VALUES OF DECOMPOSABLE FORMS AT S-INTEGER POINTS AND TORI ORBITS ON HOMOGENEOUS SPACES

GEORGE TOMANOV

Abstract. Let $\mathbf{G}$ be a reductive algebraic group defined over a number field K and $\mathcal{S}$ be a finite set of places of K containing the archimedean ones. Let $G = \prod_{v\in\mathcal{S}} \mathbf{G}(K_v)$ and Γ be an $\mathcal{S}$-arithmetic subgroup of G. Let $\mathcal{R} \subset \mathcal{S}$ and $T_{\mathcal{R}} = \prod_{v\in\mathcal{R}} T_v$ where T_v is a sub-torus of $\mathbf{G}(K_v)$ containing a maximal K_v-split torus. We prove that if G/Γ admits a closed $T_{\mathcal{R}}$-orbit then $\mathcal{R} = \mathcal{S}$ or $\mathcal{R}$ is a singleton. In addition, the closed $T_{\mathcal{R}}$-orbits are always "standard"; this generalizes the result of [To-W]. When $\#\mathcal{S} > 1$ it turns out that for $\mathcal{R} = \mathcal{S}$ there are no divergent orbits and for $\#\mathcal{R} = 1$ all closed orbits are divergent. As an application, we prove that if a collection of decomposable homogeneous forms $f_v \in K_v[x_1, \dots, x_n], v \in \mathcal{S}$, takes discrete values at $\mathcal{O}^n$, where $\mathcal{O}$ is the ring of $\mathcal{S}$-integers of K, then there exists an homogeneous form $g \in \mathcal{O}[x_1, \dots, x_n]$ such that $f_v = \alpha_v g$, $\alpha_v \in K_v^*$, for all $v \in \mathcal{S}$.

1. Introduction

Let $\mathbf{G}$ be a reductive algebraic group defined over a number field K and let $\mathcal{S}$ be a finite set of places of K containing the archimedean ones. Let $G_v = \mathbf{G}(K_v)$, where K_v is the completion of K with respect to $v \in \mathcal{S}$, and let $G = \prod_{v\in\mathcal{S}} G_v$ be a direct product of locally compact groups. The group $\mathbf{G}(K)$ is identified with its diagonal imbedding in G. We denote by Γ an $\mathcal{S}$-arithmetic subgroup of G, that is, Γ is a subgroup of $\mathbf{G}$ such that $\Gamma \cap \mathbf{G}(\mathcal{O})$ has finite index in both Γ and $\mathbf{G}(\mathcal{O})$, where $\mathcal{O}$ is the ring of $\mathcal{S}$-integers of K. For any subset $\mathcal{R}$ of $\mathcal{S}$ we let $\mathrm{rank}_{\mathcal{R}}\mathbf{G} \stackrel{def}{=} \sum_{v\in\mathcal{R}} \mathrm{rank}_{K_v}\mathbf{G}$ be the *$\mathcal{R}$-rank* of $\mathbf{G}$. (Recall that if F is a field containing K, $\mathrm{rank}_F\mathbf{G}$ is by definition the dimension of any maximal F-split torus of $\mathbf{G}$.) One of the goals of this paper is to describe the closed orbits under the action by left translations on G/Γ of tori of maximal $\mathcal{R}$-rank.

More specifically, for every $v \in \mathcal{S}$, let $\mathbf{T}_v \subset \mathbf{G}$ be a K_v-torus containing a maximal K_v-split torus of $\mathbf{G}$. We suppose that there is a maximal K-split torus $\mathbf{D}$ such that $\mathbf{T}_v \supset \mathbf{D}$ for all $v \in \mathcal{S}$. For every non-empty

$\mathcal{R} \subset \mathcal{S}$, we set $T_{\mathcal{R}} = \prod_{v\in\mathcal{R}} T_v$ and $D_{\mathcal{R}} = \prod_{v\in\mathcal{R}} D_v$, where $T_v = \mathbf{T}_v(K_v)$ and $D_v = \mathbf{D}(K_v)$. The *torus of maximal $\mathcal{R}$-rank $T_{\mathcal{R}}$* is identified with its projection in G and it acts on G/Γ by left translations

$$t\pi(g) = \pi(tg),$$

where $\pi : G \to G/\Gamma$ is the quotient map. An orbit $T_{\mathcal{R}}\pi(g)$ is called *divergent* if the orbit map $t \to t\pi(g)$ is proper, i.e. if $\{t_i\pi(g)\}$ leaves compacts of G/Γ whenever $\{t_i\}$ leaves compacts of $T_{\mathcal{R}}$. In particular, the divergent orbits are closed.

We prove the following:

Theorem 1.1. *Let* $\mathrm{rank}_{\mathcal{R}}\mathbf{G} > 0$ *and* $g \in G$.

(a) *The orbit $T_{\mathcal{R}}\pi(g)$ is closed if and only if $\mathcal{R} = \{v\}$ is a singleton and $T_v\pi(g)$ is divergent, or $\mathcal{R} = \mathcal{S}$ and*

$$g^{-1}T_{\mathcal{S}}g = C.L,$$

where C is a compact subgroup and $L = \mathbf{L}(K_{\mathcal{S}})$ with $\mathbf{L}$ a K-torus of $\mathbf{G}$;

(b) *The orbit $T_{\mathcal{R}}\pi(g)$ is divergent if and only if the following conditions are satisfied: $\mathcal{R} = \{v\}$, $\mathrm{rank}_{K_v}\mathbf{G} = \mathrm{rank}_K\mathbf{G}$ and*

$$g \in \mathcal{Z}_G(D_v)\mathbf{G}(K),$$

where D_v is identified with its natural projection in G and $\mathcal{Z}_G(D_v)$ is the centralizer of D_v in G.

Theorem 1.1 generalizes the following result by B.Weiss and the author, the second part of which has been earlier proved (though unpublished) by G.Margulis for $G = \mathrm{SL}_n(\mathbb{R})$ and $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ (cf. [To-W, Appendix]).

Theorem 1.2. ([To-W, Theorem 1.1]) *Let $\mathbf{G}$ be a reductive $\mathbb{Q}$-algebraic group, $\mathbf{T}$ an $\mathbb{R}$-torus containing a maximal $\mathbb{R}$-split torus, $T = \mathbf{T}(\mathbb{R})$ and let $x \in G$. Then:*

- *$T\pi(x)$ is a closed orbit if and only if $x^{-1}\mathbf{T}x$ is a product of a $\mathbb{Q}$-subtorus and an $\mathbb{R}$-anisotropic $\mathbb{R}$-subtorus;*
- *$T\pi(x)$ is a divergent orbit if and only if the maximal $\mathbb{R}$-split subtorus of $x^{-1}\mathbf{T}x$ is defined over $\mathbb{Q}$ and $\mathbb{Q}$-split.*

When $\#\mathcal{R} > 1$, Theorem 1.1 implies a specific phenomenon:

Corollary 1.3. *If $\#\mathcal{R} > 1$ and $T_{\mathcal{R}}\pi(g)$ is a closed orbit then either $\mathcal{R} = \mathcal{S}$ and $T_{\mathcal{R}}\pi(g)$ is never divergent, or $\mathcal{R} = \{v\}$ is a singleton and $T_v\pi(g)$ is always divergent.*

An orbit $T_{\mathcal{R}}\pi(g)$ is called *locally divergent* if $T_v\pi(g)$ is divergent for every $v \in \mathcal{R}$. Theorem 1.1 will be deduced from the next theorem about the locally divergent orbits.

Theorem 1.4. *Let* $\mathrm{rank}_{\mathcal{R}}(\mathbf{G}) > 0$. *Then the orbit* $T_{\mathcal{R}}\pi(g)$ *is closed and locally divergent if and only if the following conditions are fulfilled:*

(i) $\mathcal{R} = \mathcal{S}$ *or* $\mathcal{R}$ *is a singleton;*
(ii) $\mathrm{rank}_{\mathcal{R}}(\mathbf{G}) = \#\mathcal{R}\ \mathrm{rank}_K(\mathbf{G})$;
(iii) $g \in \mathcal{N}_G(D_{\mathcal{R}})\mathbf{G}(K)$, *where* $\mathcal{N}_G(D_{\mathcal{R}})$ *is the normalizer of* $D_{\mathcal{R}}$ *in* G.

When $\#\mathcal{R} = 1$ we can replace the normalizer $\mathcal{N}_G(D_{\mathcal{R}})$ in the formulation of Theorem 1.4 (iii) by the centralizer $\mathcal{Z}_G(D_{\mathcal{R}})$. This is not possible when $\mathcal{R} = \mathcal{S}$ (see §6).

As a consequence of Theorem 1.4, one can easily see that the locally divergent $T_{\mathcal{R}}$-orbits are also all "standard":

Corollary 1.5. *Let* $g \in G$. *The orbit* $T_{\mathcal{R}}\pi(g)$ *is locally divergent if and only if*
$$\mathrm{rank}_{\mathcal{R}}(\mathbf{G}) = \#\mathcal{R}\ \mathrm{rank}_K(\mathbf{G})$$
and
$$g \in \bigcap_{v \in \mathcal{R}} \mathcal{Z}_G(D_v)\mathbf{G}(K).$$

We also get the following result:

Corollary 1.6. (a) *If* $\mathrm{rank}_{\mathcal{R}}(\mathbf{G}) > \#\mathcal{R}\ \mathrm{rank}_K(\mathbf{G})$ *then there are no locally divergent orbits for* $T_{\mathcal{R}}$;

(b) *Let* $\mathbf{G}$ *be semisimple,* $\#\mathcal{R} > 1$ *and* $\mathrm{rank}_{\mathcal{R}}(\mathbf{G}) = \#\mathcal{R}\ \mathrm{rank}_K(\mathbf{G}) > 0$. *Then there exist locally divergent but non-closed orbits for* $T_{\mathcal{R}}$.

We apply Theorem 1.1 to obtain a characterization of the rational decomposable homogeneous forms in terms of their values at the integer points. Such forms appear in a very natural way in both the algebraic number theory and the Diophantine approximation of numbers. (See, for example, [Bor-Sha, ch.2] and [Mar, §2], respectively.)

We will first formulate our result in technically simpler particular cases. Given a commutative ring R, we denote by $R[\ \vec{x}\ ]$ the ring of polynomials in n variables $\vec{x} = (x_1, \dots, x_n)$.

Theorem 1.7. *Let* $f(\vec{x}) = l_1(\vec{x}) \dots l_m(\vec{x})$, *where* $l_1(\vec{x}), \dots, l_m(\vec{x}) \in \mathbb{R}[\ \vec{x}\ ]$ *are real linear forms. Suppose that* $l_1(\vec{x}), \dots, l_m(\vec{x})$ *are linearly independent over* $\mathbb{R}$ *and that the set* $f(\mathbb{Z}^n)$ *is discrete in* $\mathbb{R}$. *Then* $f(\vec{x}) = \alpha g(\vec{x})$, *where* $g(\vec{x}) \in \mathbb{Z}[\ \vec{x}\ ]$ *and* $\alpha \in \mathbb{R}^*$.

The hypotheses that the form $f(\vec{x})$ is decomposable and $l_1(\vec{x}),\dots,$ $l_m(\vec{x})$ are linearly independent over $\mathbb{R}$ are essential. (See §7 for simple examples.) Note that in some particular cases (as, for example, when $m = 2$) Theorem 1.7 can be easily proved without the use of Theorem 1.1. But, to the best of our knowledge, in order to tackle the general case the classical approaches are not efficient. Finally, remark that in view of [Bor-Sha, ch.2, Theorem 2], the form $g(\vec{x})$ in the formulation of the theorem is a constant multiple of a product of forms of the type $\mathrm{N}_{K/\mathbb{Q}}(x_1 + x_2\mu_2 + \ldots + x_n\mu_n)$, where $\mu_2,\dots,\mu_n$ are algebraic numbers generating a totally real number field K of degree n and $\mathrm{N}_{K/\mathbb{Q}}$ is the algebraic norm of K.

