PU(2) MONOPOLES AND LINKS OF TOP-LEVEL SEIBERG-WITTEN MODULI SPACES

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Abstract. This is the first of two articles in which we give a proof—for a broad class of four-manifolds—of Witten's conjecture that the Donaldson and Seiberg-Witten series coincide, at least through terms of degree less than or equal to $c-2$, where $c=-\frac{1}{4}(7\chi+11\sigma)$ and χ and σ are the Euler characteristic and signature of the four-manifold. In the present article, we construct virtual normal bundles for the Seiberg-Witten strata of the moduli space of PU(2) monopoles and compute their Chern classes.

1. Introduction

1.1. **Main results.** The purpose of the present article and its sequel [18], is to prove that Witten's conjecture [76] relating the Donaldson and Seiberg-Witten invariants holds in "low degrees" for a broad class of four-manifolds, using the PU(2)-monopole cobordism [64]. We shall assume throughout that X is a closed, connected, smooth four-manifold with an orientation for which $b_2^+(X)>0$. We state the simplest version of our result here; more general results are given in [18, §1]. The Seiberg-Witten (SW) invariants (see [18, §4.1]) comprise a function, $SW_X:\mathrm{Spin}^c(X)\to\mathbb{Z}$, where $\mathrm{Spin}^c(X)$ is the set of isomorphism classes of spin^c structures on X. For $w\in H^2(X;\mathbb{Z})$, one defines

$$\mathbf{SW}_X^w(h)=\sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)}(-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)}SW_X(\mathfrak{s})e^{\langle c_1(\mathfrak{s}),h\rangle},\quad h\in H_2(X;\mathbb{R}), \tag{1.1}$$

by analogy with the structure of the Donaldson series $\mathbf{D}_X^w(h)$ [44, Theorem 1.7]. There is a map $c_1:\mathrm{Spin}^c(X)\to H^2(X;\mathbb{Z})$ and the image of the support of SW_X is the set B of SW-basic classes [76]. A four-manifold X has SW-simple type when $b_1(X)=0$ if $c_1(\mathfrak{s})^2=2\chi+3\sigma$ for all $c_1(\mathfrak{s})\in B$, where χ and σ are the Euler characteristic and signature of X. We let $B^\perp\subset H^2(X;\mathbb{Z})$ denote the orthogonal complement of B with respect to the intersection form Q_X on $H^2(X;\mathbb{Z})$. Let $c(X)=-\frac{1}{4}(7\chi+11\sigma)$.

Theorem 1.1. *Let X be four-manifold with $b_1(X)=0$ and odd $b_2^+(X)\geq 3$. Assume X is abundant, SW-simple type, and effective. For any $\Lambda\in B^\perp$ and $w\in H^2(X;\mathbb{Z})$ for which $\Lambda^2=2-(\chi+\sigma)$ and $w-\Lambda\equiv w_2(X)\pmod 2$, and any $h\in H_2(X;\mathbb{R})$, one has*

$$\begin{aligned}\mathbf{D}_X^w(h)&\equiv 0\equiv\mathbf{SW}_X^w(h)\pmod{h^{c(X)-2}},\\ \mathbf{D}_X^w(h)&\equiv 2^{2-c(X)}e^{\frac{1}{2}Q_X(h,h)}\mathbf{SW}_X^w(h)\pmod{h^{c(X)}}.\end{aligned} \tag{1.2}$$

The vanishing assertion in low degrees for the series $\mathbf{D}_X^w(h)$ and $\mathbf{SW}_X^w(h)$ in equation (1.2) was proved in [16].

We shall explain below the terminology and notation in the statement of Theorem 1.1. Witten's conjecture [76] asserts that a four-manifold X with $b_1(X)=0$ and odd $b_2^+(X)\geq 3$

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has SW-simple type if and only if it has "KM-simple type", that is, simple type in the sense of Kronheimer and Mrowka (see Definition 1.4 in [44]), and that the SW-basic and "KM-basic" classes (see Theorem 1.7 in [44]) coincide; if X has simple type, then

$$\mathbf{D}^w_X(h) = 2^{2-c(X)} e^{\frac{1}{2}Q_X(h,h)} \mathbf{SW}^w_X(h), \quad h \in H_2(X;\mathbb{R}). \tag{1.3}$$

The quantum field theory argument giving equation (1.3) when $b_2^+(X) \geq 3$ has been extended by Moore and Witten [55] to allow $b_2^+(X) \geq 1$, $b_1(X) \geq 0$, and four-manifolds X of non-simple type. Recall that $b_2^+(X)$ is the dimension of a maximal positive-definite linear subspace $H^{2,+}(X;\mathbb{R})$ for the intersection pairing Q_X on $H^2(X;\mathbb{R})$.

With our stated hypotheses, Theorem 1.1 therefore proves that equation (1.3) holds mod $h^{c(X)}$ where $\delta = c(X)$. To prove the equivalence mod h^δ for $\delta > c(X)$, more work is required. For example, in [21], we use the gluing theory of [19], [20]—allowing "one bubble"— to prove that equation (1.3) holds mod $h^{c(X)+2}$ under hypotheses similar to those of Theorem 1.1. If one desires a mod h^δ relation such as (1.2) for larger values of δ relative to $c(X)$, one must allow "more bubbles" and the difficulty of the calculations rapidly increases: see [18, §1] for a more detailed discussion.

The concept of "abundance" was first introduced in [16]:

Definition 1.2. [16, p. 169] We say that a closed, oriented four-manifold X is *abundant* if the restriction of the intersection form to $B^\perp$ contains a hyperbolic sublattice.

The abundance condition is merely a convenient way of formulating the weaker, but more technical condition that one can find (for example) classes $\Lambda_j \in B^\perp$ such that $\Lambda_j^2 = 2j - (\chi + \sigma)$, for $j = 1, 2, 3$: this is the only property of $Q_X|B^\perp$ which we use to prove Theorem 1.1, while slightly different choices of classes Λ with even squares are used to prove the main results (Theorems 1.1 and 1.3) of [16]. While all compact, complex algebraic, simply-connected surfaces with $b_2^+ \geq 3$ are abundant (see Theorem A.1 and its proof in Appendix A), some examples of manifolds which are not abundant but for which one can still find classes in $B^\perp$ with the desired even squares are described in [16, §2]. It remains an interesting problem to determine whether all smooth four-manifolds have this property, whether or not they are abundant.

To see that such classes Λ_j exist when X is abundant, note that because $Q_X|B^\perp$ contains a hyperbolic factor there are classes $e_0, e_1 \in H^2(X;\mathbb{Z})$ which are orthogonal to the SW-basic classes and which satisfy $e_0^2 = 0 = e_1^2$ and $e_0 \cdot e_1 = 1$. Now set $t = \frac{1}{2}(\chi + \sigma)$ and define $\Lambda_j \in H^2(X;\mathbb{Z})$ by $\Lambda = e_0 + (j - t)e_1$, and observe that Λ_j^2 has the desired values for $j = 1, 2, 3$.

In the present article and its companion [18] we prove Theorem 1.1 using the moduli space $\mathcal{M}_\mathfrak{t}$ of PU(2) monopoles [64] to provide a cobordism between the link of the moduli space M^w_κ of anti-self-dual connections and links of moduli spaces of Seiberg-Witten moduli spaces, $M_\mathfrak{s}$, these moduli spaces being (topologically) embedded in $\mathcal{M}_\mathfrak{t}$. Let $\bar{\mathcal{M}}_\mathfrak{t}$ denote the Uhlenbeck compactification (see Theorem 2.12) of $\mathcal{M}_\mathfrak{t}$ in the space of ideal PU(2) monopoles, $\cup_{\ell=0}^\infty (\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X))$.

Definition 1.3. We say that a closed, oriented, smooth four-manifold X with $b_1(X) \geq 0$ and $b_2^+(X) \geq 1$ is *effective* if X satisfies Conjecture 3.1 in [16]. This conjecture asserts that for a Seiberg-Witten moduli space $M_\mathfrak{s}$ appearing in level $\ell \geq 0$ of $\bar{\mathcal{M}}_\mathfrak{t}$, the pairings of products of Donaldson-type cohomology classes on the top stratum of $\mathcal{M}_\mathfrak{t}/S^1$ with a link of $(M_\mathfrak{s} \times \mathrm{Sym}^\ell(X)) \cap \bar{\mathcal{M}}_\mathfrak{t}$ in $\bar{\mathcal{M}}_\mathfrak{t}$ are multiples of the Seiberg-Witten invariants for $M_\mathfrak{s}$. In particular, these pairings are zero when the Seiberg-Witten invariants for $M_\mathfrak{s}$ are trivial.

The motivation for Conjecture 3.1 of [16] and a more detailed explanation appears in [16, §3.1]; see also [23]. It is almost certainly true that this conjecture holds for all four-manifolds, based on our work in [19], [20], [22]. We verify Conjecture 3.1 in [16] by direct calculation in the present pair of articles for Donaldson invariants defined by $M_\kappa^w \hookrightarrow \mathcal{M}_\mathfrak{t}$ and Seiberg-Witten moduli spaces $M_\mathfrak{s} \hookrightarrow \mathcal{M}_\mathfrak{t}$, while in [21] we verify the conjecture for Seiberg-Witten moduli spaces $M_\mathfrak{s}$ contained in the "first level" of the Uhlenbeck compactification $\bar{\mathcal{M}}_\mathfrak{t}$. However, we strongly expect the conjecture to hold for Seiberg-Witten moduli spaces $M_\mathfrak{s}$ contained in any level of the compactification $\bar{\mathcal{M}}_\mathfrak{t}$.

Theorem 1.1 gives a relation, mod h^δ, between the Donaldson and Seiberg-Witten series when $\delta = c(X)$. Now if $\tilde X = X \# \overline{\mathbb{CP}}^2$ is the blow-up of X, then $c(\tilde X) = c(X)+1$. However, this does not imply that one can compute a term of higher degree in h in the Donaldson series $\mathbf{D}_X^w(h)$ from Theorem 1.1 merely by blowing up. To understand why, it suffices to examine the case where X also has KM-simple type. Let $\mathrm{PD}[e] \in H^2(\tilde X;\mathbb{Z})$ denote the Poincaré dual of the exceptional class $e \in H_2(\tilde X;\mathbb{Z})$. Then Proposition 1.9 in [44] implies that for all $h \in H_2(\tilde X;\mathbb{Z}) \cong \mathbb{R}[e] \oplus H_2(X;\mathbb{Z})$,

$$\mathbf{D}_{\tilde X}^{w+\mathrm{PD}[e]}(h) = -\mathbf{D}_X^w(h)\exp(-\tfrac{1}{2}e\cdot e)\sinh(e\cdot h).$$

Thus, $\mathbf{D}_{\tilde X}^{w+\mathrm{PD}[e]}$ has a zero at the origin of order one greater than that of $\mathbf{D}_X^w$.

On the other hand, If $\Lambda \in B^\perp(X)$, then we also have $\Lambda \in B^\perp(\tilde X)$ and $w+\mathrm{PD}[e]-\Lambda \equiv w_2(X)+\mathrm{PD}[e] \equiv w_2(\tilde X) \pmod 2$, while $\chi(\tilde X)+\sigma(\tilde X) = \chi(X)+\sigma(X)$, so Λ^2 still has the desired even square in the case of $\tilde X$. If X is SW-simple type and abundant then the same holds for $\tilde X$ (and as mentioned, it is almost certainly true that all four-manifolds are effective). Theorem 1.1 then yields, for all $h \in H_2(\tilde X;\mathbb{Z})$,

$$\begin{aligned}
&\mathbf{D}_{\tilde X}^{w+\mathrm{PD}[e]}(h) \equiv 0 \equiv \mathbf{SW}_{\tilde X}^{w+\mathrm{PD}[e]}(h) \pmod{h^{c(X)-1}},\\
&\mathbf{D}_{\tilde X}^{w+\mathrm{PD}[e]}(h) \equiv 2^{2-c(X)-1}\exp\left(\tfrac{1}{2}Q_{\tilde X}(h,h)\right)\mathbf{SW}_{\tilde X}^{w+\mathrm{PD}[e]}(h) \pmod{h^{c(X)+1}}.
\end{aligned}$$

One sees that the formula does not give new higher-degree terms in the Donaldson series.

As we mentioned just prior to its statement, Theorem 1.1 is the simplest formulation of the main results of the present article and its companion [18]. More general results are described in [18, §1], where we allow four-manifolds with $b_2^+(X) \geq 1$ and drop the assumptions that X has SW-simple type or is abundant and relax the constraints on the existence of classes $\Lambda \in B^\perp$ with prescribed squares; we also consider some limited cases where $b_1(X) > 0$. A full account of the case $b_1(X) > 0$ is of potential interest but is beyond the scope of the present pair of articles: we shall describe this case elsewhere.

1.2. **An outline of the proof of Theorem 1.1.** The proof of Theorem 1.1 splits into two steps. Step (i), which we carry out in this article, is to construct links of Seiberg-Witten moduli spaces in the top level of the Uhlenbeck-compactified moduli space of PU(2) monopoles, as boundaries of tubular neighborhoods in certain "thickened" or "virtual" moduli spaces of PU(2) monopoles (see Theorems 3.19 and 3.21) and then compute the Chern character and Chern classes of the vector bundles defining these tubular neighborhoods (see Theorem 3.29 and Corollary 3.30).

Step (ii), which we take up in the companion article [18], is to compute the pairings of products of cohomology classes on the moduli space of PU(2) monopoles with the links of the anti-self-dual moduli space of SO(3) connections and with links of the moduli spaces of Seiberg-Witten monopoles. These computations rely on our calculation of the Chern characters of the normal bundles of the strata of Seiberg-Witten monopoles, and a comparison of the orientations of the moduli spaces of anti-self-dual connections and Seiberg-Witten monopoles, and their links in the moduli space of PU(2) monopoles. Applying the PU(2)-monopole cobordism then yields an expression for the Donaldson invariants in terms of Seiberg-Witten invariants and hence completes the proof of Theorem 1.1.

1.3. **A guide to the article.** An index of notation appears just before §2. The present article is a revision of sections 1–3 of the preprint [17], while the companion article [18] is a revision of sections 4–7 of [17], which was distributed in December 1997.

Section 2 of this paper gathers together the principal gauge theory results developed in [15], [24] that we shall need here for the moduli space of PU(2) monopoles $\mathcal{M}_{\mathfrak{t}}$. In §2.1, we review the construction of the moduli spaces of PU(2) monopoles $\mathcal{M}_{\mathfrak{t}}$ from [24], but now phrased in the convenient and more compact framework of "spin^u structures" $\mathfrak{t}$. In §2.2 we recall our Uhlenbeck compactness and transversality results for the moduli spaces of PU(2) monopoles from [24] and [15]. In §2.3, we recall the construction of the moduli space of Seiberg-Witten monopoles $M_{\mathfrak{s}}$ as in [43], [45], [56], but with non-standard perturbations so we can directly identify these moduli spaces with strata of reducible PU(2) monopoles. Our transversality result [15] ensures that the natural stratification of the Uhlenbeck-compactified moduli space of PU(2) monopoles is smooth. In §2.4 we compute the cohomology ring of the configuration space of spin^c pairs and describe the cohomology classes arising in the definition of Seiberg-Witten invariants and in our later calculation (see §3.6) of the Chern character of certain universal families of vector bundles over X parameterized by $M_{\mathfrak{s}}$.

The construction of the links in $\mathcal{M}_{\mathfrak{t}}$ of the stratum M_κ^w of anti-self-dual connections and of the strata $M_{\mathfrak{s}}$ of *reducible* (or *Seiberg-Witten*) PU(2) monopoles occupies §3. In §3.1 we classify the possible singularities of $\mathcal{M}_{\mathfrak{t}}$. A link $\mathbf{L}_{\mathfrak{t},\kappa}^w$ of the stratum M_κ^w of anti-self-dual connections in $\mathcal{M}_{\mathfrak{t}}$ is constructed in §3.2 using the L^2 norm of the spinor components of PU(2) monopoles to define the distance to the stratum M_κ^w. In §3.3 we show that the subspaces of reducible PU(2) monopoles in $\mathcal{M}_{\mathfrak{t}}$ can be identified with Seiberg-Witten moduli spaces $M_{\mathfrak{s}}$, as defined in §2.3. As we explain in §3.4, the elliptic deformation complex for the PU(2) monopole equations (2.32) at a reducible pair splits into a *tangential deformation complex* — which can be identified with the elliptic deformation complex for the Seiberg-Witten monopole equations — and a *normal deformation complex*. Via the Kuranishi model, these deformation complexes describe the local structure of the moduli space $\mathcal{M}_{\mathfrak{t}}$ of PU(2) monopoles near a reducible solution. The description of a neighborhood of $M_{\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}}$ is complicated by the fact that we may have both a tangential deformation complex with positive index, so $\dim M_{\mathfrak{s}} > 0$, and a normal deformation complex with negative index. (This contrasts with the simpler situation considered in [12], [29, pp. 65–69], where abstract perturbations are used in conjunction with the local Kuranishi model to describe the local structure of the moduli space M_κ^w near an isolated reducible connection when the four-manifold has $b_2^+(X) = 0$: in this case the normal deformation complex has positive index.) The stratum $M_{\mathfrak{s}}$ will not in general be a smooth submanifold of $\mathcal{M}_{\mathfrak{t}}$ as reducible PU(2) monopoles cannot be shown to be regular points of the zero locus of the PU(2) monopole equations (2.32). Therefore, in §3.5, we construct an ambient finite-dimensional, smooth manifold $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ containing an open neighborhood in $\mathcal{M}_{\mathfrak{t}}$ of the stratum $M_{\mathfrak{s}}$ and containing $M_{\mathfrak{s}}$ as a smooth submanifold. We can then define a link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of $M_{\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}}/S^1$ as the S^1 quotient of the intersection with $\mathcal{M}_{\mathfrak{t}}/S^1$ of the boundary of a tubular neighborhood of $M_{\mathfrak{s}}$ in the ambient manifold $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$. In order to compute the pairings of cohomology classes on $\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1$ (the smooth locus or top stratum of $\mathcal{M}_{\mathfrak{t}}/S^1$) with the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ we shall need the Chern classes of the normal bundle $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ of the stratum $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$: we accomplish this in §3.6, using the Atiyah-Singer index theorem for families, by computing the Chern character of this normal bundle (Theorem 3.29) and then, after imposing a constraint on $H^1(X;\mathbb{Z})$ to simplify our calculations, its Chern classes (Corollary 3.30).

In Appendix A, we include a proof that all compact, complex algebraic, simply connected surfaces with $b_2^+ \geq 3$ are abundant (Theorem A.1). For minimal surfaces of general type, this fact was asserted in [16, p. 175].

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Index of Notation

2. Preliminaries

In this section we recall the framework for gauge theory for PU(2) monopoles established in [15], [24], [23], though we introduce some notational and other simplifications here. In §2.1 we describe the PU(2) monopole equations while in §2.2 we recall our Uhlenbeck compactness and transversality results from [15], [24]. We review the usual construction of the moduli space of Seiberg-Witten monopoles in §2.3, although we use the perturbation parameter $\tau \in \Omega^0(\mathrm{GL}(\Lambda^+))$ to achieve transversality for the moduli space of Seiberg-Witten monopoles rather than employ the customary perturbation parameter $\eta \in \Omega^+(X, i\mathbb{R})$.

One slightly awkward issue here then concerns the possible presence of zero-section Seiberg-Witten monopoles. We recall from [56, §6.3] that these may be avoided when $b_2^+(X) > 0$ by choosing a generic perturbation $\eta \in \Omega^+(X, i\mathbb{R})$. For trivial reasons, variation of the parameter τ has no effect on the presence or absence of zero-section solutions and so, even for a generic Riemannian metric on X, they cannot be avoided in all cases. Instead, we circumvent this problem by a employing a restriction on the second Stiefel-Whitney class of an SO(3) bundle comprising part of the definition of the "spin^u structure" introduced below, following [25], which precludes the existence of flat, reducible connections. With the aid of the blow-up trick of [57], there is no resulting loss of generality when computing Donaldson or Seiberg-Witten invariants [27], [26].

Finally, in §2.4 we compute the cohomology ring of the configuration space of Seiberg-Witten pairs and express the Seiberg-Witten μ-classes in terms of these generators.

Although we use the generic metrics and Clifford maps of [15] to achieve transversality for the moduli space of PU(2) monopoles, we note that all of the results in this article can be obtained using the holonomy perturbations of [24].

2.1. **PU(2) monopoles.**

Throughout this article, X denotes a closed, connected, oriented, smooth four-manifold. We begin our discussion with a brief review of the definition of spin^c structures and introduce the concept of a "spin^u structure", which will then allow us to give a convenient definition of the PU(2)-monopole equations.

2.1.1. *Hermitian Clifford modules, module derivations, and spin connections.* Let V be a complex vector bundle over a Riemannian manifold (X, g). A real-linear map $\rho : T^*X \to \mathrm{End}(V)$ is a *Clifford map*, which is *compatible with* g, if it satisfies

$$\rho(\alpha)\rho(\alpha) = -g(\alpha, \alpha)\mathrm{id}_V, \qquad \alpha \in C^\infty(T^*X). \tag{2.1}$$

Equation (2.1) and the universal property of Clifford algebras [49, Proposition I.1.1] imply that ρ extends to a real algebra homomorphism $\rho : \mathrm{C}\ell(T^*X) \to \mathrm{End}(V)$, where $\mathrm{C}\ell(T^*X)$ is the real Clifford algebra, and a complex algebra homomorphism, $\rho : \mathbb{C}\ell(T^*X) \to \mathrm{End}(V)$, where $\mathbb{C}\ell(T^*X) = \mathrm{C}\ell(T^*X) \otimes_{\mathbb{R}} \mathbb{C}$ is the complex Clifford algebra. In particular, the pair (ρ, V) defines a complex Clifford module. Recall that $\mathrm{C}\ell(T^*X)$ is canonically isomorphic to $\Lambda^\bullet(T^*X)$ as an orthogonal vector bundle (see [49, Proposition II.3.5] or [8, Proposition 3.5]). The fibers of V are irreducible modules for the fibers of $\mathbb{C}\ell(T^*X)$, [49, Theorem I.5.8] if and only if V has complex rank 2^n, when X has dimension $2n$. If X is even-dimensional, as we shall assume from now on, the bundle V admits a splitting $V = V^+ \oplus V^-$, where the subbundles $V^\pm$ are the eigenspaces of $\rho(\mathrm{vol})$ and are irreducible $\mathrm{C}\ell^0(T^*X)$ modules [49, p. 98–99], where $\mathrm{C}\ell^0(T^*X) \cong \Lambda^{\mathrm{even}}(T^*X)$. If V is a Hermitian vector bundle, then we require that $\rho(\alpha) \in \mathrm{End}(V)$ be skew-Hermitian for all $\alpha \in C^\infty(T^*X)$ and call (ρ, V) a *Hermitian Clifford module*.

A unitary connection A on V, where (ρ, V) is a Hermitian Clifford module, defines a *Clifford module derivation* ∇_A on $C^\infty(V)$ if

$$\nabla_\eta^A(\rho(\alpha)\Phi) = \rho(\nabla_\eta \alpha)\Phi + \rho(\alpha)\nabla_\eta^A\Phi, \tag{2.2}$$

where $\alpha \in C^\infty(T^*X)$, $\eta \in C^\infty(TX)$, and $\Phi \in C^\infty(V)$ and ∇ is an orthogonal connection on T^*X. (We follow the convention of [8, Definition 3.39].) If ∇_A is a module derivation, it preserves the splitting $V = V^+ \oplus V^-$ [49, Corollary II.4.12]. We shall also let ∇ denote the canonically induced orthogonal connections on $\Lambda^\bullet = \Lambda^\bullet(T^*X)$ and the real Clifford algebra $\mathrm{C}\ell(T^*X)$ [49, Proposition II.4.8].

Conversely, if we require that ∇_A be a Clifford module derivation, then the relation (2.2) defines an orthogonal connection ∇ on T^*X [68, §6.1]. The connection A is called *spin* if ∇ is the Levi-Civita connection on T^*X.

We assume for the remainder of the article that (X, g) is an oriented, Riemannian four-manifold. The subbundles $V^\pm$ may then be characterized by requiring that $\rho(\omega)V^- = 0$ for all $\omega \in \Omega^+(X, \mathbb{R})$ and similarly for V^+.

2.1.2. *Spin^c structures.* Here we specialize to the case where the complex Clifford module has rank four.

Definition 2.1. We call $\mathfrak{s} = (\rho, W)$ a *spin^c structure* over an oriented Riemannian four-manifold (X, g) if (ρ, W) is a Hermitian Clifford module and W has complex rank four.

Given a spin^c structure $\mathfrak{s} = (\rho, W)$, one defines

$$c_1(\mathfrak{s}) = c_1(W^+). \tag{2.3}$$

If L is a complex line bundle over X, we obtain a new spin^c structure on (X, g),

$$\mathfrak{s} \otimes L = (\rho, W \otimes L), \tag{2.4}$$

with $c_1(\mathfrak{s} \otimes L) = c_1(\mathfrak{s}) + 2c_1(L)$. If $\mathfrak{s}$, $\mathfrak{s}'$ are any two spin^c structures on (X, g) then one has $\mathfrak{s}' = \mathfrak{s} \otimes L$, for a complex line bundle L on X uniquely determined by $\mathfrak{s}$ and $\mathfrak{s}'$ [68].

There is a canonical isomorphism of orthogonal vector bundles,

$$\mathfrak{u}(W) \cong i\underline{\mathbb{R}} \oplus \mathfrak{su}(W).$$

where we abbreviate the trivial real line subbundle $i\mathbb{R}\,\mathrm{id}_W \subset \mathfrak{u}(W)$ by $i\underline{\mathbb{R}} = X \times i\mathbb{R}$. The Clifford multiplication ρ defines canonical isomorphisms $\Lambda^\pm \cong \mathfrak{su}(W^\pm)$, where $\Lambda^\pm = \Lambda^\pm(T^*X)$ are the bundles of self-dual and anti-self-dual two-forms, with respect to the Riemannian metric g on T^*X. A unitary connection ∇_A on W determines a unitary connection on $\det(W)$; conversely, a choice of orthogonal connection on $\mathfrak{su}(W)$ and a unitary connection on $\det(W)$ uniquely determine a unitary connection on W [24, §2.1.1]. In particular, any two unitary connections on W, which are both spinorial with respect to ∇, differ by an element of $\Omega^1(X, i\mathbb{R})$, since the induced connection on $\mathfrak{su}(W) \cong \Lambda^2$ is constrained by the choice of ∇ on T^*X. The Dirac operator $D = \rho \circ \nabla$ on $C^\infty(W)$ is not self-adjoint unless the connection ∇ on T^*X is torsion-free—that is, ∇ is the Levi-Civita connection—as one can see from examples.

For $k \geq 2$, we let $\mathcal{A}_\mathfrak{s}$ denote the space of L^2_k spin connections on W. From the preceding remarks (see [68, Lemma 6.1] for details), $\mathcal{A}_\mathfrak{s}$ is an affine space for the Hilbert space $L^2_k(X, i\Lambda^1)$ of imaginary one-forms: following the convention of [43, §2(i)], the action is given by

$$(B, b) \mapsto B + b\,\mathrm{id}_W. \tag{2.5}$$

Thus, denoting the trace on two-by-two complex matrices by Tr, one has

$$\operatorname{Tr}(F_{B+b}) = \operatorname{Tr}(F_B) + 2db \quad \text{and} \quad D_{B+b} = D_B + \rho(b), \tag{2.6}$$

as we later use when describing the deformation complex for the Seiberg-Witten equations.

2.1.3. *Spin*^{u} *structures.* An elegant reformulation of the PU(2) monopole equations, as discussed in [64] and [24], has been described by Mrowka [59], based on his joint work with Kronheimer and, motivated by his comments, we shall give a more invariant definition of PU(2) monopoles than the one we presented in [24].

Definition 2.2. We call $\mathfrak{t} = (\rho, V)$ a *spin*^{u} *structure* over an oriented Riemannian four-manifold (X, g) if (ρ, V) is a Hermitian Clifford module and V has complex rank eight.

In the case of even-dimensional manifolds X with spin structures and complex modules (of arbitrary dimension) for the real Clifford algebra, $\mathbb{C}\ell(T^*X)$, the following result appears as Proposition 3.35 in [8].

Lemma 2.3. *Let W and V be complex Clifford modules over a Riemannian four-manifold X, of rank four and eight respectively. Then there is a rank-two complex vector bundle E over X, unique up to isomorphism, and an isomorphism of complex Clifford modules,*

$$V \cong W \otimes E.$$

If W and V are Hermitian Clifford modules, then the bundle E can be assumed Hermitian and the isomorphism $V \cong W \otimes E$ can be taken to be an isomorphism of Hermitian Clifford modules.

Proof. Let Δ be an irreducible $\mathbb{C}\ell(\mathbb{R}^4)$ module, recalling that any such module is unique up to equivalence [49, Theorem I.5.7], with $\mathbb{C}\ell(\mathbb{R}^4) \cong M_4(\mathbb{C})$ as a complex algebra acting on $\Delta \cong \mathbb{C}^4$ by the standard representation. (We use $M_d(\mathbb{C})$ to denote the algebra of complex $d \times d$ matrices.) Hence, the only $\mathbb{C}\ell(\mathbb{R}^4)$-module endomorphisms of Δ are given by complex scalar multiplication and we have an isomorphism of complex vector spaces, $\mathbb{C} \cong \operatorname{End}_{\mathbb{C}\ell(\mathbb{R}^4)}(\Delta)$, given by $z \mapsto z\,\mathrm{id}_\Delta$. Therefore, we obtain an isomorphism of complex vector spaces

$$\mathbb{C}^2 \cong \operatorname{Hom}_{\mathbb{C}\ell(\mathbb{R}^4)}(\Delta, \Delta \oplus \Delta), \tag{2.7}$$

where $(z_1, z_2) \in \mathbb{C}^2$ is identified with the $\mathbb{C}\ell(\mathbb{R}^4)$-module homomorphism $\Delta \to \Delta \oplus \Delta$ given by $v \mapsto (z_1 v, z_2 v)$. Indeed, the map (2.7) is surjective because we can compose any homomorphism $\Delta \to \Delta \oplus \Delta$ with projection onto each factor and then use the fact that $\operatorname{End}_{\mathbb{C}\ell(\mathbb{R}^4)}(\Delta) \cong \mathbb{C}$. Moreover, the map (2.7) is injective by construction.

Given a complex Clifford module W of rank four, define

$$E = \operatorname{Hom}_{\mathbb{C}\ell(T^*X)}(W, V). \tag{2.8}$$

For every $x \in X$ and isomorphism $\mathbb{C}\ell(T^*X)|_x \cong \mathbb{C}\ell(\mathbb{R}^4)$ of complex Clifford algebras, there are isomorphisms of complex Clifford modules, $W|_x \cong \Delta$ and $V|_x \cong \Delta \oplus \Delta$. The isomorphism (2.7) then implies that E is a rank-two complex vector bundle over X. The map $W \otimes E \to V$ given by $\Phi \otimes M \to M(\Phi)$, where $\Phi \in W|_x$ and $M \in E|_x$ for some $x \in X$, is a $\mathbb{C}\ell(\mathbb{R}^4)$-module isomorphism since it is fiberwise injective, the ranks of the two bundles agree, and it is a complex Clifford module homomorphism by construction. To see that the map $W \otimes E \to V$ is injective, observe that if $M(\Phi) = 0$, then either $\Phi = 0$ or $\Phi \neq 0$ and M has a non-trivial kernel, in which case one can see from the explicit form of the map (2.7) that $M|_x$ must then be zero.

Next we show that E is unique up to isomorphism. From equation (2.9) below, we have

$$c_1(E) = \tfrac{1}{2}c_1(V^+) - c_1(W^+).$$

From equation (2.16) below we see that the bundle $\mathfrak{su}(V)$ determines the subbundle $\mathfrak{su}(E)$ up to isomorphism, independently of W, and hence $\mathfrak{su}(V)$ uniquely determines $p_1(\mathfrak{su}(E))$. Then

$$c_2(E) = -\tfrac{1}{4}(p_1(\mathfrak{su}(E)) - c_1(E)^2),$$

and E is determined up to isomorphism by $c_1(E)$ and $c_2(E)$. Finally, if W and V are Hermitian then E is Hermitian and the isomorphism $V \cong W \otimes E$ can be taken to be Hermitian. □

The presentation of V as $W \otimes E$ is not unique. If W is replaced by $W \otimes L$, where L is a complex line bundle over X, then E may be replaced by $L^{-1} \otimes E$ and so one also has $V \cong (W \otimes L) \otimes (L^{-1} \otimes E)$. By Lemma 2.3, we may always write

$$(\rho, V) = (\rho, W \otimes E),$$

for some spin^c structure $\mathfrak{s} = (\rho, W)$ and Hermitian bundle E. Although W and E are not determined by (ρ, V), the spin^u structure (ρ, V) does determine a complex line bundle, which is independent of tensor-product decomposition $V \cong W \otimes E$,

$$\det{}^{\frac{1}{2}}(V^+) = \det(W^+) \otimes \det(E), \tag{2.9}$$

noting that $\det(V^+) \cong \det(V^-)$, so $\det(V) \cong \det(V^+)^{\otimes 2}$ since $V = V^+ \oplus V^-$.

The spin^u structure (ρ, V) also defines an SO(3) bundle over X. To see this, recall that there are isometries of orthogonal vector bundles,

$$\mathfrak{u}(E) \cong i\underline{\mathbb{R}} \oplus \mathfrak{su}(E), \quad M \mapsto (\operatorname{Tr} M)\mathrm{id}_E \oplus (M - (\operatorname{Tr} M)\mathrm{id}_E), \tag{2.10}$$

where $\underline{\mathbb{R}} = \mathbb{R}\,\mathrm{id}_E$ here, and

$$\mathfrak{gl}(E) \cong i\mathfrak{u}(E) \oplus \mathfrak{u}(E), \quad M \mapsto \tfrac{1}{2}(M + M^\dagger) \oplus \tfrac{1}{2}(M - M^\dagger), \tag{2.11}$$

with similar isomorphisms for W. Since $\mathfrak{gl}(V) \cong \mathfrak{gl}(W) \otimes \mathfrak{gl}(E)$, the decomposition (2.11) yields an isometry,

$$\mathfrak{u}(V) \cong i\mathfrak{u}(W) \otimes_{\mathbb{R}} \mathfrak{u}(E). \tag{2.12}$$

Combining the identifications (2.10) and (2.12), we obtain isometries of orthogonal vector bundles, $\mathfrak{u}(V) \cong i\underline{\mathbb{R}} \oplus \mathfrak{su}(V)$ where $\underline{\mathbb{R}} = \mathbb{R}\,\mathrm{id}_V$ here, and an orthogonal decomposition,

$$\mathfrak{su}(V) \cong \mathfrak{su}(W) \oplus i\mathfrak{su}(W) \otimes \mathfrak{su}(E) \oplus \mathfrak{su}(E), \tag{2.13}$$

where we have identified $\mathbb{R}\,\mathrm{id}_W \otimes \mathfrak{su}(E)$ with $\mathfrak{su}(E)$ and $\mathfrak{su}(W) \otimes \mathbb{R}\,\mathrm{id}_E$ with $\mathfrak{su}(W)$.

Recall that the Clifford map $\rho : T^*X \to \operatorname{End}(W)$ defines isometries of orthogonal vector bundles $\rho : \Lambda^\pm \to \mathfrak{su}(W^\pm)$ and $\rho : \Lambda^2 \to \mathfrak{su}(W)$ [68, §4.8]. More generally, one has:

Lemma 2.4. *Let (ρ, W) be a spin^c structure over an oriented, Riemannian four-manifold X. Then the map $\rho : \Lambda^\bullet \otimes_{\mathbb{R}} \mathbb{C} \to \operatorname{End}(W)$ defined by composing the isomorphism $\Lambda^\bullet \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\ell(T^*X)$ and the Clifford representation $\mathbb{C}\ell(T^*X) \to \operatorname{End}(W)$ yields an isometric isomorphism of orthogonal vector bundles,*

$$\Lambda^1 \oplus \Lambda^2 \oplus i(\Lambda^3 \oplus \Lambda^4) \cong \mathfrak{su}(W). \tag{2.14}$$

Proof. The isomorphism $\Lambda^\bullet \cong \mathrm{C}\ell(T^*X)$ is given explicitly by [8, Proposition 3.5]

$$e^{i_1}\wedge\cdots\wedge e^{i_p} \mapsto \rho(e^{i_1})\cdots\rho(e^{i_p}),$$

if $\{e^1,\dots,e^4\}$ is a local oriented, orthonormal frame for T^*X. Using this isomorphism and the fact that $\rho(\alpha)^\dagger = -\rho(\alpha)$ when $\alpha \in \Omega^1(X,\mathbb{R})$, it is easy to see that the left-hand side of (2.14) is mapped into $\mathfrak{u}(W)$, the skew-Hermitian endomorphisms of W. The elements of $\rho(\Lambda^1)$ and $\rho(i\Lambda^3)$ give isomorphisms in $\mathrm{Hom}(W^\pm, W^\mp)$ and their block-matrix representations in $\mathrm{End}(W^+\oplus W^-)$ have zero components in $\mathrm{End}(W^\pm)$, so they have zero trace. As remarked earlier, the elements of $\rho(\Lambda^2)$ are known to have zero trace. Next, $\rho(\mathrm{vol})|_{W^\pm} = \pm\mathrm{id}_W$ by definition of $W^\pm$ and thus $\rho(i\Lambda^4)$ has zero trace. Hence the left-hand side of (2.14) is also mapped into $\mathfrak{sl}(W)$, the traceless endomorphisms of W.

The map $\mathbb{C}\ell(T^*X)\to\mathrm{End}(W)$ is an isomorphism [8, Proposition 3.19] of Clifford modules and in particular is injective. Since the sum of the ranks of the bundles on the left-hand side of (2.14) is 15, which is equal to the rank of $\mathfrak{su}(W)$, we see that the map (2.14) must be an isomorphism, as claimed. □

Note that $\rho:\Lambda^\bullet\otimes_{\mathbb{R}}\mathbb{C}$ assigns $\rho(1)=\mathrm{id}_V$. For convenience, we define

$$\Lambda^\sharp = \Lambda^1\oplus\Lambda^2\oplus i(\Lambda^3\oplus\Lambda^4). \tag{2.15}$$

From the decomposition (2.13), we see that the Clifford map $\rho:\Lambda^\sharp\to\mathfrak{su}(V)$ embeds $\Lambda^\sharp$ as a subbundle of $\mathfrak{su}(V)$ and $\Lambda^\pm\cong\mathfrak{su}(W^\pm)$ as subbundles of $\mathfrak{su}(V^\pm)$ via $\omega\mapsto\rho(\omega)$. In particular, the orthogonal decomposition (2.13) is equivalent to

$$\mathfrak{su}(V)\cong\rho(\Lambda^\sharp)\oplus i\rho(\Lambda^\sharp)\otimes\mathfrak{su}(E)\oplus\mathfrak{su}(E). \tag{2.16}$$

and, upon restriction to $\mathfrak{su}(V^\pm)$,

$$\mathfrak{su}(V^\pm)\cong\rho(\Lambda^\pm)\oplus i\rho(\Lambda^\pm)\otimes\mathfrak{su}(E)\oplus\mathfrak{su}(E). \tag{2.17}$$

Plainly, the decompositions (2.16) and (2.17) are independent of the tensor-product decomposition $V\cong W\otimes E$, and so the spin^u structure $\mathfrak{t}$ determines an SO(3) bundle,

$$\mathfrak{g}_{\mathfrak{t}} = \mathfrak{su}(E). \tag{2.18}$$

Indeed, the subbundle $\mathfrak{g}_{\mathfrak{t}}\subset\mathfrak{su}(V)$ can be characterized invariantly as span of the sections $\xi\in C^\infty(\mathfrak{su}(V))$ for which $[\rho(\omega),\xi]=0$ for all $\omega\in C^\infty(\Lambda^\bullet)$.

