GEOMETRIC APPLICATIONS OF SYMMETRIC POLYNOMIALS; SOME RECENT DEVELOPMENTS

Piotr Pragacz

Mathematical Institute
Polish Academy of Sciences,
Chopina 12,
87-100 Toruń, Poland

Institute of Mathematics,
Warsaw University,
Banacha 2,
02-097 Warszawa, Poland

Max-Planck-Institut für Mathematik,
Gottfried-Claren-Strasse 26,
D-5300 Bonn 3, Germany

MPI / 92-16

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by

Piotr Pragacz[2]

INTRODUCTION

The goal of this paper is to give a survey of some recent applications of the theory of symmetric polynomials to geometry. Being in the past a good part of classical algebraic knowledge, the theory of symmetric functions is rediscovered and developed nowadays (see [M]). Here we discuss only some of geometric applications of symmetric polynomials which are related to the present interest of the author.

The geometrical objects we study are : degeneracy loci of vector bundle homomorphisms, Flag varieties, Grassmannians including isotropic Grassmannians i.e. the parameter spaces for isotropic subspaces of a given vector space endowed with a symplectic or orthogonal form, Schubert varieties and the parameter spaces of complete quadrics and correlations.

The algebraical tools we use are: Schur polynomials including supersymmetric and Q-polynomials, binomial determinants and pfaffians, divided differences, reduced decompositions in the Weyl groups and Young diagrams.

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[1] This paper is an extended version of the talks given by the author at "Seminario Internacional de Algebra y sus Aplicaciones", Mexico City, January 1991.

[2] Visiting the Max-Planck Institut during the preparation of this paper.

The material surveyed here contains :

1. Polynomials supported on determinantal/degeneracy loci,
2. Formulas for the Segre classes of tensor operations and applications,
3. Divided differences and flag degeneracy loci,
4. Symplectic & orthogonal divided differences and the intersection rings of isotropic Grassmannians.

The bibliography is only a sampling of the vast literature in the subject and it doesn't pretend to be complete.

Last remark: we use a separate numbering of theorems, propositions etc. for each section.

Acknowgledgement. It was A. Lascoux who introduced me several years ago to this branch of mathematics. I wish to express to him my gratitude.

1. POLYNOMIALS SUPPORTED ON DETERMINANTAL / DEGENERACY LOCI

This Section summarizes mainly a series of results from $[P]_{1-4}$, $[P-P]_{1-3}$, $[P-R]_1$ and $[P-T]$.

Let $Mat_{mxn}(K)$ be the affine space of mxn matrices over a field K. The subvariety D_r of $Mat_{mxn}(K)$ consisting of all matrices of rank $\le r$ is called a *determinantal variety*. Algebro-geometric properties of these varieties were widely investigated in the seventies and eighties. The prototype of the results of this Section is, however, an older result - a formula of Giambelli $[G]_1$ (1903) for the degree of the *projective determinantal variety* (i.e. the class of $D_r\setminus 0$ in $\mathbb{P}(Mat_{mxn}(K))$). In order to perform his computations Giambelli used the machinery of symmetric polynomials developed by the 18-th - and 19-th - century elimination theory.

Determinantal varieties are a particular case of *degeneracy loci*

$$D_r(\varphi) = \{ x \in X \ , \ rk\ \varphi(x) \le r \}$$

$r=0,1,\dots,\min(\mathrm{rk}F,\mathrm{rk}E)$ associated with a morphism $\varphi: F \longrightarrow E$ of vector bundles on algebraic (or differentiable) variety X. The main tool of investigation of vector bundles $E \longrightarrow X$ are the *Chern classes* $c_i(E)$ $i=1,\dots$ $\dots,\mathrm{rk}E$. Let us assume that here, for simplicity, by X we will denote always a smooth variety . Recall that the *Chow group* of X, denoted $A_*(X)$, is the group of algebraic cycles on X modulo the rational equivalence. It is graded by the dimension .Thus, denoting by $A_k(X)$ the group of k-dimensional cycles modulo the rational equivalence we have $A_*(X) = \oplus A_k(X)$. Putting $A^i(X) = A_{\dim X-i}(X)$, the group $A^*(X)= \oplus A^i(X)$ has a structure of a commutative graded ring (with the multiplicative structure given by intersection theory $[F]_1$). For a vector bundle E on X, the i-th Chern class $c_i(E)$ is given by an element of $A^i(X)$.(In the case of a singular X , $c_i(E)$ is defined to be an operator on $A_*(X)$.) One of the fundamental problems in the investigation of concrete subvarieties of a given variety X is a computation of its fundamental class in terms of given generators of $A^*(X)$. For instance, Giambelli's formula mentioned above gives the fundamental class of the (projective) determinantal variety in the intersection ring $A^*(\mathbb{P}(\mathrm{Mat}_{m\times n}(K)))$[3]. In 1957 R.Thom ([T]) proved that for sufficiently general morphisms $\varphi: F \longrightarrow E$, there exists a polynomial, depending solely on $c.(E)$, $c.(F)$, which describes the fundamental class of $D_r(\varphi)$. This polynomial has been found subsequently by Porteous:

$$\det\Big[\, c_{n-r-p+q}(E-F) \,\Big]_{1\le p,q\le m-r} \qquad (*)$$

where $c_k(E-F)$ is defined by:

$$1+c_1(E-F)+c_2(E-F)+\dots = (1+c_1(E)+c_2(E)+\dots)/\ (1+c_1(F)+c_2(F) + \dots) .$$

Different variants and generalizations of (*) were considered later in [K-L], $[L]_1$, [J-L-P], $[H-T]_1$ and recently in $[F]_2$ (compare Sect.3). The second domain of research concerning (smooth) degeneracy loci is the calculation of their Chern numbers (see [H], [N] and $[H-T]_2$). Finally, the third kind of problems stems from a study of different type homology of

[3] More precisely, Giambelli calculated, in a form of a determinantal expression, the degree of the $D_r(\varphi)$ for a general map φ: $O(m_1)\oplus O(m_2)\oplus \dots \longrightarrow O(n_1)\oplus O(n_2)\oplus \dots$. As Lascoux points out, there is only a little step to pass from Giambelli's result to (*).

degeneracy loci (see a survey article [Tu]). It turns out that all these questions are closely related with the following

Problem Which polynomials in the Chern classes of E and F are universally supported on the r-th determinantal/degeneracy locus ?