If f is a decomposable homogeneous form with complex coefficients and we are considering the values of f at the Gaussian integer vectors, we get:

Theorem 1.8. *Let $f(\vec{x}) = l_1(\vec{x})\dots l_m(\vec{x})$, where $l_1(\vec{x}),\dots,l_m(\vec{x}) \in \mathbb{C}[\ \vec{x}\ ]$ are complex linear forms. Suppose that $l_1(\vec{x}),\dots,l_m(\vec{x})$ are linearly independent over $\mathbb{C}$ and that the set $f(\mathbb{Z}[\,i\,]^n)$ is discrete in $\mathbb{C}$. Then $f(\vec{x}) = \alpha g(\vec{x})$, where $g(\vec{x}) \in \mathbb{Z}[\,i\,][\,\vec{x}\,]$ and $\alpha \in \mathbb{C}^*$.*

Let K, $\mathcal{S}$ and $\mathcal{O}$ be as above. For every $v \in \mathcal{S}$, let $f_v = l_1^{(v)}\dots l_m^{(v)} \in K_v[\,\vec{x}\,]$, where $l_1^{(v)},\dots,l_m^{(v)}$ are linearly independent over K_v linear forms in $K_v[\,\vec{x}\,]$. Denote by $K_{\mathcal{S}}$ the direct product of the topological fields K_v, $v \in \mathcal{S}$. Both Theorems 1.7 and 1.8 are particular cases for $K = \mathbb{Q}$ and $K = \mathbb{Q}(\,i\,)$, respectively, of the next general theorem:

Theorem 1.9. *With the above notation, assume that $\{(f_v(\vec{z}))_{v\in\mathcal{S}} \in K_{\mathcal{S}} | \vec{z} \in \mathcal{O}^n\}$ is a discrete subset of $K_{\mathcal{S}}$. Then there exist an homogeneous form g with coefficients from $\mathcal{O}$ and an element $(\alpha_v)_{v\in\mathcal{S}} \in K_{\mathcal{S}}^*$ such that $f_v = \alpha_v g$ for all $v \in \mathcal{S}$.*

In connection with Theorem 1.9 it seems natural to formulate the following conjecture which generalizes a known conjecture for the real forms f:

Conjecture. Let $f_v, v \in \mathcal{S}$, be as in the formulation of Theorem 1.9 with $n = m > 2$ or $n = m = 2$ and $\#\mathcal{S} > 1$. Additionally, assume that there exists a neighborhood W of 0 in $K_{\mathcal{S}}$ such that $(f_v(\vec{z}))_{v\in\mathcal{S}} \notin W$ for every $\vec{z} \in \mathcal{O}^n, \vec{z} \neq 0$. Then there exist an homogeneous form g with coefficients from $\mathcal{O}$ and an element $(\alpha_v)_{v\in\mathcal{S}} \in K_{\mathcal{S}}^*$ such that $f_v = \alpha_v g$ for all $v \in \mathcal{S}$.

The above conjecture can be reformulated in terms of Theorem 1.1 as follows: If $\mathbf{G} = \mathbf{SL}_n$ and $\mathrm{rank}_{\mathcal{S}}\mathbf{G} > 1$ then $T_{\mathcal{S}}\pi(g)$ is compact whenever $T_{\mathcal{S}}\pi(g)$ is relatively compact. In the case $n = 3$ and $K = \mathbb{Q}$ the conjecture implies (see [Mar, §2]) the notable Littlewood conjecture which states that

$$\liminf_{n\to\infty} n\langle n\alpha\rangle\langle n\beta\rangle = 0$$

for all $\alpha, \beta \in \mathbb{R}$, where $\langle x\rangle$ denotes the distance from x to $\mathbb{Z}$. In a recent paper [Ei-Ka-Li], M.Einsiedler, A.Katok and E.Lindenstrauss proved that the Littlewood conjecture fails at most on a set of Hausdorff dimension zero. Similar results in the p-adic setting have just appeared in the M.Einsiedler and D.Kleinbock paper [Ei-Kl].

The paper is organized as follows. The basic notation and terminology are introduced in §2. The proofs of the results about the structure of the closed tori orbits are given in §§3 - 6. Our starting point is the paper [To-W]. In §3, using [To-W], we prove an $\mathcal{S}$-adic compactness criterium in terms of intersections of so-called quasiballs with horospherical subsets. Proposition 4.3, proved in §4, plays a crucial role in revealing the dichotomy in Corollary 1.3. In §5 we describe the locally divergent orbits in terms of minimal parabolic K-algebras. In order to do this, we have to apply more intrinsic arguments than in [To-W]. For instance, the Galois type arguments in [To-W, §5] are replaced by Proposition 5.4 which, alone, presents some interest. Theorems 1.1, 1.4 and their corollaries are proved in §6. The proof of Theorem 1.9 is presented in §7. It uses Theorem 1.1 and the main result of M.Ratner's paper [Ra].

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2. Preliminaries: notation and terminology

2.1. **Numbers.** As usual $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ denote the complex, real, rational and integer numbers, respectively.

In this paper K denotes a number field, that is, a finite extension of $\mathbb{Q}$. For every place v of K we let K_v be the completion of K with respect to v and $|.|_v$ be the *normalized* valuation on K_v (see [Ca-F, ch.2, §7]). If v is a non-archimedean place then $\mathcal{O}_v = \{x \in K_v : |\ x\ |_v \leq 1\}$ is the ring of integers of K_v.

We fix a finite set $\mathcal{S}$ of places of K containing all archimedean places of K. The latter set is denoted by $\mathcal{S}_\infty$ or, simply, ∞, if this does not lead to confusion. We also put $\mathcal{S}_f = \mathcal{S} \setminus \mathcal{S}_\infty$.

We denote by $\mathcal{O}$ the ring of $\mathcal{S}$-integers of K, i.e., $\mathcal{O} = K \bigcap (\bigcap_{v\notin\mathcal{S}} \mathcal{O}_v)$.

For any non-empty subset $\mathcal{R}$ of $\mathcal{S}$, $K_{\mathcal{R}} \stackrel{def}{=} \prod_{v\in\mathcal{R}} K_v$ is a direct product of locally compact fields. Note that $K_{\mathcal{R}}$ is a topological ring and that the diagonal imbedding of K in $K_{\mathcal{R}}$ is dense. As usual, we denote by $K_{\mathcal{R}}^*$ the multiplicative group of all invertible elements in the ring $K_{\mathcal{R}}$.

2.2. **Norms.** Let $\mathbf{V}$ be a finite dimensional vector space defined over K. For every $\mathcal{R} \subset \mathcal{S}$ (respectively $v \in \mathcal{S}$) we write $V_{\mathcal{R}}$ for $\mathbf{V}(K_{\mathcal{R}})$ (respectively, V_v for $\mathbf{V}(K_v)$). Fixing a basis of K-rational vectors $e_1, \dots, e_n$, for every K-algebra A, we identify $\mathbf{V}(A)$ with A^n. For every $v \in \mathcal{S}$ we define a *normalized* norm $\|\cdot\|_v$ on V_v as follows. If v is real (respectively, complex) then $\|\cdot\|_v$ is the standard norm on $\mathbb{R}^n$ (respectively, the square of the standard norm on $\mathbb{C}^n$). If v is non-archimedean, then $\|\cdot\|_v$ is defined by $\|\mathbf{x}\|_v = \max_i |x_i|_v$, where $(x_1, \dots, x_n)$ are the coordinates of the vector $\mathbf{x} \in V_v$ with respect to the bases $e_1, \dots, e_n$.

For $\mathbf{x} = (\mathbf{x}^{(v)})_{v\in\mathcal{S}}$ in $V_{\mathcal{R}}$ we define the norm of $\mathbf{x}$ as

$$\|\mathbf{x}\|_{\mathcal{R}} = \max_{v\in\mathcal{R}} \|\mathbf{x}^{(v)}\|_v.$$

Also, if $\mathcal{R} = \mathcal{S}$ we define the *content* of $\mathbf{x}$ as

$$\mathbf{c}_{\mathcal{S}}(\mathbf{x}) = \prod_{v\in\mathcal{S}} \|\mathbf{x}^{(v)}\|_v.$$

Since all our norms are normalized and $\prod_{v\in\mathcal{S}} |\xi|_v = 1$ for every $\xi \in \mathcal{O}^*$ [Ca-F, ch.2, Theorem 12.1], we have that

$$\mathbf{c}_{\mathcal{S}}(\mathbf{x}) = \mathbf{c}_{\mathcal{S}}(\xi\mathbf{x}), \forall \xi \in \mathcal{O}^*. \tag{1}$$

By a *pseudoball* in $V_{\mathcal{S}}$ of radius $r > 0$ centered at 0 we mean the set $\mathcal{B}_{\mathcal{S}}(r) = \{\mathbf{x} \in V_{\mathcal{S}} | \mathbf{c}_{\mathcal{S}}(\mathbf{x}) < r\}$. We preserve the notation $B_{\mathcal{S}}(r)$ to denote the usual ball in $V_{\mathcal{S}}$ of radius r centered at 0 with respect to the norm $\|.\|_{\mathcal{S}}$.

2.3. **K-algebraic groups and their Lie algebras.** We use boldface upper case letters to denote the algebraic groups and boldface lower case Gothic letters to denote their Lie algebras.

In this paper $\mathbf{G}$ is a reductive algebraic group defined over K (or, shortly, $\mathbf{G}$ is a reductive K-algebraic group). Recall that the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ is equipped with a K-structure compatible with the K-structure of $\mathbf{G}$ [Bo1, Theorem 3.4].

Given $\mathcal{R} \subset \mathcal{S}$, we usually denote $G_{\mathcal{R}} \stackrel{def}{=} \mathbf{G}(K_{\mathcal{R}})$ and $\mathfrak{g}_{\mathcal{R}} \stackrel{def}{=} \mathfrak{g}(K_{\mathcal{R}})$. The group $G_{\mathcal{R}}$ (respectively, its Lie algebra $\mathfrak{g}_{\mathcal{R}}$) is identified with the direct product $\prod_{v\in\mathcal{R}} G_v$ (respectively, $\prod_{v\in\mathcal{R}} \mathfrak{g}_v$), where $G_v \stackrel{def}{=} \mathbf{G}(K_v)$

(respectively, $\mathfrak{g}_v \overset{def}{=} \mathfrak{g}(K_v)$). If $\mathcal{R} = \mathcal{S}$ we write G (respectively, $\mathfrak{g}$) for $G_{\mathcal{S}}$ (respectively, for $\mathfrak{g}_{\mathcal{S}}$).

We will use the notation $\mathrm{pr}_{\mathcal{R}}$ to denote both the natural projections $G \to G_{\mathcal{R}}$ and $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$. (The exact use of pr_∞ will follow from the context.)

On every G_v we have a *Zariski topology* induced by the Zariski topology on $\mathbf{G}$ and a *Hausdorff topology* induced by the locally compact topology on K_v. The formal product of the Zariski (respectively, Hausdorff) topologies on G_v, $v \in \mathcal{R}$, is the Zariski (respectively, Hausdorff) topology on $G_{\mathcal{R}}$. In order to distinguish the two topologies, all topological notions connected with the first one will be used with the prefix "Zariski".

An element $g = (g_v)_{v\in\mathcal{R}} \in G_{\mathcal{R}}$ is called *unipotent* (respectively, *semisimple*) if each v-component g_v of g is unipotent (respectively, semisimple). A subgroup U of $G_{\mathcal{R}}$ is called unipotent if it consists of unipotent elements. A subalgebra $\mathfrak{u}$ of $\mathfrak{g}_{\mathcal{R}}$ is *unipotent* if it corresponds to a Zariski closed unipotent subgroup U of $G_{\mathcal{R}}$, i.e. if there exists a subgroup $U \subset G_{\mathcal{R}}$ such that $U = \prod_{v\in\mathcal{R}} U_v$, each U_v is Zariski closed in G_v, and $\mathfrak{u} = \prod_{v\in\mathcal{R}} \mathfrak{u}_v$ where $\mathfrak{u}_v$ is the Lie algebra of U_v.

If $\mathbf{H}$ is a K-subgroup of $\mathbf{G}$ then $R_u(\mathbf{H})$ denotes the unipotent radical of $\mathbf{H}$. The *unipotent radical of the Lie algebra* $\mathfrak{h}$ is by definition the Lie algebra of $R_u(\mathbf{H})$.

If P is a subgroup of G then $\mathcal{N}_G(P)$ (respectively, $\mathcal{Z}_G(P)$) denotes the normalizer (respectively, the centralizer) of P in G.

