \\ -e & i=2 \\ -e^2 & i=3 \\ -e^3 & i=4 \end{cases}$

(ii) $c(N_{B_3^{(q)}} V_3) = (1+f)(1+k-2e-f)^3$

(iii) $i_3^* P_3 = dk - 2e - f$

PROOF: (iii) is clear, as P_2 is non-singular along $B_3^{(q)}$.

For the other items, we have to produce $B_3^{(q)} \subset E_3^{(q)} = \mathbf{P}(N_{B_2^{(q)}} V_2)$ explicitly as the projectivization of a rank-2 subbundle of $N_{B_2^{(q)}} V_2$. Recall that each fiber of $B_3^{(q)}$ is spanned by two points corresponding respectively to (1) a direction transversal to E_2, and (2) a direction in E_2, transversal to the fiber of E_2. Since these two points are always distinct, $B_3^{(q)} = \mathbf{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)$, where $\mathbf{P}\mathcal{L}_1$, $\mathbf{P}\mathcal{L}_2$ give the two distinguished points on each fiber. Now, $\mathcal{L}_1 \cap N_{B_2^{(q)}} E_2$ is the zero-section in $N_{B_2^{(q)}} V_2$ (the first direction is transversal to E_2); so, with $\mathcal{L}$ as in the proof of 3.7,

$$\mathcal{L}_1 \cong N_{E_2} V_2 \cong \mathcal{L} \quad .$$

Similarly, since the second direction is transversal to the *fiber* of E_2, whose normal bundle in E_2 is trivial, $\mathcal{L}_2 \cong \mathcal{O}$; and therefore we have

$$B_3^{(q)} = \mathbf{P}(\mathcal{L} \oplus \mathcal{O}) \quad .$$

(i) As in the proof of 3.5 (i),

$$p_{3*} \sum_i (-1)^i f^i = c(\mathcal{L} \oplus \mathcal{O})^{-1} = \sum_i (-1)^i e^i$$

and (i) follows by matching dimensions.

(ii) Another pair of Euler sequences: on $B_3^{(q)}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & (\mathcal{L} \oplus \mathcal{O}) \otimes \mathcal{O}(1) & \longrightarrow & TB_3^{(q)}|B_2^{(q)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & N_{B_2^{(q)}} V_2 \otimes \mathcal{O}(1) & \longrightarrow & TE_3^{(q)}|B_2^{(q)} & \longrightarrow & 0
\end{array}$$

Since $c_1(\mathcal{O}(1)) = -f$ and E_3 is the disjoint union of the $E_3^{(q)}$:

$$\begin{aligned}
c(N_{B_3^{(q)}} E_3) &= c(N_{B_3^{(q)}} E_3^{(q)}) \\
&= c\left(\frac{N_{B_2^{(q)}} V_2}{\mathcal{L} \oplus \mathcal{O}} \otimes \mathcal{O}(1)\right) \\
&= (1 + k - 2e - f)^3
\end{aligned}$$

(the Chern roots of $N_{B_2^{(q)}} V_2$ are $e, k-2e, k-2e, k-2e, 0$ by Lemma 3.7 (iii)). Finally:

$$c(N_{B_3^{(q)}} V_3) = c(N_{E_3} V_3) c(N_{B_3^{(q)}} E_3) = (1+f)(1+k-2e-f)^3$$

as stated. ∎

Lemma 3.9 describes the situation at the fourth blow-up. The next blow-ups are built on this in the sequence described in §2.4: the center $B_j \overset{i_j}{\hookrightarrow} V_j$ of the $(j+1)$-st blow-up ($j \geq 3$) is the union

$$B_j = \bigcup_{j \ell(q) > j-2} B_j^{(q)} \quad ,$$

where each $B_j^{(q)}$ maps isomorphically down to $B_3^{(q)}$, and is disjoint from $\widetilde{E}_{i-1}$ (Lemma 2.11). The intersection ring of each $B_j^{(q)} \cong B_3^{(q)}$ is then generated by k, e, f, and the relations stated in Lemma 3.9 (i) hold, for the projection $p_{j2} : B_j^{(q)} \to B_2^{(q)}$. Denote by f_j the pull-back of E_j to $B_j^{(q)}$; Lemma 3.9 can be extended to all stages in the sequence:

LEMMA 3.9 (CONTINUED). *For $3 \le j \le f\ell(q)+1$*

$(i)_j$ $f_j = f$

$(ii)_j$ $c(N_{B_j^{(q)}} V_j) = (1+f)(1+k-2e-(j-2)f)^3$

$(iii)_j$ $i_j^* P_j = dk - 2e - (j-2)f$

PROOF: For $j = 3$ this is given by Lemma 3.9. So it suffices to show that, for $3 \le j \le f\ell(q)$, $(i)_j, (ii)_j, (iii)_j$ imply $(i)_{j+1}, (ii)_{j+1}, (iii)_{j+1}$. Consider then $B_{j+1}^{(q)} = \mathbf{P}(\mathcal{L}_{j+1}) \subset \mathbf{P}(N_{B_j^{(q)}} V_j)$. So f_{j+1} is the class of $\mathcal{O}_{B_{j+1}^{(q)}}(-1)$, i.e. of $\mathcal{L}_{j+1}$. Since $B_{j+1}^{(q)} \cap \widetilde{E}_j = \emptyset$ (Lemma 2.11 (iv)), we get by the usual argument

$$f_{j+1} = c_1(\mathcal{L}_{j+1}) = c_1(N_{E_j} V_j) = f_j \quad :$$

and $f_j = f$ by $(i)_j$; so $f_{j+1} = f$, giving $(i)_{j+1}$.

$(iii)_{j+1}$ follows then from $(iii)_j$ and $(i)_{j+1}$, since P_j is non-singular along B_j.

Finally, we use the Euler sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{L}_{j+1} \otimes \mathcal{O}(1) & \longrightarrow & TB_{j+1}^{(q)}|B_j^{(q)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & N_{B_j^{(q)}} V_j \otimes \mathcal{O}(1) & \longrightarrow & TE_{j+1}^{(q)}|B_j^{(q)} & \longrightarrow & 0
\end{array}$$

to get (since E_{j+1} is the disjoint union of the $E_{j+1}^{(q)}$)

$$\begin{aligned}
c(N_{B_{j+1}^{(q)}} E_{j+1}) &= c(N_{B_{j+1}^{(q)}} E_{j+1}^{(q)}) \\
&= c\left(\frac{N_{B_j^{(q)}} V_j}{\mathcal{L}_{j+1}} \otimes \mathcal{O}(1)\right) \\
&= \frac{(1+f-f)(1+k-2e-(j-2)f-f)^3}{(1+f-f)} \qquad \text{by } (ii)_j \\
&= (1+k-2e-(j-1)f)^3 \quad ,
\end{aligned}$$

so

$$c(N_{B_{j+1}^{(q)}} V_{j+1}) = c(N_{E_{j+1}} V_{j+1}) c(N_{B_{j+1}^{(q)}} E_{j+1}) = (1+f)(1+k-2e-(j-1)f)^3 \quad ,$$

i.e. $(ii)_{j+1}$. ∎

We get then the key term to apply Proposition 3.2 to the j-th blow up in the sequence. In fact, we can cover Lemma 3.8 as well in one statement:

LEMMA 3.10. *For $j \ge 2$*

$$\int_{B_j} \frac{(B_j + i_j^* P_j)^8}{c(N_{B_j} V_j)} = \sum_{f\ell(q) > j-2} 30j^4 - 96(d-1)j^3 + 12(d-1)(7d-11)j^2 + 84(d-1)^2 j - 7(2d-3)(22d-39).$$

PROOF: For $j = 2$, this is Lemma 3.8. For $j \geq 3$, by Lemma 3.9 this is

$$\sum_{f\ell(q)>j-2} \int_{B_j^{(q)}} \frac{(1+dk-2e-(j-2)f)^8}{(1+f)(1+k-2e-(j-2)f)^3} \quad .$$

