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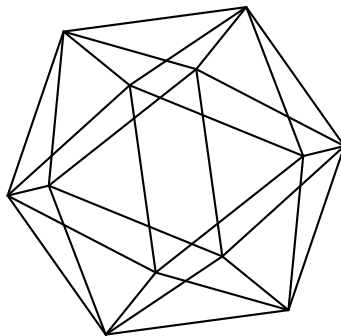
On one stable birational invariant

by

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ABSTRACT. This is an expository article in which we propose that (rational) fibrations on the projective space $\mathbb{P}^n$ by (birationally) Abelian hypersurfaces, for an arbitrary $n \geq 2$, provide an obstruction to stable rationality of algebraic varieties. We discuss the evidence for this proposition and derive some (almost straightforward) corollaries from it.

1. INTRODUCTION

1.1. Let X be an algebraic variety defined over a field $\mathbf{k}$. If not stated otherwise, char $\mathbf{k} = 0$ and X will be smooth, projective and geometrically integral. We will denote by $\mathbf{k}(X)$ the field of rational functions on X. When speaking about *stable birational geometry* of X, one is usually up to some ((bi)rational) interrelation between X and its "stabilization" $X \times \mathbb{P}^k$, with an arbitrary $k \geq 1$. More specifically, one is interested in those properties of the field $\mathbf{k}(X)$ that (dis)appear when passing to the field $\mathbf{k}(X)(t_1, \ldots, t_k)$, t_i being $\mathbf{k}(X)$-transcendental variables. For instance, one may study such classical question as (stable) rationality of linear quotients $X = V/G$ (for $V := \mathbf{k}^{\dim X}$, $G \subseteq GL(V)$ a reductive group and the G-action being free at the generic point on V), and we refer to [24] (see also [61], [78]) and references therein for an extensive overview of the state of art. In its turn, a particular (and in fact the only one) really unavoidable matter in this discussion is the need of *stable birational invariants* of X, i.e. those properties of $\mathbf{k}(X)$ that "do not change" after passage to $\mathbf{k}(X)(t_1, \ldots, t_k)$ and/or vice versa.

Our aim in this introductory note is like this. First we will give an account (mostly in this section) of some of classical stable birational invariants of X (see **1.2**, **1.4** and **1.7** below). Next we will formulate a problem which seems to be not accessible by the classical tools (see **1.6**, Question C). Then we introduce a (seemingly new) stable birational invariant of X (see **1.7**, **1.9**), conjecture one of its crucial properties (Conjecture 1.11), and after that we deduce Theorem 1.15. The latter is proved completely in Section 2. We remark that Theorem 1.15 (as well as Conjecture 1.11) has many quite strong implications, among which we mention only the solution to our initial problem (= Question C) and another interesting corollary (= Corollary 1.17), all treated in **1.14**. The rest of the paper (Sections 3 and 4) is devoted to the verification (or, if one likes, to the sketch of a proof) of Conjecture 1.11.

As a result of the overview style we have adopted, the text contains an abundance of Remarks and *Intermedias*, aiming to guide the reader through the line of arguments we have employed to attack Conjecture 1.11, as well as to draw some parallels with relevant theories (the latter are heuristics really and we apologize in advance for a loose exposition). To sum up, our arguments here mimic in a sense

(and were inspired by) those in [66], used to construct an algebraic surface (over $\mathbb{Q}(\sqrt[7]{1})$), different from $\mathbb{P}^2$ and having ample K, $K^2 = 9$, $q = p_g = 0$. In fact, this surface is "homeomorphic" to a degree 2 del Pezzo surface, when both brought to $\mathbb{Q}_2$, – the fact established in [66] by means of p-adic uniformization (cf. Intermedia in **3.23** below).

1.2. To start with, we indicate that the above mentioned classical invariants are essentially "group-theoretic". For instance, let G be a profinite group and M be a free $\mathbb{Z}$-module of finite rank, with continuous G-action (M is called G-*module* for short). Then M admits what is called *flascque resolvent* (see [23]), i.e. there is an exact sequence

$$0 \to M \to D \to F \to 0$$

of G-modules, where D carries a $\mathbb{Z}$-basis in which G acts by permutations (D is called a *permutation module*), and $\mathrm{Hom}(H^1(G', F), \mathbb{Q}/\mathbb{Z}) = 0$ for any open subgroup $G' \subseteq G$ (F is called *flascque module*).

Example 1.3. Let T be a **k**-torus and $X \supset T$ a smooth projective completion of T. Put $\bar{X} := X \times_{\mathbf{k}} \bar{\mathbf{k}}$, $\bar{T} := T \times_{\mathbf{k}} \bar{\mathbf{k}}$ and consider the group D of divisors supported on $\bar{X} \setminus \bar{T}$. Let also $\hat{T} := \mathrm{Hom}(T, \bar{\mathbf{k}}^*)$ be the group of characters of T. Then $\mathrm{Pic}(\bar{X})$ is flascque and one gets a flascque resolvent $0 \to \hat{T} \to D \to \mathrm{Pic}(\bar{X}) \to 0$ of $\hat{T}$.

Further, considering various M up to the direct sums with permutation G-modules, one gets a homomorphism $\rho : S_G \to F_G$ given by $M \mapsto \rho(M) := F$, where S_G is the semigroup (w.r.t. $\oplus$) of classes $[M \oplus P \mid P$ a permutation G-module] and F_G is the semigroup (of classes of) flascque G-modules. Now, if say $L \supseteq \mathbf{k}$ is a Galois extension with $\mathrm{Gal}(L/\mathbf{k}) = G$, then $\rho(X) := \rho(H^0(U_L, \mathcal{O}^*_{U_L})/\mathbf{k}^*) \in F_G$ is a stable (**k**-)birational invariant of X. Here $U \subset X$ is an open subset such that $\mathrm{Pic}(U_L) = 0$ for $U_L := U \times_{\mathbf{k}} L$ (see [23]). Furthermore, one gets $\rho(X) = 0$ when X is **k**-rational, thus an obstruction to stable **k**-rationality.

1.4. Let us now consider the *Brauer group* $\mathrm{Br}(\mathbf{k}(X))$ of the field $\mathbf{k}(X)$. One distinguishes a subgroup $\mathrm{Br}_v(\mathbf{k}(X)) \subset \mathrm{Br}(\mathbf{k}(X))$ (or simply Br_v if $\mathbf{k}(X)$ is clear from the context) – the *unramified Brauer group* of $\mathbf{k}(X)$ – by the property that for every element $\gamma \in \mathrm{Br}_v$ and any valuation v of $\mathbf{k}(X)$, $\gamma \in \mathrm{Br}(A_v)$ for the valuation ring A_v of v. Now, to be able to work with (actually calculate) Br_v, one interprets $\mathbf{k}(X)$ as the field of invariants in the algebraic closure $\overline{\mathbf{k}(X)}$ of the group $G := \mathrm{Gal}(\overline{\mathbf{k}(X)}/\mathbf{k}(X))$. Thus X, in a sense, is a "geometric quotient U/G" (see [7], [12], [13] for the description of the (universal) space U (be advised though that $\mathbf{k} = \overline{\mathbb{F}}_p$ in [12], [13]); we remark that the current discussion is merely a heuristics aimed to develop some intuition for Br_v rather than rigorous statements). Then, for G being a profinite group, $\mathrm{Br}(\mathbf{k}(X))$ can be interpreted as $\varinjlim H^2(G_i, \mathbb{Q}/\mathbb{Z})$ for some *finite* groups G_i (quotients of G). In fact, for each $\gamma \in H^2(G_i, \mathbb{Q}/\mathbb{Z}) = H^2(G_i, \overline{\mathbf{k}(X)}^*)$ one constructs a central extension $G_{i,\gamma}$ of G_i and then defines a $\mathbb{P}^m$-bundle (for some $m = m(\gamma) \in \mathbb{N}$) on (a smooth birational model of) X as the quotient $(\mathbb{P}^m \times X_i)/G_{i,\gamma}$, where $X_i/G_{i,\gamma} \approx X^{1)}$ and $X_i := U/\mathrm{Ker}[G \to G_i]$ for the previously mentioned U. We will write, informally, $\mathrm{Br}(\mathbf{k}(X)) = H^2(G, \overline{\mathbf{k}(X)}^*)$. One may now apply the computations from [5] and [6] to identify Br_v with the subgroup of those $\gamma \in \mathrm{Br}(\mathbf{k}(X))$ that restrict trivially on all the rank 2 finite abelian subgroups in G with

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[1] $\approx$ stands for the birational equivalence.

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cyclic image in the decomposition group of v (for all v). It also follows from the $\mathbb{P}^m$-bundle description of $\mathrm{Br}(\mathbf{k}(X))$ that this group (together with Br_v) is a stable birational invariant of X (see e. g. [85]).

Remark 1.5. Of course, there are many more (stable) birational invariants (all for $\bar{\mathbf{k}}$-rational X), such as the sets $X(\mathbf{k})/$"R-equivalence", $X(\mathbf{k})/$"Brauer equivalence" and the groups $\mathrm{CH}_0(X)$, $\mathrm{Ker}\big(\deg : \mathrm{CH}_0(X) \to \mathbb{Z}\big)$ of 0-cycles on X, as well as numerous problems on interactions between these (in the context of the Hasse principle for example). Let us stop, however, our listing of invariants at this point and proceed to the main subject of this paper.

1.6. Recall that variety X is called *stably b-infinitely transitive* if $X \times \mathbb{P}^k$ admits an infinitely transitive model (w. r. t. the group SAut) for some k (see [9] for the basic definitions). It is conjectured (= [9, Conjecture 1.4]) this property is equivalent to X being unirational (see [2] for an implication in one direction). In order to develop an approach to the conjecture one may consider the next

Question C (= [9, Remark 3.3]). Suppose X is unirational. Does then $X \times \mathbb{P}^k$, some $k \geq 0$, admit a model which is infinitely transitive and carries an algebraic group $\mathfrak{G}$ action with a Zariski dense orbit ($\dim \mathfrak{G} < \infty$)?

It is not difficult to work out positive answer to Question C in case X is a linear quotient V/G, as in **1.1**, with finite G at least (cf. [9, 3.1]). Thus, in view of the discussion started in **1.2**, it is tempting to test Question C in a more geometric set-up, namely that when X is a hypersurface (see **1.14**). But then we have to look for such a (stable) birational property of X that would "feel" unirationality (unlike those properties mentioned above).

1.7. Let us start with some motivation for the forthcoming constructions. Consider an elliptic fibration $f : X \longrightarrow B$ over a curve B and suppose that f admits a section. Form a group $\mathfrak{I}$ of all elliptic fibrations (over B) having $f : X \longrightarrow B$ their Jacobi fibration (see e. g. [84]). Set $E_b := f^{-1}(b)$ for $b \in B$. Then there is a homomorphism
$$h : \mathfrak{I} \to \oplus_{b \in B} \mathrm{Tors}(E_b)$$
obtained from the fact that every multiple fiber of any elliptic fibration from $\mathfrak{I}$ is isogeneous to some E_b. Put $\mathfrak{I}_0 := \mathrm{Ker}(h)$. This $\mathfrak{I}_0$ is isomorphic to the *Brauer group* $\mathrm{Br}(X) := H^2_{\acute{e}t}(X, \mathcal{O}_X^*)$ *of the surface* X. This is very much similar to $\mathrm{Br}(\mathbf{k}(X))$ (or, as probably more to the point, to Br_v) in **1.4** and suggests that the group $\mathrm{Br}(\mathbf{k}(X))$ for any (or at least sufficiently general in Hilb_X) variety X should admit an "intrinsic" description in terms of the (birational) geometry of X. Without going further into discussing the philosophy, let us just say that we are up to the existence of rational fibrations $f : X \dashrightarrow B$, where B is a curve and generic fiber of f is birational to an Abelian variety. Call such X *b-Hamiltonian* (or *b.-H.* for short). Then we claim that being b.-H. is a stable birational invariant of X, i. e. X is b.-H. if (iff?) $X \times \mathbb{P}^k$ is so for some (all?) $k \geq 1$. This setting, however, is too general to be true, and we now pass to imposing some constraints on X and giving the precise statements.

Remark 1.8. One may draw a parallel between the above description of the "Brauer-like" objects, namely $\mathrm{Br}(\mathbf{k}(X))$ vs. $\mathrm{Br}(X)$, and the "algebraic" approach of [1] vs. "transcendental" methods of [21], with considerations of both [1] and [21] boiling

down to that in $H^3(X)$ (see also [31], [72], [22]). Indeed, given a conic bundle (or the *Severi–Brauer*) structure $X \longrightarrow S$ (with certain restrictions on the discriminant), one naturally assigns with X a quaternion algebra A (the Azumaya algebra of rank 4) over $\mathbf{k}(S)$. This A is a 2-torsion element in $\mathrm{Br}(\mathbf{k}(S))$, which yields $H^4(X, \mu_2)$ and hence $H^3_{tors}(X, \mathbb{Z}) \neq 0$, with the latter being an obstruction to (stable) rationality of X. From the transcendental side in turn, one considers $J(X) := H^3(X, \mathbb{C})/(H^3(X, \mathbb{Z}) + H^{1,2})$ ($\dim X = 3$), the *intermediate Jacobian* of X, which is an Abelian variety with a principal polarization Θ. The fundamental property of $J(X)$ is that when X is rational, then $(J(X), \Theta) \simeq \prod_i (J(C_i), \Theta_i)$ as principally

polarized Abelian varieties, where C_i are some smooth curves with Jacobians $J(C_i)$ and theta-polarizations Θ_i. Note however that the "stable" version of $J(X)$ does not make sense in the current setting (but compare with Theorem 1.15 below).

1.9. Throughout the paper A^n denotes an Abelian variety of dimension n.

Definition 1.10. We call A^n *rully* if there exists a sequence of Abelian subvarieties $0 \subsetneq A^1 \subsetneq \ldots \subsetneq A^{n-1} \subsetneq A^n$.