For any non-empty $\mathcal{R} \subset \mathcal{S}$ the adjoint representation $\mathrm{Ad}_{\mathcal{R}} : G_{\mathcal{R}} \to \mathrm{GL}(\mathfrak{g}_{\mathcal{R}})$, where $\mathrm{GL}(\mathfrak{g}_{\mathcal{R}}) = \prod_{v\in\mathcal{R}} \mathrm{GL}(\mathfrak{g}_v)$, is the direct product of the adjoint representations $\mathrm{Ad}_v : G_v \to \mathrm{GL}(\mathfrak{g}_v)$, $v \in \mathcal{R}$. We will use the notation Ad (respectively, Ad_∞) when $\mathcal{R} = \mathcal{S}$ (respectively, $\mathcal{R} = \mathcal{S}_\infty$).

2.4. **$\mathcal{S}$-arithmetic subgroups.** Recall that Γ is an $\mathcal{S}$-*arithmetic* subgroup of G if the group $\Gamma \cap \mathbf{G}(\mathcal{O})$ has finite index in both Γ and $\mathbf{G}(\mathcal{O})$. We assume that $\mathbf{G}$ is imbedded in $\mathbf{SL}_n$ in such a way that $\mathbf{G}(\mathcal{O}) = \mathbf{SL}_n(\mathcal{O}) \cap \mathbf{G}$ and $\mathfrak{g}(\mathcal{O}) = \mathfrak{sl}_n(\mathcal{O}) \cap \mathfrak{g}$. In particular, $\mathfrak{g}(\mathcal{O})$ *is invariant under the adjoint action of* $\mathbf{G}(\mathcal{O})$. Let Γ' be a subgroup of finite index in Γ and let $\phi : G/\Gamma' \to G/\Gamma$ be the natural map. Since ϕ is a proper map it is easy to see that Theorems 1.1, 1.4 and their corollaries are valid for Γ if and only if they are valid for Γ'. Therefore, we may suppose *without* loss of generality that $\Gamma = \mathbf{G}(\mathcal{O})$.

Let $\pi : G \to G/\Gamma$ be the natural projection. For every $x \in G/\Gamma$ we introduce the following notation. If $x = \pi(g)$, $g \in G$, we denote

$$\mathfrak{g}_x = \mathrm{Ad}(g)\mathfrak{g}(\mathcal{O}).$$

Since $\mathfrak{g}(\mathcal{O})$ is $\mathrm{Ad}(\Gamma)$-invariant, $\mathfrak{g}_x$ does not depend on the choice of the element g.

3. Compactness criteria in $\mathcal{S}$-adic setting

3.1. **$\mathcal{S}$-adic Mahler's criterion.** Let $G = \mathrm{SL}_n(K_{\mathcal{S}})$, $\Gamma = \mathrm{SL}_n(\mathcal{O})$ and $\pi : G \to G/\Gamma$ be the natural projection. The group G is acting naturally on $K_{\mathcal{S}}^n$ and Γ is the stabilizer of $\mathcal{O}^n$ in G. If $r > 0$ then $B_{\mathcal{S}}(r)$ (resp., $\mathcal{B}_{\mathcal{S}}(r)$) is the ball (resp. pseudoball) in $K_{\mathcal{S}}^n$ centered in 0 and with radius r (see §2.3).

We have

Theorem 3.1. (Mahler′s criterion) *With the above notation, given a subset $M \subset G$ the following conditions are equivalent:*

(i) *$\pi(M)$ is relatively compact in G/Γ;*
(ii) *There exists $r > 0$ such that $g\mathcal{O}^n \cap \mathcal{B}_{\mathcal{S}}(r) = \{0\}$ for all $g \in M$;*
(iii) *There exists $r > 0$ such that $g\mathcal{O}^n \cap B_{\mathcal{S}}(r) = \{0\}$ for all $g \in M$.*

The equivalence between (i) and (iii) is proved in [Kl-To, Theorem 5.12] and it is obvious that (ii) implies (iii). In order to prove that (iii) implies (ii) note that, in view of the formula (1), every $\mathcal{B}_{\mathcal{S}}(r)$ is invariant under the multiplication by elements from $\mathcal{O}^*$. Now the implication easily follows from the following lemma:

Lemma 3.2. (cf. [Kl-To, Lemma 5.10]) *There exists a constant $\kappa > 1$ with the following property. Let $\boldsymbol{x} = (\boldsymbol{x}^{(v)})_{v\in\mathcal{S}} \in K_{\mathcal{S}}^n$ be such that $\boldsymbol{x}^{(v)} \neq 0$ for all $v \in \mathcal{S}$. For each $v \in \mathcal{S}$ we choose a positive real number a_v in such a way that $\mathbf{c}_{\mathcal{S}}(\boldsymbol{x}) = \prod_{v\in\mathcal{S}} a_v$. Then there exists $\xi \in \mathcal{O}^*$ such that*

$$\frac{a_v}{\kappa} \leq \|\xi \boldsymbol{x}^{(v)}\|_v \leq \kappa a_v \tag{2}$$

for all $v \in \mathcal{S}$. In particular, for every $\boldsymbol{x}$ as above there exists $\xi \in \mathcal{O}^*$ such that

$$\frac{\mathbf{c}_{\mathcal{S}}(\boldsymbol{x})^{1/m}}{\kappa} \leq \|\xi \boldsymbol{x}\|_{\mathcal{S}} \leq \kappa \mathbf{c}_{\mathcal{S}}(\boldsymbol{x})^{1/m}, \tag{3}$$

where $m = \#\mathcal{S}$.

Proof. Let $K_{\mathcal{S}}^1 = \{y = (y^{(v)}) \in K_{\mathcal{S}}^* | \prod_{v\in\mathcal{S}} |y^{(v)}|_v = 1\}$. Then $\mathcal{O}^* \subset K_{\mathcal{S}}^1$ and $K_{\mathcal{S}}^1/\mathcal{O}^*$ is compact [Ca-F, ch.2, Theorem 16.1]. Therefore there exists a constant $\kappa_0 > 1$ such that for every $y = (y^{(v)}) \in K_{\mathcal{S}}^1$ there exists $\xi \in \mathcal{O}^*$ such that

$$\frac{1}{\kappa_0} \leq |\xi y^{(v)}|_v \leq \kappa_0, \forall v \in \mathcal{S}. \tag{4}$$

Let $\mathbf{x}$ and $a_v, v \in \mathcal{S}$, be as in the formulation of the proposition. There exists a constant $c > 1$, depending only on $\mathcal{S}$, such that for every $v \in \mathcal{S}$ there exists $\alpha^{(v)} \in K_v^*$ with

$$\frac{c}{|\alpha^{(v)}|_v} \leq a_v \leq c|\alpha^{(v)}|_v \tag{5}$$

and $\prod_{v\in\mathcal{S}} |\alpha^{(v)}|_v = \prod_{v\in\mathcal{S}} a_v$. So, $\mathbf{c}_{\mathcal{S}}(\alpha^{-1}\mathbf{x}) = 1$ where $\alpha = (\alpha^{(v)})_{v\in\mathcal{S}} \in K_{\mathcal{S}}^*$. Put $\kappa = \kappa_0 c$. In view of (4) and (5) there exists $\xi \in \mathcal{O}^*$ such that

$$\frac{|\alpha^{(v)}|_v}{\kappa} \leq |\xi\mathbf{x}^{(v)}|_v \leq \kappa|\alpha^{(v)}|_v, \forall v \in \mathcal{S},$$

which proves (2).

In order to prove (3) it is enough to apply (2) with $a_v = \mathbf{c}_{\mathcal{S}}(\mathbf{x})^{1/n}$. $\square$

3.2. **Horospherical subsets.** We need to prove a compactness criterion which reflects the group structure of $\mathbf{G}$.

We generalize the notion of horospherical subset from [To-W, Definition 3.4].

Definition 3.3. Let $\mathcal{R} \subset \mathcal{S}$. A finite subset $\mathcal{M}$ of $\mathfrak{g}_{\mathcal{R}}$ is called $\mathcal{R}$-*horospherical* (or, simply, horospherical when $\mathcal{R}$ is implicit) if $\mathcal{M} = \mathrm{pr}_{\mathcal{R}}(\mathrm{Ad}(g)(\mathcal{M}_0))$, where $g \in G$ and $\mathcal{M}_0$ is a subset of $\mathbf{\mathfrak{g}}(\mathcal{O})$ which spans linearly the unipotent radical of a maximal parabolic K-subalgebra of $\mathbf{\mathfrak{g}}$.

The next proposition provides a compactness criterion in terms of the intersection of pseudo-balls (and balls) in $\mathfrak{g}$ with $\mathfrak{g}_x$, $x \in G/\Gamma$ (see 2.1 for the notation). It generalizes [To-W, Propositions 3.3 and 3.5].

Proposition 3.4. *Assume that* $\mathbf{G}$ *is a semisimple algebraic group. Then the following assertions hold:*

(a) *There exists* $r > 0$ (*respectively,* $t > 0$) *such that for any* $x = \pi(g)$ *the subalgebra of* $\mathfrak{g}$ *spanned by* $\mathcal{B}_{\mathcal{S}}(r) \cap \mathfrak{g}_x$ (*respectively,* $B_{\mathcal{S}}(t) \cap \mathfrak{g}_x$) *is unipotent;*

(b) (**Compactness Criterion**) *A subset* M *of* G/Γ *is relatively compact if and only if there exists* $r > 0$ (*respectively,* $t > 0$) *such that* $\mathcal{B}_{\mathcal{S}}(r) \cap \mathfrak{g}_x$ (*respectively,* $B_{\mathcal{S}}(t) \cap \mathfrak{g}_x$) *does not contain a horospherical subset for any* $x \in M$.

3.3. **Proof of Proposition 3.4.** For every $t > 0$ we let $r = \left(\frac{t}{\kappa}\right)^m$, where κ and m are as in the formulation of Lemma 3.2. It follows from Lemma 3.2 that

$$B_{\mathcal{S}}(t/\kappa) \subset \mathcal{B}_{\mathcal{S}}(r) \subset \mathcal{O}^* B_{\mathcal{S}}(t).$$

Now the validity of the proposition for the balls $B_{\mathcal{S}}(t)$ implies easily its validity for the pseudoballs $\mathcal{B}_{\mathcal{S}}(r)$.

Further on, the proof of the proposition breaks in two cases. (In view of 2.4, we will assume that $\Gamma = \mathbf{G}(\mathcal{O})$.)

3.3.1. *The case $\mathcal{S} = \mathcal{S}_\infty$.* Let $\mathrm{R}_{\mathrm{K}/\mathbb{Q}}$ be the Weil restriction of scalars functor. Then $\mathbf{H} = \mathrm{R}_{\mathrm{K}/\mathbb{Q}}(\mathbf{G})$ is a semisimple $\mathbb{Q}$-algebraic group and $\boldsymbol{\mathfrak{h}} = \mathrm{R}_{\mathrm{K}/\mathbb{Q}}(\boldsymbol{\mathfrak{g}})$ is its $\mathbb{Q}$-Lie algebra. Denote $\Delta = \mathbf{H}(\mathbb{Z})$, $H = \mathbf{H}(\mathbb{R})$ and $\mathfrak{h} = \boldsymbol{\mathfrak{h}}(\mathbb{R})$. The following properties of the functor $\mathrm{R}_{\mathrm{K}/\mathbb{Q}}$ are well known and easily follow from its definition (see, for example, [Pl-R, ch.2, §2.1.1]). There exist continuous isomorphisms $\mu : G \to H$ and $\nu : \mathfrak{g} \to \mathfrak{h}$ such that $\mu(\Gamma) = \Delta$, $\nu(\boldsymbol{\mathfrak{g}}(\mathcal{O})) = \boldsymbol{\mathfrak{h}}(\mathbb{Z})$ and

$$\nu(\mathrm{Ad}_G(g)x) = \mathrm{Ad}_H(\mu(g))\nu(x)$$

for all $g \in G$ and $x \in \mathfrak{g}$. Moreover, ν maps bijectively the family of the horospherical subsets of $\mathfrak{g}$ to the family of the horospherical subsets of $\mathfrak{h}$ and μ induces an homeomorphism $G/\Gamma \to H/\Delta$. Hence, when $\mathcal{S} = \mathcal{S}_\infty$ the proposition follows from the case $K = \mathbb{Q}$ considered in [To-W, Propositons 3.3 and 3.5].

3.3.2. *The case $\mathcal{S} \supsetneq \mathcal{S}_\infty$.* We introduce the topological rings $\mathcal{O}_f \overset{def}{=} \prod_{v \in \mathcal{S}_f} \mathcal{O}_v$ and $K_f \overset{def}{=} K_\infty \times \mathcal{O}_f$ (see 2.1). So, $\mathcal{O}_\infty = \mathcal{O} \cap (K_\infty \times \mathcal{O}_f)$ is the ring of integers of K.