The spin^u structure $\mathfrak{t}$ thus defines the characteristic classes,

$$c_1(\mathfrak{t}) = c_1(\det^{\frac{1}{2}}(V^+)),\quad p_1(\mathfrak{t}) = p_1(\mathfrak{g}_{\mathfrak{t}}),\quad\text{and}\quad w_2(\mathfrak{t}) = w_2(\mathfrak{g}_{\mathfrak{t}}). \tag{2.19}$$

These classes obey the constraints that $w_2(\mathfrak{t})\in H^2(X;\mathbb{Z}/2\mathbb{Z})$ has an integral lift and

$$\begin{aligned} c_1(\mathfrak{t}) - w_2(\mathfrak{t}) &\equiv w_2(X) \pmod 2,\\ p_1(\mathfrak{t}) &\equiv w_2(\mathfrak{t})^2 \pmod 4, \end{aligned} \tag{2.20}$$

recalling that $w_2(\mathfrak{su}(E))\equiv c_1(E) \pmod 2$ and $c_1(W^+)\equiv w_2(X) \pmod 2$. Note that $w_2(\mathfrak{t})$ is *not* necessarily equal to $c_1(\mathfrak{t}) \pmod 2$.

Conversely, given a triple (Λ, w, p), where $\Lambda, w\in H^2(X;\mathbb{Z})$ obey $\Lambda - w\equiv w_2(X) \pmod 2$ and $p\in\mathbb{Z}$ obeys $p\equiv w^2 \pmod 4$, there is a spin^u structure $\mathfrak{t}=(\rho,V)$ with $c_1(\mathfrak{t})=\Lambda$, $w_2(\mathfrak{t})\equiv w \pmod 2$, and $p(\mathfrak{t})=p$. Indeed, it suffices to choose a spin^c structure (ρ,W) with $c_1(W^+)=\Lambda-w$ and a U(2) bundle E with $c_1(E)=w$ and $c_2(E)=-\frac{1}{4}(p-w^2)$; one then sets $V=W\otimes E$.

2.1.4. *Spin^u structures and spin connections.* Suppose we are given a spin^u structure (ρ, V). If A is a unitary connection on V, it induces an orthogonal connection on $\mathfrak{u}(V) \cong i\underline{\mathbb{R}} \oplus \mathfrak{su}(V)$ and $\mathfrak{su}(V)$, denoted A^{ad}, by setting

$$\nabla_\eta^A(M\Phi) = (\nabla_\eta^{A^{\text{ad}}} M)\Phi + M\nabla_\eta^A \Phi, \tag{2.21}$$

where $M \in C^\infty(\mathfrak{su}(V))$, $\eta \in C^\infty(TX)$, and $\Phi \in C^\infty(V)$; more succinctly, we have

$$\nabla_\eta^{A^{\text{ad}}} M = [\nabla_\eta^A, M]. \tag{2.22}$$

Suppose A defines a Clifford module derivation ∇_A. The bundle $\mathfrak{su}(V)$ is a real Clifford module and since ∇_A is a Clifford module derivation on $C^\infty(V)$, it induces a Clifford module derivation on $C^\infty(\mathfrak{su}(V))$, denoted $\nabla_{A^{\text{ad}}}$, via equations (2.2) and (2.21), so that

$$\nabla_\eta^{A^{\text{ad}}}(\rho(\alpha)M) = \rho(\nabla_\eta \alpha)M + \rho(\alpha)\nabla_\eta^{A^{\text{ad}}} M, \tag{2.23}$$

where $\alpha \in C^\infty(T^*X)$, $\eta \in C^\infty(TX)$, and $M \in C^\infty(\mathfrak{su}(V))$. Hence, if A is a spin connection on V then A^{ad} is a spin connection on $\mathfrak{su}(V)$, inducing the Levi-Civita connection ∇ on T^*X and hence on Λ^2.

Lemma 2.5. *Let (ρ, V) be a spin^u structure over a Riemannian four-manifold (X, g) and let A be a spin connection on V. Then the induced orthogonal connection A^{ad} on $\mathfrak{su}(V)$ is spin and preserves the three subbundles in the orthogonal decomposition* (2.16) *for $\mathfrak{su}(V)$, inducing the Levi-Civita connection ∇ on the subbundle $\rho(\Lambda^\sharp)$, an $\mathrm{SO}(3)$ connection $\hat{A}$ on the subbundle $\mathfrak{su}(E)$, and the tensor-product connection $\nabla \otimes \hat{A}$ on the subbundle $i\rho(\Lambda^\sharp) \otimes \mathfrak{su}(E)$.*

Proof. The connection A^{ad} is spin by the remarks preceding the statement of the lemma. From equations (2.23) and (2.22) one can see that

$$\nabla_\eta^{A^{\text{ad}}}(\rho(\omega)\mathrm{id}_V) = \rho(\nabla_\eta \omega)\mathrm{id}_V,$$

for all $\eta \in C^\infty(T^*X)$ and $\omega \in C^\infty(\Lambda^\sharp)$, so A^{ad} preserves the subspace $\rho(\Lambda^\sharp) \subset \mathfrak{su}(V)$, inducing the Levi-Civita connection ∇.

Recall that sections ξ of $\mathfrak{su}(E) \subset \mathfrak{su}(V)$ can be characterized as sections of $\mathfrak{su}(V)$ having zero commutator with all $\rho(\omega)$, for $\omega \in C^\infty(\Lambda^\sharp)$. Thus, for any such ξ, ω, and $\eta \in C^\infty(T^*X)$, we have $[\xi, \rho(\omega)] = 0$ and

$$\begin{aligned}\nabla_\eta^{A^{\text{ad}}}[\xi, \rho(\omega)] &= [\nabla_\eta^{A^{\text{ad}}}\xi, \rho(\omega)] + [\xi, \nabla_\eta^{A^{\text{ad}}}\rho(\omega)] \\ &= [\nabla_\eta^{A^{\text{ad}}}\xi, \rho(\omega)] + [\xi, \rho(\nabla_\eta \omega)] = [\nabla_\eta^{A^{\text{ad}}}\xi, \rho(\omega)].\end{aligned}$$

Hence, $[\nabla_\eta^{A^{\text{ad}}}\xi, \rho(\omega)] = 0$ for all ω and therefore A^{ad} preserves the subspace $\mathfrak{su}(E) \subset \mathfrak{su}(V)$, on which it induces an orthogonal connection $\hat{A}$.

Sections of the subbundle $i\rho(\Lambda^\sharp) \otimes \mathfrak{su}(E)$ are linear combinations of sections of the form $i\rho(\omega)\xi$, where $\omega \in C^\infty(\Lambda^\sharp)$ and $\xi \in C^\infty(\mathfrak{su}(E))$. Since $\nabla_{A^{\text{ad}}}$ is a Clifford module derivation and induces the connections ∇ and $\hat{A}$ on the subbundles $\rho(\Lambda^\sharp)$ and $\mathfrak{su}(E)$, respectively, we have

$$\nabla_\eta^{A^{\text{ad}}}(\rho(\omega)\xi) = \rho(\nabla_\eta \omega)\xi + \rho(\omega)\nabla_\eta^{\hat{A}}\xi,$$

so the orthogonal connection induced by A on $i\rho(\Lambda^\sharp) \otimes \mathfrak{su}(E)$ is given by $\nabla \otimes \hat{A}$. □

We shall fix, once and for all, a smooth, unitary connection A_Λ on the square-root determinant line bundle, $\det^{\frac{1}{2}}(V^+)$, and henceforth require that our unitary connections A on $V = V^+ \oplus V^-$ induce the resulting unitary connection on $\det(V^+)$,

$$A^{\det} = 2A_\Lambda \text{ on } \det(V^+), \tag{2.24}$$

where we write $A^{\det}$ for the connection on $\det(V^+)$ induced by $A|_{V^+}$. If a unitary connection A on V induces a connection $A^{\det} = 2A_\Lambda$ on $\det(V^+)$, then it necessarily induces the connection A_Λ on $\det^{\frac{1}{2}}(V^+)$.

2.1.5. PU(2) *monopoles.* For $k \geq 2$, we let $\mathcal{A}_{\mathfrak{t}}$ denote the space of L^2_k spin connections on V. From the preceding subsection, $\mathcal{A}_{\mathfrak{t}}$ is an affine space for the Hilbert space $L^2_k(\Lambda^1 \otimes \mathfrak{g}_{\mathfrak{t}})$,

$$(A, a) \mapsto A + a, \tag{2.25}$$

via the inclusion (2.13) given by $\mathfrak{su}(E) \subset \mathfrak{su}(V)$, $a \mapsto \mathrm{id} \otimes a$. This descends to an action on the affine space of SO(3) connections on $\mathfrak{g}_{\mathfrak{t}}$,

$$(\hat{A}, a) \mapsto \hat{A} + \mathrm{ad}(a), \tag{2.26}$$

with $\mathrm{ad}(a) \in L^2_k(\Lambda^1 \otimes \mathfrak{so}(\mathfrak{g}_{\mathfrak{t}}))$. We have

$$\mathrm{ad}^{-1}(F_{\widehat{A+a}}) = \mathrm{ad}^{-1}(F_{\hat{A}}) + d_{\hat{A}} a + a \wedge a \quad \text{and} \quad D_{A+a} = D_A + \rho(a), \tag{2.27}$$

as we later use when describing the deformation complex for the PU(2)-monopole equations; note that the map $\mathrm{ad} : \mathfrak{g}_{\mathfrak{t}} \to \mathfrak{so}(\mathfrak{g}_{\mathfrak{t}})$ is an isomorphism.

Recall from the proof of Lemma 2.3 that the fibers of V are isomorphic to $\Delta \oplus \Delta$ as complex Clifford algebra modules, where Δ is the unique (up to equivalence) irreducible $\mathbb{C}\ell(\mathbb{R}^4)$ module. From equation (2.7) that there is an isomorphism $\mathbb{C}^2 \cong \mathrm{Hom}_{\mathbb{C}\ell(\mathbb{R}^4)}(\Delta, \Delta \oplus \Delta)$, identifying (z_1, z_2) with the homomorphism $\Delta \to \Delta \oplus \Delta$, $v \mapsto (z_1 v, z_2 v)$. Hence, there is an isomorphism of complex algebras,

$$m : M_2(\mathbb{C}) \cong \mathrm{End}_{\mathbb{C}\ell(\mathbb{R}^4)}(\Delta \oplus \Delta), \quad M \mapsto \mathrm{id}_\Delta \otimes M, \tag{2.28}$$

where $M_2(\mathbb{C})$ is the space of complex 2×2 matrices. If $u \in \mathrm{End}_{\mathbb{C}\ell(\mathbb{R}^4)}(\Delta \oplus \Delta)$, we define the *Clifford determinant* of u by

$$\det_{\mathbb{C}\ell}(u) = \det(m^{-1}(u)), \tag{2.29}$$

where $\det(m^{-1}(u))$ is the usual complex determinant of $m^{-1}(u)$ as a 2×2 complex matrix. Since the Clifford determinant (2.29) is invariant under conjugation by Clifford automorphisms of $\Delta \oplus \Delta$ and complex automorphisms of $\mathbb{C}^2$, we can therefore define the Clifford determinant, $\det_{\mathbb{C}\ell}(u)$, of a complex Clifford algebra endomorphism u of the bundle V by taking the corresponding definition on the fibers of V.

Definition 2.6. We say that u is a *spin^u automorphism* of V if it is an L^2_{k+1} unitary, complex Clifford algebra automorphism of V with Clifford-determinant one. We let $\mathcal{G}_{\mathfrak{t}}$ denote the Hilbert Lie group of spin^u automorphisms of V.

Let $\mathcal{G}^w_\kappa$ be the group of L^2_{k+1} unitary, determinant-one automorphisms of E, with Lie algebra $L^2_{k+1}(\mathfrak{su}(E))$, where $w = c_1(E)$ and $\kappa = c_2(E) - \frac{1}{4}c_1(E)^2$. While at first glance it might seem more natural to relax the requirement that $u \in \mathcal{G}_{\mathfrak{t}}$ have Clifford-determinant one to that of complex-determinant one, our refinement gives us a useful identification of $\mathcal{G}_{\mathfrak{t}}$ with $\mathcal{G}^w_\kappa$:

Lemma 2.7. *Suppose (ρ, V) is a spin^u structure with $V = W \otimes E$, for some spin^c structure (ρ, W) and rank-two Hermitian bundle E. Then the following map is an isomorphism of Hilbert Lie groups:*

$$m : \mathcal{G}^w_\kappa \cong \mathcal{G}_{\mathfrak{t}}, \quad u \mapsto \mathrm{id}_W \otimes u. \tag{2.30}$$

Proof. Plainly, the map is an injective homomorphism, so it remains to show that it is surjective. Suppose $v \in \mathcal{G}_{\mathfrak{t}}$. For any $x \in X$, the remarks preceding Definition 2.6 imply that we may write $v|_x = \mathrm{id}_{W_x} \otimes u_x$, where $u_x \in \mathrm{End}_{\mathbb{C}}(E_x)$ and so we have $v = \mathrm{id}_W \otimes u$ for some $u \in \mathrm{End}_{\mathbb{C}}(E)$. But $\det(u) = \det_{C\ell}(v) = 1$ and as $\mathrm{id}_W \otimes u^\dagger u = v^\dagger v = \mathrm{id}_V$, we must have $u^\dagger u = \mathrm{id}_E$ and thus $u \in \mathcal{G}_\kappa^w$, as desired. □

Consequently, the Hilbert Lie group $\mathcal{G}_{\mathfrak{t}}$ has Lie algebra $L^2_{k+1}(\mathfrak{g}_{\mathfrak{t}}) \subset L^2_{k+1}(\mathfrak{su}(V))$. Note that if u is a spin^u automorphism of V, then $\mathrm{Ad}\,(u)$ preserves three factors in the orthogonal decomposition (2.17) of $\mathfrak{su}(V)$; it acts as the identity on the subbundle $\rho(\Lambda^2)$, as an orthogonal gauge transformation $\hat{u}$ on $\mathfrak{su}(E)$, and as $\mathrm{id}\otimes\hat{u}$ on the subbundle $i\rho(\Lambda^2)\otimes\mathfrak{su}(E)$.

For an L^2_k section Φ of V^+, we let Φ^* denote its pointwise Hermitian dual and let $(\Phi\otimes\Phi^*)_{00}$ be the component of $\Phi\otimes\Phi^* \in i\mathfrak{u}(V^+)$ which lies in the factor $\mathfrak{su}(W^+)\otimes\mathfrak{su}(E)$ of the decomposition (2.13) of $i\mathfrak{u}(V^+) \cong \underline{\mathbb{R}} \oplus i\mathfrak{su}(V^+)$ (with V^+ in place of V). The Clifford multiplication ρ defines an isomorphism $\rho : \Lambda^+ \to \mathfrak{su}(W^+)$ and thus an isomorphism $\rho = \rho\otimes\mathrm{id}_{\mathfrak{su}(E)}$ of $\Lambda^+\otimes\mathfrak{su}(E)$ with $\mathfrak{su}(W^+)\otimes\mathfrak{su}(E)$.

The pre-configuration space of pairs on V is given by

$$\tilde{\mathcal{C}}_{\mathfrak{t}} = \mathcal{A}_{\mathfrak{t}} \times L^2_k(V^+), \tag{2.31}$$

with tangent spaces $L^2_k(\mathfrak{g}_{\mathfrak{t}}) \oplus L^2_k(V^+)$. We call a pair $(A,\Phi) \in \tilde{\mathcal{C}}_{\mathfrak{t}}$ a $\mathrm{PU}(2)$ *monopole* if

$$\mathfrak{S}(A,\Phi) = \begin{pmatrix} \mathrm{ad}^{-1}(F^+_{\hat{A}}) - \tau\rho^{-1}(\Phi\otimes\Phi^*)_{00} \\ D_A\Phi + \rho(\vartheta)\Phi \end{pmatrix} = 0, \tag{2.32}$$

where $F^+_{\hat{A}} \in L^2_{k-1}(\Lambda^+\otimes\mathfrak{so}(\mathfrak{g}_{\mathfrak{t}}))$ is the self-dual component of the curvature $F_{\hat{A}}$ of $\hat{A}$ while $D_A = \rho\circ\nabla_A : L^2_k(V^+) \to L^2_{k-1}(V^-)$ is the Dirac operator, $\tau \in L^2_{k+1}(X, \mathrm{GL}(\Lambda^+))$ and $\vartheta \in L^2_{k+1}(\Lambda^1\otimes\mathbb{C})$ are perturbation parameters. We let

$$\mathcal{M}_{\mathfrak{t}} = \{[A,\Phi] \in \mathcal{C}_{\mathfrak{t}} : \ (A,\Phi) \text{ satisfies (2.32)}\}, \tag{2.33}$$

be the moduli space of solutions to (2.32) cut out of the configuration space,

$$\mathcal{C}_{\mathfrak{t}} = \tilde{\mathcal{C}}_{\mathfrak{t}}/\mathcal{G}_{\mathfrak{t}}, \tag{2.34}$$

where $u \in \mathcal{G}_{\mathfrak{t}}$ acts by $u(A,\Phi) = (u_*A, u\Phi)$. The linearization of the map $\mathcal{G}_{\mathfrak{t}} \to \tilde{\mathcal{C}}_{\mathfrak{t}}$, $u \mapsto u(A,\Phi)$, at $\mathrm{id}_V \in \mathcal{G}_{\mathfrak{t}}$, is given by

$$\zeta \mapsto -d^0_{A,\Phi}\zeta = (-d_{\hat{A}}\zeta, \zeta\Phi), \tag{2.35}$$

with L^2-adjoint $d^{0,*}_{A,\Phi}$, where $\zeta \in L^2_{k+1}(\mathfrak{g}_{\mathfrak{t}})$.

The circle S^1 defines a family of unitary gauge transformations on V acting by scalar multiplication, so that

$$S^1 \times \tilde{\mathcal{C}}_{\mathfrak{t}} \to \tilde{\mathcal{C}}_{\mathfrak{t}}, \quad (e^{i\theta}, (A,\Phi)) \mapsto (A, e^{i\theta}\Phi). \tag{2.36}$$

Because scalar multiplication by S^1 commutes with $\mathcal{G}_{\mathfrak{t}}$, the action (2.36) descends to an action on $\mathcal{C}_{\mathfrak{t}}$ and on $\mathcal{M}_{\mathfrak{t}}$.

Remark 2.8. Note that we break here with our former convention [24], [23] of considering $\mathcal{M}_{\mathfrak{t}}$ and $\mathcal{C}_{\mathfrak{t}}$ as quotients by $S^1\times_{\{\pm1\}}\mathcal{G}_{\mathfrak{t}}$, rather than $\mathcal{G}_{\mathfrak{t}}$ as defined here.

We call a spin connection A on V *reducible* if it splits as a direct sum of connections on $V = W\oplus W'$, where $\mathfrak{s} = (\rho, W)$ and $\mathfrak{s}' = (\rho, W')$ are spin^c structures, and call A *irreducible* otherwise. We write $\mathfrak{t} = \mathfrak{s}\oplus\mathfrak{s}'$ and $A = B\oplus B$, for the induced spin connections B on W and B' on W'.

We call a connection $\hat{A}$ on an SO(3) bundle F *reducible* if it splits as a direct sum $d_{\mathbb{R}} \oplus A_L$ on $F = i\underline{\mathbb{R}} \oplus L$, where A_L is a unitary connection on a complex line bundle L over X and $d_{\mathbb{R}}$ is the product connection on $i\underline{\mathbb{R}}$, and call $\hat{A}$ *irreducible* otherwise. The following lemma summarizes the relationship between these notions of reducibility:

Lemma 2.9. *Let (ρ, V) be a spin^u structure on X. Then a spin connection A on V is reducible with respect to a splitting $V = W \oplus W'$, where (ρ, W) and (ρ, W') are spin^c structures, if and only if the induced* SO(3) *connection $\hat{A}$ on $\mathfrak{g}_{\mathfrak{t}}$ is reducible with respect to a splitting $\mathfrak{g}_{\mathfrak{t}} = i\underline{\mathbb{R}} \oplus L$, where L is a complex line bundle over X such that $W' = W \otimes L$. If $A = B \oplus B'$ is reducible, then there is a unitary connection A_L on L such that*

$$B' = B \otimes A_L \quad \textit{and} \quad \hat{A} = d_{\mathbb{R}} \oplus A_L. \tag{2.37}$$

If $A^{\det} = 2A_{\Lambda}$ on $\det(V^+)$, *then*

$$A_L = A_{\Lambda} \otimes (B^{\det})^*. \tag{2.38}$$

Remark 2.10. We write $B^{\det}$ for the connection on $\det(W^+)$ induced by $B|_{W^+}$, where B is a spin connection on $W = W^+ \oplus W^-$.

Proof. Suppose that $A = B \oplus B'$ is a reducible spin connection on V with respect to the splitting $V = W \oplus W'$. There is a unique complex line bundle L over X such that $W' = W \otimes L$. If A_L is a unitary connection on L, then both B' and $B \otimes A_L$ are spin connections on W'. But any two spin connections on W' differ by an element of $\Omega^1(X, i\mathbb{R})$, via the action (2.5), and thus we can write $B' = B \otimes A_L$ for some unitary connection A_L on L.

The unitary connection $\tilde{A} = d_{\mathbb{C}} \oplus A_L$ on $E = \underline{\mathbb{C}} \oplus L$ induces the SO(3) connection $d_{\mathbb{R}} \oplus A_L$ on $\mathfrak{su}(E) \cong i\underline{\mathbb{R}} \oplus L$. To see this, we pass to a local trivialization of E and view $\tilde{A}$ as a connection matrix one-form,

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & A_L \end{pmatrix} \in \Omega^1(\mathfrak{u}(E)).$$

The matrix $\tilde{A}$ acts on a section ξ of $\mathfrak{su}(E)$ via the adjoint representation,

$$\hat{A} \cdot \xi = \operatorname{ad}(\tilde{A})\xi = [\tilde{A}, \xi] = \begin{pmatrix} 0 & -\bar{A}_L \bar{z} \\ A_L z & 0 \end{pmatrix}, \text{ where } \xi = \begin{pmatrix} \nu & -\bar{z} \\ z & -\nu \end{pmatrix} \in C^\infty(\mathfrak{su}(E)).$$

Thus, $\hat{A} \cdot (\nu, z) = (0, A_L z)$, with respect to the isomorphism $\mathfrak{su}(E) \cong i\underline{\mathbb{R}} \oplus L$ of Lemma 3.10 and so, as a connection, we see that $\hat{A} = d_{\mathbb{R}} \oplus A_L$.

Therefore, the spin connection $A = B \otimes (d_{\mathbb{C}} \oplus A_L)$ on $W \otimes E$ induces the SO(3) connection $d_{\mathbb{R}} \oplus A_L$ on $\mathfrak{su}(E) \cong i\underline{\mathbb{R}} \oplus L$ via the decomposition (2.13),

$$\mathfrak{su}(V) \cong \mathfrak{su}(E) \oplus \mathfrak{su}(W) \oplus \mathfrak{su}(W) \otimes \mathfrak{su}(E).$$

The connections B on W and $d_{\mathbb{C}} \oplus A_L$ on E induce the connection $B^{\det} \otimes A_L$ on $\det(W^+) \otimes \det(E) = \det^{\frac{1}{2}}(V^+)$. By convention, our spin connections A on V induce the connection $2A_{\Lambda}$ on $\det(V^+)$, and thus $A_{\Lambda} = B^{\det} \otimes A_L$.

Conversely, if $\hat{A} = d_{\mathbb{R}} \oplus A_L$ is an SO(3) connection which is reducible with respect to the splitting $\mathfrak{su}(E) = i\underline{\mathbb{R}} \oplus L$, then $\hat{A}$ lifts to a U(2) connection $d_{\mathbb{C}} \oplus A_L$ on E which is reducible with respect to the splitting $E = \underline{\mathbb{C}} \oplus L$, and lifts to a spin connection $A = B \otimes (d_{\mathbb{C}} \oplus A_L) = B \oplus B'$ on V which is reducible with respect to the splitting $W \otimes (\underline{\mathbb{C}} \oplus L) = W \oplus W \otimes L$. $\square$

We say that a pair $(A,\Phi)\in\tilde{\mathcal{C}}_{\mathfrak{t}}$ is irreducible (respectively, reducible) if the connection A is irreducible (respectively, reducible). We let $\mathcal{C}_{\mathfrak{t}}^*\subset\mathcal{C}_{\mathfrak{t}}$ be the open subspace of gauge-equivalence classes of irreducible pairs. If $\Phi\equiv 0$ on X, we call (A,Φ) a *zero-section* pair. We let $\mathcal{C}_{\mathfrak{t}}^0\subset\mathcal{C}_{\mathfrak{t}}$ be the open subspace of gauge-equivalence classes of non-zero-section pairs and recall that

$$\mathcal{C}_{\mathfrak{t}}^{*,0}=\mathcal{C}_{\mathfrak{t}}^*\cap\mathcal{C}_{\mathfrak{t}}^0 \tag{2.39}$$

is a Hausdorff, Hilbert manifold [24, Proposition 2.8] represented by pairs with trivial stabilizer in $\mathcal{G}_{\mathfrak{t}}$. Let

$$\mathcal{M}_{\mathfrak{t}}^{*,0}=\mathcal{M}_{\mathfrak{t}}\cap\mathcal{C}_{\mathfrak{t}}^{*,0} \tag{2.40}$$

be the open subspace of the moduli space $\mathcal{M}_{\mathfrak{t}}$ represented by irreducible, non-zero-section PU(2) monopoles; the subspaces $\mathcal{M}_{\mathfrak{t}}^*$ and $\mathcal{M}_{\mathfrak{t}}^0$ are defined analogously.

2.1.6. *Spaces of* SO(3) *connections.* Given a class in $H^2(X;\mathbb{Z}/2\mathbb{Z})$ with an integral lift $w\in H^2(X;\mathbb{Z})$ and an integer p obeying $p^2\equiv w^2 \pmod 4$, set $\kappa=-\frac{1}{4}p$, and choose any Hermitian rank-two bundle E with $c_1(E)=w$, so $w_2(\mathfrak{su}(E))=w \pmod 2$, and $p_1(\mathfrak{su}(E))=-4\kappa$. Let $\mathcal{A}_\kappa^w(X)$ denote the affine space of L_k^2 orthogonal connections on $\mathfrak{su}(E)$ and let $\mathcal{B}_\kappa^w(X)=\mathcal{A}_\kappa^w(X)/\mathcal{G}_\kappa^w$ be the quotient, where $\mathcal{G}_\kappa^w$ acts on $\mathfrak{su}(E)$ via the adjoint action.

Let $\mathcal{A}_\kappa^{w,*}(X)$ and $\mathcal{B}_\kappa^{w,*}(X)$ be the subspace of irreducible SO(3) connections and its quotient. The moduli space of anti-self-dual connections on $\mathfrak{su}(E)$ is then defined by

$$M_\kappa^w(X)=\{[\hat{A}]\in\mathcal{B}_\kappa^w(X): F_{\hat{A}}^+=0\}, \tag{2.41}$$

with $M_\kappa^{w,*}=M_\kappa^w\cap\mathcal{B}_\kappa^{w,*}$. We follow the convention of [44] in taking the quotient by the group of determinant-one, unitary automorphisms of E rather than the group of determinant-one, orthogonal automorphisms of $\mathfrak{su}(E)$.

Lemma 2.5 implies that a spin connection A on $V=W\otimes E$ determines a unique orthogonal connection $\hat{A}$ on $\mathfrak{su}(E)$ and, conversely, that every orthogonal connection on $\mathfrak{su}(E)$ is in the image of the map $A\mapsto\hat{A}$. Hence, one can easily translate between the conventions employed in the present article and those of [24], [15], [23]:

Lemma 2.11. *Let A_Λ be the fixed unitary connection on the complex line bundle $\det^{\frac{1}{2}}(V^+)$ and let $\mathcal{A}_{\mathfrak{t}}$ denote the corresponding space of spin connections on V, as described in §2.1.4. For each* SO(3) *connection $\hat{A}$ on $\mathfrak{su}(E)$, let A denote the corresponding spin connection on V. With respect to the action of $\mathcal{G}_\kappa^w$ on $\mathcal{A}_\kappa^w$ and of $\mathcal{G}_{\mathfrak{t}}$ on $\mathcal{A}_{\mathfrak{t}}$, and the identification $\mathcal{G}_\kappa^w\cong\mathcal{G}_{\mathfrak{t}}$, the following map is a gauge-equivariant bijection:*

$$\mathcal{A}_\kappa^w\cong\mathcal{A}_{\mathfrak{t}},\quad \hat{A}\mapsto A. \tag{2.42}$$

2.2. **Uhlenbeck compactness and transversality for PU(2) monopoles.** We briefly recall our Uhlenbeck compactness and transversality results [15], [24] for the moduli space of PU(2) monopoles with the perturbations discussed in the preceding section. The moduli space of PU(2) monopoles is non-compact but has an Uhlenbeck compactification analogous to the compactification $\bar{M}_\kappa^w$ of the moduli space of anti-self-dual connections on an SO(3) bundle [14, §4.4].

Given a non-negative integer ℓ and a spin^u structure $\mathfrak{t}=(\rho,V)$, Lemma 2.3 allows us to write $V=W\otimes E$ for some choice of spin^c structure (ρ,W) and corresponding Hermitian, rank-two bundle E. Let $E_\ell\to X$ be the Hermitian, rank-two bundle with

$$c_1(E_\ell)=c_1(E)\quad\text{and}\quad c_2(E_\ell)=c_2(E)-\ell. \tag{2.43}$$

We then define a spin^u structure $\mathfrak{t}_\ell = (\rho, V_\ell)$ on X by setting

$$V_\ell = W \otimes E_\ell, \tag{2.44}$$

and observe that

$$c_1(\mathfrak{t}_\ell) = c_1(\mathfrak{t}), \quad p_1(\mathfrak{t}_\ell) = p_1(\mathfrak{t}) + 4\ell, \quad \text{and} \quad w_2(\mathfrak{t}_\ell) = w_2(\mathfrak{t}). \tag{2.45}$$

We say that a sequence of points $[A_\alpha, \Phi_\alpha]$ in $\mathcal{C}_\mathfrak{t}$ *converges* to a point $[A, \Phi, \mathbf{x}]$ in $\mathcal{C}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$ if the following hold:

- There is a sequence of $L^2_{k+1,\mathrm{loc}}$ spin^u bundle isomorphisms $u_\alpha : V|_{X \setminus \mathbf{x}} \to V_\ell|_{X \setminus \mathbf{x}}$ such that the sequence of monopoles $u_\alpha(A_\alpha, \Phi_\alpha)$ converges to (A, Φ) in $L^2_{k,\mathrm{loc}}$ over $X \setminus \mathbf{x}$, and
- The sequence of measures $|F_{A_\alpha}|^2$ converges in the weak-* topology on measures to $|F_A|^2 + 8\pi^2 \sum_{x \in \mathbf{x}} \delta(x)$.

There is a natural extension of this definition of convergence of points in $\mathcal{C}_\mathfrak{t}$ to one for sequences in the space of "ideal pairs", $\sqcup_{\ell=0}^\infty (\mathcal{C}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X))$, and this serves to define the "Uhlenbeck topology" on the space of ideal pairs. We define the topological space

$$\bar{\mathcal{M}}_\mathfrak{t} = \mathrm{Closure}(\mathcal{M}_\mathfrak{t}) \subset \bigsqcup_{\ell=0}^\infty (\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)), \tag{2.46}$$

where the closure is taken with respect to the Uhlenbeck topology on the (second countable, Hausdorff) space of *ideal* PU(2) *monopoles*, $\sqcup_{\ell=0}^\infty (\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X))$. We call the intersection of $\bar{\mathcal{M}}_\mathfrak{t}$ with $\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$ a *lower-level* of the compactification $\bar{\mathcal{M}}_\mathfrak{t}$ if $\ell > 0$ and call $\mathcal{M}_\mathfrak{t}$ the *top* or *highest level*.

Theorem 2.12. [24] *Let X be a closed, oriented, Riemannian, smooth four-manifold with spin^u structure $\mathfrak{t}$. Then there is a positive integer N, depending at most on the curvature of the fixed unitary connection on $\det(V^+)$ together with $p_1(\mathfrak{t})$, such that the topological space $\bar{\mathcal{M}}_\mathfrak{t}$ is second countable, Hausdorff, compact, and given by the closure of $\mathcal{M}_\mathfrak{t}$ in the space of ideal* PU(2) *monopoles, $\sqcup_{\ell=0}^N (\mathcal{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X))$, with respect to the Uhlenbeck topology.*

Theorem 2.12 is a special case of the more general result proved in [24] for the moduli space of solutions to the PU(2) monopole equations in the presence of holonomy perturbations. The existence of an Uhlenbeck compactification for the moduli space of solutions to the unperturbed PU(2) monopole equations (2.32) was announced by Pidstrigatch [63] and an argument was outlined in [64]. A similar argument for the equations (2.32) (without perturbations) was outlined by Okonek and Teleman in [61]. An independent proof of Uhlenbeck compactness for (2.32) and other perturbations of these equations is also given in [73].

We recall from [24, Equation (2.37)] that the elliptic deformation complex for the moduli space $\mathcal{M}_\mathfrak{t}$ is given by

$$L^2_{k+1}(\mathfrak{g}_\mathfrak{t}) \xrightarrow{d^0_{A,\Phi}} \begin{matrix} L^2_k(\Lambda^1 \otimes \mathfrak{g}_\mathfrak{t}) \\ \oplus \\ L^2_k(V^+) \end{matrix} \xrightarrow{d^1_{A,\Phi}} \begin{matrix} L^2_{k-1}(\Lambda^+ \otimes \mathfrak{g}_\mathfrak{t}) \\ \oplus \\ L^2_{k-1}(V^-) \end{matrix} \tag{2.47}$$

with elliptic deformation operator

$$\mathcal{D}_{A,\Phi} = d^{0,*}_{A,\Phi} + d^1_{A,\Phi} \tag{2.48}$$

and cohomology $H^\bullet_{A,\Phi}$. Here, $d^1_{A,\Phi}$ is the linearization at the pair (A,Φ) of the gauge-equivariant map $\mathfrak{S}$ defined by the equations (2.32), so

$$d^1_{A,\Phi}(a,\phi) = (D\mathfrak{S})_{A,\Phi}(a,\phi) = \begin{pmatrix} d^+_{\hat{A}}a - \tau\rho^{-1}(\phi\otimes\Phi^* + \Phi\otimes\phi^*)_{00} \\ (D_A+\rho(\vartheta))\phi + \rho(a)\Phi \end{pmatrix}, \tag{2.49}$$

while $-d^0_{A,\Phi}$ is the differential of the map $\mathcal{G}_{\mathfrak{t}} \to \tilde{\mathcal{C}}_{\mathfrak{t}}$, $u \mapsto u(A,\Phi) = (A-(d_Au)u^{-1}, u\Phi)$, so

$$d^0_{A,\Phi}\zeta = (d_A\zeta, \zeta\Phi). \tag{2.50}$$

The space $H^0_{A,\Phi} = \operatorname{Ker} d^0_{A,\Phi}$ is the Lie algebra of the stabilizer in $\mathcal{G}_{\mathfrak{t}}$ of a pair (A,Φ) and $H^1_{A,\Phi}$ is the Zariski tangent space to $\mathcal{M}_{\mathfrak{t}}$ at a point $[A,\Phi]$. If $H^2_{A,\Phi}=0$, then $[A,\Phi]$ is a regular point of the zero locus of the PU(2) monopole equations (2.32) on $\mathcal{C}_{\mathfrak{t}}$.

We now turn to the question of transversality. Let $D_{A,\nabla}$ and D_A be the Dirac operators on V^+ defined, respectively, by unitary connections A on V which are spin with respect to an SO(4) connection ∇ and spin, in the usual sense, with respect to the Levi-Civita connection on T^*X: the two operators differ by an element $\rho(\vartheta') \in \operatorname{Hom}(V^+,V^-)$ [15, Lemma 3.1], where $\vartheta' \in \Omega^1(X,\mathbb{C})$. Even though a unitary connection on V will not necessarily induce a torsion-free connection on T^*X for generic pairs (g,ρ) of Riemannian metrics and compatible Clifford maps, we can assume that the Dirac operator D_A in (2.32) is defined using the Levi-Civita connection for the metric g by absorbing the difference term $\rho(\vartheta')$ into the perturbation term $\rho(\vartheta)$. Given any fixed pair (g_0,ρ_0) satisfying (2.1) and automorphism $f \in C^\infty(\mathrm{GL}(T^*X))$, then $(g,\rho) = (f^*g_0, f^*\rho_0)$ is again a compatible pair; the pair (g,ρ) is *generic* if f is generic.

Theorem 2.13. [15] *Let X be a closed, oriented, smooth four-manifold with spin^u structure $\mathfrak{t}$. Then for a generic, C^∞ pair (g,ρ) satisfying* (2.1) *and generic, C^∞ parameters (τ,ϑ), the moduli space $\mathcal{M}^{*,0}_{\mathfrak{t}} = \mathcal{M}^{*,0}_{\mathfrak{t}}(g,\rho,\tau,\vartheta)$ of* PU(2) *monopoles is a smooth manifold of the expected dimension,*

$$\dim \mathcal{M}^{*,0}_{\mathfrak{t}} = d_a + 2n_a,$$

where

$$\begin{aligned} d_a(\mathfrak{t}) &= -2p_1(\mathfrak{t}) - \tfrac{3}{2}(\chi+\sigma), \\ n_a(\mathfrak{t}) &= \tfrac{1}{4}(p_1(\mathfrak{t}) + c_1(\mathfrak{t})^2 - \sigma). \end{aligned} \tag{2.51}$$

In equation (2.51), the quantity $d_a(\mathfrak{t})$ is the expected dimension of the moduli space of anti-self-dual connections on $\mathfrak{g}_{\mathfrak{t}}$, while $n_a(\mathfrak{t})$ is the complex index of the Dirac operator on $C^\infty(V^+)$. Theorem 2.13 was proved independently, using a somewhat different method, by A. Teleman [73].