The subject of primary interest for us in the present narrative will be the following:

Conjecture 1.11. *Projective space $\mathbb{P}^n$ is b.-H. for any $n \geq 2$. More precisely, there exists a rational fibration $\mathbb{P}^n \dashrightarrow \mathbb{P}^1$ with generic fiber $\approx A^{n-1}$ for some rully Abelian variety A^{n-1}.*

Let μ_n be the cyclic group of order n. An elementary manifestation of Conjecture 1.11 is the rational surface $(E \times \mathbb{P}^1)/\mu_2$ fibred by elliptic curves $\simeq E$ (cf. Example 1.12 below). At the same time, one can not extend the same construction verbatim to the case of the quotient $(E^n \times \mathbb{P}^1)/\mu_n^{\oplus n}$, $n \geq 2$, with $\mu_n^{\oplus n}$ acting diagonally (and componentwise on E^n) via $\pm$. However, we will reincarnate similar, "geometric", approach to Conjecture 1.11 later in Section 4.

Let us now collect some examples and heuristics in support of Conjecture 1.11 (the reader will find more discussion in Section 3).

Example 1.12. Given an elliptic curve E and nine points $P_1, \ldots, P_9 \in E$, the condition $m \sum_{i=1}^9 P_i = 0$ in the group $E(\mathbb{C})$, for some $m \geq 1$, is equivalent to the existence of a curve $Z \subset \mathbb{P}^2$ having $\deg Z = 3m$ and $\mathrm{mult}_{P_i}(Z) = m$ for all i. If $f = 0, g = 0$ are equations of E, Z respectively, then generic curve on $\mathbb{P}^2$ given by $\lambda f^m + \mu g = 0$, $\lambda, \mu \in \mathbb{C}$, is birational to an elliptic curve. This is the example of a *Halphen pencil* on $\mathbb{P}^2$ (thus Conjecture 1.11 is trivial when $n = 2$) and in fact any Halphen pencil (= 1-dimensional linear system with generic element birational to an elliptic curve) on $\mathbb{P}^2$ is reduced to this one (for an appropriate m) via Cremona transformations. This is a classical result (*Dolgachev–Bertini theorem*), proved in [30], which provides one with a huge source of problems of similar type for other rationally connected varieties, such as existence and (explicit) description of Halphen pencils (i. e. generic hypersurface in the pencil is required to have Kodaira dimension 0) on these varieties (compare with Conjecture 1.11 or Theorem 1.16 below for instance). Unfortunately, the case of $\mathbb{P}^2$ seems to be rather an exception (a gem if one likes), since in general one should not expect similar neat (or "contact") description of special fibrations on $\mathbb{P}^n$ say, $n \geq 3$. In the latter case for example, if A^{n-1} is as in Conjecture 1.11, then its image in $\mathbb{P}^n$ is necessarily a non-normal (!)

hypersurface, as one easily elaborates via the Lefschetz-type theorem(s) (see e.g. [**?**, 3.1]), together with [52, Theorem 11] and [41, Corollaries 1.3, 1.5].[2)]

To overcome the difficulty pointed out in Example 1.12, we develop the above observation with $(E \times \mathbb{P}^1)/\mu_2$ further. The idea is to use (annoying) similarity between two "torus-like" objects: $A^n(\mathbb{C}) = \mathbb{C}^n/\mathbb{Z}^n$ and $(\mathbb{C}^*)^n$. We would like to stress that this similarity between two different types of tori becomes even stronger when the ground field $\mathbf{k} = \mathbb{C}$ is replaced by a field having char $= p > 0$. Namely, if $\mathbf{k}$ is the global field $\mathbb{F}_p(t)$ say, and A^n is *ordinary* (see **3.10** below), we indicate in Section 3 that (after completing and closing $\mathbf{k}$ further) the underlying topological spaces $A^n(\mathbf{k})$ and $\mathbb{P}^n(\mathbf{k})$ are *homeomorphic*. This is done in a framework very much typical to the one developed along the uniformization theories for modular varieties (see Intermedia in **3.23** for discussion and references). The only difference is that instead of considering moduli (cf. **1.7**) we stick to the patching data inscribed into the one solid A^n. This is a familiar type of duality one meets in Kähler geometry for instance, where variation of Kähler structure $(V, \omega), \omega \in H^2(V, \mathbb{Z})$, on a compact complex manifold V may equally be seen in terms of the action of the symplectomorphism group $\mathrm{SDiff}(V, \omega)$ on V (see e.g. [32]). Again we postpone the discussion of further analogies until Section 3 and go on with applications of Conjecture 1.11.

Remark 1.13. Take the 3-dimensional hyperbolic space $\mathbb{H} := \mathbb{C} \times \mathbb{R}_{\geq 0}$ acted (isometrically) by a free subgroup $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ in $g \geq 2$ generators. The action of Γ extends to the one on the ideal boundary of $\mathbb{H}$ identified with $\mathbb{P}^1(\mathbb{C})$. Let $\Lambda_\Gamma \subset \mathbb{P}^1$ be the closure of the attractive and repulsive fixed points for all the elements $\gamma \in \Gamma$. Then for the complement $\Omega_\Gamma := \mathbb{P}^1 \setminus \Lambda_\Gamma$, the quotient $X := \Omega_\Gamma/\Gamma$ is a Riemann surface of genus g, and the covering $\Omega_\Gamma \longrightarrow X$ is called *Schottky uniformization* of X (in fact every (oriented) Riemann surface admits a Schottky uniformization). Again, in line with the previous discussion let us note that $X \subset \mathbb{H}//\Gamma$, a 3-dim handlebody of genus g. We refer to [60] (especially, to a beautiful parallel with the p-adic case, treated in [67], [68]) for further illustrations and applications of this "introducing-extra-dimension" principle. This is another supplement for treating geometric objects over a global field of char > 0 (cf. **3.23** below).

1.14. The main result actually proved in this paper is the following:

Theorem 1.15. *Assuming Conjecture 1.11, suppose X is* stably rational, *i.e.* $X \times \mathbb{P}^k \approx \mathbb{P}^{n+1}$ *for some k and $n = \dim X + k - 1$. Then one (or both) of the following holds:*

- *X is rationally fibred by hypersurfaces of negative Kodaira dimension;*
- *X is b.- H.*

Given Theorem 1.15, it is tempting to produce examples of non-stably rational rationally connected varieties, in addition to linear quotients and conic bundles discussed above.[3)] Let us first mention the following result (see also [17], [19]):

[2)]We should mention here a construction, which is due to J. Kollár, justifying Conjecture 1.11 in the case $n = 3$. Namely, let E be an elliptic curve with the μ_3-complex multiplication. Then the quotient $X := (E \times E \times \mathbb{P}^1)/\mu_3$ by (non-trivial) diagonal μ_3-action is a rational 3-fold with a map $X \dashrightarrow \mathbb{P}^1$ whose general fiber $= E \times E$.

[3)]Note that any cubic 3-fold $X_3 \subset \mathbb{P}^4$ is b.-H., since $X \approx \mathbb{P}^3/\tau$ for a birational involution $\tau \in \mathrm{Bir}(\mathbb{P}^3)$, and so the arguments in the proof of Corollary 1.19 apply.

Theorem 1.16 (see [45], [25], [18])**.** *Any smooth quartic 3-fold $X_4 \subset \mathbb{P}^4$ is not b.-H. and can not be rationally fibred by hypersurfaces of negative Kodaira dimension.*

Theorems 1.15, 1.16 and Conjecture 1.11 yield

Corollary 1.17. *Any smooth quartic 3-fold $X_4 \subset \mathbb{P}^4$ is not stably rational.*

Theorem 1.15 is proved in Section 2 (see Remark 2.18 for an outline). Let us indicate that Theorem 1.15 relies mostly on Theorem 11 in [52] (and is similar, in a way, to Theorem 14 in *loc. cit.*) and on [51, Theorem 11.1], and these are the only (essentially) non-trivial results used in the proof. We will return to the exposition of [52] later in Section 4.

Remark 1.18. In the setting of Conjecture 1.11, it is easy to modify the arguments of Section 2 in such a way that the assertion of Theorem 1.15 still holds, with $\mathbb{P}^n$ replaced by some *unirational* b.-H. variety $\mathbb{P}$ with a transitive regular action of the group $\mathrm{Aut}(U)$ on a Zariski open subset $U \subseteq \mathbb{P}$ (as one just needs irreducibility of intersections of $x \times \mathbb{P}^k$, for various (generic) $x \in X$, with (birationally) rully Abelian hypersurfaces in $X \times \mathbb{P}^k$).

Corollary 1.19. *Let X be as in Question C. Then the same options as in Theorem 1.15 hold in this case.*

Proof. Indeed, since X is considered up to stable birational equivalence, we may assume that $X \times \mathbb{P}^k = \mathfrak{G}/H$ for some k, where $\mathfrak{G}$ is a connected algebraic group and $H \subset \mathfrak{G}$ is a finite subgroup (see [9], [8]). Then notice that $\mathfrak{G} \approx \mathbb{P}^n$ (see [20]), with $n \gg 1$, so that one finds a 1-parameter family $\{A_t\}_{t \in \mathbb{C}}$ of (birationally) Abelian hypersurfaces in $\mathfrak{G}$ (with $A_t \sim A_t'$ (linearly equivalent) for all t, t', as Conjecture 1.11 predicts). Pick a generic point $P \in \mathfrak{G}$ and some $A_t \ni P$. It suffices to establish that $(H \cdot P) \cap A_t = \{P\}$ (scheme-theoretically) for the H-orbit of P. Indeed, if this is the case, then for generic t' the hypersurface $A_{t'}$ maps isomorphically into $\mathbb{P} := \mathfrak{G}/H$ under the quotient morphism $q : \mathfrak{G} \longrightarrow \mathbb{P}$, and the cycles $q_*(A_t)$, $q_*(A_{t'})$ are rationally equivalent. But $\mathbb{P}$ is rationally connected, which implies that in fact $q_*(A_t) \sim q_*(A_{t'})$, and we are done by Remark 1.18.

Now, in order to achieve the identity $(H \cdot P) \cap A_t = \{P\}$, we take an appropriate $\sigma \in \mathrm{Bir}(\mathfrak{G})$ such that $\sigma(P) = P$ and $\sigma(H \cdot P \setminus \{P\}) \cap A_t = \emptyset$ (cf. **1.6**). It remains to replace A_t by $\sigma^{-1}(A_t)$. $\qquad\square$

The next example provides negative answer to Question C:

Example 1.20. The quartic $X_4 \subset \mathbb{P}^4$ with equation $x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_0 x_4^3 + x_1^3 x_4 - 6 x_2^2 x_3^2 = 0$ is smooth and unirational (see e. g. [45]). One concludes via Corollary 1.19 and Theorem 1.16.

2. Proof of Theorem 1.15

2.1. We use notations from **1.9**. Throughout this section A^n is supposed to be rully. Recall that there is a generically 1-to-1 (onto its image) map from A^n to $\mathbb{P}^{n+1}$ (since Conjecture 1.11 holds by our assumption). Then, composing the natural projection $X \times \mathbb{P}^k \longrightarrow X$ with a birational isomorphism $X \times \mathbb{P}^k \approx \mathbb{P}^{n+1}$, we obtain a rational dominant map $\phi : A^n \dashrightarrow X$.

Lemma 2.2. *The general fiber of ϕ is irreducible ($k \geq 2$).*

Proof. Identify A^n with a hypersurface in $X \times \mathbb{P}^k$. Then ϕ coincides with the restriction to A^n of the projection $X \times \mathbb{P}^k \longrightarrow X$. The fibers of the induced morphism $A^n \longrightarrow X$ are all of the form $(\mathrm{pt} \times \mathbb{P}^k) \cap A^n$. Thus it suffices to show that $(\mathrm{pt} \times \mathbb{P}^k) \cap A^n$ is irreducible for generic $\mathrm{pt} \in X$.

Recall that A^n varies in a pencil $\mathcal{P}$ on $X \times \mathbb{P}^k$. There is also a Zariski open subset $U \subseteq X \times \mathbb{P}^k$ with a transitive regular $\mathrm{Aut}(U)$-action. Then we have $A^n \sim g_* A^n$ for all $g \in G$ and an open subgroup $G \subseteq \mathrm{Aut}(U)$. In particular, since $\dim G \gg 1$, there is a surface $\Pi \subseteq |A^d|$ such that the line $\mathcal{P}$ varies in a 2-dimensional family on Π. Hence $\Pi = \mathbb{P}^2$ and we may replace $\mathcal{P} \ni A^n$ by a net. Then (generic) A^n is smooth in codimension 1, hence normal, and so ϕ has irreducible general fiber. $\square$

Let us resolve the indeterminacies of ϕ:

$$
\begin{array}{ccc}
 & Y & \\
\sigma \swarrow & & \searrow \sigma' \\
A^n \dashrightarrow_{\phi} & & X.
\end{array}
$$

Here we take σ to be composed of the blow-ups at smooth centers. In particular, we have $K_Y = \sum a_i E_i$, where E_i are σ-exceptional divisors and $a_i > 0$ for all i.[4] Furthermore, taking generic $A^{d+1} \subset A^n$, $d := \dim X$, we may stick to the case $n = d + 1$ and so the general fiber F of ϕ is an irreducible curve (see Lemma 2.2).

Lemma 2.3. *There is a proper subscheme $\widetilde{Y} \subset \mathbb{P}(\sigma'_* \omega_{Y/X})$ such that $\widetilde{Y} \approx Y$ and $\widetilde{Y} \simeq Y$ near F.*

Proof. Indeed, the curve $\sigma_*^{-1} F$ is smooth, l. c. i. and canonically polarized by $K_Y = \sum a_i E_i$, which gives a subscheme in $\mathbb{P}(\sigma'_* \omega_{Y/X})$ (restriction of the Hilbert family) birational to Y. $\square$

Let $\widetilde{Y}$ be as in Lemma 2.3 and $\tilde{\sigma} : \widetilde{Y} \longrightarrow \widetilde{X}$ be the induced (flat) morphism ($\widetilde{X} \approx X$). We may replace $\widetilde{X}$ by its resolution so that $\tilde{\sigma}$ remains flat. We will also suppose $\widetilde{Y}$ to be normal and CM (the general case can be reduced to this setting).

