Explicit forms for Koecher-Maass series and their applications

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Abstract

The author gives a survey of one aspect about Koecher-Maass series mainly from his results on explicit forms for Koecher-Maass series and their applications.

1 Introduction

In this survey, one aspect about Koecher-Maass series will be presented. In particular the author would like to explain his works on explicit forms for Koecher-Maass series and their applications. As for more general and wide topics, there exist a valuable proceedings "On Koecher-Maass series" [25] and beautiful papers "A survey on the new proof of Saito-Kurokawa lifting after Duke and Imamoglu" [23] by T. Ibukiyama, "Bemerkungen über die Dirichletreichen von Koecher und Maass" [7] by S. Böcherer. We recommend these articles to anyone interested in Koecher-Maass series. Initially this series was introduced by Maass [39] and Koecher [33] for holomorphic Siegel modular forms. Let

$$F(Z) = \sum_{T \geq O} A(T,F) \exp(2\pi i tr(TZ))$$

be a holomorphic Siegel modular form of degree n, where the summation extends over all half-integral positive semi-definite symmetric matrices T of degree n. Then the Koecher-Maass series $\eta_F(s)$ associated with $F(Z)$ is defined by

$$\eta_F(s) = \sum_{\{T>O\}/SL_n(\mathbf{Z})} \frac{A(T,F)}{\epsilon(T)(\det T)^s},$$

where the summation extends over all half-integral positive definite symmetric matrices T of degree n modulo the action $T \to T[U] = {}^tUTU$ of the group $SL_n(\mathbf{Z})$ and $\epsilon(T) = \sharp\{U \in SL_n(\mathbf{Z}); T[U] = T\}$ is the order of the unit group of T. Some standard analytical properties of $\eta_F(s)$ are well known. The series $\eta_F(s)$ converges absolutely for $\Re(s)$ sufficiently large. It has a meromorphic continuation to the whole complex s-plane and satisfies a functional equation. An expression for its principal part of poles was obtained by Arakawa [2]. See [38] and [31] for detail expositions.

In this paper we give a survey about some special topics about explicit forms for Koecher-Maass series and their applications, mainly from the present author's results. In case the degree is two, our approach is based on the following observation: if the Fourier coefficients $A(T,F)$ satisfy a Maass type relation then the Koecher-Maass series $\eta_F(s)$ is a convolution product of two modular forms and can be studied by Zagier's Rankin-Selberg method independent of the modularity with respect to the inversion $Z \to -Z^{-1}$ on the Fourier series $F(Z)$. Let us explain these process more precisely. In section 2 we prepare some basic tools. Imai discovered how one can apply the spectral theory on the upper half-plane to Siegel modular forms of degree two. Take a Fourier series on the Siegel upper half-space. Suppose that its Fourier coefficients are unimodular invariant and have a reasonable growth condition. Let $Z = it^{1/2}W$ be a variable on the Siegel upper half-space, where $t > 0$ and W is a positive definite real symmetric matix of size two whose determinant is one. Identifying W with a variable τ on the upper half-plane, we have a Roelcke-Selberg spectral decomposition of the Fourier series as a function of τ. Then each of the spectral coefficients with respect to spectral eigenfunctions is the inverse Mellin transform of the Koecher-Maass series twisted by the eigenfunction. Thus we can analyze the Fourier series on the Siegel upper half-space by studying each spectral coefficient, in other words by studying each Koecher-Maass series. Using this principle Imai could formulate a converse theorem for Siegel modular forms. In order to study each Koecher-Maass series (each spectral coefficient) without assuming the modularity with respect to the inversion on the Fourier series, we use an explicit form of the Koecher-Maass series discovered by Böcherer and independently by Duke-Imamoglu. Their result implies that if the Fourier coefficients satisfy a Maass type relation then the Koecher-Maass series is a convolution product of certain Dirichlet series. At the first glance it is just a convolution product. To regard this convolution product as the Rankin-Selberg convolution of modular forms, Duke-Imamoglu employ Katok-Sarnak's correspondence for Maass forms. This allows us applying the Rankin-Selberg method to study the Koecher-Maass series. In our applications, the convolution products of two Eisenstein series arise frequently and we need to introduce Zagier's Rankin-Selberg method as the final key tool. Under these preparations, in section 3 we present two applications of these tools to holomorphic Siegel modular forms. In particular we will explain Duke-Imamoglu's new proof of Saito-Kurokawa lift and an explicit determination for the Fourier coefficients of Siegel-Eisenstein series on some congruence subgroups by the present author. Section 4 is devoted to give an application to non-holomorphic case. An application to the Koecher-Maass series for a real analytic Siegel-Eisenstein series will be given based on the works by Ibukiyama-Katsurada and the present author. In each section, we will add some related works.

2 Some basic tools

Let $\mathcal{P}_2$ be the set of all positive definite real symmetric matrices of size two and $\mathcal{PS}_2$ be the determinant one surface of $\mathcal{P}_2$. We identify $\mathcal{PS}_2$ with the upper half-plane H_1 by

$$\begin{pmatrix} v^{-1} & -uv^{-1} \\ -uv^{-1} & v^{-1}(u^2+v^2) \end{pmatrix} \to \tau = u + iv.$$

We mean by a Maass form of weght 0 any function $\mathcal{U}(\tau)$ on $H_1 = \{\tau = u + iv; v > 0\}$ satisfying the following three conditions.

(i)$\mathcal{U}(\gamma\tau) = \mathcal{U}(\tau)$ for all $\gamma \in SL_2(\mathbf{Z})$.
(ii)$\mathcal{U}(\tau)$ is a C^2-function on H_1 with respect to $u = \Re\tau, v = \Im\tau$ which verifies a differential equation $\Delta\mathcal{U} = -\lambda\mathcal{U}$ with some $\lambda \in \mathbf{C}$, where $\Delta = v^2(\frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial u^2})$ is the Laplacian on H_1.
(iii)$\mathcal{U}(\tau)$ is of polynomial growth as $v = \Im\tau$ tends to ∞.

A Maass form $\mathcal{U}(\tau)$ can be extended to a function on $\mathcal{P}_2$ by setting $\mathcal{U}(T) = \mathcal{U}(\tau_T)$, where τ_T corresponds to $(\det T)^{-1/2}T$, in other words $T \in \mathcal{P}_2$ is identified with $\tau_T \in H_1$ by

$$T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \to \tau_T = \frac{-b + i\sqrt{\det 2T}}{2a}.$$

2.1 Koecher-Maass series as spectral coefficients (Maass, Imai)

For any $Y \in \mathcal{P}_2$ put $Y = t^{1/2}W$, where $t > 0$ and $W \in \mathcal{PS}_2$. Take any Fourier series $F(Z) = \sum_{T \in L_2^+} A(T,F)e(\mathrm{tr}(TZ))$ on the Siegel upper half-space $H_2 = \{Z =^t Z \in M_2(\mathbf{C}); \Im Z > O\}$, where the summation extends over all $T \in L_2^+$, the set of all half-integral positive definite symmetric matrices of size two and $e(x) = e^{2\pi i x}$. Set

$$\tilde{F}_s(\tau) = \int_0^\infty F(it^{1/2}W)t^{s-1}dt, \quad \tau = \tau_W.$$