2.3. **Moduli spaces of Seiberg-Witten monopoles.** We recall the definition of the space of Seiberg-Witten monopoles. Let $\mathfrak{s} = (\rho, W)$ be a spin^c structure on X and let

$$\tilde{\mathcal{C}}_{\mathfrak{s}} = \mathcal{A}_{\mathfrak{s}} \times L^2_k(W^+) \tag{2.52}$$

be the pre-configuration space of pairs (B,Ψ), with tangent space $L^2_k(X,i\mathbb{R}) \oplus L^2_k(W^+)$ due to the affine structure (2.5). We call a unitary automorphism of W a *spin^c automorphism* if it is a complex Clifford module endomorphism of W. Thus, spin^c automorphisms induce the identity on the factor $\mathfrak{su}(W) \cong \Lambda^2(T^*X)$ of the orthogonal decomposition $\mathfrak{u}(W) = i\underline{\mathbb{R}} \oplus \mathfrak{su}(W)$. We let $\mathcal{G}_{\mathfrak{s}}$ denote the group of L^2_{k+1} spin^c automorphisms of W and observe that there is an isomorphism [56, §§4.3–4.5]

$$L^2_{k+1}(X,S^1) \to \mathcal{G}_{\mathfrak{s}}, \quad s \mapsto s\cdot \mathrm{id}_W.$$

Hence, the group $\mathcal{G}_{\mathfrak{s}}$ has Lie algebra $L^2_{k+1}(X, i\mathbb{R})$ via the identification

$$L^2_{k+1}(X, i\mathbb{R}) \cong T_{\mathrm{id}}\mathcal{G}_{\mathfrak{s}}, \quad f \mapsto f \cdot \mathrm{id}_W,$$

and acts smoothly by pushforward on $\tilde{\mathcal{C}}_{\mathfrak{s}}$,

$$(s, (B, \Psi)) \mapsto (s_* B, s\Psi) = (B - s^{-1}ds, s\Psi). \tag{2.53}$$

We thus obtain a configuration space

$$\mathcal{C}_{\mathfrak{s}} = \tilde{\mathcal{C}}_{\mathfrak{s}}/\mathcal{G}_{\mathfrak{s}} \tag{2.54}$$

and let $\tilde{\mathcal{C}}^0_{\mathfrak{s}} \subset \tilde{\mathcal{C}}_{\mathfrak{s}}$ denote the open subspace of pairs (B, Ψ) with $\Psi \not\equiv 0$, where $\mathcal{G}_{\mathfrak{s}}$ acts freely on $\tilde{\mathcal{C}}^0_{\mathfrak{s}}$ with quotient $\mathcal{C}^0_{\mathfrak{s}}$, a smooth Hilbert manifold.

We call a pair $(B, \Psi) \in \tilde{\mathcal{C}}_{\mathfrak{s}}$ a *Seiberg-Witten monopole* if

$$\mathfrak{S}(B, \Psi) = \begin{pmatrix} \mathrm{Tr}(F^+_B) - \tau\rho^{-1}(\Psi \otimes \Psi^*)_0 - \eta \\ D_B\Psi + \rho(\vartheta)\Psi, \end{pmatrix} = 0, \tag{2.55}$$

where $F^+_B \in L^2_{k-1}(\Lambda^+ \otimes \mathfrak{u}(W^+))$ is the self-dual component of the curvature F_B of B and $\mathrm{Tr}(F^+_B) \in L^2_{k-1}(\Lambda^+ \otimes \mathfrak{su}(W^+))$ is the trace-free part, $D_B = \rho \circ \nabla_B : C^\infty(W^+) \to C^\infty(W^-)$ is the Dirac operator defined by the spin connection B, the perturbation terms τ and ϑ are as defined in our version of the PU(2) monopole equations (2.32), and where $\eta \in C^\infty(i\Lambda^+)$ is an additional perturbation (see Remark 2.14). The quadratic term $\Psi \otimes \Psi^*$ lies in $C^\infty(i\mathfrak{u}(W^+))$ and $(\Psi \otimes \Psi^*)_0$ denotes the traceless component lying in $C^\infty(i\mathfrak{su}(W^+))$, so $\rho^{-1}(\Psi \otimes \Psi^*)_0 \in C^\infty(i\Lambda^+)$.

Remark 2.14. We note that in the usual presentation of the Seiberg-Witten equations [43], [60], one takes $\tau = \mathrm{id}_{\Lambda^+}$ and $\vartheta = 0$, while η is a generic perturbation. However, we shall see in Lemma 3.12 that in order to identify solutions to the Seiberg-Witten equations (2.55) with reducible solutions to the PU(2) monopole equations (2.32), we need to employ the perturbations given in equation (2.55) and choose

$$\eta = F^+_{A_\Lambda}, \tag{2.56}$$

where A_Λ is the fixed unitary connection on the line bundle $\det^{\frac{1}{2}}(V^+)$ with Chern class denoted by $c_1(\mathfrak{t}) = \Lambda \in H^2(X; \mathbb{Z})$ and represented by the real two-form $(1/2\pi i)F_{A_\Lambda}$. In particular, this choice of η is *not generic* in the sense of [56, Proposition 6.3.1 & Corollary 6.3.2] if $c_1(\mathfrak{s}) - \Lambda \in H^2(X; \mathbb{Z})$ is a torsion class. If $c_1(\mathfrak{s}) - \Lambda$ is not torsion, then this class is represented by $(1/2\pi i)(\mathrm{Tr}(F_B) - F_{A_\Lambda})$ and, if $b^+_2(X) > 0$ and the metric g is generic, then there are no zero-section solutions to the Seiberg-Witten equations (2.55) by [56, Proposition 6.3.1].

The moduli space of Seiberg-Witten monopoles is defined by

$$\begin{aligned} M_{\mathfrak{s}} &= \{(B, \Psi) \in \tilde{\mathcal{C}}_{\mathfrak{s}} : \mathfrak{S}(B, \Psi) = 0\}/\mathcal{G}_{\mathfrak{s}}. \\ M^0_{\mathfrak{s}} &= M_{\mathfrak{s}} \cap \mathcal{C}^0_{\mathfrak{s}}. \end{aligned} \tag{2.57}$$

The usual Seiberg-Witten moduli space—obtained from our definition with $\tau = \mathrm{id}_{\Lambda^+}$ and $\vartheta = 0$—is compact [43, Corollary 3], [56, Proposition 6.4.1] and, for generic η, the open subspace $M^0_{\mathfrak{s}}$ is a smooth manifold of the expected dimension [43, Lemma 5], [56, Proposition 6.2.2]. In the subsections below we describe transversality and compactness properties for the space $M_{\mathfrak{s}}$, as defined here.

2.3.1. *The deformation complex.* Given $(B,\Psi)\in\tilde{\mathcal{C}}_{\mathfrak{s}}$, the smooth map $\mathcal{G}_{\mathfrak{s}}\to\tilde{\mathcal{C}}_{\mathfrak{s}}$ defined by (2.53) has differential at id_W,

$$f\mapsto -d^0_{B,\Psi}f=(-df,f\Psi),\qquad f\in L^2_{k+1}(X,i\mathbb{R}). \tag{2.58}$$

For all $(b,\psi)\in L^2_k(i\Lambda^1)\oplus L^2_k(W^+)$ and $f\in L^2_{k+1}(X,i\mathbb{R})$ we have

$$\begin{aligned}((df,-f\Psi),(b,\psi))_{L^2}&=(df,b)_{L^2}+(-f\Psi,\psi)_{L^2}\\&=(f,d^*b)_{L^2}-\left(f,\overline{\langle\Psi,\psi\rangle}\right)_{L^2}\\&=(f,d^*b)_{L^2}-(f,\langle\psi,\Psi\rangle)_{L^2}\\&=(f,d^*b)_{L^2}-(f,i\operatorname{Im}\langle\psi,\Psi\rangle)_{L^2},\end{aligned}$$

noting that $\Omega^0(X,i\mathbb{R})$ has a real L^2 inner product, and thus (compare [56, Lemma 4.5.5])

$$d^{0,*}_{B,\Psi}(b,\psi)=d^*b-i\operatorname{Im}\langle\psi,\Psi\rangle. \tag{2.59}$$

The Seiberg-Witten equations (2.55) define a $\mathcal{G}_{\mathfrak{s}}$-equivariant map $\mathfrak{S}:\tilde{\mathcal{C}}_{\mathfrak{s}}\to L^2_{k-1}(i\Lambda^+)\times L^2_{k-1}(W^-)$ with differential at the point (B,Ψ) given by

$$d^1_{B,\Psi}(b,\psi)=(D\mathfrak{S})_{B,\Psi}(b,\psi)=\begin{pmatrix}2d^+b-\tau\rho^{-1}(\psi\otimes\Psi^*+\Psi\otimes\psi^*)_0\\ D_B\psi+\rho(\vartheta)\psi+\rho(b)\Psi\end{pmatrix}. \tag{2.60}$$

Since $\mathfrak{S}(s(B,\Psi))=(\mathfrak{S}_1(B,\Psi),s\mathfrak{S}_2(B,\Psi))=s\cdot\mathfrak{S}(B,\Psi)$, the differential of the composition $\mathcal{G}_{\mathfrak{s}}\to L^2_k(i\Lambda^+)\times L^2_k(W^-)$ is given by

$$f\mapsto d^1_{B,\Psi}\circ d^0_{B,\Psi}f=(0,f\mathfrak{S}_2(B,\Psi)),\qquad f\in L^2_{k+1}(X,i\mathbb{R}),$$

and so $d^1_{B,\Psi}\circ d^0_{B,\Psi}=0$ if and only if $\mathfrak{S}_2(B,\Psi)=0$. In particular, if $\mathfrak{S}(B,\Psi)=0$ we have an elliptic deformation complex

$$L^2_{k+1}(X,i\mathbb{R})\xrightarrow{d^0_{B,\Psi}}\begin{array}{c}L^2_k(i\Lambda^1)\\ \oplus\\ L^2_k(W^+)\end{array}\xrightarrow{d^1_{B,\Psi}}\begin{array}{c}L^2_{k-1}(i\Lambda^+)\\ \oplus\\ L^2_{k-1}(W^-)\end{array} \tag{2.61}$$

with cohomology $H^\bullet_{B,\Psi}$. We write

$$\mathcal{D}_{B,\Psi}=d^{0,*}_{B,\Psi}+d^1_{B,\Psi} \tag{2.62}$$

for the rolled-up deformation operator of the complex (2.61).

The deformation complex (2.61) is the same, up to slight differences in the zeroth order terms of the differential $d^1_{B,\Psi}$, as the deformation complex [56, §4.6] for the usual Seiberg-Witten equations and so has the same index. Therefore, provided the zero locus $M^0_{\mathfrak{s}}$ of the section $\mathfrak{S}$ is regular we have

$$d_s(\mathfrak{s})=\dim M^0_{\mathfrak{s}}=\tfrac{1}{4}\left(c_1(\mathfrak{s})^2-2\chi-3\sigma\right). \tag{2.63}$$

With the deformation complex in place, we now turn to the issues of compactness and transversality.

2.3.2. *Compactness and transversality.* We first observe that the standard arguments [43], [56], [76] establishing that the Seiberg-Witten equations as stated in those references define a compact moduli space carry over to show that the moduli space $M_{\mathfrak{s}}$ defined by (2.55) is compact; the only slight change is the requirement that τ be C^0-close to the identity on Λ^+:

Proposition 2.15. *Let $\mathfrak{s} = (\rho, W)$ be a spin^c structure on a closed, oriented, Riemannian four-manifold, (X, g), parameters $\eta \in \Omega^0(i\Lambda^+)$ and $\vartheta \in \Omega^1(X, \mathbb{C})$, and perturbation $\tau \in \Omega^0(\mathrm{GL}(\Lambda^+))$ such that $\|\tau - \mathrm{id}_{\Lambda^+}\|_{C^0} < \frac{1}{64}$. Then the moduli space $M_\mathfrak{s}(g, \eta, \tau, \vartheta)$ is compact.*

Our next task is to establish an transversality result for $M^0_\mathfrak{s}(g, \eta, \tau, \vartheta)$ with generic parameter τ analogous to the one in [43] where η is generic.

Proposition 2.16. *Let $\mathfrak{s} = (\rho, W)$ be a spin^c structure on a closed, oriented, Riemannian four-manifold, (X, g). Then for any fixed parameters $\eta \in \Omega^0(i\Lambda^+)$ and $\vartheta \in \Omega^1(X, \mathbb{C})$ there is a first-category subset of $\Omega^0(\mathrm{GL}(\Lambda^+))$ such that for all C^∞ parameters τ in the complement of this subset, the moduli space $M^0_\mathfrak{s}(g, \tau, \eta, \vartheta)$ is a smooth manifold of the expected dimension.*

Proof. We closely follow the method described in [24, §5.1 & §5.2]. Our Banach space of parameters is given by

$$\mathcal{P} = C^r(\mathrm{GL}(\Lambda^+)).$$

We define an extended $\mathcal{G}_\mathfrak{s}$-equivariant map,

$$\underline{\mathfrak{S}} = (\underline{\mathfrak{S}}_1, \underline{\mathfrak{S}}_2) : \mathcal{P} \times \tilde{\mathcal{C}}_\mathfrak{s} \to L^2_{k-1}(i\Lambda^+) \times L^2_{k-1}(W^-),$$

by setting

$$\underline{\mathfrak{S}}(\tau, B, \Psi) = \begin{pmatrix} \mathrm{Tr}(F^+_B) - \tau\rho^{-1}(\Psi \otimes \Psi^*)_0 - \eta \\ D_B\Psi + \rho(\vartheta)\Psi \end{pmatrix}.$$

The parametrized moduli space $\mathfrak{M}_\mathfrak{s}$ is then $(\underline{\mathfrak{S}})^{-1}(0)/\mathcal{G}_\mathfrak{s} \subset \mathcal{C}_\mathfrak{s}$ and $\mathfrak{M}^0_\mathfrak{s} = \mathfrak{M}_\mathfrak{s} \cap (\mathcal{P} \times \mathcal{C}^0_\mathfrak{s})$. The map $\underline{\mathfrak{S}}$ has differential at the point $(\tau, [B, \Psi]) \in \mathfrak{M}^0_\mathfrak{s}$ given by

$$D\underline{\mathfrak{S}}(\delta\tau, b, \psi) = \begin{pmatrix} 2d^+b - \tau\rho^{-1}(\psi \otimes \Psi^* + \psi \otimes \Psi^*)_0 + (\delta\tau)\tau\rho^{-1}(\Psi \otimes \Psi^*)_0 \\ (D_B + \rho(\vartheta))\psi + \rho(b)\psi \end{pmatrix},$$

where $(b, \psi) \in L^2_k(i\Lambda^1) \oplus L^2_k(W^+)$. Suppose

$$(c, \varphi) \in L^2_{k-1}(i\Lambda^+) \oplus L^2_{k-1}(W^-)$$

is L^2 orthogonal to the image of $D\underline{\mathfrak{S}}$. We may assume without loss of generality that (τ, B, Ψ) is a C^r representative of the point $(\tau, [B, \Psi])$ (see, for example, [24, Proposition 3.7]) and so, as $(D\underline{\mathfrak{S}})(D\underline{\mathfrak{S}})^*(c, \varphi) = 0$, elliptic regularity implies that (c, φ) is C^r. Then

$$0 = (D\underline{\mathfrak{S}}(\delta\tau, 0, 0), (c, \varphi))_{L^2} = ((\delta\tau)\tau(\Psi \otimes \Psi^*)_0, c)_{L^2}, \tag{2.64}$$

for all $\delta\tau \in C^r(\mathfrak{gl}(\Lambda^+))$, which yields the pointwise identity

$$\langle (\delta\tau)_x\tau_x(\Psi_x \otimes \Psi^*_x)_0, c_x\rangle_x = 0, \qquad x \in X. \tag{2.65}$$

If $c_x \neq 0$ for some $x \in X$ (and thus $c \neq 0$ on a non-empty open neighborhood in X), then (2.65) implies that $\Psi = 0$ on the non-empty open subset $\{c \neq 0\} \subset X$. Aronszajn's theorem then implies that $\Psi \equiv 0$ on X, contradicting our assumption that (τ, B, Ψ) is a point in $\mathfrak{M}^0_\mathfrak{s}$. Hence, we must have $c \equiv 0$ on X, so (2.64) now yields

$$0 = (D\underline{\mathfrak{S}}(0, b, 0), (0, \varphi))_{L^2} = (\rho(b)\Psi, \varphi)_{L^2},$$

for all $b \in L^2_k(i\Lambda^1)$. Since $\Psi \not\equiv 0$, this implies that $\varphi \equiv 0$, just as in the proof of [43, Lemma 5]. Hence, $\mathrm{Coker}(D\underline{\mathfrak{S}})_{\tau,[B,\Psi]} = 0$ at each point $(\tau, [B, \Psi])$ in $(\underline{\mathfrak{S}})^{-1}(0)$ and so the parametrized moduli space $(\underline{\mathfrak{S}})^{-1}(0) = \mathfrak{M}^0_\mathfrak{s}$ is a smooth Banach submanifold of $\mathcal{P} \times \mathcal{C}^0_\mathfrak{s}$. Therefore, the Sard-Smale theorem, in the form of [14, Proposition 4.3.11] (see the proof of Corollary 5.3 in [24]), implies that there is a first-category subset of $C^r(\mathrm{GL}(\Lambda^+))$ such that for all C^r parameters τ in the complement of this subset, the zero locus $M^0_\mathfrak{s}(\tau)$ is a regular submanifold of $\mathcal{C}^0_\mathfrak{s}$. Lastly, we can constrain the parameter τ to be C^∞ (and not just C^r) by the argument used in [24, §5.1.2]—see [68, §8.4].) $\square$

The following lemma relates the Seiberg-Witten moduli spaces defined with the perturbations of [43], [56], [76] to those used in this article.

Lemma 2.17. *Let (X,g) be a closed, oriented, Riemannian four-manifold with $b_2^+(X)>0$ and spin^c structure $\mathfrak{s}$.*

1. *Suppose that the moduli spaces $M_{\mathfrak{s}}^0(g,\mathrm{id}_{\Lambda^+},\eta',0)$ and $M_{\mathfrak{s}}^0(g,\tau,\eta,\vartheta)$ are regular zero loci of the Seiberg-Witten maps $\underline{\mathfrak{S}}$ in (2.55) for these perturbation parameters. If $b_+^2(X)>1$, then these moduli spaces are related by an oriented, smooth cobordism.*
2. *Suppose that $M_{\mathfrak{s}}(g,\mathrm{id}_{\Lambda^+},\eta',0)$ contains no zero-section solutions. Then $M_{\mathfrak{s}}(g,\tau,\eta,\vartheta)$ contains no zero-section solutions for any triple (τ,ϑ,η) which is C^r-close enough to $(\mathrm{id}_{\Lambda^+},0,\eta')$.*

Proof. If $\mathcal{P}$ denotes the Banach space of C^r perturbation parameters (τ,η,ϑ), then a generic, smooth path in $\mathcal{P}$ joining $(\mathrm{id}_{\Lambda^+},\eta',0)$ to (τ,η,ϑ) induces a smooth, oriented cobordism in $\mathcal{P}\times\mathcal{C}_{\mathfrak{s}}^0$ joining $M_{\mathfrak{s}}^0(g,\mathrm{id}_{\Lambda^+},\eta',0)$ to $M_{\mathfrak{s}}^0(g,\tau,\eta,\vartheta)$. This proves Assertion (1).

If the compact space $M_{\mathfrak{s}}(g,\mathrm{id}_{\Lambda^+},\eta',0)$ is contained in the open subset $\mathcal{C}_{\mathfrak{s}}^0\subset\mathcal{C}_{\mathfrak{s}}$, then the same holds for $M_{\mathfrak{s}}(g,\tau,\eta,\vartheta)$, that is, the latter subspace also contains no zero-section pairs, for any triple (τ,ϑ,η) which is C^r-close enough to $(\mathrm{id}_{\Lambda^+},0,\eta')$. This proves Assertion (2). □

2.4. **Cohomology ring of the configuration space of spin^c pairs.** We next compute the cohomology ring of the configuration space $\mathcal{C}_{\mathfrak{s}}^0$ and describe the cohomology classes used to define Seiberg-Witten invariants and arising in the calculation (see §3.6) of the Chern classes of certain vector bundles over $M_{\mathfrak{s}}$. For this purpose we may make use of the fact since U(1) is Abelian, we have a global slice theorem for $\mathcal{C}_{\mathfrak{s}}$ modulo the harmonic gauge transformations. Computations of the cohomology ring of $\mathcal{C}_{\mathfrak{s}}^0$ have already appeared in, for example [68, Chapter 7] and [62, Lemma 5]. The version we present here is convenient for the computation of the Chern class of the universal line bundle in Lemma 2.24.

2.4.1. *Harmonic gauge transformations.* A gauge transformation $s\in\mathrm{Map}(X,S^1)$ is *harmonic* if $s^{-1}ds\in L_k^2(i\Lambda)$ satisfies $d^*(s^{-1}ds)=0$ (we always have $d(s^{-1}ds)=0$). Every component of $\mathrm{Map}(X,S^1)$ contains a harmonic representative which is unique up to multiplication by a constant element of S^1. The harmonic elements of $\mathrm{Map}(X,S^1)$ form a subgroup,

$$\hat{\mathcal{G}}_{\mathfrak{s}}=\{s\in\mathrm{Map}(X,S^1):d^*(s^{-1}ds)=0\}\cong S^1\times H^1(X,i\mathbb{Z}), \tag{2.66}$$

where, given a point $p\in X$, the isomorphism is defined by $s\mapsto(s(p),s^{-1}ds)$. Conversely, we have a map

$$\mathbf{h}_p:H^1(X;i\mathbb{Z})\to\mathrm{Map}(X,S^1)$$

defined by setting $\mathbf{h}_p(\beta)$ equal to the unique harmonic gauge transformation satisfying $\mathbf{h}_p(\beta)(p)=1$ (see, for example, [68, §5.4]). The group $S^1\times H^1(X;i\mathbb{Z})$ acts on $\tilde{\mathcal{C}}_{\mathfrak{s}}$ by

$$\Big((e^{i\theta},\beta),(B,\Psi)\Big)\mapsto(B-\beta\,\mathrm{id}_W,e^{i\theta}\mathbf{h}_p(\beta)\Psi), \tag{2.67}$$

where $(e^{i\theta},\beta)\in S^1\times H^1(X;i\mathbb{Z})$ and $(B,\Psi)\in\tilde{\mathcal{C}}_{\mathfrak{s}}$. We then have the following global "slice result" for $\mathcal{C}_{\mathfrak{s}}$:

Lemma 2.18. *Let B_0 be an L_k^2 spin connection on W. Then the inclusion*

$$\hat{\mathcal{C}}_{\mathfrak{s}}=(B_0+(\mathrm{Ker}\,d^*)\mathrm{id}_W)\times L_k^2(W^+)\hookrightarrow\tilde{\mathcal{C}}_{\mathfrak{s}}, \tag{2.68}$$

where $\operatorname{Ker} d^* \subset L^2_k(i\Lambda^1)$, *is equivariant with respect to the action* (2.67) *of* $S^1 \times H^1(X;i\mathbb{Z}) \subset \mathcal{G}_\mathfrak{s}$ *and descends to a homeomorphism*

$$(B_0 + (\operatorname{Ker} d^*)\mathrm{id}_W) \times_{S^1\times H^1(X;i\mathbb{Z})} L^2_k(W^+) \to \mathcal{C}_\mathfrak{s},$$

and a diffeomorphism on $\hat{\mathcal{C}}^0_\mathfrak{s}$, *the complement in* $\hat{\mathcal{C}}_\mathfrak{s}$ *of the space of zero-section pairs.*

Proof. Any L^2_k spin connection on W can be written as $B = B_0 + b\,\mathrm{id}_W$, where $b \in L^2_k(i\Lambda^1)$. An element s of the gauge group $\mathcal{G}_\mathfrak{s}$ acts on this representation of B by sending b to $b - s^{-1}ds$. The argument in [68, §7.2] (also see [14, pp. 54–55]) implies that there is a solution of the equation $d^*(b - s^{-1}ds) = 0$, which is unique up to a harmonic gauge transformation, so the map on quotients is surjective. If (B_j, Ψ_j), $j = 1,2$, are pairs in $\tilde{\mathcal{C}}_\mathfrak{s}$ such that $d^*(B_j - B_0) = 0$ and $s \in \mathcal{G}_\mathfrak{s}$ satisfies $s(B_1,\Psi_1) = (B_2,\Psi_2)$, that is

$$(B_1 - s^{-1}ds\,\mathrm{id}_W, s\Psi_1) = (B_2,\Psi_2),$$

then $d^*(B_2 - B_1) = 0$ implies $d^*(s^{-1}ds) = 0$, so s is harmonic and the map on quotients is injective. □

2.4.2. *Cohomology ring of the configuration space.* We now use the reduction of the gauge group from $\mathcal{G}_\mathfrak{s}$to $\hat{\mathcal{G}}_\mathfrak{s}$ to compute the integral cohomology ring of $\mathcal{C}^0_\mathfrak{s}$. We begin by defining the universal complex line bundle over $\mathcal{C}^0_\mathfrak{s} \times X$,

$$\mathbb{L}_\mathfrak{s} = \tilde{\mathcal{C}}^0_\mathfrak{s} \times_{\mathcal{G}_\mathfrak{s}} \underline{\mathbb{C}} \cong \hat{\mathcal{C}}^0_\mathfrak{s} \times_{\hat{\mathcal{G}}_\mathfrak{s}} \underline{\mathbb{C}}. \tag{2.69}$$

where $\underline{\mathbb{C}} = X \times \mathbb{C}$ and the action of $\mathcal{G}_\mathfrak{s}$ is given for $s \in \mathcal{G}_\mathfrak{s}$, $x \in X$ and $z \in \mathbb{C}$ by

$$(s,(B,\Psi),(x,z)) \to \left(s_*(B,\Psi),(x,s(x)^{-1}z)\right). \tag{2.70}$$

As in [68, §7.4], we define cohomology classes on $\mathcal{C}^0_\mathfrak{s}$ (the "Seiberg-Witten μ-classes") by

$$\mu_\mathfrak{s} : H_\bullet(X;\mathbb{R}) \to H^{2-\bullet}(\mathcal{C}^0_\mathfrak{s};\mathbb{R}), \qquad \mu_\mathfrak{s}(\alpha) = c_1(\mathbb{L}_\mathfrak{s})/\alpha, \tag{2.71}$$

where α is either the positive generator $x \in H_0(X;\mathbb{Z})$ or a class $\gamma \in H_1(X;\mathbb{R})$. Let $\mathcal{G}^p_\mathfrak{s}$ be the subgroup of maps $s \in \mathcal{G}_\mathfrak{s}$ with $s(p) = 1$ and note that

$$\mathcal{G}_\mathfrak{s} = \mathcal{G}^p_\mathfrak{s} \times S^1. \tag{2.72}$$

The following alternative characterization of $\mu_\mathfrak{s}(x)$ in appears in much of the literature on Seiberg-Witten invariants.

Lemma 2.19. *Let* $\mathcal{C}^{p,0}_\mathfrak{s} = \tilde{\mathcal{C}}^0_\mathfrak{s}/\mathcal{G}^p_\mathfrak{s}$ *be the configuration space of framed pairs. Then*

$$c_1(\mathcal{C}^{p,0}_\mathfrak{s} \times_{S^1} \mathbb{C}) = \mu_\mathfrak{s}(x). \tag{2.73}$$

Proof. The class $\mu_\mathfrak{s}(x)$ is the first Chern class of $\mathbb{L}_\mathfrak{s}$ restricted to $\mathcal{C}^0_\mathfrak{s} \times \{p\}$. Equation (2.73) then follows from the splitting $\mathcal{G}_\mathfrak{s} = \mathcal{G}^p_\mathfrak{s} \times S^1$. □

Using the isomorphism of universal line bundles in (2.69), we shall now give another description of the cohomology classes $\mu_\mathfrak{s}(x)$ and $\mu_\mathfrak{s}(\gamma)$, and show they generate the cohomology ring $H^*(\mathcal{C}^0_\mathfrak{s};\mathbb{Z})$. We define

$$\operatorname{Jac}(X) = H^1(X;i\mathbb{R})/H^1(X;i\mathbb{Z}). \tag{2.74}$$

We then have:

Lemma 2.20. *Let* $\mathfrak{s}$ *be a spin*^{c} *structure on a smooth, closed, oriented four-manifold* X. *Then, there is an* S^1*-equivariant retraction* $\mathbf{r} : \mathcal{C}^{p,0}_\mathfrak{s} \to \operatorname{Jac}(X) \times \left(L^2_k(W^+) - \{0\}\right)$ *where* S^1 *acts trivially on* $\operatorname{Jac}(X)$ *and by complex multiplication on* $L^2_k(W^+)$.

Proof. From the Hodge decomposition,

$$L^2_k(i\Lambda^1) = \mathbf{H}^1(X;i\mathbb{R}) \oplus \operatorname{Ran} d \oplus \operatorname{Ran} d^*,$$

where $\mathbf{H}^1(X;i\mathbb{R})$ denotes the harmonic, imaginary-valued one-forms on X, we see that

$$\operatorname{Ker} d^* = (\operatorname{Ran} d)^\perp = \mathbf{H}^1(X;i\mathbb{R}) \oplus \operatorname{Ran} d^*.$$

Thus, we can write the pre-configuration space $\hat{\mathcal{C}}^0_{\mathfrak{s}}$ as

$$\hat{\mathcal{C}}^0_{\mathfrak{s}} = \left(B_0 + \mathbf{H}^1(X;i\mathbb{R}) \oplus \operatorname{Ran} d^*\right) \times \left(L^2_k(W^+) - \{0\}\right).$$

As usual, the factor S^1 of the harmonic gauge group $\hat{\mathcal{G}}_{\mathfrak{s}} = S^1 \times H^1(X;i\mathbb{Z})$ acts trivially on $B_0 + \mathbf{H}^1(X;i\mathbb{R}) \oplus \operatorname{Ran} d^*$ and by complex multiplication on $L^2_k(W^+) - \{0\}$. An element $\beta \in \mathbf{H}^1(X;i\mathbb{Z}) \cong H^1(X;i\mathbb{Z})$ acts on an element $(B,\Psi) = (B_0 + b' + b'', \Psi)$ of $\hat{\mathcal{C}}^0_{\mathfrak{s}}$ (where $b' \in \mathbf{H}^1(X;i\mathbb{R})$ and $b'' \in \operatorname{Ran} d^*$) by

$$(\beta, (B_0 + b' + b'', \Psi)) \mapsto (B_0 + (b' + \beta) + b'', \mathbf{h}_p(\beta)\Psi).$$

Thus, replacing $b'' \in \operatorname{Ran} d^*$ with $tb'', 0 \le t \le 1$, defines a $\hat{\mathcal{G}}_{\mathfrak{s}}$-equivariant retraction of $\hat{\mathcal{C}}^0_{\mathfrak{s}}$ onto

$$\left(B_0 + \mathbf{H}^1(X;i\mathbb{R})\right) \times \left(L^2_k(W^+) - \{0\}\right).$$

This yields a deformation retraction of $\hat{\mathcal{C}}^0_{\mathfrak{s}}$ onto the projectivization of the complex vector bundle

$$\left(B_0 + \mathbf{H}^1(X;i\mathbb{R})\right) \times_{H^1(X;i\mathbb{Z})} L^2_k(W^+) \to \operatorname{Jac}(X). \tag{2.75}$$

By Kuiper's theorem (see Corollary 3.18) this vector bundle is trivial, which completes the proof. □

Definition 2.21. Let $\{\gamma_i\}$ be a basis for $H_1(X;\mathbb{Z})/\operatorname{Tor}$ and let $\{\gamma_i^*\}$ be the dual basis for $H^1(X;\mathbb{Z})$, so $\langle \gamma_i^*, \gamma_j\rangle = \delta_{ij}$. Then $\{\sqrt{-1}\gamma_j^*\}$ generates $\pi_1(\operatorname{Jac}(X))$. We will write these elements of $\pi_1(\operatorname{Jac}(X))$ as γ_j^J to avoid confusion. Let $\gamma_j^{J,*} \in H^1(\operatorname{Jac}(X);\mathbb{Z}) = \operatorname{Hom}(\pi_1(\operatorname{Jac}(X)),\mathbb{Z})$ be defined by $\langle \gamma_i^{J,*}, \gamma_j^J\rangle = \delta_{ij}$. Given such a basis $\{\gamma_i\}$ for $H_1(X;\mathbb{Z})/\operatorname{Tor}$, we call $\{\gamma_i^J\}$ and $\{\gamma_i^{J,*}\}$ the *related generators* and *basis* for $\pi_1(\operatorname{Jac}(X))$ and $H^1(\operatorname{Jac}(X);\mathbb{Z})$ respectively.

Note that $H^1(X;\mathbb{Z})$ and $H^1(\operatorname{Jac}(X);\mathbb{Z})$ are free $\mathbb{Z}$-modules by the universal coefficient theorem [70, Theorem 5.5.3].

Corollary 2.22. *Continue the notation and assumptions of Lemma 2.20. Let $\mathbf{r} : \mathcal{C}^0_{\mathfrak{s}} \to \operatorname{Jac}(X) \times \left(L^2_k(W^+) - \{0\}\right)/S^1$ be the retraction given by Lemma 2.20. Then the cohomology ring of $\mathcal{C}^0_{\mathfrak{s}}$ is generated by $\{\mathbf{r}^*h, \mathbf{r}^*\gamma_i^{J,*}\}$, where $\{\gamma_1^{J,*}, \dots, \gamma_{b_1}^{J,*}\}$ is a basis for $H^1(\operatorname{Jac}(X);\mathbb{Z})$ and h the first Chern class of the S^1 bundle given by*

$$\begin{array}{c} \left(B_0 + \mathbf{H}^1(X;i\mathbb{R})\right) \times_{H^1(X;i\mathbb{Z})} \left(L^2_k(W^+) - \{0\}\right) \\ \downarrow \\ \left(B_0 + \mathbf{H}^1(X;i\mathbb{R})\right) \times_{H^1(X;i\mathbb{Z})} \left(L^2_k(W^+) - \{0\}\right)/S^1. \end{array}$$

2.4.3. *Seiberg-Witten μ-classes and generators of the cohomology ring of the configuration space.* Finally, we show that the Seiberg-Witten μ-classes can be expressed in terms of the classes $\mathbf{r}^*h$ and $\mathbf{r}^*\gamma_i^*$ defined in Corollary 2.22. For this purpose, we use the complex line bundle,

$$\boldsymbol{\Delta} = \mathbf{H}^1(X;i\mathbb{R}) \times X \times_{H^1(X;i\mathbb{Z}),\mathbf{h}_p} \mathbb{C} \to \operatorname{Jac}(X) \times X, \tag{2.76}$$

whose first Chern class we now compute.

Lemma 2.23. *Continue the notation of Definition 2.21. Then* $c_1(\boldsymbol{\Delta}) = \sum_{i=1}^{b_1(X)} \gamma_i^{J,*} \times \gamma_i^*$.

Proof. Since $H^1(X;\mathbb{Z})$ and $H^1(\operatorname{Jac}(X);\mathbb{Z})$ are free, we can write $c_1(\boldsymbol{\Delta}) = c_{0,2} + c_{1,1} + c_{2,0}$ with respect to the decomposition given by the Kunneth formula:

$$H^2(\operatorname{Jac}(X)\times X;\mathbb{Z}) \cong \bigoplus_{i=0}^{2} H^i(\operatorname{Jac}(X);\mathbb{Z}) \otimes H^{2-i}(X;\mathbb{Z}),$$
$$c_{i,2-i} \in H^i(\operatorname{Jac}(X);\mathbb{Z}) \otimes H^{2-i}(X;\mathbb{Z}).$$

For $[B] \in \operatorname{Jac}(X)$ and $p \in X$, the restrictions of $\boldsymbol{\Delta}$ to $\{[B]\}\times X$ and to $\operatorname{Jac}(X)\times\{p\}$ are trivial so $c_1(\boldsymbol{\Delta}) = c_{1,1}$. We now calculate the first Chern class of $\boldsymbol{\Delta}$ restricted to the tori $\gamma_i^J \times \gamma_j$. Let $\tilde\gamma_i^J(t)$ be the path in $\mathbf{H}^1(X;i\mathbb{R})$, starting at zero and covering γ_i^J. Let $s : X \to S^1$ be an element of $\hat{\mathcal{G}}_{\mathfrak{s}}$ satisfying $s(p) = 1$ and $s^{-1}ds = \tilde\gamma_i^J(1)$. Thus, the degree of the map $s : \gamma_j \to S^1$ is δ_{ij}. The restriction of the line bundle $\boldsymbol{\Delta}$ to $\gamma_i^J \times \gamma_j$ is given by taking the trivial bundle over $I \times \gamma_j$ and making the identification

$$(0,q,z) \sim (1,q,s^{-1}(q)(z)), \quad \text{where} \quad q \in \gamma_j, z \in \mathbb{C}.$$

Since the degree of the map $s : \gamma_j \to S^1$ is δ_{ij}, we see

$$c_1(\boldsymbol{\Delta}|_{\gamma_i^J\times\gamma_j}) = \delta_{ij}\gamma_i^{J,*} \times \gamma_j^*.$$

This completes the proof. ☐

We now describe the relation between the cohomology classes of Corollary 2.22 and the $\mu_{\mathfrak{s}}$-map.

Lemma 2.24. *Let $\{\gamma_i\}$ be a basis for $H_1(X;\mathbb{Z})/\operatorname{Tor}$, let $\{\gamma_i^*\}$ be the related basis for $H^1(\operatorname{Jac}(X);\mathbb{Z})$, and let $x \in H_0(X;\mathbb{Z})$ be a generator. Let $\mathbf{r}_J : \mathcal{C}^0_{\mathfrak{s}} \to \operatorname{Jac}(X)$ be the composition of the retraction $\mathbf{r}$ defined in Lemma 2.20 with the projection to $\operatorname{Jac}(X)$. Then*

$$c_1(\mathbb{L}_{\mathfrak{s}}) = h \times 1 + (\mathbf{r}_J \times \mathrm{id}_X)^* c_1(\boldsymbol{\Delta}), \tag{2.77}$$

and thus

$$\mu_{\mathfrak{s}}(x) = \mathbf{r}^*h, \qquad \mu_{\mathfrak{s}}(\gamma_i) = \mathbf{r}_J^*\gamma_i^{J,*}.$$

Proof. Because $\mathbf{r}$ is a retraction, it suffices to compute the restriction of the universal line bundle $\mathbb{L}_{\mathfrak{s}}$ to the image of the retraction where $\mathbb{L}_{\mathfrak{s}}$ is given by

$$\mathbf{H}^1(X;i\mathbb{R}) \times \left(L^2_k(W^+) - \{0\}\right) \times_{S^1\times H^1(X;i\mathbb{Z})} \underline{\mathbb{C}}. \tag{2.78}$$

The bundle

$$\begin{array}{c} \mathbf{H}^1(X;i\mathbb{R}) \times_{H^1(X;i\mathbb{Z})} \left(L^2_k(W^+) - \{0\}\right) \times_{S^1} \mathbb{C} \\ \downarrow \\ \mathbf{H}^1(X;i\mathbb{R}) \times_{H^1(X;i\mathbb{Z})} \left(L^2_k(W^+) - \{0\}\right)/S^1 \end{array} \tag{2.79}$$

is the restriction of the framed configuration space, $\mathcal{C}_{\mathfrak{s}}^{p,0} \times_{S^1} \mathbb{C}$ to the image of $\mathbf{r}$ and thus has first Chern class h. Then the restriction of $\mathbb{L}_{\mathfrak{s}}$ in equation (2.78) is isomorphic to the tensor product of the bundle (2.79) with the pullback of the bundle $\mathbf{\Delta}$ by the map $\mathbf{r}_J \times \mathrm{id}_X$. Thus $c_1(\mathbb{L}_{\mathfrak{s}}) = h \times 1 + (\mathbf{r}_J \times \mathrm{id}_X)^* c_1(\mathbf{\Delta})$. The computations of $\mu_{\mathfrak{s}}(\beta)$ then follow from the definition of the map $\mu_{\mathfrak{s}}$ in equation (2.71) and the computation of $c_1(\mathbf{\Delta})$ in Lemma 2.23. □

3. Singularities and their links

As we shall see in this section, the moduli space $\mathcal{M}_{\mathfrak{t}}$ of solutions to the perturbed PU(2) monopole equations (2.32) is a smoothly stratified space. According to Theorem 2.13, the subspace $\mathcal{M}_{\mathfrak{t}}^{*,0}$ of solutions to (2.32) which are neither zero-section nor reducible is a smooth manifold of the expected dimension and this comprises the top stratum of $\mathcal{M}_{\mathfrak{t}}$. In §3.1 we classify the PU(2) monopoles where S^1 acts trivially, noting that these are given by the zero-section or reducible solutions and comprise the singular points of $\mathcal{M}_{\mathfrak{t}}$. We exclude the possibility of PU(2) monopoles with flat associated SO(3) connections by a suitable choice of $w_2(\mathfrak{t})$. We then analyze these lower strata of solutions in $\mathcal{M}_{\mathfrak{t}}$ which are either zero-section or reducible or both and show that they are identified with the moduli space M_κ^w of anti-self-dual connections on the associated SO(3) bundle $\mathfrak{g}_{\mathfrak{t}}$ and moduli spaces of Seiberg-Witten monopoles, $M_{\mathfrak{s}}$ (see §3.3). We describe the link in $\mathcal{M}_{\mathfrak{t}}^{*,0}$ of the stratum M_κ^w in §3.2 and the links in $\mathcal{M}_{\mathfrak{t}}^{*,0}$ of the strata $M_{\mathfrak{s}}$ in §3.4 and §3.5. For M_κ^w, it will suffice to describe the normal cone (in the sense of [35, p. 41]) at a generic point in M_κ^w, while in the case of the link of a stratum $M_{\mathfrak{s}}$, a global description is required. Finally, in §3.6 we compute the Chern character of the normal bundle of $M_{\mathfrak{s}}$ with respect to a finite-dimensional, open, S^1-equivariant smooth manifold containing a neighborhood of the image of $M_{\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}}$.