Lemma 2.4. *The induced birational map $\widetilde{Y} \dashrightarrow Y \xrightarrow{\sigma} A^n$ is contracting.*[5]

Proof. We may assume w. l. o. g. there is a birational morphism $f : Y \longrightarrow \widetilde{Y}$. Note that

$$
K_{\widetilde{Y}} \equiv \sum a_i E_i' + \sum b_j F_j
$$

(numerically) on $\widetilde{Y}$ for some $b_j \in \mathbb{Q}$ with $F_j \cdot (f \circ \sigma)_*^{-1} F = 0$ for all j. Here E_i' are the proper transforms of (some of the) E_i from the above formula for K_Y (a_i are the same). Further, since $\sigma_* K_Y = 0$ and $K_Y - f^*(K_{\widetilde{Y}}) \subseteq \mathrm{Exc}(f)$, we obtain that $f_*^{-1} F_j = E_i'$ for some $i = i(j)$ and every j whenever $b_j \neq 0$. Hence we may assume that $b_j = 0$ for all j. This implies that $f_* K_Y = K_{\widetilde{Y}}$ and so $\widetilde{Y}$ has rational singularities. The result now follows from the weak factorization theorem for birational morphisms (as A^n does not contain rational curves). $\square$

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[4] To be more precise, one has to include the case $a_j = 0$ for some j as well. But then $Y = A^n$, $\sigma = \mathrm{id}$, ϕ is regular and it is easy to see that $\kappa(X) = 0$, a contradiction.

[5] We refer to [53] for standard notions and facts from the minimal model theory.

2.5. Consider the morphism $\pi : Y \xrightarrow{\sigma} A^n \longrightarrow A^n/A^{d-1}$. Put $S := \sigma'^{-1}\sigma'(\Omega) \subset Y$ for $\Omega := \sigma_*^{-1}(A^{d-1})$. Note that Ω is a fiber of π. It follows from Lemma 2.4 that the induced map $\pi \circ f^{-1} : \widetilde{Y} \dashrightarrow A^n/A^{d-1}$ is defined near $f_*\Omega$ for generic (varying) A^{d-1}. This allows us restrict to the case $\widetilde{Y} = Y$ (the arguments below work literarilly in the general setting).

Lemma 2.6. *The linear system $|S|$ is basepoint-free on Y.*

Proof. By construction, the cycles Ω (with varying A^{d-1}) are all algebraically equivalent on Y, since Ω is a fiber of π. Then the cycles $\sigma'_*(\Omega)$ on X are also algebraically equivalent. Moreover, since X is smooth and rationally connected, thus having $\mathrm{Alb}(X) = 0$, all $\sigma'_*(\Omega)$ are in fact linearly equivalent divisors. In particular, since σ' is flat, the divisors $\sigma'^{-1}\sigma'(\Omega) \subset Y$ are linearly equivalent as well. Finally, since all $\sigma'^{-1}\sigma'(\Omega)$ belong to $|S|$ and A^{d-1} was chosen arbitrarily, the assertion follows. $\square$

Choose $\Sigma \in |S|$ generic. Note that Σ is smooth. Also, blowing up X and making a base change for σ', we may assume that $\dim |S| = 1$

Proposition 2.7. *The hypersurface Σ contains a fiber of the morphism $\pi\big|_\Sigma : \Sigma \longrightarrow A^n/A^{d-1} =: A^2$ birational to Ω.*

Proof. Suppose not. Then all fibers of $\pi\big|_{\sigma(\Sigma)}$ have dimension $\leq d-2$. Cutting with $A^3 \subset A^n$ we reduce to the case Σ is a smooth surface such that $\pi\big|_{\sigma(\Sigma)} : \sigma(\Sigma) \longrightarrow A^2$ is finite.

Lemma 2.8. *Let K_Σ be not σ-nef. Then X is (birationally) fibred by hypersurfaces of negative Kodaira dimension.*

Proof. Notice that $\sigma\big|_\Sigma$ is a resolution of indeterminacies of the map $\phi\big|_{\sigma(\Sigma)}$. Also, since $\Sigma \in |S|$ is generic, the intersections $\Sigma \cap E_i$ are irreducible for all i. Let $C \subset \Sigma$ be a (-1)-curve contracted by σ. Then we have $C \subseteq E_j$ for some j and $-1 = (C^2) = mE_j \cdot C$ for the (infinitely close) multiplicity $m = 1$ of $\sigma(\Sigma)$ (w.r.t. E_j). In particular, for $\sigma(C)$ is a base point of $\phi\big|_{\sigma(\Sigma)}$, the curve C is a multisection of the fibration $\sigma'\big|_\Sigma$ and so $\phi(C)$ is a rational curve on X.

Thus, if $d = 2$, then there is a rational fibration on X by rational curves $\phi(\Sigma)$. (Note that at this point we do not need any assumption on the map $\pi\big|_{\sigma(\Sigma)}$.) Further, varying $A^3 \subset A^n$ in our initial setting (with $\dim \Sigma = d$) and assuming $K_{\Sigma \cap A^3}$ to be not σ-nef for (most of) these A^3, we find a bunch of rational curves $\phi(\Sigma \cap A^3)$ which cover the hypersurface $\phi(\Sigma)$. This yields a fibration on X as claimed. $\square$

Let us now consider the case of K_Σ being σ-nef. Here we use the assumption on $\pi\big|_{\sigma(\Sigma)}$ from the beginning. We get

$$K_\Sigma = \sigma^* K_{\sigma(\Sigma)} - \sum b_j E_j \big|_\Sigma.^{6)}$$

Here $E_j\big|_\Sigma$ are $\sigma\big|_\Sigma$-exceptional so that $b_j \geq 0$ for all j. On the other hand, we have $K_\Sigma = \sum a_i E_i\big|_\Sigma = K_Y\big|_\Sigma$, which gives

$$(2.9) \qquad\qquad \sum (a_i + b_i) E_i\big|_\Sigma = \sigma^* K_{\sigma(\Sigma)}$$

---

[6] We assume for simplicity that $\sigma(\Sigma)$ is normal and $\mathbb{Q}$-Gorenstein.

(with $b_i = 0$ for non-$\sigma\big|_\Sigma$-exceptional $E_i\big|_\Sigma$). Furthermore, the Hurwitz formula gives

$$K_{\sigma(\Sigma)} = \pi^*(D)$$

for some effective cycle $D \subset A^2$, the ramification locus of $\pi\big|_\Sigma$.

Lemma 2.10. *If the cycle $E_j\big|_\Sigma$ is not $\sigma\big|_\Sigma$-exceptional, then $E_j\big|_\Sigma = \pi^*[0\text{-cycle on } A^2]$, all j.*

Proof. Notice that $\sigma(S)$ is smooth near $\sigma(\Omega)$. This gives $(\sigma(\Sigma)\big|_{\sigma(S)}) \cdot \sigma(\Omega) = 0$ and $\sigma(\Omega)^2 = 0$. Indeed, we have $K_{\sigma(\Sigma)} = \pi^*(D)$, $\sigma(\Sigma)\big|_{\sigma(S)} = K_{\sigma(S)}$ near $\sigma(\Omega)$, and so $K_{\sigma(S)} \equiv S \cdot \pi^*[\text{cycle on } A^2]$ (as $S \sim \Sigma$). In particular, $K_{\sigma(S)} \cdot \sigma(\Omega) = 0$ (for Ω is a fiber of π), thus $(\sigma(\Sigma)\big|_{\sigma(S)}) \cdot \sigma(\Omega) = 0$ and $\sigma(\Omega)^2 = 0$.

Now $\sigma(E_j)$ equals (set-theoretically) $\pi^{-1}[0\text{-cycle on } A^2]$ and the result follows. $\square$

It follows from Lemma 2.10 that the left hand side of (2.9) is negative on all the $E_j\big|_\Sigma$ that are $\sigma\big|_\Sigma$-exceptional. This shows that $b_j = 0$ for all j.

Lemma 2.11. *The rational map $\phi\big|_{\sigma(\Sigma)}$ is everywhere defined.*

Proof. Indeed, given $b_j = 0$ for all j there are either no $\sigma\big|_\Sigma$-exceptional curves $E_j\big|_\Sigma$, or all of them are (-2)-curves on Σ. We are done in the former case. In the latter case, arguing as in the proof of Lemma 2.8 we find that $E_j\big|_\Sigma$ are multisections of $\sigma'\big|_\Sigma$, thus X is (birationally) fibred by hypersurfaces of negative Kodaira dimension. $\square$

According to Lemma 2.11 we may identify $\sigma(\Sigma)$ with its normalization Σ (and $\phi\big|_{\sigma(\Sigma)}$ with $\sigma'\big|_\Sigma$). Suppose that the ramification cycle D is big (i.e. $D^2 > 0$).

Lemma 2.12. $\phi(\Sigma) \simeq A^1$.

Proof. Recall that $\phi\big|_\Sigma : \Sigma \longrightarrow X$ is a flat morphism with connected 1-dimensional fibers. Let $|F|$ be the corresponding linear system on Σ. Note that $F = \pi^*(E)$ for an elliptic curve $E \subset A^2$ (as $(F^2) = 0$ and $D \cap E \neq \emptyset$). This gives a morphism $\Sigma \longrightarrow A^2 \longrightarrow A^2/E =: A^1$ with connected fibers $\sim F$. But this means that $\phi(\Sigma) \simeq A^1$. $\square$

Further, cutting with (varying) $n-3$ hypersurfaces from $\phi^*\mathcal{O}_X(1)$ in our initial setting (with $\dim \Sigma = d$) and applying the arguments in the proof of Lemma 2.12 to the morphism $\Sigma \longrightarrow A^d \longrightarrow A^d/E =: A^{d-1}$ (for again $K_\Sigma = \pi^*(D)$, D is big, etc.), we obtain that $\phi(\Sigma) \approx A^{d-1}$. Thus X is b.- H. in this case.

Now let $D^2 = 0$. Then $(K_\Sigma^2) = 0$ and the linear system $|K_\Sigma| = |\pi^*(D)|$ is basepoint-free.

Lemma 2.13. *The cycle $S \cdot \Sigma \subset A^n$ is supported on the fibers of $\phi\big|_\Sigma$.*

Proof. We have $S \cdot \Sigma \in |K_\Sigma|$ and also $S \cdot \Sigma \supseteq F$ (see the proof of Lemma 2.12). But $S \cdot \Sigma$ is a disjoint union of elliptic curves C_i by the assumption on K_Σ. The assertion now follows because $(F^2) = C_i \cdot F = 0$ for all i. $\square$

Again, cutting with (varying) $n-3$ hypersurfaces from $\phi^*\mathcal{O}_X(1)$ in our initial setting (with $\dim \Sigma = d$) and applying Lemma 2.13, we get $\sigma(\Sigma)^n = 0$. The latter

means that $\sigma'\big|_\Sigma$ factors through the morphism $A^n \longrightarrow A^n/A^1$. This also holds for $\sigma'\big|_S$ (as $S \sim \Sigma$). Furthermore, since $\Omega \subset S$, the morphism $\sigma'\big|_S$ admits a (rational) section. This gives $S \approx \sigma(S) \simeq A^1 \times A^{d-1}$. Then S maps onto a curve under the morphism π. Hence the same holds for $\Sigma \in |S|$.

Proposition 2.7 is proved. $\qquad\square$

We may assume that $\Omega \subset \Sigma$ due to Lemma 2.7. Now, if $\kappa(\sigma'(\Omega)) = -\infty$, then X is (birationally) fibred by hypersurfaces of negative Kodaira dimension.

Proposition 2.14. *Let* $\kappa(\sigma'(\Omega)) \geq 0$. *Then* $\sigma'(\Omega) \approx$ *an Abelian variety.*

Proof. Notice first that $\kappa(\sigma'(\Omega)) = 0$ because $\kappa(\Omega) \geq \kappa(\sigma'(\Omega))$ for generically finite dominant $\sigma'\big|_\Omega : \Omega \longrightarrow \sigma'(\Omega)$. Also, since σ' is flat with 1-dimensional fibers, the morphism $\sigma'\big|_\Omega$ is finite. Then, for $\sigma'(\Omega)$ is smooth (cf. Lemma 2.6) and not uniruled, the result of [52, Theorem 11] implies that $\sigma'\big|_\Omega$ is étale.[7] This yields the estimate

$$d - 1 = q(\Omega) \leq q(\sigma'(\Omega))$$

for irregularities.

On the other hand, the Leray spectral sequence (applied to $\sigma'\big|_\Sigma$) shows that

$$(2.15) \qquad\qquad q(\Sigma) = q(\sigma'(\Omega)) + g,$$

where g is the genus of a fiber of $\sigma'\big|_\Sigma$. Furthermore, restricting the morphism $\pi : Y \xrightarrow{\sigma} A^n \longrightarrow A^n/A^2 =: A^{d-1}$ to Σ, we obtain a morphism $f : \Sigma \longrightarrow A^{d-1}$. Note that the fibers of f are connected because $\Omega \subset \Sigma$ is (generically) its section. In particular, we get

$$(2.16) \qquad\qquad q(\Sigma) = q(\Omega) + g',$$

where g' is the genus of a fiber of f, and so $g \leq g'$.

Lemma 2.17. *The estimate* $g \geq g'$ *holds.*

Proof. We may replace $\sigma'(\Omega)$ by its étale cover Ω, make a base change for $\sigma'\big|_\Sigma$, reduce to the case of $\sigma'\big|_\Sigma$ being a fibration over Ω (with connected fibers and a section), and thus arrive at $q(\Sigma) = q(\Omega) + g$ (in place of (2.15)). Then (2.16) turns into $q(\Sigma) = q(\Omega) + (g' - 1)m + 1$ for some $m \geq 1$. This gives $g \geq g'$. $\qquad\square$

Lemma 2.17 implies that $q(\sigma'(\Omega)) = d - 1$ as well. The assertion now follows from [51, Theorem 11.1]. $\qquad\square$

Proposition 2.14 finishes the proof of Theorem 1.15.