Assume that $A(T,F) = A(T[U],F)$ for any $U \in GL_2(\mathbf{Z})$ and there exists a positive constant α so that $A(T,F) = O((\det T)^\alpha)$. Then for $\Re(s)$ sufficiently large, $\tilde{F}_s(\tau)$ has a Roelcke-Selberg spectral decomposition as a function of τ [58], [27], [34], [59]. Each spectral coefficient with respect to a spectral eigenfunction $\mathcal{U}(\tau)$ on H_1 is given by

$$D^*(F,\overline{\mathcal{U}},s) = \int_{SL_2(\mathbf{Z})\backslash H_1} \tilde{F}_s(\tau)\overline{\mathcal{U}}(\tau)\frac{dudv}{v^2}.$$

Up to a gamma factor, this equals the Dirichlet series

$$D(F,\overline{\mathcal{U}},s) = \sum_{T \in L_2^+/SL_2(\mathbf{Z})} \frac{A(T,F)\overline{\mathcal{U}}(T)}{\epsilon(T)(\det T)^s},$$

where the summation extends over all $T \in L_2^+$ modulo the usual action $T \to T[U] = {}^tUTU$ of the group $SL_2(\mathbf{Z})$ and $\epsilon(T) = \sharp\{U \in SL_2(\mathbf{Z}); T[U] = T\}$ is the order of the unit group of T. This Dirichlet series was introduced by Maass [39] and now called by Koecher-Maass series twisted by a Maass form $\mathcal{U}(\tau)$. Key tools to formulate Imai's converse theorem are the spectral decomposition on the upper half-plane and the Koecher-Maass series as the spectral coefficients. Roughly speaking, for the Fourier series $F(Z)$ as above, the modularity of $F(Z)$ with respect to the inversion $Z \to -Z^{-1}$ is equivalent to a holomorphy and a functional equation of the Koecher-Maass series twisted by every spectral eigenfunctions. Note that the unimodular invariance and translation invariance are already assumed for the Fourier series, and these transformations combined with the inversion generate the full Siegel modular group.

Theorem 1. *(Imai) Let k be an even natural number. If $D^*(F,\mathcal{U},s)$ is entire, bounded in every vertical strip in s and satisfies*

$$D^*(F,\mathcal{U},s) = D^*(F,\mathcal{U},k-s)$$

for any even spectral eigenfunction $\mathcal{U}(\tau)$ on H_1, then $F(Z)$ is a holomorphic Siegel modular form of degree two and weight k.

See [23] by Ibukiyama for very detail exposition. The view point "Koecher-Maass series as spectral coefficients" is very useful, in particular to investigate the question when two Siegel modular forms coincide. In fact using a similar approach to the converse theorem, Breulman-Kohnen [6] showed that Hecke eigen cuspforms of degree two coincide if their T-th Fourier coefficients coincide for every $T = nT_1$ with any primitive T_1 and square-free natural number n. We will see one more application of this view point in section 3.2. As for Weissauer's converse theorem, a generalization of the converse theorem to higher degree with level, see Sugano's exposition in [25]. A general theory of Koecher-Maass series associated with modular forms on tube domains is obtained in [24].

2.2 Explicit forms for Koecher-Maass series (Böcherer, Ibukiyama-Katsurada)

In order to analyze each spectral coefficient without assuming the modularity with respect to the inversion on the Fourier series, the following result due to Böcherer gives a starting point (see Satz 3 [7] p.20). This is also observed by Duke-Imamoglu (Lemma 3 [10] p.350). This implies that if the Fourier coefficients satisfy a Maass type relation then the Koecher-Maass series (the spectral coefficient) is a certain convolution product.

Theorem 2. *(Böcherer, Duke-Imamoglu) Suppose that there exists a function $c(n), n \in \mathbf{N}$ on the set of all positive integers such that*

$$A(T,F) = \sum_{d|e(T)} d^{k-1} c\left(\frac{\det 2T}{d^2}\right),$$

where $e(T) = (n,r,m)$ is the content of $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$. Then one has

$$D(F,\mathcal{U},s) = 2^{2s}\zeta(2s-k+1)\sum_{n=1}^{\infty}\frac{c(n)b(-n)n^{3/4}}{n^s}, \tag{1}$$

where

$$b(-n) = n^{-3/4} \sum_{T\in L_2^+/SL_2(\mathbf{Z}),\ \det 2T=n} \mathcal{U}(T)\epsilon(T)^{-1}.$$

This is the origin of the theory of explicit forms. In [7], Böcherer also obtained an explicit form of associated one for Klingen-Eisenstein series of degree two. The result turned out to be Kohnen-Zagier's type Dirichlet series. These were generalized to any degree by Ibukiyama-Katsurada [16], [17], [18], [19]. They gave explicit forms of the Koecher-Maass series associated with Siegel-Eisenstein series, Klingen-Eisenstein series and the images of Ikeda lift by a finite sum (at most two) of products of shifted Riemann zeta functions and Dirichlet series defined from modular forms of one variable. In the case of even degree, the later Dirichlet series are similar to those appeared in Böcherer's degree two case. For automorphic forms on other types of tube domains, Krieg [32] showed that the Koecher-Maass series associated with quaternionic Siegel-Eisenstein series of degree two is a product of four shifted Riemann zeta functions with an additional 2-factor. For Hermitian-Eisenstein series of degree two over imaginary quadratic fields whose class number is one, the present author [45] showed that the Koecher-Maass series can be expressed by a finite sum of products of four shifted Dirichlet L-functions. Also we can add results by Lippa [35], [36] who gave explicit forms of the Koecher-Maass series for some Hecke eigenforms of degree two twisted by the non-holomorphic Eisenstein series $E(\tau,s)$ on $SL_2(\mathbf{Z})$.

2.3 Katok-Sarnak correspondence (Katok-Sarnak, Duke-Imamoglu)

In some actual applications, at least as far as we know, one can assume that $c(n)$ in (1) are the Fourier coefficients of a modular form. We are interested in $b(-n)$ which itself is an important object. Katok-Sarnak's correspondence implies that $b(-n)$ are also the Fourier coefficients of a (real analytic) modular form. To state the correspondence, we introduce Maass wave form of weight 1/2. Let

$$j(\gamma,\tau)=\frac{\theta(\gamma\tau)}{\theta(\tau)},\quad \theta(\tau)=\sum_{n\in\mathbf{Z}}e^{2\pi in^2\tau},\quad \gamma\in\Gamma_0(4).$$

For $r\in\mathbf{C}$ let T_r^+ denote the vector space consisting of all functions $g(\tau)$ on the upper half-plane H_1 which satisfy the following conditions.

(i) Each $g(\tau)$ is a C^2-function of $u=\Re\tau$ and $v=\Im\tau$ satisfying

$$g(\gamma\tau)=g(\tau)j(\gamma,\tau)|c\tau+d|^{-1/2}$$

for all $\gamma\in\Gamma_0(4)$ and it is of polynomial growth at all cusps of $\Gamma_0(4)$.
(ii)$g(\tau)$ has a Fourier expansion of the form $g(\tau)=\sum_{n\in\mathbf{Z}}B(n,v)e(nu)$, where the Fourier coefficients $B(n,v)$ for $n\neq 0$ are given by

$$B(n,v)=b(n)W_{sign(n)/4,ir/2}(4\pi|n|v).$$

Here $W_{\alpha,\beta}(v)$ is the usual Whittaker function.
(iii)If $n\equiv 2,3 \pmod 4$ then $B(n,v)=0$.

