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**Abstract**

We give the first examples of smooth Fano and Calabi–Yau varieties violating the (narrow) canonical strip hypothesis. They are given by moduli spaces of rank 2 bundles with fixed odd-degree determinant on curves of sufficiently high genus, hence our Fano examples have Picard rank 1, index 2, are rational, and have moduli. The hypotheses also fail for several other closely related varieties.

## 1 The canonical strip hypotheses

Associated to a polarisation of a smooth projective variety $X$ we can consider its Hilbert polynomial. The complex roots of this polynomial satisfy a symmetry property induced by Serre duality. In [7] Golyshev introduced further constraints on these roots: the (narrow) canonical strip hypothesis. The motivation for these restrictions comes from Yau's inequalities on characteristic numbers. At the end of this introduction we give a quick summary of the positive results regarding these hypotheses.

To state (and generalise) the canonical strip hypothesis we will use the following definition.

**Definition 1.** A pair $(X, H)$ of a normal variety and an ample line bundle is said to be *monotone of index r* if

$$(1)\qquad \mathrm{c}_1(X) = -\mathrm{K}_X \equiv rH,$$

where the symbol $\equiv$ denotes numerical equivalence of divisors.

The case of $H = -\mathrm{K}_X$ (resp. $H = \mathrm{K}_X$) as considered in [7] for a Fano variety (resp. variety with $\mathrm{K}_X$ ample) has index 1 (resp. $-1$). We will also consider polarised Calabi–Yau varieties, for which $r = 0$.

By Serre duality we have that

$$(2)\qquad \chi(nH) = (-1)^{\dim X}\chi(-(r+n)H).$$

Hence the roots of the Hilbert polynomial are symmetric around the line $-r/2$. Golyshev introduced the following further constraints on the real parts of the roots of the Hilbert polynomial.

**Definition 2.** Let $X$ be a smooth projective variety, and $H$ an ample line bundle, such that $(X, H)$ is monotone polarised of index $r$. Let $\alpha_1, \ldots, \alpha_{\dim X}$ be the real parts of the roots of the Hilbert polynomial associated to $H$. Then we say that $X$ satisfies

**(CL)** the *canonical line hypothesis* if

$$\alpha_i = r/2, \tag{3}$$

**(NCS)** the *narrow canonical strip hypothesis* if $r \leq 0$ and

$$\alpha_i \in \left[-r + \frac{r}{\dim X + 1}, -\frac{r}{\dim X + 1}\right] \tag{4}$$

if $r \geq 0$, and

$$\alpha_i \in \left[\frac{-r}{\dim X + 1}, -r - \frac{r}{\dim X + 1}\right] \tag{5}$$

otherwise,

**(CS)** the *canonical strip hypothesis* if $r \leq 0$ and

$$\alpha_i \in [-r, 0] \tag{6}$$

if $r \geq 0$ and

$$\alpha_i \in [0, -r] \tag{7}$$

otherwise,

for all $i = 1, \ldots, \dim X$.

It is clear that

$$\text{(CL)} \Rightarrow \text{(NCS)} \Rightarrow \text{(CS)}. \tag{8}$$

If $X$ is a Fano variety, and $Y \hookrightarrow X$ is a (normal) anticanonical divisor, we can consider the monotone polarised variety $(Y, -\mathrm{K}_X|_Y)$. By [7, theorem 4] we know that if $(X, -\mathrm{K}_X)$ satisfies (CS) then $(Y, -\mathrm{K}_X|_Y)$ satisfies (CL).

The goal of this paper is to give the first examples of

1. Fano varieties which violate the (narrow) canonical strip hypothesis;
2. embedded Calabi–Yau varieties which violate the canonical line hypothesis.

The question whether such varieties exist was raised by Golyshev in [7, §5.A]. The examples we give are moduli spaces $\mathrm{M}_C(2, \mathcal{L})$ of vector bundles of rank 2 with fixed determinant $\mathcal{L}$ of odd degree on a curve $C$ of genus $g \gg 2$.

**Theorem 3.** We have the following examples violating the (narrow) canonical strip hypothesis.

- Let $g \geq 8$, then $\mathrm{M}_C(2, \mathcal{L})$ does not satisfy the narrow canonical strip hypothesis.
- Let $g \geq 10$, then $\mathrm{M}_C(2, \mathcal{L})$ does not satisfy the canonical strip hypothesis.

- Let $g \geq 11$ then an anticanonical Calabi–Yau hypersurface inside $\mathrm{M}_C(2, \mathcal{L})$ does not satisfy the canonical line hypothesis[1].