More precisely let $i_r : D_r(\varphi) \longrightarrow X$ be the inclusion and let $(i_r)_*: A_*(D_r(\varphi)) \longrightarrow A_*(X)$ be the induced morphism of the Chow groups. Fix integers $m,n > 0$ and $r \geq 0$. Introduce m+n variables $c_1,\dots,c_n$; $c'_1,\dots,c'_m$ such that $\deg c_i = \deg c'_i = i$. Let $\mathbb{Z}[c.,c'.] = \mathbb{Z}[c_1,\dots,c_n,c'_1,\dots,c'_m]$ be the polynomial algebra. Following $[P]_{1,3}$ we say that $P \in \mathbb{Z}[c.,c'.]$ *is universally supported on the r-th degeneracy locus* if

$$P(c_1(E),\dots,c_n(E),c_1(F),\dots,c_m(F)) \in \mathrm{Im}(i_r)_*$$

for any morphism $\varphi: F \longrightarrow E$ of vector bundles on X such that $n=\mathrm{rk}E$, $m=\mathrm{rk}F$. Denote by $\mathcal{P}_r$ the set (ideal) of all polynomials universally supported on r-th determinantal/degeneracy locus. The polynomials (*) describing $D_i(\varphi)$ for $i \leq r$ belong to $\mathcal{P}_r$, but they do not generate this ideal. An analogous problem can be stated for symmetric (resp. skew-symmetric) morphisms: $F = E^\vee$, $\varphi^\vee = \varphi$ (resp. $\varphi^\vee = -\varphi$). In this case the corresponding ideal $\mathcal{P}^s_r$ (resp. $\mathcal{P}^{as}_r$ r-even) is contained in $\mathbb{Z}[c_1,\dots,c_n] = \mathbb{Z}[c.]$.

It follows from the main theorem on symmetric polynomials that for a sequence of variables $A = (a_1,\dots,a_n)$ the assignment

$$c_i \longmapsto (\text{i-th elementary symmetric polynomial in } A)$$

defines an isomorphism of $\mathbb{Z}[c.]$ and $\mathcal{S}ym(A)$ - the ring of symmetric polynomials in A. Similarly, by considering an analogous assignment for the c'_j 's and a second sequence of variables $B = (b_1,\dots,b_m)$, we get an isomorphism of $\mathbb{Z}[c.,c'.]$ with $\mathcal{S}ym(A,B)$ - the ring of symmetric polynomials in A and B separately. Therefore we can treat $\mathcal{P}_r$, $\mathcal{P}^s_r$ (resp. $\mathcal{P}^{as}_r$) as ideals in $\mathcal{S}ym(A,B)$ and $\mathcal{S}ym(A)$.

Their description requires two families of symmetric polynomials.

(i) Let $I=(i_1,\dots,i_k)$ be a sequence of integers. We define

$$s_I(A-B) = \det\left[s_{i_p-p+q}(A-B)\right]_{1\le p,q\le k},$$

where $s_i(A-B)$ is a homogenous polynomial of degree i such that

$$\sum_{i=-\infty}^{\infty} s_i(A-B) = \prod_{i=1}^{n} (1-a_i)^{-1} \prod_{j=1}^{m} (1-b_j) .$$

(ii) Let $I=(i_1,\dots,i_k)$ be a sequence of nonnegative integers. Assume that k is even (we put $i_k=0$, if necessary). Define

$$Q_I(A) = \text{Pfaffian}\left[Q_{i_p,i_q}(A)\right]_{1\le p,q\le k} \quad , \text{ where for}$$

$$Q(t) = \sum_{i=-\infty}^{\infty} Q_i(A)\ t^i = \prod_{i}^{n} (1+ta_i)(1-ta_i)^{-1} \text{ , we put}$$

$$Q_{i,j}(A) = Q_i(A)\ Q_j(A) + 2\sum_{p=1}^{j} (-1)^p\ Q_{i+p}(A)\ Q_{j-p}(A) .$$

(The above matrix is skew-symmetric because $Q(t)Q(-t)=1$.) The members of the first family are often called *supersymmetric polynomials* - for a particularly simple account to their properties we refer to [P-T]. The members of the second family are called *(Schur) Q-polynomials.*

Now, let E and F be two vector bundles on X. Then $s_I(E-F)$ is an element of $A^*(X)$, which is obtained from $s_I(A-B)$ via the specialization $c_i := c_i(E)$, $i=1,\dots,n$; $c'_j := c_j(F)$, $j=1,\dots,m$. Similarly we define $Q_I(E) \in A^*(X)$.

Recall that by a *partition* we understand a sequence of integers $I=(i_1,\dots,i_k)$, where $i_1\ge i_2\ge\dots\ge i_k\ge 0$. For partitions I, J we write $I\supset J$ if $i_1\ge j_1$, $i_2\ge j_2$, ...; the partition $(i,\dots i)$ (r-times) is denoted by $(i)^r$; finally the partition $(k,k-1,\dots,2,1)$ is denoted by ρ_k .

<u>Theorem 1</u> $[P]_{1,3}$ (i) The ideal $\mathcal{P}_r$ is generated by $s_I(A-B)$, where I runs over all partitions $I \supset (m-r)^{n-r}$.

(ii) The ideal $\mathcal{P}^s_r$ is generated by $Q_I(A)$, where I runs over all partitions $I \supset \rho_{n-r}$.

A similar description exists in the skew-symmetric case. The statement of the theorem remains true, when we consider also singular varieties.

For further investigations we need the following definitions. For two partitions I,J by $I\pm J$ we denote the sequence $(i_1\pm j_1, i_2\pm j_2, \ldots)$, and by I,J the sequence $(i_1, i_2, \ldots, j_1, j_2, \ldots)$.

To prove that the quoted polynomials belong to $\mathcal{P}_r$, $\mathcal{P}^s_r$, the key tools are the factorization formula the push-forward formula in the Grassmannian bundle.

<u>Proposition 2</u> (Factorization Formula) Let $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_p)$ be two partitions, $j_1 \le m$. Then

(i)
$$s_{(m)^n+I,J}(A-B) = s_I(A)\, s_{(m)^n}(A-B)\, s_J(-B)$$
$$= (-1)^{|J|}\, s_I(A)\, s_{(m)^n}(A-B)\, s_{J\sim}(B)^4$$

(ii)
$$Q_{\rho_{n-1}+I}(A) = Q_{\rho_{n-1}}(A)\, s_I(A)\ .$$

((ii) is a result of Stanley, references for (i) are discussed in $[L]_2$).

<u>Proposition 3</u> Let π: $G=G^q(E) \longrightarrow X$ be the Grassmannian bundle parametrizing q-quotients of E. Let $0 \leftarrow Q \leftarrow E_G \leftarrow R \leftarrow 0$ be the tautological exact sequence of vector bundles on G.

(i) ([J-L-P], $[P]_3$) For every vector bundle H on X,

$$\pi_* \Big[\, s_I(Q-H_G)\ s_J(R-H_G)\, \Big] = s_{I-(n-q)^q,J}(E-H)\ ,$$

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[4] For a given partition I, we write $|I| := \sum i_p$ - the sum of parts of I, and $I\sim$ denotes the partition $(h_1, h_2, \ldots)$ where $h_p = \mathrm{card}\{q : i_q \ge p\}$.