If p_{j2} denotes the projection $B_j^{(q)} \longrightarrow B_2^{(q)}$, (and p_{20} is the map $B_2^{(q)} \longrightarrow \check{\mathbf{P}}^2 \times \{q\} \cong \mathbf{P}^2$, as in §3.3), this can be computed as

$$\sum_{f\ell(q)>j-2} \int_{\mathbf{P}^2} p_{20*} p_{j2*} \frac{(1+dk-2e-(j-2)f)^8}{(1+f)(1+k-2e-(j-2)f)^3} \quad ,$$

which is evaluated by using the projection formula , 3.9 (i) and 3.7 (ii).[2] ∎

§3.5. The degree of $\overline{O_C}$. Computing the predegree of O_C is now a straightforward application of Proposition 3.2 and Lemmas 3.4, 3.6 and 3.10: by Proposition 3.2

$$\widetilde{P}^8 = P^8 - \sum_{j\geq 0} \int_{B_j} \frac{(B_j - i_j^* P_j)^8}{c(N_{B_j} V_j)}$$

(where $B_0 = B$, etc.), and the terms in the summation have been computed in sections 3.1–3.4. This gives

PROPOSITON 3.11. *The predegree of O_C is*

$$\begin{aligned} d^8 - d(10d-9)(14d^2-33d+21) - d(2d-3)(322d^2-1257d+1233) \\ - \sum_{j\geq 2} \sum_{\substack{q\in C \\ f\ell(q)>j-2}} 30j^4 - 96(d-1)j^3 + 12(d-1)(7d-11)j^2 \\ + 84(d-1)^2 j - 7(2d-3)(22d-39) \quad . \end{aligned}$$

This result can be given in handier forms. For example:

[2]The reason why this works for $j = 2$ as well is that

$$p_{3*} \frac{(1+dk-2e)^8}{(1+f)(1+k-2e)^3} = \frac{(1+dk-2e)^8}{(1+k-2e)^3} p_{3*} \sum_i (-1)^i f^i$$

expanding $1/(1+f)$ and applying the projection formula

$$\begin{aligned} &= \frac{(1+dk-2e)^8}{(1+k-2e)^3} \sum_i (-1)^i e^i \qquad \text{by Lemma 3.9(i)} \\ &= \frac{(1+dk-2e)^8}{(1+e)(1+k-2e)^3} \end{aligned}$$

as needed for Lemma 3.8.

THEOREM III(A). *The predegree of O_C is*

$$d(d-2)(d^6+2d^5+4d^4+8d^3-1356d^2+5280d-5319)-\sum_{q\in C} f\ell(q)(f\ell(q)-1)$$
$$(6f\ell(q)^3+(75-24d)f\ell(q)^2+(28d^2-240d+393)f\ell(q)+196d^2-960d+1125)$$

PROOF: Invert the order of summations in Proposition 3.11, then use the fact that $\sum_{q\in C} f\ell(q)=3d(d-2)$ (the number of flexes of C, counted with multiplicity). ∎

Or, in another form:

THEOREM III(B). *Denote by $f_C^{(r)}$ the sum $\sum_{q\in C} f\ell(q)^r$. Then the predegree of O_C is*

$$d^8-8d(98d^3-492d^2+843d-486)-(168d^2-720d+732)f_C^{(2)}$$
$$-(28d^2-216d+318)f_C^{(3)}-(69-24d)f_C^{(4)}-6f_C^{(5)}$$

By Theorem III(B), if C is smooth then *the predegree of O_C depends only on the degree d of C and on the four numbers $f_C^{(2)}$, $f_C^{(3)}$, $f_C^{(4)}$ and $f_C^{(5)}$.*

If C only has simple flexes, then $f\ell(q)=0$ or 1 for all $q\in C$, so Theorem III(A) gives

COROLLARY. *If all flexes of C are simple, then the predegree of O_C is*

$$d(d-2)(d^6+2d^5+4d^4+8d^3-1356d^2+5280d-5319)$$
$$=d^8-1372d^4+7992d^3-15879d^2+10638d \quad .$$

Denoting this polynomial in d by $P(d)$, we remark that it gives the degree of the orbit closure of the general smooth plane curve of degree $d\geq 4$ (indeed, such a curve C has no non-trivial automorphisms, so by Lemma 3.1 the degree of $\overline{O_C}$ equals the predegree).

REMARK. Denoting by $f_k(d)$ the (negative) contribution to the predegree arising from a flex of order k on a curve of degree d, we have, as an immediate consequence of Theorem III(A):

$$f_k(d)=-k(k-1)((28k+196)d^2-(24k^2+240k+960)d+(6k^3+75k^2+393k+1125)).$$

One checks that $f_k(d)<0$ for all $d\geq k+2\geq 4$. This says that the predegree is maximal for a curve with only simple flexes. Here are two examples:

—Call 'hyperflex' a flex of order 2. If C has n hyperflexes, and all other flexes of C are simple, then the predegree of O_C is

$$P(d)+n\cdot f_2(d)=d(d-2)(d^6+2d^5+4d^4+8d^3-1356d^2+5280d-5319)$$
$$-6n(84d^2-512d+753) \quad .$$

At the other end of the spectrum, suppose all flexes have maximal order:

—If all flexes of C have order $d-2$, then the predegree of O_C is

$$P(d)+3d\cdot f_{d-2}(d)=d^2(d-2)(d^5+2d^4-26d^3-7d^2+192d-192) \quad .$$

Since the Fermat curves are examples, it is from this result that we get the degree of the trisecant variety to the d-uple Veronese embedding of $\mathbf{P}^2$, as mentioned in the introduction.

Before giving some more examples, we remind the reader that the degree of the orbit closure will be computed from the predegree by dividing it by the order of the group of automorphisms of C induced from PGL(3), by Lemma 3.1.

EXAMPLES. First we list, for some small values of d, the numbers (and their factorizations) we get from the corollary to Theorem III:

d	$P(d)$	$P(d)$ factored
3	216	$2^3 \cdot 3^3$
4	14280	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 17$
5	188340	$2^2 \cdot 3 \cdot 5 \cdot 43 \cdot 73$
6	1119960	$2^3 \cdot 3^3 \cdot 5 \cdot 17 \cdot 61$
7	4508280	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 1789$
8	14318256	$2^4 \cdot 3 \cdot 317 \cdot 941$
9	38680740	$2^2 \cdot 3^6 \cdot 5 \cdot 7 \cdot 379$
10	92790480	$2^4 \cdot 3 \cdot 5 \cdot 59 \cdot 6553$

So for $d = 3$ we get 216 for the predegree of the orbit of any smooth plane cubic curve. This gives the well-known numbers 12, resp. 6, resp. 4 for the degree of the orbit closure of a smooth plane cubic with $j \neq 0, 1728$, resp. $j = 1728$, resp. $j = 0$. E.g., the group of projective automorphisms of a general cubic consists of 18 elements: the 9 translations over points of order 3 and the 9 reflections in a flex of the curve. For cubics with $j = 1728$, resp. $j = 0$, the projective automorphism group is twice, resp. thrice, as large.

For $d = 4$ we get 14280 for the predegree of the orbit of a smooth plane quartic with only simple flexes. An example of such a curve is the Klein curve $x^3y + y^3z + z^3x$; it has 168 automorphisms, so the degree of its orbit closure is $14280/168 = 85$.

If a smooth quartic has n hyperflexes, the predegree of its orbit equals $14280 - 294n$. E.g., the degree of the orbit closure of the Fermat quartic is 112, as there are 96 automorphisms. As an other example, consider the curve $x^4 + xy^3 + yz^3$. It has 1 hyperflex and 9 automorphisms, so the degree of its orbit closure is $(14280 - 294)/9 = 1554$.

Finally, one more example for all $d \geq 5$: the curve $x^{d-1}y + y^{d-1}z + z^{d-1}x$. It has three $(d-1)$-flexes; the remaining $3(d^2 - 3d + 3)$ flexes are simple, form one orbit for the action of the automorphism group and have trivial stabilizer. (The automorphism group is the semidirect product of the diagonal matrices with entries $(1, \zeta, \zeta^{2-d})$, where ζ is a $(d^2 - 3d + 3)$-rd root of unity, and the even permutations of the coordinates.) So the degree of the orbit closure is

$$\frac{P(d) + 3f_{d-3}(d)}{3(d^2 - 3d + 3)} = \frac{1}{3}(d^6 + 3d^5 + 6d^4 - 21d^3 - 1354d^2 + 5463d - 5508) \quad .$$

The conclusion to draw from these examples (more precisely, from Lemma 3.1) is that the existence of a curve with (many) automorphisms creates a congruence condition to be satisfied by the formula for the predegree of its orbit. An amusing example: if the number of automorphisms of a smooth quartic is divisible by 32, the quartic necessarily has 12 hyperflexes—the Fermat quartic is the only such curve. In fact, in [**Vermeulen**] there is a complete list of the automorphism groups that occur for a quartic with a given number of hyperflexes—the reader may be assured that the numbers 14280 and 294 satisfy all implied congruences.