Observe that there exist smooth anticanonical hypersurfaces, by the very ampleness of $\Theta$ [3] and the Bertini theorem.

In section 2 we give the proof of this theorem, and discuss related constructions, giving more families of examples vilating Golyshev's hypotheses. Before we do this we give an overview of the positive results in the literature. In [7] Golyshev explains how

1. the canonical line hypothesis holds for smooth projective curves (with the elliptic curve being embedded in $\mathbb{P}^2$);
2. the narrow canonical strip hypothesis holds for del Pezzo surfaces and surfaces of general type, and the canonical line hypothesis holds for embedded K3 surfaces;
3. the narrow canonical strip hypothesis holds for Fano 3-folds and minimal threefolds of general type.

Moreover it is explained how all Grassmannians (not just projective spaces) satisfy the narrow canonical strip hypothesis.

In [8] Manivel shows that for $G$ a simple affine algebraic group and $P$ a maximal parabolic subgroup

1. $G/P$ satisfies the tight[2] strip hypothesis;
2. Fano complete intersections in $G/P$ satisfy the tight canonical strip hypothesis;
3. general type complete intersections in $G/P$ satisfy the canonical line hypothesis;
4. Calabi–Yau complete intersections in $G/P$ satisfy the canonical line hypothesis.

Miyaoka's celebrated pseudo-effectivity theorem [9] implies that the embedded canonical line hypothesis holds for smooth Calabi–Yau threefolds[3].

Another case that can be checked is that of smooth toric Fano $n$-folds, for $n = 4, \ldots, 7$. By [5, proposition 9.4.3] we have that the Hilbert polynomial for the anticanonical bundle is the Ehrhart polynomial of the moment polytope. In [1] we have computed these Ehrhart polynomials, based on the classification of the toric Fano polytopes up to dimension 7. It turns out there are no examples violating the canonical strip hypothesis. In other words, we can add the following proposition to the list of positive examples.

**Proposition 4.** Let $X$ be a smooth toric Fano variety of dimension at most 7. Then $X$ satisfies the canonical strip hypothesis[4].

The maximal value $m_d$ of the real parts of the roots of the Hilbert polynomials for

[1] Hence for $g = 10$ we have that $\mathrm{M}_C(2, \mathcal{L})$ violates the canonical strip hypothesis, yet an anticanonical Calabi–Yau hypersurface still satisfies the embedded canonical line hypothesis. See also table 1 for more information.

[2] A strengthening of the narrow canonical strip hypothesis for Fano varieties involving the index $\iota_X$ of $X$, i.e. with the notation of definition 2 one asks for $\alpha_i \in [-1 + 1/\iota_X \leq -1/\iota_X$, when $H = -\mathrm{K}_X$.

[3] As well as for threefolds with numerically trivial canonical bundle, and terminal Gorenstein singularities that admit crepant resolution.

[4] The narrow canonical strip hypothesis is violated starting in dimension 4.

smooth toric Fano varieties of dimension $d$ is given as

$$\begin{aligned}
m_2 &= -0.3333333333\ldots \\
m_3 &= -0.2500000000\ldots \\
m_4 &= -0.1394448724\ldots \\
m_5 &= -0.0868988066\ldots \\
m_6 &= -0.0566708554\ldots \\
m_7 &= -0.0354049073\ldots
\end{aligned} \tag{9}$$

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## 2 Examples violating the hypotheses

An interesting class of Fano varieties is given by moduli spaces of vector bundles on a curve. We will restrict ourselves to the case of rank 2. Let $\mathcal{L}$ be a line bundle of odd degree on a smooth projective curve $C$ of genus $g$. Then the moduli space $\mathrm{M}_C(2,\mathcal{L})$ of rank 2 bundles with determinant $\mathcal{L}$ is a smooth projective variety of dimension $3g-3$, of rank 1 and index 2, i.e. $\operatorname{Pic}\mathrm{M}_C(2,\mathcal{L}) \cong \mathbb{Z}\Theta$, and $\omega_{\mathrm{M}_C(2,\mathcal{L})} \cong \Theta^{\otimes -2}$, see [6].