(ii) ($[P]_{3,4}$) Let $I=(i_1,\dots,i_h)$, $J=(j_1,\dots,j_k)$ be two sequences of positive integers, $h\le q$, $k\le n-q$. Then

$$\pi_* \left[c_{q(n-q)}(R\otimes Q)\ Q_I(Q)\ Q_J(R) \right] = Q_{I,J}(E) \ .$$

A proof that the ideals $\mathcal{P}_r$ and $\mathcal{P}^s_r$ are actually generated by the above polynomials is based on the investigation of the universal tautological determinantal variety $\mathbf{D}_r \subset \mathbf{Hom}(F,E)$ (the fiber of $\mathbf{D}_r$ over a point $x \in X$ is equal to $\{ f \in \mathrm{Hom}(F(x),E(x)) \mid \dim \mathrm{Im}\, f \le r \}$. The bundles E and F occuring in this construction are some "universal enough" vector bundles over the product GG of two Grassmannians (see $[P]_3$). In fact, in $[P]_3$, two proofs of this assertion are given. The first one uses a certain desingularization of $\mathbf{D}_r$. The second one goes by induction on r with the help of exact sequence

$$A(\mathbf{D}_{r-1}) \longrightarrow A(\mathbf{D}_r) \longrightarrow A(\mathbf{D}_r-\mathbf{D}_{r-1}) \longrightarrow 0 \quad ,$$

and a detailed analysis of $A(\mathbf{D}_r-\mathbf{D}_{r-1})$.

Theorem 1 is valid also for singular homology, Borel-Moore homology and étale homology instead of the Chow groups. Since the same applies to Proposition 3, the proof that the quoted polynomials belong to $\mathcal{P}_r$ and $\mathcal{P}^s_r$, is the same. On the other hand the proof that the ideals $\mathcal{P}_r$, $\mathcal{P}^s_r$ are generated by the above polynomials is based on the following compactification of $\mathbf{D}_r$. Let us embed the above $\mathbf{Hom}(F,E)$ into a Grassmannian bundle $\mathbf{G}= G_m(F\oplus E)$ by assigning fiberwise to $f\in \mathrm{Hom}(F(x),E(x))$ its (graph of f) $\in G_m(\ F(x)\oplus E(x)\ )$, x belonging to the base space GG. On $\mathbf{G}$ there exists a natural tautological extension of the universal morphism on $\mathbf{Hom}(F,E)$ and its degeneracy loci together with their desingularization serve to prove the assertion. An important advantage of the above compactification is the vanishing of its odd homology groups - this is not the case of $\mathbf{D}_r$. For details see $[P\text{-}R]_1$.

Propositions 2 and 3 allow one to prove the following results concerning the structure of the ideals in question.

Theorem 4 $[P]_3$ (i) $\mathcal{P}_r = \left[s_{(m-r)^{n-r}+I}(A-B)\ ,\ I \subset (r)^{n-r} \right]$,

$$\text{(ii) } \mathcal{P}^s_r = \left[Q_{\rho_{n-r}+I}(A)\ ,\ I \subset (r)^{n-r} \right] ,$$

- thus these ideals are generated by $\binom{n}{r}$ elements.

Note that it is still an open problem to show that these sets form minimal sets of generators of $\mathcal{P}_r$ and $\mathcal{P}^s_r$, for m≥n.

The additive structure of the ideal $\mathcal{P}_r$ is described in:

Theorem 5 $[P]_3$ The polynomials $s_{I_k}(A-B)\, s_{J_k}(A)$ where I_k contains $(m-k)^{n-k}$, and does not contain $(m-k+1)^{n-k+1}$, $\ell(J_k) \le k$ [5] , form a $\mathbb{Z}$-basis of $\mathcal{P}_r$.

Moreover, the ideal $\mathcal{P}_r$ is prime ($[P]_2$) , and is a set-theoretical complete intersection (is equal to the radical of an ideal generated by a regular sequence of length r+1).

The methods developped in the proof of Theorem 1 (especially Proposition 3) allow us to obtain the following applications.

a) An algorithm for computation of the Chern numbers of the Kernel and Cokernel bundle and the Chern numbers of $D_r(\varphi)$.

Let $\varphi: F \longrightarrow E$ be a general morphism[6] of C^∞-vector bundles on a complex manifold X. Assume that $D_{r-1}(\varphi) = \emptyset$. Then, Ker$\varphi$ and Cokerφ are vector bundles on $D_r(\varphi)$ of ranks respectively m-r and n-r. In $[H\text{-}T]_2$, the authors posed the following problem : "Calculate the Chern numbers of Kerφ and Cokerφ " (whenever we speak on Chern numbers we assume X compact, as it is standard). In fact, in loc.cit., an algorithm is constructed, which, however, leads to the necessity of performing of many cancellations of

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[5] For a given partition I, we denote by $\ell(I)$ the number of its nonzero parts.

[6] We say that φ: $F \to E$ is general if the induced section $X \to \mathrm{Hom}(F,E)$ is transwerse to all tautological degeneracy loci.

pairwise opposite elements. The algorithm constructed in $[P]_3$ has a combinatorial character (it requires no cancellations) and is based on the following

Theorem 6 $[P]_3$:

(i) $(i_r)_* s_I(C)\, s_J(-K) = s_{(m-r)^{n-r}+I,J}(E-F)$,

(ii) If $F=E^\vee$ and $\varphi : E^\vee \longrightarrow E$ is symmetric then

$$(i_r)_* s_I(C) = Q_{\rho_{n-r}+I}(E) .$$

A similar formula exists for the skew-symmetric morphism (see $[P]_3$).

Using Theorem 6 one constructs, for a manifold X and holomorphic morphism φ , an effective algorithm for the calculation of the Chern numbers of $D_r(\varphi)$ ($D_r(\varphi)$ is smooth under the above assumptions). In particular, if $d = \dim D_r(\varphi)$, then (assuming $m \geq n$) one obtains a closed formula $c_d(TD_r(\varphi)) \cap [D_r(\varphi)]$ (= Euler characteristic of $D_r(\varphi)$):

$$\sum (-1)^{|I|+|J|}\, D_{I,J\sim}\, s_{(m-r)^{n-r}+I,J}(E-F)\, c_{d-|I|-|J|}(X) \qquad (\#)$$

where the sum is over partitions I,J with $\ell(I) \leq n-r$, $\ell(J) \leq n-r$; and the coefficient $D_{I,J}$ is the following determinant of binomial coefficients

$$D_{I,J} = \operatorname{Det}\left[\binom{i_p+j_q+m+n-2r-p-q}{i_p+n-r-p}\right] .$$

Similar algorithms (involving Q-polynomials) were found in $[P]_3$ to compute Chern numbers in the symmetric and skew-symmetric cases. Since the Segre classes of $- \otimes -$, $S^2(-)$ and $\Lambda^2(-)$ play a significant role in the calculations of the Chern number $c_d(TD_r(\varphi))$, we will come back to this problem in Section 2.