Remark 2.18. It is instructive to illustrate the preceding arguments on a model example, that is of $X := \mathbb{P}^2$, $A^n := A^3$. One proceeds along the following steps:

- start with a rational map $\phi : A^3 \dashrightarrow X$ with irreducible fibers, resolve ϕ via $\sigma : Y \longrightarrow A^3$ as in **2.1**, arrive at a regular fibration $\sigma' : Y \longrightarrow X$ and reduce to the case when σ' is flat (see the beginning of **2.5**);
- each surface $S = \sigma'^{-1}\sigma'(\Omega)$ carries an elliptic curve $\Omega \subset Y$ which is a multisections of the fibration $\sigma'\big|_S$;

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[7] Note that $\sigma'_{\ \Omega}$ is not ramified along the $\sigma_{\ \Omega}$-exceptional locus by generality of $\Sigma \supset \Omega$.

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- as σ' is flat, the linear system $|S|$ is the pullback of a linear system on X, and we may then assume $|S|$ to be a basepoint-free pencil (as before Proposition 2.7);

- for general $\Sigma \in |S|$, we observe that if the divisor K_Σ is not $\sigma\big|_\Sigma$-nef, then $\phi(\Sigma)$ is a rational curve, which gives a pencil of rational curves on X (see Lemma 2.8);

- in turn, if K_Σ *is* $\sigma\big|_\Sigma$-nef and the factorization map $A^3 \longrightarrow A^3/A^1 = A^2$ (where $A^1 = \sigma(\Omega)$) induces a *finite* morphism $\sigma(\Sigma) \longrightarrow A^2$, then the map $\phi\big|_{\sigma(\Sigma)}$ is everywhere defined, and we may assume that $\sigma(\Sigma) = \Sigma$ (see Lemmas 2.10, 2.11);

- furthermore, if the above map $\sigma(\Sigma) \longrightarrow A^2$ has *big* ramification locus, then we obtain (Lemma 2.12) that $\phi(\Sigma)$ is an elliptic curve, and so X is b.-H. in this case;

- in the remaining cases, the surfaces $\Sigma \in |S|$ carry elliptic curves $\equiv \Omega$ on Y, and these give either a rational or an elliptic fibration on $X = \mathbb{P}^2$ (cf. Proposition 2.14).

3. Evidence for Conjecture 1.11: an algebraic approach

3.1. Elliptic curves. The most direct way of relating A^d to $\mathbb{P}^d$ is via the *uniformization* of A^d. Namely, one introduces a number of (transcendental) parameters, having their source in the affine space $\mathbf{k}^d$ and the target in A^d. For instance, classically one has $A^d(\mathbb{C}) = \mathbb{C}^d/\Lambda$, where $\Lambda \subset \mathbb{C}^d$ is a full sublattice. We are up to a similar picture over (almost) arbitrary ground field $\mathbf{k}$. In what follows, we will be mostly concerned with the case $d = 1$, that is of $A^1 =: E$ being an elliptic curve (this should suffice to develop the necessary intuition for any A^d).

Recall that E is given by an equation (in the Weierstrass normal form)

$$y^2 = 4x^3 - g_2 x - g_3$$

on the (x, y)-plane, with some $g_2, g_3 \in \mathbf{k}$, where we suppose $p := \operatorname{char} \mathbf{k} \neq 2, 3$. Adding a new variable z, we identify E with a cubic curve in $\mathbb{P}^2$ (the point $[0 : 1 : 0] \in E$ being 0 w.r.t. the group law on $E(\bar{\mathbf{k}})$). Let $\widehat{E}$ be the completion of E at 0 (cf. Example 3.5 below). This is simply the formal scheme $\operatorname{Spec} \mathbf{k}[[t]]$ for a $\mathbf{k}$-transcendental variable t. Furthermore, the group law $E \times E \longrightarrow E$ leads to a Hopf algebra structure on $\mathbf{k}[[t]]$, thus making $\widehat{E}$ into a *formal* (aka *local, quasi, etc.) group* (see e.g. [57], [59], [82], [73], as well as **3.3** below, for an account of the basic theory).

More specifically, letting

$$\tilde{x} := -\frac{2x}{y}, \ \tilde{y} := -\frac{2}{y}$$

in the above discussion we get E defined by the equation

$$\tilde{y} = \tilde{x}^3 + \frac{g_2}{4} \tilde{x}\tilde{y}^2 + \frac{g_3}{4} \tilde{y}^3$$

on the $(\tilde{x}, \tilde{y})$-plane, and so

$$\tilde{y} = \tilde{x}^3(1 + A_1\tilde{x} + A_2\tilde{x}^2 + \ldots),$$

where A_i are homogeneous polynomials in $\mathbb{Z}[g_2/4, g_3/4]$. On the other hand, for any two points $P_1, P_2 \in E$, $P_i := (x_i, y_i)$, $x_i, y_i \in \mathbf{k}$, one gets the x-coordinate of the point $P_3 := P_1 + P_2$ in the form

$$-x_1 - x_2 + \frac{1}{4}\Big(\frac{y_1 - y_2}{x_1 - x_2}\Big)^2.$$

Now, substituting the preceding (formal) series expression for $\tilde{y}_i$, we find that the formal group law on $\widehat{E} = \mathbf{k}[[t]]$ is given by an element $G(x_1, x_2) \in \mathbf{k}[[x_1, x_2]]$, which carries the following properties:

(G_1) $G(x_1, G(x_2, x_3)) = G(G(x_1, x_2), x_3)$ (associativity),

(G_2) $G(x_1, x_2) = G(x_2, x_1)$ (commutativity),

(G_3) $G(x, i_G(x)) = 0$ for a unique $i_G(x) \in \mathbf{k}[[x]]$ (existence of an inverse),

(G_4) $G(x, 0) = G(0, x) = x$ (i.e. $x = 0$ is the identity w.r.t. G).

The series $G(x_1, x_2)$ is called *formal (abelian, over $\mathbf{k}$) group* (see **3.3** for further recollections).

Remark 3.2. It is $\widehat{E}$ (and $G(*, *)$) that we are going to use as a source of "uniformizers" for E. But before gluing E out of $\widehat{E}$'s (= constructing entire (multi-valued) functions on $E(\bar{\mathbf{k}})$), one has to equip E with extra symmetries, in line with the complex-analytic case of $E(\mathbb{C})$ above. For instance, heuristically at least, one has to have a counterpart of the complex conjugation on the set $E(\bar{\mathbf{k}})$ in order to be able to speak about "entire" (or "holomorphic") parametrization of E. The latter forces the topology of $\mathbf{k}$ to be fruitful as well. We will develop all these matters starting from **3.10**.

3.3. Formal groups. Here we recall some standard definitions and facts about (abelian) formal groups. Let us begin with two examples:

Example 3.4. Take the polynomial $G(x, y) := x + y + xy \in \mathbf{k}[x, y]$ and add up any two elements $a, b \in \mathbf{k}$ via $a +_G b := G(a, b)$. This defines the *multiplicative formal group* $G_m := (\mathbf{k}, +_G)$ (cf. (G_1) – (G_4) above). One can easily check the series $\phi(x) := \log(1 + x) = x - \dfrac{x^2}{2} + \dots$ gives an isomorphism between G_m and the *additive group* $G_a := (\mathbf{k}, +)$ (note that the latter can also be regarded as a formal group with the group law $G(x, y) := x + y$). More generally, we set $G_m^{\oplus d} := (k, +_{G_1}) \times \underbrace{\dots}_{d \text{ times}} \times (k, +_{G_d})$, with $G_i(x, y) = x + y + xy$ for all i, and call this formal group *direct sum of d copies of G_m*.

Example 3.5. Construction of $\widehat{E}$ in **3.1** can be extended to the case of an arbitrary Abelian variety A^n. In fact, replacing A^n by its formal neighborhood at 0 one gets a formal group $\widehat{A^n}$. The latter is defined similarly to $G(x, y)$ in (G_1) – (G_4), with x replaced by the n-string $(x_1, \dots, x_n)$ (the same with y_i for y) and G replaced by the n-string $(G_1, \dots, G_n)$, $G_i \in \mathbf{k}[[x_1, \dots, x_n, y_1, \dots, y_n]]$. Thus $\widehat{A^n}$ is a counterpart of the local group of a ($\mathbb{C}$- or $\mathbb{R}$-) Lie group.

Regard the affine space $\mathbb{A}^n$ as its formal completion at the origin and fix two formal group structures $G_1(x, y)$, $G_2(x, y)$ on $\mathbb{A}^n$ (as defined in Example 3.5).

Definition 3.6. A formal endomorphism $f : \mathbb{A}^n \longrightarrow \mathbb{A}^n$, $f := (f_1, \dots, f_n)$, $f_i \in \mathbf{k}[[x]]$, is called *homomorphism* of formal groups G_i (specifically, from G_1 to G_2) if

$f(G_1(x, y)) = G_2(f(x), f(y))$. Further, f is an *isomorphism*, and we put $G_1 \overset{f}{\simeq} G_2$ (or simply $G_1 \simeq G_2$ (or $G_1 \overset{\sim}{\to} G_2$) if no confusion is likely), if there exists an inverse f^{-1} ($f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}_{\mathbf{k}^n}$) which is also a homomorphism. For $G_1 = G_2 =: G$, we denote by $\mathrm{End}(G)$ the set of all group endomorphisms $G \to G$, with $\mathrm{Aut}(G) \subset \mathrm{End}(G)$ being the collection of all automorphisms.

If $p = \mathrm{char}\,\mathbf{k} > 0$, then according to [29] after a (formal) coordinate change on G_1 and G_2, respectively, one can write $f_i = x_i^{p^k}$, $n - s_k < i \leq n - s_{k+1}$, for some $0 \leq s_m \leq \ldots \leq s_1 \leq s_0 := n$, $m \in \mathbb{Z}_{\geq 0}$.

Definition 3.7. f is called an *isogeny* if $s_m = 0$.

Example 3.8. In the previous notations, let $G_2 := G_1^{(p^k)}$, $k \geq 1$, be a formal group with the composition law $G_2(x, y)$ obtained by first replacing $G_1(x, y)$ by $(G_1(x, y))^{p^k}$ and then substituting x_i, y_i in place of $x_i^{p^k}, y_i^{p^k}$, respectively, for all $1 \leq i \leq n$. Then $f := \mathrm{Fr}_p^k$, with $f_i := x_i^{p^k}$ for all i, is an isogeny (the kth iteration of the Frobenius Fr_p). Note that G_2 (as a formal scheme) carries the same underlying topological space as G_1, but the structure sheaf $\mathcal{O}_{G_2}$ (as a sheaf of $\mathbf{k}^{p^k}$-algebras) is identified with $\mathcal{O}_{G_1}^{p^k}$ (as a sheaf of $\mathbf{k}$-algebras).

In view of Definition 3.7 and Example 3.8, the groups G_1, G_2 are called *isogeneous* (and we put $G_1 \sim G_2$) if there exists an isogeny $f : G_1 \longrightarrow \mathrm{Fr}_p^k(G_2) = G_2^{(p^k)}$ for some k. Notice that $\sim$ is an equivalence relation.

Remark 3.9. Replacing $\mathbf{k}^n$ by its dual, one may associate the *dual group* G^* to the formal group G. Consider Fr_p acting on G^*, and its conjugate Fr_p^* acting on G. The group G is thus equipped (canonically) with the *multiplication-by-p endomorphism* $p \cdot \mathrm{Id}_G := \mathrm{Fr}_p \circ \mathrm{Fr}_p^* = \mathrm{Fr}_p^* \circ \mathrm{Fr}_p$ (see [59, Proposition 1.4]). For instance, if $G = \widehat{A^n}$ (see Example 3.5), then $p \cdot \mathrm{Id}_G$ is the localization of the usual multiplication-by-p endomorphism of A^n. Note that in the latter case $p \cdot \mathrm{Id}_G$ is obviously an isogeny – la propriété incontournable in the classification of formal groups (cf. Theorem 3.13 below).

3.10. Ordinary Abelian varieties. Let $\mathbf{k}$ be as above ($p = \mathrm{char}\,\mathbf{k} > 0$). Recall that an Abelian variety $A := A^d$ (over $\mathbf{k}$) is called *p-ordinary* (or simply *ordinary* if no confusion is possible) if one of the following equivalent conditions holds:

- A contains p^d points of order p;
- the Hasse–Witt matrix $\mathrm{Fr}_p^* : H^1(A^{(p)}, \mathcal{O}_{A^{(p)}}) \to H^1(A, \mathcal{O}_A)$ is invertible.

It is impossible to give here an extensive account of the beautiful theory of ordinary Abelian varieties (and the interested reader is addressed to the papers [26], [49], [28], [83] for which we have tried to be in line with in our current exposition). That is why we restrict ourselves to simply recalling some basic technical facts which will be used further (see e. g. **3.23** and the end of **3.29**).

Let us start with the case $d = 1$:

Example 3.11. Consider an elliptic curve E given by equation $y^2 = x(x-1)(x-\lambda)$ (on the affine piece ($z \neq 0$) of $\mathbb{P}^2 = \mathrm{Proj}\,\mathbb{F}_p\,[x, y, z]$) for some $\lambda \in \mathbb{F}_p \setminus \{0, 1\}$. If E is not ordinary, then it is called *supersingular* – the property characterized by the vanishing $h_E := \sum_{i=0}^{m} \binom{m}{i}^2 \lambda^i = 0$, where $m := (p-1)/2$ (see [42, Corollary

4.22] for example). In particular, once E is the mod p reduction of an elliptic curve $\widetilde{E}$ defined over a number field K, there is a place $\mathfrak{p} \subset K$ such that $\widetilde{E}$ admits an ordinary reduction modulo $\mathfrak{p}$. Note also that if $p \neq q$ are primes and E, E' are elliptic curves over $\mathbb{Q}$ say, so that $h_E \equiv 0 \bmod p$ but $h_E \not\equiv 0 \bmod q$, and similarly for E' with p, q interchanged (we suppose the mod p and q reductions of both E, E' are non-singular), then E and E' are *not* isogeneous (cf. Definition 3.7).

Remark 3.12. The setting of Example 3.11 can be generalized as follows. Let A be as above (not necessarily ordinary). Suppose A is the mod $\mathfrak{p}$ reduction of an Abelian variety $\widetilde{A}$ defined over some number field K (with $\mathfrak{p} \cap \mathbb{Z} = (p)$). Then, applying [28, Corollary 2.8], one may choose $\mathfrak{p}$ (after enlarging K if necessary) to be such that A is ordinary.