By Katok-Sarnak [28] for cusp forms and Duke-Imamoglu [10] (Theorem 4 p.350) for non-cusp forms, we have the following.

Theorem 3. *(Katok-Sarnak, Duke-Imamoglu) Let $\mathcal{U}(\tau)$ be a Maass form of weight 0 in the set consisting of a constant $\sqrt{3/\pi}$, the non-holomorphic Eisenstein series $E(\tau,1/2+ir)$ ($r\in\mathbf{R}$) on $SL_2(\mathbf{Z})$ and even cusp forms. Assume that $\Delta\mathcal{U}=-\left(\frac{1}{4}+r^2\right)\mathcal{U}$ with some $r\in\mathbf{C}$. Then there exists $g(\tau)\in T_r^+$ such that*

$$b(-n)=n^{-3/4}\sum_{T\in L_2^+/SL_2(\mathbf{Z}),\ \det 2T=n}\mathcal{U}(T)\epsilon(T)^{-1},\qquad (n>0).$$

One might recall Zagier's Eisenstein series of weight 3/2 whose Fourier coefficients are the Hurwitz-Kronecker class numbers [15]. Recall that the Hurwitz-Kronecker class number is a (weighted) class number of positive definite binary quadratic forms of discriminant $-n<0$, and equals $b(-n)n^{3/4}$ when $\mathcal{U}(\tau)$ is a suitable constant function. In [41], [42] we obtain an analogous result to Theorem 3 for automorphic functions on the three dimensional hyperbolic space. See also [40], [46]. This has some applications to Hermitian modular forms.

2.4 Zagier's Rankin-Selberg method (Zagier, Dutta Gupta, [44])

Assuming that $c(n)$ in (1) are the Fourier coefficients of a modular form, Theorem 3 implies that the series in (1) is a Rankin-Selberg convolution of two modular forms. Note that the usual Rankin-Selberg method can not work frequently, because both modular forms are not always cusp forms. If $\mathcal{U}(\tau)$ is a non-cusp form then $g(\tau)$ in Theorem 3 comes from a real analytic Cohen's Eisenstein series $F(k,\sigma,\tau)$ (see section 4). Also $c(n)$ might be Fourier coefficients of Eisenstein series. Thus we present Zagier's Rankin-Selberg method.

Theorem 4. *(Zagier, Dutta Gupta, [44]) Even if a real analytic modular form is not of rapid decay at all cusps, under suitable assumptions we can conclude a similar consequence about analytical properties for the Mellin transform of its constant term as if we apply the usual Rankin-Selberg method.*

For precise formulations, see [60], [12], [13], [14] and [44]. In Theorem 4, the Mellin transform is of course renormalized as in the case of Hecke on Mellin transfrms of non-cusp forms. In other words something wrong for the convergence of the integral transform are subtracted from the constant term to take its Mellin transformation. Also note that, for a product of two (real analytic) modular forms, the Mellin transform of its constant term is associated convolution product. Kudla simplified Zagier's "suitable assumptions" to the assumption that a real analytic modular form applied a suitable invariant differential operator (a polynomial of the Laplacian) is of rapid decay at all cusps. If we apply Kudla's method then we can take the integral transform directly without renormalization. The remaining process is completely the same as the usual Rankin-Selberg method. See [44] for example. Zagier's method is a basic tool to study Koecher-Maass series as we will see in the following sections. For example if we treat Koecher-Maass series for Siegel type Eisenstein series, we naturally arrive at a Rankin-Selberg convolution of two Eisenstein series. Thus Zagier's method is essential. As a related result, see [8] by Chiera.

3 Applications to holomorphic case

3.1 Saito-Kurokawa lift (Duke-Imamoglu)

Take a cusp form $f(\tau)$ in the Kohnen plus space $S^+_{k-1/2}(k:\text{even})$. By definition this has a Fourier expansion of the form

$$f(\tau) = \sum_{l\geq 1,\ l\equiv 0,3 \pmod 4} c(l)e(l\tau) \in S^+_{k-1/2}.$$

Duke-Imamoglu gave a new proof of Saito-Kurokawa lift.

Theorem 5. *(Maass, Zagier, Duke-Imamoglu) For $f(\tau) \in S^+_{k-1/2}$ as above, we define a function $F(Z)$ on H_2 by*

$$F(Z) = \sum_{T\in L_2^+} \left(\sum_{d|e(T)} d^{k-1} c\left(\frac{\det 2T}{d^2}\right)\right) e(\mathrm{tr}(TZ)),$$

Then $F(Z)$ is a Siegel modular form of degree two and even weight k.

See [11] for more details about Saito-Kurokawa lift. Especially the original proof and Hecke equivalence are given. In their analytic proof, Duke-Imamoglu used some tools given in section 2. See [23] once more for very detail exposition. Roughly speaking, by Theorems 2 and 3 the Koecher-Maass series equals the convolution product of $f(\tau)$ and $g(\tau)$, where $g(\tau)$ is as in Theorem 3. Then the Rankin-Selberg method implies the analytic conditions assumed in the converse theorem (Theorem 1). Note that Zagier's method is not necessary since $f(\tau)$ is a cusp form. In [4] a converse theorem for not necessarily cuspidal Siegel modular forms was given and Zagier's Rankin-Selberg method was used. As for odd weight case, see [3].

3.2 Explicit form for the Fourier coefficients of Siegel-Eisenstein series ([47])

The tools given in section 2 are also useful to analyze each spectral coefficient in another direction. These can be used for an explicit determination of the spectral coefficients of Siegel-Eisenstein series on certain congruence subgroups without knowing its Fourier coefficients. Note that so far any explicit form of Koecher-Maass series (=spectral coefficients) was obtained by using some informations of its Fourier coefficients. On the other hand, after working out this calculation of every spectral coefficients for the Siegel-Eisenstein series, we can construct a Siegel modular form which has the same spectral coefficients by Maass lift. As a consequence, it turned out that the Siegel-Eisenstein series coincides with the image of Maass lift and this implies an explicit form of the Fourier coefficients of the Siegel-Eisenstein series.

For any integer $k > 3$ and Dirichlet character χ mod N satisfying $\chi(-1) = (-1)^k$, the Siegel-Eisenstein series $E^{(2)}_{k,\overline{\chi}}$ of weight k, level N and character χ is defined by

$$E^{(2)}_{k,\overline{\chi}}(Z) = \sum_{\{C,D\}} \overline{\chi}(\det D)\det(CZ+D)^{-k}, \qquad Z \in H_2,$$

where the sum is taken over all pairs $\{C, D\}$ which occur as the second matrix row of representatives of $\Gamma^{(2)}_\infty \backslash \Gamma^{(2)}_0(N)$, $\Gamma^{(2)}_0(N) = \{\gamma \in Sp_2(\mathbf{Z}); C \equiv O_2 \pmod N\}$, $\Gamma^{(2)}_\infty = \{\gamma \in Sp_2(\mathbf{Z}); C = O_2\}$.