If $\widetilde{\mathbf{G}}$ is the simply connected covering of the algebraic group $\mathbf{G}$ then $\widetilde{G}/\widetilde{\Gamma}$ is naturally homeomorphic to G/Γ, where $\widetilde{G} = \widetilde{\mathbf{G}}(K_{\mathcal{S}})$ and $\widetilde{\Gamma} = \widetilde{\mathbf{G}}(\mathcal{O})$. In view of this and of Theorem 4.1 below, we may (and will) assume without loss of generality that $\mathbf{G}$ is simply connected and without K-anisotropic factors. Then the diagonal imbedding of Γ into $\prod_{v \in \mathcal{S}_f} \mathbf{G}(K_v)$ is dense. (This fact follows immediately from the strong approximation theorem [Pl-R, Theorem 7.12].) Therefore

$$G = \mathbf{G}(K_f)\Gamma.$$

Every $g \in G$ can be writhen in the following way

$$g = g_\infty g_f \gamma, \tag{6}$$

where $g_\infty \in G_\infty, g_f \in \mathbf{G}(\mathcal{O}_f)$ and $\gamma \in \Gamma$. Let $\Gamma_\infty = \mathbf{G}(K_f) \cap \Gamma$. Then G/Γ is homeomorphic to $\mathbf{G}(K_f)/\Gamma_\infty$ and the projection of $\mathbf{G}(K_f)$ on G_∞ yields the following map

$$\varphi : G/\Gamma \to G_\infty/\Gamma_\infty,\ \varphi(\pi(g)) \overset{def}{=} \pi_\infty(g_\infty), \forall g \in G,$$

where $\pi_\infty : G_\infty \to G_\infty/\Gamma$ is the natural map. In view of the compactness of $\mathbf{G}(\mathcal{O}_f)$, φ is a proper continuous map.

Let A be a subset of $\mathfrak{g}_\infty$ and $x = \pi(g)$ for some $g \in G$. Set $A_f = A \times \mathfrak{g}(\mathcal{O}_f)$. Using (6) and the fact that $\mathfrak{g}(\mathcal{O})$ is invariant under the adjoint action of Γ, we obtain

$$\begin{aligned} &\mathrm{pr}_\infty\big(\mathfrak{g}_x \cap A_f\big) = \mathrm{pr}_\infty\Big(\mathrm{Ad}_{\mathcal{S}}(g)\big(\mathfrak{g}(\mathcal{O})\big) \cap A_f\Big) = \\ &\mathrm{pr}_\infty\Big(\mathrm{Ad}_{\mathcal{S}}(g_\infty g_f)\big(\mathfrak{g}(\mathcal{O}) \cap A_f\big)\Big) = \mathfrak{g}_{\infty,y} \cap A, \end{aligned} \tag{7}$$

where $y = \varphi(x)$ and $\mathfrak{g}_{\infty,y} = \mathrm{Ad}_\infty(g_\infty)\mathfrak{g}(\mathcal{O}_\infty)$. (Recall that pr_∞ denotes the natural projection $\mathfrak{g} \to \mathfrak{g}_\infty$.)

Let $\tilde{B}(t) = B_\infty(t) \times \mathfrak{g}(\mathcal{O}_f)$. Applying (7) with $A = B_\infty(t)$, we get

$$\mathrm{pr}_\infty\big(\mathfrak{g}_x \cap \tilde{B}(t)\big) = \mathfrak{g}_{\infty,y} \cap B_\infty(t).$$

Since the restriction of pr_∞ to $\mathfrak{g}_x$ is injective, we obtain that the subalgebra spanned by $\mathfrak{g}_x \cap \tilde{B}(t)$ is unipotent if and only if the subalgebra spanned by $\mathfrak{g}_{\infty,y} \cap B_\infty(t)$ is unipotent. This, in view of 3.3.1, proves (a).

Let us prove (b). If M is compact, it follows from the continuity of the adjoint action that if $t > 0$ is sufficiently small then $B_{\mathcal{S}}(t) \cap \mathfrak{g}_x$ does not contain horospherical subsets for all $x \in G/\Gamma$. In order to prove the inverse implication, let $M \subset G/\Gamma$ and $t > 0$ be such that $B_{\mathcal{S}}(t) \cap \mathfrak{g}_x$ does not contain horospherical subsets for any $x \in M$. Assume the contrary, that is, that there exists a divergent sequence $\{x_i\}$ of elements in M. Then the sequence $\{y_i = \varphi(x_i)\}$ is divergent in G_∞/Γ_∞ (because φ is proper). Since the proposition is true for G_∞/Γ_∞, for every $\varepsilon > 0$ there exists $i \gg 0$ such that $B_\infty(\varepsilon) \cap \mathfrak{g}_{\infty,y_i}$ contains a horospherical subset. Set $\tilde{B}(\varepsilon) = B_\infty(\varepsilon) \times \mathfrak{g}(\mathcal{O}_f)$. By (7) (applied with $A = B_\infty(\varepsilon)$) and the injectivity of the restriction of pr_∞ to $\mathfrak{g}_x$, we obtain that $\tilde{B}(\varepsilon) \cap \mathfrak{g}_{x_i}$ contains a horospherical subset. Now, using Lemma 3.2, we conclude that $B_{\mathcal{S}}(t) \cap \mathfrak{g}_{x_i}$ contains horospherical subsets for all sufficiently large i. Contradiction. □

3.4. **Expanding transformations.** For every $v \in \mathcal{S}$, we fix a maximal K_v-split torus $\mathbf{T}_v$ of $\mathbf{G}$. We denote $T_v = \mathbf{T}_v(K_v)$ and $T_{\mathcal{R}} = \prod_{v \in \mathcal{R}} T_v$ where $\mathcal{R}$ is a non empty subset of $\mathcal{S}$.

Proposition 3.5. *With the above notation, for every real $\tau > 1$ there exists a finite set $t_1, \dots, t_s$ of elements in $T_{\mathcal{R}}$ such that if $\mathfrak{u}$ is a unipotent subalgebra of $\mathfrak{g}_{\mathcal{R}}$ then there exists an element t_i such that*

$$\|\mathrm{Ad}(t_i)(\mathbf{x})\|_{\mathcal{R}} \geq \tau \|\mathbf{x}\|_{\mathcal{R}} \tag{8}$$

for all $\mathbf{x} \in \mathfrak{u}$.

Proof. It is easy to see that it is enough to prove the proposition when $\mathcal{R}$ is a singleton. Let $\mathcal{R} = \{v\}$. If v is real then the proposition is proved in [To-W, Proposition 4.1]. Here we present a shorter proof for an arbitrary v.

Let $\mathfrak{u}_v^+$ and $\mathfrak{u}_v^-$ be invariant under the adjoint action of T_v maximal unipotent subalgebras of $\mathfrak{g}_v$ which are opposite to each other. Then

$$C_v = \{d \in T_v | \lim_{n\to+\infty} \mathrm{Ad}(d^n)x = \infty, \forall x \in \mathfrak{u}_v^+\} \tag{9}$$

is the interior of the Weil chamber corresponding to $\mathfrak{u}_v^+$ (see [Bo1]). Denote by U_v^+ and U_v^- the unipotent subgroups of G_v with Lie algebras $\mathfrak{u}_v^+$ and $\mathfrak{u}_v^-$, respectively.

Now let $\mathfrak{u}_v$ be any maximal unipotent subalgebra of $\mathfrak{g}_v$. There exists $g \in G_v$ such that $\mathrm{Ad}(g)\mathfrak{u}_v^+ = \mathfrak{u}_v$. By Bruhat decomposition $g = a\omega b$, where $\omega \in \mathcal{N}_{G_v}(T_v)$, a and $b \in U_v^+$ and $\omega^{-1}a\omega \in U_v^-$. We can write $\mathfrak{u}_v = \mathrm{Ad}(\omega a^-)\mathfrak{u}_v^+$, where $a^- = \omega^{-1}a\omega$. Let $x \in \mathfrak{u}_v^+$ and $f_v \in C_v$. We put $y = \mathrm{Ad}(\omega a^-)x$ and $d_v = \omega f_v \omega^{-1}$. Using (9) and the fact that $\lim_{n\to+\infty} f_v^n a^- f_v^{-n} = 0$, we get

$$\lim_{n\to+\infty} \mathrm{Ad}(d_v^n)y = \lim_{n\to+\infty} \mathrm{Ad}(\omega(f_v^n a^- f_v^{-n})) \circ \mathrm{Ad}(f_v^n)(x) = \infty.$$

Therefore, taking $t = d^n$ with n sufficiently large, we obtain that

$$\|\mathrm{Ad}(t)z\|_v > \tau\|z\|_v \tag{10}$$

for all non-zero $z \in \mathfrak{u}_v$.

Since the stabilizer of every maximal unipotent subalgebra is a minimal parabolic subgroup and all minimal parabolic subgroups are conjugated, the set of all maximal unipotent subalgebras can be identified with the compact homogeneous space G_v/P_v^+, where P_v^+ is the parabolic subgroup of G_v with Lie algebra $\mathfrak{u}_v^+$. It is easy to see that (10) is true for all subalgebras in a neighborhood of $\mathfrak{u}_v$. Now the existence of the elements $t_1, \dots, t_s$ as in the formulation of the theorem follows from the compactness of G_v/P_v^+ by a standard argument. $\square$

4. Closed orbits of reductive K-groups

4.1. **Reductive groups.** Recall the $\mathcal{S}$-adic version of a well-known theorem of Borel and Harish-Chandra. (As usual, $G = \mathbf{G}(K_{\mathcal{S}})$ and $\Gamma = \mathbf{G}(\mathcal{O})$.)

Theorem 4.1. (*cf.*[Pl-R, Theorem 5.7]) *Let* $\mathbf{G}$ *be a reductive* K*-group and let* $\mathrm{X}_K(\mathbf{G})$ *be the group of* K*-rational characters of* $\mathbf{G}$. *Then*

(a) G/Γ *has a finite invariant volume if and only if* $\mathrm{X}_K(\mathbf{G}) = \{1\}$;
(b) G/Γ *is compact if and only if* $\mathbf{G}$ *is anisotropic over* K.

Because of the lack of appropriate reference we will prove the following known proposition.

Proposition 4.2. *With the above notation, let* $\mathbf{H}$ *be a reductive subgroup of* $\mathbf{G}$ *defined over* K *and* $H = \mathbf{H}(K_{\mathcal{S}})$. *Then* $H\pi(e)$ *is closed in* G/Γ.

Proof. Using the Weil restriction of scalars, one can reduce the proof to the case when $K = \mathbb{Q}$. In view of [Bo2, Proposition 7.7] there exists a $\mathbb{Q}$-rational action of $\mathbf{G}$ on an affine $\mathbb{Q}$-variety $\mathbf{V}$ admitting an element $a \in \mathbf{V}(\mathbb{Z})$ such that $\mathbf{H} = \{g \in \mathbf{G} | ga = a\}$. Since the map $\mathbf{G} \to \mathbf{V}$, $g \to ga$, is polynomial with rational coefficients, there exists a non-zero integer n such that $\gamma na \in \mathbf{V}(\mathcal{O})$ for all $\gamma \in \Gamma$. Therefore ΓH is closed in G, equivalently, $H\pi(e)$ is closed. $\square$

4.2. **Algebraic tori.** We will need the following

Proposition 4.3. *Let* $\mathbf{T}$ *be a* K*-torus in* $\mathbf{G}$ *and let* $\mathcal{R}$ *be a non-empty subset of* $\mathcal{S}$. *Suppose that* $T_{\mathcal{R}}$ *is not compact. Then the orbit* $T_{\mathcal{R}}\pi(e)$ *is divergent if and only if the following conditions are fulfilled:*

(i) $\mathcal{R} = \{v_\circ\}$ *is a singleton, and*

(ii) $\operatorname{rank}_K \mathbf{T} = \operatorname{rank}_{K_{v_\circ}} \mathbf{T} > 0$.

Proof. In view of Proposition 4.2 the orbit $\mathbf{T}(K_{\mathcal{S}})\pi(e)$ is closed and, therefore, homeomorphic to $\mathbf{T}(K_{\mathcal{S}})/(\mathbf{T}(K_{\mathcal{S}}) \cap \Gamma)$. So, we may suppose, with no loss or generality, that $\mathbf{T} = \mathbf{G}$.