In the other direction, our formulas create necessary conditions for the existence of automorphisms of smooth plane curves. We give one example: from the corollary to Theorem III (compare the list above) one concludes that a smooth plane curve of degree $d \equiv 3 \pmod 5$ with only simple flexes cannot have an automorphism of order 5. However, it is not hard to prove this result in a more direct way.

§4. Boundaries of orbits and families of orbits

§4.1. Boundaries of orbits. In this subsection we will study the 'boundaries' of the PGL(3)-orbits of nonsingular plane curves of degree $d \geq 3$. I.e., for C such a curve, we analyse the locus $\overline{O_C} - O_C$. This locus is the disjoint union of orbits of (very) singular curves; our purpose is to write down which types of curves occur, depending on C.

Consider the rational map $c : \mathbf{P}^8 \dashrightarrow \mathbf{P}^N$. For $\alpha \in \mathrm{PGL}(3)$, the image $c(\alpha)$ is just the translation of C by α, an element of the orbit O_C.

The base locus of the map c is supported on the set $\{\alpha \in \mathbf{P}^8 : \mathrm{im}\,\alpha \subset C\}$. Since C does not contain a line, these α necessarily have rank 1, and their image is a point on C.

Now let α have rank 2. By the above, the image $c(\alpha)$ is defined, and is an element of the boundary. It is the curve corresponding to the form $C \circ \alpha$, which is the composition

$$\mathbf{C}^3 \xrightarrow{\alpha} \mathbf{C}^3 \xrightarrow{C} \mathbf{C} \quad .$$

As far as C goes, only the restriction of C to the image of α is important; in projective terms, this is a d-tuple of points on the line $\mathrm{im}\,\alpha$ (namely, the intersection of that line with the curve C). To get the curve corresponding to $C \circ \alpha$, we have to 'pull-back' this d-tuple along α; we get a d-tuple of lines, all passing through the point $\ker\alpha$. Note that the two d-tuples are isomorphic: giving a rank-2 matrix is equivalent to giving a point in the source $\mathbf{P}^2$ (its kernel), a line in the target $\mathbf{P}^2$ (its image), and an element of PGL(2) identifying the $\mathbf{P}^1$ of lines through the kernel with the image $\mathbf{P}^1$.

Using the above description of rank-2 matrices, it follows that the 5-dimensional locus of rank-2 matrices with given image line has as its image under c a generally 5-dimensional orbit of curves that are a d-tuple of lines through one point. Here only the isomorphism class of the d-tuple is fixed; namely, it is that of the d-tuple of points of C on the image line. (If the d-tuple contains at least 3 distinct points, the orbit is 5-dimensional; if it is supported on 2 distinct points, the orbit is 4-dimensional; if all d points are equal, the orbit is 2-dimensional.)

Therefore, to find the image under c of all of the locus of rank-2 matrices, we have to study the natural rational map from the g^2_d on C (the 2-dimensional linear system of (collinear) d-tuples of points of C corresponding to the dual $\check{\mathbf{P}}^2$) to the 'moduli space of d-tuples' (of points on a line). Namely, it is quite possible that distinct lines give isomorphic d-tuples. E.g., for $d = 3$ all 3-tuples of distinct points are isomorphic, and the image under any (nonsingular) C of the 7-dimensional locus of rank-2 matrices is the (closed) 5-dimensional locus of curves consisting of 3 lines through a point. (The image is closed because a cubic curve has flexes as well as points of ordinary tangency.)

As for $d = 4$, the moduli space of 4-tuples of points on $\mathbf{P}^1$ is the j-line.

LEMMA 4.1. *Let C be a smooth quartic curve. The rational map $j\colon g^2_d(C) \dashrightarrow \mathbf{P}^1$ is dominant.*

PROOF: The j-invariant of a 4-tuple of points is undefined exactly when at least three of the four points coincide. Therefore the base locus of the map j consists of finitely many points, corresponding to the flex lines of C. We have to exclude the possibility that the image of the irreducible open set where j is defined, is a point. This is easy: there are ordinary tangent lines as well as lines which are not tangent lines. ∎

For $d \geq 5$ the moduli space of d-tuples of points on $\mathbf{P}^1$ is the quotient (in the sense of geometric invariant theory) $(\mathbf{P}^d)_{ss}/\mathrm{PGL}(2)$. Here $\mathbf{P}^d$ is the projective space of d-tuples, and $(\mathbf{P}^d)_{ss}$ stands for the locus of semistable d-tuples. Note (see [**Mumford**, p. 45]) that a d-tuple is stable (resp. semistable) if and only if all multiplicities are $< d/2$ (resp. $\leq d/2$). The geometric quotient $(\mathbf{P}^d)_{ss}/\mathrm{PGL}(2)$ is a projective variety which we denote with M_d.

PROPOSITION 4.2. *Let C be a smooth plane curve of degree $d \geq 5$. The natural rational map $f_C\colon g^2_d(C) \dashrightarrow M_d$ is generically finite.*

PROOF: We first observe that for many (possibly singular) curves, the result is easy to prove. Assume C to be a reduced plane curve of degree $d \geq 5$. Assume moreover that the dual *curve* exists (i.e., the closure of the locus of tangent lines at smooth points is a curve—this happens exactly if C is not a union of lines) and that there is at least one *special* tangent line giving a *semistable* d-tuple. Then the result follows. Indeed, a general point of the dual curve gives a d-tuple with one point of multiplicity two and otherwise simple points, whereas a special tangent line produces a d-tuple with at least one multiplicity ≥ 3 or at least two multiplicities ≥ 2. Since we required that one special tangent tuple be semistable, whereas the general tangent tuple is stable (here we use that $d \geq 5$), the image of the dual curve is a curve. But the general line in $\mathbf{P}^2$ produces a d-tuple with d points of multiplicity one: a point *not* on said curve. So the (closure of the) image contains a closed curve and a point not on that curve. Since the image is irreducible, it is 2-dimensional. So the map f_C is indeed generically finite.

To finish the proof of the proposition, we sketch an argument that on a smooth plane curve there always exists a semistable special tangent line. We count flexes and bitangents, with multiplicities. An unstable tangent line has one multiplicity $\geq \frac{d+1}{2}$, so counts for at least $\frac{d-3}{2}$ flexes. On the other hand, it counts for at most $\binom{d-2}{2}$ bitangents, as this is the number of bitangents 'disappearing' in a d-flex. So the number of disappearing bitangents is at most $d-2$ times the number of disappearing flexes, for any unstable tangent line. Now the *total* number of bitangents is $\frac{d^2-9}{6}$ times the total number of flexes (both numbers with multiplicities). Since

$$\frac{d^2-9}{6} > d-2$$

for $d \geq 6$, it follows that for these values of d not all special tangent lines can give unstable d-tuples. For $d = 5$ it suffices to remark that the estimate $d - 2$ above can be improved: an unstable special tangent line counts for at most twice as many bitangents as flexes in this case (cf. the 5-tuples of the form $(3,2)$). Since there are 45 flexes and 120 bitangents (counted with multiplicities), there exist at least 30 *ordinary* bitangents. ∎

REMARK 4.3. If f_C is a morphism, i.e., at each point the intersection multiplicity of the curve and the tangent line is $\leq d/2$ (so that $d \geq 6$), then it is a finite morphism. (An infinite fibre other than the dual curve intersects the dual curve, contradiction; and the dual curve is not a fibre since there are non-ordinary tangent lines.)

So far for the images of rank-2 matrices. If α is a rank-1 matrix with image a point not on C, one checks that the curve $C \circ \alpha$ is a d-fold line; the line is the kernel of α.

As said before, the remaining matrices—those of rank 1 with image a point on C—are in the base locus of the map c. To find the remaining points in the boundary $\overline{O_C} - O_C$, one can proceed in several ways.