Theorem 6. *Let $k > 3$ be an integer, $N > 1$ a square-free odd natural number and χ a primitive Dirichlet character mod N satisfying $\chi(-1) = (-1)^k$. Then for any positive definite $T \in L_2^+$, the T-th Fourier coefficient of $E^{(2)}_{k,\overline{\chi}}$ is given by*

$$A(T, E^{(2)}_{k,\overline{\chi}}) = \frac{(-2\pi i)^k \tau_N(\overline{\chi})}{N^k\Gamma(k)L(k,\overline{\chi})} \sum_{d|e(T)} \chi(d)d^{k-1}e^\infty_{\overline{\chi}}\left(\frac{-\det 2T}{d^2}\right),$$

where $\tau_N(\overline{\chi})$ is the Gauss sum $\tau_N(\overline{\chi}) = \sum_{r=1}^N \overline{\chi}(r)e^{2\pi i r/N}$, $\Gamma(s)$ is the gamma function, $L(s,\overline{\chi})$ is the Dirichlet L-function of $\overline{\chi}$, $e(T)$ is the content of T and $e^\infty_{\overline{\chi}}(D)$ has the form

$$e^\infty_{\overline{\chi}}(D) = \frac{\pi^{k-1/2}\overline{\chi}(-4)}{i^k 2^{k-2}\Gamma(k-1/2)}|D|^{k-3/2}\frac{L(k-1,\chi_K\overline{\chi})}{L(2k-2,\overline{\chi}^2)}$$
$$\times \prod_{prime\ p|N}\left\{\sum_{e=1}^{1+ord_pD} \frac{\overline{\chi^*_p}(p^e)}{p^{(k-1/2)e}}\epsilon^3_{p^e}C^\infty_{\overline{\chi},p}(D,p^e)\right\}\sum_{d|f}\mu(d)\chi_K(d)\overline{\chi}(d)d^{1-k}\sigma_{3-2k,\overline{\chi}^2}(f/d).$$

Here we use the following notations. Let μ be the Möbius function, ord_pD an integer such that p^{ord_pD} is the exact power of p dividing D and $\sigma_{s,\overline{\chi}^2}(f) = \sum_{d|f}\overline{\chi}^2(d)d^s$. The natural number f is defined by $D = D_Kf^2$ with the discriminant D_K of $K = \mathbf{Q}(\sqrt{D})$ and $\chi_K(*) = \left(\frac{D_K}{*}\right)$ is the Kronecker symbol of K. We denote by χ_p the primitive characters mod p so that $\chi = \prod_{prime\ p|N}\chi_p$ and define χ^*_p by $\chi^*_p = \prod_{prime\ q|(N/p)}\chi_q$. As usual $\epsilon_d = 1$ or i according to $d \equiv 1$ or $3 \pmod 4$. If we denote by $\tau_p(\chi)$ the Gauss sum $\tau_p(\chi) = \sum_{r=1}^p \chi(r)e^{2\pi i r/p}$ for any character χ mod p, $\left(\frac{*}{p}\right)$ the Legendre symbol and $m = ord_pD$, then $C^\infty_{\overline{\chi},p}(D,p^e)$ are given by
(a)for $e \le m$,

$$C^\infty_{\overline{\chi},p}(D,p^e) = \begin{cases} p^{e-1}(p-1), & \chi_p = \left(\frac{*}{p}\right) \text{ and } e \text{ is odd},\\ 0, & \text{otherwise},\end{cases}$$

(b)for $e = m+1$,

$$C^\infty_{\overline{\chi},p}(D,p^e) = \chi_p(D/p^m)\left(\frac{D/p^m}{p}\right)^{m+1} p^m\tau_p(\overline{\chi_p}\left(\frac{*}{p}\right)^{m+1}),$$

(c)for $e \ge m+2$, $C^\infty_{\overline{\chi},p}(D,p^e) = 0$.

In the case of level $N = 1$, we know at least two methods to obtain explicit forms of the Fourier coefficients. The first is due to Kaufhold [30] and Maass [39]. They used Siegel's formula which gives an Euler product expression for the Fourier coefficients and calculated each Euler factor to get their formulas. The second is due to Eichler-Zagier [11]. They showed that the Saito-Kurokawa lift of Cohen's Eisenstein series coincides with the Siegel-Eisenstein series of level one by employing a characterization of the series in terms of the Hecke operators and zero-th coefficient. Our method in the case of level $N > 1$ is completely different from theirs. However in some sense we owe several results and ideas to Kaufhold and Eichler-Zagier. First of all we construct a Siegel modular form $\mathcal{M}E^{\infty}_{k,1,\overline{\chi}}$ by Maass lift of a Jacobi Eisenstein series $E^{\infty}_{k,1,\overline{\chi}}$ for the cusp ∞ on $\Gamma_0(N) \ltimes \mathbf{Z}^2$. Then Theorem 6 follows from the coincidence of two Siegel modular forms $E^{(2)}_{k,\overline{\chi}}$ and $\mathcal{M}E^{\infty}_{k,1,\overline{\chi}}$ up to a scalar multiple. Note that the right hand side on the main formula in Theorem 6 is exactly the Fourier coefficient of the image of Maass lift and $e^{\infty}_{\overline{\chi}}(D)$ is the Fourier coefficient of $E^{\infty}_{k,1,\overline{\chi}}$. One might recall that in Eichler-Zagier's level one case, the Saito-Kurokawa lift of Cohen's Eisenstein series coincides with the Maass lift of Jacobi Eisenstein series on $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$. To prove the desired coincidence of two Siegel modular forms, unlike Eichler-Zagier's case, we show a coincidence of associated Koecher-Maass series twisted by an eigenfunction $\mathcal{U}$ (=spectral coefficient with respect to $\mathcal{U}$) up to a scalar multiple for each $\mathcal{U}$ as follows. $D^*(\mathcal{M}E^{\infty}_{k,1,\overline{\chi}}, \mathcal{U}, s)$ can be determined easily by a character analogue of Theorem 2. To calculate $D^*(E^{(2)}_{k,\overline{\chi}}, \mathcal{U}, s)$, we start to determine $D^*(F^{(2)}_{k,\overline{\chi}}, \mathcal{U}, s)$ for the twisted Siegel-Eisenstein series $F^{(2)}_{k,\overline{\chi}}(Z) = N^{-k} \det Z^{-k} E^{(2)}_{k,\overline{\chi}}(-(NZ)^{-1})$. An explicit formula for the Fourier coefficients of $F^{(2)}_{k,\overline{\chi}}$ is available by the explicit form of the Siegel series [30]. This combined with Theorem 3 implies that $D^*(F^{(2)}_{k,\overline{\chi}}, \mathcal{U}, s)$ can be regarded as the Rankin-Selberg transform of a certain modular form on $\Gamma_0(N)$. By a general theory, we know $D^*(E^{(2)}_{k,\overline{\chi}}, \mathcal{U}, k-s) = (-1)^k D^*(F^{(2)}_{k,\overline{\chi}}, \mathcal{U}, s)$. Hence we can compute $D^*(E^{(2)}_{k,\overline{\chi}}, \mathcal{U}, s)$ from the explicit formula of $D^*(F^{(2)}_{k,\overline{\chi}}, \mathcal{U}, s)$ by Zagier's Rankin-Selberg method (Theorem 4). We remark that involved modular forms are not always cuspidal depending on Maass forms $\mathcal{U}(\tau)$. Thus we can not apply the usual Rankin-Selberg method and Zagier's method is necessary. Roughly speaking, the coincidence of the Koecher-Maass series (=the spectral coefficients) implies the coincidence of two Siegel modular forms $E^{(2)}_{k,\overline{\chi}}$ and $\mathcal{M}E^{\infty}_{k,1,\overline{\chi}}$ up to a scalar multiple. The Fourier coefficients of the Maass lift $\mathcal{M}E^{\infty}_{k,1,\overline{\chi}}$ can be described easily in terms of those of $E^{\infty}_{k,1,\overline{\chi}}$ as given in the right hand side on the main formula. Then our formula follows from an explicit calculation of the Fourier coefficients $e^{\infty}_{\overline{\chi}}(D)$ of $E^{\infty}_{k,1,\overline{\chi}}$. Strictly speaking we used the fact that two Siegel modular forms have the same Fourier coefficients for any degenerate T. This follows from applying the Siegel operator and ensures that a difference of two forms has a Fourier expansion supported by positive definite $T \in L^+_2$. This difference turned out to be zero by above observation on the Koecher-Maass series and thus we get the coincidence of two Siegel modular forms.