3.1. **Classification of fixed points under the circle action.** In this section we classify the fixed points in $\mathcal{M}_{\mathfrak{t}}$ under the circle action induced by scalar multiplication on V.

For pairs (A,Φ) in $\tilde{\mathcal{C}}_{\mathfrak{t}}$, it is useful to distinguish between two kinds of circle action. First, S^1 can act on V by scalar multiplication:

$$S^1 \times V \to V, \quad (e^{i\theta},\Phi) \mapsto e^{i\theta}\Phi. \tag{3.1}$$

Second, if $V = W \oplus W \otimes L$ is a direct sum of Clifford modules, then S^1 can act by scalar multiplication on the factor $W \otimes L$ and the identity on W:

$$S^1 \times V \to V, \quad (e^{i\theta},\Psi \oplus \Psi') \mapsto \Psi \oplus e^{i\theta}\Psi', \tag{3.2}$$

where $\Psi \in C^\infty(W)$ and $\Psi' \in C^\infty(W\otimes L)$. With respect to the splitting $V = W \oplus W \otimes L$, the two actions are related by

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{pmatrix} = e^{i\theta} u, \text{ where } u = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in \mathcal{G}_{\mathfrak{t}},$$

and so, when we pass to the induced circle actions on the quotient $\mathcal{C}_{\mathfrak{t}} = \tilde{\mathcal{C}}_{\mathfrak{t}}/\mathcal{G}_{\mathfrak{t}}$, we obtain the same space $\mathcal{C}_{\mathfrak{t}}/S^1$ for both circle actions.

If we write $V = W \otimes E$, where $E = \underline{\mathbb{C}} \oplus L$, the circle action (3.2) on V is equivalent to the one induced by scalar multiplication on the complex line bundle L and the trivial action on $\underline{\mathbb{C}}$. Hence, the induced circle action (3.3) on $\mathfrak{su}(E) \subset \mathfrak{su}(V)$ with respect to the decomposition (2.13) takes the form

$$S^1 \times \mathfrak{su}(E) \to \mathfrak{su}(E), \quad (e^{i\theta},(\nu,z)) \mapsto (\nu, e^{i\theta}z), \tag{3.3}$$

with respect to the isomorphism $\mathfrak{su}(E) \cong i\underline{\mathbb{R}} \oplus L$ in Lemma 3.10: the circle acts as the identity on $i\underline{\mathbb{R}}$ and by scalar multiplication on the complex line bundle L. On the other hand, the circle action (3.1) on V induces the trivial action on both $\mathfrak{su}(V)$ and $\mathfrak{su}(E)$.

Proposition 3.1. *Let $\mathfrak{t} = (\rho, V)$ be a spin^u structure over a closed, oriented, smooth four-manifold X, with $b_2^+(X) \geq 1$ and generic Riemannian metric. If $\hat{A}$ is a non-flat connection on $\mathfrak{g}_{\mathfrak{t}}$, then $[A, \Phi] \in \mathcal{M}_{\mathfrak{t}}$ is a fixed point with respect to the S^1 action on $\mathcal{M}_{\mathfrak{t}}$ if and only if one of the following hold:*

1. *The connection $\hat{A}$ is anti-self-dual, irreducible, and $\Phi \equiv 0$. The pair (A, Φ) is a fixed point with respect to the circle action (3.1).*
2. *The pair (A, Φ) is reducible with respect to a splitting $V = W \oplus W \otimes L$ but $\Phi \not\equiv 0$,*
$$(A, \Phi) = (B \oplus B \otimes A_L, \Psi \oplus 0).$$
The pair (A, Φ) is a fixed point with respect to the circle action (3.2). The connection $\hat{A}$ on $\mathfrak{g}_{\mathfrak{t}}$ is reducible with respect to the splitting $\mathfrak{g}_{\mathfrak{t}} \cong i\underline{\mathbb{R}} \oplus L$, with $\hat{A} = d_{\mathbb{R}} \oplus A_L$.

Proof. Suppose $[A, \Phi] \in \mathcal{M}_{\mathfrak{t}}$ is a fixed point of the S^1 action on $\mathcal{M}_{\mathfrak{t}}$. Consequently, there is an element $e^{i\theta} \neq \pm 1$ such that $e^{i\theta}[A, \Phi] = [A, \Phi]$ and hence a gauge transformation $u \in \mathcal{G}_{\mathfrak{t}}$ such that
$$(A, e^{i\theta}\Phi) = u(A, \Phi) = (u(A), u\Phi).$$
Thus, u is in the stabilizer of A, and hence that of $\hat{A}$, and $e^{i\theta}\Phi = u\Phi$.

(1) If $\hat{A}$ has the trivial stabilizer $\{\pm \mathrm{id}_V\}$ in $\mathcal{G}_{\mathfrak{t}}$, then $e^{i\theta}\Phi = \pm\Phi$ and so $\Phi \equiv 0$ because $e^{i\theta} \neq \pm 1$. The connection $\hat{A}$ is irreducible [29, Theorem 10.8] since it is an SO(3) connection with minimal stabilizer in $\mathcal{G}_{\mathfrak{t}}$. The curvature equation in (2.32) implies that $F^+_{\hat{A}} = 0$, so $\hat{A}$ is anti-self-dual. The pair (A, Φ) is fixed by the S^1 action (3.1).

(2) If $\hat{A}$ has non-trivial stabilizer in $\mathcal{G}_{\mathfrak{t}}$, then $\hat{A}$ is reducible with respect to a splitting $\mathfrak{g}_{\mathfrak{t}} \cong i\underline{\mathbb{R}} \oplus L$ [29, Theorem 10.8], for some complex line bundle L, and takes the form $\hat{A} = d_{\mathbb{R}} \oplus A_L$. The connection $\hat{A}$ has stabilizer $\mathrm{SO}(2) \cong S^1$ acting on L by complex multiplication and trivially on $i\underline{\mathbb{R}}$. Lemma 2.9 implies that A is reducible with respect to the splitting $V = W \oplus W \otimes L$, taking the form $A = B \oplus B \otimes A_L$.

If $u = \pm \mathrm{id}_V$, then $\Phi \equiv 0$ just as in case (1) and so $\hat{A}$ would be a reducible, anti-self-dual SO(3) connection on $\mathfrak{g}_{\mathfrak{t}}$. But $b_2^+(X) > 0$ and the Riemannian metric g on X is generic, so this possibility is excluded by Corollary 4.3.15 in [14] (the four-manifold X does not need to be simply connected).

Hence, we must have $u \neq \pm \mathrm{id}_V$ and $\Phi \not\equiv 0$. Because $\mathrm{Ad}\,(u)$ stabilizes $\hat{A}$, it induces an action on L via complex multiplication by some $e^{-i2\mu} \neq 1$ and the trivial action on $i\underline{\mathbb{R}}$, as in (3.3). Hence, with respect to the splitting $V = W \oplus W \otimes L$, the gauge transformation u takes the form
$$u = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix}.$$
Because $e^{-i\theta}u\Phi = \Phi$ and writing $\Phi = \Psi \oplus \Psi'$ with respect to the splitting $V = W \oplus W \otimes L$, the section Φ is fixed by
$$e^{-i\theta}u = \begin{pmatrix} e^{i(\mu-\theta)} & 0 \\ 0 & e^{-i(\mu+\theta)} \end{pmatrix}.$$
If $\Psi \not\equiv 0$, then $\mu = \theta \pmod{2\pi}$ and we must have $\Psi' \equiv 0$; conversely, if $\Psi' \not\equiv 0$, then $\mu = -\theta$ $\pmod{2\pi}$ and we must have $\Psi \equiv 0$. The two cases differ only by how the factors in the splitting of V are labeled, so we can assume that $\Psi' \equiv 0$ and $\Psi \in C^\infty(W^+)$. In particular, the pair (A, Φ) is fixed by the S^1 action (3.2). □

Our proof that Φ is a section of W^+ when $\hat{A}$ is a non-flat reducible connection could be replaced by an appeal to Lemma 5.22 in [24], but the argument here seems more direct.

Proposition 3.1 only classifies the fixed points $[A,\Phi] \in \mathcal{M}_{\mathfrak{t}}$ of the circle action under the assumption that $\hat{A}$ is not flat. However, the following lemma gives a simple condition which guarantees that there will be no pairs in $\mathcal{M}_{\mathfrak{t}}$ with flat associated SO(3) connections. Because it relies only on a choice of integral class $w \pmod 2$, the lemma applies simultaneously to all spin^u structures $\mathfrak{t} = (\rho, V)$ with a fixed $w_2(\mathfrak{t}) \equiv w \pmod 2$ and all oriented, orthogonal 3-plane bundles F with $w_2(F) \equiv w \pmod 2$, and thus simultaneously to all levels of the Uhlenbeck compactifications $\bar{M}_\kappa^w$ and $\bar{\mathcal{M}}_{\mathfrak{t}}$.

Lemma 3.2. [57, p. 226] *Let X be a closed, oriented four-manifold. Then*

- *If $w \in H^2(X;\mathbb{Z})$ and $e \in H_2(X;\mathbb{Z})$ is a spherical class such that $\langle w, e\rangle \not\equiv 0 \pmod 2$, then no oriented, orthogonal 3-plane bundle F over X with $w_2(F) \equiv w \pmod 2$ admits a flat connection.*
- *In particular, if $w \in H^2(X;\mathbb{Z})$ and $\tilde{X} = X\#\overline{\mathbb{CP}}^2$ is the blow-up, with exceptional class $e = [\overline{\mathbb{CP}}^1] \in H_2(X;\mathbb{Z})$ and Poincaré dual $e^* \in H^2(X;\mathbb{Z})$, then no SO(3) bundle F over $\tilde{X}$ with $w_2(F) = w + e^* \pmod 2$ admits a flat connection.*

Therefore, if X has a spherical class $e \in H_2(X;\mathbb{Z})$ such that $\langle w, e\rangle \neq 0 \pmod 2$, then Lemma 3.2 implies that there are no pairs in $\mathcal{M}_{\mathfrak{t}}$ with flat SO(3) connections when $w_2(\mathfrak{t}) \equiv w \pmod 2$. In this situation, Proposition 3.1 implies that the only fixed points of the circle action on $\mathcal{M}_{\mathfrak{t}}$ are given by either points $[A,\Phi]$ with A irreducible and $\Phi \equiv 0$ or A reducible and $\Phi \not\equiv 0$.

Suppose $w \in H^2(X;\mathbb{Z})$ is any integral class and that $\mathfrak{t} = (\rho, V)$ is a spin^u structure over $\tilde{X}$ with $w_2(\mathfrak{t}) \equiv w + e^* \pmod 2$. Then the SO(3) bundle $\mathfrak{g}_{\mathfrak{t}}$ over $\tilde{X}$ obeys the Morgan-Mrowka criterion of Lemma 3.2. Because the Donaldson and Seiberg-Witten invariants of $\tilde{X}$ determine and are determined by those of X, (see [26], [27], [57]), no information about these invariants is lost by passing to the blow-up in this way.

In Lemma 3.13 we show that the moduli space of Seiberg-Witten monopoles, $M_{\mathfrak{s}}^0$, can be identified with the subspace of PU(2) monopoles in $\mathcal{M}_{\mathfrak{t}}^0$ which are reducible with respect to the splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$, while zero-section points in $M_{\mathfrak{s}}$ are mapped to zero-section reducibles in $\mathcal{M}_{\mathfrak{t}}$ (though not necessarily injectively). If $w_2(\mathfrak{t})$ obeys the Morgan-Mrowka criterion then Proposition 3.1 implies that reducible PU(2) monopoles cannot be zero-section pairs, leading to the following

Corollary 3.3. *If $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$ is a Seiberg-Witten moduli subspace and $w_2(\mathfrak{t})$ obeys the Morgan-Mrowka criterion of Lemma 3.2, then $M_{\mathfrak{s}}$ contains no zero-section solutions.*

3.2. **The link of the stratum of anti-self-dual PU(2) monopoles.** In this section we describe the Kuranishi model for a neighborhood of a zero-section solution in the PU(2) monopole moduli space and define a link of the stratum M_κ^w in $\mathcal{M}_{\mathfrak{t}}$. The differential $D\mathfrak{S}$ will not be surjective at all the zero-section monopoles $(A,0)$, but we will show that the cokernel of $(D\mathfrak{S})_{A,0}$, namely $H^2_{A,0}$, can be identified with the cokernel of the perturbed Dirac operator, $D_A + \rho(\vartheta)$.

If we are given a decomposition, $V = W \otimes E$, Lemma 2.7 provides a canonical identification of automorphism groups, $\mathcal{G}_\kappa^w \cong \mathcal{G}_{\mathfrak{t}}$, and with respect to this identification and choice of fixed connection A_Λ on $\det^{\frac{1}{2}}(V^+)$, Lemma 2.11 provides a canonical gauge-equivariant isomorphism

$$\mathcal{A}_\kappa^w \cong \mathcal{A}_{\mathfrak{t}}, \quad \hat{A} \mapsto A,$$

where the space $\mathcal{A}_\kappa^w$ of SO(3) connections on $\mathfrak{su}(E) \cong \mathfrak{g}_\mathfrak{t}$ was defined in §2.1.6, with $p_1(\mathfrak{g}_\mathfrak{t}) = -4\kappa$ and $w_2(\mathfrak{g}_\mathfrak{t}) \equiv w \pmod 2$.

Let $\mathcal{B}_\mathfrak{t} = \mathcal{A}_\mathfrak{t}/\mathcal{G}_\mathfrak{t}$ be the quotient of the space of L_k^2 spin connections on V. By the preceding discussion there is a canonical homeomorphism,

$$\mathcal{B}_\kappa^w \cong \mathcal{B}_\mathfrak{t}, \quad [\hat{A}] \mapsto [A],$$

restricting to diffeomorphisms on smooth strata. There are a canonical smooth embedding $\mathcal{A}_\mathfrak{t} \hookrightarrow \tilde{\mathcal{C}}_\mathfrak{t}$ and a "smoothly stratified embedding" $\mathcal{B}_\mathfrak{t} \hookrightarrow \mathcal{C}_\mathfrak{t}$—that is, a topological embedding restricting to smooth embeddings on smooth strata—given by $A \mapsto (A,0)$ and $[A] \mapsto [A,0]$. Combining these identifications and embeddings, we obtain a gauge-equivariant smooth embedding and a smoothly stratified embedding,

$$\iota : \mathcal{A}_\kappa^w \hookrightarrow \tilde{\mathcal{C}}_\mathfrak{t}, \quad \hat{A} \mapsto (A,0) \quad \text{and} \quad \iota : \mathcal{B}_\kappa^w \hookrightarrow \mathcal{C}_\mathfrak{t}, \quad [\hat{A}] \mapsto [A,0]. \tag{3.4}$$

Hence, we can identify the image of the induced smoothly stratified embedding

$$\iota : M_\kappa^w(X) \hookrightarrow \mathcal{C}_\mathfrak{t}, \quad [\hat{A}] \mapsto [A,0]. \tag{3.5}$$

with the subspace of $\mathcal{M}_\mathfrak{t}$ given by the zero-section solutions to the PU(2) monopole equations (2.32). Therefore, we shall refer to pairs or connections representing points in M_κ^w as *zero-section* PU(2) *monopoles* or anti-self-dual connections, depending on the context.

For a generic Riemannian metric on X, the moduli space $M_\kappa^{w,*}$ of irreducible anti-self-dual connections is a smooth manifold [14, Corollary 4.3.18], [29] of the expected dimension $d_a(\mathfrak{t})$ given in equation (2.51).

At a zero-section PU(2) monopole $(A,0)$, the elliptic deformation complex (2.47) splits into a direct sum of complexes:

$$\begin{gathered} C^\infty(\mathfrak{g}_\mathfrak{t}) \xrightarrow{d_{\hat{A}}} C^\infty(\Lambda^1 \otimes \mathfrak{g}_\mathfrak{t}) \xrightarrow{d_{\hat{A}}^+} C^\infty(\Lambda^+ \otimes \mathfrak{g}_\mathfrak{t}), \\ C^\infty(V^+) \xrightarrow{D_A + \rho(\vartheta)} C^\infty(V^-). \end{gathered} \tag{3.6}$$

The first complex in (3.6) is simply the elliptic deformation complex for the moduli space M_κ^w of anti-self-dual connections, with cohomology $H_{\hat{A}}^\bullet$. The following lemma is then a clear consequence of the preceding decomposition:

Lemma 3.4. *If* $(A,0)$ *is a zero-section* PU(2) *monopole then there are canonical isomorphisms,*

$$\begin{aligned} H_{A,0}^0 &\cong H_{\hat{A}}^0, \\ H_{A,0}^1 &\cong H_{\hat{A}}^1 \oplus \operatorname{Ker}(D_A + \rho(\vartheta)), \\ H_{A,0}^2 &\cong H_{\hat{A}}^2 \oplus \operatorname{Coker}(D_A + \rho(\vartheta)). \end{aligned}$$

If the manifold X is not simply connected, there exist anti-self-dual connections which, while not globally reducible, become reducible when restricted to certain open sets in X: this complicates the construction of the restriction maps used in the definition of geometric representatives on moduli spaces of anti-self-dual connections in [44] and for PU(2) monopoles here. Therefore, we recall some of the technical points from [44, pp. 586–588] which we shall need to address these additional complications when X is not simply connected. A connection $\hat{A}$ on an SO(3) bundle F over X is called a *twisted reducible* if it preserves a splitting $F = \lambda \oplus N$, where λ is a non-trivial real line bundle and N is an O(2) bundle. The curvature $F_{\hat{A}}$ of a twisted reducible connection $\hat{A}$ has rank one even though $\hat{A}$ is not reducible. When restricted to open sets in X over which the real line bundle λ is trivial, the connection $\hat{A}$ becomes reducible. Let $H^i(X;\lambda)$ denote the cohomology group of dimension $b^i(\lambda)$ with coefficients in the local system given by λ. The bundle $\lambda\otimes\lambda$ is trivial and so the cup product gives a pairing $H^2(X;\lambda)\otimes H^2(X;\lambda)\to H^4(X;\mathbb{R})\cong\mathbb{R}$. We define $b^+(\lambda)$ and $b^-(\lambda)$ to be the maximum dimensions of subspaces of $H^2(X;\lambda)$ on which this pairing is respectively positive or negative definite. If F admits a reduction $\lambda\oplus N$, then the corresponding subspace of twisted reducibles in M^w_κ (where $p_1(F)=-4\kappa$ and $w_2(F)\equiv w \pmod 2$) defined by this splitting is a torus of dimension $b^1(\lambda)$. Whether or not such connections exist in M^w_κ for a generic metric depends on $b^+(\lambda)$, as described in the following lemma.

Lemma 3.5. [44, Lemma 2.4 & Corollary 2.5] *Let X be a smooth four-manifold, λ a non-trivial real line bundle on X, and F an oriented, orthogonal three-plane bundle for which F admits a reduction $F=\lambda\oplus N$. Suppose $p_1(F)=-4\kappa\neq 0$ and $w_2(F)\equiv w \pmod 2$, where $w\in H^2(X;\mathbb{Z})$.*

1. *If $b^+(\lambda)=0$, then $b^1(\lambda)=-(b^+-b^1+1)$ and M^w_κ contains twisted reducibles corresponding to $F=\lambda\oplus N$ for all Riemannian metrics on X. If, in addition, $b^1(\lambda)\geq 0$, then $-2p_1(F)-3(1-b^1+b^+)>b^1(\lambda)$ and if the Riemannian metric on X is generic, each anti-self-dual connection $\hat A$, which is a twisted reducible with respect to $F=\lambda\oplus N$, has $H^2_{\hat A}=0$.*
2. *If $b^+(\lambda)\geq 1$, then for generic Riemannian metrics on X, the space M^w_κ contains no twisted reducibles for the splitting $F=\lambda\oplus N$.*

Therefore, when $b_2^+(X)>0$, the only twisted reducibles appearing in M^w_κ have codimension at least $-2p_1(\mathfrak{g}_{\mathfrak{t}})-2(b_2^+-b_1+1)$ and are smooth points. The Kuranishi lemma, [14, Proposition 4.2.19] or [29, Lemma 4.7], gives the following description of a neighborhood of a zero-section monopole in $\mathcal{M}_{\mathfrak{t}}$.

Corollary 3.6. *Let $\mathfrak{t}$ be a spin^u structure on a closed, oriented, smooth four-manifold X with $b_2^+(X)>0$ and generic Riemannian metric. Let $[A,0]$ be a point in the image of $M^w_\kappa\hookrightarrow\mathcal{M}_{\mathfrak{t}}$, so $H^0_{\hat A}=0=H^2_{\hat A}$ for $[\hat A]\in M^w_\kappa$. Then there are*

- *An open, S^1-invariant neighborhood $\mathcal{O}_A$ of the origin in $T_{\hat A}M^w_\kappa\oplus\operatorname{Ker}(D_A+\rho(\vartheta))$ together with a smooth, S^1-equivariant embedding*
$$\boldsymbol{\gamma}_A:\mathcal{O}_A\subset T_{\hat A}M^w_\kappa\oplus\operatorname{Ker}(D_A+\rho(\vartheta))\hookrightarrow\tilde{\mathcal{C}}_{\mathfrak{t}},$$
with $\boldsymbol{\gamma}_A(0,0)=(A,0)$ and $\mathcal{M}_{\mathfrak{t}}\cap\boldsymbol{\gamma}_A(\mathcal{O}_A)$ an open neighborhood of $[A,0]$ in $\mathcal{M}_{\mathfrak{t}}$, and
- *A smooth, S^1-equivariant map*
$$\boldsymbol{\varphi}_A:\mathcal{O}_A\subset T_{\hat A}M^w_\kappa\oplus\operatorname{Ker}(D_A+\rho(\vartheta))\to\operatorname{Coker}(D_A+\rho(\vartheta))$$
such that $\boldsymbol{\gamma}_A$ restricts to an S^1-equivariant, smoothly-stratified diffeomorphism from $\boldsymbol{\varphi}_A^{-1}(0)\cap\mathcal{O}_A$ onto $\mathcal{M}_{\mathfrak{t}}\cap\boldsymbol{\gamma}_A(\mathcal{O}_A)$.

If $n_a\leq 0$, then at a generic point $[A,0]\in\mathcal{M}_{\mathfrak{t}}$ where $\operatorname{Ker}(D_A+\rho(\vartheta))=\{0\}$, (assuming the map from M^w_κ to the space of Fredholm operators of index n_a is transverse to the "jumping lines strata" as described in [41]) the Kuranishi model in Lemma 3.6 shows that a neighborhood of $[A,0]\in\mathcal{M}_{\mathfrak{t}}$ contains no elements of $M^0_{\mathfrak{t}}$. Thus, such points in the anti-self-dual moduli space are isolated from the subspace $\mathcal{M}^0_{\mathfrak{t}}$ of non-zero-section points. For this reason, we will restrict our attention to the cases where $n_a>0$.

Although we can only describe a neighborhood of the anti-self-dual connections locally, because of the problem of spectral flow, we can still introduce a global, codimension-one subspace of the compactification $\bar{\mathcal{M}}_{\mathfrak{t}}$ which will serve as a link. This space might not have a fundamental class because it is not known to have locally finite topology near the lower strata of the Uhlenbeck compactification $\bar{\mathcal{M}}_{\mathfrak{t}}$. However, we shall see that the local Kuranishi model in Corollary 3.6 will suffice to define intersections under some additional assumptions.

Definition 3.7. The *link of* $\bar{M}^w_\kappa$ in $\bar{\mathcal{M}}_{\mathfrak{t}}$ is given by

$$\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa} = \{[A,\Phi,\mathbf{x}] \in \bar{\mathcal{M}}_{\mathfrak{t}}/S^1 : \|\Phi\|^2_{L^2} = \varepsilon\}.$$

We write $\mathbf{L}^w_{\mathfrak{t},\kappa}$ when the positive constant ε is understood.

Lemma 3.8. *For generic $\varepsilon > 0$, the link $\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}$ is closed under the S^1 action, is a smoothly stratified, closed subspace of $\bar{\mathcal{M}}_{\mathfrak{t}}$, and has codimension one in every stratum of $\bar{\mathcal{M}}_{\mathfrak{t}}$ which it intersects.*

Proof. That $\mathbf{L}^{w,\varepsilon}_{\mathfrak{t},\kappa}$ is closed under the S^1 action follows directly from the definitions.

It is clear that the function

$$\boldsymbol{\ell} : \bar{\mathcal{M}}_{\mathfrak{t}} \to \mathbb{R}, \qquad [A,\Phi,\mathbf{x}] \mapsto \|\Phi\|^2_{L^2},$$

is smooth on each stratum of $\bar{\mathcal{M}}_{\mathfrak{t}}$: we claim it is continuous on $\bar{\mathcal{M}}_{\mathfrak{t}}$. Let $[A_\alpha,\Phi_\alpha]$ be a sequence of points in $\bar{\mathcal{M}}_{\mathfrak{t}}$ which converge to $[A_\infty,\Phi_\infty,\mathbf{x}]$. We may assume, by an appropriate choice of a sequence of L^2_{k+1} spin^u gauge transformations of V, that the sequence $\{\Phi_\alpha\}$ converges to Φ_∞ in C^∞ on $X - B(\mathbf{x},r)$, where we define $B(\mathbf{x},r) = \cup_{x\in\mathbf{x}} B(x,r)$. Therefore,

$$\begin{aligned}\left| \|\Phi_\alpha\|^2_{L^2(X)} - \|\Phi_\infty\|^2_{L^2(X)} \right| &\le \|\Phi_\alpha - \Phi_\infty\|^2_{L^2(X-B(\mathbf{x},r))} + \|\Phi_\alpha - \Phi_\infty\|^2_{L^2(B(\mathbf{x},r))} \\ &\le \|\Phi_\alpha - \Phi_\infty\|^2_{L^2(X-B(\mathbf{x},r))} + Cr^4,\end{aligned}$$

where the second inequality follows from the universal *a priori* C^0 bound for the sequence Φ_α and Φ_∞ given by [24, Lemma 4.4]. Thus,

$$\limsup_{\alpha\to\infty} \left| \|\Phi_\alpha\|^2_{L^2(X)} - \|\Phi_\infty\|^2_{L^2(X)} \right| \le Cr^4, \quad \text{for all } r > 0,$$

and so

$$\lim_{\alpha\to\infty} \|\Phi_\alpha\|^2_{L^2(X)} = \|\Phi_\infty\|^2_{L^2(X)},$$

as desired. A generic $\varepsilon > 0$ is a regular value for the function $\boldsymbol{\ell}$ on each smooth stratum of $\bar{\mathcal{M}}_{\mathfrak{t}}$. For such an ε, the preimage $\boldsymbol{\ell}^{-1}(\varepsilon)$ is a smooth submanifold of each stratum and because the function $\boldsymbol{\ell} : \bar{\mathcal{M}}_{\mathfrak{t}} \to \mathbb{R}$ is continuous, these smooth submanifolds fit together to form a smoothly stratified subspace of $\bar{\mathcal{M}}_{\mathfrak{t}}$ (see Remark 3.3 in [18]). □

Remark 3.9. In defining a link, it might seem more natural to work with the image of the ε-sphere in the normal bundle given by $\operatorname{Ker}(D_A + \rho(\vartheta))$, at least on the image of $M^w_\kappa \hookrightarrow \mathcal{M}_{\mathfrak{t}}$ where the cokernel of the Dirac operator vanishes. This definition would have the disadvantage of not being a global object because of the jumping-line problem (that is, spectral flow). However, it can be shown that the two functions defined on the open set $\mathcal{O}$ of the Kuranishi model of $[A,0]$ in Corollary 3.6, one given by the L^2 norm of the element of $\operatorname{Ker}(D_A + \rho(\vartheta))$, the other defined by $\boldsymbol{\ell}\circ\boldsymbol{\gamma}$ are C^1 close as ε goes to zero, as they differ by the difference of $\boldsymbol{\gamma}$ and the identity. As we shall see in §3.4.2 in [18] it is sufficient that these two links are cobordant.

3.3. **Reducible PU(2) monopoles and their identification with Seiberg-Witten monopoles.** In this section we show that the subspaces of reducible PU(2) monopoles in $\mathcal{M}_{\mathfrak{t}}$ can be identified with the moduli spaces of Seiberg-Witten monopoles defined in §2.3.

3.3.1. *Decomposing* PU(2) *bundles.* We now describe the canonical isomorphisms of associated bundles induced from a splitting $E = L_1 \oplus L_2$ of a Hermitian two-plane bundle E; the lemmas below are of course elementary and we just state them in order to make our conventions clear.

Lemma 3.10. *If E is a Hermitian two-plane bundle and L_1, L_2 are Hermitian line bundles over a manifold X such that $E = L_1 \oplus L_2$, then there is a canonical isometry of* SO(3) *bundles, $\mathfrak{su}(E) \cong i\underline{\mathbb{R}} \oplus (L_2 \otimes L_1^*)$, given by*

$$\begin{pmatrix} \nu & -\bar{z} \\ z & -\nu \end{pmatrix} \longmapsto (\nu, z). \tag{3.7}$$

Proof. With respect to the decomposition $E = L_1 \oplus L_2$, any element $M \in \mathfrak{gl}(E)$ takes the form

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where $M_{jk} \in L_j \otimes L_k^*$, for $1 \le j, k \le 2$. Thus, $M_{11}, M_{22} \in \mathbb{C}$, after identifying $L_1 \otimes L_1^* \cong L_2 \otimes L_2^* \cong \mathbb{C}$. If $M \in \mathfrak{su}(E)$, then $M_{11} \in i\mathbb{R}$ and $M_{12} \in L_1 \otimes L_2^*$ with $M_{22} = -M_{11}$ and $M_{21} = -\bar{M}_{12}$, so any element of $\mathfrak{su}(E)$ takes the shape

$$M = \begin{pmatrix} \nu & -\bar{z} \\ z & -\nu \end{pmatrix}, \qquad \nu \in \mathbb{C}, \quad z \in L_2 \otimes L_1^*,$$

and the desired isomorphism $\mathfrak{su}(E) \cong i\underline{\mathbb{R}} \oplus (L_2 \otimes L_1^*)$ is given by $M \mapsto (\nu, z)$.

We recall from [24, §2.4] that the induced fiber inner product on $\mathfrak{gl}(E)$ is defined by $\langle M, M' \rangle = \frac{1}{2}\operatorname{tr}(M'M^\dagger)$. Thus, if $M', M \in C^\infty(\mathfrak{su}(E))$ correspond to $(\nu', z'), (\nu, z) \in C^\infty(i\Lambda^0)) \oplus C^\infty(L)$, respectively, we see that

$$\begin{aligned} \langle M', M \rangle &= \tfrac{1}{2}\operatorname{tr}(M'M^\dagger) = \tfrac{1}{2}\operatorname{tr}\begin{pmatrix} \nu' & -\bar{z}' \\ z' & -\nu' \end{pmatrix}\begin{pmatrix} \bar{\nu} & -\bar{z} \\ z & -\bar{\nu} \end{pmatrix} \\ &= \nu'\bar{\nu} + \tfrac{1}{2}(z'\bar{z} + \bar{z}'z) = \nu'\bar{\nu} + \operatorname{Re}\langle z', z\rangle_{\mathbb{C}}, \end{aligned} \tag{3.8}$$

and so the isomorphism of SO(3) bundles is an isometry. □

3.3.2. *The identification of reducible* PU(2) *monopoles with Seiberg-Witten monopoles.* Next, we explain how reducible pairs in the moduli space of PU(2) monopoles may be identified with Seiberg-Witten monopoles.

Recall that $\mathcal{C}_{\mathfrak{t}}$ is the quotient space of pairs whose spin^u connections induce the fixed unitary connection $2A_\Lambda$ on the complex line bundle $\det(V^+)$.

Lemma 3.11. *Let $\mathfrak{s} = (\rho, W)$ be a spin^c structure over an oriented, Riemannian four-manifold X and let $\mathfrak{t} = (\rho, V)$ be a spin^u structure with $V = W \oplus W \otimes L$. Then*

$$\iota : \tilde{\mathcal{C}}_{\mathfrak{s}} \hookrightarrow \tilde{\mathcal{C}}_{\mathfrak{t}}, \qquad (B, \Psi) \mapsto (B \oplus B \otimes A_L, \Psi \oplus 0), \tag{3.9}$$

is a smooth embedding, where $A_L = A_\Lambda \otimes (B^{\det})^*$, and is gauge-equivariant with respect to

$$\varrho : \mathcal{G}_{\mathfrak{s}} \hookrightarrow \mathcal{G}_{\mathfrak{t}}, \qquad s \mapsto s\,\mathrm{id}_W \oplus s^{-1}\,\mathrm{id}_{W\otimes L}, \tag{3.10}$$

so that

$$\iota(s(B, \Psi)) = \varrho(s)\iota(B, \Psi).$$

The image of the map (3.9) contains all pairs in $\tilde{\mathcal{C}}_{\mathfrak{t}}$ fixed by the action (3.2) of S^1 on V. The induced map,

$$\iota : \mathcal{C}_{\mathfrak{s}} \hookrightarrow \mathcal{C}_{\mathfrak{t}}, \qquad [B, \Psi] \mapsto [B \oplus B \otimes A_L, \Psi \oplus 0], \tag{3.11}$$

is continuous and, when restricted to $\mathcal{C}^0_{\mathfrak{s}}$, a topological embedding. If $w_2(\mathfrak{t}) \neq 0$, then (3.11) *is a topological embedding of $\mathcal{C}_{\mathfrak{s}}$. If $w_2(\mathfrak{t}) = 0$ then the map* (3.11) *takes zero-section points in $\mathcal{C}_{\mathfrak{s}}$ to zero-section reducibles in $\mathcal{C}_{\mathfrak{t}}$, although this identification of zero-section points need not be injective if $b_1(X) > 0$.*

Proof. The map $\iota : \tilde{\mathcal{C}}_{\mathfrak{s}} \to \tilde{\mathcal{C}}_{\mathfrak{t}}$ is clearly a C^∞ embedding. Furthermore,

$$\varrho(s)\iota(B, \Psi) = (s_* B \oplus (s^{-1})_*(B \otimes A_L), s\Psi \oplus 0).$$

Since $A_L = A_\Lambda \otimes (B^{\det})^*$, we see that $s \in \mathcal{G}_{\mathfrak{s}}$ acts on A_L as $(s^{-2})_* A_L$ (see also §3.4.2) and so $(s^{-1})_*(B \otimes A_L) = s_* B \otimes (s^{-2})_* A_L$. Thus, $\varrho(s)\iota(B, \Psi) = \iota(s_* B, s\Psi)$, as desired.

Next, we characterize the image of the map $\iota : \tilde{\mathcal{C}}_{\mathfrak{s}} \to \tilde{\mathcal{C}}_{\mathfrak{t}}$. Suppose $(A, \Phi) \in \tilde{\mathcal{C}}_{\mathfrak{t}}$ is fixed by the S^1 action (3.2) on V: this action descends to the action (3.3) on $\mathfrak{g}_{\mathfrak{t}} = i\underline{\mathbb{R}} \oplus L$, which fixes the induced connection $\hat{A}$. As in the proof of Assertion (2) of Proposition 3.1, the connection $\hat{A}$ must then be reducible with respect to this splitting, taking the form $\hat{A} = d_{\mathbb{R}} \oplus A_L$ on $\mathfrak{g}_{\mathfrak{t}}$ and $A = B \oplus B \otimes A_L$ on V. Thus, (A, Φ) is in the image of ι.

If $\mathcal{U} \subset \mathcal{C}_{\mathfrak{s}}$ is an open subset, so it is easy to see that $\iota(\mathcal{U})$ is open in $\iota(\mathcal{C}_{\mathfrak{s}})$ with respect to the subspace topology induced by $\mathcal{C}_{\mathfrak{t}}$.

We now show $\iota : \mathcal{C}^0_{\mathfrak{s}} \to \mathcal{C}_{\mathfrak{t}}$ is injective and that $\iota : \mathcal{C}_{\mathfrak{s}} \to \mathcal{C}_{\mathfrak{t}}$ is injective when $w_2(\mathfrak{t}) \not\equiv 0$. Suppose (B, Ψ) and (B', Ψ') are pairs in $\tilde{\mathcal{C}}^0_{\mathfrak{s}}$ and that $u \in \mathcal{G}_{\mathfrak{t}}$ satisfies $u\iota(B, \Psi) = \iota(B', \Psi')$, and thus $[\iota(B, \Psi)] = [\iota(B', \Psi')]$ in $\mathcal{C}_{\mathfrak{t}}$. With respect to the decomposition $V = W \otimes E$ and splitting $E = \underline{\mathbb{C}} \oplus L$, we can consider u to be a gauge transformation of $E = \underline{\mathbb{C}} \oplus L$ via the isomorphism $\mathcal{G}^w_\kappa \cong \mathcal{G}_{\mathfrak{t}}$ in Lemma 2.7 (and acting as the identity on W) and write

$$u = \begin{pmatrix} s & v \\ -\bar{v} & \bar{s} \end{pmatrix}, \quad \text{where } s \in \mathrm{Map}(X, \mathbb{C}) \text{ and } v \in C^\infty(\mathrm{Hom}(L, \mathbb{C})),$$

noting that $u^{-1} = u^\dagger$ and $\det(u) = 1$. As in the proof of Lemma 2.9, we may write $\iota(B) = B \otimes A_E$ and $\iota(B') = B \otimes A'_E$, where A_E and $A'_E = u(A_E)$ are U(2) connections on E which are reducible with respect to the splitting $E = \underline{\mathbb{C}} \oplus L$. Let $\varrho_L : S^1 \to \mathrm{End}(E)$ denote the action of S^1 on E by the trivial action on $\underline{\mathbb{C}}$ and scalar multiplication on L. Then $\varrho_L(S^1)$ fixes $u(A_E)$ and thus for all real θ

$$u^{-1}\varrho_L(e^{i\theta})u \in \mathrm{Stab}_{\mathrm{U}(2)}(A_E) = S^1 \times S^1,$$

where $\mathrm{Stab}_{\mathrm{U}(2)}(A_E)$ is the stabilizer of A_E in the space of unitary automorphisms of E. Consequently, for every real θ there are real constants μ and ν such that

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} s & v \\ -\bar{v} & \bar{s} \end{pmatrix} = \begin{pmatrix} s & v \\ -\bar{v} & \bar{s} \end{pmatrix} \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{i\nu} \end{pmatrix},$$

and we can assume $e^{i\theta} \neq 1$. Simplifying, this becomes

$$\begin{pmatrix} s & v \\ -e^{i\theta}\bar{v} & e^{i\theta}\bar{s} \end{pmatrix} = \begin{pmatrix} e^{i\mu}s & e^{i\nu}v \\ -e^{i\mu}\bar{v} & e^{i\nu}\bar{s} \end{pmatrix}.$$

If $s \not\equiv 0$, then we must have $e^{i\mu} = 1$ and thus $v = 0$ because $e^{i\theta}\bar{v} = \bar{v}$. Since $\det(u) = |s|^2 + |v|^2 = 1$, we see that $s \in \mathrm{Map}(X, S^1) \cong \mathcal{G}_{\mathfrak{s}}$. Hence, $u = \varrho(s)$ and $[B, \Psi] = [B', \Psi'] \in \mathcal{C}_{\mathfrak{s}}$ because $s(B, \Psi) = (B', \Psi')$.