Discussion started in Remark 3.9 finds its further development in the following fundamental

Theorem 3.13 (see [58, Proposition 2, Theorem 2])**.** *Let A^d be a p-ordinary Abelian variety over $\mathbf{k}$. Then*

$$\widehat{A^d} \sim G_m^{\oplus d}$$

(cf. Example 3.4). More precisely, there is an isogeny $\Theta : \widehat{A^d} \longrightarrow G_m^{\oplus d}$, defined up to Frobenius twist, such that $\Theta \circ p \cdot Id_{\widehat{A^d}} = Fr_p \circ \Theta$ (cf. Example 3.8).

Remark 3.14. Theorem 3.13 may be considered as the first crucial manifestation of a "torus-like" uniformization of an (ordinary) Abelian variety we are up to (cf. the discussion after Example 1.12). Elaborating further the heuristics of Remark 3.2, let us collect more evidence for such kind of a uniformization, thus making a road map for the upcoming constructions. The strategy is like this: one first "localizes" A in some way (to $\widehat{A}$ as above, or to the *Barsotti–Tate group* $A[p^\infty]$, say); after that the "rigidity" of A comes into play (like the Frobenius action above) to help one handle the patching data for various "localizations" of A.

3.15. Elliptic modules. We will now stick to the case of a *global* ground field $\mathbf{k}$ having $p = \operatorname{char} \mathbf{k} > 0$. Thus, $\mathbf{k}$ is a finite extension of the field $\mathbb{F}_p(t)$, where t is a transcendental parameter.

We are going to introduce another enhancement of formal groups (cf. Remark 3.9), called *formal module* (structure), as well as its refinement, called *elliptic* (or *Drinfel'd*) *module*.[8] We will be mostly following the papers [33], [34].

Let $G \in \mathbf{k}[[x]]$ be a formal group as above. All the previous definitions/ constructions for G may be equally carried over some coefficient ring B in place of $\mathbf{k}$ (but still char $B = p$). Now, given $f \in \operatorname{End}(G) \subset B[[x]]$ we put $D(f) := f'(0)$, thus defining a ring homomorphism (*differential*) $D : \operatorname{End}(G) \to B$. Assume in addition that B is an R-algebra with a ring homomorphism $e : R \to B$ for the ring of integers $R \subset \mathbf{k}$.

Definition 3.16 (cf. [33, §1])**.** One calls the pair (G, φ) a *formal R-module over* B for $\varphi : R \to \operatorname{End}(G)$ being a ring homomorphism such that $D \circ \varphi = e$. (The same definition goes over for $\mathbf{k}$ in place of R. Also, when speaking about formal

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[8]Our interest here is merely a consumer's one (for simply to establish the canonicity property of the isomorphism Θ in **3.23** below, using Proposition 3.19 and Lemma 3.22) and the motivated reader should turn to more fruitful expositions such as [27], [54] or [69] for instance.

R-modules over a field $K \supseteq \mathbf{k}$, we will be silently assuming that $e = $ the embedding $R \subset \mathbf{k}$.)

Example 3.17. If G is the additive group of B, $G(x, y) = x + y$ (cf. Example 3.4), and $\varphi(f) := fx$ for all $f \in R$, then one gets the *additive formal R-module* $\mathfrak{A}$ (over B). The crucial property of $\mathfrak{A}$ is that for any formal R-module $\mathfrak{B}$ over B there exists an isomorphism $\mathfrak{B} \overset{\iota}{\simeq} \mathfrak{A}$ of formal $\mathbf{k}$-modules. Moreover, if one restricts to the case of $D(\iota) = 1$, then such ι is unique (see [33, Proposition 1.2]).

For the additive (algebraic) group G of B, one has $\text{End}(G) = B\{\{\text{Fr}_p\}\}$, the ring of all formal power series $\sum_{i=0}^{\infty} b_i \cdot \text{Fr}_p^i$, $b_i \in B$, where $b_0 \cdot \text{Fr}_p^0 := [\text{multiplication by } b_0]$, so that $\text{Fr}_p \cdot b = b^p \cdot \text{Fr}_p$ for all $b \in B$.[9] Define the embedding $\varepsilon : B \hookrightarrow B\{\{\text{Fr}_p\}\}$ via $b \mapsto b \cdot \text{Fr}_p^0$. Let also $D : B\{\{\text{Fr}_p\}\} \to B$ be the differential homomorphism as above (with $D\left(\sum_{i=0}^{\infty} b_i \cdot \text{Fr}_p^i \right) := b_0$). Finally, we introduce the subring $B\{\text{Fr}_p\} \subset$ $B\{\{\text{Fr}_p\}\}$ of all "polynomials" $\sum_{i=0}^{\infty} b_i \cdot \text{Fr}_p^i$, i.e. $b_i = 0$ for $i \gg 1$.

From now on B is a field.

Definition 3.18 (cf. [33, §2]). A formal R-module (G, φ) is called *elliptic* if $\varphi(R) \subseteq B\{\text{Fr}_p\}$ and $\varphi(R) \not\subset \varepsilon(B)$.

Elliptic modules come along with their *rank* $d \in \mathbb{N}$ defined by $p^{\deg \varphi(f)} = |f|^d$, $f \in R$, for the absolute value $|\cdot|$ w.r.t. the point ∞ of $\mathbf{k}$ (see [33, Propositions 2.1, 2.2]). Further, let $\mathbf{k}_\infty^s$ be the separable closure of the completion $\mathbf{k}_\infty$ of the field $\mathbf{k}$ at ∞, and set $B := \mathbf{k}_\infty^s$ in what follows. Consider $\Lambda \subset B$, a *(period) lattice of dimension d*, i.e. Λ is a discrete $\text{Gal}\,(\mathbf{k}_\infty^s/\mathbf{k}_\infty)$-invariant R-module with d generators. Then we define

$$u(x) := x \prod_{0 \neq \alpha \in \Lambda} \left(1 - \frac{x}{\alpha}\right).$$

This is an entire function on $\mathbb{A}^1$ such that $u(x_1 + x_2) = u(x_1) + u(x_2)$.[10]

The essential feature of elliptic modules is the next fundamental

Proposition 3.19 (see [33, §3]). *Let (G, φ) be an elliptic module over $\mathbf{k}_\infty^s$ of rank d. Then, in the notations of Example 3.17, there exist $\Lambda =: \Lambda_{(G,\varphi)}$ and $u =: u_{(G,\varphi)}$ as above such that $\iota = u$ (for $(G, \varphi) = \mathfrak{B}$). Furthermore, if $f : G_1 \to G_2$ is a homomorphism of elliptic modules (G_i, φ_i) (i.e. $f \in B[x]$ and satisfies $\varphi_2 \circ f = f \circ \varphi_1$), with the associated lattices $\Lambda^i =: \Lambda^i_{(G_i, \varphi_i)}$,[11] then $w\Lambda^1 \subset \Lambda^2$ for some $w \in \mathbf{k}_\infty^s$. In particular, the correspondence $(G, \varphi) \rightsquigarrow \Lambda$ is an equivalence of categories.*

---

[9] When the (maximal) constant subfield of $\mathbf{k}$ is $\mathbb{F}_q$, with $q := p^k$ for some power $k \geq 1$, one should rather take $\text{Fr}_q := \text{Fr}_p^k$ (and replace $\mathbb{F}_p$ by $\mathbb{F}_q$).

[10] Here the affine line $\mathbb{A}^1$ is identified with the topological space $\mathbb{A}^1(\mathbf{k}_\infty^s)$ carrying the rigid topology induced from $\mathbf{k}_\infty$.

[11] Note that Λ^i are necessarily of the same dimension.

Example 3.20. With the formal group G_m from Example 3.4 we will (canonically) associate an elliptic module over $\mathbf{k}_\infty^s$. This together with Theorem 3.13 is the glimpse of a uniformization we are looking for (cf. Remarks 3.2, 3.14). Take $f := tx + x^p \in \text{End}(G_a)$ and consider the group $\mathbb{Z}_f$ of all $x \in \mathbf{k}_\infty^s$ such that $f^k(x) = 0$ for some (variable) $k \geq 1$ (here $f^k := f \circ \underbrace{\ldots}_{k \text{ times}} \circ f$ is the kth iteration of the endomorphism f). Then for the isomorphism $\phi : G_m \to G_a$ the lattice $\Lambda_m := \phi^{-1}(\mathbb{Z}_f)$ determines an elliptic module (G, φ) with $\text{End}(G) \simeq \text{End}(G_m)$ canonically. Indeed, one can see that $\varphi(t)(x) = \phi \circ f \circ \phi^{-1} = \log(1 + e^f - 1) = f$ (hence ϕ is an entire function by Proposition 3.19) and $G_m(\bar{x}, \bar{y}) = u_{(G,\varphi)}(x + y)$ as (additive) group laws, where $\bar{x}, \bar{y}$ are the classes of x, y modulo Λ_m. One may thus think of G_m as being the "elliptic curve" G_a/Λ_m.

Remark 3.21. In light of the above discussion, any formal group G can also be identified with its (profinite) *Dieudonné module* (see [59, Ch. I, §4] for instance), which yields an equivalence between the category of formal groups and discrete modules (over the ring of Witt vectors enhanced with Fr_p and $p \cdot \text{Id}_G$), similar to $\rightsquigarrow$ in Proposition 3.19. However, as Example 3.20 suggests (see also the discussion in **3.23** below), the equivalence $\rightsquigarrow$ is more fruitful.

We conclude this subsection by proving the following:

Lemma 3.22. *Let (G, φ) be an elliptic module and $f \in \text{Aut}(G) \subset \text{End}(G)$ its (formal) automorphism (i. e. $f \circ \varphi \circ f^{-1} = \varphi$ and $D(f) = 1$). Then $f = \text{Id}_G$.*

Proof. We may regard G as a formal $\mathbf{k}$-module $\mathfrak{B}$ (see Example 3.17). Then we have $(f \circ \iota) \circ \varphi \circ (f \circ \iota)^{-1}(r) = \iota^{-1} \circ \varphi \circ \iota(r) = [\text{multiplication by r}]$ for all $r \in R$. Since $D(\iota) = D(f) = D(f \circ \iota) = 1$, by uniqueness we get $f \circ \iota = \iota$, which implies that $f = \text{Id}_G$. $\qquad\square$

3.23. Uniformization. We keep on with the previous notations. Let us assume in addition that A^d is ordinary and defined over $\mathbf{k}$ (as in **3.15**). Also, since the objects of primary interest to us are rully A^d's, especially those equal to $\prod_{i=1}^d E_i$ for some elliptic curves $E_i/\mathbf{k}$ (cf. **1.9**), we will assume w. l. o. g. that A^d is a mod p reduction of some Abelian variety/$\overline{\mathbb{Q}}(t)$.

Fix some local analytic coordinates $x_1, \ldots, x_d$ on A^d near 0 and consider an isomorphism of formal schemes

$$\widehat{A^d} \xrightarrow{\ \Theta\ }_{\sim} \text{Spec } \mathbf{k}[[x_1, \ldots, x_d]].$$

Replacing $\widehat{A^d}$ by $\widehat{A^d}^{(p^k)}$ for some $k \geq 1$, we may additionally assume that there is an isogeny $\widehat{A^d} \longrightarrow G_m^{\oplus d}$, which factors through Θ (see Theorem 3.13). Moreover, replacing each summand in $G_m^{\oplus d}$ by its Frobenius twist (see Definition 3.7 and Example 3.8), we reduce to the case of Θ as in Theorem 3.13. Thus Θ is an isomorphism of formal groups $G := \widehat{A^d}$ and $G_m^{\oplus d}$.

For another choice of local analytic coordinates x_i' on A^d, say $x' = \tau(x)$ for some $\tau \in \mathbf{k}[[x]]$, so we get a formal group G' together with an isomorphism Θ', defined similarly as Θ. In order to extend Θ onto the entire A^d (or, if one likes, to uniformize A^d), it suffices to show that $\Theta = \Theta'$ (cf. the proof of Corollary 3.26 below).

First we need the following:

Lemma 3.24. τ *induces an isomorphism* $G \simeq G'$ *of formal groups.*

Proof. Note that $\widehat{A^d} = \varinjlim_n \widehat{A^d}/n$ for $\widehat{A^d}/n := \widehat{A^d} \otimes_{\mathbf{k}} \operatorname{Spec}(\mathcal{O}_{0,A^d}/\mathfrak{m}^n)$ and the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{0,A^d}$. Furthermore, the components of $\tau(x)$ are all locally convergent, by the choice of x'. So it is enough to prove the assertion for some fixed $\widehat{A^d}/n$ (defined over an Artin ring). But the latter is evident because the group law on $\widehat{A^d}/n$ is induced by the composition $A^d \times A^d \longrightarrow A^d$. $\qquad\square$

From Lemma 3.24 we obtain a commutative diagram

$$
\begin{array}{ccc}
G & \overset{\tau}{\simeq} & G' \\
\downarrow{\scriptstyle\Theta} & & \downarrow{\scriptstyle\Theta'} \\
G_m^{\oplus d} & = & G_m^{\oplus d}.
\end{array}
$$

Here the equality $G_m^{\oplus d} = G_m^{\oplus d}$ reads as $\Theta^{-1*}(x_i) = \Theta'^{-1*}(x'_{\sigma(i)})$ on $G_m^{\oplus d}$ (for all i and some fixed $\sigma \in S_d$) due to the next

Proposition 3.25. $\Theta' \circ \tau \circ \Theta^{-1} = Id_{G_m^{\oplus d}}$ *(up to permutations of summands in* $G_m^{\oplus d}$*).*

Proof. The case $d = 1$ follows from Lemma 3.22. Indeed, the elliptic module structure φ associated with G_m is canonically determined by the group law on G_m (see Example 3.20), which implies that $f \circ \varphi \circ f^{-1} = \varphi$ for every automorphism $f \in \operatorname{Aut}(G_m)$. On the other hand, since τ preserves the complex volume form on A^d, we have $D(f) = 1$ for $f = \Theta' \circ \tau \circ \Theta^{-1}$.