One way would be to follow closely the process of resolving the map c, as it is done in §2. There the map is resolved by a sequence of blow-ups; the center of each blow-up is the support of the base locus of the induced map. If one determines at each stage the image under the induced map of the exceptional divisor (minus the new base locus), one finds step-by-step all points in the boundary of the orbit of C. Here we will follow another method. We want to use the set-up of [**Kempf**, proof of Theorem (1.4)]. In order to be able to do so, we need a slight change of perspective: we will study (the orbits of) the action of $G = GL(3)$ on $\mathbf{C}^{N+1} = \mathbf{C}^{\binom{d+2}{2}}$. This just amounts to considering the affine cones over the orbits instead of the orbits themselves.

So consider the smooth plane curve C of degree d as an element of $\mathbf{C}^{N+1}$. According to (loc.cit.), in order to find all points in the closure of the G-orbit of C, we have to determine the limits $\lim_{t \to 0} C \cdot p(t)$ that exist, where $p\colon \mathbf{C}((t)) \to G$ is a rational map. By the Cartan-Iwahori decomposition (loc.cit.; cf. the footnote on p. 53 of [**Mumford-Fogarty**]) one has $p = h_1 \cdot \lambda \cdot h_2$, where $h_1, h_2 \in G(\mathbf{C}[[t]])$ and λ is a one-parameter subgroup (1-PS) of G (see also [**Mumford**, pp. 42–43]). So it suffices to determine the limits

$$\lim_{t \to 0} C \cdot h_1 \cdot \lambda$$

that exist. Note that we are interested only in the *orbits* that occur in the boundary, so we ignore the effect of h_2 (a translation by $h_2(0)$). For the same reason we may and will assume that $h_1(0)$ is the identity matrix.

Now we choose coordinates so that λ is diagonal:

$$\begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}$$

with $a \leq b \leq c$ integers.

LEMMA 4.4. *Let*

$$h_1 = \begin{pmatrix} u_1 & b_1 & c_1 \\ a_2 & u_2 & c_2 \\ a_3 & b_3 & u_3 \end{pmatrix}$$

be an element of $G(\mathbf{C}[[t]])$ *with* $h_1(0) = I_3$. *Then* h_1 *can be written as a product* $h_1 = h \cdot j$ *with*

$$h = \begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ r & s & 1 \end{pmatrix} \in G(\mathbf{C}[t]) \quad (!), \qquad j = \begin{pmatrix} v_1 & e_1 & f_1 \\ d_2 & v_2 & f_2 \\ d_3 & e_3 & v_3 \end{pmatrix} \in G(\mathbf{C}[[t]]),$$

satisfying

(1) $h(0) = j(0) = I_3$;
(2) $deg(q) < b - a$, $deg(r) < c - a$, $deg(s) < c - b$;
(3) $d_2 \equiv 0 \pmod{t^{b-a}}$, $d_3 \equiv 0 \pmod{t^{c-a}}$, $e_3 \equiv 0 \pmod{t^{c-b}}$.

(Here we define the degree of the zero polynomial to be $-\infty$.*)*

PROOF: Obviously $v_1 = u_1, e_1 = b_1$ and $f_1 = c_1$. Use division with remainder to write

$$v_1^{-1} a_2 = D_2 t^{b-a} + q$$

with $deg(q) < b - a$, and let $d_2 = v_1 D_2 t^{b-a}$ (so that $qv_1 + d_2 = a_2$). This defines q and d_2, and uniquely determines v_2 and f_2. (Note that $q(0) = d_2(0) = f_2(0) = 0$ and that $v_2(0) = 1$.)

Similarly, we let r be the remainder of

$$(v_1 v_2 - e_1 d_2)^{-1} (v_2 a_3 - d_2 b_3)$$

under division by t^{c-a}; and s be the remainder of

$$(v_1 v_2 - e_1 d_2)^{-1} (v_1 b_3 - e_1 a_3)$$

under division by t^{c-b}.

Then $deg(r) < c - a$, $deg(s) < c - b$ and $r(0) = s(0) = 0$; moreover, we have

$$v_1 r + d_2 s \equiv a_3 \pmod{t^{c-a}}, \qquad e_1 r + v_2 s \equiv b_3 \pmod{t^{c-b}},$$

so we take $d_3 = a_3 - v_1 r - d_2 s$, $e_3 = b_3 - e_1 r - v_2 s$. This defines r, s, d_3 and e_3, and uniquely determines v_3. ∎

It suffices therefore to consider the limits

$$\lim_{t \to 0} C \cdot h \cdot \lambda$$

with h as in the lemma. The reason is that by (3) above we have $j \cdot \lambda = \lambda \cdot k$ for a $k \in G(\mathbf{C}[[t]])$, and the effect of k can be ignored (note that $k(0)$ is a lower triangular matrix with 1's on the diagonal).

THEOREM IV(1). *Let C be a smooth plane curve of degree $d \geq 3$. The boundary components of the PGL(3)-orbit O_C of C in $\mathbf{P}^N$ are all 7-dimensional. The following hold:*

(1) *For $3 \leq k \leq d$, the closure of the orbit of the curve $x^{d-k}y^k + x^{d-1}z$ is a boundary component if and only if C has a k-flex.*
(2) *For $d \geq 5$ there is exactly one other component, the closure of the images of the rank-2 matrices. For $d = 3$ or 4 there are no other components.*

REMARK 4.5. Suppose that C has a k-flex in $(1:0:0)$ along $z = 0$. We may assume that the coefficients of $x^{d-k}y^k$ and of $x^{d-1}z$ are both 1. Then the curve $x^{d-k}y^k + x^{d-1}z$ is equal to the limit $\lim_{t\to 0} C \cdot \lambda$ where λ is the 1-PS

$$\begin{pmatrix} t^{-k} & 0 & 0 \\ 0 & t^{d-k} & 0 \\ 0 & 0 & t^{dk-k} \end{pmatrix}.$$

In other words, the boundary components of type (1) are orbit closures of 1-PS limits of C. A direct proof of this statement, perhaps also for certain singular C, would be very welcome. (Note: it is *not* true that all curves in the boundary of an orbit are 1-PS limits of (translates of) C. For the general C of degree $d \geq 6$, the curve $(x^2+y^2)x^{d-2}$ may serve as an example. Moreover, recently we have found examples of (rather special) singular curves whose orbit closure has a (7-dimensional) boundary component that is *not* contained in the Zariski closure of the set of 1-PS limits of (translates of) C. The only reference we have been able to find that touches upon questions of this kind is [**Kraft**, III.2.3 Bemerkung 1, p. 178].)

PROOF OF THE THEOREM: We have to determine all limits

$$\lim_{t\to 0} C \cdot h \cdot \lambda$$

that exist, with h and λ as before.

We start with the following observation. Suppose that $\lim_{t\to 0} C \cdot h \cdot \lambda$ exists, and suppose that a certain monomial $x^k y^l z^m$ (with $k+l+m = d$) appears in C with non-zero coefficient. Then $ka + lb + mc \geq 0$, and this monomial appears in the limit if and only if $ka + lb + mc = 0$. However, it is possible that other monomials—those that don't appear in C—do appear in the limit. The reason is that for all terms $t^w x^k y^l z^m$ in $C \cdot h \cdot \lambda$ the *weight* w is at least $ka + lb + mc$, and there is a term with this minimum weight if and only if $x^k y^l z^m$ has non-zero coefficient in C.

It's easy to analyse the cases where $a \geq 0$:

(1) If $a > 0$, the limit is 0.
(2) If $a = 0 < b$, the limit is $C(x,0,0)$, which is a multiple of x^d.
(3) If $a = b = 0 < c$, the limit is $C(x,y,0)$. This is a rank-2 image.
(4) If $a = b = c = 0$, the limit is C itself.

Note that in all these cases h has no effect whatsoever.

Next we assume $a < 0$. Since C does not contain a line, one of the monomials $x^{d-k}y^k$ has non-zero coefficient. Using the above observation, one first sees that the coefficient of x^d *is* 0 (since $a < 0$) and then that $b \geq 0$ (so in particular $b > a$).

Again using the above observation, if C has at most a simple flex in $(1:0:0)$ along $z=0$ (i.e., the coefficient of $x^{d-k}y^k$ is $\neq 0$ for a $k \leq 3$), the limit is in the span of x^d, $x^{d-1}y$, $x^{d-2}y^2$, $x^{d-3}y^3$ and $x^{d-1}z$. (Note: if the coefficient of $x^{d-1}y$ is 0, the coefficient of $x^{d-1}z$ is $\neq 0$ since C is smooth.) One immediately checks that this span is contained in the closure of the orbit of the curve $x^{d-3}y^3+x^{d-1}z$.