Let us denote the expression (#) by $g(r)$ and drop now the assumption $D_{r-1}(\varphi)=\emptyset$.

Theorem 7 $[P\text{-}P]_{1,2}$ Let φ be a general holomorphic morphism of vector bundles over a complex compact manifold. Then, the (topological) Euler characteristic of $D_r(\varphi)$ is

$$\sum_{k=0}^{r} (-1)^k \binom{n-r+1-k}{k} g(r-k) .$$

Theorem 6 is not valid if the codimension of $D_r(\varphi)$ has the expected value, $(m-r)(n-r)$, even if $D_r(\varphi)$ is a divisor - compare $[P\text{-}P]_{1,3}$.

b) A calculation of the Chow groups of the determinantal schemes.

A prototype of these results is the following result of Bruns ([B]). Let R be a normal, noetherian ring, X - a m×n matrix of indeterminates, I - the ideal generated by (r+1)-minors of X. Then, the divisor class groups satisfy: $Cl(R[X]/I) \cong Cl(R)\oplus\mathbb{Z}$.

The geometric analogue of Cl is A^1 (the Chow group of codimension 1 cycles modulo the rational equivalence).

Let $Mat_{m\times n}(K)$ be the affine space of m×n matrices over a field K. Let $D_r \subset Mat_{m\times n}(K)$ be the subscheme defined by the ideal generated by all minors of order r+1.

<u>Theorem 8</u> $[P]_{1,3}$ If $m\geq n$ then the Chow group of D_r graded by the codimension, is isomorphic to the Chow group of the Grassmannian $G_r(K^n)$.

There exists a relative analogue of this Theorem (see $[P]_3$ Proposition 4.3), which gives, in particular the following corollary. For every K-scheme X, $A^k(X\times D_r) \simeq \oplus A^{k-|I|}(X)$, the sum over all partitions $I \subset (r)^{n-r}$. For k=1 this is a geometric analogue of the result of Bruns.

We end this Section with following algebraic digression.

c) A generalization of the resultant of two polynomials.

$$\text{Let}\quad A(x) = x^n + \sum_{i=1}^{n} c_i x^{n-i} \quad , \quad B(x) = x^m + \sum_{j=1}^{m} c'_j\, x^{m-j}$$

be two polynomials in one variable with generic coefficients. It follows from the classical algebra, that there exists a polynomial in $\{c_i\}$, $\{c'_i\}$ called the *resultant* whose vanishing (after a specialization of $\{c_i\},\{c'_i\}$ in an algebraically closed field) implies that the corresponding polynomials have a common root (see $[L]_2$, for instance).

Now, let $\mathcal{I}_r$ be the ideal of all $P \in \mathbb{Z}[c.,c'.]$, which vanish if, after a specialization in a field, $A(x)$ and $B(x)$ have $r+1$ common roots. Surprisingly (or not) we have

Theorem 9 $[P]_{2,4}$:

$$\mathcal{I}_r = \left[s_{(m-r)^{n-r}+I}(A-B) \ , \ I \subset (r)^{n-r} \right] .$$

In other words $\mathcal{I}_r = \mathcal{P}_r$ in the above notation. It would be interesting to have an *intrinsic proof* of this equality. A similar interpretation is given in $[P]_4$ for the ideal $\mathcal{P}^s_r$ generated by Q-polynomials.

2. FORMULAS FOR THE SEGRE CLASSES OF TENSOR OPERATIONS AND APPLICATIONS

This Section summarizes some results from [L-L-T] and $[P]_3$.

Let E, F be vector bundles of ranks n and m respectively. Assume $m \geq n$. We state

Theorem 1 (i) ([L-L-T]) The total Segre class of the tensor product $E \otimes F$ is given by

$$s(E\otimes F) = \sum D^{n,m}_{I,J}\ s_I(E)\ s_J(F) \qquad ,$$

where the sum is over partitions I, J of length $\leq n$ and

$$D^{m,n}_{I,J} = \mathrm{Det}\left[\begin{pmatrix} i_p + j_q + m+n-p-q \\ i_p+n-p \end{pmatrix} \right]_{1\leq p,q\leq n} .$$

(ii) ([L-L-T] & $[P]_3$) The total Segre class of the second symmetric power S^2E is given by

$$s(S^2E) = \sum ((I+\rho_{n-1}))\ s_I(E) \qquad ,$$

where the sum is over all partitions I and the definition of $((J))$ is as follows. If $\ell = \ell(J)$ is even, define $((J))$ to be the Pfaffian of the $\ell \times \ell$ skew-symmetric matrix $[a_{p,q}]$ where

$$a_{p,q} = \sum \begin{pmatrix} i_p + j_q \\ j \end{pmatrix} \qquad (\text{ the sum over } j_q < j < j_p) ,$$

and if ℓ is odd, then $((J)) := \sum (-1)^{p-1} 2^{j_p} ((J \setminus \{j_p\}))$.

(iii) ([L-L-T] & $[P]_3$) The total Segre class of the second exterior power $\Lambda^2 E$ is given by

$$s(\Lambda^2 E) = \sum \ [I+\rho_{n-1}]\ s_I(E) \qquad ,$$

where the sum is over all partitions I and the definition of [J] is as follows. If $\ell=\ell(J)$ is even, define [J] to be the Pfaffian of the $\ell\times\ell$-skew-symmetric matrix $[\ (j_p+j_q-1)!/\ j_p!\ j_q!\ ]$; if ℓ is odd then [J]=0 unless $j_\ell=0$ where $[J] = [j_1,\dots,j_{\ell-1}]$.

Remark 2 See [L-L-T] for other approaches to the numbers ((J)) and [J]. The history of formulas for $s(S^2E)$ and $s(\Lambda^2E)$ is as follows. At first, one of the authors of [L-L-T] has informed the author about recursive formulas for ((J)) and [J] obtained with the help of divided differences. (We will explain and use this extremely powerful technique in Section 3 and 4.) Using this recursion the author has found and proved the above pfaffian-formulas in $[P]_3$. Finally, the authors of [L-L-T] managed to give a self-cointained, full and elegant account of all these formulas based on an interplay between the original recursive formulas, pfaffian expressions from $[P]_3$ and formulas which present ((J)) and [J] as sums of minors in some matrices of binomial numbers. Consequently, there are no divided differences in the final version of [L-L-T]. ("The power was eliminated by the elegance"!)

As it was mentioned in Section 1, Theorem 1 can be applied in the calculation of the Euler characteristic of $D_r(\varphi)$. Besides the formula (#) after Theorem 6 in Sect.1 one has also (in the notation of Sect.1):

Theorem 3 $[P]_3$ Assume X is compact, φ is general and $D_{r-1}(\varphi)=\emptyset$.