3.2.1 An example ([48], [49])

For primes dividing the level N, each Euler factor looks unusual. If the level N is a prime p and a suitable linear combination of two Eisenstein series is chosen, then we can get a simple formula. Let λ_p be the Legendre symbol, $k > 3$ an integer such that $k \equiv (p-1)/2 \pmod 2$. Let $F^{(2)}_{k,\lambda_p}(Z) = p^{-k} \det Z^{-k} E^{(2)}_{k,\lambda_p}(-(pZ)^{-1})$ as before. Then one has $E^{(2)}_{k,\lambda_p}, F^{(2)}_{k,\lambda_p} \in M_k(\Gamma^{(2)}_0(p), \lambda_p)$, the space of all Siegel modular forms of degree two, weight k and character λ_p on $\Gamma^{(2)}_0(p)$. A linear combination of Siegel-Eisenstein series is defined by

$$\mathcal{E}^{(2)}_{k,\lambda_p} = \frac{-B_{k,\lambda_p}}{2k}\{E^{(2)}_{k,\lambda_p} + (-1)^k p^{1-k} F^{(2)}_{k,\lambda_p}\}.$$

Recall that the Bernoulli numbers B_m and the generalized Bernoulli numbers $B_{m,\chi}$ are defined by

$$\sum_{m=0}^{\infty} B_m \frac{t^m}{m!} = \frac{te^t}{e^t-1}, \qquad \sum_{m=0}^{\infty} B_{m,\chi}\frac{t^m}{m!} = \sum_{a=1}^{f} \frac{\chi(a)te^{at}}{e^{ft}-1}$$

for any primitive character χ of conductor f.

Theorem 7. *$\mathcal{E}^{(2)}_{k,\lambda_p}$ has a Fourier expansion*

$$\mathcal{E}^{(2)}_{k,\lambda_p}(Z) = \sum_{T\in L_2,\ p|\det 2T} \mathcal{A}_k(T) e(\mathrm{tr}(TZ)),$$

where the summation extends over all $T \in L_2$ (the set of all half-integral positive semi-definite symmetric matrices of size two) such that $\det 2T$ is divisible by p and $\mathcal{A}_k(T)$ are given as follows. For degenerate T, the Fourier coefficients come from a degree one Eisenstein series, in other words $\mathcal{A}_k(O_2) = -B_{k,\lambda_p}/(2k)$ and $\mathcal{A}_k(T) = \sum_{d|e(T)} \lambda_p(d)d^{k-1}$ for $rkT = 1$. For $T > O$ such that $\det 2T$ is divisible by p, we have

$$\mathcal{A}_k(T) = \frac{2B_{k-1,\chi_{D_K}}}{B_{2k-2}} \sum_{d|e(T)} \lambda_p(d)d^{k-1} \sum_{a|f/d} \mu(a)\chi_{D_K}(a)a^{k-2}\sigma_{2k-3}\left(\frac{f/d}{a}\right),$$

where $B_{k-1,\chi}$ and B_{2k-2} are the Bernoulli numbers, if we put $p^* = (-1)^{(p-1)/2}p$ then D_K is the discriminant of $K = \mathbf{Q}\left(\sqrt{\frac{-\det 2T}{p^*}}\right)$, the natural number f is defined by $-\det 2T = p^* D_K f^2$, χ_{D_K} is the Kronecker symbol of K, $e(T)$ is the content of T, $\sigma_s(n) = \sum_{d|n} d^s$ and μ is the Möbius function.

This formula has some applications to a question raised by Nagaoka [53], [29]: if every Fourier coefficients of Eisenstein series (of level one) are replaced by their p-adic limit, then what is the resulting formal power series? Is it also a modular form? In [29], Katsurada-Nagaoka introduced a p-adic Siegel-Eisenstein series $\tilde{G}^{(2)}_{(k,\frac{p+2k-1}{2})}$ of weight k. Let $E^{(2)}_k = E^{(2)}_{k,\chi_0}$ be the Siegel-Eisenstein series of degree two, weight k, level 1 and the principal character χ_0. This has a Fourier expansion with respect to $e(tr(TZ))$ indexed by $T \in L_2$. Take a prime p and a natural number k such that $p > 2k$ and $k \equiv \frac{p-1}{2} \pmod 2$. Put $k_m = k + p^{m-1}(p-1)/2$. If there exsist $B(T)$ such that $inf_{T\in L_2}\{ord_p(B(T) - A_m(T))\} \to \infty$ as $m \to \infty$, then a p-adic convergence is defined by

$$\lim_{m\to\infty}\left\{\sum_{T\in L_2} A_m(T)e(\mathrm{tr}(TZ))\right\} = \sum_{T\in L_2} B(T)e(\mathrm{tr}(TZ)) \ \ (p-adically).$$

They defined a p-adic Siegel-Eisenstein series by

$$\tilde{G}^{(2)}_{(k,\frac{p+2k-1}{2})} = \lim_{m\to\infty}\frac{-B_{k_m}}{2k_m}E^{(2)}_{k_m} \ \ (p-adically)$$

and described it mainly by some genus theta series. Our description is as follow and Theorem 7 was important.

Theorem 8. *For $k > 3$, we have*

$$\tilde{G}^{(2)}_{(k,\frac{p+2k-1}{2})} = \frac{-B_{k,\lambda_p}}{2k}\left\{E^{(2)}_{k,\lambda_p} + (-1)^k \frac{p^{k-2}(1-p)}{p^{2k-3}-1} F^{(2)}_{k,\lambda_p}\right\}.$$

See [49] for the proof. Also we can add one more example for a similar phenomenon [48].