It remains to consider the case $s \equiv 0$, for which we must then have $v\bar{v} = 1$. First suppose $\Psi \not\equiv 0$ and observe that

$$u(\Psi \oplus 0) = s\Psi \oplus -\bar{v}\Psi = \Psi' \oplus 0,$$

so $\bar{v}\Psi \equiv 0$ on X and hence $\Psi \equiv 0$, contradicting our hypothesis that $[B, \Psi] \in \mathcal{C}^0_{\mathfrak{s}}$. Therefore the case $s \equiv 0$ cannot occur in this situation. Thus, $\iota : \mathcal{C}^0_{\mathfrak{s}} \to \mathcal{C}_{\mathfrak{t}}$ is injective.

Otherwise, if $s \equiv 0$, suppose $w_2(\mathfrak{t}) \neq 0$ and observe that the automorphism $u \in \mathrm{Aut}(\underline{\mathbb{C}} \oplus L)$ induces an isomorphism $L \cong \underline{\mathbb{C}}$. But $c_1(E) = c_1(L)$ and $c_1(E) \equiv w_2(\mathfrak{t}) \pmod 2$: by the hypothesis in the final statement of the lemma, $w_2(\mathfrak{t}) \neq 0$, so this contradicts $L \cong \underline{\mathbb{C}}$ and therefore the case $s \equiv 0$ cannot occur in this situation either. Thus, if $w_2(\mathfrak{t}) \neq 0$, the map $\iota : \mathcal{C}_{\mathfrak{s}} \to \mathcal{C}_{\mathfrak{t}}$ is injective. □

The following lemma highlights the key property of the PU(2) monopole equations:

Lemma 3.12. *Let (ρ, V) be a spin^u structure on X. Suppose $(A, \Phi) = (B \oplus B \otimes A_L, \Psi \oplus 0)$ is a reducible pair with respect to a splitting $V = W \oplus W \otimes L$, where B is a spin connection on W, A_L is a unitary connection on a complex line bundle L, and Ψ is a section of W^+. Then (A, Φ) solves the* PU(2) *monopole equations* (2.32) *if and only if (B, Ψ) solves the Seiberg-Witten equations* (2.55) *with perturbation $\eta = F^+_{A_\Lambda}$.*

Proof. Suppose that (A, Φ) solves the PU(2) monopole equations (2.32). With respect to the splitting $\mathfrak{g}_{\mathfrak{t}} = i\underline{\mathbb{R}} \oplus L$, Lemma 2.9 implies that $\hat{A} = d_{\mathbb{R}} \oplus A_L$, where $d_{\mathbb{R}}$ is the product connection on $i\underline{\mathbb{R}}$ and $A_L = A_\Lambda \otimes (B^{\det})^*$ is a unitary connection on the complex line bundle $L = \det^{\frac{1}{2}}(V^+) \otimes \det(W^+)^*$. Let $E = \underline{\mathbb{C}} \oplus L$ and observe that $\tilde{A} = d_{\mathbb{C}} \oplus A_L$ is a reducible unitary connection on E which is a lift of $\hat{A}$ on $\mathfrak{su}(E)$. Then

$$F^+_{\tilde{A}} = \begin{pmatrix} 0 & 0 \\ 0 & F^+_{A_L} \end{pmatrix} = F^+_{A_L} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in C^\infty(\Lambda^+ \otimes \mathfrak{u}(E)),$$

and therefore, since $F^+_{\hat{A}} \in C^\infty(\Lambda^+ \otimes \mathfrak{so}(\mathfrak{su}(E)))$, we have

$$\mathrm{ad}^{-1}(F^+_{\hat{A}}) = (F^+_{\tilde{A}})_0 = -F^+_{A_L} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in C^\infty(\Lambda^+ \otimes \mathfrak{su}(E)). \tag{3.12}$$

If $\Phi = \Psi \oplus 0$, where $\Psi \in \Omega^0(W^+)$ and writing $\mathfrak{u}(V^+) = \mathfrak{u}(W^+) \otimes \mathfrak{u}(E)$, we have

$$\Phi \otimes \Phi^* = \begin{pmatrix} \Psi \otimes \Psi^* & 0 \\ 0 & 0 \end{pmatrix} = (\Psi \otimes \Psi^*) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in C^\infty(\mathfrak{u}(W^+) \otimes \mathfrak{u}(E)).$$

Hence, projecting to $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$, we see that

$$(\Phi \otimes \Phi^*)_{00} = (\Psi \otimes \Psi^*)_0 \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in C^\infty(\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)). \tag{3.13}$$

Furthermore, with respect to the splitting $V^+ = W^+ \oplus W^+ \otimes L$, we clearly have

$$(D_A + \rho(\vartheta))\Phi = ((D_B + \rho(\vartheta))\Psi, 0). \tag{3.14}$$

Then, combining equations (2.32), (3.12), (3.13), and (3.14), and noting that

$$F_{A_L} = F_{A_\Lambda} - F_{B^{\det}} = F_{A_\Lambda} - \mathrm{Tr}(F_B),$$

shows that the pair (B, Ψ) solves

$$\begin{aligned} \mathrm{Tr}(F^+_B) - \tau\rho^{-1}(\Psi \otimes \Psi^*)_0 - F^+_{A_\Lambda} &= 0, \\ (D_B + \rho(\vartheta))\Psi &= 0. \end{aligned} \tag{3.15}$$

Comparing (3.15) with the Seiberg-Witten equations (2.55) concludes the proof. □

Lemma 3.13. *Let X be a closed, oriented, Riemannian four-manifold with spin^u structure $\mathfrak{t}$ having a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$. If $w_2(\mathfrak{t}) \neq 0$ then the map* (3.11) *of $\mathcal{C}_{\mathfrak{s}}$ into $\mathcal{C}_{\mathfrak{t}}$ restricts to a topological embedding*

$$\iota : M_{\mathfrak{s}} \hookrightarrow \mathcal{C}_{\mathfrak{t}}, \tag{3.16}$$

whose image is the subspace of $\mathcal{M}_{\mathfrak{t}}$ represented by pairs (A,Φ) which are reducible with respect to the splitting $V = W \oplus W \otimes L$, with $\Phi \in L^2_k(W^+)$ and $\mathfrak{t} = (\rho, V)$ and $\mathfrak{s} = (\rho, W)$. If $w_2(\mathfrak{t}) = 0$ then the map (3.11) takes zero-section points in $M_{\mathfrak{s}}$ to zero-section reducibles in $\mathcal{M}_{\mathfrak{t}}$, although this identification of zero-section points need not be injective when $b_1(X) > 0$.

Proof. Given Lemma 3.11, we need only characterize the image of ι. If (A,Φ) represents a point in $\mathcal{M}_{\mathfrak{t}}$ and is reducible with respect to the splitting $V = W \oplus W \otimes L$, with $\Phi \in L^2_k(W^+)$, then Lemma 3.11 implies that $(A,\Phi) = \iota(B,\Psi)$, for some pair $(B,\Psi) \in \tilde{\mathcal{C}}_{\mathfrak{s}}$. Then Lemma 3.12 implies that (B,Ψ) satisfies the Seiberg-Witten equations (2.55) since (A,Φ) satisfies the PU(2) monopole equations (2.32). □

3.4. **The link of a stratum of reducible monopoles: local structure.** The construction of the link in $\mathcal{M}_{\mathfrak{t}}$ of the stratum $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$ of reducible PU(2) monopoles occupies this and the next subsection. Although for generic perturbations the locus $M_{\mathfrak{s}}$ of reducible solutions in $\mathcal{M}_{\mathfrak{t}}$ defined by a reduction $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$ will be a smooth manifold, the linearization of the map $\mathfrak{S}$ defined by the PU(2) monopole equations (2.32) need not be surjective at a reducible solution and so the points of $M_{\mathfrak{s}}$ might not be regular points of $\mathcal{M}_{\mathfrak{t}}$. Moreover, unlike the case of the link of M^w_κ, a local model of the link does not suffice as only one of our cohomology classes extend over $M_{\mathfrak{s}}$ (and that one vanishes in many cases), so we cannot use geometric representatives to cut down to a generic point in $M_{\mathfrak{s}}$.

We begin with a definition of link of a stratum in smoothly stratified space, essentially following Mather [52] and Goresky-MacPherson [35]. We need only consider the relatively simple case of a stratified space with two strata since the lower strata in

$$\mathcal{M}_{\mathfrak{t}} \cong \mathcal{M}^{*,0}_{\mathfrak{t}} \cup M^w_\kappa \cup \bigcup_{\mathfrak{s}} M_{\mathfrak{s}} \tag{3.17}$$

do not intersect when $\mathcal{M}_{\mathfrak{t}}$ *contains no reducible, zero-section solutions*. (The finite union above over $\mathfrak{s}$ is over the subset of all spin^c structures for which $M_{\mathfrak{s}}$ is non-empty and for which there is a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s}'$.)

Definition 3.14. Let Z be a closed subset of a smooth, Riemannian manifold M, and suppose that $Z = Z_0 \cup Z_1$, where Z_0 and Z_1 are locally closed, smooth submanifolds of M and $Z_1 \subset \bar{Z}_0$. (That is, Z is a smoothly stratified space with two strata in the sense of [58, Chapter 11].) Let N_{Z_1} be the normal bundle of $Z_1 \subset M$ and let $\mathcal{O}' \subset N_{Z_1}$ be an open neighborhood of the zero section $Z_1 \subset N_{Z_1}$ such that there is a diffeomorphism γ, commuting with the zero section of N_{Z_1} (so $\gamma|_{Z_1} = \mathrm{id}_{Z_1}$), from $\mathcal{O}'$ onto an open neighborhood $\gamma(\mathcal{O}')$ of $Z_1 \subset M$. Let $\mathcal{O} \Subset \mathcal{O}'$ be an open neighborhood of the zero section $Z_1 \subset N_{Z_1}$, where $\bar{\mathcal{O}} = \mathcal{O} \cup \partial\mathcal{O} \subset \mathcal{O}'$ is a smooth manifold-with-boundary. Then $L_{Z_1} = Z_0 \cap \gamma(\partial\mathcal{O})$ is a *link of Z_1 in Z_0*.

In the preceding definition, Z_0 will be the intersection of $\mathcal{M}^{*,0}_{\mathfrak{t}}$ with a neighborhood of the image of $M_{\mathfrak{s}}$ and Z_1 will be identified with $M_{\mathfrak{s}}$. Although the ambient manifold M in Definition 3.14 it is not required to be finite-dimensional, we shall impose this constraint as we need to define a fundamental class for the sphere bundle of N_{Z_1}. Thus, we will need to define a finite-dimensional submanifold of $\mathcal{C}_{\mathfrak{t}}$ containing—as smooth submanifolds—the image of $M_{\mathfrak{s}}$ and an open neighborhood in $\mathcal{M}_{\mathfrak{t}}$ of the image of $M_{\mathfrak{s}}$. We fulfill these requirements in the next subsection by using a globalized, stabilized version of the local Kuranishi model [14, §4.2.4], [47] defined by the deformation complex of the PU(2) monopole equations (2.32). For the remainder of this subsection, we describe the splitting of this deformation complex, at a reducible PU(2) monopole, into deformation complexes which are "tangential" and "normal" to the stratum $M_{\mathfrak{s}}$.

3.4.1. *Decomposing the* PU(2) *elliptic deformation sequence at reducible pairs.* In this and the next two sub-subsections we describe how the elliptic deformation complex for $\mathcal{M}_\mathfrak{t}$ at a point $[A,\Phi]$ in the image of $M_\mathfrak{s} \hookrightarrow \mathcal{M}_\mathfrak{t}$ can be split into normal and tangential components, where $\mathfrak{s} = (\rho, W)$ and $\mathfrak{t} = (\rho, V)$ with $V = W \oplus W \otimes L$ as in Lemma 3.11.

Let $(A,\Phi) = \iota(B,\Psi)$ be a pair in $\tilde{\mathcal{C}}_\mathfrak{t}$, although not necessarily a solution to the PU(2) monopole equations. We begin by considering the deformation sequence (2.47), namely

$$C^\infty(F_0) \xrightarrow{d^0_{A,\Phi}} C^\infty(F_1) \xrightarrow{d^1_{A,\Phi}} C^\infty(F_2), \tag{3.18}$$

at a point $(A,\Phi) = (B \oplus B \otimes A_L, \Psi \oplus 0)$, where $A_L = A_\Lambda \otimes (B^{\det})^*$ and the vector bundles F_j, $j = 0,1,2$, are defined by

$$\begin{aligned} F_0 &= \mathfrak{g}_\mathfrak{t}, \\ F_1 &= \Lambda^1 \otimes \mathfrak{g}_\mathfrak{t} \oplus V^+, \\ F_2 &= \Lambda^+ \otimes \mathfrak{g}_\mathfrak{t} \oplus V^-. \end{aligned} \tag{3.19}$$

We shall also use $L^2_{k+1-j}(F_j)$, the Hilbert spaces of L^2_{k+1-j} sections for $j = 0,1,2$, when applying these sequences. (The sequence is a complex if and only $\mathfrak{S}(A,\Phi) = 0$.) It will be convenient to define vector bundle splittings,

$$F_j \cong F_j^t \oplus F_j^n, \quad j = 0,1,2, \tag{3.20}$$

using the canonical isomorphism $\mathfrak{g}_\mathfrak{t} \cong i\underline{\mathbb{R}} \oplus L$ of Lemma 3.10:

$$\begin{array}{lll} F_0^t = i\Lambda^0 & \text{and} & F_0^n = \Lambda^0 \otimes_\mathbb{R} L = L, \\ F_1^t = i\Lambda^1 \oplus W^+ & \text{and} & F_1^n = \Lambda^1 \otimes_\mathbb{R} L \oplus W^+ \otimes L, \\ F_2^t = i\Lambda^+ \oplus W^- & \text{and} & F_2^n = \Lambda^+ \otimes_\mathbb{R} L \oplus W^- \otimes L. \end{array} \tag{3.21}$$

Conversely, the decompositions (3.20) yield inclusions which we write as

$$\iota : F_j^t \hookrightarrow F_j \quad \text{and} \quad \iota : F_j^n \hookrightarrow F_j, \quad j = 0,1,2. \tag{3.22}$$

The motivation for the splitting is due to the fact that the component $C^\infty(F_1^t)$ will contain vectors tangent to $M_\mathfrak{s}$ while $C^\infty(F_1^n)$ will contain those vectors normal to $M_\mathfrak{s}$. We note in passing that when the above vector bundles are given their natural fiber inner products, the splittings $F_j = F_j^t \oplus F_j^n$ define isomorphisms of Riemannian vector bundles when the bundles $F_j^t \oplus F_j^n$ are given their direct sum fiber inner products.

3.4.2. *Decomposing the group actions on the deformation sequence bundles.* The embedding $\mathcal{G}_\mathfrak{s} \hookrightarrow \mathcal{G}_\mathfrak{t}$ defined in (3.10) and two S^1 actions on $V = W \oplus W \otimes L$ of interest to us, namely (3.1) and (3.2), induce corresponding group actions on the bundles F_j^t, F_j^n in (3.21), arising in the decomposition (3.34) of the PU(2)-monopole deformation sequence (3.18). For ease of later reference, we record the induced group actions here on F_j^t and F_j^n.

Writing $V = W \otimes E$ and $E = \underline{\mathbb{C}} \oplus L$, we defined an embedding (3.10) of gauge groups,

$$\varrho : \mathcal{G}_\mathfrak{s} \hookrightarrow \mathcal{G}_\mathfrak{t}, \quad s \mapsto \mathrm{id}_W \otimes \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}.$$

Then the homomorphism $\mathrm{Ad} : \mathrm{Aut}(E) \to \mathrm{Aut}(\mathfrak{su}(E))$, $u \mapsto u(\cdot)u^{-1}$, induces an action of $\varrho(s)$ on $\mathfrak{su}(E) \cong i\underline{\mathbb{R}} \oplus L$ via the isomorphism $\xi \mapsto (\nu, z)$ in (3.7),

$$\mathrm{Ad}\,(\varrho(s))\xi = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} \nu & -\bar{z} \\ z & \nu \end{pmatrix} \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} \nu & -s^2\bar{z} \\ s^{-2}z & \nu \end{pmatrix},$$

where $\nu \in C^\infty(X, i\mathbb{R})$ and $z \in C^\infty(X, \mathbb{C})$. Hence, the induced action of $\varrho(s)$ on $\mathfrak{su}(E) \cong i\underline{\mathbb{R}} \oplus L$—via the composition of the embedding $\varrho : \mathcal{G}_\mathfrak{s} \hookrightarrow \mathcal{G}_\mathfrak{t}$ given by (3.10), the map

Ad : $\mathrm{Aut}(V) \to \mathrm{Aut}(\mathfrak{su}(V))$, and the projection $\mathfrak{su}(V) \to \mathfrak{su}(E) = \mathfrak{g}_{\mathfrak{t}}$ given by (2.13) —is trivial on the factor $i\underline{\mathbb{R}}$ and acts as scalar multiplication by s^{-2} on the complex line bundle L:

$$\mathrm{Ad} \circ \varrho : \mathcal{G}_{\mathfrak{s}} \times (i\underline{\mathbb{R}} \oplus L) \to i\underline{\mathbb{R}} \oplus L, \quad (s, (\nu, z)) \mapsto (\nu, s^{-2}z). \tag{3.23}$$

Hence, the induced action of $s \in \mathcal{G}_{\mathfrak{s}}$ is trivial on the bundles F_j^t while on F_j^n it acts as s^{-2} on the factors $\Lambda^j \otimes_{\mathbb{R}} L$ and as s^{-1} on the factors $W^\pm \otimes L$.

The action (3.1) of S^1 on $V = W \oplus W \otimes L$ by scalar multiplication induces the trivial action on $\mathfrak{g}_{\mathfrak{t}} = \mathfrak{su}(E)$. Hence, S^1 acts trivially on the factors $i\Lambda^j$ and $\Lambda^j \otimes_{\mathbb{R}} L$ of F_j^t, F_j^n, whereas it acts scalar multiplication on the factors $W^\pm$ and $W^\pm \otimes L$.

Finally, we consider the action (3.2) of S^1 on $V = W \oplus W \otimes L$ by the identity on the factor W and as scalar multiplication on $W \otimes L$. The induced action of S^1 is trivial on the factors F_j^t while it acts by scalar multiplication on the factors F_j^n.

3.4.3. *Decomposing the differentials in the deformation sequence at reducible pairs.* We now describe the splitting of the differentials $d^j_{A,\Phi}$ in the sequence, when $(A, \Phi) = \iota(B, \Psi)$ is a reducible pair in $\tilde{\mathcal{C}}_{\mathfrak{t}}$.

Recall from equation (2.47) that

$$d^1_{A,\Phi}(a, \phi) = (D\mathfrak{S})_{A,\Phi}(a, \phi) = \begin{pmatrix} d^+_{\hat{A}} a - \tau\rho^{-1}(\Phi \otimes \phi^* + \phi \otimes \Phi^*)_{00} \\ (D_A + \rho(\vartheta))\phi + \rho(a)\Phi \end{pmatrix}. \tag{3.24}$$

If $\Phi = \Psi \oplus 0$ and $\phi = \psi \oplus \psi'$, the quadratic term takes the form

$$\Phi \otimes \phi^* + \phi \otimes \Phi^* = \begin{pmatrix} \Psi \otimes \psi^* + \psi \otimes \Psi^* & \Psi \otimes \psi'^{,*} \\ \psi' \otimes \Psi^* & 0 \end{pmatrix},$$

and so

$$\begin{aligned} &\tau\rho^{-1}(\Phi \otimes \phi^* + \phi \otimes \Phi^*)_{00} \\ &\quad = \begin{pmatrix} \frac{1}{2}\tau\rho^{-1}(\Psi \otimes \psi^* + \psi \otimes \Psi^*)_0 & \tau\rho^{-1}(\Psi \otimes \psi'^{,*})_0 \\ \tau\rho^{-1}(\psi' \otimes \Psi^*)_0 & -\frac{1}{2}\tau\rho^{-1}(\Psi \otimes \psi^* + \psi \otimes \Psi^*)_0 \end{pmatrix}, \end{aligned}$$

where the diagonal term $\frac{1}{2}\tau\rho^{-1}(\Psi \otimes \psi^* + \psi \otimes \Psi^*)_0$ is in $C^\infty(i\Lambda^+)$ while the off-diagonal term $\tau\rho^{-1}(\psi' \otimes \Psi^*)_0$ is in $C^\infty(\Lambda^+ \otimes L)$. Indeed, we have

$$(\psi' \otimes \Psi^*)_0 \in C^\infty(\mathfrak{sl}(W^+) \otimes_{\mathbb{C}} L) = C^\infty(\mathfrak{su}(W^+) \otimes_{\mathbb{R}} L),$$

since $\mathfrak{sl}(W^+) = \mathfrak{su}(W^+) \otimes_{\mathbb{R}} \mathbb{C}$. The complex-linear Clifford map restricts to a real-linear isomorphism $\rho : \Lambda^+ \to \mathfrak{su}(W^+)$, so we have $\rho^{-1}(\psi' \otimes \Psi^*)_0 \in C^\infty(\Lambda^+ \otimes L)$.

The Dirac-operator term $D_A + \rho(\vartheta) + \rho(a) : C^\infty(V^+) \to C^\infty(V^-)$ splits as

$$(D_A + \rho(\vartheta))\phi + \rho(a)\Phi = \begin{pmatrix} (D_B + \rho(\vartheta))\psi + \rho(\alpha)\Psi \\ (D_{B\otimes A_L} + \rho(\vartheta))\psi' + \rho(\beta)\Psi \end{pmatrix},$$

noting that

$$a = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & -\alpha \end{pmatrix}, \quad \text{with} \quad \alpha \in C^\infty(i\Lambda^1) \quad \text{and} \quad \beta \in C^\infty(\Lambda^1 \otimes L).$$

By Lemma 2.9, $\hat{A} = d_{\mathbb{R}} \oplus A_L$. The term $d^+_{\hat{A}} a \in C^\infty(\Lambda^+ \otimes \mathfrak{g}_{\mathfrak{t}})$ then splits as

$$d^+_{\hat{A}} a = \begin{pmatrix} d^+\alpha & -d^+_{A_L^*}\bar{\beta} \\ d^+_{A_L}\beta & -d^+\alpha \end{pmatrix}, \tag{3.25}$$

where the diagonal term $d^+\alpha$ is in $C^\infty(i\Lambda^+)$, and off-diagonal term $d^+_{A_L}\beta$ is in $C^\infty(\Lambda^+ \otimes L)$.

With respect to these identifications, the linear operator $d^1_{\iota(B,\Psi)} : C^\infty(F_1) \to C^\infty(F_2)$ diagonalizes to give

$$d^1_{\iota(B,\Psi)} = \begin{pmatrix} d^{1,t}_{\iota(B,\Psi)} & 0 \\ 0 & d^{1,n}_{\iota(B,\Psi)} \end{pmatrix} : C^\infty(F_1^t) \oplus C^\infty(F_1^n) \to C^\infty(F_2^t) \oplus C^\infty(F_2^n),$$

where $d^{1,t}_{\iota(B,\Psi)} : C^\infty(F_1^t) \to C^\infty(F_2^t)$ and $d^{1,n}_{\iota(B,\Psi)} : C^\infty(F_1^n) \to C^\infty(F_2^n)$ are the "tangential" and "normal" components of $d^1_{\iota(B,\Psi)}$. More explicitly, the operator

$$d^1_{\iota(B,\Psi)} : \begin{array}{c} C^\infty(i\Lambda^1) \oplus C^\infty(W^+), \\ \oplus \\ C^\infty(\Lambda^1 \otimes L) \oplus C^\infty(W^+ \otimes L) \end{array} \longrightarrow \begin{array}{c} C^\infty(i\Lambda^+) \oplus C^\infty(W^-), \\ \oplus \\ C^\infty(\Lambda^+ \otimes L) \oplus C^\infty(W^- \otimes L) \end{array}$$

diagonalizes and the tangential component of $d^1_{\iota(B,\Psi)}$ is given by

$$d^{1,t}_{\iota(B,\Psi)} : C^\infty(i\Lambda^1) \oplus C^\infty(W^+) \longrightarrow C^\infty(i\Lambda^+) \oplus C^\infty(W^-),$$

where we define

$$d^{1,t}_{\iota(B,\Psi)}(\alpha,\psi) = \begin{pmatrix} d^+\alpha - \frac{1}{2}\tau\rho^{-1}(\Psi \otimes \psi^* + \psi \otimes \Psi^*)_0 \\ (D_B + \rho(\vartheta))\psi + \rho(\alpha)\Psi \end{pmatrix}, \tag{3.26}$$

Note that $d^{1,t}_{\iota(B,\Psi)}$ matches the Seiberg-Witten differential $d^1_{B,\Psi}$ in equation (2.60), aside from a factor of 2 in the d^+ component: the scaling factor has no significance since we are only interested in the kernels and cokernels of these operators. The normal component of $d^1_{\iota(B,\Psi)}$ is given by

$$d^{1,n}_{\iota(B,\Psi)} : C^\infty(\Lambda^1 \otimes L) \oplus C^\infty(W^+ \otimes L) \longrightarrow C^\infty(\Lambda^+ \otimes L) \oplus C^\infty(W^- \otimes L),$$

where we define

$$d^{1,n}_{\iota(B,\Psi)}(\beta,\psi') = \begin{pmatrix} d^+_{A_L}\beta - \tau\rho^{-1}(\psi' \otimes \Psi^*)_0 \\ (D_{B\otimes A_L} + \rho(\vartheta))\psi' + \rho(\beta)\Psi \end{pmatrix}. \tag{3.27}$$

This completes our description of the diagonalization of the differential $d^1_{\iota(B,\Psi)}$.

It remains to discuss the rather simpler diagonalization of the differential $d^0_{\iota(B,\Psi)}$ with respect to the splitting of $C^\infty(F_0)$ and $C^\infty(F_1)$ into normal and tangential components. From Lemma 3.10 we see that elements $\zeta = \zeta_0$ of $C^\infty(\mathfrak{g}_\mathfrak{t})$, recalling that $L^2_{k+1}(\mathfrak{g}_\mathfrak{t}) = T_{\mathrm{id}}\mathcal{G}_\mathfrak{t}$, may be uniquely written as

$$\zeta = \begin{pmatrix} f & -\bar\kappa \\ \kappa & -f \end{pmatrix}, \tag{3.28}$$

where $f \in C^\infty(X, i\mathbb{R})$ is an imaginary-valued function and $\kappa \in C^\infty(L)$. Thus,

$$\zeta = f \oplus \kappa \in C^\infty(F_0^t) \oplus C^\infty(F_0^n).$$

Recall from equation (2.50) that the differential $d^0_{A,\Phi}$ is defined by

$$d^0_{A,\Phi}\zeta = (d_{\hat A}\zeta, -\zeta\Phi) \in C^\infty(\Lambda^1 \otimes \mathfrak{g}_\mathfrak{t}) \oplus C^\infty(V^+), \qquad \zeta \in C^\infty(\Lambda^1 \otimes \mathfrak{g}_\mathfrak{t}).$$

Note that

$$\zeta\Phi = \begin{pmatrix} f & -\bar\kappa \\ \kappa & -f \end{pmatrix} \cdot \begin{pmatrix} \Psi \\ 0 \end{pmatrix} = \begin{pmatrix} f\Psi \\ \kappa\Psi \end{pmatrix},$$

$$d_{\hat A}\zeta = \begin{pmatrix} df & -d_{A_L^*}\bar\kappa \\ d_{A_L}\kappa & -df \end{pmatrix}.$$

So, at the pair $(A,\Phi)=\iota(B,\Psi)=(B\oplus B\otimes A_L,\Psi\oplus 0)$ we have

$$d^0_{\iota(B,\Psi)}\zeta=\left(\begin{pmatrix} df & -d_{A_L^*}\bar\kappa \\ d_{A_L}\kappa & -df\end{pmatrix},\begin{pmatrix}-f\Psi\\ -\kappa\Psi\end{pmatrix}\right).$$

With respect to these identifications, the linear operator $d^0_{\iota(B,\Psi)}:C^\infty(F_0)\to C^\infty(F_1)$ diagonalizes to give

$$d^0_{\iota(B,\Psi)}=\begin{pmatrix} d^{0,t}_{\iota(B,\Psi)} & 0\\ 0 & d^{0,n}_{\iota(B,\Psi)}\end{pmatrix}:C^\infty(F_0^t)\oplus C^\infty(F_0^n)\longrightarrow C^\infty(F_1^t)\oplus C^\infty(F_1^n),$$

where $d^{0,t}_{\iota(B,\Psi)}:C^\infty(F_0^t)\to C^\infty(F_1^t)$ and $d^{0,n}_{\iota(B,\Psi)}:C^\infty(F_0^n)\to C^\infty(F_1^n)$ are the "tangential" and "normal" components of $d^0_{\iota(B,\Psi)}$. More explicitly, the operator

$$d^0_{\iota(B,\Psi)}:\begin{matrix} C^\infty(i\Lambda^0)) \\ \oplus \\ C^\infty(L)\end{matrix}\longrightarrow \begin{matrix} C^\infty(i\Lambda^1)\oplus C^\infty(W^+)\\ \oplus \\ C^\infty(\Lambda^1\otimes L)\oplus C^\infty(W^+\otimes L)\end{matrix}$$

diagonalizes and the tangential component of $d^0_{\iota(B,\Psi)}$ is given by

$$d^{0,t}_{\iota(B,\Psi)}:C^\infty(i\Lambda^0)\longrightarrow C^\infty(i\Lambda^1)\oplus C^\infty(W^+), \tag{3.29}$$

where we define

$$d^{0,t}_{\iota(B,\Psi)}f=(df,-f\Psi).$$

Note that $d^{0,t}_{\iota(B,\Psi)}$ matches the gauge group differential $d^0_{B,\Psi}$ in equation (2.58). The normal component of $d^0_{\iota(B,\Psi)}$ is given by

$$d^{0,n}_{\iota(B,\Psi)}:C^\infty(L)\longrightarrow C^\infty(\Lambda^1\otimes L)\oplus C^\infty(W^+\otimes L), \tag{3.30}$$

where we define

$$d^{0,n}_{\iota(B,\Psi)}\kappa=(d_{A_L}\kappa,-\kappa\Psi).$$

This completes our description of the diagonalization of the differential $d^0_{\iota(B,\Psi)}$.

Recall from [24, Equation (2.38)] that the L^2 adjoint of $d^0_{A,\Phi}$ is given by

$$d^{0,*}_{A,\Phi}(a,\phi)=d^*_{\hat A}a-(\cdot\Phi)^*\phi,\qquad (a,\phi)\in C^\infty(\Lambda^1\otimes\mathfrak{g}_{\mathfrak{t}})\oplus C^\infty(V^+). \tag{3.31}$$

The L^2 adjoint $d^{0,*}_{\iota(B,\Psi)}:C^\infty(F_1)\to C^\infty(F_0)$ then diagonalizes to give

$$d^{0,*}_{\iota(B,\Psi)}=\begin{pmatrix} d^{0,t,*}_{\iota(B,\Psi)} & 0\\ 0 & d^{0,n,*}_{\iota(B,\Psi)}\end{pmatrix}:C^\infty(F_1^t)\oplus C^\infty(F_1^n)\to C^\infty(F_0^t)\oplus C^\infty(F_0^n),$$

where $d^{0,t,*}_{\iota(B,\Psi)}:C^\infty(F_1^t)\to C^\infty(F_0^t)$ is given explicitly by

$$d^{0,t,*}_{\iota(B,\Psi)}:C^\infty(i\Lambda^1)\oplus C^\infty(W^+)\to C^\infty(i\Lambda^0),$$

with

$$d^{0,t,*}_{\iota(B,\Psi)}(\alpha,\psi)=d^*\alpha-\operatorname{Im}\langle\Psi,\psi\rangle, \tag{3.32}$$

and $d^{0,n,*}_{\iota(B,\Psi)}:C^\infty(F_1^n)\to C^\infty(F_0^n)$ is explicitly given by

$$d^{0,n,*}_{\iota(B,\Psi)}:C^\infty(\Lambda^1\otimes L)\oplus C^\infty(W^+\otimes L)\to C^\infty(L),$$

with

$$d^{0,n,*}_{\iota(B,\Psi)}(\beta,\psi') = d^*_{A_L}\beta + \psi'\otimes\Psi^*. \tag{3.33}$$

Of course, these L^2 adjoints are defined in the usual way by

$$\begin{aligned}(d^*\alpha - \mathrm{Im}\langle\Psi,\psi\rangle, f)_{L^2} &= (\alpha, df)_{L^2} - (\psi, f\Psi)_{L^2},\\ (d^*_{A_L}\beta + \psi'\otimes\Psi^*,\kappa)_{L^2} &= (\beta, d_{A_L}\kappa)_{L^2} + (\psi',\kappa\Psi)_{L^2},\end{aligned}$$

for all $f\in C^\infty(i\Lambda^0)$ and $\kappa\in C^\infty(L)$, where

$$\begin{aligned}(f\Psi,\psi)_{L^2} &= (f, \overline{\langle\psi,\Psi\rangle})_{L^2} = (f,\langle\Psi,\psi\rangle)_{L^2},\\ (\kappa\Psi,\psi')_{L^2} &= (\kappa,\psi'\otimes\Psi^*)_{L^2}.\end{aligned}$$

This completes our description of the diagonalization of the L^2 adjoint $d^{0,*}_{\iota(B,\Psi)}$.

The above discussion implies that the deformation sequence (3.18) splits when $(A,\Phi) = \iota(B,\Psi)$:

$$\begin{aligned}C^\infty(F_0^t) &\xrightarrow{d^{0,t}_{\iota(B,\Psi)}} C^\infty(F_1^t) \xrightarrow{d^{1,t}_{\iota(B,\Psi)}} C^\infty(F_2^t),\\ C^\infty(F_0^n) &\xrightarrow{d^{0,n}_{\iota(B,\Psi)}} C^\infty(F_1^n) \xrightarrow{d^{1,n}_{\iota(B,\Psi)}} C^\infty(F_2^n).\end{aligned} \tag{3.34}$$

These sequences form complexes if and only if $\mathfrak{S}(A,\Phi)=0$, in which case they give the *tangential* and *normal deformation complexes* for the PU(2) monopole equations at a reducible solution, so it is convenient to consider the rolled-up sequences

$$\begin{aligned}C^\infty(F_1^t) &\xrightarrow{\mathcal{D}^t_{\iota(B,\Psi)}} C^\infty(F_0^t)\oplus C^\infty(F_2^t),\\ C^\infty(F_1^n) &\xrightarrow{\mathcal{D}^n_{\iota(B,\Psi)}} C^\infty(F_0^n)\oplus C^\infty(F_2^n),\end{aligned}$$

comprising the tangential and normal components of the rolled-up deformation sequence

$$\mathcal{D}_{\iota(B,\Psi)} : C^\infty(F_1)\to C^\infty(F_0)\oplus C^\infty(F_2).$$

for the PU(2) monopole equations. The tangential and normal components of the deformation operator $\mathcal{D}_{\iota(B,\Psi)} = d^{0,*}_{\iota(B,\Psi)} + d^1_{\iota(B,\Psi)}$ are given by

$$\mathcal{D}^t_{\iota(B,\Psi)} = d^{0,t,*}_{\iota(B,\Psi)} + d^{1,t}_{\iota(B,\Psi)} \quad\text{and}\quad \mathcal{D}^n_{\iota(B,\Psi)} = d^{0,n,*}_{\iota(B,\Psi)} + d^{1,n}_{\iota(B,\Psi)},$$

so that we have a decomposition

$$\mathcal{D}_{\iota(B,\Psi)} = \mathcal{D}^t_{\iota(B,\Psi)} \oplus \mathcal{D}^n_{\iota(B,\Psi)}. \tag{3.35}$$

More explicitly, we see from (3.26) and (3.32) that the tangential component

$$\mathcal{D}^t_{\iota(B,\Psi)} : C^\infty(i\Lambda^1)\oplus C^\infty(W^+) \longrightarrow C^\infty(i\Lambda^0)\oplus C^\infty(i\Lambda^+)\oplus C^\infty(W^-) \tag{3.36}$$

of $\mathcal{D}_{\iota(B,\Psi)}$ is given by

$$\mathcal{D}^t_{\iota(B,\Psi)}(\alpha,\psi) = \begin{pmatrix} d^*\alpha - \mathrm{Im}\langle\Psi,\psi\rangle \\ d^+\alpha - \frac{1}{2}\tau\rho^{-1}(\Psi\otimes\psi^* + \psi\otimes\Psi^*)_0 \\ (D_B + \rho(\vartheta))\psi + \rho(\alpha)\Psi \end{pmatrix},$$

while from (3.27) and (3.33) the normal component

$$\mathcal{D}^n_{\iota(B,\Psi)} : C^\infty(\Lambda^1\otimes L)\oplus C^\infty(W^+\otimes L) \longrightarrow C^\infty(L)\oplus C^\infty(\Lambda^+\otimes L)\oplus C^\infty(W^-\otimes L) \tag{3.37}$$

of $\mathcal{D}_{\iota(B,\Psi)}$ is given by

$$\mathcal{D}^n_{\iota(B,\Psi)}(\beta,\psi') = \begin{pmatrix} d^*_{A_L}\beta + \psi'\otimes\Psi^* \\ d^+_{A_L}\beta - \tau\rho^{-1}(\psi'\otimes\Psi^*)_0 \\ (D_{B\otimes A_L} + \rho(\vartheta))\psi' + \rho(\beta)\Psi \end{pmatrix}.$$

Note the operators $d^{0,n}_{\iota(B,\Psi)}$ and $d^{1,n}_{\iota(B,\Psi)}$, and thus $\mathcal{D}^n_{\iota(B,\Psi)}$, are complex linear when the vector bundles F^n_j, $j=0,1,2$, are given the natural complex structures induced by the S^1 action on L.