(i) If φ is symmetric, d:= dimX-(n-r)(n-r+1)/2 then

$$\chi(D_r(\varphi))=\sum (-1)^{|I|}\ ((I+\rho_{n-r-1}))\ Q_{\rho_{n-r}+I}(E)\ c_{d-|I|}(X).$$

(i) If φ is skew-symmetric, d:= dimX-(n-r)(n-r-1)/2 , r-even, then

$$\chi(D_r(\varphi))=\sum (-1)^{|I|}\ [I+\rho_{n-r-1}]\ P_{\rho_{n-r-1}+I}(E)\ c_{d-|I|}(X).$$

The proof is based on Theorem 1 (ii),(iii) and Theorem 6 from Sect.1.

Another application of Theorem 1 was given in [L-L-T] to enumerative properties of complete correlations and quadrics. Let us limit ourselves to the latter case.

Let us fix a positive integer r and a projective space $\mathbb{P}$. By a *complete quadric of rank r* we understand a sequence $Q.: Q_1 \subset Q_2 \subset \ldots \subset Q_n$ (n can vary) of quadrics in $\mathbb{P}$, such that

1) Q_1 is smooth,
2) the linear span $L(Q_i)$ of Q_i is the vertex of Q_{i+1} , $i=1,\ldots,n-1$,
3) $\dim L(Q_n) = r-1$.

There exists a natural structure of a smooth algebraic projective variety on $CQ(r)$ - the set of all rank r complete quadrics (see e.g. [L-L-T]). Let $\mu_i \in A_*(CQ(r))$ $(i=1,\ldots,r)$ be the class of the locus of all complete quadrics Q. such that Q_n is tangent to a given (codimension i)-plane in $\mathbb{P}$.

Now let $G= G_r(\mathbb{P})$ be tha Grassmannian parametrizing (r-1)-dimensional linear subspaces of $\mathbb{P}$. Fix a sequence $I=(1\le i_1<i_2<\ldots<i_r\le \dim\mathbb{P})$ of integers and consider the flag $L.: L_1\subset L_2\subset\ldots\subset L_r$ of linear subspaces in $\mathbb{P}$ where $\dim L_j=i_j$, $j=1,\ldots,r$. Let $\Omega(I)$ be the class in $A_*(G)$ of the Schubert cycle $\{L\in G: \dim(L\cap L_j)\ge j-1 , j=1,\ldots,r\}$. We have a map $f: CQ(r) \longrightarrow G$ such that $f(Q.) = L(Q_n)$. Let $\omega(I) := f^*\Omega(I)$.

Classics of Enumerative geometry like Schubert, Giambelli ...were interested in computation of the number of complete quadrics Q. such that Q_n is tangent to m_j fixed planes of codimension j in general position in $\mathbb{P}$ and such that $\dim (L(Q_n)\cap L_j)\ge j-1$ for each member of a flag L. as above. This question makes sense if $i_1+\ldots+i_r+r-1 = m_1+\ldots+m_r$ because then $\mu_1^{m_1}\mu_2^{m_2}\ldots\mu_r^{m_r}\cdot\omega(I)$ in $A_*(CQ(r))$ represents a 0-dimensional cycle. The answer to the question (under the above assumption) needs besides the numbers ((J)) defined at the begining of this section, also the function $\alpha(p;k,j)$ defined by $\alpha(p;k,j):= \binom{k}{0} + \binom{k}{1} p + \ldots + \binom{k}{j} p^j$ if $j\ge 0$, - 0 otherwise. In fact, the following result answers a more general question:

Theorem 4 [L-L-T] Assume that p is a number such that $0 \le p < r$ and $m_1+\dots+m_q > i_r+i_{r-1}+\dots+i_{r-q+1}+q-1$ for $q=1,\dots,p-1$. Then

$$\mu_1^{m_1}\mu_2^{m_2}\dots\mu_{p+1}^{m_{p+1}}\cdot\omega(I) =$$

$$1^{m_1}2^{m_2}\dots p^{m_p}\,[\,(p+1)^{m_{p+1}}((I)) - \sum\alpha(p;m_{p+1},m_{p+1}-|J|-(r-p)\,\varepsilon_J((J))\cdot((J'))\,]$$

where the sum is over all (card r-p) - subsequences J in I ; $J'=I\setminus J$ and $\varepsilon_J=\mathrm{sign}(J,J')$.

There is similar formula for complete correlations which, in turn, uses the numbers $D^{n,m}_{I,J}$ (see [L-L-T]).

3. DIVIDED DIFFERENCES AND FLAG DEGENERACY LOCI

This Section summarizes some of the results of $[F]_2$.

Consider the flagged vector bundles

$$F_1 \subset F_2 \subset \dots \subset F_m = F \qquad \text{and} \qquad E = E_n \twoheadrightarrow \dots \twoheadrightarrow E_2 \twoheadrightarrow E_1$$

over a variety X and let φ: $F \longrightarrow E$ be map of vector bundles. Assume that a function $\mathbf{r}:\{1,\dots,n\}\times\{1,\dots,m\}\longrightarrow \mathbb{N}$ is given (we will refer to **r** as to *rank fonction*). Define

$$D_{\mathbf{r}}(\varphi) = \{\ x\in X,\ \mathrm{rk}\ (\ F_q(x)\longrightarrow E_p(x)\ ) \le \mathbf{r}(p,q)\ \forall p,q\ \}$$

In $[F]_2$ the author gives the conditions on **r** , which guarantee that for "generic" φ, $D_{\mathbf{r}}(\varphi)$ is irreducible. Then, a natural problem arises, to find for such a **r** and φ a formula expressing $[D_{\mathbf{r}}(\varphi)]$ in terms of the Chern classes of E. and F..

It turns out that the crucial case is the case of complete flags i.e. $\mathrm{rk}E_i=\mathrm{rk}F_i=i$ and $m=n$. The desired formula in every other case can be deduced from that one. In this situation the degeneracy loci $D_{\mathbf{r}}(\varphi)$ are parametrized by permutations $\mu\in S_n$, and

$$r_\mu(p,q) = \mathrm{card}\{\ i\le p,\ \mu(i)\le q\ \}.$$

Let $X_\mu(E.,F.) = D_{r_\mu}(\varphi)$. Then the expected (i.e. maximum, if nonempty) codimension of $X_\mu(E.,F.)$ is $\ell(\mu)$ (the length of μ). In order to describe a formula for the fundamental class of $X_\mu(E.,F.)$ associated with a generic φ we need some algebraic tools developed in [B-G-G], [D] and $[L\text{-}S]_2$.