Assume that the orbit $T_{\mathcal{R}}\pi(e)$ is divergent. Let $\mathbf{T}_a$ (respectively, $\mathbf{T}_d$) be the largest K-anisotropic (respectively, split over K) subtorus of $\mathbf{T}$. It is well known that $\mathbf{T}$ is an almost direct product of $\mathbf{T}_a$ and $\mathbf{T}_d$. This implies that if there exists $v \in \mathcal{R}$ such that $\operatorname{rank}_{K_v} \mathbf{T} > \operatorname{rank}_K \mathbf{T}$ then $\mathbf{T}_a(K_{\mathcal{R}})$ is not compact. But $\mathbf{T}_a(K_{\mathcal{S}})\pi(e)$ is compact (Theorem 4.1). Therefore, $T_{\mathcal{R}}\pi(e)$ can not be divergent, a contradiction. So, $\operatorname{rank}_{K_v} \mathbf{T} = \operatorname{rank}_K \mathbf{T}$ for all $v \in \mathcal{R}$. In this case $\mathbf{T}_a(K_{\mathcal{R}})$ is compact and, since $T_{\mathcal{R}}$ is not compact, $\mathbf{T}_d$ is not trivial. Note that $T_{\mathcal{R}}\pi(e)$ is divergent if and only if $\mathbf{T}_d(K_{\mathcal{R}})\pi(e)$ is divergent.

In order to prove (i) consider the character group $\mathrm{X}_K(\mathbf{T})$ of $\mathbf{T}$. It is well known that $\mathrm{X}_K(\mathbf{T})$ is a free $\mathbb{Z}$-module of rank equal to $\dim \mathbf{T}_d$ (cf. [Bo1, 8.15]). Let $\chi_1, \dots, \chi_r$ be a basis of $\mathrm{X}_K(\mathbf{T})$. Define a homomorphism of K-algebraic groups $\chi = (\chi_1, \dots, \chi_r) : \mathbf{T} \to \mathbf{G}_m^r$, where $\mathbf{G}_m$ denotes the one-dimensional K-split torus. Let $T = \mathbf{T}(K_{\mathcal{S}})$ and $T_\circ = \{(t_v)_{v \in \mathcal{S}} \in T | \prod_{v \in \mathcal{S}} |\chi_i(t_v)|_v = 1 \text{ for all } i\}$. It follows from [Ca-F, ch.2, Theorem 16.1] that Γ is a co-compact lattice in $T_\circ$. Set $\varphi : T \to \mathbb{R}^r$, $\varphi((t_v)_{v \in \mathcal{S}}) = \big(\log(\prod_{v \in \mathcal{S}} |\chi_1(t_v)|_v), \dots, \log(\prod_{v \in \mathcal{S}} |\chi_r(t_v)|_v)\big)$. It is clear that φ is a continuous surjective homomorphism of locally compact topological groups with $\ker(\varphi) = T_\circ$. Since $T_\circ/\Gamma$ is compact, φ induces a proper homomorphism $\psi : T/\Gamma \to T/T_\circ$. Now let $\mathcal{R}$ contain two different valuations v_1 and v_2. It is easy to find sequences $\{a_i\}$ in $K_{v_1}^*$ and $\{b_i\}$ in $K_{v_2}^*$ such that $\log|a_i|_{v_1} \to +\infty$, $\log|b_i|_{v_2} \to -\infty$ and the sequence $\{\log|a_i|_{v_1} + \log|b_i|_{v_2}\}$ is bounded. We define a sequence $\{s_i = (s_i^{(v)})_{v\in\mathcal{R}}\}$ in $T_\mathcal{R}$ as follows:

$$s_i^{(v)} = \begin{cases} 1, \text{ if } v \in \mathcal{R} \setminus \{v_1, v_2\}; \\ \chi_1(s_i^{(v_1)}) = a_i \text{ and } \chi_j(s_i^{(v_1)}) = 1 \text{ for all } j > 1; \\ \chi_1(s_i^{(v_2)}) = b_i \text{ and } \chi_j(s_i^{(v_2)}) = 1 \text{ for all } j > 1. \end{cases}$$

We have that $\{s_i\}$ is unbounded and that $\{\varphi(s_i)\}$ is bounded. (Recall that $T_\mathcal{R}$ is considered as a subgroup of T, so that the notation $\varphi(s_i)$ makes sense.) Since ψ is proper, $s_i\pi(e)$ is bounded. Therefore the orbite $T_\mathcal{R}\pi(e)$ is not divergent. This contradiction completes the proof of (i).

Assume that $\mathcal{R}$ contains only one valuation $v_\circ$ and that $\mathrm{rank}_K\mathbf{T} = \mathrm{rank}_{K_{v_\circ}}\mathbf{T} > 0$. It follows from the above definition of φ and the fact that χ is an homomorphism with compact kernel, that if a sequence $\{t_i\}$ in $T_\mathcal{R}$ diverges then $\{\varphi(t_i)\}$ does too. Therefore $T_\mathcal{R}\pi(e)$ is a divergent orbit. □

Proposition 4.3 implies:

Proposition 4.4. *Let $\mathbf{T}$ be a K-torus and let $\mathcal{R}$ be a non-empty subset of $\mathcal{S}$. Then the orbit $T_\mathcal{R}\pi(e)$ is closed if and only if one of the following conditions holds:*

(1) $\mathcal{R} = \mathcal{S}$;
(2) $\mathrm{rank}_{K_v}\mathbf{T} = 0$ *for all* $v \in \mathcal{R}$, *equivalently,* $T_\mathcal{R}$ *is compact;*
(3) $\mathcal{R} = \{v_\circ\}$ *and* $\mathrm{rank}_K\mathbf{T} = \mathrm{rank}_{K_{v_\circ}}\mathbf{T}$.

Proof. Note that if $\mathcal{R} \neq \mathcal{S}$ and $T_\mathcal{R}$ is not compact then $T_\mathcal{R}\pi(e)$ is closed if and only if it is divergent. Now the proposition follows easily from Proposition 4.3. □

5. Parabolic subgroups and divergent orbits

5.1. **Main proposition.** Recall that, given a subset $\mathcal{R} \subset \mathcal{S}$, we use the notation $\mathrm{pr}_\mathcal{R}$ to denote depending of the context the projection $G \to G_\mathcal{R}$ or the projection $\mathfrak{g} \to \mathfrak{g}_\mathcal{R}$.

The goal of this section is to prove the following

Proposition 5.1. *Let $\mathbf{G}$ be a reductive K-algebraic group, $\mathcal{R}$ be a non-empty subset of $\mathcal{S}$, $g = (g_v)_{v\in\mathcal{S}} \in G$ and $x = \pi(g)$. Assume that* $\mathrm{rank}_K\mathbf{G} > 0$ *and that for every minimal parabolic K-subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing the Lie algebra of $\mathbf{D}$ there exists a horospherical subset $\mathcal{M}_{\mathfrak{b}}$ of $\mathfrak{g}_{\mathcal{R}}$ such that $\mathcal{M}_{\mathfrak{b}} \subset \mathrm{pr}_{\mathcal{R}}(\mathfrak{g}_x) \cap \mathfrak{b}_{\mathcal{R}}$. Then the following assertions hold:*

(a) *For every $v \in \mathcal{R}$ the orbit $D_v\pi(g)$ is divergent;*
(b) *If $g_{\mathcal{R}} = \mathrm{pr}_{\mathcal{R}}(g)$ then*

$$g_{\mathcal{R}} \in \mathcal{Z}_{G_{\mathcal{R}}}(D_{\mathcal{R}})\mathrm{pr}_{\mathcal{R}}(\mathbf{G}(K)); \tag{11}$$

(c) *There exists a maximal K-split torus $\mathbf{S}$ of $\mathbf{G}$ such that*

$$S_v = {g_v}^{-1}D_v g_v \tag{12}$$

for all $v \in \mathcal{R}$, where $S_v = \mathbf{S}(K_v)$.

In order to prove Proposition 5.1 we will need some facts from the algebraic group theory.

5.2. **Intersections of parabolic subgroups.** The next three propositions remain valid for any field K.

Proposition 5.2. [Bo1, Propositions 14.22 and 21.13] *Let $\mathbf{P}$ and $\mathbf{Q}$ be parabolic K-subgroups of $\mathbf{G}$.*

(i) *$(\mathbf{P} \cap \mathbf{Q})R_u(\mathbf{P})$ is a parabolic K-subgroup;*
(ii) *If $\mathbf{Q}$ is conjugate to $\mathbf{P}$ and contains $R_u(\mathbf{P})$ then $\mathbf{Q} = \mathbf{P}$.*

We also have

Proposition 5.3. [To-W, Proposition 5.2] *For every minimal parabolic K-subgroup $\mathbf{B}$ containing $\mathbf{D}$ we let $\mathbf{P}_{\mathbf{B}}$ be a proper parabolic K-subgroup containing $\mathbf{B}$. Then*

$$\bigcap_{\mathbf{B}} \mathbf{P}_{\mathbf{B}} = \mathcal{Z}_{\mathbf{G}}(\mathbf{D}). \tag{13}$$

Keeping the notation and assumptions of Proposition 5.3, we prove:

Proposition 5.4. *Let $n \in \mathcal{N}_{\mathbf{G}}(\mathcal{Z}_{\mathbf{G}}(\mathbf{D}))$. Assume that for every $\mathbf{B}$ the group $n\mathbf{P}_{\mathbf{B}}n^{-1}$ is defined over K. Then $n \in \mathcal{N}_{\mathbf{G}}(\mathbf{D})$. The projection of n into the Weyl group $\mathrm{W}_K = \mathcal{N}_{\mathbf{G}}(\mathbf{D})/\mathcal{Z}_{\mathbf{G}}(\mathbf{D})$ is uniquely defined by the map $\mathbf{B} \to n\mathbf{P}_{\mathbf{B}}n^{-1}$.*

Proof. The uniqueness of the projection of n into W_K follows immediately from Proposition 5.3 and the fact that every parabolic subgroup coincides with its normalizer.

We will assume that for every $\mathbf{B}$ the group $\mathbf{P_B}$ is minimal among the parabolic K-subgroups $\mathbf{P}$ containing $\mathbf{B}$ and such that $n\mathbf{P}n^{-1}$ is defined over K.

Assume that there exists $\mathbf{B}$ such that $\mathbf{B} = \mathbf{P_B}$. Let $\mathbf{B}' = n\mathbf{B}n^{-1}$. Since all minimal parabolic K-subgroups are conjugated under the action of W_K and $\mathcal{N}_\mathbf{G}(\mathbf{D}) = \mathcal{N}_\mathbf{G}(\mathbf{D})(K)\mathcal{Z}_\mathbf{G}(\mathbf{D})$ [Bo1, Theorem 21.2], there exists $n_\circ \in \mathcal{N}_\mathbf{G}(\mathbf{D})(K)$ such that $\mathbf{B} = n_\circ\mathbf{B}'n_\circ^{-1}$. Therefore, $\mathbf{B} = n_\circ n\mathbf{B}(n_\circ n)^{-1}$ which implies that $n_\circ n \in \mathbf{B}$. Since $\mathcal{N}_\mathbf{G}(\mathbf{D}) \subset \mathcal{N}_\mathbf{G}(\mathcal{Z}_\mathbf{G}(\mathbf{D}))$, we get $n_\circ n \in \mathcal{N}_\mathbf{B}(\mathcal{Z}_\mathbf{G}(\mathbf{D}))$. Now, the proposition follows from the fact that $\mathcal{Z}_\mathbf{G}(\mathbf{D}) = \mathcal{N}_\mathbf{B}(\mathcal{Z}_\mathbf{G}(\mathbf{D}))$ [Bo1, Corollary 14.19].