To compute the Hilbert polynomial we can use the celebrated Verlinde formula, which gives an expression for $\dim_k \mathrm{H}^0(\mathrm{M}_C(2,\mathcal{L}),\Theta^{\otimes k})$, see [2, 10] for a survey. It reads

$$\dim_k \mathrm{H}^0(\mathrm{M}_C(2,\mathcal{L}),\Theta^{\otimes k}) = (k+1)^{g-1}\sum_{j=1}^{2k+1}\frac{(-1)^{j-1}}{\sin^{2g-2}\frac{j\pi}{2k+2}}. \tag{10}$$

Rather than this version of the Verlinde formula we will use an alternative form, taken from [10]. Namely item (x) in theorem 1 of op. cit. gives the formula

$$\dim_k \mathrm{H}^0(\mathrm{M}_C(2,\mathcal{L}),\Theta^{\otimes k}) = \frac{2^g \det M_{r,s}}{\prod_{j=1}^{g}(2j)!} \tag{11}$$

where the matrix $(M_{r,s})_{r,s=0,\ldots,g-1}$ is given by

$$M_{r,s} = \begin{cases} 1 & r = 0 \\ (k+1+r)^{2s+2} - (k+1-r)^{2s+2} & r \geq 0 \end{cases}. \tag{12}$$

The benefit of using this expression is that it can be computed in an exact fashion in computer algebra.

Using this formula one computes the first $3g$ coefficients of the Hilbert series, and from this we can obtain the Hilbert polynomial of $\mathrm{M}_C(2,\mathcal{L})$ with respect to $\Theta$, i.e. we consider the monotone polarisation given by $H = \Theta$ for $\mathrm{M}_C(2,\mathcal{L})$. Two implementations of the computations (one in Pari/GP, another in Sage) can be found at [4]. The implementation computes the maximum of the real parts of the complex roots of the Hilbert polynomial, so we are interested in knowing when these are negative, but close to 0, or positive. From these computations we get theorem 3 as in the introduction.

**Remark 5.** The values in the column labeled $M_C(2, \mathcal{L})$ in table 1 suggest an interesting convergence behaviour for the maximum of the real part of the complex roots of the Hilbert polynomial. More generally one can compute that the collection of all roots of the Hilbert polynomial seems to exhibit a pattern where the limiting behaviour involves the complex hull of the roots for increasing genera. A visualisation of this is given in fig. 1. In the picture we have omitted the root at $t = -1$, which in all the examples we computed is of multiplicity $g-1$, but we have no proof of this. We suggest these questions for future work.

**Related constructions** Besides an anticanonical Calabi–Yau hypersurface constructed out of $M_C(2, \mathcal{L})$ there are other Fano and Calabi–Yau varieties we can construct out of it. These are

**Fano$^1$** the $3g-4$-dimensional Fano variety given by a linear section;

**Fano$^2$** the $3g-3$-dimensional Fano variety given by a double cover branched in $2\Theta$;

**CY$^2$** the $3g-5$-dimensional Calabi–Yau variety given by a linear section of codimension 2;

**CY$^3$** the $3g-3$-dimensional Calabi–Yau variety given by a double cover branched in $4\Theta$;

**CY$^4$** the $3g-3$-dimensional Calabi–Yau variety given by the cone over the embedding given by $\Theta$, intersected with a cubic hypersurface;

**CY$^5$** the $3g-3$-dimensional Calabi–Yau variety given by the join with a line intersected with two quadric hypersurfaces;

**CY$^6$** the $3g-3$-dimensional Calabi–Yau variety given by a smoothing of a linear section of a join with an elliptic curve of degree 1.

For all of these the canonical strip (resp. line) hypothesis eventually fails, as checked in [4]. In table 1 we have collected the maximum over the real parts of the complex roots of the Hilbert polynomial, where the columns are labelled as in this remark. The Calabi–Yau variety denoted $CY^1$ is the anticanonical section of $M_C(2, \mathcal{L})$ as considered in theorem 3.

We have not found counterexamples with ample canonical bundle: the canonical line hypothesis was satisfied for all constructions we considered.