Let $A=(a_1,\dots,a_n)$, $B=(b_1,\dots,b_n)$ be two sequence of independent and commuting variables. We have divided differences

$$\partial_i : \mathbb{Z}[AB] \longrightarrow \mathbb{Z}[AB] \qquad \text{(of degree } -1)$$

defined by

$$\partial_i(f)=(f - s_i f)/(a_i - a_{i+1}) \qquad i=1,\dots,n-1,$$

where s_i denotes the i-th simple transposition. For every reduced decomposition $\mu= s_{i_1} \dots s_{i_k}$ [7] one can define $\partial_\mu = \partial_{i_1} \circ \dots \circ \partial_{i_k}$ - an operator on $\mathbb{Z}[AB]$ of degree $-\ell(\mu)$. In fact ∂_μ does not depend on the reduced decomposition chosen. Finally, for a permutation $\mu \in S_n$, we give, following $[L\text{-}S]_2$, $[L]_3$:

<u>Definition 1 of a *(double) Schubert polynomial*</u> $X_\mu(A,B)$.

$$X_\mu(A,B) = \partial_{\mu^{-1}\omega}\left(\prod_{i+j\le n}(a_i-b_j)\right),$$

where ω is the permutation with biggest length in S_n. Note that the operators act here on A-variables; however it can be shown that

$$X_\mu(A,B) = (-1)^{\ell(\mu)} X_{\mu^{-1}}(B,A).$$

Specialize now

$$a_i := c_1(\mathrm{Ker}\ E_i \longrightarrow E_{i-1}) \quad \text{and} \quad b_i := c_1(F_i / F_{i-1}),$$

and assume X is smooth, for simplicity. Then we have

---

[7] Writing here and in the sequel $s_{i_1} s_{i_2} \dots s_{i_k}$ we mean that we perform first s_{i_1}, then - s_{i_2} etc.

Theorem 2 $[F]_2$ If $\operatorname{codim}_X(X_\mu(E.,F.))=\ell(\mu)$ then

$$[X_\mu(E.,F.)] = X_\mu(A,B) \qquad \text{in } A_*(X).$$

The key point in the proof of Theorem 2 in $[F]_2$ is a geometric interpretation of the divided differences with the help of the correspondences in the Flag bundles.

This theorem generalizes in an uniform way the formulas for the fundamental classes of Schubert varieties in the flag varieties from [B-G-G], [D] and $[L\text{-}S]_1$, and -with the help of some algebra of Schubert polynomials ($[L\text{-}S]_2$) -other formulas for degeneracy loci like the Giambelli-Thom-Porteous formula (see Section 1) as well as determinantal formulas for flag degeneracy loci from [K-L], $[L]_1$ and $[P,(8.3)]_3$.

Example 3 In the above notation we put $E=E_i$ for all i , and consider the locus:

$$D = \{ x\in X : \dim \operatorname{Ker} (F_j(x) \longrightarrow E(x)) \geq j \ , \ j=1,\dots,m \}$$

then Theorem 2 specializes to the Kempf-Laksov formula asserting that the fundamental class of **D** is

$$\operatorname{Det} [c_{\mathrm{rk}E-\mathrm{rk}F_i+j}(E-F_i)]_{1\leq i,j\leq m} .$$

A combination of Theorem 4 with $[G]_2$ gives some interesting formulas for specialization of Schubert polynomials.

Finally, note that (double) Schubert polynomials are a useful tool in computation of Chern classes of the tangent vector bundles to the Flag and Grassmannian varieties (see $[L]_3$).

4. SYMPLECTIC & ORTHOGONAL DIVIDED DIFFERENCES AND THE INTERSECTION RINGS OF ISOTROPIC GRASSMANNIANS

In this Section we summarize some results from [H-B], $[P,\text{Sect.}6]_4$ and $[P\text{-}R]_2$.

Let G denote the Grassmannian of n-dimensional isotropic subspaces in

$\mathbb{C}^{2n}$ with respect to a non-degenerate symplectic form on $\mathbb{C}^{2n}$. Let F denote the flag variety of (total) isotropic flags in $\mathbb{C}^{2n}$ (with respect to the same symplectic form). By ρ we will denote the partition $(n,\dots,2,1)$. Let $I\subset\rho$ be a strict partition $I=(i_1>i_2>\dots>i_k>0)$. We associate to I the element w_I of the symplectic Weyl group W :

$$w_I = s_{n-i_k+1}\dots s_{n-1}s_n \dots s_{n-i_2+1}\dots s_{n-1}s_n\, s_{n-i_1+1}\dots s_{n-1}s_n$$

where s_j stands for the j-th simple transposition in W (see [H-B, Sect.2] for details about W) [8].

From the theory in [B-G-G] and [D] we get a Schubert cycle $X_{w_I}\in A^{|I|}(F)$ which in fact belongs to $A^{|I|}(G)\subset A^{|I|}(F)$. Denote this element in $A^{|I|}(G)$ by $\sigma(I)$, for short.

As usual, we will associate to a partition I a diagram D_I. The elements of the D_I will be boxes (and not dots). This will allow us to speak about "connected components" of differences between diagrams without misunderstandings.

The following result was proved originally in [H-B].

Theorem 1 Let $I=(i_1,\dots,i_k)\subset\rho$ be a strict partition. The following equality holds in $A^*(G)$ $(p=1,\dots,n)$:

$$\sigma(I)\ \sigma(p) = \sum 2^{m(I,J)}\ \sigma(J)\ ,$$

where the sum is over strict partitions J such that $i_{h-1}\ge j_h\ge i_h$ ($i_0=n$, $i_{k+1}=0$) , $|J|=|I|+p$ and $m(I,J)$ is the number of connected components of $D_J\setminus D_I$ not meeting the first column.

Example 2 n=7
$\sigma(632)\ \sigma(5) = 2\ \sigma(763) + 2^2\ \sigma(7531) + 2\ \sigma(7621) + 2\ \sigma(7432) + \sigma(6532).$

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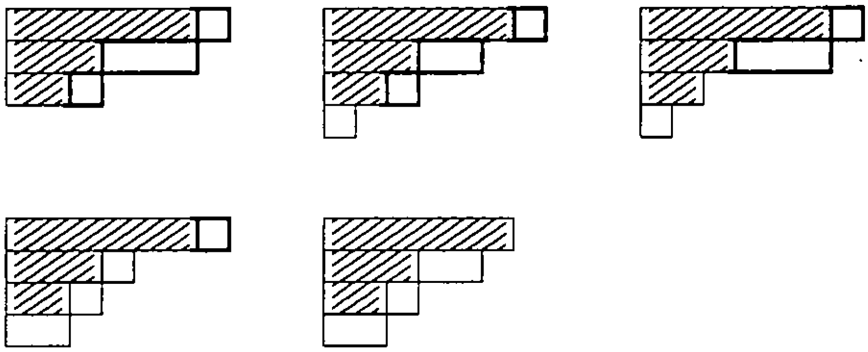
[8] Footnote 7 applies here with S_n replaced by W.



