On the homotopy category of Moore spaces and the cohomology of the category of abelian groups

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ON THE HOMOTOPY CATEGORY OF MOORE SPACES AND THE COHOMOLOGY OF THE CATEGORY OF ABELIAN GROUPS

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An abelian group A determines the Moore space $M(A) = M(A,2)$ which up to homotopy equivalence is the unique simply connected CW-space X with homology groups $H_2X = A$ and $H_iX = 0$ for $i > 2$. Since $M(A)$ can be chosen to be a suspension the set of homotopy classes $[M(A), M(B)]$ is a group which is part of a classical central extension of groups

$$(1) \qquad Ext(A, \Gamma B) \rightarrowtail [M(A), M(B)] \twoheadrightarrow Hom(A, B)$$

due to Barratt. It is known that (1) in general is not split, for example $[M(\mathbb{Z}/2), M(\mathbb{Z}/2)] = \mathbb{Z}/4$. We here are not interested in this additive structure of the sets $[M(A), M(B)]$ but in the multiplicative structure given by the composition of maps, in particular in the extension of groups

$$(2) \qquad Ext(A, \Gamma A) \rightarrowtail \mathcal{E}(M(A)) \twoheadrightarrow Aut(A)$$

where $\mathcal{E}(M(A))$ is the group of homotopy equivalences of the space $M(A)$. The extension (2) determines the cohomology class

$$(3) \qquad \{\mathcal{E}(M(A))\} \in H^2(Aut(A), Ext(A, \Gamma A))$$

Though the group $\mathcal{E}(M(A))$ is defined in an "easy" range of homotopy theory the cohomology class (3) is not yet computed for all abelian groups A. In this paper we prove a nice algebraic formula for the class (3) if A is a product of cyclic groups and we show that $\{\mathcal{E}(M(A))\}$ is trivial if $Ext(A, \Gamma A)$ has no 2-torsion; see (3.6) and (5.2). Moreover we compute for all abelian groups A the image of the class (3) under the surjection of coefficients

$$(4) \qquad Ext(A, \Gamma A) \twoheadrightarrow Ext(A, H(\Gamma A)).$$

Here $H(\Gamma A)$ is the image of $H : \Gamma A \to A \otimes A$; see (4.2). We do such computations not in the cohomology of groups but more distinctly in the cohomology of categories. In fact the homotopy category $\underline{\underline{M}}^2$ of Moore spaces $M(A)$ leads to a topological

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"characteristic class" in the cohomology of the category $\underline{\underline{Ab}}$ of abelian groups; see (2.2). It is the computation of such topologically defined cohomology classes which motivated the results in this paper. For example the topological James-Hopf invariant on the category $\underline{\underline{M}}^2$ or the "chains on the loop space" functor $C_*\Omega$ on $\underline{\underline{M}}^2$ have interesting interpretations on the level of the cohomology of the category $\underline{\underline{Ab}}$; see (4.11). As an application we describe algebraically the image category $(C_*\Omega)(\underline{\underline{M}}^2)$ in the homotopy category of chain algebras showing fundamental differences between the homotopy category of spaces and chain algebras respectively; see (4.12). This implies that the image of the group $\mathcal{E}(M(A))$ under the functor $C_*\Omega$ is part of an extension

$$(5) \qquad Ext(A, H(\Gamma A)) \rightarrowtail (C_*\Omega)\mathcal{E}(M(A)) \twoheadrightarrow Aut(A)$$

which we compute explicitly in terms of A for all abelian groups A.

§ 1 Linear extensions of categories and the cohomology of categories

An extension of a group G by a G-module A is a short exact sequence of groups

$$0 \to A \underset{i}{\longrightarrow} E \underset{p}{\longrightarrow} G \to 0$$

where i is compatible with the action of G. Two such extensions E and E' are equivalent if there is an isomorphism $\epsilon : E \cong E'$ of groups with $p'\epsilon = p$ and $\epsilon i = i'$. It is well known that the equivalence classes of extensions are classified by the cohomology $H^2(G, A)$.

We now recall from [2] basic notation on the cohomology of categories. We describe linear extensions of a small category $\underline{\underline{C}}$ by a "natural system" D. The equivalence classes of such extensions are classified by the cohomology $H^2(\underline{\underline{C}}, D)$. A natural system D on a category $\underline{\underline{C}}$ is the appropriate generalization of a G-module.

(1.1) *Definition.* Let $\underline{\underline{C}}$ be a category. The category of factorizations in $\underline{\underline{C}}$, denoted by $F\underline{\underline{C}}$, is given as follows. Objects are morphisms $f, g, \ldots$ in $\underline{\underline{C}}$ and morphisms $f \to g$ are pairs (α, β) for which

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ B & \xleftarrow{\beta} & B' \end{array}$$

commutes in $\underline{\underline{C}}$. Here $\alpha f \beta$ is factorization of g. Composition is defined by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. We clearly have $(\alpha, \beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$. A natural system (of abelian groups) on $\underline{\underline{C}}$ is a functor $D : F\underline{\underline{C}} \to \underline{\underline{Ab}}$. The functor D carries the object f to $D_f = D(f)$ and carries the morphism $(\alpha, \beta) : f \to g$ to the induced homomorphism

$$D(\alpha,\beta) = \alpha_*\beta^* : D_f \to D_{\alpha f \beta} = D_g$$

Here we set $D(\alpha, 1) = \alpha_*$, $D(1,\beta) = \beta^*$.