| | Fano | | | Calabi–Yau | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $g$ | Fano[1] | Fano[2] | $\mathrm{M}_C(2,\mathcal{L})$ | CY[1] | CY[2] | CY[3] | CY[4] | CY[5] | CY[6] |
| 2 | −0.5 | −0.5 | −1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | −0.5 | −0.5 | −0.706 640 539 5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | −0.5 | −0.5 | −0.477 001 948 8 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | −0.289 050 709 8 | −0.313 172 706 4 | −0.309 498 927 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | −0.179 205 632 6 | −0.206 390 561 0 | −0.191 196 178 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | −0.104 714 434 0 | −0.111 984 402 5 | −0.108 353 678 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | −0.050 040 882 5 | −0.049 987 964 3 | −0.050 040 972 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | −0.008 887 509 0 | −0.008 135 607 4 | −0.008 509 422 5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0.021 353 423 8 | 0.021 620 187 9 | 0.021 486 936 1 | 0 | 0.037 953 952 1 | 0.038 169 563 0 | 0.038 276 745 3 | 0.038 383 517 2 | 0.038 061 966 6 |
| 11 | 0.043 439 254 9 | 0.043 469 996 3 | 0.043 454 600 3 | 0.061 436 909 1 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0.059 750 706 4 | 0.059 741 223 1 | 0.059 745 965 2 | 0.039 947 163 2 | 0.073 179 436 1 | 0.073 174 523 6 | 0.073 172 066 9 | 0.073 169 609 9 | 0.073 176 980 0 |
| 13 | 0.071 960 067 7 | 0.071 955 094 1 | 0.071 957 581 0 | 0.080 107 739 3 | 0.067 567 715 6 | 0.067 562 078 2 | 0.067 559 258 8 | 0.067 556 439 1 | 0.067 564 897 1 |
| 14 | 0.081 189 939 6 | 0.081 189 060 3 | 0.081 189 499 9 | 0.081 930 543 0 | 0.084 573 517 3 | 0.084 573 049 0 | 0.084 572 814 8 | 0.084 572 580 7 | 0.084 573 283 1 |
| 15 | 0.088 212 105 2 | 0.088 212 142 3 | 0.088 212 123 8 | 0.087 924 527 3 | 0.090 774 334 4 | 0.090 774 282 6 | 0.090 774 256 7 | 0.090 774 230 8 | 0.090 774 308 5 |
| 16 | 0.093 573 807 3 | 0.093 573 864 6 | 0.093 573 835 9 | 0.096 525 889 1 | 0.091 120 060 4 | 0.091 120 123 6 | 0.091 120 155 1 | 0.091 120 186 7 | 0.091 120 092 0 |
| 17 | 0.097 671 125 5 | 0.097 671 138 7 | 0.097 671 132 1 | 0.094 622 277 9 | 0.100 367 508 4 | 0.100 367 521 1 | 0.100 367 527 5 | 0.100 367 533 9 | 0.100 367 514 8 |
| 18 | 0.100 794 936 1 | 0.100 794 936 8 | 0.100 794 936 5 | 0.102 973 719 9 | 0.098 184 901 6 | 0.098 184 901 9 | 0.098 184 902 0 | 0.098 184 902 2 | 0.098 184 901 8 |
| 19 | 0.105 885 924 9 | 0.105 886 335 8 | 0.105 886 130 4 | 0.105 101 938 1 | 0.107 002 849 0 | 0.107 003 186 0 | 0.107 003 354 6 | 0.107 003 523 1 | 0.107 003 017 5 |
| 20 | 0.114 639 348 4 | 0.114 639 352 4 | 0.114 639 350 4 | 0.114 407 509 1 | 0.115 050 095 7 | 0.115 050 115 0 | 0.115 050 124 7 | 0.115 050 134 4 | 0.115 050 105 3 |
| 21 | 0.121 885 049 8 | 0.121 885 016 4 | 0.121 885 033 1 | 0.122 573 559 5 | 0.121 199 207 4 | 0.121 199 171 2 | 0.121 199 153 2 | 0.121 199 135 1 | 0.121 199 189 3 |
| 22 | 0.127 891 132 5 | 0.127 891 119 9 | 0.127 891 126 2 | 0.127 282 948 0 | 0.128 469 880 7 | 0.128 469 868 6 | 0.128 469 862 5 | 0.128 469 856 4 | 0.128 469 874 7 |
| 23 | 0.132 872 201 6 | 0.132 872 199 7 | 0.132 872 200 6 | 0.133 234 607 5 | 0.132 497 558 6 | 0.132 497 556 6 | 0.132 497 555 6 | 0.132 497 554 6 | 0.132 497 557 6 |
| 24 | 0.137 001 216 5 | 0.137 001 216 7 | 0.137 001 216 6 | 0.136 830 612 4 | 0.137 171 671 4 | 0.137 171 671 6 | 0.137 171 671 7 | 0.137 171 671 9 | 0.137 171 671 5 |
| 25 | 0.140 418 474 5 | 0.140 418 474 7 | 0.140 418 474 6 | 0.140 462 979 8 | 0.140 376 163 0 | 0.140 376 163 2 | 0.140 376 163 3 | 0.140 376 163 4 | 0.140 376 163 1 |
| dim | $3g-4$ | $3g-3$ | $3g-3$ | $3g-4$ | $3g-5$ | $3g-3$ | $3g-3$ | $3g-3$ | $3g-3$ |