Now let $d = 2$ (the case of arbitrary $d \geq 3$ differs only by more involved notations). Given $f = \Theta' \circ \tau \circ \Theta^{-1}$, we write $f = (f_1, f_2)$, $f_i \in \mathbf{k}[[x_1, x_2]]$, and denote by $G_{m,i}$ the ith summand of $G_m^{\oplus d}$. After restricting f to $G_{m,1}$, projecting $\operatorname{Im} f$ onto $G_{m,1}$ (via $(x_1, x_2) \mapsto x_1$) and applying Lemma 3.22 as above, we obtain that either $f_1 = 0$ or x_1, both modulo the ideal (x_2).[12] Similarly we have $f_2 = x_2$ or 0 mod (x_1). Thus, we may assume that $f_i = x_i \tilde{f}_i$ for some $\tilde{f}_i \in \mathbf{k}[[x_1, x_2]]$, $\tilde{f}_i = 1$ mod (x_1, x_2).

Further, letting $x_2 := \lambda x_1$ for an arbitrary $\lambda \in \mathbf{k}$ and applying Lemma 3.22 once again, we obtain that both of the automorphisms of $G_{m,1}$, given by

$$
x_1 \mapsto (x_1 \tilde{f}_1(x_1, \lambda x_1), \ \lambda x_1 \tilde{f}_2(x_1, \lambda x_1)) \overset{\mathrm{pr}}{\mapsto} x_1 \ (\text{resp. } x_2),
$$

are trivial. This implies that $\tilde{f}_i = 1$ and completes the proof of Proposition 3.25. $\quad\square$

Further, we endow variety A^d with a *rigid analytic* structure (over $\mathbf{k}_\infty$), as defined in [15] or [4] for instance. The preceding discussion condensates to the next

Corollary 3.26. *The components of the map* Θ *are entire analytic functions on* A^d.

Proof. Proposition 3.25 implies that $\Theta' = \Theta \circ \tau^{-1}$ is an analytic continuation of Θ. More precisely, $(\Theta^{-1} \circ \tau^{-1})^*(x_i)$ are (formal) analytic functions of x, for all i. Thus, it suffices to show the components $\Theta_i(x)$ of Θ are entire functions of x, i.e. $\Theta_i(x)$ converges on an analytic x-chart $U_x \subset A^d, 1 \leq i \leq d$. We will treat only the case $d = 1$ (the case $d \geq 2$ is similar).

---

[12] Up to interchanges $x_i \leftrightarrow x_j$.

Pick an n-torsion point $P \in U_x \setminus \{0\}$ for some $n \geq 2$. Then we have $n\Theta_i(P) \in \Lambda_m$ (see Example 3.20) by construction of Θ. This shows that $\Theta_i(x)$ is convergent on U_x. □

Intermedia. In the preceding considerations, we have used essentially two kinds of arguments, very much typical to the (rigid) uniformization theories. The first one makes it possible to treat only the formal situation (i.e. that of formal schemes etc.), employing the rigid topology of a complete ground field $\mathbf{k}$ when needed (see [67], [68], [62], [64], [37], [74]). A model example is the *Tate curve* $E_q := \mathbf{k}^*/q^{\mathbb{Z}}$, $q \in \mathbf{k}$, $0 < |q| < 1$ (see [88, §6]), which is an elliptic curve with the j-invariant equal to $\frac{1}{q} + 744 + 196884q + \ldots$ (this example had gotten further development in [67], [68] (cf. Remark 1.13 above), where the rigid uniformization was established for the *totally degenerate* curves and Abelian varieties). The second type of arguments, dealing with char $\mathbf{k} > 0$ and bringing a finer analysis into play, uses the "inner symmetry" of $\mathbf{k}$ (as compared to the complex conjugation on $\mathbf{k} = \mathbb{C}$), that is the Frobenius $x \mapsto x^p$, $x \in \mathbf{k}$ (cf. Example 3.8 and Remarks 3.2, 3.14). Classical illustration in this case is the *p-adic* (or *Serre – Tate* if one prefers) uniformization found in [35] (compare with constructions in [33], [49], [16]). Namely, there is a formal scheme $\hat{\Omega}^d$, parameterizing *special* formal O_D-modules over various (nilpotent) R-algebras (cf. **3.15**). Here D is a central division algebra over $\mathbf{k}$ with invariant $1/d$ and $O_D \subset D$ is the ring of integers. Universal formal O_D-module over $\hat{\Omega}^d$ admits a (canonical) system of finite subgroups $\Gamma_k, k \geq 1$, flat over $\hat{\Omega}^d$, which are the kernels of the multiplication by $t^k \in R$ (cf. Example 3.20). Now, taking D to be the quaternion ring over $\mathbb{Q}$, with maximal order O_D, the schemes Γ_k can be considered (up to compact factors) as (formal) modular curves, the latter parameterizing *special* Abelian 2-dimensional O_D-schemes (*Cherednik's theorem*).

At this point, let us return to our model case $d = 1$, $A^d = E$ being an elliptic curve. This is done just to simplify the notations and the reader should consult the discussion of the general ($d \geq 2$) case at the end of this section. Further, we identify E with its ordinary mod p reduction defined over $\overline{\mathbb{F}}_p$ (cf. **3.10**), thus regarding the surface $S := E \times \mathbb{P}^1$ as the curve E over the field $\mathbf{k}$ (the one at the beginning of **3.15**).

Fix another prime $\ell \neq p$ and consider the ℓ-primary component $G_S^\ell \subset \mathrm{Gal}(S) =: \mathrm{Gal}(\overline{\mathbb{F}_p(S)}/\mathbb{F}_p(S))$. Using "topological" Corollary 3.26 (applied to $E/\mathbf{k}$) we are going to describe G_S^ℓ (more or less) explicitly (see Proposition 3.28 below).

3.27. Structure of G_S^ℓ. Recall that we have constructed a one-to-one correspondence between the points from $E(\mathbf{k}_\infty^s)$ and some compact subset in $\mathbb{A}^1(\mathbf{k}_\infty^s)$. For $(x, y) \in E$, the correspondence in question is given by an entire function $\Theta(x, y)$ on $E(\mathbf{k}_\infty^s)$, which yields a *homeomorphism* $\Theta : E(\mathbf{k}_\infty^s) \xrightarrow{\sim} \mathbb{P}^1(\mathbf{k}_\infty^s)$.

Any Galois ℓ-cover $c : \widetilde{E} \longrightarrow E$ (over $\mathbf{k}^s$) is determined by its ramification divisor $o_1 + \ldots + o_l$ and the monodromy group G_c. Also, given a $\mathbf{k}$-variety X, we have $\mathrm{Gal}(X) = \varinjlim_U \pi_1^{\mathrm{alg}}(U)$ for various Zariski open subsets $U \subset X$. This results in an exact sequence

$$1 \to \mathrm{Gal}(X \otimes \mathbf{k}^s) \to \mathrm{Gal}(X) \to \mathrm{Gal}(\mathbf{k}^s/\mathbf{k}) \to 1$$

(see e. g. [40, §8]) and a *bijection*

$$\hbar : G_S^\ell \to G_{\mathbb{P}^2}^\ell$$

between the Galois ℓ-covers of $E/\mathbf{k}$ ($\approx S$) and $\mathbb{P}^1/\mathbf{k}$ ($\approx \mathbb{P}^2$). Indeed, given $c : \widetilde{E} \longrightarrow E$ as above and the fact that $G_{\mathbb{P}^1}^\ell$ is pro-ℓ-free (see [75]), one constructs the Galois cover $\hbar(c) : \widetilde{\mathbb{P}^1} \longrightarrow \mathbb{P}^1$ (over $\mathbf{k}^s$) as follows. First, letting $E(\mathbf{k}_\infty^s) = \Theta(\mathbb{P}^1(\mathbf{k}_\infty^s))$, we identify c with a *topological* covering of $\mathbb{P}^1$. Hence $\mathbb{P}^1$ is (topologically) a G_c-quotient of $\widetilde{E}$. Then we choose $\hbar(c)$ to have the Galois group equal G_c and coincide with c over an open subset of $\mathbb{P}^1$.[13] Finally, using the presentation $\mathbb{P}^1 = \widetilde{E}/G_c$, one can see the definition of $\hbar$ is correct.

The next assertion is straightforward (its proof is left to the reader as an exercise):

Proposition 3.28. *$\hbar$ is a group homomorphism.*

Thus we obtain an (outer) isomorphism $G_S^\ell \simeq G_{\mathbb{P}^2}^\ell$ of groups. This creates the grounds for applying results from *anabelian birational geometry* (see **3.29** and [11] for relevant topics).

Intermedia. The previously constructed Θ points out (again) the "pathological" nature of the char > 0 world (cf. [36], [70], [71], [63], [87]). Indeed, Θ provides an "analytic" isomorphism between E and $\mathbb{P}^1$, though it does not induce an isomorphism of the underlying rigid analytic structures – otherwise $E \simeq \mathbb{P}^1$ as algebraic curves by the (rigid) GAGA principle.[14] In particular, the function Θ does not belong to the *Tate algebra* of the local ring $\mathcal{O}_{0,E}$, i.e. if $x \in \mathcal{O}_{0,E}$ is a local parameter and $\Theta = \sum a_i x^i$, $a_i \in \mathbf{k}_\infty^s$, then $|a_i| \not\to 0$ as $i \to \infty$. However, in line with Intermedia at the end of **3.23**, our construction of the above isomorphism $\hbar$ is (a sort of) a reminiscent of the approach developed in [75] to prove the (geometric) Shafarevich's (resp. Abhyankar's) Conjecture. Indeed, in [75] one proceeds from local to global, starting with a unit (rigid) disk D (a *germ of local behavior*) in a given affine curve $U/\bar{\mathbf{k}}$, after what one constructs a Galois covering of D (*rigid Galois cover*) and glues (analytically) all this data together, for various $D \subset U$, to obtain a prescribed element in $\pi_1^{\mathrm{alg}}(U)$. Canonicity (aka correctness) of this construction is ensured by the existence of Galois cyclic covers of $\mathbb{P}^1 \setminus \{0, \infty\}$, ramified only at 0 and some other $c \in \bar{\mathbf{k}}$, with $|c| < 1$. The latter, in turn, follows simply from the Kummer and Artin–Schreier theories – in contrast with the "analytic" arguments of **3.23**.

3.29. Concluding discussion. We now return to the ground field $\mathbf{k} = \overline{\mathbb{F}}_p$.

The isomorphism $G_S^\ell \overset{\hbar}{\simeq} G_{\mathbb{P}^2}^\ell$ and the main theorem of [76] yield a field F together with (finite) Galois ℓ-extensions $F \supseteq \mathbf{k}(S)$, $F \supseteq \mathbf{k}(\mathbb{P}^2)$. More precisely, according to [76] both $\mathbf{k}(S)$ and $\mathbf{k}(\mathbb{P}^2)$ have common purely inseparable closure, F^{pi} say. Then, using the canonical bijection $\mathrm{Out}\big(G_S^\ell, G_{\mathbb{P}^2}^\ell\big) \leftrightarrow \mathrm{Aut}(F^{\mathrm{pi}})$ between the outer isomorphisms of Galois groups and the field automorphisms$/\mathbf{k}$ (up to Frobenius twists), we pick $G_S^\ell \simeq G_{\mathbb{P}^2}^\ell$ corresponding to $\mathrm{id} \in \mathrm{Aut}(F^{\mathrm{pi}})$, thus finding our F.

---

[13] But the ramification divisor of $\hbar(c)$ need not coincide with $\Theta(o_1) + \ldots + \Theta(o_l)$.

[14] This principle (almost tautologically) leads to an isomorphism $G_S^\ell \simeq G_{\mathbb{P}^2}^\ell$ when $E = \mathbf{k}^*/q^{\mathbb{Z}}$ is the Tate curve. So Proposition 3.28 (as well as its proof) may be considered as a generalization of this fact (without direct appeal to the rigid GAGA).

This results in a diagram

$$
\begin{array}{ccc}
 & W & \\
 {}^{r}\swarrow & & \searrow^{r'} \\
 S & & \mathbb{P}^2,
\end{array}
$$

where both r, r' are some Galois ℓ-covers (with isomorphic Galois groups acting birationally on W), superseded probably by p.i. maps. Furthermore, since there is no a priori preference in the choice of the extension $F \supseteq \mathbf{k}(S)$, we pick the latter to be such that r is unramified over generic fiber of the projection $S \longrightarrow \mathbb{P}^1$ (we are neglecting the Frobenius action for the sheer transparency).

Fix some (generic) $t_0 \in \mathbf{k}$ and consider $E_{t_0} = E \times t_0 \subset S$. Let $E_{t_0}^r$ be the normalization of the scheme preimage $r^{-1}(E_{t_0})$ in $\mathbf{k}(W)$.

Lemma 3.30. $E_{t_0}^r$ *is an elliptic curve.*

Proof. Suppose first that r is p.i. of degree p^k for some $k \geq 1$. Let us assume for simplicity that $k = 1$. Then the extension $\mathbf{k}(W) \supseteq \mathbf{k}(S)$ is given by an equation $z^p = f(x, y, t)$. Here t is a coordinate on $\mathbb{P}^1$, x, y are coordinates on the affine model of E (cf. **3.1**), z is an extra variable and $f \in \mathbf{k}(x, y, t)$. Replacing $E = E_{t_0}$ by $E^{(p)}$ (cf. Example 3.8), we get a morphism $r_1 : W \longrightarrow E^{(p)} \times \mathbb{P}^1 =: S_1$, which is a prolongation of r via the natural (Frobenius) map $S \longrightarrow S_1$. Now, extension $\mathbf{k}(W) \supseteq \mathbf{k}(S_1)$ is given by the equation $z^p = f(x^p, y^p, t)$, which implies the normalization of $r_1^{-1}(E^{(p)})$ $(= E_{t_0}^r)$ is birational to $E^{(p)}$. Indeed, $r_1^{-1}(E^{(p)})$ coincides with the locus $z^p = f(x^p, y^p, t_0)$, so that its normalization is $\approx$ [graph of the function $f(x, y, t_0)$ on E].