To finish the proof of the first part of the theorem, we have to analyse the situation at the higher flexes. So assume that C has a k-flex ($k \geq 4$) at $(1:0:0)$ along $z=0$. Then the coefficients of x^d, $x^{d-1}y$, $\ldots$, $x^{d-k+1}y^{k-1}$ are all 0, while the coefficients of $x^{d-k}y^k$ and of $x^{d-1}z$ are $\neq 0$. This implies

$$(*) \qquad (d-k)a+kb \geq 0,$$

$$(**) \qquad (d-1)a+c \geq 0.$$

In order that we will be able to determine which of the monomials x^d, $\ldots$, $x^{d-k+1}y^{k-1}$ occur in the limit, we will write down for each monomial the numbers that could *a priori* be the minimum weight of that monomial in the expression $C \cdot h \cdot \lambda$. Since these monomials $x^{d-j}y^j$ (with $0 \leq j \leq k-1$) do not appear in C, their possible minimum weights are $> (d-j)a+jb$; to find the possible minimum weights we repeatedly replace in $x^{d-j}y^j$ either an x by a y, or an x by a z, or a y by a z, until we arrive at a monomial (call it M) that could appear in C; we then write down the weight of the original monomial as it arises from M in the expression $M \cdot h \cdot \lambda$. Replacing an x by a y, we have to go all the way up to $x^{d-k}y^k$; as for replacing an x or a y by a z, we have to do only one such substitution, but unless the result is $x^{d-1}z$, we cannot guarantee that the resulting monomial has non-zero coefficient in C. This accounts for the $\geq$-signs in the list below. Note that the original monomial arises in a unique way from M if we replaced an x by a y or a y by a z; if we replaced an x by a z, there are two ways to reproduce the original monomial: either we let the one z 'take care' of an x, or the z produce a y and a y produce an x. In order that the latter possibility make sense, we need that M contain a y; however, the weight (of the original monomial) gotten in that case is strictly larger than the weight one gets by just replacing a y by a z. So we will ignore this second possibility.

Before we write down the list, we introduce some notation: we let l, resp. m, resp. n be the valuation (with respect to t) of q, resp. r, resp. s (with $v(0)=+\infty$).

Now for the list. The possible minimum weights of x^d:

$$(0\text{-}1) \qquad (d-k)a+k(a+l)=da+kl,$$

$$(0\text{-}2) \qquad (d-1)a+(a+m)=da+m.$$

The possible minimum weights of $x^{d-1}y$:

$$(1\text{-}1) \qquad (d-k)a+(k-1)(a+l)+b=(d-1)a+b+(k-1)l,$$

$$(1\text{-}2) \qquad \geq (d-2)a+b+(a+m)=(d-1)a+b+m,$$

$$(1\text{-}3) \qquad (d-1)a+(b+n)=(d-1)a+b+n.$$

The possible minimum weights of $x^{d-j}y^j$ (with $2 \leq j \leq k-1$):

$$(j\text{-}1) \qquad (d-k)a+(k-j)(a+l)+jb=(d-j)a+jb+(k-j)l,$$

$$(j\text{-}2) \qquad \geq (d-j-1)a+jb+(a+m)=(d-j)a+jb+m,$$

$$(j\text{-}3) \qquad \geq (d-j)a+(j-1)b+(b+n)=(d-j)a+jb+n.$$

Since we assumed that the limit $\lim_{t\to 0} C \cdot h \cdot \lambda$ exists, it follows that each monomial $x^{d-j}y^j$ (with $0 \leq j \leq k-1$) occurs with nonnegative weight in the expression $C \cdot h \cdot \lambda$. This leaves two possibilities for the possible minimum weights of such a monomial as written down in the list above: either all possible minimum weights are ≥ 0, or at least two of them are < 0 *and equal*, and the third weight (if it exists) is greater than or equal to these two. The second possibility reflects the phenomenon of *cancellation*: *a priori* it is possible that terms arising from distinct monomials M cancel each other out by having equal negative weights and coefficients adding up to 0, in which case the true minimum weight of the original monomial is larger than the smallest weight in the table above; in particular, the true minimum weight could be greater than or equal to zero.

Hoping that the reader is still with us at this point, we will now analyse which limits occur. Remember that $a < 0 \leq b$.

If $l = \infty$ (i.e., $q = 0$), then from the above it follows that $da + m \geq 0$. So all other second terms are > 0, so $(d-1)a + b + n \geq 0$, so all other third terms are > 0. The limit is in the span of x^d, $x^{d-1}y$, $x^{d-1}z$ and $x^{d-k}y^k$, thus in the closure of the orbit of $x^{d-k}y^k + x^{d-1}z$.

If $q \neq 0$, then by the lemma $0 < l \leq deg(q) < b - a$. We distinguish two cases.

Case I.

$$da + kl \geq 0, \qquad da + m \geq 0.$$

Since $b - a - l > 0$, all other first terms are > 0, and in $(*)$ we get strict inequality. As before we get that all other second terms are > 0, so (1-3) is ≥ 0, so all other third terms are > 0. The limit is in the span of x^d, $x^{d-1}y$ and $x^{d-1}z$. This span is contained in the closure of the (projectively) 4-dimensional orbit of two distinct lines, one of which has multiplicity $d - 1$.

Case II.

$$da + kl = da + m < 0.$$

This implies $(d-1)a + b + (k-1)l < (d-1)a + b + m$, so we are left with two subcases.

Case II-1.

$$(d-1)a + b + (k-1)l \geq 0, \qquad (d-1)a + b + n \geq 0.$$

Clearly all terms with $j \geq 2$ are > 0, and the term in $(*)$ is > 0 as well. So the limit is again in the span of x^d, $x^{d-1}y$ and $x^{d-1}z$.

Case II-2.

$$(d-1)a + b + (k-1)l = (d-1)a + b + n < 0.$$

Then $(d-2)a + 2b + (k-2)l < (d-2)a + 2b + n$, so necessarily the first term is ≥ 0. One easily sees that the limit is contained in the span of x^d, $x^{d-1}y$, $x^{d-1}z$ and $x^{d-2}y^2$. This span is contained in the closure of the (projectively) 6-dimensional orbit of a smooth conic with a $(d-2)$-fold tangent line. Note that this 6-dimensional locus is contained in the orbit closure of the curve $x^{d-k}y^k + x^{d-1}z$. (Choose coordinates so that $(1:0:0)$ is a smooth point where $z = 0$ is simply tangent, and apply the diagonalized 1-PS with entries $(t^{-2}, t^{d-2}, t^{2d-2})$ to get the result.)

Note that the above arguments apply when $m = \infty$ and/or $n = \infty$.

All in all, we proved that if C has a k-flex at $(1:0:0)$ along $z = 0$ in the coordinates chosen to diagonalize λ, and if $a < 0$, then the limits $\lim_{t \to 0} C \cdot h \cdot \lambda$ that exist are contained in the orbit closure of the curve $x^{d-k}y^k + x^{d-1}z$.

Using Remark 4.5, we see that the first part of the theorem is now proved. To prove the second part we have to deal with the limits of 1-PS's for which $a \geq 0$. It is immediate from the (easy) analysis of these limits (see above) that this gives rise to at most one boundary component, the closure of the locus of images of rank-2 matrices. By Proposition 4.2, this locus is 7-dimensional for $d \geq 5$, and thus its closure forms a boundary component. For $d = 3$ (resp. 4) the locus has dimension 5 (resp. 6), and its closure is the locus of all d-tuples of lines through a point. One easily checks that this locus is contained in the boundary components of type (1).

This finishes the proof of Theorem IV(1). ∎

§4.2. Families of orbits. In this subsection we will study one-dimensional families of orbits. Let $C(u)$ be a family of smooth plane curves of degree d over $\mathbf{A}^1_u$, with central fibre $C = C(0)$ a curve with one k-flex ($4 \leq k \leq d$) and otherwise simple flexes, and general fibre a curve with only simple flexes. Assume for simplicity that C has no non-trivial automorphisms. Denote by X the scheme-theoretic closure in $\mathbf{P}^N \times \mathbf{A}^1_u$ of the corresponding family of orbits, and by X_0 the special fibre of X.