If $\mathfrak{S}(A,\Phi)=0$, the two sequences in (3.34) define cohomology groups $H^{\bullet,t}_{\iota(B,\Psi)}$ and $H^{\bullet,n}_{\iota(B,\Psi)}$, respectively, which we may compare with the cohomology groups $H^\bullet_{\iota(B,\Psi)}$ and $H^\bullet_{B,\Psi}$ of the PU(2) monopole deformation complex (2.47) at the pair $\iota(B,\Psi)\in\tilde{\mathcal{M}}_{\mathfrak{t}}$ and the cohomology groups $H^\bullet_{B,\Psi}$ of the Seiberg-Witten deformation complex (2.61) at the pair $(B,\Psi)\in\tilde{M}_{\mathfrak{s}}$:

Lemma 3.15. *Continue the above notation and require that $\iota(B,\Psi)$ be a (reducible)* PU(2) *monopole. For the elliptic deformation complex* (2.47) *defined by the* PU(2) *monopole equations* (2.32) *and the automorphism group $\mathcal{G}_{\mathfrak{t}}$, we have the following canonical isomorphisms of cohomology groups,*

$$H^{\bullet,t}_{\iota(B,\Psi)} \cong H^\bullet_{B,\Psi},$$

and hence,

$$H^\bullet_{\iota(B,\Psi)} \cong H^\bullet_{B,\Psi}\oplus H^{\bullet,n}_{\iota(B,\Psi)}.$$

Moreover, if $\Psi\not\equiv 0$ then $H^0_{B,\Psi}=0$, while if $H^2_{B,\Psi}=0$ then $H^2_{\iota(B,\Psi)}\cong H^{2,n}_{\iota(B,\Psi)}$.

Proof. The isomorphisms identifying the cohomology of the tangential deformation complex with that of the Seiberg-Witten complex follow immediately from a comparison of these two complexes.

If $\Psi\not\equiv 0$ then $H^0_{B,\Psi}=0$ and (3.30) implies $H^{0,n}_{\iota(B,\Psi)}=0$ and so $H^0_{\iota(B,\Psi)}=0$. Moreover, if $H^2_{B,\Psi}=0$, then $H^2_{\iota(B,\Psi)}=H^{2,n}_{\iota(B,\Psi)}$. □

Transversality for the Seiberg-Witten equations (Proposition 2.16) implies $H^2_{B,\Psi}=0$.

3.5. **The link of a stratum of reducible monopoles: global structure.** Our task in this subsection is to construct an ambient finite-dimensional, smooth submanifold $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})\subset\mathcal{C}_{\mathfrak{t}}$ containing $M_{\mathfrak{s}}$ as a smooth submanifold, as required by the definition of the link in Definition 3.14. Recall that $\mathfrak{S}$ is the section of the S^1-equivariant, infinite-rank "obstruction bundle"

$$\mathfrak{V} = \tilde{\mathcal{C}}_{\mathfrak{t}}\times_{\mathcal{G}_{\mathfrak{t}}} L^2_{k-1}(F_2)\to\mathcal{C}_{\mathfrak{t}} \tag{3.38}$$

defined by the PU(2) monopole equations (2.32) and this section need not vanish transversely along $M_{\mathfrak{s}}$.

To motivate the construction of the ambient manifold $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ given in this subsection, suppose temporarily that the cokernel of $\mathcal{D}_{A,\Phi}$ has constant rank as $[A,\Phi]$ varies in the image of $M_{\mathfrak{s}}\hookrightarrow\mathcal{M}_{\mathfrak{t}}$ (that is, no spectral flow occurs). Then we obtain a finite-rank, S^1-equivariant vector bundle $\operatorname{Coker}\boldsymbol{\mathcal{D}}$ over $M_{\mathfrak{s}}$, with fibers $\operatorname{Coker}\mathcal{D}_{A,\Phi}$. Let 2ν be the least positive eigenvalue of the Laplacian $\Delta_{A,\Phi}=\mathcal{D}_{A,\Phi}\mathcal{D}^*_{A,\Phi}$ as $[A,\Phi]$ varies along the image of the compact manifold $M_{\mathfrak{s}}$ and let $\Pi_{A,\Phi;\nu}$ denote the L^2 orthogonal projection from $L^2_{k-1}(F_2)$ onto the subspace spanned by the eigenvectors of $\Delta_{A,\Phi}$ with eigenvalue less than ν. The vector bundle $\operatorname{Coker}\boldsymbol{\mathcal{D}}$ over $\iota(M_{\mathfrak{s}})\subset\mathcal{C}_{\mathfrak{t}}$ then extends to a vector bundle $\Xi_\nu=\operatorname{Ker}\Pi^\perp_\nu\boldsymbol{\Delta}=\operatorname{Coker}\Pi^\perp_\nu\boldsymbol{\mathcal{D}}$, where $\Pi^\perp_\nu=\mathrm{id}-\Pi_\nu$, of the same rank over an open neighborhood of $\iota(M_{\mathfrak{s}}) \subset \mathcal{C}_{\mathfrak{t}}$. Arguing as in Lemma 3.16, one can see that the space $M_{\mathfrak{s}}$ would be a smooth submanifold of $\mathcal{C}_{\mathfrak{t}}$. Then both $M_{\mathfrak{s}}$ and an open neighborhood in $\mathcal{M}_{\mathfrak{t}}^{*,0}$ of $\iota(M_{\mathfrak{s}})$ would be smooth submanifolds of the S^1-invariant "thickened" moduli space,

$$\mathcal{M}_{\mathfrak{t}}(\Xi_\nu, \mathfrak{s}) = (\Pi_\nu^\perp \mathfrak{S})^{-1}(0) \subset \mathcal{C}_{\mathfrak{t}},$$

a finite-dimensional, smooth, S^1-invariant manifold which serves as the ambient, finite-dimensional, smooth manifold "M" of Definition 3.14. Ambient manifolds of this form, defined by spectral projections as above, have been used by Donaldson [13], Friedman-Morgan [31], and Taubes [71], [72] to describe neighborhoods of the stratum $\{[\Theta]\} \times \mathrm{Sym}^\kappa(X)$ containing the trivial connection Θ in the Uhlenbeck compactification of the moduli space M_κ^w of anti-self-dual connections when $w = 0$.

In practice, one cannot guarantee that $\operatorname{Coker} \boldsymbol{\mathcal{D}}$ will either vanish or even have constant rank due to spectral flow, so we must resort to a more general construction of an S^1-equivariant vector bundle Ξ over an open neighborhood of the image of $M_{\mathfrak{s}}$ in $\mathcal{C}_{\mathfrak{t}}$ which "spans" $\operatorname{Coker} \boldsymbol{\mathcal{D}}$ along $M_{\mathfrak{s}}$. The method we employ is an extension of one used by Atiyah and Singer to construct the index bundle or determinant-line bundle of a family of elliptic operators [6, pp. 122–127], [14, §5.1.3 & §5.2.1]. The Atiyah-Singer method has also been exploited by Furuta [32], T-J. Li & Liu [51], J. Li & Tian [50], and Ruan [65], [66], [67] to construct certain "global Kuranishi models" (or "virtual" or "thickened" moduli spaces) parameterizing spaces of Seiberg-Witten monopoles or pseudo-holomorphic curves; related ideas are contained in [11], [69]. The principal difference between our construction and those of Li-Liu, Li-Tian, or Ruan is that the vector bundle replacing $\operatorname{Coker} \boldsymbol{\mathcal{D}}$ is defined over an open neighborhood in the original configuration space $\mathcal{C}_{\mathfrak{t}}$ rather than on an artificial, augmented configuration space, such as $\mathbb{C}^r \times \mathcal{C}_{\mathfrak{t}}$. In this sense, our construction is closer to that of [13], [31], [71], [72] and corresponds to the alternative construction of index bundles discussed in [4, pp. 153–166], [9, §1.7]; we find this second stabilization technique more convenient when constructing links of lower-level strata of reducibles via gluing in [19], [20], [21].

3.5.1. *Smooth embeddings.* We recall that our abstract definition of a link of the lower stratum of a two-stratum space (see Definition 3.14) requires an ambient smooth manifold containing the strata. As a first step in the construction of a *finite-dimensional* ambient manifold, we use the decompositions of §3.4 to show that the topological embedding $M_{\mathfrak{s}} \hookrightarrow \mathcal{C}_{\mathfrak{t}}^0$ of Lemma 3.13 is smooth, so $M_{\mathfrak{s}}$ is a smooth submanifold of $\mathcal{C}_{\mathfrak{t}}^0$.

Lemma 3.16. *Let $\mathfrak{t}$ be a spin^u structure on a closed, oriented, smooth four-manifold X, with reduction $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$. Let $\iota : \mathcal{C}_{\mathfrak{s}}^0 \hookrightarrow \mathcal{C}_{\mathfrak{t}}^0$ be the map in Lemma 3.13. Then the following hold:*

- *The map $\iota : \mathcal{C}_{\mathfrak{s}}^0 \to \mathcal{C}_{\mathfrak{t}}^0$ is a smooth immersion, and*
- *The map $\iota : M_{\mathfrak{s}}^0 \hookrightarrow \mathcal{C}_{\mathfrak{t}}^0$ is a smooth embedding, so $M_{\mathfrak{s}}^0$ is a submanifold of $\mathcal{C}_{\mathfrak{t}}^0$.*

Proof. Let $[B, \Psi]$ be a point in $\mathcal{C}_{\mathfrak{s}}^0$: by the slice theorem for $\mathcal{C}_{\mathfrak{s}}^0$, the restriction of the projection map $\tilde{\mathcal{C}}_{\mathfrak{s}}^0 \to \mathcal{C}_{\mathfrak{s}}^0 = \tilde{\mathcal{C}}_{\mathfrak{s}}^0/\mathcal{G}_{\mathfrak{s}}$ to a small enough open neighborhood of (B, Ψ) in the slice $(B, \Psi) + \operatorname{Ker} d_{B,\Psi}^{*,0}$ gives a smooth parameterization of an open neighborhood of $[B, \Psi]$ in $\mathcal{C}_{\mathfrak{s}}^0$. Similarly, by the slice theorem [24, Proposition 2.8] for $\mathcal{C}_{\mathfrak{t}}^0$, the restriction of the projection map $\tilde{\mathcal{C}}_{\mathfrak{t}}^0 \to \mathcal{C}_{\mathfrak{t}}^0 = \tilde{\mathcal{C}}_{\mathfrak{t}}^0/\mathcal{G}_{\mathfrak{t}}$ to a small enough open neighborhood of $\iota(B, \Psi)$ in the slice $\iota(B, \Psi) + \operatorname{Ker} d_{\iota(B,\Psi)}^{*,0}$ gives a smooth parameterization of an open neighborhood of $[\iota(B, \Psi)]$ in $\mathcal{C}_{\mathfrak{t}}^0$. Comparing the operators $d_{B,\Psi}^{0,*}$ in equation (2.59) and $d_{\iota(B,\Psi)}^{0,t,*}$ in equation

(3.32), we see that the differential of the smooth embedding

$$\iota : \tilde{\mathcal{C}}^0_{\mathfrak{s}} \to \tilde{\mathcal{C}}^0_{\mathfrak{t}}, \qquad (B,\Psi) \mapsto (B \oplus B \otimes A_L, \Psi \oplus 0),$$

of Lemma 3.11 restricts to an isomorphism

$$D\iota : \operatorname{Ker} d^{*,0}_{B,\Psi} \cong \operatorname{Ker} d^{*,0,t}_{\iota(B,\Psi)} \subset \operatorname{Ker} d^{0,t,*}_{A,\Phi} \oplus \operatorname{Ker} d^{0,n,*}_{A,\Phi} = \operatorname{Ker} d^{*,0}_{B,\Psi}.$$

Hence, the differential $D\iota : T_{B,\Psi}\mathcal{C}^0_{\mathfrak{s}} \to T_{\iota(B,\Psi)}\mathcal{C}^0_{\mathfrak{t}}$ is injective and the induced maps $\iota : \mathcal{C}^0_{\mathfrak{s}} \to \mathcal{C}^0_{\mathfrak{t}}$ and $\iota : M^0_{\mathfrak{s}} \to \mathcal{C}^0_{\mathfrak{t}}$ are smooth immersions. Since the map $\iota : M^0_{\mathfrak{s}} \to \mathcal{C}^0_{\mathfrak{t}}$ is a topological embedding according to Lemma 3.13, and is a smooth immersion, the map $\iota : M^0_{\mathfrak{s}} \to \mathcal{C}^0_{\mathfrak{t}}$ is a smooth embedding. □

3.5.2. *Construction of the stabilized cokernel bundle.* Again, we assume that there are no zero-section reducible monopoles in $M_{\mathfrak{s}}$. For any representative (A,Φ) of a point in the image of $\iota : M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, we have $\operatorname{Coker}(D\mathfrak{S})_{A,\Phi} = \operatorname{Coker} d^{1,n}_{A,\Phi} \subset L^2_{k-1}(F^n_2)$, since $(D\mathfrak{S})_{A,\Phi} = d^1_{A,\Phi}$ (by definition) and $\operatorname{Coker} d^{1,t}_{A,\Phi} = 0$. The splitting of the last term of the elliptic deformation complex (3.18) at a reducible PU(2) monopole into tangential and normal deformation components (3.34) yields a splitting of the corresponding complexes of Hilbert bundles,

$$\mathfrak{V} = \tilde{\mathcal{C}}^0_{\mathfrak{t}} \times_{\mathcal{G}_{\mathfrak{t}}} L^2_{k-1}(F_2) \cong \tilde{\mathcal{C}}^0_{\mathfrak{t}} \times_{\mathcal{G}_{\mathfrak{t}}} L^2_{k-1}(F^t_2 \oplus F^n_2), \tag{3.39}$$

when restricted to $\iota(M_{\mathfrak{s}})$. This splitting can be seen by using a reduction of the structure group $\mathcal{G}_{\mathfrak{t}}$ to $\mathcal{G}_{\mathfrak{s}}$:

$$\begin{aligned} \mathfrak{V}|_{\iota(M_{\mathfrak{s}})} &\cong \iota(\tilde{M}_{\mathfrak{s}}) \times_{\mathcal{G}_{\mathfrak{s}}} L^2_{k-1}(F_2) \cong \mathfrak{V}^t \oplus \mathfrak{V}^n, \\ \text{where} \quad \mathfrak{V}^t &= \iota(\tilde{M}_{\mathfrak{s}}) \times_{\mathcal{G}_{\mathfrak{s}}} L^2_{k-1}(F^t_2), \\ \mathfrak{V}^n &= \iota(\tilde{M}_{\mathfrak{s}}) \times_{\mathcal{G}_{\mathfrak{s}}} L^2_{k-1}(F^n_2) \end{aligned} \tag{3.40}$$

The action (3.2) of S^1 on $V = W \oplus W \otimes L$ induces the trivial action of S^1 on the bundle $\mathfrak{V}^t$ and the standard action by complex multiplication on $\mathfrak{V}^n$; see the action (3.3) of S^1 on $\mathfrak{g}_{\mathfrak{t}} \cong i\underline{\mathbb{R}} \oplus L$ induced by the action of S^1 on V.

Note that although the splitting (3.40) is not well-defined away from the stratum $\iota(M_{\mathfrak{s}})$, since the full gauge group $\mathcal{G}_{\mathfrak{t}}$ does not preserve the splitting of the fiber, $L^2_{k-1}(F_2) \cong L^2_{k-1}(F^t_2) \oplus L^2_{k-1}(F^n_2)$, the circle action on $L^2_{k-1}(F_2)$ induced by the circle action (3.2) on V does preserve this splitting, with the circle acting trivially on $L^2_{k-1}(F^t_2)$ and by complex multiplication on $L^2_{k-1}(F^n_2)$. Hence, the vector bundle (3.39) is S^1 equivariant with respect to the circle action on the fiber $L^2_{k-1}(F^t_2 \oplus F^n_2)$ induced by the circle action (3.2) on V.

We use the preceding observation below to define a finite-rank, real subbundle $\Xi \subset \mathfrak{V}$ with an almost complex structure when restricted to $\iota(M_{\mathfrak{s}})$. (If the requirement that $\Xi \to \iota(M_{\mathfrak{s}})$ has an almost complex structure were dropped then we would not require the splitting into tangential and normal components in order to construct Ξ.) We shall need the following well-known consequence of Kuiper's result that the unitary group of a Hilbert space is contractible [46, Theorem 3] (see also [9, §I.7, p. 67] or [49, p. 208]):

Theorem 3.17. [46, p. 29] *Let M be a compact topological space or a space which has the homotopy type of a CW complex. Then every vector bundle over M, with fiber an infinite-dimensional, real or complex separable Hilbert space $\mathfrak{H}$ and structure group* $\mathrm{GL}(\mathfrak{H})$*, is trivial.*

We have the following version of Kuiper's result in the smooth category:

Corollary 3.18. *Let M be a smooth manifold which is compact or has the homotopy type of a CW complex. Then every C^∞ vector bundle over M, with fiber an infinite-dimensional, real or complex separable Hilbert space $\mathfrak{H}$ and structure group $\mathrm{GL}(\mathfrak{H})$, has a global C^∞ trivialization.*

Proof. If $\mathfrak{W}$ is a vector bundle over M with fiber a Hilbert space $\mathfrak{H}$, then Theorem 3.17 yields a C^0 trivialization $\boldsymbol{\tau}:\mathfrak{W}\to M\times\mathfrak{H}$, that is, a C^0 section $\boldsymbol{\tau}$ of the C^∞ Hilbert bundle $\mathrm{Hom}(\mathfrak{W}, M\times\mathfrak{H})$ over M which gives a linear isomorphism on each fiber. Now suppose that $\boldsymbol{\tau}_\infty$ is a C^∞ section of the bundle $\mathrm{Hom}(\mathfrak{W}, M\times\mathfrak{H})$. If $\boldsymbol{\tau}_\infty$ is chosen so that $\|\boldsymbol{\tau}(p)-\boldsymbol{\tau}_\infty(p)\|$ is sufficiently small for each $p\in M$, then $\boldsymbol{\tau}_\infty$ is an isomorphism on each fiber and gives the desired global, C^∞ trivialization. □

Note that if we are given a vector bundle $\mathfrak{W}$ over a space M with fiber a separable Hilbert space $\mathfrak{H}$ and structure group $G\subset \mathrm{U}(\mathfrak{H})$, then we have an isomorphism of vector bundles $\mathfrak{W}\cong \mathrm{Fr}(\mathfrak{W})\times_{\mathrm{U}(\mathfrak{H})}\mathfrak{H}$, where $\mathrm{Fr}(\mathfrak{W})$ is the principal $\mathrm{U}(\mathfrak{H})$ bundle of unitary frames for $\mathfrak{W}$. Recall that a *quasi-subbundle* $V'\to M$ of a smooth vector bundle $\pi:V\to M$ over a manifold M is a closed subspace $V'\subset V$ such that $\pi|_{V'}$ is still surjective and each fiber of $\pi|_{V'}$ is a linear subspace [40, Definition 1.2].

Theorem 3.19. *Assume that the moduli space $M_\mathfrak{s}$ contains no zero-section pairs. Then there is an open neighborhood $\mathcal{U}$ of the subspace $\iota(M_\mathfrak{s})$ in $\mathcal{C}_\mathfrak{t}$, which does not contain any zero-section pairs or other reducibles, and a finite-rank, smooth, trivial, vector subbundle $\Xi\to\mathcal{U}$ of $\mathfrak{V}\to\mathcal{U}$, which is S^1 equivariant with respect to the circle action (3.2), such that the following hold:*

1. *The restriction of Ξ to $\iota(M_\mathfrak{s})$ is a complex vector bundle.*
2. *The smooth bundle map $\Pi_{\Xi^\perp}:\mathfrak{V}\to\Xi^\perp$ defined by the fiberwise L^2-orthogonal projection $\Pi_{\Xi^\perp}$ onto the subbundle $\Xi^\perp\to\mathcal{U}$ restricts to a surjective fiber map $\Pi_{\Xi^\perp}:\mathrm{Ran}(D\mathfrak{S})_{A,\Phi}\to\Xi^\perp_{A,\Phi}$ for any point $[A,\Phi]\in\mathcal{U}$.*
3. *If (A,Φ) is an L^2_ℓ representative of a point in $\mathcal{U}$, for some integer $k\le\ell\le\infty$, then the fiber $\Xi_{A,\Phi}$ is contained in $L^2_\ell(F_2)=L^2_\ell(\Lambda^+\otimes\mathfrak{g}_\mathfrak{t})\oplus L^2_\ell(V^-)$.*

Proof. By Corollary 3.18, there is a smooth trivialization of $\mathfrak{V}^n$ given by a smooth isomorphism of complex Hilbert bundles,

$$\boldsymbol{\tau}:\mathfrak{V}^n\cong\iota(M_\mathfrak{s})\times L^2(F_2^n),$$

as the space $\mathcal{G}_\mathfrak{s}$ (of L^2_{k+1} gauge transformations) is a subgroup of the unitary group of the Hilbert space $L^2(F_2^n)$. We have an equality of L^2-orthogonal complements,

$$(\mathrm{Ran}\,D\mathfrak{S}_{A,\Phi})^\perp=(\mathrm{Ran}\,d^{1,n}_{A,\Phi})^\perp\subset L^2(F_2^n),$$

for every $(A,\Phi)\in\iota(\tilde M_\mathfrak{s})$, by Lemma 3.15 and the fact that $\mathrm{Ran}\,d^{1,t}_{A,\Phi}=L^2(F_2^t)$ by Proposition 2.16, for generic parameters τ. Thus, the family of operators $\{d^{1,n}_{A,\Phi}:(A,\Phi)\in\iota(\tilde M_\mathfrak{s})\}$ defines quasi-subbundles $\mathrm{Ran}\,\mathbf{d}^{1,n}$ and $(\mathrm{Ran}\,\mathbf{d}^{1,n})^\perp$ of $\mathfrak{V}^n\to\iota(M_\mathfrak{s})$ with fibers

$$\mathrm{Ran}\,\mathbf{d}^{1,n}|_{[A,\Phi]}=\{(A',\Phi',\mathrm{Ran}\,d^{1,n}_{A',\Phi'}):(A',\Phi')\in[A,\Phi]\}/\mathcal{G}_\mathfrak{t}.$$

For each point $[A,\Phi]\in\iota(M_\mathfrak{s})$, we define

$$\mathbf{V}^\perp_{[A,\Phi]}=\boldsymbol{\tau}(\mathrm{Ran}\,\mathbf{d}^{1,n}|_{[A,\Phi]})\subset L^2(F_2^n).$$

Observe that there is an open neighborhood $U_{[A,\Phi]}\subset\iota(M_\mathfrak{s})$ with the property that for all $[A',\Phi']\in U_{[A,\Phi]}$, the map

$$\boldsymbol{\tau}(\mathrm{Ran}\,\mathbf{d}^{1,n}|_{[A',\Phi']})\to\mathbf{V}^\perp_{[A,\Phi]}$$

defined by L^2-orthogonal projection onto $\mathbf{V}_{[A,\Phi]}^{\perp}$ is surjective. Since $\iota(M_{\mathfrak{s}})$ is compact, it has a finite subcover $U_{[A_\alpha,\Phi_\alpha]}$ of such neighborhoods. If $\mathbf{V}=\oplus_\alpha \mathbf{V}_{[A_\alpha,\Phi_\alpha]}$, then $\mathbf{V}$ is a finite-dimensional, complex subspace of $L^2(F_2^n)$. If we define $\Xi''\to\iota(M_{\mathfrak{s}})$ by setting

$$\Xi''=\boldsymbol{\tau}^{-1}\left(\iota(M_{\mathfrak{s}})\times\mathbf{V}\right),$$

then Ξ will be a complex, finite-dimensional, trivial subbundle of $\mathfrak{V}^n|_{\iota(M_{\mathfrak{s}})}$ such that fiberwise L^2-orthogonal projection onto the subbundle $\Xi^{\perp}\subset\mathfrak{V}^n|_{\iota(M_{\mathfrak{s}})}$ is surjective when restricted to the quasi-subbundle $\operatorname{Ran}\mathbf{d}^{1,n}$.

We now extend the bundle $\Xi''\to\iota(M_{\mathfrak{s}})$ to a subbundle of $\mathfrak{V}$ over an open neighborhood $\mathcal{U}$ of $\iota(M_{\mathfrak{s}})$ in $\mathcal{C}_{\mathfrak{t}}^0$, which does not contain any other reducible or zero-section pairs, and which is S^1 equivariant with respect to the circle action (3.2). The space $M_{\mathfrak{s}}$ is a smooth submanifold of the Riemannian manifold $\mathcal{C}_{\mathfrak{t}}^0$ by Lemma 3.16 and so it has an S^1-equivariant normal bundle $\pi:\mathfrak{N}\to\iota(M_{\mathfrak{s}})$ and an S^1-invariant tubular neighborhood given by an S^1-equivariant diffeomorphism

$$\mathbf{g}:\mathcal{O}\subset\mathfrak{N}\to\mathcal{U}\subset\mathcal{C}_{\mathfrak{t}}^0,$$

from an open, S^1-invariant neighborhood $\mathcal{O}$ of the zero section of $\mathfrak{N}$ onto an open, S^1-invariant neighborhood of $\iota(M_{\mathfrak{s}})$ in $\mathcal{C}_{\mathfrak{t}}^0$ (see, for example, [10, p. 306]). The bundle projection π and the diffeomorphism $\mathbf{g}$ define an S^1-equivariant, C^∞ retraction,

$$\mathbf{r}=\pi\circ\mathbf{g}^{-1}:\mathcal{U}\subset\mathcal{C}_{\mathfrak{t}}^0\to\iota(M_{\mathfrak{s}}).$$

The complex vector bundle $\Xi''\to\iota(M_{\mathfrak{s}})$ then extends to a vector bundle $\mathbf{r}^*\Xi''\to\mathcal{U}$, a subbundle of $\mathbf{r}^*(\mathfrak{V}|_{\iota(M_{\mathfrak{s}})})\to\mathcal{U}$, which are both S^1 equivariant with respect to the circle action (3.2). Since the map $\mathbf{r}$ is an S^1-equivariant, C^∞ retraction, there is an S^1-equivariant, C^∞ isomorphism [37, Theorem 4.1.5], [39],

$$\mathbf{f}:\mathfrak{V}|_{\mathcal{U}}\to\mathbf{r}^*(\mathfrak{V}|_{\iota(M_{\mathfrak{s}})}).$$

We obtain a C^∞ subbundle of the vector bundle $\mathfrak{V}|_{\mathcal{U}}$ by setting

$$\Xi'=\mathbf{f}^{-1}(\mathbf{r}^*\Xi'')\to\mathcal{U},$$

both S^1 equivariant with respect to the action (3.2). Because $\Xi''\to\iota(M_{\mathfrak{s}})$ is isomorphic to $\iota(M_{\mathfrak{s}})\times\mathbb{C}^{r_\Xi}$ as a complex vector bundle, for some $r_\Xi\in\mathbb{N}$, we obtain an isomorphism of S^1-equivariant vector bundles,

$$\Xi'\cong\mathcal{U}\times\mathbb{C}^{r_\Xi},$$

since the maps $\mathbf{f}$ and $\mathbf{r}$ are S^1 equivariant; the circle acts non-trivially on $\mathcal{U}$, except along the stratum $\iota(M_{\mathfrak{s}})$, and acts by complex multiplication on $\mathbb{C}^{r_\Xi}$.

By construction, the fiberwise L^2-orthogonal projection $\mathfrak{V}|_{A,\Phi}\to(\Xi'_{A,\Phi})^{\perp}$ restricts to a surjective map $\operatorname{Ran}d^1_{A,\Phi}\to(\Xi'_{A,\Phi})^{\perp}$ for any pair (A,Φ) representing a point in $\mathcal{U}$, after shrinking $\mathcal{U}$ if necessary.

Given an L^2_k pair (A,Φ) representing a point in $\mathcal{U}\subset\mathcal{C}_{\mathfrak{t}}^0$, our construction yields a subspace

$$\Xi'_{A,\Phi}\subset L^2(F_2)=L^2(\Lambda^+\otimes\mathfrak{g}_{\mathfrak{t}})\oplus L^2(V^-).$$

If $\ell\geq k$ is an integer, it does not necessarily follow that $\Xi'_{A,\Phi}$ is contained in the subspace of $L^2_{\ell-1}$ pairs when (A,Φ) is an L^2_ℓ pair. However, for any $t>0$ the heat operator

$$\exp(-t(1+d^1_{A,\Phi}d^{1,*}_{A,\Phi})):L^2(\Lambda^+\otimes\mathfrak{g}_{\mathfrak{t}})\oplus L^2(V^-)\to L^2_{\ell-1}(\Lambda^+\otimes\mathfrak{g}_{\mathfrak{t}})\oplus L^2_{\ell-1}(V^-)$$

is a bounded, $\mathcal{G}_{\mathfrak{t}}$-equivariant, S^1-equivariant, linear map and we can define

$$\Xi_{A,\Phi}=\exp(-t(1+d^1_{A,\Phi}d^{1,*}_{A,\Phi}))\Xi'_{A,\Phi}.$$

For small enough $t = t(\mathcal{U})$, the approximation properties of the heat kernel (see [24, Lemma A.1] for a similar application), ensure that

- $\Xi \to \mathcal{U}$ is a trivial, S^1-equivariant vector bundle, with the same rank as Ξ', and
- L^2-orthogonal projection $\mathfrak{V}|_{A,\Phi} \to \Xi^{\perp}_{A,\Phi}$ restricts to a surjective map $\operatorname{Ran} d^1_{A,\Phi} \to \Xi^{\perp}_{A,\Phi}$ for any pair (A,Φ) representing a point in $\mathcal{U}$.

We let $\Pi_\Xi : \mathfrak{V} \to \Xi$ be the C^∞ bundle map defined by fiberwise L^2-orthogonal projection. This completes the proof. □

3.5.3. *Definition of the thickened moduli space and the link.* With Theorem 3.19 in place, we can finally construct the required ambient, finite-dimensional, smooth manifold of Definition 3.14 and the link of the singular stratum of reducibles.

Definition 3.20. Assume $M_{\mathfrak{s}}$ contains no zero-section pairs. Let Ξ be a finite-rank, smooth, S^1-equivariant vector bundle over an open neighborhood of $\iota(M_{\mathfrak{s}})$ in $\mathcal{C}_{\mathfrak{t}}$ such that, as in Theorem 3.19, L^2-orthogonal projection gives a surjective map of quasi vector bundles $\operatorname{Ran} D\mathfrak{S} \to \Xi^\perp$ over $M_{\mathfrak{s}}$. We say that Ξ is a *stabilizing bundle* for $\operatorname{Ran} D\mathfrak{S}$ and call

$$\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s}) = (\Pi_{\Xi^\perp}\mathfrak{S})^{-1}(0) \subset \mathcal{C}_{\mathfrak{t}} \tag{3.41}$$

the *thickened moduli space* defined by Ξ. If $(A,\Phi) \in \tilde{\mathcal{C}}_{\mathfrak{t}}$ represents $[A,\Phi] \in \mathcal{C}_{\mathfrak{t}}$, we write $\Xi|_{(A,\Phi)}$ for the subspace of $L^2_{k-1}(F_2)$ representing the fiber of Ξ over $[A,\Phi]$. We then define

$$N_{\mathfrak{t}}(\Xi,\mathfrak{s}) = \operatorname{Ker}(\Pi_{\Xi^\perp}\boldsymbol{\mathcal{D}}^n), \tag{3.42}$$

a complex, finite-rank, smooth vector bundle over $M_{\mathfrak{s}}$ with fibers

$$N_{\mathfrak{t}}(\Xi,\mathfrak{s})|_{[A,\Phi]} \cong \operatorname{Ker}\left(d^{0,n,*}_{A,\Phi} + \Pi_{\Xi^\perp} d^{1,n}_{A,\Phi}\right),$$

noting that $\mathcal{D}^n_{A,\Phi} = d^{0,n,*}_{A,\Phi} + d^{1,n}_{A,\Phi}$ for $[A,\Phi] \in \iota(M_{\mathfrak{s}})$.

Theorem 3.21. *Suppose that the spin^u structure $\mathfrak{t}$ admits a reduction $\mathfrak{t} = \mathfrak{s}\oplus\mathfrak{s}\otimes L$. Assume $M_{\mathfrak{s}}$ contains no zero-section pairs. Then the following hold:*

1. *There is an S^1-invariant, open neighborhood $\mathcal{U}$ of $\iota(M_{\mathfrak{s}})$ in $\mathcal{C}_{\mathfrak{t}}$ such that the zero locus $\mathcal{U} \cap \mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is regular and so a manifold of dimension $\dim \mathcal{M}_{\mathfrak{t}} + \operatorname{rank}_{\mathbb{R}} \Xi$.*
2. *The space $M_{\mathfrak{s}}$ is a smooth, S^1-invariant submanifold of $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$.*
3. *The bundle $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is a normal bundle for the submanifold $\iota : M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ and the tubular map is equivariant with respect to the circle action on $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ given by the trivial action on the base $M_{\mathfrak{s}}$ and complex multiplication on the fibers, and the circle action on $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ induced from the S^1 action (3.2).*
4. *The restriction of the section $\mathfrak{S}$ to $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ takes values in Ξ and vanishes transversely on $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s}) - \iota(M_{\mathfrak{s}})$.*

Proof. The space $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is the zero locus of the section $\mathfrak{T} = \Pi_{\Xi^\perp}\mathfrak{S}$ of a vector subbundle $\Xi^\perp \to \mathcal{U} \subset \mathcal{C}_{\mathfrak{t}}$ of $\mathfrak{V}|_{\mathcal{U}}$ constructed in Theorem 3.19, for some open neighborhood $\mathcal{U}$ of $\iota(M_{\mathfrak{s}})$ in $\mathcal{C}_{\mathfrak{t}}$. For any point $[A,\Phi] \in \iota(M_{\mathfrak{s}}) \subset \mathfrak{S}^{-1}(0)$, we have

$$(D\mathfrak{T})_{A,\Phi} = \Pi_{\Xi^\perp}(D\mathfrak{S})_{A,\Phi}.$$

(The differential of Π_Ξ does not appear here since $\mathfrak{S}[A,\Phi] = 0$.) According to Definition 3.20, the L^2-orthogonal projection $\Pi_{\Xi^\perp}$ gives a surjective map

$$\Pi_{\Xi^\perp} : \operatorname{Ran}(D\mathfrak{S})_{A,\Phi} \to \Xi^\perp_{A,\Phi}$$

and thus $(D\mathfrak{T})_{A,\Phi}$ is surjective at all points $[A,\Phi]$ in the image of $M_{\mathfrak{s}}$. Surjectivity is an open condition, so we may assume that $D\mathfrak{T}$ is surjective on the open neighborhood $\mathcal{U}$ of

$\iota(M_{\mathfrak{s}}) \subset \mathcal{C}_{\mathfrak{t}}$, after shrinking $\mathcal{U}$ if necessary. The zero locus of $\mathfrak{T}$ in this open set is regular and thus a smooth submanifold of $\mathcal{C}_{\mathfrak{t}}$, which gives Assertion (1).

By Lemma 3.16 the space $M_{\mathfrak{s}}$ is a smooth submanifold of $\mathcal{C}_{\mathfrak{t}}$ and as its image is contained in $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$, it is also a smooth submanifold of $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ and Assertion (2) follows, as the zero locus of an S^1-equivariant section is S^1 invariant.

We observe that $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is the normal bundle of $M_{\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$, since

$$
\begin{aligned}
T_{A,\Phi}\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s}) &= \operatorname{Ker} d^{0,*}_{A,\Phi} \cap \operatorname{Ker}\Pi_{\Xi^\perp}(D\mathfrak{S})_{A,\Phi} \\
&= \left(\operatorname{Ker} d^{0,t,*}_{A,\Phi} \oplus \operatorname{Ker} d^{0,n,*}_{A,\Phi}\right) \cap \left(\operatorname{Ker} d^{1,t}_{A,\Phi} \oplus \operatorname{Ker}\Pi_{\Xi^\perp} d^{1,n}_{A,\Phi}\right) \\
&= \operatorname{Ker}\mathcal{D}^t_{A,\Phi} \oplus \operatorname{Ker}\Pi_{\Xi^\perp}\mathcal{D}^n_{A,\Phi} \\
&= T_{A,\Phi}\iota(M_{\mathfrak{s}}) \oplus N_{\mathfrak{t}}(\Xi,\mathfrak{s})|_{A,\Phi}.
\end{aligned}
\tag{3.43}
$$

In the second equality above, we make use of the fact that the fibers of the vector bundle $\Xi \to \iota(M_{\mathfrak{s}})$ are contained in $L^2_{k-1}(F_2^n)$ and so (by definition) $\Pi_\Xi = 0$ on $L^2_{k-1}(F_0^t) \oplus L^2_{k-1}(F_2^t) \supset \operatorname{Ran}\mathcal{D}^t_{A,\Phi}$. Also, the splitting (3.43) of tangent spaces corresponds to the splitting of Hilbert spaces $L^2_k(F_1) \cong L^2_k(F_1^t) \oplus L^2_k(F_1^n)$, defined by the subspaces (3.21). Hence, the isomorphism (3.43) is equivariant with respect to the circle action (3.2) on the subspace $T_{A,\Phi}\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ of $L^2_k(F_1)$, the trivial action on the subspace $T_{A,\Phi}\iota(M_{\mathfrak{s}})$ of $L^2_k(F_1^t)$, and complex multiplication on the subspace $N_{\mathfrak{t}}(\Xi,\mathfrak{s})|_{A,\Phi}$ of $L^2_k(F_1^n)$. This proves Assertion (3).

Because $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is given by $\mathcal{U}\cap(\Pi_{\Xi^\perp}\mathfrak{S})^{-1}(0) = \mathcal{U}\cap\mathfrak{S}^{-1}(\Xi)$, we see that $\mathfrak{S}$ takes values in Ξ on $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ and so $\Pi_{\Xi^\perp}\mathfrak{S} = \mathfrak{S}$ on $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$. According to Theorem 2.13, our transversality result for the section $\mathfrak{S}$ of $\mathfrak{V}|_{\mathcal{U}} = \Xi^\perp \oplus \Xi$ defined by the PU(2) monopole equations (2.32), the section $\mathfrak{S}$ vanishes transversely on $\mathcal{U} - \iota(M_{\mathfrak{s}})$ with zero locus $(\mathcal{U}\cap\mathcal{M}_{\mathfrak{t}}) - \iota(M_{\mathfrak{s}})$. This implies that for each $[A,\Phi]$ in $(\mathcal{U}-\iota(M_{\mathfrak{s}}))\cap\mathfrak{S}^{-1}(0)$, the differential

$$
(D\mathfrak{S})_{A,\Phi} : T_{[A,\Phi]}\mathcal{C}^{*,0}_{\mathfrak{t}} \to \mathfrak{V}|_{[A,\Phi]}
$$

is surjective. But we have the identifications

$$
\begin{aligned}
T_{[A,\Phi]}\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s}) &= \operatorname{Ker}(D(\Pi_{\Xi^\perp}\mathfrak{S}))_{[A,\Phi]} \\
&= \operatorname{Ker}(\Pi_{\Xi^\perp} D\mathfrak{S})_{[A,\Phi]} \quad (\text{since } \mathfrak{S}(A,\Phi)=0) \\
&= (D\mathfrak{S})^{-1}_{[A,\Phi]}(\Xi|_{[A,\Phi]}),
\end{aligned}
$$

and so one has a surjective differential

$$
(D\mathfrak{S})_{[A,\Phi]} : T_{[A,\Phi]}\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s}) \to \Xi|_{[A,\Phi]},
$$

for $[A,\Phi] \in (\mathcal{U}-\iota(M_{\mathfrak{s}}))\cap\mathfrak{S}^{-1}(0)$. Thus, the sections $\Pi_{\Xi^\perp}\mathfrak{S}$ and $\mathfrak{S}$—which are equal on $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$—vanish transversely when restricted to $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$, proving Assertion (4). □

The equivariant tubular neighborhood theorem (see [10, p. 306] for the finite-dimensional, G-equivariant case and [48] for the infinite-dimensional case) provides an embedding

$$
\gamma : \mathcal{O} \hookrightarrow \mathcal{C}_{\mathfrak{t}},
\tag{3.44}
$$

mapping an open, S^1-invariant neighborhood $\mathcal{O}$ of the zero-section $M_{\mathfrak{s}} \subset N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ onto an open neighborhood of the submanifold $\iota : M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ which covers the embedding ι; the map (3.44) is S^1-equivariant with respect to scalar multiplication on the fibers of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ and the circle action induced by (3.2) on $\mathcal{C}_{\mathfrak{t}}$.