We go on to the case of arbitrary r. One may assume w.l.o.g. that r is separable. Then, by construction, the covering $r : E_{t_0}^r \longrightarrow E$ is unramified, thus $E_{t_0}^r$ is an elliptic curve. $\square$

Replace E_{t_0} by a Frobenius twist of $E_{t_0}^r$ if necessary to obtain the morphism $r'\big|_E$ (cf. Lemma 3.30) is smooth. Again, assuming w.l.o.g. r' to be separable, we find that $r'(E) = E/G$ for the Galois group G of the covering r'. Finally, since G is an ℓ-group, with $\ell \gg 1$ say, the quotient E/G is an elliptic curve. This shows (again) that $\mathbb{P}^2$ is b.-H. (over $\overline{\mathbb{F}}_p$).

Intermedia. One may spot a similarity between the preceding considerations and those of paper [10]. Recall that for any two smooth curves $C, \bar{C}$ over $\mathbf{k} = \overline{\mathbb{F}}_p$, $p \geq 3$, an *isomorphism of pairs* $(C, J) \to (\bar{C}, \bar{J})$ implies the corresponding Jacobians J and $\bar{J}$ are isogeneous (see [10, Theorem 1.2]). In particular, given a curve C of genus > 4, the maximal abelian quotient $\mathcal{G}_K^a$ of the prime-to-p part of the Galois group $\mathrm{Gal}(\overline{K}/K)$, $K := \mathbf{k}(C)$, together with the collection $\mathcal{I} := \{I_\nu\}_\nu$ of inertia subgroups $I_\nu \subset \mathcal{G}_K^a$, determines the isogeny class of the Jacobian of C (see [10, Theorem 9.3]). We recall that in the latter setting the isomorphism of pairs $\phi : (C, J) \to (\bar{C}, \bar{J})$ implies that two natural actions of the Frobenius Fr_p^k, $k \gg 1$ (when both considered in $\mathrm{End}(C)$), on the Tate modules $T_\ell(J), T_\ell(\bar{J})$ commute. The idea then is to reduce to the case $k = 1$ and reconstruct an isogeny $J \longrightarrow \bar{J}$ via [86]. Now, given two isomorphic data $(\mathcal{G}_K^a, \mathcal{I})$ and $(\mathcal{G}_{\overline{K}}^a, \bar{\mathcal{I}})$ for C and $\bar{C}$, respectively, one essentially recovers an isomorphism of pairs ϕ from isomorphic Div^0's.

Remark 3.31. Note that the morphism r' can not be purely inseparable. Indeed, otherwise the surface W (hence also S) is rationally connected, which forces S to be rational (for S is *ruled*). This shows in particular that the condition $\mathrm{Pic}(X) = \mathrm{NS}(X)$ in [12, Theorem 1] *is* actually necessary for the claim to be true as stated (compare with [13]). Let us also remark that W may be considered as a 2-dimensional analog of the *universal curve* found in [14]. This is another (supporting) side of the "ideology" adopted in the present section in order to relate the function fields $\mathbf{k}(\mathbb{P}^2)$ and $\mathbf{k}(S)$.

Up to this end all the arguments apply literarilly to $\prod_{i=1}^n E_i$ in place of E (respectively, to $\prod_{i=1}^n E_i \times \mathbb{P}^1$ in place of S, $\mathbb{P}^{n+1}$ in place of $\mathbb{P}^2$, etc.). Hence Conjecture 1.11 would be proven once we showed the birational embedding r' : $\prod_{i=1}^n E_i \times t_0 \dashrightarrow \mathbb{P}^{n+1}$, a priori defined over $\overline{\mathbb{F}}_p$, can be lifted to char 0.

Note at this point that r' corresponds to a $(n+1)$-dimensional linear subsystem $\subseteq H^0\big(\prod_{i=1}^n E_i, \mathcal{L}\big)$ for some $\mathcal{L} \in \mathrm{Pic}\big(\prod_{i=1}^n E_i \times_{\mathbf{k}} \overline{\mathbb{F}}_p\big)$. Then, since r' is separable and generically 1-to-1 on $\prod_{i=1}^n E_i$, it suffices to show that $\mathcal{L}$ admits a char 0 lifting. The latter is easy when $n = 1$, for $\mathrm{Pic}(E) = \mathbb{Z} \oplus \mathrm{Pic}^0(E)$, and in the case $n \geq 2$ it suffices to take E_i to be pair-wise non-isogeneous for all i, so that $\mathrm{Pic}\big(\prod_{i=1}^n E_i\big) = \prod_{i=1}^n \mathrm{Pic}(E_i)$. We leave this as an exercise to the reader. (Hint: use Example 3.11.)

4. Miscellania

4.1. We would like to conclude by discussing another, geometric, variant of the point of view we have tried to advocate throughout the paper, namely that "an Abelian variety A^n with many symmetries is (birationally) close to $\mathbb{P}^n$, after one mods out A^n by these symmetries." As a model illustration, we took $A^n := E_1 \times \ldots \times E_n$, with elliptic curves E_i defined over a global field $\mathbf{k}$ having char $\mathbf{k} = p > 0$, and acted this A^n by (various powers of) the Frobenius. Then the upshot of our thesis in this case are the results of subsections **3.23**, **3.27** and **3.29**. Geometric counterpart of this ideology is as follows.

Consider $A^n := E^n$, where E is an elliptic curve over $\mathbf{k} = \mathbb{C}$, and identify A^n with $\mathrm{Hom}(\Lambda, E)$ for a full lattice $\Lambda \subset \mathbb{R}^n$. Thus there is a natural $GL(n, \mathbb{Z})$-action on A^n. More specifically, we consider $R \subset GL(n, \mathbb{Z})$, a reflection group, for which one can form the quotient $Q := E^n/R$. Then it is expected (in accordance with the edifice from the last paragraph) that variety Q should be "close to rational."

Example 4.2. Classically, when Λ is the lattice w. r. t. the (reduced irreducible) root system for a Weyl group R, it was shown in the beautiful paper [56] that Q is a weighted projective space. More precisely, let Λ span the linear space V, so that $n = \dim V$ is the rank of the corresponding root system. Let $\alpha \in V$ be the highest root and $\alpha^\vee \in V^*$ be its dual. This $\alpha^\vee$ is a root in the dual root system and we can write $\alpha^\vee = \sum_{i=1}^n g_i \alpha_i$ for the basic roots α_i and some $g_i \in \mathbb{N}$. Then $Q = \mathbb{P}(1, g_1, \ldots, g_n)$.

The setting of Example 4.2 was generalized further in [52]. Namely, it was shown in [52] (among other things) that the quotient of an Abelian variety A^n by a finite group $G \subset \mathrm{Aut}(A^n)$ is *rationally connected* iff the G-action on the tangent space $T_0(A^n)$ is irreducible and violates the *Reid – Tai property* (see [52, Corollary 25]).

In particular, Weyl groups from Example 4.2 (with $A^n = E^n$) are like this, but at the same time the following question seems to be out of reach (cf. [52, Section 5]):

Question K-L. Let $A^n = E^n$ and G be as in Example 4.2, but suppose now that G is *any* finite group acting (isometrically, say) on Λ, with induced G-action on $T_0(A^n)$ irreducible and non-RT. Is then the quotient $X := A^n/G$ rational (or at least unirational)?

Note that positive answer to Question K-L might give an alternative way of proving Conjecture 1.11 (for one may choose a general Abelian hypersurface $A_t \subset A^n$ and proceed as in the proof of Corollary 1.19). Let us outline how one could answer Question K-L for *complex crystallographic reflection groups*.

Fix the quotient map $q : Y := E^n \longrightarrow Y/G = X$ and write

$$(4.3) \qquad 0 = K_Y \equiv q^*\Big(K_X + \sum_i \frac{r_i - 1}{r_i} R_i\Big)$$

for some $r_i \in \mathbb{Z}_{\geq 0}$ and $R_i \in \text{Weil}(X)$ (Hurwitz formula). Note that $\#R_i = n + 1$ by the assumption on G (see the list in [77]).

Consider a G-invariant divisor $H := q^*(\mathcal{O}_X(1)) \in \text{Pic}(Y)$. The Appell–Humbert theorem (see [65]) identifies H with a positive definite Hermitian form I on $\mathbb{C}^n$ such that $\text{Im}(I)$ is skew-symmetric and $\mathbb{Z}$-valued on the lattice $\Lambda \otimes_\mathbb{Z} H^1(E,\mathbb{Z})$ $(= H^1(Y,\mathbb{Z})$ canonically). Note also that I is G-invariant by construction. In particular, since the G-representation $\mathbb{C}^n = H^1(Y,\mathbb{Z}) \otimes \mathbb{C}$ is irreducible, the form I is unique (up to a constant multiple), which implies that $\text{Pic}(X) = \mathbb{Z}$ and X is a *log Fano* (see [46]). Our intention here is to prove that X is *toric*. In order to do this we employ the following:

Conjecture 4.4 (V. V. Shokurov)**.** *Let X be a normal $\mathbb{Q}$-factorial algebraic variety with a $\mathbb{Q}$-boundary divisor $D = \sum d_i D_i$ ($0 \leq d_i \leq 1$ and D_i are prime Weil divisors) such that*

- *the divisor $-(K_X + D)$ is nef,*
- *the pair (X, D) is log canonical.*

Then the estimate $\sum d_i \leq \text{rk } \text{Pic}(X) + \dim X$ holds. Moreover, the equality is achieved iff the pair $(X, \llcorner D \lrcorner)$ is toric, i. e. X is toric and $\llcorner D \lrcorner$ is its boundary.

Recall that Conjecture 4.4 had been verified in the case $\text{Pic}(X) = \mathbb{Z}$ (see [79, Corollary 2.8]) and so we can apply it to our $X = Y/G$ (see [47] for further discussion of Conjecture 4.4 and its variations). Namely, fix some generator $\mathcal{O}_X(1)$ of $\text{Pic}(X)$, so that $R_i = \mathcal{O}_X(r_i m_i)$ in (4.3) for some integers $m_i \geq 1$ and all i. Then to show that X is toric it suffices to establish the pair $(X, \sum_i R_{i,X})$ is log canonical

for generic $R_{i,X} \in |\mathcal{O}_X(r_i - 1)|$ (recall that $\#R_i = n + 1$, i.e. we handle all the conditions in Conjecture 4.4, except for the lc one). The latter, in turn, is easily seen to be equivalent to the same statement about Y in place of X and generic G-invariant divisors $R_{i,Y} \in |H^{r_i - 1}|$ in place of $R_{i,X}$, respectively. Then we can construct G-invariant theta characteristics θ_i corresponding to $R_{i,Y}$ exactly as in [56, §4], taking I to be the Coxeter matrix w.r.t. G, say. At this point, one can do without knowing algebraic relations between θ_i in the ring $\bigoplus_{m \geq 0} H^0(Y, H^m)$ (as it was in [56]), and all one has to check is that the loci $\theta_i = 0$ (for all i and generic θ_i) determine a log canonical pair.

Remark 4.5. Obviously, the remaining assertion one is left with (in order to prove that X is toric) is easier to deal with and applies in the more general setting, compared to the the proof of Theorem 3.4 in [56]. However, the preceding arguments can only show that X is a *fake weighted projective space* (see [48]), lacking the explicit description of X in terms of G as in Example 4.2.

4.6. Regrettably, due to the lack of space and the author's competence we could not touch many other topics, concerning (both birational and biregular) geometry of Abelian fibrations. We thus restrict ourselves to simply stating some of these (together with formulation of relevant problems), hoping the technique (and results) of the present paper (in interaction with methods of the articles we are going to mention) will bring further insight and development to this beautiful subject.

(A) In view of Conjecture 1.11 and constructions in Section 3, it would be interesting to find out whether an Abelian fibration $f : \mathbb{P}^n \dashrightarrow B$, with $\dim X \geq \dim B + 2$, is isotrivial, and if it is not, does it possess a section. Again, comparing with the $n = 2$ situation, one should not expect "many" (up to birational equivalence) non-isotrivial f's, while for isotrivial ones there should exist a complete (effective?) description (cf. Example 1.12). Leaving the non-isotrivial problem (which is of the existence-type and should be the most difficult) silently aside, for isotrivial f the case of particular interest is when $\dim B$ is small, say $\dim B \leq 2$. If this holds, one may apply the technique of [55] for example to estimate the Mordell–Weil group, discriminant locus, etc. for the desingularized f.

(B) The method of Section 3 is implicit and it would be interesting to construct Abelian fibrations on rational varieties explicitly (as we have attempted to do in **4.1**). Classically, there are Horrocks–Mumford Abelian surfaces in $\mathbb{P}^4$ (see [43]). This result was extended in [44] to find Abelian surfaces in $\mathbb{P}^1 \times \mathbb{P}^3$, then in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)))$ (see [80]), etc. In the beautiful papers [38], [39], one finds pencils of Abelian surfaces lying on Calabi–Yau 3-folds, and for a birational construction (of rational fibrations on $\mathbb{P}^5$ by degree 3 nodal surfaces) we refer to [89]. On this way, one obtains a source of explicit examples of Abelian fibrations on $\mathbb{P}^n$, $n \gg 1$.

(C) We would like to distinguish/characterize b.-H. structures on a rationally connected X ($= \mathbb{P}^n$ for instance) by existence of a certain sheaf $\mathcal{L}$ on X with particular values of $c_i(\mathcal{L})$, rk $\mathcal{L}$, etc. Also we would like to know how b.-H. behaves under small deformations. See [50] for relevant discussion and results.

(D) Finally, it might be of some interest (and use) to compare the matters in (A), (B), (C) with similar ones for symplectic (or Poisson, or ...) manifolds in place of rational varieties (see [3], [81] for an overview of related problems).

References

[1] M. Artin and D. Mumford, Some elementary examples of unirational varieties which are not rational, Proc. London Math. Soc. (3) **25** (1972), 75 − 95.

[2] I. Arzhantsev et al., Flexible varieties and automorphism groups, Duke Math. J. **162** (2013), no. 4, 767 − 823.