Since X is flat over $\mathbf{A}^1_u$ the degree of the fibres is constant. Now the general fibre is the orbit closure of a curve with only simple flexes and no automorphisms. Its degree was computed in the Corollary to Theorem III, §3, and equals that of the orbit closure of a general plane curve. The special fibre contains the orbit closure of C. This orbit closure has degree strictly less than the degree of the general fibre, since not all flexes of C are simple (see the remark following Theorem III). As a consequence, X_0 must have other components. We will prove that in a situation as above, where the family is general in a sense to be specified later, there is only one other component, the orbit closure of a certain, well-determined singular curve.

Thus the degree of the orbit (closure) of a general plane curve is the sum of the degrees of two other orbits; or, what we have to subtract from the general degree to get the degree of the orbit of a curve with a k-flex, is in fact the degree of the orbit of another, singular, curve; in some sense the two curves are 'dual' to each other. (Perhaps 'liaison' gives a better description of the situation.)

At present we are not able to compute the degrees of the orbits of these singular curves directly, except in the case $k = 4$, by an ad hoc method. For other (less) singular curves, we can compute the degrees of the orbits. We hope to publish these results in a second paper, in which we will also exhibit other examples of the 'duality' observed above.

In the following lemma, $C(u)$ may be any family of smooth plane curves of degree d over $\mathbf{A}^1_u$. Again we denote by X the scheme-theoretic closure in $\mathbf{P}^N \times \mathbf{A}^1_u$ of the corresponding family of orbits, and by X_0 the central fibre of X. Furthermore, let Y_0 be the affine cone over X_0, and let $G = GL(3)$.

LEMMA 4.6. *As sets,*

$$Y_0 = \{ \lim_{t \to 0} C(u) \cdot g(t) : g \in G(\mathbf{C}((t))), u = t^e, e \in \mathbf{N}_+ \},$$

where we ignore the limits that don't exist.

PROOF: Again, we follow the set-up of [**Kempf**, proof of Theorem (1.4)]. Let $U = \{ C(u) \cdot h : u \neq 0, h \in G \}$, which locally around 0 equals the union of the orbits outside the central fibre. Then Y_0 consists of the points in the closure of U that lie above $u = 0$. Pick such a point. We may find a curve S in U that contains this point in its closure. This curve comes with a dominant map to $\mathbb{A}^1_u$. Writing down for each $u \neq 0$ the elements in G that map $C(u)$ to one of the points of S above u,we find a curve S' in G. Finally, let S'' be a smooth complete curve normalizing the curve S' in G; it comes with a map to G and with a dominant map to $\mathbb{A}^1_u$. Localize at $u = 0$: let t be a local parameter at a point of S'' lying above $u = 0$. Then $u = t^e$ (up to a unit, which can be ignored) for some $e \in \mathbf{N}_+$. Considering that S'' gives us a rational map from $\mathbf{C}[[t]]$ to G, or in other words, an element of $G(\mathbf{C}((t)))$, we arrive at the statement of the lemma. ∎

Returning to the beginning of this subsection, let $C(u)$ be a family with central fibre $C = C(0)$ a smooth curve with one k-flex ($k \geq 4$) and otherwise simple flexes, and general fibre a curve with only simple flexes.

We use Lemma 4.6 to determine the components of X_0. As before, by the Cartan-Iwahori decomposition and since we are interested only in the *orbits* that occur, we are reduced to computing the limits $\lim_{t \to 0} C(u) \cdot h_1(t) \cdot \lambda(t)$ that exist, with $u = t^e$ for an $e \in \mathbf{N}_+$ and h_1 an element of $G(\mathbf{C}[[t]])$ with $h_1(0) = I_3$. Choose coordinates so that λ is a diagonal matrix with entries (t^a, t^b, t^c), with $a \leq b \leq c$. If $a \geq 0$, the limit is contained in the orbit closure of C, so we assume $a < 0$. As before it follows that $b \geq 0 > a$.

To get a limit that is not in the orbit closure of C, we have to assume that the k-flex of C is in $(1 : 0 : 0)$ along $z = 0$. So C is of the form $x^{d-k}y^k + x^{d-1}z + \ldots$, where the dots indicate terms that have higher (thus positive) λ-weight. Denote by $a_j(u)$ the coefficient of $x^{d-j}y^j$ in $C(u)$, for $0 \leq j \leq k-1$. Each of these k polynomials has a zero at $u = 0$. We may assume that $a_0(u)$ and $a_1(u)$ are identically zero: let $b(u)$ be the coefficient of $x^{d-1}z$ in $C(u)$, let $g(u)$ be the change of coordinates fixing x and y and sending z to $z - a_0(u)b(u)^{-1}x - a_1(u)b(u)^{-1}y$, and consider $C(u) \cdot g(u)$ as the new $C(u)$ and $g(u)^{-1}h_1(t)$ as the new $h_1(t)$.

Now we restrict our attention to a *general* family, in the following sense: we assume that $a_2(u)$ has a *simple* zero at $u = 0$. In other words, we assume that $C(u)$ represents a first-order deformation of C in which the point $(1 : 0 : 0)$ is not a flex. One checks that $C(u)$ can be written in the form $x^{d-k}y^k + x^{d-1}z + ux^{d-2}y^2 + \ldots$, where the dots indicate terms that have higher *total* weight (the sum of the λ-weight of a monomial and the t-valuation of its coefficient).

THEOREM IV(2). *Assume that $C(u)$ is a general family, as above. Then the central fibre X_0 of the closure X of the corresponding family of $PGL(3)$-orbits consists of two components, the orbit closure of $C = C(0)$ and the orbit closure of the curve $x^{d-k}y^k + x^{d-2}y^2 + x^{d-1}z$.*

REMARK. The curve $y^k + x^{k-2}y^2 + x^{k-1}z$ has one singular point and $k-2$ *simple* flexes.

PROOF OF THE THEOREM: We continue our analysis of the limits $\lim_{t\to 0} C(u) \cdot h_1(t) \cdot \lambda(t)$, with $C(u)$ in the form $x^{d-k}y^k + x^{d-1}z + ux^{d-2}y^2 + \dots$. Applying Lemma 4.4, we see that it suffices to determine the limits $\lim_{t\to 0} C(u) \cdot h(t) \cdot \lambda(t)$ with h as in that lemma. Once again we write down the possible minimum (total) weights of the monomials $x^{d-j}y^j$, for $0 \le j \le k-1$; we use the same notations as in the proof of Theorem IV(1).

The weights of x^d:

$$(0\text{-}1) \qquad (d-k)a + k(a+l) = da + kl,$$
$$(0\text{-}2) \qquad (d-1)a + (a+m) = da + m,$$
$$(0\text{-}3) \qquad (d-2)a + 2(a+l) + e = da + 2l + e.$$

The weights of $x^{d-1}y$:

$$(1\text{-}1) \qquad (d-k)a + (k-1)(a+l) + b = (d-1)a + b + (k-1)l,$$
$$(1\text{-}2) \qquad \ge (d-2)a + b + (a+m) = (d-1)a + b + m,$$
$$(1\text{-}3) \qquad (d-2)a + (a+l) + b + e = (d-1)a + b + l + e,$$
$$(1\text{-}4) \qquad (d-1)a + (b+n) = (d-1)a + b + n.$$

The weights of $x^{d-2}y^2$:

$$(2\text{-}1) \qquad (d-k)a + (k-2)(a+l) + 2b = (d-2)a + 2b + (k-2)l,$$
$$(2\text{-}2) \qquad \ge (d-3)a + 2b + (a+m) = (d-2)a + 2b + m,$$
$$(2\text{-}3) \qquad (d-2)a + 2b + e = (d-2)a + 2b + e,$$
$$(2\text{-}4) \qquad \ge (d-2)a + b + (b+n) = (d-2)a + 2b + n.$$

The weights of $x^{d-j}y^j$, for $3 \le j \le k-1$:

$$(j\text{-}1) \qquad (d-k)a + (k-j)(a+l) + jb = (d-j)a + jb + (k-j)l,$$
$$(j\text{-}2) \qquad \ge (d-j-1)a + jb + (a+m) = (d-j)a + jb + m,$$
$$(j\text{-}3) \qquad \ge (d-j)a + jb + e = (d-j)a + jb + e,$$
$$(j\text{-}4) \qquad \ge (d-j)a + (j-1)b + (b+n) = (d-j)a + jb + n.$$

Let us analyse the various possibilities. If $l = +\infty$, then $da + m \ge 0$, thus $(d-1)a + b + n \ge 0$, thus $(d-2)a + 2b + e \ge 0$; all other weights are > 0. The limit is in the span of x^d, $x^{d-1}y$, $x^{d-2}y^2$, $x^{d-k}y^k$ and $x^{d-1}z$. This gives the new 8-dimensional orbit closure. (Apply the 1-PS with $(a,b,c) = (-k, d-k, dk-k)$ to the family with $e = d(k-2)$.)