Assume that $\mathbf{P_B} \supsetneqq \mathbf{B}$ for all $\mathbf{B}$. Choose a $\mathbf{P_B}$ with the minimal dimension and set $\mathbf{P} = \mathbf{P_B}$. Let $\Phi(\mathbf{D}, \mathbf{G})$ be the relative root system of $\mathbf{G}$ with respect to $\mathbf{D}$. (See [Bo1, 21.1 and 8.17] for the standard definition of a system of K-roots.) Since $\mathbf{P} \supsetneqq \mathbf{B}$, there exists a long root $\alpha \in \Phi(\mathbf{D}, \mathbf{G})$ such that $\pm\alpha$ are roots of the group $\mathbf{P}$ with respect to $\mathbf{D}$. Recall that all roots of the same length in $\Phi(\mathbf{D}, \mathbf{G})$ are conjugated under the action of W_K [Hu, 10.4, Lemma C and 10.3, Theorem]. Therefore there exists a minimal parabolic K-subgroup $\mathbf{B}^+$ containing $\mathbf{D}$ such that α is a maximal long root of $\mathbf{B}^+$ relative to $\mathbf{D}$. Let Δ^+ be the set of simple roots corresponding to $\mathbf{B}^+$. Then in the expression of α as a linear combination of the roots in Δ^+ all coefficients are strictly positive [Hu, 10.4, Lemma A]. It follows from the explicit description of the standard parabolic K-subgroups (see [Bo1, 21.11]), that $-\alpha$ is not a root of any parabolic K-subgroup containing $\mathbf{B}^+$. Similarly, α is not a root of any parabolic K-subgroup containing $\mathbf{B}^-$, where $\mathbf{B}^-$ is the minimal parabolic K-subgroup opposite to $\mathbf{B}^+$. As a consequence, one of the K-subgroups $(\mathbf{P}_{\mathbf{B}^+} \cap \mathbf{P})R_u(\mathbf{P})$ or $(\mathbf{P}_{\mathbf{B}^-} \cap \mathbf{P})R_u(\mathbf{P})$ is strictly smaller than $\mathbf{P}$. Let $\mathbf{P} \neq (\mathbf{P}_{\mathbf{B}^+} \cap \mathbf{P})R_u(\mathbf{P})$. Since $(\mathbf{P}_{\mathbf{B}^+} \cap \mathbf{P})R_u(\mathbf{P})$ is a parabolic K-subgroup (Proposition 5.2(i)) and $n(\mathbf{P}_{\mathbf{B}^+} \cap \mathbf{P})R_u(\mathbf{P})n^{-1}$ is defined over K. The latter contradicts the choice of $\mathbf{P}$, which completes our proof. $\square$

Remark 5.5. In connection with the above proposition, let us note that in many cases $\mathcal{N}_\mathbf{G}(\mathbf{D}) \subsetneq \mathcal{N}_\mathbf{G}(\mathcal{Z}_\mathbf{G}(\mathbf{D}))$. As a simple example one can consider the special unitary group $\mathbf{SU}_3(h)$, where h is an hermitian form with coefficients from K of indice 1. This is a quasisplit group of type A_2. Therefore $\mathcal{N}_\mathbf{G}(\mathcal{Z}_\mathbf{G}(\mathbf{D}))/\mathcal{Z}_\mathbf{G}(\mathbf{D})$ is isomorphic to the symmetric group S_3 and $\mathcal{N}_\mathbf{G}(\mathbf{D})/\mathcal{Z}_\mathbf{G}(\mathbf{D})$ is a group of order two.

5.3. **Proof of Proposition 5.1.** We start the proof with a general remark. With the proposition assumptions and notation, for every

$\mathfrak{b}$ there exists a finite subset $\mathcal{M}^{\bullet}_{\mathfrak{b}}$ of $\mathfrak{g}(\mathcal{O})$ which spans linearly the unipotent radical of a maximal parabolic K-subgroup $\mathbf{P}^{\bullet}_{\mathfrak{b}}$ of $\mathbf{G}$ and such that $\mathcal{M}_{\mathfrak{b}} = \mathrm{pr}_{\mathcal{R}}(\mathrm{Ad}(g)(\mathcal{M}^{\bullet}_{\mathfrak{b}})$. So, if $v \in \mathcal{R}$, we have

$$g_v R_u(\mathbf{P}^{\bullet}_{\mathfrak{b}})(K_v) g_v^{-1} \subset \mathbf{B}(K_v),$$

where $\mathbf{B}$ is the K-algebraic subgroup of $\mathbf{G}$ the Lie algebra of which is $\mathfrak{b}$. It follows from Proposition 5.2(ii) that there exists a parabolic K-subgroup $\mathbf{P}_{\mathfrak{b}}$ containing $\mathbf{B}$ such that

$$\mathbf{P}_{\mathfrak{b}} = g_v \mathbf{P}^{\bullet}_{\mathfrak{b}} g_v^{-1} \tag{14}$$

for all $v \in \mathcal{R}$.

Let us prove (a). (Remark that (a) follows *a posteriori* from (b) and Proposition 4.3.) Fix $v \in \mathcal{R}$. We want to prove that the orbit $D_v \pi(g)$ diverges. Let $\{d_i\}$ be a divergent sequence in D_v. Put $s_i = g_v^{-1} d_i g_v$. It is enough to prove that the sequence $\{s_i \pi(e)\}$ is divergent. Passing to a subsequence we may assume that $\{d_i^{-1}\}$ belongs to the Weyl chamber corresponding to some minimal parabolic K-subgroup $\mathbf{B}$. Let $\mathfrak{u}$ be the Lie algebra of $R_u(\mathbf{P}^{\bullet}_{\mathfrak{b}})$. Let m be the dimension of $\mathfrak{u}$ and let $\bigwedge^m \mathrm{Ad}$ be the adjoint representation of $\mathbf{G}$ on the m-th exterior power $\bigwedge^m \mathfrak{g}$. Since $\mathfrak{u}$ is defined over K, there exists a non-zero K-*rational* vector $z \in \bigwedge^m \mathfrak{g}$ corresponding to $\mathfrak{u}$. It is known (see the proof of Proposition 5.4) that if α is a maximal root of $\mathbf{B}$ with respect to $\mathbf{D}$ then α is a root of every standard parabolic subgroup containing $\mathbf{B}$ and, given the choice of $\{d_i\}$, $\lim_{i\to\infty} \alpha(d_i) = 0$. Since $\mathbf{P}_{\mathfrak{b}} = g_v \mathbf{P}^{\bullet}_{\mathfrak{b}} g_v^{-1}$ and $\mathbf{P}_{\mathfrak{b}}$ is a parabolic containing $\mathbf{B}$, we obtain that

$$\lim_{i\to\infty} \| \bigwedge^m \mathrm{Ad}(d_i) g_v z\|_v = 0.$$

This implies

$$\lim_{i\to\infty} \mathbf{c}_{\mathcal{S}}(\bigwedge^m \mathrm{Ad}(s_i) z) = 0.$$

It follows from Theorem 3.1 (ii) that $\{s_i \pi(e)\}$ diverges. This completes the proof of (a).

Note that (c) follows immediately from (b). So, it remains to prove (b). Let $\mathbf{P}^{\bullet}_{\mathfrak{b}}$ be as above. Set $\mathbf{H} = \bigcap_{\mathfrak{b}} \mathbf{P}^{\bullet}_{\mathfrak{b}}$. Since $\mathbf{P}_{\mathfrak{b}}$ is a K-parabolic subgroup of $\mathbf{G}$ containing $\mathbf{B}$, in view of Proposition 5.3, we get that

$$\mathbf{H} = \bigcap_{\mathfrak{b}} g_v^{-1} \mathbf{P}_{\mathfrak{b}} g_v = g_v^{-1}\big(\bigcap_{\mathfrak{b}} \mathbf{P}_{\mathfrak{b}}\big) g_v = g_v^{-1} \mathcal{Z}_{\mathbf{G}}(\mathbf{D}) g_v \tag{15}$$

for all $v \in \mathcal{R}$.

Note that the groups $\mathcal{Z}_{\mathbf{G}}(\mathbf{D})$ and $\mathbf{H}$ are reductive and defined over K. Let $\mathbf{Z}$ (respectively, $\mathbf{Z}^{\bullet}$) be the Zariski connected component of the center of $\mathcal{Z}_{\mathbf{G}}(\mathbf{D})$ (respectively, $\mathbf{H}$). It follows from (15) that

$$\mathbf{Z}^\bullet = g_v^{-1}\mathbf{Z}g_v \tag{16}$$

for all $v \in \mathcal{R}$. Since $\mathbf{D}$ is a maximal K-split torus of $\mathbf{G}$, we have that $\mathbf{D} = \mathbf{Z}_d$, where $\mathbf{Z}_d$ is the largest K-split subtorus of $\mathbf{Z}$.

Denote by $\mathbf{Z}_d^\bullet$ the largest K-split subtorus of $\mathbf{Z}^\bullet$ and assume that $\mathbf{Z}_d^\bullet$ is not maximal in $\mathbf{G}$. Let $\mathbf{Z}_a^\bullet$ be the largest K-anisotropic subtorus of $\mathbf{Z}^\bullet$. Fix $v \in \mathcal{R}$. Since every K-torus is an almost direct product over K of its largest K-split and its largest K-anisotropic subtori [Bo1, Proposition 8.15], it follows from (16) that there exists an element $t \in \mathbf{Z}_a^\bullet(K_v) \cap {g_v}^{-1}\mathbf{D}(K_v)g_v$ such that $\{t^n | n \in \mathbb{N}\}$ is a divergent sequence. In view of (a), $\{g_v t^n {g_v}^{-1}\pi(g)\}$, and therefore $\{t^n\pi(e)\}$, are also divergent sequences. The latter contradicts the fact that the orbit $\mathbf{Z}_a^\bullet(K_{\mathcal{R}})\pi(e)$ is compact (see Theorem 4.1). Therefore $\mathbf{Z}_d^\bullet$ is a maximal K-split torus of $\mathbf{G}$.

Since the maximal K-split tori are conjugated under $\mathbf{G}(K)$ [Bo1, Theorem 20.9], there exists $q \in \mathbf{G}(K)$ such that $\mathbf{Z}_d^\bullet = q^{-1}\mathbf{D}q$. Also, $\mathcal{Z}_{\mathbf{G}}(\mathbf{Z}_d^\bullet) = q^{-1}\mathcal{Z}_{\mathbf{G}}(\mathbf{D})q$, $\mathcal{Z}_{\mathbf{G}}(\mathbf{Z}_d^\bullet) \supset \mathbf{H}$ and $\dim \mathbf{H} = \dim \mathcal{Z}_{\mathbf{G}}(\mathbf{D})$. Therefore,

$$\mathbf{H} = q^{-1}\mathcal{Z}_{\mathbf{G}}(\mathbf{D})q.$$

In view of (15), we have

$$g_v q^{-1} \in \mathcal{N}_{\mathbf{G}}(\mathcal{Z}_{\mathbf{G}}(\mathbf{D})), \forall v \in \mathcal{R}.$$

Given $v \in \mathcal{R}$, the group

$$qg_v^{-1}\mathbf{P}_{\mathfrak{b}}(qg_v^{-1})^{-1} = q\mathbf{P}_{\mathfrak{b}}^\bullet q^{-1}$$

is defined over K for every $\mathfrak{b}$. It follows from Proposition 5.4 that there exists $n \in \mathcal{N}_{\mathbf{G}}(\mathbf{D})(K)$ such that

$$nqg_v^{-1} \in \mathcal{Z}_{\mathbf{G}}(\mathbf{D}), \forall v \in \mathcal{R}. \tag{17}$$

Since n is the same for all $v \in \mathcal{R}$, (17) implies (11), which completes the proof. □

6. Proofs of Theorem 1.4 and of its corollaries

6.1. **Proof of Theorem 1.4.** Let the conditions (i)-(iii) in the formulation of the theorem hold. Since $\mathrm{rank}_{K_v}\mathbf{G} \geq \mathrm{rank}_K\mathbf{G}$, it follows from (ii) that $\mathrm{rank}_{K_v}\mathbf{G} = \mathrm{rank}_K\mathbf{G}$ for all $v \in \mathcal{R}$. Therefore, $T_{\mathcal{R}}/D_{\mathcal{R}}$ is compact. So, $T_{\mathcal{R}}\pi(g)$ is closed and locally divergent if and and only if $D_{\mathcal{R}}\pi(g)$ has this property. In view of (iii), $g^{-1}D_{\mathcal{R}}g = \widetilde{D}_{\mathcal{R}}$, where $\widetilde{D}_{\mathcal{R}} = \widetilde{\mathbf{D}}(K_{\mathcal{R}})$ and $\widetilde{\mathbf{D}}$ is a K-split torus. Using (i) and Proposition 4.3, it is easy to see that $\widetilde{D}_{\mathcal{R}}\pi(g)$, and therefore $D_{\mathcal{R}}\pi(g)$, are closed locally divergent orbits.