We sketch now a proof of this Theorem due to Jan Ratajski and the author in $[P\text{-}R]_2$. This proof is much simpler than the proof in [H-B]; it uses essentially the symplectic divided differences from [B-G-G] and [D].

Let $A=(a_1,\dots,a_n)$ be independent variables. It follows from [B-G-G] and [D] that $A^*(F)$ is identified with $\mathbb{Z}[A]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by symmetric polynomials in $a_1^2,\dots,a_n^2$ without constant term. Also, $A^*(G)$ is identified with $(\mathbb{Z}[A]/\mathcal{I})^{S_n}$ i.e. with the quotient of the symmetric polynomials modulo $\mathcal{I}$ restricted to the ring of symmetric polynomials.

We have "symplectic divided differences":

$$\partial_i : \mathbb{Z}[A] \longrightarrow \mathbb{Z}[A] \qquad \text{(of degree } -1), \ i=1,\dots,n, \qquad \text{defined by}$$

$$\partial_i(f)=(f - s_i f)/(a_i - a_{i+1}) \qquad i=1,\dots,n-1,$$

$$\partial_n(f) = (f - s_n f)/2a_n \quad .$$

The key tool for our purposes is a Leibnitz-type formula:

$$\partial_i(f\cdot g) = (\partial_i f)\cdot g + (s_i f)\cdot(\partial_i g).$$

For every reduced decomposition $w=s_{i_1}\dots s_{i_k}$ one can define $\partial_w = \partial_{i_1}\circ\dots\circ\partial_{i_k}$ - an operator on $\mathbb{Z}[A]$ of degree $-\ell(w)$. In fact ∂_w does not depend on the reduced decomposition chosen. There exists a ring homomorphism

$$c:\ \mathbb{Z}[A] \longrightarrow A^{\bullet}(F) \quad ,$$

(called the characteristic map) defined for a homogeneous $f \in \mathbb{Z}[A]$ by

$$c(f) = \sum_{\ell(w)=\deg f} \partial_w(f)\, X_w \quad .$$

For instance, denoting by e_p the p-th elementary symmetric polynomial in A , we have

$$c(e_p) = \sigma(p) = X_{s_{n-p+1} \cdots s_{n-1} s_n} \in A^p(G)$$

([H-B, Lemma 2.13']).

The operators ∂_w give rise to operators on $A^{\bullet}(F)$ (denoted by the same letters) and these two families of operators commute with c . Moreover for w, v , $\partial_w(X_v) = 1$ iff $w=v$.

Let f_I be such that $c(f_I) = \sigma(I)$. Our goal is to find coefficients m_J appearing in

$$c(\, f_I \cdot e_p) = \sum m_J\, \sigma(J) \ .$$

Consider $D \subset D_J$. The boxes in D_J which belong to D will be called D-boxes; the boxes in $D_J \backslash D$ will be called non D-boxes. We associate with D the following operators $\bar{\partial}^D_J$ and $\partial^D_{\underline{J}}$. For technical reasons we will use, from now on, the following coordinates for indexing boxes in $J \subset \rho$:

$$\begin{array}{c|ccccc}
 & n & n-1 & \dots & 2 & 1 \\
\hline
1 & & & & & \\
2 & & & & & \\
\vdots & & & & & \\
n & & & & &
\end{array}$$

In Definitions 3,4 we read D_J row by row from left to right starting ting from the first row.

<u>Definition 3 of $\partial^D_{\underline{J}}$</u> : Read D_J. Every D-box in the i-th column gives us the s_i. Every non D-box in the i-th column gives the ∂_i. Then $\partial^D_{\underline{J}}$ is the composition of the so obtained s_i's and ∂_i's (the composition written from right to left).

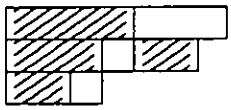
<u>Definition 4 of r_D</u> : Read D_J. Every D-box in the i-th column gives us

the s_i. Non D-boxes have no influence on r_D. Then r_D is the word obtained by writing the so obtained s_i's from right to left.

Definition 5 of $\bar{\partial}^D_J$: $\bar{\partial}^D_J := \partial_{r_D}$.

Example 6 J=(763), n=7.

7 6 5 4 3 2 1



(D-boxes are "dark" here)

$$\underline{\partial}^D_J = \partial_5\circ s_6\circ s_7\circ s_2\circ s_3\circ\partial_4\circ s_5\circ s_6\circ s_7\circ\partial_1\circ\partial_2\circ\partial_3\circ s_4\circ s_5\circ s_6\circ s_7 \,,$$

$$r_D = s_6s_7s_2s_3s_5s_6s_7s_4s_5s_6s_7 \,,$$

$$\bar{\partial}^D_J = \partial_6\circ\partial_7\circ\partial_2\circ\partial_3\circ\partial_5\circ\partial_6\circ\partial_7\circ\partial_4\circ\partial_5\circ\partial_6\circ\partial_7 \,.$$

Proposition 7 In the above notation,

$$m_J = \sum \bar{\partial}^D_J(f_I)\cdot\underline{\partial}^D_J(e_p) \,,$$

where the sum is over all $D\subset D_J$ such that $r_D\in R(w_I)$[9] and $\underline{\partial}^D_J(e_p)\neq 0$.

This is a consequence of consecutive applications of the Leibnitz rule used in this way: we apply only the ∂_i's (and the identity operators) to f_I ; and both the s_i's and ∂_i's to e_p.

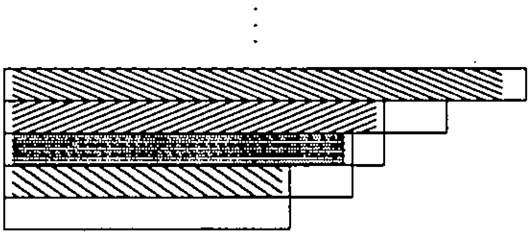
One proves that if $\ell(J) > \ell(I) + 1$ or $j_{h+1} > i_h$ for some h , then $\underline{\partial}^D_J(E)=0$ for every $D\subset D_J$ such that $r_D\in R(w_I)$.

Moreover, fix a strict partition $I\subset\rho$. Let J be a strict partition such that $I\subset J\subset\rho$, $\ell(J)\le\ell(I)+1$, $j_{h+1}\le i_h$ for every h. Then there exists exactly one $D^{I,J}\subset D_J$ such that $r_D\in R(w_I)$ and $\underline{\partial}^D_J(E)\neq 0$ for $D=D^{I,J}$.

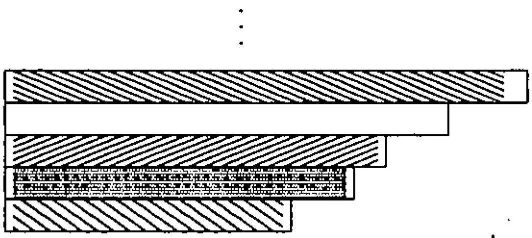
---

[9] By R(w) for w in W, we denote the set of reduced decompositions of w.