We have a canonical forgetful functor $\pi : F\underline{\underline{C}} \to \underline{\underline{C}}^{op} \times \underline{\underline{C}}$ so that each bifunctor $D : \underline{\underline{C}}^{op} \times \underline{\underline{C}} \to \underline{\underline{Ab}}$ yields a natural system $D\pi$, as well denoted by D. Such a bifunctor is also called a $\underline{\underline{C}}$ -bimodule. In this case $D_f = D(B, A)$ depends only on the objects A, B for all $f \in \underline{\underline{C}}(B, A)$. Two functors F, $G : \underline{\underline{Ab}} \to \underline{\underline{Ab}}$ yield the $\underline{\underline{Ab}}$ -bimodule

$$Hom(F,G) : \underline{\underline{Ab}}^{op} \times \underline{\underline{Ab}} \to \underline{\underline{Ab}}$$

which carries (A, B) to the group of homomorphisms $Hom(FA, GB)$. If F is the identity functor we write $Hom(-, G)$. Similarly we define the $\underline{\underline{Ab}}$ -bimodule $Ext(F, G)$.

For a group G and a G-module A the corresponding natural system D on the group G, considered as a category, is given by $D_g = A$ for $g \in G$ and $g_*a = g \cdot a$ for $a \in A$, $g^*a = a$. If we restrict the following notion of a "linear extension" to the case $\underline{\underline{C}} = G$ and $D = A$ we obtain the notion of a group extension above.

(1.2) Definition. Let D be a natural system on $\underline{\underline{C}}$. We say that

$$D \overset{+}{\rightarrowtail} \underline{\underline{E}} \overset{p}{\twoheadrightarrow} \underline{\underline{C}}$$

is a linear extension of the category $\underline{\underline{C}}$ by D if (a), (b) and (c) hold.

- (a) $\underline{\underline{E}}$ and $\underline{\underline{C}}$ have the same objects and p is a full functor which is the identity on objects.
- (b) For each $f : A \to B$ in $\underline{\underline{C}}$ the abelian group D_f acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\underline{\underline{E}}$. We write $f_0 + \alpha$ for the action of $\alpha \in D_f$ on $f_0 \in p^{-1}(f)$.
- (c) The action satisfies the linear distributivity law:

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_*\beta + g^*\alpha.$$

Two linear extensions $\underline{\underline{E}}$ and $\underline{\underline{E}}'$ are equivalent if there is an isomorphism of categories $\epsilon : \underline{\underline{E}} \cong \underline{\underline{E}}'$ with $p'\epsilon = p$ and with $\epsilon(f_0 + \alpha) = \epsilon(f_0) + \alpha$ for $f_0 \in \mathrm{Mor}(\underline{\underline{E}})$, $\alpha \in D_{pf_0}$. The extension $\underline{\underline{E}}$ is split if there is a functor $s : \underline{\underline{C}} \to \underline{\underline{E}}$ with $ps = 1$. Let $M(\underline{\underline{C}}, D)$ be the set of equivalence classes of linear extensions of $\underline{\underline{C}}$ by $\underline{\underline{D}}$. Then there is a canonical bijection

$$\psi : M(\underline{\underline{C}}, D) \cong H^2(\underline{\underline{C}}, D) \tag{1.3}$$

which maps the split extension to the zero element, see [2] and IV § 6 in [4]. Here $H^n(\underline{\underline{C}}, D)$ denotes the cohomology of $\underline{\underline{C}}$ with coefficients in D which is defined below. We obtain a representing cocycle Δ_t of the cohomology class $\{\underline{\underline{E}}\} = \psi(\underline{\underline{E}}) \in$

$H^2(\underline{\underline{C}}, D)$ as follows. Let t be a "splitting" function for p which associates with each morphism $f : A \to B$ in $\underline{\underline{C}}$ a morphism $f_0 = t(f)$ in $\underline{\underline{E}}$ with $pf_0 = f$. Then t yields a cocycle Δ_t by the formula

$$(1.4) \qquad t(gf) = t(g)t(f) + \Delta_t(g, f)$$

with $\Delta_t(g, f) \in D(gf)$. The cohomology class $\{\underline{\underline{E}}\} = \{\Delta_t\}$ is trivial if and only if $\underline{\underline{E}}$ is a split extension.