So we may assume $0 < l < b - a$. We distinguish four cases.

CASE I.

$$da + kl \ge 0, \qquad da + m \ge 0, \qquad da + 2l + e \ge 0.$$

Then $(d-1)a + b + n \ge 0$, all other weights are > 0. The limit is in the orbit closure of $x^{d-1}y$.

CASE II.

$$da + 2l + e > da + kl = da + m < 0.$$

Then (1-1) is less than (1-2) and (1-3). There are two subcases:

CASE II-1.

$$(d-1)a+b+(k-1)l \geq 0, \qquad (d-1)a+b+n \geq 0.$$

The limit is in the orbit closure of $x^{d-1}y$.

CASE II-2.

$$(d-1)a+b+(k-1)l = (d-1)a+b+n < 0.$$

Then $0 \leq (d-2)a+2b+(k-2)l < (d-2)a+2b+n$. The limit is in the orbit closure of $x^{d-2}y^2 + x^{d-1}z$.

CASE III.

$$da+m \geq da+2l+e = da+kl < 0.$$

Then (1-1) equals (1-3). If this number is ≥ 0, then the limit is in the orbit closure of $x^{d-1}y$. Else (1-4) is greater than or equal to it. Then (2-4) is greater than (2-1), which equals (2-3). Finally, it follows then that (3-3) is greater than (3-1), which is ≥ 0. The limit is in the orbit closure of $x^{d-3}y^3 + x^{d-1}z$.

CASE IV.

$$da+kl > da+m = da+2l+e < 0.$$

Then (1-1) and (1-2) are both greater than (1-3). Again, two subcases:

CASE IV-1.

$$(d-1)a+b+n \geq 0, \qquad (d-1)a+b+l+e \geq 0.$$

The limit is in the orbit closure of $x^{d-1}y$.

CASE IV-2.

$$(d-1)a+b+n = (d-1)a+b+l+e < 0.$$

Then $0 \leq (d-2)a+2b+e < (d-2)a+2b+n$. The limit is the orbit closure of $x^{d-2}y^2 + x^{d-1}z$.

This finishes the proof of the theorem. ∎

§5. AN APPLICATION.

In this section we present an application of the results so far. First we introduce some notation. Let $d \geq 4$ be an integer, let $N = \binom{d+2}{2} - 1$ and let $g = \binom{d-1}{2}$. Denote by $\mathbf{P}^N$ the projective space of plane curves of degree d, by $\mathcal{M}_g$ the moduli space of smooth, irreducible curves of genus g, and by $\mathcal{M}_d^2$ the locally closed subvariety of $\mathcal{M}_g$ of curves that can be embedded as smooth plane curves of degree d. (Note that these curves have a unique g_d^2, compare [**ACGH**, App. A, Exc. 18(iii)].) Denote by V the closed (irreducible) subvariety, of codimension one, of $\mathcal{M}_d^2$ of curves with at least one k-flex for a $k \geq 4$ (in other words, curves for which not all flexes are simple).

THEOREM V(1). *For $d=4$ or 5, the Chow ring (with $\mathbf{Q}$-coefficients) of $\mathcal{M}_d^2 - V$ is trivial. I.e., for $\ell \geq 1$*

$$A^\ell(\mathcal{M}_d^2 - V)_{\mathbf{Q}} = (0).$$

PROOF: From the proof of Theorem IV(1) we know that the curves appearing in the boundary of an orbit of a smooth plane curve of degree d are either of the form 'd lines through a point' or have an equation of the form $x^{d-k}y^k + x^{d-1}z$ with $2 \leq k \leq d$.

Let X be the closed set in $\mathbf{P}^N$ which is the union of the closures of the loci of such curves. The various loci have dimension either $d+2$ or 6 or 7, so (since $d=4$ or 5) each component of X has dimension ≤ 7 (in fact, equality holds).

Let P be a general linear subspace of codimension 8 in $\mathbf{P}^N$; in fact, any such subspace that has empty intersection with X. Let $\Delta \subset \mathbf{P}^N$ be the discriminant hypersurface parametrizing the singular plane curves, and let $U = P - \Delta$ be the (non-empty, as we will see) open subset of P of smooth plane curves. Then (by the definition of $\mathcal{M}_g$) there is a natural morphism $\phi\colon U \to \mathcal{M}_d^2$ which sends a point in U to the isomorphism class of the curve corresponding to it. We claim:

(1) ϕ is surjective and quasi-finite;

(2) ϕ is finite (exactly) above $\mathcal{M}_d^2 - V$.

Once the claim is proven, the theorem follows by applying the (easy) Lemma A in [**Faber**, Introduction]. (Note that the Chow ring (with $\mathbf{Q}$-coefficients) of U is trivial, cf. [**Fulton**, §1.8].)

PROOF OF CLAIM: To prove the first part, note that P has non-empty intersection with every 8-dimensional closed subvariety of $\mathbf{P}^N$, so in particular $P \cap \overline{O_C} \neq \emptyset$ for a smooth C. Since $P \cap X = \emptyset$, we find

$$(*) \qquad \emptyset \neq P \cap \overline{O_C} = P \cap O_C = U \cap O_C,$$

so ϕ is surjective. Since U is affine and the intersection is a closed subset of P, it consists of finitely many points, so that ϕ is quasi-finite.

To prove the theorem, it suffices to prove that ϕ is proper above $\mathcal{M}_d^2 - V$, the locus of curves with only simple flexes, since ϕ is quasi-finite ([**EGA III**, 4.4.2]). Let $R = \mathbf{C}[[t]]$, let $K = \mathbf{C}((t))$ be its quotient field, and let $i\colon Spec\,K \to Spec\,R$ be the map induced by the inclusion $R \subset K$. Consider a family over $Spec\,R$ where both fibres are smooth plane curves with only simple flexes. Let $m\colon Spec\,R \to \mathcal{M}_d^2 - V$ be the morphism we get by taking isomorphism classes. Suppose we have a map $n\colon Spec\,K \to U$ such that $\phi \cdot n = m \cdot i$. Since P is proper over $\mathbf{C}$, the map n can be extended to $Spec\,R$ to give a map $p\colon Spec\,R \to P$. Denoting by C the special fibre of the family, we know that the image of the closed point is contained in $\overline{O_C}$ (it is here that we use that the family consists of curves with only simple flexes; the special fibre in the closure of the corresponding family of orbits is $\overline{O_C}$). Using $(*)$, we conclude that the image of p is contained in U. This proves that ϕ is proper above $\mathcal{M}_d^2 - V$.

Next we show that in fact ϕ is not proper above V. A curve is in V if and only if it has at least one k-flex ($4 \leq k \leq d$). Write down a general family (see Theorem IV(2)) of smooth plane curves with central fibre a curve C with a k-flex and general fibre a curve with only simple flexes: $C(u) = x^{d-k}y^k + x^{d-1}z + ux^{d-2}y^2 + \ldots$. Let λ be the diagonalized 1-PS with entries $(t^{-k}, t^{d-k}, t^{dk-k})$, let $e = d(k-2)$, and consider the family $D(t) = C(t^e) \cdot \lambda(t)$. Taking isomorphism classes, we see that $E(t) = C(t^e)$ gives a map $m\colon Spec\,R \to \mathcal{M}_d^2$. Assume for a moment that $D(t)$ gives a map $n\colon Spec\,K \to U$. Then $\phi \cdot n = m \cdot i$ since the generic fibres of $D(t)$ and of $E(t)$ are isomorphic. But there doesn't exist a map $p\colon Spec\,R \to U$ with $p \cdot i = n$, since $\lim_{t\to 0} D(t)$ is the *singular* curve $L = x^{d-k}y^k + x^{d-1}z + x^{d-2}y^2$. It follows that ϕ is not proper above V.