4 Applications to non-holomorphic case (Ibukiyama-Katsurada, [50], [51], [52])

In his lecture note [38], Maass raised a question "whether it is possible to attach Dirichlet series by means of integral transforms to the non analytic Eisenstein series" and also said that "already in the case degree is two difficulties come up which show that one can not proceed in the usual way". Because of complexity of Fourier expansions of non-holomorphic Siegel modular forms unlike the holomorphic case, there are at least two main problems to study associated Koecher-Maass series. The first is about the Mellin transform of Whittaker functions, and the second is how one can delete the degenerate Fourier coefficients. In the holomorphic case these are solved by Siegel's formula on the Mellin transform of exponential functions and Maass' differential operator. For a real analytic Siegel-Eisenstein series whose degree exceeds two, the associated Koecher-Maass series was first introduced by Arakawa [1] directly from the Siegel series (an arithmetic part of the Fourier coefficients). Some basic analytical properties were established by Arakawa [1] and Ibukiyama-Katsurada [19] combined with our results in [50]. The later is also an application of the theory of explicit form and contains a simplification of Arakawa's functional equations. Ibukiyama and Katsurada got explicit forms of the Koecher-Maass series as a finite sum of products of shifted Riemann zeta functions and the Rankin-Selberg convolution of certain real analytic Eisenstein series of half-integral weight. Hence we can concentrate our attention on this convolution product. By the tools in section 2 we can show a meromorphic continuation and a simple functional equation for this Rankin-Selberg convolution. In this section we would like to present some flavor of our study more precisely in the case of degree two as a typical example (see [19] and [50] for the case degree exceeds two). Moreover this degree two case has independent interests in the following sense. It seems natural to expect defining associated Koecher-Maass series by

$$L^{(i)}(s,\sigma) = \sum_{T \in L_2^{(i)}/SL_2(\mathbf{Z})} \frac{a(T,\sigma)\mu(T)}{|\det T|^s},$$

where $a(T,\sigma)$ is proportional to the Siegel series (an arithmetic part of the Fourier coefficients), the summation extends over all half-integral symmetric matrices of size two and signature $(i, 2-i)$, $\mu(T)$ is a certain volume associated with T introduced by Siegel. See [57], [22], [19] for the precise definition for $\mu(T)$ and note that in case $i = 2$, it is proportional to $\epsilon(T)^{-1}$. In case $i = 1$ it is known that if $-\det T$ is a square of a rational number then $\mu(T)$ is not finite. This surves a serious difficulty to define Koecher-Maass series reasonably. The same difficulty comes up when we treat the prehomogeneous zeta function associated with the space of two by two symmetric matrices. This was solved by Shintani [57]. There is another approach due to Ibukiyama-Saito [23], [22] as an application of explicit forms of the zeta functions. They proved all of Shintani's results by using real analytic Cohen's Eisenstein series of half-integral weight. Their method gives us a reasonable definition of the zeta function very naturally. It seems possible to define and treat associated Koecher-Maass series by a "convolution version" of their approach as we will see below. The purpose of this section is to explain our attempt in the case of degree two and give a meromorphic continuation and a functional equation of the Koecher-Maass series for positive definite Fourier coefficients. In the future we will use the results given below to get a reasonable definition of Koecher-Maass series for the indefinite Fourier coefficients.

Let k be an even integer and σ a complex number such that $2\Re\sigma + k > 3$. A real analytic Siegel-Eisenstein series of degree two and weight k is defined by

$$E_{2,k}(Z,\sigma) = \sum_{\{C,D\}} \det(CZ+D)^{-k}|\det(CZ+D)|^{-2\sigma}, \qquad Z \in H_2,$$

where the sum is taken over all pairs $\{C, D\}$ which occur as the second matrix row of representatives of $\Gamma_\infty^{(2)} \setminus Sp_2(\mathbf{Z})$. A Fourier expansion is given by

$$E_{2,k}(Z,\sigma) = \sum_{T \in L_2^*} C(T,\sigma,Y)e(\mathrm{tr}(TX)), \qquad Z = X + iY \in H_2,$$

where the summation extends over all $T \in L_2^*$ (the set of all half-integral symmetric matrices of size two). If $\det T \neq 0$ then the Fourier coefficients can be written as $C(T,\sigma,Y) = b(T, k+2\sigma)\xi(Y,T,\sigma+k,\sigma)$, a product of the Siegel series $b(T, k+2\sigma)$ and a certain function $\xi(Y,T,\sigma+k,\sigma)$ (essentially the confluent hypergeometric function of degree two). See [55], [56], [38], [30]. Then following Ibukiyama-Katsurada [19], the Koecher-Maass series for positive definite Fourier coefficients is defined by

$$L_{2,k}^{(2)}(s,\sigma) = \sum_{T \in L_2^+/SL_2(\mathbf{Z})} \frac{a_{2,k}(T,\sigma)}{\epsilon(T)(\det T)^s},$$

$$a_{2,k}(T,\sigma) = \gamma_2(k+2\sigma)|\det 2T|^{k+2\sigma-3/2}2^2 b(T,k+2\sigma), \quad \gamma_2(\sigma) = e^{\pi i\sigma}\frac{\pi^{2\sigma-1/2}}{\Gamma(\sigma)\Gamma(\sigma-1/2)}.$$

Note that $b(T, k+2\sigma)$ has a meromorphic continuation to all σ.

Theorem 9. *Suppose that $\sigma \notin 1/4 + \mathbf{Z}/2$. Then the Koecher-Maass series*

$$L_{2,k}^*(s,\sigma) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-2\sigma-k+3/2)L_{2,k}^{(2)}(s,\sigma)$$

can be meromorphically continued to the whole s-plane. It satisfies a functional equation

$$\begin{aligned} & L_{2,k}^*(k+2\sigma-s,\sigma) = L_{2,k}^*(s,\sigma) \\ + \quad & 2\pi^{-k-2\sigma+1/2}\frac{\gamma_2(k+2\sigma)\zeta(k+2\sigma-1)}{\zeta(k+2\sigma)\zeta(2k+4\sigma-2)} \\ \times \quad & \frac{\sin\pi\sigma\sin\pi(s-\sigma)}{\cos\pi s\sin\pi(s-2\sigma)}\frac{\Gamma(s)\Gamma(s-2\sigma-k+3/2)}{\Gamma(s-1/2)\Gamma(s-2\sigma-k+1)}\zeta^*(2s-1)\zeta^*(2s-4\sigma-2k+2), \end{aligned}$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Our starting point is an explicit form of the Koecher-Maass series. An explicit formula of the Siegel series [30] combined with Theorem 2 implies

$$\begin{aligned} & L_{2,k}^{(2)}(s,\sigma) = 2^{2s+2}\gamma_2(k+2\sigma) \\ & \times\frac{\zeta(2s-k-2\sigma+1)}{\zeta(k+2\sigma)\zeta(2k+4\sigma-2)}\sum_{d>0}H(d)L_{-d}(k+2\sigma-1)d^{k+2\sigma-3/2-s}. \end{aligned}$$

Here $H(d)=\sum_{T\in L_2^+/SL_2(\mathbf{Z}),\ \det 2T=d}\epsilon(T)^{-1}$,

$$L_D(s)=\begin{cases}\zeta(2s-1), & D=0\\ L(s,\chi_{D_K})\sum_{a|f}\mu(a)\chi_{D_K}(a)a^{-s}\sigma_{1-2s}(f/a), & D\neq 0,\ D\equiv 0,1 \pmod 4\\ 0, & D\equiv 2,3 \pmod 4),\end{cases}$$

where the natural number f is defined by $D=D_Kf^2$ with the discriminant D_K of $K=\mathbf{Q}(\sqrt{D})$, χ_{D_K} is the Kronecker symbol, μ is the Möbius function and $\sigma_s(n)=\sum_{d|n}d^s$.