Let the orbit $T_{\mathcal{R}}\pi(e)$ be closed and locally divergent. In view of Theorem 4.1(b), $\mathrm{rank}_K\mathbf{G} > 0$. Moreover, since every $\mathbf{T}_v$ is a product of a maximal K_v-split torus and a compact, we can suppose without loss of generality that $\mathbf{T}_v$ is a maximal K_v-split torus.

Denote by $\mathbf{S}$ the connected component of the Zariski closure of $g^{-1}T_{\mathcal{R}}g \cap \Gamma$ in $\mathbf{G}$. Suppose that $\mathbf{S}$ is not trivial. Then $\mathcal{R} = \mathcal{S}$. Set $S = \mathbf{S}(K_{\mathcal{S}})$. Since S is not compact, $S\pi(e)$ is locally divergent and $\mathbf{S}$ is K_v-split, $v \in \mathcal{S}$, it follows from Proposition 4.3 that $\mathbf{S}$ is K-split. Set $\mathbf{H} = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$, $H = \mathbf{H}(K_{\mathcal{S}})$ and $\Delta = H \cap \Gamma$. Let $\pi_H : H \to H/\Delta$ be the natural projection. Remark that $\mathbf{H}$ is a reductive group [Bo1, 13.17, Corollary 2]. Choose a maximal K-split torus $\widetilde{\mathbf{S}}$ of $\mathbf{H}$. Then $\widetilde{\mathbf{S}} \supset \mathbf{S}$ and there exists $q \in \mathbf{G}(K)$ such that

$$\widetilde{\mathbf{S}} = q^{-1}\mathbf{D}q. \tag{18}$$

Denote $\widetilde{S}_v = \widetilde{\mathbf{S}}(K_v)$, $v \in \mathcal{S}$, and $\widetilde{S} = \widetilde{\mathbf{S}}(K_{\mathcal{S}})$. There exists $h = (h_v)_{v\in\mathcal{S}} \in H$ such that $h_v^{-1}\widetilde{S}_v h_v \subseteq g_v^{-1}T_v g_v$ for every $v \in \mathcal{S}$. Denote $\widetilde{T}_v = h_v g_v^{-1}T_v g_v h_v^{-1}$ and $\widetilde{T} = \prod_{v\in\mathcal{S}}\widetilde{T}_v$. Then $\widetilde{S} \subset \widetilde{T} \subset H$ and $\widetilde{T}\pi_H(h)$ is a closed locally divergent orbit. Suppose for a moment that the theorem is valid for $\mathbf{H}$. Then the conditions (i) and (ii) in the formulation of the theorem are automatically fulfilled because $\mathrm{rank}_K\mathbf{G} = \mathrm{rank}_K\mathbf{H}$ and $\mathrm{rank}_{K_v}\mathbf{G} = \mathrm{rank}_{K_v}\mathbf{H}$, $v \in \mathcal{S}$. Since $h = zd$, where $z \in \mathcal{N}_H(\widetilde{S})$ and $d \in \mathbf{H}(K)$, using (18), we obtain

$$\begin{aligned} D = gh^{-1}\widetilde{S}hg^{-1} =& gd\widetilde{S}d^{-1}g^{-1} = \\ =& gdq^{-1}Dqd^{-1}g^{-1}. \end{aligned}$$

Therefore, $g \in \mathcal{N}_G(D)\mathbf{G}(K)$, which proves (iii). The above discussion reduces the proof to the case when $\mathbf{S}$ is a central K-split torus in $\mathbf{G}$. In this case $\mathbf{G}$ is an almost direct product over K of $\mathbf{S}$ and a reductive K-group. Factorizing by $\mathbf{S}$, we can further reduce the proof to the case when $\mathbf{S}$ is trivial.

So, in order to complete the proof of the theorem, it is enough to consider the case when $T_{\mathcal{R}}\pi(g)$ is a divergent orbit. The rest of the proof breaks in two cases according to whether or not the assumptions in the formulation of Proposition 5.1 are satisfied.

Assume that for every K-subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathrm{Lie}(\mathbf{D})$ the intersection $\mathrm{pr}_{\mathcal{R}}(\mathfrak{g}_x) \cap \mathfrak{b}_{\mathcal{R}}$, where $x = \pi(g)$, contains a horospherical subset. Then (iii) follows from Proposition 5.1(b), and (ii) from Proposition 5.1(c) and Theorem 4.1(b). The condition (i) follows easily from (ii), (iii) and Proposition 4.3.

Now assume the contrary, that is, that there exists a minimal parabolic K-subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathrm{Lie}(\mathbf{D})$ and such that $\mathrm{pr}_{\mathcal{R}}(\mathfrak{g}_x)\cap\mathfrak{b}_{\mathcal{R}}$

does not contain a horospherical subset. We will prove that this assumption leads to contradiction. (As in [To-W], our argument is inspired by Margulis' one, cf.[To-W, Appendix].) Let $\mathfrak{u}^-$ be the unipotent radical of the minimal parabolic K-subalgebra opposite to $\mathfrak{b}$. For every positive integer n we let B_n be a ball of radius n in $\mathfrak{g}$. Since $\mathfrak{g}_x$ is discrete in $\mathfrak{g}$, the family of the horospherical subsets in $\mathrm{pr}_{\mathcal{R}}(\mathfrak{g}_x) \cap \mathfrak{b}_{\mathcal{R}}$ is finite. In view of this and the assumption that $\mathrm{pr}_{\mathcal{R}}(\mathfrak{g}_x) \cap \mathfrak{b}_{\mathcal{R}}$ does not contain horospherical subsets, for every n there exists an element $s_n \in D_{\mathcal{R}}$ such that $\mathrm{Ad}(s_n)$ acts as an expansion on $\mathfrak{u}^-_{\mathcal{R}}$ and

$$\mathrm{Ad}(s_n)\mathcal{M} \not\subseteq B_n \tag{19}$$

for every horospherical subset $\mathcal{M} \subset \mathfrak{g}_x \cap B_n$.

Using Proposition 3.4(a), we fix a compact neighborhood W_0 of 0 in $\mathfrak{g}$ such that $W_0 \subset B_n$ and for every $x \in G/\Gamma$ the subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{g}_x \cap W_0$ is unipotent.

Proposition 3.5 and the choice of W_0 imply that there exist a constant $\tau > 1$ and a finite set $t_1, \dots, t_l$ in $D_{\mathcal{R}}$ such that for every $y \in G/\Gamma$ there exists $t \in \{t_1, \dots, t_l\}$ satisfying

$$\|\mathrm{Ad}(t)a\|_{\mathcal{R}} \geq \tau \|a\|_{\mathcal{R}}, \forall a \in \mathfrak{g}_y \cap W_0. \tag{20}$$

We put

$$W = W_0 \bigcap \big(\bigcap_{i=1}^{l} \mathrm{Ad}(t_i) W_0 \big).$$

Given a positive $n \in \mathbb{N}$, we define inductively a *finite* sequence $p_0, p_1, \dots, p_{r_n}$ as follows. We put $p_0 = s_n$. Assume that $p_0, p_1, \dots, p_i$ are already defined. If $\mathrm{Ad}(p_i \dots p_0)(\mathfrak{g}_x) \cap W$ does not contain a horospherical subset then $p_0, p_1, \dots, p_i$ is the required sequence. If not, we put $p_{i+1} = t$, where t satisfies (20) with $y = p_i \dots p_0 x$. With the same y and p_{i+1}, remark that if $b \in \mathfrak{g}_y$ and $b \notin W_0$ then $\mathrm{Ad}(p_{i+1})b \notin W$. This and (20) imply the following

Claim: If $p_0, p_1, \dots, p_r$ are already defined, $0 \leq i < r$, $y = p_i \dots p_0 x$, $b \in \mathfrak{g}_y$ and $b \notin W_0$ then $\mathrm{Ad}(p_j \dots p_{i+1})b \notin W$ for every j such that $i \leq j \leq r$.

The claim implies that the cardinality of $\mathrm{Ad}(p_i \dots p_0)(\mathfrak{g}_x) \cap W$ does not increase with i and, moreover, the sequence $\{p_i\}$ is finite. Put $g_n = p_{r_n} \dots p_1 p_0$. It follows from Proposition 3.4(b) that the sequence $\{g_n x\}$ is bounded in G/Γ. Since the orbit $T_{\mathcal{R}} x$ is divergent, the sequence $\{g_n\}$ is bounded in $T_{\mathcal{R}}$. Also note that, given the above definition of s_n, the sequence $\{s_n\}$ is unbounded. Again by Proposition 3.4(b), passing to a subsequence, we may assume that $r_n > 0$ for all n.

Let $h_n = p_{r_n}^{-1} g_n$ and $\mathcal{M}_n$ be a horospherical subset of $\mathrm{Ad}(h_n)(\mathfrak{g}_x) \cap W$. Assume that $\mathrm{Ad}(h_n^{-1})(\mathcal{M}_n) \subset B_n$. Then it follows from (19) that $\mathrm{Ad}(p_0 h_n^{-1})(\mathcal{M}_n) \not\subseteq B_n$. The Claim implies that $\mathcal{M}_n \not\subseteq W$, which contradicts the choice of $\mathcal{M}_n$. Therefore,

$$\mathrm{Ad}(h_n^{-1})(\mathcal{M}_n) \not\subseteq B_n.$$

Since $\mathcal{M}_n \subset W$ and W is compact, the sequence $\{h_n^{-1}\}$ is not bounded. Therefore, $\{g_n\}$ is not either. Contradiction. $\square$

6.2. **Remarks.** (a) It follows from the proof of Theorem 1.4 that if $\#\mathcal{S} > 1$ and the orbit Tx (where $T = T_{\mathcal{S}}$) is closed and locally divergent then the Zariski closure of $g^{-1}Tg \cap \Gamma$ in $\mathbf{G}$ contains a maximal K-split torus.

(b) Since $\mathcal{N}_{\mathbf{G}}(\mathbf{D})(K)$ meets every coset of the quotient $\mathcal{N}_{\mathbf{G}}(\mathbf{D})/\mathcal{Z}_{\mathbf{G}}(\mathbf{D})$, we have that $\mathcal{Z}_G(D_v)\mathbf{G}(K) = \mathcal{N}_G(D_v)\mathbf{G}(K)$ for every v. On the other hand, it is easy to see that $\mathcal{Z}_G(D_{\mathcal{R}})\mathbf{G}(K) \subsetneqq \mathcal{N}_G(D_{\mathcal{R}})\mathbf{G}(K)$ whenever $\#\mathcal{R} > 1$ and $\mathbf{G}$ is a semisimple K-isotropic group.

6.3. **Proof of Theorem 1.1.** Let us prove (a). The implication $\Leftarrow$ is trivial. Assume that $T_{\mathcal{R}}\pi(g)$ is closed. Let $\mathbf{S}$ be the connected component of the Zariski closure of $g^{-1}Tg \cap \Gamma$. If $\mathcal{R} \neq \mathcal{S}$ then $\mathbf{S} = \{1\}$ and $T_{\mathcal{R}}\pi(g)$ is divergent. Let $\mathcal{R} = \mathcal{S}$. Set $\mathbf{H} = \mathcal{Z}_{\mathbf{G}}(\mathbf{S})$, $H = \mathbf{H}(K_{\mathcal{S}})$ and $\Delta = \mathbf{H}(\mathcal{O})$. Since $\mathbf{H}$ is an almost direct product over K of $\mathbf{S}$ and of a reductive K-group, the proof can be easily reduced to the case when $\mathbf{S}$ is trivial, i.e., when $T_{\mathcal{R}}\pi(g)$ is divergent. The case when $T_{\mathcal{R}}\pi(g)$ is divergent follows from (b).

The part (b) of the theorem follows easily from Theorem 1.1 and Proposition 4.3. $\square$

6.4. **Proof of Corollaries 1.3, 1.5 and 1.6.** Corollary 1.3 follows from Theorem 1.1 (a) and Remark 6.2 (a), and Corollary 1.5 follows from Theorem 1.4 and Remark 6.2 (b).