The map $\boldsymbol{\gamma}$ then descends to a homeomorphism, and a diffeomorphism on smooth strata, from the zero locus $\boldsymbol{\varphi}^{-1}(0)/S^1$ in $N_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1$ onto an open neighborhood of $\iota(M_{\mathfrak{s}})$ in the actual moduli space, $\mathcal{M}_{\mathfrak{t}}$, where we define

$$\boldsymbol{\varphi} = \Pi_{\Xi}\mathfrak{S}\circ\boldsymbol{\gamma}, \tag{3.45}$$

to be an S^1-equivariant *obstruction section* over $\mathcal{O}\subset N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ of the S^1 equivariant *obstruction bundle*

$$\boldsymbol{\gamma}^*\Xi \to N_{\mathfrak{t}}(\Xi,\mathfrak{s}). \tag{3.46}$$

This descends to a vector bundle

$$(\boldsymbol{\gamma}^*\Xi)/S^1 \to N_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1$$

on the complement of the zero section, $M_{\mathfrak{s}}\subset N_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1$, whose Euler class may be computed from

$$(\boldsymbol{\gamma}^*\Xi)/S^1 \cong (\pi_N^*\Xi)/S^1 \to N_{\mathfrak{t}}(\Xi,\mathfrak{s})/S^1,$$

where $\pi_N : N_{\mathfrak{t}}(\Xi,\mathfrak{s}) \to M_{\mathfrak{s}}$ is the projection.

We can now construct the link of $M_{\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}}$, following Definition 3.14.

Definition 3.22. Assume $M_{\mathfrak{s}}$ contains no zero-section pairs. Let $N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s})\subset N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ be the sphere bundle of fiber vectors of length ε and set

$$\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s}) = N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s})/S^1.$$

The *link of the stratum* $\iota(M_{\mathfrak{s}})\subset\mathcal{M}_{\mathfrak{t}}$ of reducible PU(2) monopoles is defined by

$$\mathbf{L}_{\mathfrak{t},\mathfrak{s}} = \boldsymbol{\gamma}\left(\boldsymbol{\varphi}^{-1}(0)\cap\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})\right)\subset\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1. \tag{3.47}$$

The orientation for $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ is defined by the orientation on $M_{\mathfrak{s}}$ and in turn from the homology orientation Ω (see [56, §6.6]) and the complex structure on the fibers of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ given by the S^1 action. We treat the quotient $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ as the complex projectivization of a complex vector bundle; we use the complex orientation on the obstruction bundle $\boldsymbol{\gamma}^*\Xi$ in (3.46).

The following lemma shows that the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ can be represented homologically.

Lemma 3.23. *Assume $M_{\mathfrak{s}}$ contains no zero-section pairs. Then for generic ε, the section $\boldsymbol{\varphi}$ of the obstruction bundle $\boldsymbol{\gamma}^*\Xi$ in* (3.46) *vanishes transversely on $N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s})$.*

Proof. From the final statement of Theorem 3.21, we see that $\boldsymbol{\varphi}$ vanishes transversely on $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})-\iota(M_{\mathfrak{s}})$ and so $\boldsymbol{\varphi}$ cuts out the zero locus, $\mathcal{M}_{\mathfrak{t}}-\iota(M_{\mathfrak{s}})$, as a regular submanifold of $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})-\iota(M_{\mathfrak{s}})$. Then for generic values of ε, the zero locus will intersect $N_{\mathfrak{t}}^{\varepsilon}(\Xi,\mathfrak{s})$ transversely. □

Lemma 3.23 implies that

$$[\mathbf{L}_{\mathfrak{t},\mathfrak{s}}] = e\left((\boldsymbol{\gamma}^*\Xi)/S^1\right)\cap[\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})] \tag{3.48}$$

is the homology class of the link in Definition 3.22.

3.5.4. *Group actions and lifts of the normal bundle embedding to the pre-configuration space.* In [18] we shall need a lift of the S^1-equivariant diffeomorphism $\boldsymbol{\gamma}$ from a neighborhood $\mathcal{O} \subset N_{\mathfrak{t}}(\Xi, \mathfrak{s})$ of the zero-section $M_{\mathfrak{s}}$ onto on open neighborhood of $\iota(M_{\mathfrak{s}})$ in the thickened moduli space $\mathcal{M}_{\mathfrak{t}}(\Xi, \mathfrak{s}) \subset \mathcal{C}_{\mathfrak{t}}$,

$$\tilde{\boldsymbol{\gamma}} : \tilde{\mathcal{O}} \subset \tilde{N}_{\mathfrak{t}}(\Xi, \mathfrak{s}) \to \tilde{\mathcal{M}}_{\mathfrak{t}}(\Xi, \mathfrak{s}), \tag{3.49}$$

where $\tilde{\mathcal{O}} \subset \tilde{N}_{\mathfrak{t}}(\Xi, \mathfrak{s})$ is an S^1 and $\mathcal{G}_{\mathfrak{s}}$ invariant open neighborhood of $\tilde{M}_{\mathfrak{s}}$ and $\tilde{\mathcal{M}}_{\mathfrak{t}}(\Xi, \mathfrak{s}) = \pi^{-1}(\mathcal{M}_{\mathfrak{t}}(\Xi, \mathfrak{s}))$. It is convenient to describe the construction here. As usual, we need only consider the case where $M_{\mathfrak{s}}$ contains no zero-section pairs.

To see what should be the "correct" equivariance properties of the lift (3.49), we first consider the obvious extension $\iota : N_{\mathfrak{t}}(\Xi, \mathfrak{s}) \to \mathcal{C}_{\mathfrak{t}}$ of the embedding $\iota : M_{\mathfrak{s}} \hookrightarrow \mathcal{C}_{\mathfrak{t}}$, since $\boldsymbol{\gamma}$ approximates this extension on a small open neighborhood of the zero section, $M_{\mathfrak{s}}$. Let $\tilde{M}_{\mathfrak{s}} = \pi^{-1}(M_{\mathfrak{s}}) \subset \tilde{\mathcal{C}}_{\mathfrak{s}}$ be the preimage of $M_{\mathfrak{s}} \subset \mathcal{C}_{\mathfrak{s}}$ under the projection $\pi : \tilde{\mathcal{C}}_{\mathfrak{s}} \to \tilde{\mathcal{C}}_{\mathfrak{s}}/\mathcal{G}_{\mathfrak{s}}$, and let

$$\tilde{N}_{\mathfrak{t}}(\Xi, \mathfrak{s}) = \pi^* N_{\mathfrak{t}}(\Xi, \mathfrak{s}) \to \tilde{M}_{\mathfrak{s}}$$

be the $\mathcal{G}_{\mathfrak{s}}$-equivariant pullback bundle, so

$$\tilde{N}_{\mathfrak{t}}(\Xi, \mathfrak{s}) \subset \tilde{M}_{\mathfrak{s}} \times F_1^n,$$

where $F_1^n \subset F_1 \cong F_1^t \oplus F_1^n$ is the complex Hilbert subspace in (3.21). The bundle $\tilde{N}_{\mathfrak{t}}(\Xi, \mathfrak{s})$ is complex, with trivial circle action on $\tilde{M}_{\mathfrak{s}}$ and action by complex multiplication on F_1^n.

The S^1-equivariant map $\iota : N_{\mathfrak{s}}(\Xi, \mathfrak{s}) \to \mathcal{C}_{\mathfrak{t}}$, $[B, \Psi, \eta] \mapsto [\iota(B, \Psi) + \eta]$, is covered by

$$\iota : \tilde{N}_{\mathfrak{t}}(\Xi, \mathfrak{s}) \to \tilde{\mathcal{C}}_{\mathfrak{t}}, \quad (B, \Psi, \eta) \mapsto \iota(B, \Psi) + \eta. \tag{3.50}$$

This map is equivariant with respect to the embedding $\varrho : \mathcal{G}_{\mathfrak{s}} \hookrightarrow \mathcal{G}_{\mathfrak{t}}$ in definition (3.10), for the following domain and range $\mathcal{G}_{\mathfrak{s}}$ actions:

- The action of $s \in \mathcal{G}_{\mathfrak{s}}$ on $\tilde{N}_{\mathfrak{t}}(\Xi, \mathfrak{s})$ given by the usual gauge group action on $(B, \Psi) \in \tilde{M}_{\mathfrak{s}}$ and the action of $\mathcal{G}_{\mathfrak{s}}$ on $\eta = (\beta, \psi') \in F_1^n$ induced by the isomorphism $F_1 \cong F_1^t \oplus F_1^n$ and the embedding $\varrho : \mathcal{G}_{\mathfrak{s}} \hookrightarrow \mathcal{G}_{\mathfrak{t}}$, as described in §3.4.2, so
$$(s, (B, \Psi, \beta, \psi')) \mapsto (s(B, \Psi), s^{-2}\beta, s^{-1}\psi').$$
- The action of $\varrho(s) \in \mathcal{G}_{\mathfrak{t}}$ on $\tilde{\mathcal{C}}_{\mathfrak{t}}$ and F_1 in definition (3.10), so
$$(s, \iota(B, \Psi) + \eta) \mapsto \varrho(s)(\iota(B, \Psi) + \eta).$$

  The fact that $\varrho(s)\iota(B, \Psi) = \iota(s(B, \Psi))$ was noted in Lemma 3.11, while we see that $\iota(s^{-2}\beta, s^{-1}\psi') = \varrho(s)\iota(\beta, \psi')$ by the remarks following equation (3.23) in §3.4.2.

The map (3.50) is also S^1 equivariant for the following domain and range circle actions:

- The trivial S^1 action on $\tilde{M}_{\mathfrak{s}}$ and action by complex multiplication on F_1^n,
$$(e^{i\theta}, (B, \Psi, \eta)) \mapsto (B, \Psi, e^{i\theta}\eta).$$
- The S^1 action (3.2) on $\tilde{\mathcal{C}}_{\mathfrak{t}}$, induced by the trivial action on the factor W of $V = W \oplus W \otimes L$ and complex multiplication on $W \otimes L$, so on the image (3.50) one has
$$(e^{i\theta}, \iota(B, \Psi) + \eta) \mapsto \iota(B, \Psi) + e^{i\theta}\eta,$$
  recalling that the points $\iota(B, \Psi)$ are fixed by the action (3.2).

We now turn to the construction of the map (3.49), which we shall require to have the same $\mathcal{G}_{\mathfrak{s}}$ and S^1 equivariance properties as the map (3.50). If $[B, \Psi] \in M_{\mathfrak{s}}$, then Proposition 2.8 in [24] yields an open neighborhood $\tilde{U}_{\iota(B,\Psi)}$ of $\iota(B, \Psi)$ in the slice $\iota(B, \Psi) + \operatorname{Ker} d^{0,*}_{\iota(B,\Psi)}$, such that projection onto $\pi(\tilde{U}_{\iota(B,\Psi)}) = U_{\iota(B,\Psi)}$ gives a local parameterization for an open neighborhood of $[\iota(B,\Psi)]$ in $\mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$; the pair $\iota(B,\Psi)$ has trivial stabilizer in $\mathcal{G}_{\mathfrak{t}}$ since $\Psi \not\equiv 0$. Because $M_{\mathfrak{s}}$ is compact we can assume (by shrinking $\mathcal{O}$ if necessary) that

$$\boldsymbol{\gamma}[B,\Psi,\eta] \in U_{[\iota(B,\Psi)]},$$

for all $[B,\Psi,\eta] \in \mathcal{O} \subset N_{\mathfrak{t}}(\Xi,\mathfrak{s})$. Hence,

$$\boldsymbol{\gamma}[B,\Psi,\eta] = [\iota(B,\Psi) + \boldsymbol{\gamma}_0(B,\Psi,\eta)]$$

for a slice element $\boldsymbol{\gamma}_0(B,\Psi,\eta) \in \operatorname{Ker} d^{0,*}_{\iota(B,\Psi)}$ uniquely determined by $\iota(B,\Psi,\eta)$, and we can therefore set

$$\tilde{\boldsymbol{\gamma}}(B,\Psi,\eta) = \iota(B,\Psi) + \boldsymbol{\gamma}_0(B,\Psi,\eta), \quad (B,\Psi,\eta) \in \tilde{\mathcal{O}} \subset \tilde{N}_{\mathfrak{t}}(\Xi,\mathfrak{s}), \tag{3.51}$$

where $\tilde{\mathcal{O}}$ is the preimage of $\mathcal{O} = \tilde{\mathcal{O}}/\mathcal{G}_{\mathfrak{s}}$. Note that $\boldsymbol{\gamma}_0(B,\Psi,\eta) \approx \eta$. If $s \in \mathcal{G}_{\mathfrak{s}}$, then the preceding equation yields

$$\varrho(s)\tilde{\boldsymbol{\gamma}}(B,\Psi,\eta) = \iota(s(B,\Psi)) + \varrho(s)\boldsymbol{\gamma}_0(B,\Psi,\eta), \tag{3.52}$$

and we again have $\varrho(s)\boldsymbol{\gamma}_0(B,\Psi,\eta) \in \operatorname{Ker} d^{0,*}_{\iota(s(B,\Psi))}$, by the $\mathcal{G}_{\mathfrak{t}}$-equivariance of the slice condition and the fact that $\varrho(s)\iota(B,\Psi) = \iota(s(B,\Psi))$. On the other hand,

$$\tilde{\boldsymbol{\gamma}}(s(B,\Psi,\eta)) = \iota(s(B,\Psi)) + \boldsymbol{\gamma}_0(s(B,\Psi,\eta)), \tag{3.53}$$

while $\boldsymbol{\gamma}[s(B,\Psi,\eta)] = \boldsymbol{\gamma}[B,\Psi,\eta] \in \mathcal{O} = \tilde{\mathcal{O}}/\mathcal{G}_{\mathfrak{s}}$. Hence, comparing equations (3.52) and (3.53), we see that we must have

$$\tilde{\boldsymbol{\gamma}}(s(B,\Psi,\eta)) = \varrho(s)\tilde{\boldsymbol{\gamma}}(B,\Psi,\eta).$$

Therefore, the map $\tilde{\boldsymbol{\gamma}}$ has the same $\mathcal{G}_{\mathfrak{s}}$-equivariance properties as the map $\iota : \tilde{N}_{\mathfrak{t}}(\Xi,\mathfrak{s}) \to \tilde{\mathcal{C}}_{\mathfrak{t}}$ in (3.50); a very similar argument shows that it has the same S^1-equivariance properties.

3.6. **The link of a stratum of reducible monopoles: Chern character of the normal bundle.** To compute intersection pairings on the space $\mathbb{P}N_{\mathfrak{t}}(\Xi,\mathfrak{s})$, we need to know the Chern classes of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$. The Chern character of the vector bundle $N_{\mathfrak{t}}(\Xi,\mathfrak{s}) \to M_{\mathfrak{s}}$ is computed by observing that, as elements of $K(M_{\mathfrak{s}})$, we have $[N_{\mathfrak{t}}(\Xi,\mathfrak{s})] = \operatorname{Index}\boldsymbol{\mathcal{D}}^n + [\Xi]$, where $\boldsymbol{\mathcal{D}}^n$ is the normal component (3.37) of the family of deformation operators $\boldsymbol{\mathcal{D}}$ parameterized by $M_{\mathfrak{s}}$. For convenience in this section, we shall often omit explicit mention of the embedding map $\iota : M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ and write the bundles $\iota^* N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ and $\iota^*(\operatorname{Index}\boldsymbol{\mathcal{D}}^n)$ over $M_{\mathfrak{s}}$ simply as $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ and $\operatorname{Index}\boldsymbol{\mathcal{D}}^n$, respectively. We then use the Atiyah-Singer index theorem for families to express $\operatorname{ch}(N_{\mathfrak{t}}(\Xi,\mathfrak{s}))$ in terms of the cohomology classes on $H^\bullet(\mathcal{C}^0_{\mathfrak{s}};\mathbb{R})$ described in §2.4.2.

From (3.37) we obtain a family of elliptic differential operators

$$\begin{array}{ccc} C^\infty(\mathbf{E}) & \xrightarrow{\boldsymbol{\mathcal{D}}^n} & C^\infty(\mathbf{F}) \\ \downarrow & & \downarrow \\ M_{\mathfrak{s}} \times X & \xrightarrow{\mathrm{id}} & M_{\mathfrak{s}} \times X \end{array}$$

where $C^\infty(\cdot)$ denotes the space of smooth sections of the families of finite-rank vector bundles $\mathbf{E}$, $\mathbf{F}$ over $M_\mathfrak{s} \times X$ defined by

$$
\begin{aligned}
\mathbf{E} &= \tilde{M}_\mathfrak{s} \times_{\mathcal{G}_\mathfrak{s}} F_1^n \\
&= \tilde{M}_\mathfrak{s} \times_{\mathcal{G}_\mathfrak{s}} \left((\Lambda^1 \otimes L) \oplus W^+ \otimes L\right), \\
\mathbf{F} &= \tilde{M}_\mathfrak{s} \times_{\mathcal{G}_\mathfrak{s}} (F_0^n \oplus F_2^n) \\
&= \tilde{M}_\mathfrak{s} \times_{\mathcal{G}_\mathfrak{s}} \left((\Lambda^0 \oplus \Lambda^+) \otimes L \oplus W^- \otimes L\right).
\end{aligned} \tag{3.54}
$$

Recall from the paragraph following (3.23) that an element $s \in \mathcal{G}_\mathfrak{s}$ acts on the bundles listed in (3.54) as multiplication by s^{-2} on the factors $\Lambda^j \otimes L$ and as multiplication by s^{-1} on the factors $W^\pm \otimes L$.

Recall that $N_\mathfrak{t}(\Xi, \mathfrak{s}) = \operatorname{Ker} \Pi_{\Xi^\perp} \boldsymbol{\mathcal{D}}^n$ by definition (3.42). On the other hand, by equation (3.37), we see that

$$\operatorname{Coker} \Pi_{\Xi^\perp} \mathcal{D}^n_{\iota(B,\Psi)} = \operatorname{Coker} d^{0,n,*}_{\iota(B,\Psi)} \oplus \operatorname{Coker} \Pi_{\Xi^\perp} d^{1,n}_{\iota(B,\Psi)}.$$

For any $[B, \Psi]$, we have $\operatorname{Coker} d^{0,*}_{\iota(B,\Psi)} = 0$, as the stabilizer of $\iota(B, \Psi)$ is *trivial in* $\mathcal{G}_\mathfrak{t}$, so $\operatorname{Coker} d^{0,n,*}_{\iota(B,\Psi)} = 0$, while Lemma 3.15 and Proposition 2.16 implies that $\operatorname{Coker} d^{1,t}_{\iota(B,\Psi)} = 0$. Hence,

$$\operatorname{Coker} \Pi_{\Xi^\perp} d^{1,t}_{\iota(B,\Psi)} = 0$$

and

$$
\begin{aligned}
\operatorname{Coker} \Pi_{\Xi^\perp} \mathcal{D}^n_{\iota(B,\Psi)} &= \operatorname{Coker} \Pi_{\Xi^\perp} d^{1,n}_{\iota(B,\Psi)} \oplus \operatorname{Coker} \Pi_{\Xi^\perp} d^{1,t}_{\iota(B,\Psi)} \\
&= \operatorname{Coker} \Pi_{\Xi^\perp} d^{1}_{\iota(B,\Psi)} = \operatorname{Coker} \Pi_{\Xi^\perp} (D\mathfrak{S})_{\iota(B,\Psi)}.
\end{aligned}
$$

Because L^2-orthogonal projection from $\operatorname{Ran}(D\mathfrak{S})_{\iota(B,\Psi)}$ surjects onto $\Xi^\perp_{\iota(B,\Psi)}$ for all points $[B, \Psi] \in M_\mathfrak{s}$, we obtain the identity

$$\operatorname{Coker} \Pi_{\Xi^\perp} \boldsymbol{\mathcal{D}}^n = \operatorname{Coker} \Pi_{\Xi^\perp} \cong \Xi.$$

The subbundle Ξ of $\mathfrak{V} \to M_\mathfrak{s}$ is trivial by the construction of Theorem 3.19. From the stabilization construction of [9, §I.7.B] and the preceding remarks we see that, as elements of the K-theory group $K(M_\mathfrak{s})$,

$$
\begin{aligned}
\operatorname{Index} \boldsymbol{\mathcal{D}}^n &= [\operatorname{Ker} \Pi_{\Xi^\perp} \mathcal{D}^n] - [\operatorname{Coker} \Pi_{\Xi^\perp} \mathcal{D}^n] \\
&= [N_\mathfrak{t}(\Xi, \mathfrak{s})] - [\Xi] \\
&= [N_\mathfrak{t}(\Xi, \mathfrak{s})] - [M_\mathfrak{s} \times \mathbb{C}^{r_\Xi}],
\end{aligned} \tag{3.55}
$$

where $r_\Xi = \operatorname{rank}_\mathbb{C} \Xi$. To compute $\operatorname{ch}(\operatorname{Index} \boldsymbol{\mathcal{D}}^n)$ we shall apply the Atiyah-Singer index theorem for families of Dirac operators.

Proposition 3.24. [6], [14, Theorem 5.1.16] *Let X be a closed, oriented, smooth four-manifold with spin^c structure (ρ, W). Suppose $\mathcal{E} \to T \times X$ is a locally trivial family of complex vector bundles over X, parameterized by a compact space T, with a connection A_t on the bundle $\mathcal{E}_t = \mathcal{E}|_{\{t\} \times X}$ for all $t \in T$. Then the Chern character of the index bundle of the family of Dirac operators parameterized by T,*

$$
\begin{array}{ccc}
C^\infty(\mathcal{E} \otimes W^+) & \xrightarrow{\mathbf{D}} & C^\infty(\mathcal{E} \otimes W^-) \\
\downarrow & & \downarrow \\
T \times X & \xrightarrow{\mathrm{id}} & T \times X
\end{array}
$$

given by $D_{A_t} : C^\infty(\mathcal{E}_t \otimes W^+) \to C^\infty(\mathcal{E}_t \otimes W^-)$, $t \in T$, *is*

$$\begin{aligned}\operatorname{ch}(\operatorname{Index}(\mathbf{D}, \mathcal{E}, W)) &= \operatorname{ch}(\mathcal{E}) \operatorname{ch}(\operatorname{Index}(D, W)) \\ &= \operatorname{ch}(\mathcal{E}) e^{\frac{1}{2} c_1(W^+)} \hat{\mathcal{A}}(X)/[X],\end{aligned}$$

where $D : C^\infty(W^+) \to C^\infty(W^-)$ *is the Dirac operator defined by the given spin*^{c} *structure.*

The index of the family of operators in Proposition 3.24 defines a group homomorphism [14, p. 184],

$$\operatorname{Index}(\mathbf{D}, \cdot\ ) : K(T \times X) \to K(T),$$

by taking the element $[\mathcal{E}]$ of $K(T \times X)$ to $\operatorname{Index}(\mathbf{D}, \mathcal{E}, W) = [\operatorname{Ker} \mathbf{D}] - [\operatorname{Coker} \mathbf{D}]$ in $K(T)$, a virtual vector bundle over the parameter space T. We now compute $\operatorname{ch}(N_{\mathfrak{t}}(\Xi, \mathfrak{s}))$ in the following steps:

- Identify $\boldsymbol{\mathcal{D}}^n$ with the sum of a pair of families of Dirac operators, $\mathbf{D}'$ and $\mathbf{D}''$.
- Compute $\operatorname{ch}(N_{\mathfrak{t}}(\Xi, \mathfrak{s}))$, using Proposition 3.24 to compute the Chern characters of the index bundles of these families of Dirac operators.

The first step is accomplished in part by introducing the following families of operators,

$$\begin{array}{ccccccc} C^\infty(\mathbf{E}') & \xrightarrow{\boldsymbol{\delta}} & C^\infty(\mathbf{F}') & & C^\infty(\mathbf{E}'') & \xrightarrow{\mathbf{D}''} & C^\infty(\mathbf{F}'') \\ \downarrow & & \downarrow & \text{and} & \downarrow & & \downarrow \\ M_{\mathfrak{s}} \times X & \xrightarrow{\mathrm{id}} & M_{\mathfrak{s}} \times X & & M_{\mathfrak{s}} \times X & \xrightarrow{\mathrm{id}} & M_{\mathfrak{s}} \times X \end{array} \tag{3.56}$$

where $\mathbf{E} = \mathbf{E}' \oplus \mathbf{E}''$ and $\mathbf{F} = \mathbf{F}' \oplus \mathbf{F}''$, and we have defined

$$\begin{aligned} \mathbf{E}' &= \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (\Lambda^1 \otimes L) &\text{and}\quad \mathbf{E}'' &= \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (W^+ \otimes L), \\ \mathbf{F}' &= \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} ((\Lambda^0 \oplus \Lambda^+) \otimes L) &\text{and}\quad \mathbf{F}'' &= \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (W^- \otimes L). \end{aligned} \tag{3.57}$$

Explicitly, if (B, Ψ) is a pair representing a point $[B, \Psi]$ in $M_{\mathfrak{s}}$, so the spin connection in the pair $\iota(B, \Psi) \in \tilde{\mathcal{M}}_{\mathfrak{t}}$ is given by $B \oplus B \otimes A_L$ and $A_L = A_\Lambda \otimes (B^{\det})^*$ is the induced unitary connection on the subbundle L of $\mathfrak{g}_{\mathfrak{t}} \cong i\underline{\mathbb{R}} \oplus L$, the operators on the fibers defined by these families are then given by

$$\begin{aligned} \delta_{A_L} \equiv d_{A_L}^* + d_{A_L}^+ &: C^\infty(\Lambda^1 \otimes L) \to C^\infty((\Lambda^0 \oplus \Lambda^+) \otimes L), \\ D''_{B \otimes A_L} &: C^\infty(W^+ \otimes L) \to C^\infty(W^- \otimes L), \end{aligned} \tag{3.58}$$

where $D''_{B \otimes A_L}$ is the Dirac operator. We use the preceding families to rewrite $\operatorname{Index} \boldsymbol{\mathcal{D}}^n$ in terms of index bundles whose Chern characters are more readily computable:

Lemma 3.25. *Continue the above notation. Then, as elements of* $K(M_{\mathfrak{s}})$,

$$\operatorname{Index} \boldsymbol{\mathcal{D}}^n = \operatorname{Index} \boldsymbol{\delta} + \operatorname{Index} \mathbf{D}''.$$

Proof. As an element of $K(M_{\mathfrak{s}})$, the index bundle $\operatorname{Index} \boldsymbol{\mathcal{D}}^n$ depends only on the homotopy class of the leading symbol [49, Theorem III.8.6]. Thus, the index bundle of the family $(B, \Psi) \mapsto \mathcal{D}^n_{B,\Psi}$ is equivalent to that of the family $(B, \Psi) \mapsto \mathcal{D}^n_{B,0}$, where $\mathcal{D}^n_{B,0} = \delta_{A_L} + D''_{B \otimes A_L}$. □

We now identify the family of operators $\boldsymbol{\delta}$ in (3.56) with a family of Dirac operators. (Identifications of this type were used in [5] to compute the index of the elliptic deformation complex for the anti-self-dual equation.) The vector bundle $\operatorname{End}(W) \cong W \otimes W^*$ is a Clifford module by Clifford multiplication on the factor W in the tensor product and so we obtain a Dirac operator,

$$D : C^\infty(W^\pm \otimes W^*) \to C^\infty(W^\mp \otimes W^*), \tag{3.59}$$

on $\operatorname{End}(W)$ [49, p. 122-123]. Under the identification $\Lambda^\bullet \otimes_{\mathbb{R}} \mathbb{C}$ with $\operatorname{End}(W)$ given by Clifford multiplication, the operator $d^* + d$ on $\Lambda^\bullet$ is identified with the Dirac operator (3.59) (see [49, Theorem II.5.12]). Restricting the domain of Clifford multiplication and tensoring with the line bundle L gives isomorphisms

$$\begin{aligned} \Lambda^1 \otimes_{\mathbb{R}} L &\cong \operatorname{Hom}(W^+, W^-) \otimes_{\mathbb{C}} L, \\ (\Lambda^0 \oplus \Lambda^+) \otimes_{\mathbb{R}} L &\cong \operatorname{End}(W^+) \otimes_{\mathbb{C}} L, \end{aligned} \tag{3.60}$$

and therefore isomorphisms

$$\mathbf{E}' \cong \mathbf{E}'_- \quad \text{and} \quad \mathbf{F}' \cong \mathbf{E}'_+,$$

where the bundles $\mathbf{E}'_\pm$ are defined by

$$\begin{aligned} \mathbf{E}'_- &= \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (\operatorname{Hom}(W^+, W^-) \otimes L), \\ \mathbf{E}'_+ &= \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (\operatorname{End}(W^+) \otimes L). \end{aligned} \tag{3.61}$$

The operator $d^*_{A_L} + d^+_{A_L}$ is the restriction of $d^*_{A_L} + d_{A_L}$ to $\Lambda^1 \otimes_{\mathbb{R}} L$ composed with the projection from $(\Lambda^0 \oplus \Lambda^2) \otimes_{\mathbb{R}} L$ to $(\Lambda^0 \oplus \Lambda^+) \otimes_{\mathbb{R}} L$. Thus, the isomorphisms (3.60) identify the family of operators $\boldsymbol{\delta}$ in (3.56) with the family of Dirac operators:

$$\begin{array}{ccc} C^\infty(\mathbf{E}'_-) & \xrightarrow{\mathbf{D}'} & C^\infty(\mathbf{E}'_+) \\ \downarrow & & \downarrow \\ M_{\mathfrak{s}} \times X & \xrightarrow{\mathrm{id}} & M_{\mathfrak{s}} \times X \end{array} \tag{3.62}$$

Explicitly, if (B, Ψ) is a pair representing a point $[B, \Psi]$ in $M_{\mathfrak{s}}$, the Dirac operator on the fiber given by this family,

$$D'_{B \otimes A_L} : C^\infty(\operatorname{Hom}(W^+, W^-) \otimes L) \to C^\infty(\operatorname{End}(W^+) \otimes L), \tag{3.63}$$

is defined by the spin^c connection on $\operatorname{End}(W)$ induced by the spin^c connection B on W and the unitary connection $A_L = A_\Lambda \otimes (B^{\det})^*$ on L. The preceding identification yields:

Lemma 3.26. *Continue the above notation. Then, as elements of $K(M_{\mathfrak{s}})$,*

$$\operatorname{Index} \boldsymbol{\delta} = \operatorname{Index} \mathbf{D}'.$$

We now begin the second step, which is to compute the Chern characters of the index bundles of the families of Dirac operators in (3.56) and (3.62). The following technical lemma helps to identify the universal bundles $\mathbf{E}'_\pm$, $\mathbf{E}''$, and $\mathbf{F}''$.

Lemma 3.27. *If $Q_i \to M$, $i = 1, 2$, are S^1 bundles over a manifold M and $L_i = Q_i \times_{S^1} \mathbb{C}$, $i = 1, 2$ are the associated complex line bundles, and $k \in \mathbb{Z}$, then the following hold:*

1. *If $V \to M$ is a complex vector bundle, and $e^{i\theta} \in S^1$ acts on the fiber product $Q_1 \times_M V$ by $e^{i\theta} \cdot (q_1, v) = (e^{i\theta} q_1, e^{ik\theta} v)$, then $(Q_1 \times_M V)/S^1 \cong L_1^{-k} \otimes V$.*
2. *If $e^{i\theta} \in S^1$ acts on the fiber product $Q_1 \times_M Q_2$ by $e^{i\theta} \cdot (q_1, q_2) = (e^{i\theta} q_1, e^{ik\theta} q_2)$, then the first Chern class of the S^1-bundle $(Q_1 \times_M Q_2)/S^1 \to M$ is $c_1(Q_2) - kc_1(Q_1)$, where the action of S^1 on $(Q_1 \times_M Q_2)/S^1$ is induced by the S^1 action on Q_2 of weight one.*

Proof. The associated line bundles $L_i = Q_i \times_{S^1} \mathbb{C}$ are given by the quotients of $Q_i \times \mathbb{C}$ by the relation $(q_i, z) \sim (e^{-i\theta} q_i, e^{i\theta} z)$, for $(q_i, z) \in Q_i \times \mathbb{C}$ and $e^{i\theta} \in S^1$; under the same relation, $Q_1 \times_{S^1} S^1 = Q_1$. A tensor product of a complex line bundle L with a complex vector bundle V is given by the quotient of the fiber product $L \times_M V$ by the relation $(z, v) \sim (w^{-1} z, wv)$, for $z \in L, v \in V$, and $w \in \mathbb{C}^*$. Hence,

$$
\begin{aligned}
L_1^{-k} \otimes V &= \{([q_1, z], v) \in L_1 \times_M V : ([q_1, z], v) \sim ([q_1, wz], w^k v), w \in \mathbb{C}^*\} \\
&= \{([q_1, e^{i\mu}], v) \in Q_1 \times_M V : ([q_1, e^{i\mu}], v) \sim ([q_1, e^{i\theta} e^{i\mu}], e^{ik\theta} v), e^{i\theta} \in S^1\} \\
&= \{(p_1, v) \in Q_1 \times_M V : (p_1, v) \sim (e^{i\theta} p_1, e^{ik\theta} v), e^{i\theta} \in S^1\} \quad (\text{where } p_1 = [q_1, e^{i\mu}]) \\
&= (Q_1 \times_M V)/S^1,
\end{aligned}
$$

where in the second line above we can assume without loss that $z \neq 0$. This proves Assertion (1) and Assertion (2) follows trivially from this. $\square$

Since it will not cause confusion, we will write $\mathbb{L}_{\mathfrak{s}}$ for the restriction of the universal bundle of equation (2.69) to $M_{\mathfrak{s}} \times X$,

$$\mathbb{L}_{\mathfrak{s}} = \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \underline{\mathbb{C}},$$

and define

$$\mathbb{L}_{\mathfrak{s},x} = \mathbb{L}_{\mathfrak{s}}|_{M_{\mathfrak{s}} \times \{x\}}. \tag{3.64}$$

By construction, $c_1(\mathbb{L}_{\mathfrak{s},x}) = c_1(\mathbb{L}_{\mathfrak{s}})/x = \mu_{\mathfrak{s}}(x)$ where $x \in H_0(X; \mathbb{Z})$ is a generator. Let $\pi_M : M_{\mathfrak{s}} \times X \to M_{\mathfrak{s}}$ and $\pi_X : M_{\mathfrak{s}} \times X \to X$ be the projections.

Lemma 3.28. *Assume $M_{\mathfrak{s}}$ contains no zero-section pairs. If $\mathbf{r}^* \mathbf{\Delta} \to M_{\mathfrak{s}} \times X$ is the restriction of the pullback by the retraction $\mathbf{r}$ of Lemma 2.20 of the line bundle* (2.76) *then, using* $\operatorname{Hom}(W^+, W^-) \cong W^- \otimes W^{+,*}$ *and* $\operatorname{End}(W^+) \cong W^+ \otimes W^{+,*}$, *there are isomorphisms,*

$$
\begin{aligned}
\mathbf{E}'_- &\cong \mathcal{E}' \otimes \pi_X^* W^-, \qquad & \mathbf{E}'_+ &\cong \mathcal{E}' \otimes \pi_X^* W^+, \\
\mathbf{E}'' &\cong \mathcal{E}'' \otimes \pi_X^* W^+, \qquad & \mathbf{F}'' &\cong \mathcal{E}'' \otimes \pi_X^* W^-,
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
\mathcal{E}' &= (\pi_M^* \mathbb{L}_{\mathfrak{s},x} \otimes \mathbf{r}^* \mathbf{\Delta})^{\otimes 2} \otimes \pi_X^* (W^{+,*} \otimes L), \\
\mathcal{E}'' &= \pi_M^* \mathbb{L}_{\mathfrak{s},x} \otimes \mathbf{r}^* \mathbf{\Delta} \otimes \pi_X^* L.
\end{aligned} \tag{3.65}
$$

Proof. There are isomorphisms,

$$\mathbf{E}'_\pm \cong (\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L) \otimes \operatorname{Hom}(W^+, W^\pm), \tag{3.66}$$

where, from equation (3.23), the action of $s \in \mathcal{G}_{\mathfrak{s}}$ on $\tilde{M}_{\mathfrak{s}} \times L$ is given by $(B, \Psi, z) \mapsto (s(B, \Psi), s^{-2} z)$. Let $\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (X \times S^1)$ be the unit sphere bundle of $\mathbb{L}_{\mathfrak{s}}$. If $e^{i\mu} \in S^1$ acts on

$$\left(\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (X \times S^1)\right) \times_{M_{\mathfrak{s}} \times X} \pi_X^* L$$

by $[(B, \Psi), x, e^{i\theta}, z] \mapsto [(B, \Psi), x, e^{i\mu} e^{i\theta}, e^{-i2\mu} z]$—where $(B, \Psi) \in \tilde{M}_{\mathfrak{s}}$, $x \in X$, $e^{i\theta} \in S^1$ and $z \in \pi_X^* L$—then we obtain an isomorphism of complex line bundles

$$
\begin{aligned}
&\left(\left(\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (X \times S^1)\right) \times_{M_{\mathfrak{s}} \times X} \pi_X^* L\right)/S^1 \to \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L, \\
&\qquad\qquad [(B, \Psi), x, e^{i\theta}, z] \mapsto [(B, \Psi), e^{2i\theta} z].
\end{aligned} \tag{3.67}
$$

Lemma 3.27 then implies that

$$\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} L \cong \mathbb{L}_{\mathfrak{s}}^{\otimes 2} \otimes \pi_X^* L \tag{3.68}$$

and equation (3.68) gives

$$\mathbf{E}'_{\pm} \cong \mathbb{L}_{\mathfrak{s}}^{\otimes 2} \otimes \pi_X^*(L \otimes W^{+,*} \otimes W^{\pm}). \tag{3.69}$$

The desired expression for $\mathbf{E}'_{\pm}$ then follows from the isomorphism $\mathbb{L}_{\mathfrak{s}} \cong \mathbb{L}_{\mathfrak{s},x} \otimes \mathbf{r}^*\mathbf{\Delta}$ given by Lemma 2.24.