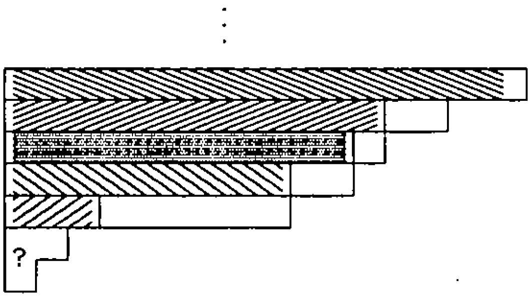
The idea of constructing such a $D^{I,J}$ can be easily explained pictorially. The boxes from $D_I \subset D_J$ are shadowed on the picture below. A part :



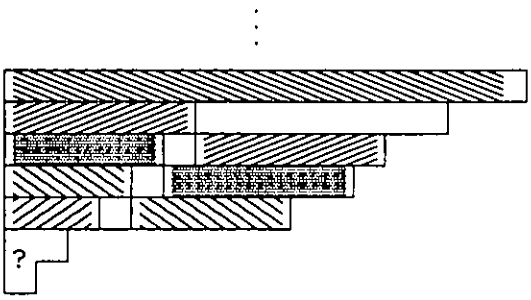
of the diagram $D_I \subset D_J$ is deformed to



On the other side, a part:



of the diagram $D_I \subset D_J$ is deformed to



The deformations are performed in the direction South $\rightarrow$ North.

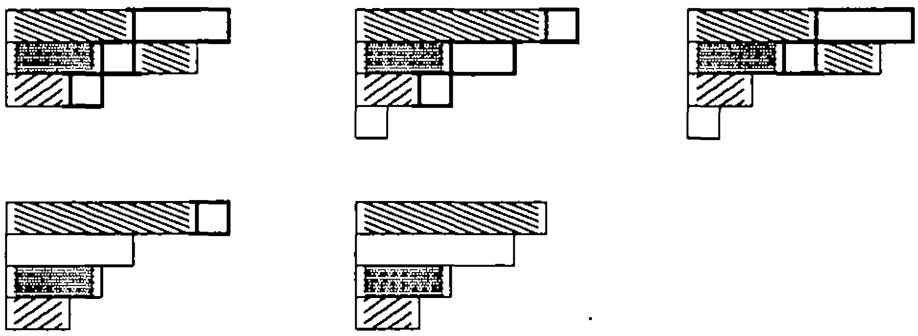
Fix a strict partition $I\subset\rho$ and a number $p = 1,\dots,n$. Let J be a strict partition such that $I\subset J\subset\rho$, $|J|=|I|+p$, $\ell(J)\le\ell(I)+1$, $j_{h+1}\le i_h$ for every h. Let $D=D^{I,J}$. Every ∂_i involved in $\underline{\partial}_J^D$ is associated to a box in $D_J\setminus D$. It turns out that the connected components of $D_J\setminus D$ play a crucial role in the computation of $\underline{\partial}_J(e_p)$. Namely, in the above notation

$$\underline{\partial}_J^D(e_p) = 2^{m(I,J)} \quad ,$$

where $m(I,J)$ is the number of connected components of $D\setminus D_J$ not meeting the n-th component. Changing the numbering of columns to the usual order, this can be easily restated as: $m(I,J)$ is the number of connected components of $D_J\setminus D_I$ not meeting the first column.

This finishes the sketch of the proof - for details see $[P\text{-}R]_2$.

<u>Example 8</u> The diagrams $D^{(632),J}$ for partitions J appearing in the decomposition $\sigma(632)\ \sigma(5)$, are :



We end with a geometric interpretation of the $\sigma(I)$'s. Let V be a 2n-dimensional vector space endowed with a symplectic nondegenerate form ϕ: $V\times V \longrightarrow \mathbb{C}$. Let $(v_1,\dots,v_n)$ be a basis of an isotropic n-subspace of V. Let $V_1\subset V_2\subset \dots \subset V_n$ be a flag of isotropic subspaces spanned by the first i vectors in the sequence $(v_1,\dots,v_n)$. Then $\sigma(i_1,\dots,i_k)$ is the class in $A^{|I|}(G)$ of the cycle of all isotropic n-subspaces L in V such that $\dim\,(\,L\cap V_{n+1-i_h}\,)\ \ge h,\ h=1,\dots,k.$

The Schubert Calculus for usual Grassmannians is based on three main theorems: Pieri's formula, Giambelli's determinantal formula and the Basis theorem (see for example $[F]_1$). In the case of the isotropic Grassmannian G, a Pieri-type formula is described in Theorem above.

In $[P, Sect.6]_4$ the author has deduced from Theorem 1 the following Giambelli-type formula.

<u>Theorem 9</u> $[P]_4$ Let $I=(i_1,\dots,i_k) \subset \rho$ be a strict partition, k -even (we can always assume it by putting $i_k=0$ if necessary). Then

$$\sigma(I) = \text{Pfaffian} [\sigma(i_p,i_q)]_{1\le p<q\le k} ,$$

where $\sigma(i_p,i_q) = \sigma(i_p)\sigma(i_q) + 2 \sum_{h=1}^{q} (-1)^h \sigma(i_p+h) \sigma(i_q-h)$, and where $\sigma(i_p,0) = \sigma(i_p)$.

This formula is deduced in $[P, Sect.6]_4$ from Theorem 1 using the properties of *Schur Q-Polynomials*.

A Basis-type theorem can be formulated as

<u>Theorem 10</u> :

$$A_*(G) = \oplus \mathbb{Z} \sigma(I) ,$$

the sum over all strict partitions $I\subset\rho$.

This result can be deduced from a general theory of the cellular Schubert/Bruhat decompositions of homogeneous spaces (see [B-G-G], [D]). The cellular decomposition in the case of G was described in details in $[P, Sect.6]_4$. An another simple, conceptual proof of Theorem 10 is given in $[P-R]_2$.

Using exactly the same method one can prove Pieri's formula for the Grassmannian of n-dimensional isotropic subspaces of (2n+1)-dimensional vector space endowed with an orthogonal nondegenerate form (for the precise Pieri-type formula in this case - see [H-B] ; and a Giambelli-type formula - see $[P, Sect.6]_4$. For analogous results in the case of Grassmannian of n-dimensional isotropic subspaces in an 2n-dimensional vector space endowed with an orthogonal nondegenerate form - see $[P, Sect.6]_4$. Finally, note that a "triple Pieri intersection theorem" for Grassmannians of (not necessary top-dimensional) isotropic spaces, in the orthogonal case, has been obtained recently in [S].

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