Following Ibukiyama-Saito [22], for an odd integer k, $\sigma\in\mathbf{C}$ such that $-k+2\Re\sigma-4>0$ and $\tau\in H_1$, the Cohen type Eisenstein series $F(k,\sigma,\tau)$ is defined by

$$F(k,\sigma,\tau)=E(k,\sigma,\tau)+2^{k/2-\sigma}(e^{2\pi i\frac{k}{8}}+e^{-2\pi i\frac{k}{8}})E(k,\sigma,-1/(4\tau))(-2i\tau)^{k/2},$$

where

$$E(k,\sigma,\tau)=(\Im\tau)^{\sigma/2}\sum_{d=1,odd}^{\infty}\sum_{c=-\infty}^{\infty}(\frac{4c}{d})\epsilon_d^{-k}(4c\tau+d)^{k/2}|4c\tau+d|^{-\sigma}$$

and $j(\gamma,\tau)=(\frac{4c}{d})\epsilon_d^{-1}(4c\tau+d)^{1/2}$ is the same as in section 2.3. This is a real analytic modular form of weight $-k/2$ on $\Gamma_0(4)$ and has a Fourier expansion

$$F(k,\sigma,\tau)=v^{\sigma/2}+v^{\sigma/2}\sum_{d=-\infty}^{\infty}c(d,\sigma,k)e^{2\pi idu}\tau_d(v,\frac{\sigma-k}{2},\frac{\sigma}{2}),\qquad \tau=u+iv,$$

where $\tau_d(v,\alpha,\beta)=\int_{-\infty}^{\infty}e^{-2\pi idu}\tau^{-\alpha}\overline{\tau}^{-\beta}du$ ([54], [43]). The d-th Fourier coefficient is given by

$$c(d,\sigma,k)=2^{k+3/2-2\sigma}e^{(-1)^{(k+1)/2}\frac{\pi i}{4}}\frac{L_{(-1)^{(k+1)/2}d}(\sigma-\frac{k+1}{2})}{\zeta(2\sigma-k-1)}.$$

As one can see from these facts, the Koecher-Maass series $L_{2,k}^{(2)}(s,\sigma)$ is the Rankin-Selberg convolution of two Cohen type Eisenstein series. Thus our main tool to prove Theorem 9 is Zagier's Rankin-Selberg method (Theorem 4). The gamma like factor is the Mellin transformation of a product of two $\tau_d(v,\alpha,\beta)$ and involves generalized hypergeometric series ${}_3F_2$. Its treatment is rather difficult and we employed two types of functional equations of ${}_3F_2$ to remove these unusual factors. Only the usual gamma functions remain in the gamma factor as stated in Theorem 9. To apply Zagier's Rankin-Selberg method, a useful observation is that there is a correspondence between a real analytic Jacobi Eisenstein series of index one and the Cohen type Eisenstein series by a similar way as in the holomorphic case. See [9] for the case $k=-3$ and [51] for general case including a skew-holomorphic analogue. As a bonus, this correspondence combined with the results in [5] implies a meromorphic continuation and a functional equation of $F(k,\sigma,\tau)$ with respect to σ as given in [51].

As a corollary of Theorem 9, we can deduce that the Dirichlet series

$$\xi(s)=\pi^{-2s}\Gamma(s)\Gamma(s-1/2)\zeta(2s-1)\sum_{d=1}^{\infty}H(d)^2d^{-s}$$

can be meromorphically continued to the whole s-plane, and satisfies a functional equation

$$\xi(2-s)=\xi(s)+2^{-3}\pi^{-3/2}\frac{\Gamma(s)}{\cos\pi s\Gamma(s-1)}\zeta^*(2s-1)\zeta^*(2s-2).$$

In fact $\xi(s)$ is the Koecher-Maass series for (non-holomorphic) Siegel-Eisenstein series of degree two and weight two ($k=2,\sigma=0$).

References

[1] T. Arakawa, Dirichlet series related to the Eisenstein series on the Siegel upper half-plane. Comment. Math. Univ. St. Paul. **27** (1978/79), no. 1, 29–42.

[2] T. Arakawa, Dirichlet series corresponding to Siegel's modular forms. Math. Ann. **238**, 157-173 (1978)

[3] T. Arakawa, Saito-Kurokawa lifting for odd weights. Comment. Math. Univ. St. Paul. **49** (2000), no. 2, 159–176.

[4] T. Arakawa, I. Makino, F. Sato, Converse theorem for not necessarily cuspidal Siegel modular forms of degree 2 and Saito-Kurokawa lifting. Comment. Math. Univ. St. Paul **50**, (2001), no. 2, 197–234.

[5] T. Arakawa, Real analytic Eisenstein series for the Jacobi group. Abh. Math. Sem. Univ. Hamburg **60** (1990), 131–148.

[6] S. Breulmann, W. Kohnen, Twisted Maass-Koecher series and spinor zeta functions. Nagoya Math. J. **155**, (1999), 153–160.

[7] S. Böcherer, Bemerkungen über die Dirichletreichen von Koecher und Maass. Math.Gottingensis des Schrift.des SFB. Geometry and Analysis Heft **68** (1986)

[8] F. L. Chiera, On Petersson products of not necessarily cuspidal modular forms. J. Number Theory **122**, (2007), 13–24.

[9] Y. Choie, Correspondence among Eisenstein series $E_{2,1}(\tau, z)$, $H_{3/2}(\tau)$ and $E_2(\tau)$. Manuscripta Math. **93** (1997), no. 2, 177–187.

[10] W. Duke, O. Imamoglu, A converse theorem and the Saito-Kurokawa lift. Internat. Math. Res. Notices 1996, no. 7, 347–355.

[11] M. Eichler, D. Zagier, The theory of Jacobi forms. Progress in Mathematics, 55. Birkhauser Boston, Inc., Boston, MA, 1985. v+148 pp.

[12] S. Dutta Gupta, The Rankin-Selberg method on congruence subgroups. Illinois J. Math. **44** (2000), no. 1, 95–103.

[13] S. Dutta Gupta, On the Rankin-Selberg method for functions not of rapid decay on congruence subgroups. J. Number Theory **62** (1997), no. 1, 115–126.

[14] S. Dutta Gupta, Xiaotie She, The GL(2) Rankin-Selberg method for functions not of rapid decay. J. Number Theory **71** (1998), no. 2, 159-165.

[15] F. Hirzebruch, D. Zagier, Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. Invent. Math. **36** (1976), 57–113.