Let us prove Corollary 1.6. The part (a) is immediate from Theorem 1.4. In order to prove (b), remark that $\big(\mathcal{N}_{\mathbf{G}}(\mathbf{D}) \times \mathcal{N}_{\mathbf{G}}(\mathbf{D})\big)\mathrm{diag}(\mathbf{G}) \subsetneqq \mathbf{G} \times \mathbf{G}$, where $\mathrm{diag}(\mathbf{G})$ is the diagonal imbedding of $\mathbf{G}$ into $\mathbf{G} \times \mathbf{G}$. Therefore, there exists $(g_1, g_2) \in (\mathbf{G} \times \mathbf{G})(K)$ such that $(g_1, g_2) \notin \big(\mathcal{N}_{\mathbf{G}}(\mathbf{D}) \times \mathcal{N}_{\mathbf{G}}(\mathbf{D})\big)\mathrm{diag}(\mathbf{G})$. Let v_1 and v_2 be two different valuations in $\mathcal{S}$ and let $g = (g_v)_{v \in \mathcal{S}} \in G$ be such that $g_{v_1} = g_1$, $g_{v_2} = g_2$ and $g_v = 1$ for all $v \in \mathcal{S} \setminus \{v_1, v_2\}$. It follows from Theorem 1.4 (iii) and Proposition 4.3 that the orbit $T_{\mathcal{R}}\pi(g)$ is locally divergent but not closed. $\square$

6.5. **Remark.** In connection with Corollary 1.6 (a), note that if $K = \mathbb{Q}$ and D_∞ is a real split torus of $G = \mathbf{G}(\mathbb{R})$ with $\dim D_\infty > \mathrm{rank}_{\mathbb{Q}} G$, it was proved by B.Weiss [W] that there are no divergent orbits for the action of T_∞. The following generalization of [W] should be also true: If v is any place of K and $\mathbf{D}_v$ is a K_v-split torus of $\mathbf{G}$ such that $\dim \mathbf{D}_v > \mathrm{rank}_K \mathbf{G}$ then G/Γ, where G and Γ are as in the formulation of Theorem 1.1, does not admit divergent orbits for the action of $D_v = \mathbf{D}_v(K_v)$.

7. Number theoretical application

Let $K_{\mathcal{S}}[\,\vec{x}\,]$ be the ring of polynomials in n variables $\vec{x} = (x_1, \dots, x_n)$ with coefficients from the topological ring $K_{\mathcal{S}}$. Let $f(\vec{x}) = l_1(\vec{x}) \dots l_m(\vec{x}) \in K_{\mathcal{S}}[\,\vec{x}\,]$, where $l_1(\vec{x}), \dots, l_m(\vec{x})$ are linearly independent over $K_{\mathcal{S}}$ linear forms.

The following is a reformulation of Theorem 1.9 from the Introduction:

Theorem 7.1. *With the above notation and assumptions, suppose that $f(\mathcal{O}^n)$ is a discrete subset of $K_{\mathcal{S}}$. Then $f(\vec{x}) = \alpha g(\vec{x})$ for some $\alpha \in K_{\mathcal{S}}^*$ and some $g(\vec{x}) \in \mathcal{O}[\,\vec{x}\,]$.*

The following examples show that the hypotheses in the formulations of Theorem 7.1 are essential and can not be omitted.

Examples. Let $\alpha \in \mathbb{R}$ be a badly approximable number, i.e. there exists a $c = c(\alpha) > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^2}$$

for all $p/q \in \mathbb{Q}$. (Recall that the quadratic irrationals, such as $\sqrt{2}$, and the golden ratio $(\sqrt{5}+1)/2$ are badly approximable.) Consider the form $f(x, y) = x^2(\alpha x - y)$. Then the set of values of f at the integer points is discrete *but* f is not a multiple of a form with rational coefficients. The reason is that f is a product of linearly *dependent* linear forms.

The hypothesis that f is decomposable is also essential. In order to see this it is enough to consider a form $f(x, y) = x^2 + \beta y^2$ where β is a positive irrational real number. It is obvious that $f(\mathbb{Z}^2)$ is discrete in $\mathbb{R}$.

We put $\mathbf{G} = \mathbf{SL}_n$. So, $G = \mathbf{SL}_n(K_{\mathcal{S}})$ and $\Gamma = \mathbf{SL}_n(\mathcal{O})$.) The group G is acting on $K_{\mathcal{S}}[\,\vec{x}\,]$ according to the low $(\sigma f)(\vec{x}) = f(\sigma^{-1}\vec{x})$, where $\sigma \in G$ and $f \in K_{\mathcal{S}}[\,\vec{x}\,]$. We denote $f_0(\vec{x}) = x_1 x_2 ... x_m$. It is clear that if $f \in K_{\mathcal{S}}[\ \vec{x}\ ]$ is as in the formulation of Theorem 7.1 then $f(\vec{x}) = \alpha(\sigma f_0)(\vec{x})$ for some $\sigma \in G$ and $\alpha \in K_{\mathcal{S}}^*$. We will denote by H_f the stabilizer of f in G.

We precede the proof of Theorem 7.1 by the following general proposition.

Proposition 7.2. *Let $f(\vec{x}) = (\sigma f_0)(\vec{x})$ for some $\sigma \in G$. Assume that $f(\mathcal{O}^n)$ is a discrete subset of $K_{\mathcal{S}}$. Then $H_f\pi(e)$ is closed in G/Γ.*

Proof. Let $\pi(a)$, $a \in G$, belong to the closure of $H_f\pi(e)$. Fix a sequence $h_i \in H_f$ such that $\lim_{i\to\infty} h_i\pi(e) = \pi(a)$. There exist $\gamma_i \in \Gamma$ and $b_i \in G$ such that $\lim_{i\to\infty} b_i = e$ and $h_i\gamma_i = b_i a$. Since $f(\mathcal{O}^n)$ is discrete, for every $\vec{z} \in \mathcal{O}^n$ there exists a real number $c(z) > 0$ such that

$$f(\gamma_i\vec{z}) = f(h_i\gamma_i\vec{z}) = f(\beta_i a\vec{z}) = f(a\vec{z}) \in f(a\mathcal{O}^n) \cap f(\mathcal{O}^n) \tag{21}$$

for all $i > c(z)$.

Let $\chi_1, \chi_2, ..., \chi_l \in K[\ \vec{x}\ ]$ be the set of all monomials of degree m. We consider $\chi_1, \chi_2, ..., \chi_l$ as homomorphisms of multiplicative groups $K^{*n} \to K^*$. Since $\chi_1, \chi_2, ..., \chi_l$ are linearly independent over K, i.e. whenever we have a relation

$$\alpha_1\chi_1 + \alpha_2\chi_2 + \ldots + \alpha_l\chi_l = 0,$$

with $\alpha_i \in K$ then all $\alpha_i = 0$, there exist $\vec{z}_1, \vec{z}_2, ..., \vec{z}_l \in \mathcal{O}^n$ such that $\det(\chi_k(\vec{z}_s)) \neq 0$. In view of (21), there exists $c > 0$ such that

$$f(b_i a\vec{z}_s) = f(a\vec{z}_s) \tag{22}$$

for all s and $i > c$.

The form f can be regarded as a collection of forms $f_v \in K_v[\ \vec{x}\ ], v \in \mathcal{S}$. Since $\det(\chi_k(\vec{z}_l)) \neq 0$, using (22), we get that

$$f_v(b_{iv}a_v\vec{x}) = f_v(a\vec{x})$$

for all $v \in \mathcal{S}$ and $i > c$, where b_{iv} is the v-component of b_i and a_v is the v-component of a. Hence $b_i \in H_f$ for all $i > c$. So, we obtain that

$$\pi(a) = b_i^{-1}h_i\pi(e) \in H_f\pi(e),$$

which proves that $H_f\pi(e)$ is closed. □

Given a subgroup L of G, we will write L_u for the subgroup generated by the Zariski closed in G unipotent subgroups of L.

The following is a particular case of Theorem 3 from [To].

Proposition 7.3. *Let L be a closed (for the Euclidean topology) subgroup of G. Assume that $L\pi(e)$ is closed and $L_u\pi(e)$ is dense in $L\pi(e)$. Let $\mathbf{P}$ be the connected component of the Zariski closure of $L \cap \Gamma$ in $\mathbf{G}$ and let $P = \mathbf{P}(K_{\mathcal{S}})$. Then*

(i) *$P \supset L_u$ and there exists a subgroup of finite index P' in P such that $L\pi(e) = P'\pi(e)$;*

(ii) *If $\mathbf{Q}$ is a proper normal K-subgroup of $\mathbf{P}$, there exists $v \in \mathcal{S}$ such that $(\mathbf{P}/\mathbf{Q})(K_v)$ contains a unipotent element different from the identity.*

Proof of Theorem 7.1. Let H_0 be the Zariski connected component of H_{f_0}. It is easy to see that

(23)
$$H_0 = \left\{ \begin{pmatrix} d & a \\ 0 & s \end{pmatrix} \mid d \in D_m,\ a \in \mathrm{M}_{m\times(n-m)}(K_{\mathcal{S}}) \text{ and } s \in \mathrm{SL}_{n-m}(K_{\mathcal{S}}) \right\},$$

where D_m is the group of all diagonal matrices in $\mathrm{SL}_m(K_{\mathcal{S}})$. Since $f = \sigma f_0$, we have that

$$H = \sigma H_0 \sigma^{-1}$$

is the Zariski connected component of H_f.

Let $\mathcal{F}_m$ be the $K_{\mathcal{S}}$-module of all homogeneous polynomials of degree m in $K_{\mathcal{S}}[\vec{x}]$. A simple calculation shows that $K_{\mathcal{S}} f_0$ is the submodule of all H_0-invariant elements in $\mathcal{F}_m$. Therefore,

$$K_{\mathcal{S}} f = \{h \in \mathcal{F}_m | \sigma h = h, \forall \sigma \in H\}. \tag{24}$$

It follows from [Ra, Theorem 2] that there exists a closed subgroup L of G such that $L\pi(e) = \overline{H_u\pi(e)}$. Let $\mathbf{P}$ be the connected component of the Zariski closure of $L \cap \Gamma$ in $\mathbf{G}$ and let $P = \mathbf{P}(K_{\mathcal{S}})$. By Proposition 7.3, $L\pi(e) = P'\pi(e)$ where P' is a subgroup of finite index in P. On the other hand, since $H_f\pi(e)$ is closed (Proposition 7.2) and H has finite index in H_f, $H\pi(e)$ is also closed. Therefore, $P' \subset H$. Since $H_u \subset P'$, it follows from Proposition 7.3 (ii) and from the description (23) of H_0 that $H_u = P$ and $L\pi(e) = P\pi(e)$.

Let $\mathbf{Q}$ be the commutator subgroup of $\mathcal{N}_{\mathbf{G}}(\mathbf{P})$. It follows from (23) that $\mathbf{Q}$ is a semidirect product over K of $\mathbf{P}$ and of an algebraic group $\mathbf{R}$ defined over K which is isomorphic over K_v to $\mathbf{SL}_m$ for all $v \in \mathcal{S}$. (Note that $\mathbf{R}$ is isomorphic to $\mathbf{SL}_m$ over a finite extension of K but, in general, $\mathbf{R}$ is not isomorphic to $\mathbf{SL}_m$ over K itself.) Let $R = \prod_{v\in\mathcal{S}} \mathbf{R}_v(K_v)$ and $T = R \cap H$. Then $T = \prod_{v\in\mathcal{S}} \mathbf{T}_v(K_v)$, where $\mathbf{T}_v$ is a maximal K_v-split torus in $\mathbf{R}$, and $H = TP$. Since the projection of H into $Q/(Q \cap \Gamma)$, where $Q = \mathbf{Q}(K_{\mathcal{S}})$, is closed, the projection of T into $R/(R \cap \Gamma)$ is closed too. Applying Theorem 1.1, we get a torus $\mathbf{T}$ in $\mathbf{R}$ defined over K such that $T = \mathbf{T}(K_{\mathcal{S}})$. Therefore, $H = \mathbf{H}(K_{\mathcal{S}})$, where $\mathbf{H} = \mathbf{TP}$ is an algebraic group defined over K.

It follows from the above that $\mathbf{H}(K)$ is Zariski dense in H. Note that given $\sigma \in \mathbf{H}(K)$ the coefficients of all $h \in \mathcal{F}_m$ such that

$$\sigma h = h$$

can be regarded as the space of solutions of a system of linear equations with coefficients from K. Therefore, in view of (24), there exist $g(\vec{x}) \in \mathcal{O}[\vec{x}]$ and $\alpha \in K_{\mathcal{S}}^*$ such that $f(\vec{x}) = \alpha g(\vec{x})$. $\square$

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Institut Girard Desargues, Université Claude Bernard - Lyon I, Bâtiment de Mathématiques, 43, Bld. du 11 Novembre 1918, 69622 Villeurbanne Cedex, France tomanov@igd.univ-lyon1.fr