In the bundles

$$\mathbf{E}'' \cong \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (W^+ \otimes L) \quad \text{and} \quad \mathbf{F}'' \cong \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} (W^- \otimes L),$$

an element $s \in \mathcal{G}_{\mathfrak{s}}$ acts on $\tilde{M}_{\mathfrak{s}} \times W^{\pm} \otimes L$ by $((B,\Psi),\Psi') \mapsto (s(B,\Psi), s^{-1}\Psi')$, where $\Psi' \in C^\infty(W^{\pm} \otimes L)$, as noted in the remark following equation (3.23). Lemma 3.27 and the argument yielding equation (3.68) then imply that there are isomorphisms of complex vector bundles

$$\mathbf{E}'' \cong \mathbb{L}_{\mathfrak{s}} \otimes \pi_X^*(W^+ \otimes L) \quad \text{and} \quad \mathbf{F}'' \cong \mathbb{L}_{\mathfrak{s}} \otimes \pi_X^*(W^- \otimes L). \tag{3.70}$$

The isomorphisms in the conclusion of the lemma now follow from equation (3.70) and the isomorphism $\mathbb{L}_{\mathfrak{s}} \cong \pi_M^*\mathbb{L}_{\mathfrak{s},x} \otimes \mathbf{r}^*\mathbf{\Delta}$ implied by equation (2.77). □

Given Lemma 3.28, the index bundles of the families of Dirac operators in (3.56) and (3.62) now take the shape:

$$\begin{array}{ccccccc}
C^\infty(\mathcal{E}' \otimes W^-) & \xrightarrow{\mathbf{D}'} & C^\infty(\mathcal{E}' \otimes W^+) & & C^\infty(\mathcal{E}'' \otimes W^+) & \xrightarrow{\mathbf{D}''} & C^\infty(\mathcal{E}'' \otimes W^-) \\
\downarrow & & \downarrow & \text{and} & \downarrow & & \downarrow \\
M_{\mathfrak{s}} \times X & \xrightarrow{\text{id}} & M_{\mathfrak{s}} \times X & & M_{\mathfrak{s}} \times X & \xrightarrow{\text{id}} & M_{\mathfrak{s}} \times X
\end{array}$$

We abuse notation slightly by continuing to denote the families of Dirac operators on these isomorphic bundles by $\mathbf{D}'$ and $\mathbf{D}''$.

The final step in the computation of the Chern character of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is to compute the Chern characters of these families.

Given the decomposition (3.37) of $\mathcal{D}^n_{\iota(B,\Psi)}$, it will be convenient to define

$$\operatorname{Index}_{\mathbb{C}} \mathcal{D}^n_{\iota(B,\Psi)} = n_s = n'_s + n''_s, \quad \text{where} \quad \begin{aligned} n'_s &= \operatorname{Index}_{\mathbb{C}}(d^*_{A_L} + d^+_{A_L}), \\ n''_s &= \operatorname{Index}_{\mathbb{C}} D''_{B\otimes A_L}. \end{aligned} \tag{3.71}$$

Viewing $E = \underline{\mathbb{C}} \oplus L$ and $\mathfrak{su}(E) = i\underline{\mathbb{R}} \oplus L$, we can compute the complex index of $d^*_{A_L} + d^+_{A_L}$ from the real index of $d^*_{\hat{A}} + d^+_{\hat{A}}$ on $C^\infty(\Lambda^1 \otimes \mathfrak{su}(E))$ (for example, see [14, Equation (4.2.22)]) and the fact (Lemma 2.9) that $d^*_{\hat{A}} + d^+_{\hat{A}}$ decomposes as the direct sum of $d^* + d^+$ on $C^\infty(i\Lambda^1)$ and $d^*_{A_L} + d^+_{A_L}$ on $C^\infty(\Lambda^1 \otimes L)$; the complex index of the spin^c Dirac operator is given in [56, p. 47]. Thus,

$$\begin{aligned} n'_s(\mathfrak{t},\mathfrak{s}) &= -\left(c_1(\mathfrak{s}) - c_1(\mathfrak{t})\right)^2 - \tfrac{1}{2}(\chi + \sigma), \\ n''_s(\mathfrak{t},\mathfrak{s}) &= \tfrac{1}{8}\left((2c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 - \sigma\right). \end{aligned} \tag{3.72}$$

We can now state and prove one of the main results of our article:

Theorem 3.29. *Let $\mathfrak{t}$ be a spin^u structure over a closed, oriented, Riemannian, smooth four-manifold X. Assume $\mathfrak{t}$ admits a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$, where L is a complex line bundle. Suppose that there are no zero-section pairs in $M_{\mathfrak{s}}$. Let $\mu_{\mathfrak{s}}$ be the Seiberg-Witten μ-map, as in definition (2.71). Let $x \in H_0(X;\mathbb{Z})$ be the positive generator and for $\{\gamma_i\}$ a basis for $H_1(X;\mathbb{Z})/\operatorname{Tor}$, let $\{\gamma_i^*\}$ be the dual basis for $H^1(X;\mathbb{Z})$ introduced in Definition*

2.21. Let $r_\Xi = \operatorname{rank}_{\mathbb{C}} \Xi$ denote the rank of the stabilizing bundle. Then the Chern character of the normal bundle $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$ of the stratum $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}(\Xi,\mathfrak{s})$ is

$$
\begin{aligned}
\operatorname{ch}(N_{\mathfrak{t}}(\Xi,\mathfrak{s})) &= r_\Xi + n_s'' e^{\mu_{\mathfrak{s}}(x)} + n_s' e^{2\mu_{\mathfrak{s}}(x)} \\
&\quad - 8\sum_{i<j} \langle \gamma_i^*\gamma_j^*(c_1(\mathfrak{t}) - c_1(\mathfrak{s})), [X]\rangle e^{2\mu_{\mathfrak{s}}(x)} \mu_{\mathfrak{s}}(\gamma_i\gamma_j) \\
&\quad + \tfrac{1}{2}\textstyle\sum_{i<j} \langle \gamma_i^*\gamma_j^*(2c_1(\mathfrak{t}) - c_1(\mathfrak{s})), [X]\rangle e^{\mu_{\mathfrak{s}}(x)} \mu_{\mathfrak{s}}(\gamma_i\gamma_j) \\
&\quad + (e^{\mu_{\mathfrak{s}}(x)} - 32 e^{2\mu_{\mathfrak{s}}(x)}) \sum_{i<j<k<\ell} \langle \gamma_i^*\gamma_j^*\gamma_k^*\gamma_\ell^*, [X]\rangle \mu_{\mathfrak{s}}(\gamma_i\gamma_j\gamma_k\gamma_\ell).
\end{aligned} \tag{3.73}
$$

where $\mu_{\mathfrak{s}}(\gamma_i\gamma_j) = \mu_{\mathfrak{s}}(\gamma_i)\mu_{\mathfrak{s}}(\gamma_j)$ and similarly for $\mu_{\mathfrak{s}}(\gamma_i\gamma_j\gamma_k\gamma_\ell)$.

Proof. For convenience in the proof, we write $\mu = \mu_{\mathfrak{s}}(x)$ and $\gamma_i^{J,*} = \mu_{\mathfrak{s}}(\gamma_i)$, as in Lemma 2.24. The K-theory identification (3.55) of $N_{\mathfrak{t}}(\Xi,\mathfrak{s})$, Lemmas 3.25 and 3.26, and the homomorphism property of the Chern character (see, for example, [49, Proposition III.11.16]) imply that

$$
\begin{aligned}
\operatorname{ch}(N_{\mathfrak{t}}(\Xi,\mathfrak{s})) &= \operatorname{ch}(\operatorname{Index} \boldsymbol{\mathcal{D}}^n) + \operatorname{ch}(\Xi) \\
&= \operatorname{ch}\big(\operatorname{Index} \mathbf{D}'\big) + \operatorname{ch}(\operatorname{Index} \mathbf{D}'') + r_\Xi.
\end{aligned} \tag{3.74}
$$

Recall that $\operatorname{Index}(D^*, \mathcal{E}, W)$ denotes the index bundle of the family of operators obtained, as in Proposition 3.24, by twisting the Dirac operator $D^* : C^\infty(W^-) \to C^\infty(W^+)$ by a family of connections on the bundle $\mathcal{E}$. Note that $\operatorname{ch}(\operatorname{Index}(D^*, W)) = -\operatorname{ch}(\operatorname{Index}(D, W))$, where $D^* : C^\infty(W^-) \to C^\infty(W^+)$. We now use Lemma 3.28 and Proposition 3.24 to partly compute the Chern characters of the index bundle $\operatorname{Index} \mathbf{D}'$. Bundles over X will be considered to be bundles over $M_{\mathfrak{s}} \times X$; the pullback π_X^* will be omitted.

$$
\begin{aligned}
\operatorname{ch}\big(\operatorname{Index} \mathbf{D}'\big) &= \operatorname{ch}(\operatorname{Index}(D^*, \mathcal{E}', W)) = \operatorname{ch}(\mathcal{E}') \operatorname{ch}(\operatorname{Index}(D^*, W)) \\
&= -\operatorname{ch}\big((\pi_M^* \mathbb{L}_{\mathfrak{s},x} \otimes \mathbf{r}^*\boldsymbol{\Delta})^{\otimes 2} \otimes W^{+,*} \otimes L\big) \operatorname{ch}(\operatorname{Index}(D, W)) \\
&= -\pi_M^* \operatorname{ch}(\mathbb{L}_{\mathfrak{s},x})^2 \mathbf{r}^* \operatorname{ch}(\boldsymbol{\Delta})^2 \operatorname{ch}(W^{+,*}) e^{c_1(L)} e^{\frac{1}{2}c_1(W^+)} \hat{\mathcal{A}}(X)/[X].
\end{aligned} \tag{3.75}
$$

Similarly, we use Lemma 3.28 and Proposition 3.24 to partly compute the Chern character of the index bundle $\operatorname{Index} \mathbf{D}''$:

$$
\begin{aligned}
\operatorname{ch}(\operatorname{Index} \mathbf{D}'') &= \operatorname{ch}(\operatorname{Index}(D, \mathcal{E}'', W)) = \operatorname{ch}(\mathcal{E}'') \operatorname{ch}(\operatorname{Index}(D, W)) \\
&= \operatorname{ch}(\pi_M^* \mathbb{L}_{\mathfrak{s},x} \otimes \mathbf{r}^*\boldsymbol{\Delta} \otimes L) e^{\frac{1}{2}c_1(W^+)} \hat{\mathcal{A}}(X)/[X] \\
&= \pi_M^* \operatorname{ch}(\mathbb{L}_{\mathfrak{s},x}) \mathbf{r}^* \operatorname{ch}(\boldsymbol{\Delta}) e^{c_1(L)} e^{\frac{1}{2}c_1(W^+)} \hat{\mathcal{A}}(X)/[X].
\end{aligned} \tag{3.76}
$$

In the first lines of (3.75) and (3.76) we simply rewrite the index bundles using the notation of Proposition 3.24.

For the calculation of $\operatorname{ch}(\operatorname{Index} \mathbf{D}'')$ in (3.76), we compute $\operatorname{ch}(\boldsymbol{\Delta})$ using the expression for $\mathbf{r}^* c_1(\boldsymbol{\Delta})$ in Lemmas 2.23 and 2.24:

$$
\mathbf{r}^* \operatorname{ch}(\boldsymbol{\Delta}) = \prod_{i=1}^{b_1(X)} e^{\gamma_i^{J,*} \times \gamma_i^*} = \prod_{i=1}^{b_1(X)} \left(1 + \gamma_i^{J,*} \times \gamma_i^*\right).
$$

Because $c_1(\mathbb{L}_{\mathfrak{s},x}) = \mu$ as noted before Lemma 3.28, we see $\pi_M^* \operatorname{ch}(\mathbb{L}_{\mathfrak{s},x}^*) = \pi_M^* e^{\mu}$. We shall write μ for $\pi_M^*\mu$. Applying this to (3.76) and noting that $c_1(L) + \frac{1}{2}c_1(W^+) = c_1(\mathfrak{t}) - \frac{1}{2}c_1(\mathfrak{s})$, we obtain

$$
\begin{aligned}
&\mathrm{ch}(\mathrm{Index}\,\mathbf{D}'') \\
&= e^{\mu}\left(\prod_{i=1}^{b_1(X)} e^{\gamma_i^{J,*}\times\gamma_i^*}\right) e^{c_1(\mathfrak{t})-\frac{1}{2}c_1(\mathfrak{s})}\left(1-\tfrac{1}{24}p_1(X)\right)/[X] \\
&= e^{\mu}\left(1+\sum_{i<j}(\gamma_i^{J,*}\gamma_j^{J,*})\times(\gamma_i^*\gamma_j^*)+\sum_{i<j<k<\ell}(\gamma_i^{J,*}\gamma_j^{J,*}\gamma_k^{J,*}\gamma_\ell^{J,*})\times(\gamma_i^*\gamma_j^*\gamma_k^*\gamma_\ell^*)\right) \\
&\quad\times\left(1+\tfrac{1}{2}(2c_1(\mathfrak{t})-c_1(\mathfrak{s}))+\tfrac{1}{8}(2c_1(\mathfrak{t})-c_1(\mathfrak{s}))^2\right)\left(1-\tfrac{1}{24}p_1(X)\right)/[X],
\end{aligned}
$$

and therefore, using $\sigma=\frac{1}{3}\langle p_1(X),[X]\rangle$ (see [7, Theorem I.3.1])

$$
\begin{aligned}
\mathrm{ch}(\mathrm{Index}\,\mathbf{D}'') = e^{\mu}\Big(&\tfrac{1}{8}(2c_1(\mathfrak{t})-c_1(\mathfrak{s}))^2-\tfrac{1}{8}\sigma \\
&+\tfrac{1}{2}\textstyle\sum_{i<j}(\gamma_i^{J,*}\gamma_j^{J,*})\langle\gamma_i^*\gamma_j^*(2c_1(\mathfrak{t})-c_1(\mathfrak{s})),[X]\rangle \\
&+\sum_{i<j<k<\ell}(\gamma_i^{J,*}\gamma_j^{J,*}\gamma_k^{J,*}\gamma_\ell^{J,*})\langle(\gamma_i^*\gamma_j^*\gamma_k^*\gamma_\ell^*),[X]\rangle\Big),
\end{aligned}
\tag{3.77}
$$

completing the calculation of $\mathrm{ch}(\mathrm{Index}\,\mathbf{D}'')$.

We now complete the calculation of $\mathrm{ch}(\mathrm{Index}\,\boldsymbol{\delta})$ in (3.75). To compute $\mathrm{ch}(W^{+,*})$ we observe that $c_1(W^{+,*})=-c_1(W^+)$ while $c_2(W^{+,*})=c_2(W^+)$ and so, using this and the isomorphism $\mathfrak{su}(W^+)\cong\Lambda^+$, we have

$$
\begin{aligned}
\mathrm{ch}(W^{+,*}) &= 2+c_1(W^{+,*})+\tfrac{1}{2}\left(c_1(W^{+,*})^2-2c_2(W^{+,*})\right) \\
&= 2-c_1(W^+)+\tfrac{1}{2}c_1(W^+)^2+\tfrac{1}{4}\left(p_1(\mathfrak{su}(W^+))-c_1(W^+)^2\right) \\
&= 2-c_1(W^+)+\tfrac{1}{4}\left(p_1(\Lambda^+)+c_1(W^+)^2\right).
\end{aligned}
$$

Applying this to the terms involving W^+ in (3.75) and writing $c_1(W^+)=c_1(\mathfrak{s})$ yields

$$
\begin{aligned}
&-\mathrm{ch}(W^{+,*})e^{\frac{1}{2}c_1(W^+)}\hat{\mathcal{A}}(X)/[X] \\
&\quad= -\left(2-c_1(\mathfrak{s})+\tfrac{1}{4}(c_1(L)^2+p_1(\Lambda^+))\right)\left(1+\tfrac{1}{2}c_1(\mathfrak{s})+\tfrac{1}{8}c_1(L)^2\right)\left(1-\tfrac{1}{24}p_1(X)\right)/[X] \\
&\quad= -(2+\tfrac{1}{4}p_1(\Lambda^+))\left(1-\tfrac{1}{24}p_1(X)\right)/[X].
\end{aligned}
$$

Hence, using the preceding expression in (3.75) and $c_1(L)=c_1(\mathfrak{t})-c_1(\mathfrak{s})$, we see that

$$
\begin{aligned}
&\mathrm{ch}(\mathrm{Index}\,\boldsymbol{\delta}) \\
&= -\pi_M^*\,\mathrm{ch}(\mathbb{L}_{\mathfrak{s},x})^2\mathbf{r}^*\,\mathrm{ch}(\boldsymbol{\Delta})^2\,\mathrm{ch}(L)(2+\tfrac{1}{4}p_1(\Lambda^+))\left(1-\tfrac{1}{24}p_1(X)\right)/[X] \\
&= -e^{2\mu}\left(\prod_{i=1}^{b_1(X)}\left(1+2\gamma_i^{J,*}\times\gamma_i^*\right)\right)\left(1+(c_1(\mathfrak{t})-c_1(\mathfrak{s}))+\tfrac{1}{2}(c_1(\mathfrak{t})-c_1(\mathfrak{s}))^2\right) \\
&\quad\times\left(2+\tfrac{1}{4}p_1(\Lambda^+)\right)\left(1-\tfrac{1}{24}p_1(X)\right)/[X].
\end{aligned}
$$

Simplifying the preceding expression for $\operatorname{ch}(\operatorname{Index}\boldsymbol{\delta})$ and recalling that $p_1(\Lambda^+) = 2\chi + 3\sigma$ (from [38, Satz 1.5]), yields

$$\begin{aligned}
&\operatorname{ch}(\operatorname{Index}\boldsymbol{\delta}) \\
&= -e^{2\mu}\Big(1 + 4\sum_{i<j}(\gamma_i^{J,*}\gamma_j^{J,*}) \times (\gamma_i^*\gamma_j^*) + 16\sum_{i<j<k<\ell}(\gamma_i^{J,*}\gamma_j^{J,*}\gamma_k^{J,*}\gamma_\ell^{J,*}) \times (\gamma_i^*\gamma_j^*\gamma_k^*\gamma_\ell^*)\Big) \\
&\qquad \times \left(1 + (c_1(\mathfrak{t}) - c_1(\mathfrak{s})) + \tfrac{1}{2}(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2\right)\left(2 + \tfrac{1}{4}p_1(\Lambda^+)\right)\left(1 - \tfrac{1}{24}p_1(X)\right)/[X],
\end{aligned}$$

and thus,

$$\begin{aligned}
\operatorname{ch}(\operatorname{Index}\boldsymbol{\delta}) = e^{2\mu}\Big(&-(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 - \tfrac{1}{2}\sigma - \tfrac{1}{2}\chi \\
&- 8\sum_{i<j}(\gamma_i^{J,*}\gamma_j^{J,*})\langle(\gamma_i^*\gamma_j^*)(c_1(\mathfrak{t}) - c_1(\mathfrak{s})), [X]\rangle \\
&- 32\sum_{i<j<k<\ell}(\gamma_i^{J,*}\gamma_j^{J,*}\gamma_k^{J,*}\gamma_\ell^{J,*})\langle\gamma_i^*\gamma_j^*\gamma_k^*\gamma_\ell^*, [X]\rangle\Big).
\end{aligned} \tag{3.78}$$

The desired expression for $\operatorname{ch}(N_\mathfrak{t}(\Xi,\mathfrak{s}))$ follows from (3.72), (3.74), (3.77), and (3.78). □

Finally, we calculate the total Chern class $c(N_\mathfrak{t}(\Xi,\mathfrak{s}))$, as an element of rational cohomology, under a simplifying assumption.

Corollary 3.30. *Continue the hypotheses of Theorem 3.29 and assume that $\alpha \smile \alpha' = 0$, for every $\alpha, \alpha' \in H^1(X;\mathbb{Z})$. Then, as elements of $H^\bullet(M_\mathfrak{s};\mathbb{R})$,*

$$c(N_\mathfrak{t}(\Xi,\mathfrak{s})) = (1 + 2\mu_\mathfrak{s}(x))^{n_s'}(1 + \mu_\mathfrak{s}(x))^{n_s''}.$$

Proof. For convenience, we write $\mu = \mu_\mathfrak{s}(x)$. The Chern character determines the Chern polynomial as an element of rational cohomology (see the formula in [53, Problem 16-A] or [3, pp. 156–157]), so $\operatorname{ch}(N_\mathfrak{t}(\Xi,\mathfrak{s}))$ determines

$$c_t(N_\mathfrak{t}(\Xi,\mathfrak{s})) = \sum_{i=0}^{r_\Xi} c_i(N_\mathfrak{t}(\Xi,\mathfrak{s}))t^i = \prod_{j=1}^{r_\Xi}(1 + \alpha_i t),$$

where the α_i are the Chern roots of $N_\mathfrak{t}(\Xi,\mathfrak{s})$. Theorem 3.29 implies that, with the constraint on $H^1(X;\mathbb{Z})$,

$$\operatorname{ch}(N_\mathfrak{t}(\Xi,\mathfrak{s})) = \sum_{j=1}^{r_\Xi}\exp(\alpha_j) = r_\Xi + n_s'e^{2\mu} + n_s''e^{\mu}.$$

Suppose $n_s' \geq 0$ and $n_s'' \geq 0$. If $\prod_{j=1}^{r_\Xi}(1 + \alpha_i t) = (1 + 2\mu t)^{n_s'}(1 + \mu t)^{n_s''}$, then $\alpha_j = +2\mu$ for $1 \leq j \leq n_s'$ and $\alpha_j = \mu$ for $n_s' < j \leq n_s' + n_s''$, while $\alpha_j = 0$ for $n_s' + n_s'' < j \leq r_\Xi$. Then one can easily see that $\operatorname{ch}(N_\mathfrak{t}(\Xi,\mathfrak{s}))$ is equal to the Chern character associated to the Chern polynomial $(1 + 2\mu t)^{n_s'}(1 + \mu t)^{n_s''}$. The case where either n_s' or n_s'' is negative follows from the observation that if $G \in K(M_\mathfrak{s})$, then $\operatorname{ch}(-G) = \operatorname{ch}(G)$ while $c(-G) = c(G)^{-1}$. □

Appendix A. Abundant four-manifolds

Our goal in this section is to prove the

Theorem A.1. *Every compact, complex algebraic, simply connected surface with $b_2^+ \geq 3$ is abundant.*

In [16, p. 175] we asserted without further explanation that simply connected, minimal, complex algebraic surfaces of general type were abundant. On the other hand, some of the fake K3-surfaces of [33] fail to be abundant. If log transforms are performed on tori in three distinct nuclei then the intersection form on $B^\perp$ is a degenerate form with three-dimensional radical and having an $-E_8 \oplus -E_8$ summand [16, p. 175]. We apply Theorem A.1 to produce classes $\Lambda \in H^2(X;\mathbb{Z})$ which are orthogonal to the SW-basic classes, with square equal to prescribed even integers. As far as we can tell, one can always find such classes Λ for compact four-manifolds with $b_2^+ \geq 3$, even if non-abundant: it is interesting problem to determine if indeed this is true.

We are extremely grateful to András Stipsicz for describing a proof of Theorem A.1 in the case where X is a minimal surface of general type with odd intersection form: see Lemmas A.4 and A.6, as well as the ideas for the case of odd minimal surfaces of general type in §A.2—these are all due to him. The argument for Lemma A.4 relies on the "odd four-square theorem" (Lemma A.5), for whose proof we are indebted to A. Agboola [1].

For the remainder of this section, unless further restrictions are mentioned, we suppose X is a compact, connected, smooth four-manifold with an orientation for which $b_2^+(X) > 0$.

A.1. **Four-manifolds whose basic classes are multiples of a fixed class.** We consider the cases of even and odd intersection forms separately, beginning with the even case:

Lemma A.2. *Suppose (L,Q) is an indefinite, integral, unimodular lattice and that $\kappa \in L$. If Q is even and $|\sigma(Q)| \leq \operatorname{rank}(Q) - 4$ then $\kappa^\perp$, the Q-orthogonal complement of κ in L, contains a hyperbolic sublattice.*

Proof. We may assume without loss that κ is primitive because a sublattice H is orthogonal to $\kappa = d\kappa'$ if and only if it is orthogonal to κ', where $d \in \mathbb{Z}$ and κ' is primitive.

From the classification of indefinite, integral, unimodular forms (for example, see Theorem 1.2.21 in [34]) we have

$$(L,Q) \cong \tfrac{1}{8}\sigma(Q)E_8 \oplus \tfrac{1}{2}(\operatorname{rank}(Q) - |\sigma(Q)|)H. \tag{A.1}$$

By hypothesis, $\operatorname{rank}(Q) - |\sigma(Q)| \geq 4$, so if $R = H \oplus H$ then $(R, Q|_R)$ is a sublattice of (L,Q). Let κ_R denote the component of κ in R. Because R is even, we have $Q(\kappa_R,\kappa_R) = 2h$ for some $h \in \mathbb{Z}$. Let e_1, e_2 and f_1, f_2 be bases for the two hyperbolic sublattices of R, so $Q(e_1,e_2) = Q(f_1,f_2) = 1$ and all other pairings of these four vectors vanish. Define $v = e_1 + he_2$, so $v^2 = 2h$ and v is primitive. The hyperbolic sublattice $F = \mathbb{Z}f_1 + \mathbb{Z}f_2 \subset R$ is orthogonal to v. According to Theorem 1 in [75], the orthogonal group of $(R, Q|_R)$ acts transitively on primitive vectors of a given square. Hence, there is an automorphism A of R such that $Q(Ax,Ay) = Q(x,y)$ for all $x,y \in R$ and $Av = \kappa_R$. Because F is contained in $v^\perp$, then $A(F) \subset \kappa_R^\perp$. Hence, the hyperbolic sublattice $A(F)$ is contained in $\kappa^\perp$. □

Corollary A.3. *Let X be a simply-connected spin four-manifold with $b_2^+(X) \geq 3$. If the SW-basic classes of X are multiples of a class $K \in H^2(X;\mathbb{Z})$, then X is abundant.*

Proof. We consider the intersection form Q to be a form on $H^2(X;\mathbb{Z})$. By Rochlin's theorem (see Theorem 1.2.29 in [34]), we know that $\sigma(X) \equiv 0 \pmod{16}$ and so the classification (A.1) of forms yields $(H^2(X;\mathbb{Z}),Q) \cong 2kE_8 \oplus lH$ with $k,l \in \mathbb{Z}$. A result of Furuta [34, Theorem 1.2.31] then implies that $l \geq 2|k|+1$, where $k = \frac{1}{16}\sigma(X)$ and $l = \frac{1}{2}(b_2(X) - \sigma(X))$. If $\sigma(X) = 0$ and $b_2^+(X) \geq 3$, then $b_2(X) \geq 4$ and so $0 \leq \operatorname{rank}(Q) - 4$; if $\sigma(X) \neq 0$, then $k \neq 0$ and Furuta's theorem implies $l \geq 2$ and so again $|\sigma(Q)| \leq \operatorname{rank}(Q) - 4$. Therefore, $K^\perp$ contains a hyperbolic sublattice by Lemma A.2. □

We turn to the more complicated case, where the intersection form is odd.

Lemma A.4. *Suppose (L,Q) is an indefinite, integral, unimodular lattice. If Q is odd with $b_2^+ \geq 5$ and $b_2^- \geq 3$, and $\kappa \in L$ is characteristic, then $\kappa^\perp$ contains a hyperbolic sublattice.*

Proof. We may assume without loss that κ is primitive for, if not, write $\kappa = d\kappa'$ where $d \in \mathbb{Z}$ and $\kappa' \in L$ is primitive. Then $Q(\kappa,x) = dQ(\kappa',x) \equiv Q(x,x) \pmod 2$ for all $x \in L$ since κ is characteristic. If d were even we would have $Q(x,x) \equiv 0 \pmod 2$ for all x, contradicting our hypothesis that Q is an odd form. Hence, d is odd and $dQ(\kappa',x) \equiv Q(\kappa',x) \equiv Q(x,x) \pmod 2$ for all x and thus κ' is also characteristic. Then a sublattice H is orthogonal to κ if and only if it is orthogonal to κ'.

The classification of indefinite, integral, unimodular forms shows that, because Q is odd, we can find a basis $\{e_i, f_j\}$ for L for which $e_i^2 = 1$, $f_i^2 = -1$, and

$$(L,Q) \cong \left(\bigoplus_{i=1}^{b_2^+} \mathbb{Z}e_i\right) \oplus \left(\bigoplus_{j=1}^{b_2^-} \mathbb{Z}f_j\right).$$

For odd integers a_1, a_2, a_3, a_4, c_1 yet to be determined, choose

$$\ell = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + \sum_{i=5}^{b_2^+} e_i + c_1f_1 + \sum_{j=2}^{b_2^-} f_j \in L.$$

The element ℓ is primitive since at least one basis coefficient is equal to one. By hypothesis, $b_2^+ \geq 5$ and $b_2^- \geq 3$, so we can define

$$H = \mathbb{Z}(e_5 + f_2) + \mathbb{Z}(e_5 + f_3)$$

and observe that $(H, Q|_H)$ is hyperbolic and orthogonal to ℓ. Then

$$\begin{aligned} Q(\ell,\ell) &= a_1^2 + a_2^2 + a_3^2 + a_4^2 + (b_2^+ - 4) - c_1^2 - (b_2^- - 1) \\ &= a_1^2 + a_2^2 + a_3^2 + a_4^2 + \sigma(Q) - c_1^2 - 3, \end{aligned}$$

and so

$$Q(\ell,\ell) - \sigma(Q) + c_1^2 + 3 = a_1^2 + a_2^2 + a_3^2 + a_4^2. \tag{A.2}$$

Since the coefficients of ℓ are odd, we have $Q(\ell, e_i) \equiv Q(\ell, f_j) \equiv 1 \pmod 2$ for all i, j; thus ℓ is characteristic and we have $\ell^2 \equiv \sigma(Q) \pmod 8$. As c_1 is odd, so we can write $c_1 = 2u+1$ for some $u \in \mathbb{Z}$. The left-hand side of the preceding equation therefore yields

$$Q(\ell,\ell) - \sigma(Q) + c_1^2 + 3 \equiv (2u+1)^2 + 3 \equiv 4 + 4u(u+1) \equiv 4 \pmod 8.$$

Now any positive integer which is congruent to $4 \pmod 8$ can be written as the sum of four odd squares (see Lemma A.5). So we select any odd integer c_1 for which $Q(\kappa,\kappa) - \sigma(Q) + c_1^2 + 3 > 0$ and then choose odd integers a_1, a_2, a_3, a_4 so that

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = Q(\kappa,\kappa) - \sigma(Q) + c_1^2 + 3. \tag{A.3}$$

Therefore equations (A.2) and (A.3) give

$$Q(\ell,\ell) = Q(\kappa,\kappa).$$

A result of Wall, (see [34, Proposition 1.2.18]) implies that the orthogonal group of (L,Q) acts transitively on the primitive, characteristic elements with a given square. Hence, we can find an orthogonal automorphism A of (L,Q) with $A\ell = \kappa$. Then $A(H)$ is a hyperbolic sublattice of L which is orthogonal to κ. ∎

Lagrange's theorem tells us that every positive integer k is the sum of four integral squares [36, Theorem 20.5]. While the following refinement of this result must surely be well known, we cannot find references to it in standard texts on elementary number theory, so we include a proof here which was generously supplied to us by Adebisi Agboola [1]. András Stipsicz has pointed out to us that Lemma A.5 was used by Dieter Kotschick in [42] to show that every finitely presentable group is the fundamental group of a closed, almost complex four-manifold.

Lemma A.5. *A positive integer is the sum of four odd squares if and only if it is congruent to* 4 *modulo* 8.

Proof. Let k be a positive integer. The set of squares modulo 8 is $0, 1, 4$. Hence, if k is the sum of four odd squares, then we must have $k \equiv 4 \pmod 8$.

Conversely, suppose that $k \equiv 4 \pmod 8$. By Lagrange's theorem we can write

$$k = w^2 + x^2 + y^2 + z^2.$$

Since k is even, we must have one of the following possibilities (up to rearranging the terms on the right-hand side):

(a) w, x are even, and y, z are odd.
(b) w, x, y, z are all even.
(c) w, x, y, z are all odd (as desired).

Now since $k \equiv 4 \pmod 8$, it is not hard to check that case (a) cannot happen. In fact, it is easy to see that if case (a) is true, then $k \equiv 2$ or $6 \pmod 8$.

To see that case (c) can occur, write $k = 8m + 4$, where m is a non-negative integer, so

$$k = 1^2 + (8m + 3).$$

Legendre's theorem tells us that a positive integer is the sum of three integral squares if and only if it cannot be written in the form $4^a(8b + 7)$, for some a, b (see [36, §20.10] for the statement and [2] for a proof). Now 1 is odd, and since $8m + 3$ cannot be of the form $4^a(8b + 7)$, we can express $8m + 3$ as a sum of three squares. So we can write

$$k = 1^2 + x^2 + y^2 + z^2.$$

Since 1 is odd, and we have ruled out case (a) above, it follows that x, y, z are all odd. □

It remains to consider an "unstable range", where $b_2^+(Q)$ and $b_2^-(Q)$ are small. Define $\chi(Q) = 2 + b_2(Q)$ and note that $2\chi(Q) + 3\sigma(Q) = 4 + 5b_2^+(Q) - b_2^-(Q)$.

Lemma A.6. *Suppose (L, Q) is an integral, indefinite, unimodular lattice and that $\kappa \in L$. Assume Q is odd and κ is characteristic. If one of the following hold, then $\kappa^\perp$ contains a hyperbolic sublattice:*

(a) $b_2^+ = 3$ *and* $b_2^- \geq 5$,
(b) $b_2^+ = 3$ *and* $2 \leq b_2^- \leq 4$, *and κ has square* $2\chi(Q) + 3\sigma(Q)$.

Proof. Interchanging the role of b_2^+ and b_2^- in the proof of Lemma A.4 takes care of case (a). Thus we need only consider case (b).

Continuing the notation of the proof of Lemma A.4, we choose

$$\ell = 3e_1 + 3e_2 + e_3 + \sum_{j=1}^{b_2^-} f_j \in L.$$

Plainly, ℓ is characteristic, primitive, and is orthogonal to the hyperbolic sublattice $H = \mathbb{Z}(e_3 + f_1) + \mathbb{Z}(e_3 + f_2)$ of (L, Q), while (as $2\chi(Q) + 3\sigma(Q) = 19 - b_2^-$)

$$Q(\ell, \ell) = 19 - b_2^- = Q(\kappa, \kappa).$$

Since Q is odd, we cannot have $b_2^- = 3$ (which would give $Q(\kappa, \kappa) = 16$) so $b_2^- = 2$ or $b_2^- = 4$, which gives $Q(\kappa, \kappa) = 17$ or 15, respectively, neither of which is divisible by d^2, $d \in \mathbb{Z}$, unless $d = \pm 1$. Thus κ is primitive.

Just as in the proof of Lemma A.4, Wall's theorem implies that we can find an orthogonal automorphism A of (L, Q) with $A\ell = \kappa$, since κ, ℓ are both primitive, characteristic, and have equal square. Then $A(H)$ is a hyperbolic sublattice of $\kappa^\perp$. This takes care of case (b) and completes the proof of the lemma. □

Lemmas A.4 and A.6 thus yield:

Corollary A.7. *Let X be a simply-connected four-manifold having odd intersection form Q_X, with $b_2^+(X) \geq 5$ and $b_2^-(X) \geq 3$ or $b_2^+(X) = 3$ and $b_2^-(X) \geq 2$. Suppose that the SW-basic classes of X are integer multiples of a class $K \in H^2(X; \mathbb{Z})$, where K is characteristic. Then X is abundant.*

Combining Corollaries A.3 and A.7 yields:

Proposition A.8. *Let X be a simply-connected four-manifold. Suppose the SW-basic classes of X are multiples of a class $K \in H^2(X; \mathbb{Z})$, where K is characteristic. If any one of the following hold, then X is abundant:*

(a) *Q_X is even with $b_2^+(X) \geq 3$,*
(b) *Q_X is odd with $b_2^+(X) \geq 5$ and $b_2^-(X) \geq 3$,*
(c) *Q_X is odd with $b_2^+(X) = 3$ and $b_2^-(X) \geq 5$,*
(d) *Q_X is odd with $b_2^+(X) = 3$ and $2 \leq b_2^-(X) \leq 4$, and $K^2 = 2\chi(X) + 3\sigma(X)$.*

A.2. **Compact, complex algebraic, simply connected surfaces.** We now combine the results of the preceding subsection to prove the principal result of this appendix:

Proof of Theorem A.1. Let X be a compact, complex algebraic, simply connected surface with $b_2^+(X) \geq 3$. Suppose X is minimal. The Enriques-Kodaira classification then implies that X is one of the following (see [34, §3.4]):

(a) A $K3$ surface,
(b) An elliptic surface,
(c) A surface of general type.

The cases where X is diffeomorphic to $\mathbb{CP}^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, or $\mathbb{CP}^1 \times \mathbb{CP}^1$ (when X has Kodaira dimension $-\infty$, see [34, p. 88 & Theorem 3.4.13]) are eliminated by our requirement that $b_2^+(X) \geq 3$.

If X is elliptic, then it is diffeomorphic to $E(n)_{p,q}$, for some $n, p, q \in \mathbb{N}$, $p \leq q$, $\gcd(p, q) = 1$ (see Theorems 3.4.12 and 3.4.13 in[34]); if X is a $K3$ surface, then it is diffeomorphic to the surface $E(2)$ (see Theorem 3.4.9 in [34]), which is included in this family as $E(n)_{1,1} = E(n)$ (see [34, p. 83] for the construction of this family).

Let f be the homology class of a regular fiber of $E(n)_{p,q}$ and observe that $f_{pq} = \frac{1}{pq} f$ is a primitive, integral homology class. According to [28] (see [34, Theorem 3.3.6]), the SW-basic classes of $E(n)_{p,q}$ are multiples of the Poincaré dual $\mathrm{PD}[f_{pq}]$. Thus, X is abundant by Proposition A.8 (cases (a), (b), and (c)) and the observation that $b_2^+(E(n)_{p,q}) = 2n - 1$ and $b_2^-(E(n)_{p,q}) = 10n - 1$ (see [34, Lemma 3.3.4]), so $b_2^+(X) \geq 3$ and $b_2^-(X) \geq 19$ for $n \geq 2$.

If X is a minimal algebraic surface of general type, then its SW-basic classes are $\pm K_X$ by [30], where K_X is the canonical class. If Q_X is even, then X is abundant by Proposition A.8. If Q_X is odd, the Bogomolov-Miyaoka-Yau inequality, $c_1^2(X) \leq 3c_2(X)$ (see [7, Theorem VII.1.1 (iii)]), implies that

$$3\sigma \leq \chi,$$

since $c_1^2(X) = K_X^2 = 2\chi + 3\sigma$ and $c_2(X) = \chi$. Thus

$$b_2^+(X) \leq 2b_2^-(X) + 1. \tag{A.4}$$

Equality in (A.4), or $c_1^2(X) = 3c_2(X)$, holds only if the universal covering space of X is the closed unit ball in $\mathbb{C}^2$ by [34, Theorem 7.2.24] (or see [7, Corollary I.15.5] and the discussion in [7, p. 230] or [54, Theorem 4]). Since X is simply-connected and closed by hypothesis, we must have

$$b_2^+(X) < 2b_2^-(X) + 1.$$

If $b_2^+(X) = 3$, then the preceding inequality yields $b_2^-(X) \geq 2$ while if $b_2^+(X) \geq 5$, it yields $b_2^-(X) \geq 3$. Hence, for Q_X odd, X is again abundant by Proposition A.8.

This proves the theorem for minimal surfaces. If X is an abundant, smooth four-manifold, there is a hyperbolic sublattice $H \subset H^2(X;\mathbb{Z})$ such that K is orthogonal to H if K is an SW-basic class. The blow-up $X\#\overline{\mathbb{CP}}^2$ is also abundant, since we may view H as a hyperbolic sublattice of $H^2(X\#\overline{\mathbb{CP}}^2;\mathbb{Z})$, all SW-basic classes of $X\#\overline{\mathbb{CP}}^2$ have the form $K \pm \mathrm{PD}[e]$ (see [26]), and such classes are again orthogonal to H. Hence, if X is abundant, all its blow-ups are abundant too. This completes the proof of the theorem. □

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