Table 1: Maximum value of real parts of complex roots of Hilbert polynomial

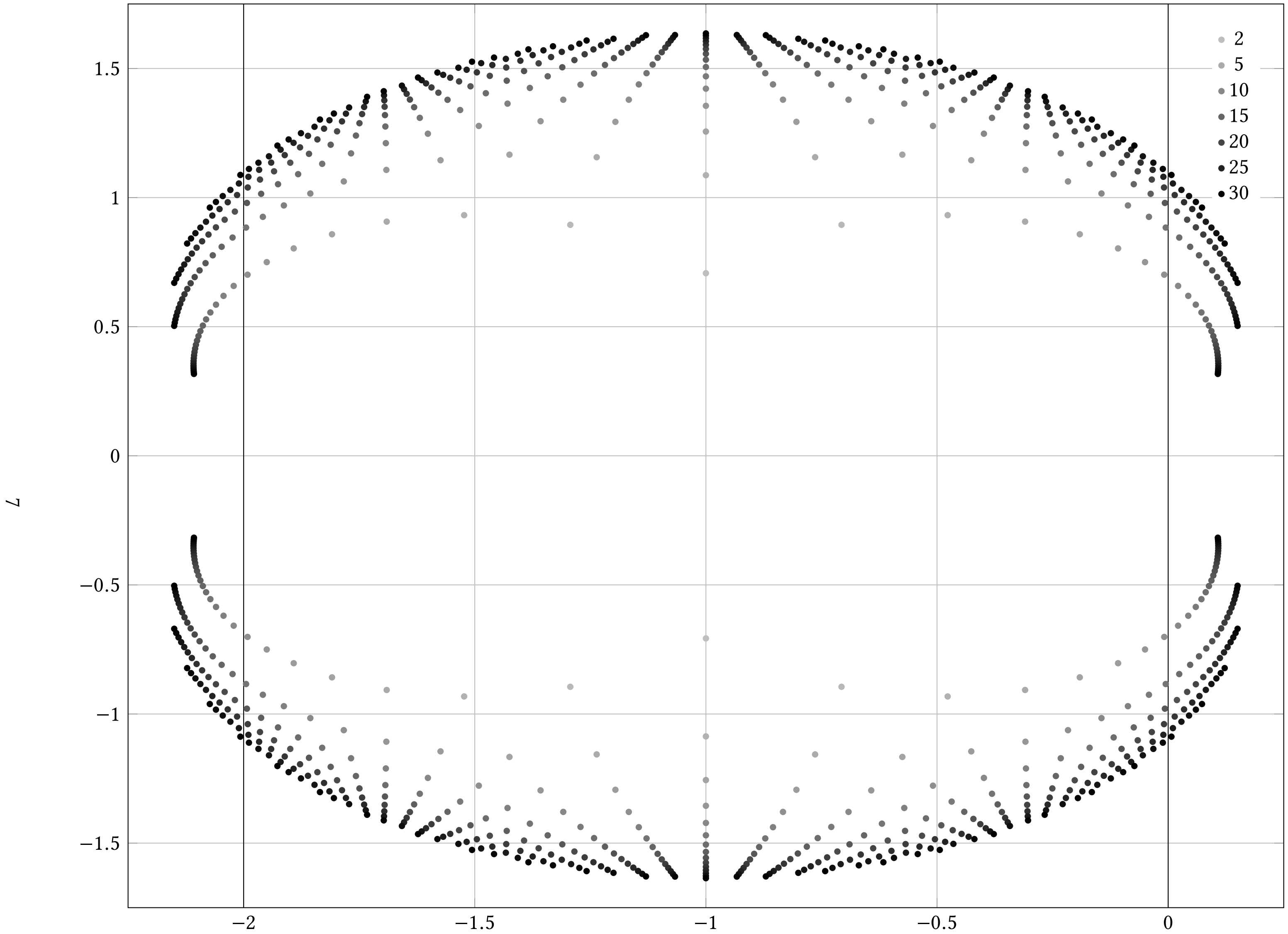


Figure 1: Complex roots of Hilbert polynomials of $\mathrm{M}_C(2, \mathcal{L})$, for $g = 2, \ldots, 30$

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