[3] A. Beauville, Holomorphic symplectic geometry: a problem list, in *Complex and differential geometry*, 49 − 63, Springer Proc. Math., 8 Springer, Heidelberg.

[4] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, 33, Amer. Math. Soc., Providence, RI, 1990.

[5] F. A. Bogomolov, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), no. 3, 485 − 516, 688; translation in Math. USSR-Izv. **30** (1988), no. 3, 455 − 485.

[6] F. A. Bogomolov, Mat. Sb. **180** (1989), no. 2, 279 − 293; translation in Math. USSR-Sb. **66** (1990), no. 1, 285 − 299.

[7] F. Bogomolov, Stable cohomology of finite and profinite groups, in *Algebraic groups*, 19 − 49, Universitätsverlag Göttingen, Göttingen.

[8] F. A. Bogomolov, Stable rationality of quotient spaces for simply connected groups, Mat. Sb. (N.S.) **130(172)** (1986), no. 1, 3 − 17, 128.

[9] F. Bogomolov, I. Karzhemanov, and K. Kuyumzhiyan, Unirationality and existence of infinitely transitive models, in *Birational Geometry, Rational Curves, and Arithmetic* (Fedor Bogomolov, Brendan Hassett, Yuri Tschinkel, eds.), Simons Symposia, Springer New York (2013), 77 − 92.

[10] F. Bogomolov, M. Korotiaev and Y. Tschinkel, A Torelli theorem for curves over finite fields, Pure Appl. Math. Q. **6** (2010), no. 1, Special Issue: In honor of John Tate. Part 2, 245 − 294.

[11] F. Bogomolov and Y. Tschinkel, Introduction to birational anabelian geometry, in *Current developments in algebraic geometry*, 17 − 63, Math. Sci. Res. Inst. Publ., 59 Cambridge Univ. Press, Cambridge.

[12] F. Bogomolov and Y. Tschinkel, Reconstruction of function fields, Geom. Funct. Anal. **18** (2008), no. 2, 400 − 462.

[13] F. Bogomolov and Y. Tschinkel, Reconstruction of higher-dimensional function fields, Mosc. Math. J. **11** (2011), no. 2, 185 − 204, 406.

[14] F. Bogomolov and Y. Tschinkel, Unramified correspondences, in *Algebraic number theory and algebraic geometry*, 17 − 25, Contemp. Math., 300 Amer. Math. Soc., Providence, RI.

[15] S. Bosch and W. Lütkebohmert, Formal and rigid geometry. I. Rigid spaces, Math. Ann. **295** (1993), no. 2, 291 − 317.

[16] C.-L. Chai, Families of ordinary abelian varieties: canonical coordinates, p-adic monodromy, Tate-linear subvarieties and Hecke orbits, Preprint (2003), available at *http://www.math.upenn.edu/∼chai/papers.html*.

[17] I. A. Cheltsov, Mat. Zametki **68** (2000), no. 1, 131 − 138; translation in Math. Notes **68** (2000), no. 1-2, 113 − 119.

[18] I. Cheltsov and I. Karzhemanov, Halphen pencils on quartic threefolds, Adv. Math. **223** (2010), no. 2, 594 − 618.

[19] I. Cheltsov and J. Park, Halphen pencils on weighted Fano threefold hypersurfaces, Cent. Eur. J. Math. **7** (2009), no. 1, 1 − 45.

[20] C. Chevalley, On algebraic group varieties, J. Math. Soc. Japan **6** (1954), 303 − 324.

[21] C. H. Clemens and P. A. Griffiths, The intermediate Jacobian of the cubic threefold, Ann. of Math. (2) **95** (1972), 281 − 356.

[22] J.-L. Colliot-Thélène and M. Ojanguren, Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford, Invent. Math. **97** (1989), no. 1, 141 − 158.

[23] J.-L. Colliot-Thélène and J.-J. Sansuc, La *R*-équivalence sur les tores, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 2, 175 − 229.

[24] J.-L. Colliot-Thélène and J.-J. Sansuc, The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group), in *Algebraic groups and*

homogeneous spaces, 113 − 186, Tata Inst. Fund. Res. Stud. Math, Tata Inst. Fund. Res., Mumbai.

[25] A. Corti, Singularities of linear systems and 3-fold birational geometry, in *Explicit birational geometry of 3-folds*, 259 − 312, London Math. Soc. Lecture Note Ser., 281 Cambridge Univ. Press, Cambridge.

[26] P. Deligne, Variétés abéliennes ordinaires sur un corps fini, Invent. Math. **8** (1969), 238 − 243.

[27] P. Deligne and D. Husemoller, Survey of Drinfel'd modules, in *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, 25 − 91, Contemp. Math., 67 Amer. Math. Soc., Providence, RI.

[28] P. Deligne et al., *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, 900, Springer, Berlin, 1982.

[29] J. Dieudonné, Groupes de Lie et hyperalgébres de Lie sur un corps de caractéristique $p > 0$, Comment. Math. Helv. **28** (1954), 87 − 118.

[30] I. V. Dolgachev, Rational surfaces with a pencil of elliptic curves, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 1073 − 1100.

[31] I. Dolgachev and M. Gross, Elliptic threefolds. I. Ogg-Shafarevich theory, J. Algebraic Geom. **3** (1994), no. 1, 39 − 80.

[32] S. K. Donaldson, Symmetric spaces, Kähler geometry and Hamiltonian dynamics, in *Northern California Symplectic Geometry Seminar*, 13 − 33, Amer. Math. Soc. Transl. Ser. 2, 196 Amer. Math. Soc., Providence, RI.

[33] V. G. Drinfel'd, Elliptic modules, Mat. Sb. (N.S.) **94(136)** (1974), 594 − 627, 656.

[34] V. G. Drinfel'd, Elliptic modules. II, Mat. Sb. (N.S.) **102(144)** (1977), no. 2, 182 − 194, 325.

[35] V. G. Drinfel'd, Coverings of p-adic symmetric domains, Funkcional. Anal. i Priložen. **10** (1976), no. 2, 29 − 40.

[36] H. Esnault and A. Hogadi, On the algebraic fundamental group of smooth varieties in characteristic $p > 0$, Trans. Amer. Math. Soc. **364** (2012), no. 5, 2429 − 2442.

[37] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 22, Springer, Berlin, 1990.

[38] M. Gross and S. Popescu, Calabi-Yau threefolds and moduli of abelian surfaces. I, Compositio Math. **127** (2001), no. 2, 169 − 228.

[39] M. Gross and S. Popescu, Calabi-Yau three-folds and moduli of abelian surfaces II, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3573 − 3599.

[40] A. Grothendieck, Géométrie formelle et géométrie algébrique, in *Séminaire Bourbaki, Vol. 5*, Exp. 182, 193 − 220, errata p. 390, Soc. Math. France, Paris.

[41] C. D. Hacon and J. Mckernan, On Shokurov's rational connectedness conjecture, Duke Math. J. **138** (2007), no. 1, 119 − 136.

[42] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.

[43] G. Horrocks and D. Mumford, A rank 2 vector bundle on $\mathbf{P}^4$ with $15,000$ symmetries, Topology **12** (1973), 63 − 81.

[44] K. Hulek, Abelian surfaces in products of projective spaces, in *Algebraic geometry (L'Aquila, 1988)*, 129 − 137, Lecture Notes in Math., 1417 Springer, Berlin.

[45] V. A. Iskovskih and Yu. I. Manin, Three-dimensional quartics and counterexamples to the Lüroth problem, Mat. Sb. (N.S.) **86(128)** (1971), 140 − 166.

[46] V. A. Iskovskikh and Yu. G. Prokhorov, Fano varieties, in *Algebraic geometry, V*, 1 − 247, Encyclopaedia Math. Sci., 47 Springer, Berlin.

[47] I. Karzhemanov, On characterization of toric varieties, Preprint arXiv:1306.4131.

[48] A. M. Kasprzyk, Bounds on fake weighted projective space, Kodai Math. J. **32** (2009), no. 2, 197 − 208.

[49] N. Katz, Serre − Tate local moduli, in *Algebraic surfaces (Orsay, 1976 − 78)*, 138 − 202, Lecture Notes in Math., 868 Springer, Berlin.

[50] J. Kollár, Deformations of elliptic Calabi–Yau manifolds, Preprint arXiv:1206.5721.

[51] J. Kollár, Shafarevich maps and plurigenera of algebraic varieties, Invent. Math. **113** (1993), no. 1, 177 − 215.

[52] J. Kollár and M. Larsen, Quotients of Calabi-Yau varieties, in *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, 179 – 211, Progr. Math., 270 Birkhäuser Boston, Boston, MA.

[53] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, translated from the 1998 Japanese original, Cambridge Tracts in Mathematics, 134, Cambridge Univ. Press, Cambridge, 1998.

[54] L. Lafforgue, Chtoucas de Drinfeld et correspondance de Langlands, Invent. Math. **147** (2002), no. 1, 1 – 241.

[55] A. Libgober, On Mordell-Weil group of isotrivial abelian varieties over function fields, Preprint arXiv:1209.4106.

[56] E. Looijenga, Root systems and elliptic curves, Invent. Math. **38** (1976/77), no. 1, 17 – 32.

[57] J. Lubin and J. Tate, Formal moduli for one-parameter formal Lie groups, Bull. Soc. Math. France **94** (1966), 49 – 59.

[58] Ju. I. Manin, On the theory of Abelian varieties over a field of finite characteristic, Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 281 – 292.

[59] Yu. I. Manin, Theory of commutative formal groups over fields of finite characteristic, Uspehi Mat. Nauk **18** (1963), no. 6 (114), 3 – 90.

[60] Yu. I. Manin, Three-dimensional hyperbolic geometry as ∞-adic Arakelov geometry, Invent. Math. **104** (1991), no. 2, 223 – 243.

[61] Yu. I. Manin and M. A. Tsfasman, Rational varieties: algebra, geometry, arithmetic, Uspekhi Mat. Nauk **41** (1986), no. 2(248), 43 – 94.

[62] J. P. McCabe, *P-ADIC THETA-FUNCTIONS*, ProQuest LLC, Ann Arbor, MI, 1968.

[63] V. B. Mehta and V. Srinivas, Varieties in positive characteristic with trivial tangent bundle, Compositio Math. **64** (1987), no. 2, 191 – 212.

[64] H. Morikawa, Theta functions and abelian varieties over valuation fields of rank one. I, Nagoya Math. J. **20** (1962), 1 – 27.

[65] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5 Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[66] D. Mumford, An algebraic surface with K ample, $(K^2) = 9$, $p_g = q = 0$, Amer. J. Math. **101** (1979), no. 1, 233 – 244.

[67] D. Mumford, An analytic construction of degenerating curves over complete local rings, Compositio Math. **24** (1972), 129 – 174.

[68] D. Mumford, An analytic construction of degenerating abelian varieties over complete rings, Compositio Math. **24** (1972), 239 – 272.

[69] D. Mumford, An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg deVries equation and related nonlinear equation, in *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, 115 – 153, Kinokuniya Book Store, Tokyo.

[70] D. Mumford, Pathologies of modular algebraic surfaces, Amer. J. Math. **83** (1961), 339 – 342.

[71] D. Mumford, Further pathologies in algebraic geometry, Amer. J. Math. **84** (1962), 642 – 648.

[72] J. P. Murre, Reduction of the proof of the non-rationality of a non-singular cubic threefold to a result of Mumford, Compositio Math. **27** (1973), 63 – 82.

[73] F. Oort, Newton polygons and formal groups: conjectures by Manin and Grothendieck, Ann. of Math. (2) **152** (2000), no. 1, 183 – 206.

[74] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, Galois theory of transcendental extensions and uniformization, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 671 – 704.

[75] F. Pop, Étale Galois covers of affine smooth curves. The geometric case of a conjecture of Shafarevich. On Abhyankar's conjecture, Invent. Math. **120** (1995), no. 3, 555 – 578.

[76] F. Pop, Pro-ℓ birational anabelian geometry over algebraically closed fields I, Preprint (2003), available at *http://www.math.upenn.edu/~pop/Research/Papers.html*.

[77] V. L. Popov, *Discrete complex reflection groups*, Communications of the Mathematical Institute, Rijksuniversiteit Utrecht, 15, Rijksuniv. Utrecht, Utrecht, 1982.

[78] Yu. G. Prokhorov, Fields of invariants of finite linear groups, in *Cohomological and geometric approaches to rationality problems*, 245 – 273, Progr. Math., 282 Birkhäuser Boston, Boston, MA.

[79] Yu. G. Prokhorov, Tr. Mat. Inst. Steklova **241** (2003), Teor. Chisel, Algebra i Algebr. Geom., 210 – 217; translation in Proc. Steklov Inst. Math. **2003**, no. 2 (241), 195 – 201.

[80] G. K. Sankaran, Abelian surfaces in toric 4-folds, Math. Ann. **313** (1999), no. 3, 409 – 427.

[81] J. Sawon, Abelian fibred holomorphic symplectic manifolds, Turkish J. Math. **27** (2003), no. 1, 197 – 230.

[82] J.-P. Serre, *Abelian l-adic representations and elliptic curves*, McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute W. A. Benjamin, Inc., New York, 1968.

[83] J.-P. Serre and J. Tate, Good reduction of abelian varieties, Ann. of Math. (2) **88** (1968), 492 – 517.

[84] I. R. Shafarevich, Principal homogeneous spaces defined over a function field, Trudy Mat. Inst. Steklov. **64** (1961), 316 – 346.

[85] I. R. Shafarevich, The Lüroth problem, Trudy Mat. Inst. Steklov. **183** (1990), 199 – 204, 229.

[86] J. Tate, Endomorphisms of abelian varieties over finite fields, Invent. Math. **2** (1966), 134 – 144.

[87] J. Tate, Genus change in inseparable extensions of function fields, Proc. Amer. Math. Soc. **3** (1952), 400 – 406.

[88] J. T. Tate, The arithmetic of elliptic curves, Invent. Math. **23** (1974), 179 – 206.

[89] B. Totaro, Hilbert's 14th problem over finite fields and a conjecture on the cone of curves, Compos. Math. **144** (2008), no. 5, 1176 – 1198.

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