[16] T. Ibukiyama, H. Katsurada, Squared Mobius function for half-integral matrices and its applications. J. Number Theory **86**, 78-117, (2001)

[17] T. Ibukiyama, H. Katsurada, An explicit formula for Koecher-Maass Dirichlet series for Eisenstein series of Klingen type. J. Number Theory **102**, 223–256 (2003)

[18] T. Ibukiyama, H. Katsurada, An explicit formula for the Koecher-Maass Dirichlet series for the Ikeda lifting. Abh. Math. Sem. Univ. Hamburg. **74**, 101–121 (2004)

[19] T. Ibukiyama, H. Katsurada, Koecher-Maass series for real analytic Siegel Eisenstein series, to appear in "Automorphic Forms and Zeta Functions, Proceedings of the conference in memory of Tsuneo Arakawa", 170–197, World Scientific (2006)

[20] T. Ibukiyama, H. Saito, On zeta functions associated to symmetric matrices. I. An explicit form of zeta functions. Amer. J. Math. **117** (1995), no. 5, 1097–1155.

[21] T. Ibukiyama, On Shintani's zeta functions of ternary zero forms (third approach). Research on prehomogeneous vector spaces (Japanese) (Kyoto, 1996). Surikaisekikenkyusho Kokyuroku No. 999 (1997), 10–20.

[22] T. Ibukiyama, H. Saito, On zeta functions associated to symmetric matrices (II), MPI preprint 97-37.

[23] T. Ibukiyama, A survey on the new proof of Saito-Kurokawa lifting after Duke and Imamoglu (in Japanese), in Report of the fifth summer school of number theory "Introduction to Siegel modular forms" (1997), 134-176.

[24] T. Ibukiyama, Koecher-Maass series on tube domains (in Japanese), in Proceedings of the first Autumn workshop on number theory "On Koecher-Maass series" (1999), 1-46.

[25] T. Ibukiyama ed, Proceedings of the first Autumn workshop on number theory "On Koecher-Maass series" (1999).

[26] K. Imai, Generalization of Hecke's correspondence to Siegel modular forms. Amer. J. Math. **102** (1980), no. 5, 903–936.

[27] H. Iwaniec, Spectral methods of automorphic forms. Second edition. Graduate Studies in Mathematics, 53. American Mathematical Society, Providence, RI; Revista Matematica Iberoamericana, Madrid, 2002. xii+220 pp.

[28] S. Katok, P. Sarnak, Heegner points, cycles and Maass forms, Israel J. Math. **84** (1993), no. 1-2, 193–227.

[29] H. Katsurada, S. Nagaoka, On some p-adic properties of Siegel-Eisenstein series. J. Number Theory **104** (2004), no. 1, 100–117.

[30] G. Kaufhold, Dirichletsche Reihe mit Funktionalgleichung in der Theorie der Modulfunktion 2. Grades. Math. Ann. **137** 1959 454–476.

[31] H. Klingen, Introductory lectures on Siegel modular forms. Cambridge Studies in Advanced Mathematics, 20. Cambridge University Press, Cambridge, 1990. x+162 pp.

[32] A. Krieg, Koecher-Maass-series for modular forms of quaternions. Manuscripta Math. **66**, 431-451 (1990)

[33] M. Koecher, Über Dirichlet-Reihen mit Funktionalgleichung. J. Reine Angew. Math. **192**, 1–23 (1953)

[34] T. Kubota, Elementary theory of Eisenstein series. Kodansha Ltd., Tokyo; Halsted Press, New York-London-Sydney, 1973. xi+110 pp.

[35] E. Lippa, On Fourier coefficients of Siegel modular forms of degree two. Amer. J. Math. **97**, 829–846 (1975)

[36] E. Lippa, A Dirichlet series associated to Eisenstein series of degree two. Math. Z. **137**, 5–8 (1974)

[37] H. Maass, Über die Fourierkoeffizienten der Eisensteinreihen zweiten Grades. Mat.-Fys. Medd. Danske Vid. Selsk. **38**, no. 14, 13 pp. (1972).

[38] H. Maass, Siegel's modular forms and Dirichlet series. Dedicated to the last great representative of a passing epoch. Carl Ludwig Siegel on the occasion of his seventy-fifth birthday. Lecture Notes in Mathematics, Vol. 216. Springer-Verlag, Berlin-New York, 1971. v+328 pp.

[39] H. Maass, Modulformen zweiten Grades und Dirichletreihen. Math. Ann. **122**, 90-108 (1950)

[40] R. Matthes, On some Poincare-series on hyperbolic space. Forum Math. **11** (1999), no. 4, 483–502.

[41] R. Matthes, Y. Mizuno, Theta lifting and formulas of Sarnak-Katok type, preprint.

[42] R. Matthes, Y. Mizuno, Two applications of the spectral theory on the three dimensional hyperbolic space to Hermitian modular forms, preprint.

[43] T. Miyake, Modular forms. Translated from the Japanese by Yoshitaka Maeda. Springer-Verlag, Berlin, 1989. x+335 pp.

[44] Y. Mizuno, The Rankin-Selberg convolution for Cohen's Eisenstein series of half integral weight. Abh. Math. Sem. Univ. Hamburg **75** (2005), 1–20.

[45] Y. Mizuno, An explicit formula of the Koecher-Maass series associated with Hermitian modular forms belonging to the Maass space. Manuscripta Math. **119** (2006), no. 2, 159–181.

[46] Y. Mizuno, On Fourier coefficients of Eisenstein series and Niebur Poincare series of integral weight, J. Number Theory, accepted for publication.

[47] Y. Mizuno, An explicit arithmetic formula for Fourier coefficients of Siegel-Eisenstein series of degree two with square-free odd levels, submitted to Math. Z.

[48] Y. Mizuno, A p-adic limit of Siegel-Eisenstein series of prime level q, Int. J. Number Theory, accepted for publication.

[49] Y. Mizuno, On p-adic Siegel-Eisenstein series of weight k, Acta. Arith, accepted for publication.

[50] Y. Mizuno, The Rankin-Selberg convolution for real analytic Cohen's Eisenstein series of half integral weight, submitted to J. London Math. Soc.

[51] Y. Mizuno, Real analytic Jacobi Eisenstein series of index one and Cohen's Eisenstein series with a parameter, submitted to Adv. Stud. Pure Math.

[52] Y. Mizuno, The Koecher-Maass series associated with real analytic Siegel-Eisenstein series of degree two, preprint.

[53] S. Nagaoka, A remark on Serre's example of p-adic Eisenstein series. Math. Z. **235** (2000), 227–250.

[54] G. Shimura, On the holomorphy of certain Dirichlet series. Proc. London Math. Soc. (3) **31** (1975), no. 1, 79–98.

[55] G. Shimura, Confluent hypergeometric functions on tube domains. Math. Ann. **260** (1982), no. 3, 269–302.

[56] G. Shimura, On Eisenstein series. Duke Math. J. **50** (1983), no. 2, 417–476.

[57] T. Shintani, On zeta-functions associated with the vector space of quadratic forms. J. Fac. Sci. Univ. Tokyo Sect. I A Math. **22** (1975), 25–65.

[58] A. Terras, Harmonic analysis on symmetric spaces and applications. I. Springer-Verlag, New York, 1985. xiv+341 pp.

[59] A. B. Venkov, Spectral theory of automorphic functions and its applications. Translated from the Russian by N. B. Lebedinskaya. Mathematics and its Applications (Soviet Series), 51. Kluwer Academic Publishers Group, Dordrecht, 1990. xiv+176 pp.

[60] D. Zagier, The Rankin-Selberg method for automorphic functions which are not of rapid decay, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), 415-437.

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