To see that we may assume that $D(t)$ gives a map $n\colon Spec\,K \to U$, the crucial point is to find a $g \in \mathrm{PGL}(3)$ such that $L \cdot g \in P$. Once we have that, we can also find a $g(t) \in G(R)$, with $g(0) = g$, such that $D(t) \cdot g(t) \in P(R)$. The generic fibre of this family is then in U. To find g, note that $\overline{O_L} \cap P \neq \emptyset$ (of course) and that $\overline{O_L} - O_L \subset X$. Namely, in the proof of Theorem IV(2) we found all points in $\overline{O_L}$. These points are either in $\overline{O_C}$, in which case they are in X, or (for certain coordinates x, y and z) they are in the span of x^d, $x^{d-1}y$, $x^{d-2}y^2$, $x^{d-k}y^k$ and $x^{d-1}z$. We have to check that the *boundary* points in this span are all in X. If the coefficient of $x^{d-1}z$ is 0, this is the case since X contains all curves of type 'd lines through a point'. If the coefficient of $x^{d-1}z$ is $\neq 0$, we may assume that the coefficients of x^d and of $x^{d-1}y$ are 0. Since we are considering boundary points, at least one of the coefficients of $x^{d-k}y^k$ and of $x^{d-2}y^2$ is 0; the resulting curves are in X.

The claim and the theorem have been proven. ∎

REMARKS. (1) The degree of ϕ is

$$d(d-2)(d^6 + 2d^5 + 4d^4 + 8d^3 - 1356d^2 + 5280d - 5319).$$

This is a direct consequence of the corollary to Theorem III: the degree of ϕ equals the number of points in the intersection of P and the orbit closure of a general plane curve of degree d.

(2) The reader is invited to compare the theorem with the construction in [**Faber**, Proposition (1.1)], where an analogous result is proved using a *special* $\mathbf{P}^6$ in the $\mathbf{P}^{14}$ of plane quartics.

It is rather involved to extend the theorem to higher values of d. The problem is that the locus $Q \subset \mathbf{P}^N$ of all curves consisting of d lines through a point has dimension $d+2$, so for $d \geq 6$ it has non-empty intersection with any linear subspace P of dimension $N-8$. Nevertheless it turns out to be possible to prove that for $d \geq 7$ the complement in $\mathcal{M}_d^2$ of *two* irreducible divisors has trivial Chow ring. The idea of the construction of the second divisor is as follows. Starting with the isomorphism class $[q]$ of a d-tuple q of points on a line, one may construct the subvariety of $\mathcal{M}_d^2$ of isomorphism classes $[C]$ of curves C for which a line m in $\mathbf{P}^2$ realizes the class $[q]$ (i.e., $[m \cap C] = [q]$). Roughly, the second divisor is a union of such subvarieties, where $[q]$ runs over the isomorphism classes of the d-tuples (of lines through a point, this time) occuring in $P \cap Q$.

To describe the construction more precisely, we first need a lemma. Let S be the locus of d-fold lines; this is the d-uple embedding of $\check{\mathbf{P}}^2$ into $\mathbf{P}^N$.

LEMMA 5.2. *As sets, $Sing\,Q = S$.*

PROOF: Given a point in $\mathbf{P}^2$, the locus of curves consisting of d lines through that point forms a $\mathbf{P}^d$ which is linearly embedded in $\mathbf{P}^N$. Globally this gives a

$\mathbf{P}^d$-bundle T over $\mathbf{P}^2$, inside $\mathbf{P}^N \times \mathbf{P}^2$. Then $Q = p_1(T)$ where $p_1 : \mathbf{P}^N \times \mathbf{P}^2 \to \mathbf{P}^N$ is the first projection. The fibres of the morphism $p_1 : T \to Q$ consist of single points above $Q - S$, and are $\mathbf{P}^1$'s above S. We claim that p_1 is an isomorphism above $Q - S$.

To prove the claim, consider the map $q : Q - S \to \mathbf{P}^2$ which sends a curve consisting of d (not all equal) lines through a point, to that point. Then the map $(i, q) : Q - S \to T \subset \mathbf{P}^N \times \mathbf{P}^2$ (where $i : Q - S \to \mathbf{P}^N$ is the inclusion) gives an inverse morphism to p_1 above $Q - S$. This shows that $Q - S$ is non-singular (which is all we need for the sequel). It's not hard to check that Q is in fact singular along S. ∎

REMARK. One can prove that Q is the locus of curves for which the Hessian is identically zero.

We assume $d \geq 7$ from now on. Denote by Q_{ss} the locus of curves in Q for which the corresponding d-tuple of points on $\mathbf{P}^1$ is semistable (i.e., has no points of multiplicity $> d/2$). Then Q_{ss} is smooth. Let P be a general linear subspace of $\mathbf{P}^N$ of dimension $N - 8$. Then $P \cap Q_{ss}$ has dimension $d - 6$. Applying Bertini, we see that it is smooth and connected (since $d \geq 7$), hence irreducible.

Let (as in §4.1) $M_d = (\mathbf{P}^d)_{ss}/\mathrm{PGL}(2)$ be the moduli space of (semistable) d-tuples of points on $\mathbf{P}^1$, and let Q' be the image of $P \cap Q_{ss}$ in M_d. One verifies that Q' is also irreducible of dimension $d - 6$.

As the next step in our construction we produce another map to M_d. Denote by $F \subset \mathbf{P}^N$ the open set of smooth curves with only simple flexes. Consider the map $f : F \times \check{\mathbf{P}}^2 \to M_d$ which sends a pair (C, l) to the isomorphism class of the d-tuple $C \cap l$. (Note that these d-tuples are in fact stable.)

We claim that the fibres of f over isomorphism classes of d-tuples consisting of d distinct points are irreducible of dimension $N - d + 5$. Fix such a d-tuple on a line l. One checks that the curves containing those d points form a (linear) $\mathbf{P}^{N-d}$. To get the fibre of f over the isomorphism class of the d-tuple, we first carry around this $\mathbf{P}^{N-d}$ with PGL(2), by varying the d-tuple in its orbit while fixing the line l; next we vary l in $\check{\mathbf{P}}^2$. The fibre is thus irreducible of the said dimension.

Since P was chosen generally, Q' contains isomorphism classes of d-tuples with d distinct points. We conclude that $D'' = f^{-1}(Q')$ is irreducible of dimension $(N - d + 5) + (d - 6) = N - 1$. Denote by $p_1 : F \times \mathbf{P}^2 \to F$ the first projection, and let $D' = p_1(D'')$. Then D' is the locus in F of curves C for which the intersection $P \cap \overline{O_C}$ contains a point in the boundary of $\overline{O_C}$. (Since P is general, it has empty intersection with all the possible boundary components of type (1) in Theorem IV(1), so a point in the intersection of P and the boundary of $\overline{O_C}$ is necessarily in the boundary component of type (2), the closure of the locus of rank-2 images.) We claim that D' is an (irreducible) divisor in F. First, the dimension of D' is $\leq N - 1$ since D'' has dimension $N - 1$. So all we need to show is that D' meets a general one-dimensional subvariety K of F. Consider the corresponding 9-dimensional union of orbit closures. The boundary components of type (2) form an 8-dimensional subvariety. Now P meets this subvariety; let C be a curve (corresponding to the point $[C]$ of K) such that P meets the boundary of O_C. By the above, $[C]$ is in D'. So D' intersects K. This proves the claim.

Finally, let $D = \phi(D')$, which is an irreducible divisor in $\mathcal{M}_d^2$. (Note that D' is

$PGL(3)$-invariant.)

THEOREM V(2). *For $d \geq 7$, $\mathcal{M}_d^2 - V - D$ has trivial Chow ring.*

PROOF: Consider the natural map $\phi\colon P \cap F \to \mathcal{M}_d^2 - V$. This morphism is proper exactly above the locus of curves C for which the intersection of P and $\overline{O_C}$ consists of points corresponding to *smooth* curves. By the construction above, this locus is the complement of D. Since for these curves $P \cap \overline{O_C} = P \cap O_C$, the map is also quasi-finite, thus finite, above this locus. Applying as before [**Faber**, Lemma A] we finish the proof of the theorem. ∎

Unfortunately, the method above does not work as well for $d = 6$. The reason is that $P \cap Q_{,,}$ is no longer irreducible, so the divisor D in $\mathcal{M}_d^2 - V$ will not be irreducible either. However, it seems likely that the complement in $\mathcal{M}_d^2 - V$ of any component of D has trivial Chow ring. It would be of some interest to examine this, since the moduli space of plane sextic curves is birational to the moduli space of K3-surfaces with a polarization of degree two.

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