(1.5) Definition. Let $\underline{\underline{C}}$ be a small category and let $N_n(\underline{\underline{C}})$ be the set of sequences $(\lambda_1, \ldots, \lambda_n)$ of n composable morphisms in $\underline{\underline{C}}$ (which are the n-simplices of the nerve of $\underline{\underline{C}}$). For $n = 0$ let $N_0(\underline{\underline{C}}) = \mathrm{Ob}(\underline{\underline{C}})$ be the set of objects in $\underline{\underline{C}}$. The cochain group $F^n = F^n(\underline{\underline{C}}, D)$ is the abelian group of all functions

$$(1) \qquad c : N_n(\underline{\underline{C}}) \to \left(\bigcup_{g \in \mathrm{Mor}(\underline{\underline{C}})} D_g \right) = D$$

with $c(\lambda_1, \ldots, \lambda_n) \in D_{\lambda_1 \circ \ldots \circ \lambda_n}$. Addition in F^n is given by adding pointwise in the abelian groups D_g. The coboundary $\partial : F^{n-1} \to F^n$ is defined by the formula

$$(2) \qquad \begin{aligned} (\partial c)(\lambda_1, \ldots, \lambda_n) &= (\lambda_1)_* c(\lambda_2, \ldots, \lambda_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i c(\lambda_1, \ldots, \lambda_i \lambda_{i+1}, \ldots, \lambda_n) \\ &+ (-1)^n (\lambda_n)^* c(\lambda_1, \ldots, \lambda_{n-1}) \end{aligned}$$

For $n = 1$ we have $(\partial c)(\lambda) = \lambda_* c(A) - \lambda^* c(B)$ for $\lambda : A \to B \in N_1(\underline{\underline{C}})$. One can check that $\partial c \in F^n$ for $c \in F^{n-1}$ and that $\partial\partial = 0$. Hence the cohomology groups

$$(3) \qquad H^n(\underline{\underline{C}}, D) = H^n(F^*(\underline{\underline{C}}, D), \delta)$$

are defined, $n \geq 0$. These groups are discussed in [2] and [4]. By change of the universe cohomology groups $H^n(\underline{\underline{C}}, D)$ can also be defined if $\underline{\underline{C}}$ is not a small category. A functor $\phi : \underline{\underline{C}}' \to \underline{\underline{C}}$ induces the homomorphism

$$(4) \qquad \phi^* : H^n(\underline{\underline{C}}, D) \to H^n(\underline{\underline{C}}', \phi^* D)$$

where $\phi^* D$ is the natural system given by $(\phi^* D)_f = D_{\phi(f)}$. On cochains the map ϕ^* is given by the formula

$$(\phi^* f)(\lambda_1', \ldots, \lambda_n') = f(\phi\lambda_1', \ldots, \phi\lambda_n')$$

where $(\lambda', \ldots, \lambda'_n) \in N_n(\underline{\underline{C}}')$. If ϕ is an equivalence of categories then ϕ^* is an isomorphism. A natural transformation $\tau : D \to D'$ between natural systems induces a homomorphism

$$(5) \qquad \tau_* : H^n(\underline{\underline{C}}, D) \to H^n(\underline{\underline{C}}, D')$$

by $(\tau_* f)(\lambda_1, \ldots, \lambda_n) = \tau_\lambda f(\lambda_1, \ldots, \lambda_n)$ where $\tau_\lambda : D_\lambda \to D'_\lambda$ with $\lambda = \lambda_1 \circ \ldots \circ \lambda_n$ is given by the transformation τ. Now let

$$D'' \overset{\iota}{\rightarrowtail} D \overset{\tau}{\twoheadrightarrow} D'$$

be a short exact sequence of natural systems on $\underline{\underline{C}}$. Then we obtain as usual the natural long exact sequence

$$(1.6) \qquad \longrightarrow H^n(\underline{\underline{C}}, D') \xrightarrow{\iota_*} H^n(\underline{\underline{C}}, D) \xrightarrow{\tau_*} H^n(\underline{\underline{C}}, D'') \xrightarrow{\beta} H^{n+1}(\underline{\underline{C}}, D') \longrightarrow$$

where β is the Bockstein homomorphism. For a cocycle c'' representing a class $\{c''\}$ in $H^n(\underline{\underline{C}}, D'')$ we obtain $\beta\{c''\}$ by choosing a cochain c as in (1.5) (1) with $\tau c = c''$. This is possible since τ is surjective. Then $\iota^{-1}\delta c$ is a cocycle which represents $\beta\{c''\}$.

(1.7) Remark. The cohomology (1.5) generalizes the cohomology of a group. In fact, let G be a group and let $\underline{\underline{G}}$ be the corresponding category with a single object and with morphisms given by the elements in G. A G-module A yields a natural system D. Then the classical definition of the cohomology $H^n(G, A)$ coincides with the definition of

$$H^n(\underline{\underline{G}}, D) = H^n(G, A)$$

given by (1.5). Further results and applications of the cohomology of categories can be found in [2], [3], [4], [5], [13], [14].

§ 2 The homotopy category $\underline{\underline{M}}^2$ of Moore spaces in degree 2

Let A be an abelian group. A Moore space $M(A, n)$, $n \geq 2$, is a simply connected CW-space X with (reduced) homology groups $H_n X = A$ and $H_i X = 0$ for $i \neq n$. An Eilenberg-Mac Lane space $K(A, n)$ is a CW-space Y with homotopy groups $\pi_n Y = A$ and $\pi_i Y = 0$ for $i \neq n$. Such spaces exist and their homotopy type is well defined by (A, n). The homotopy category of Eilenberg-Mac Lane spaces $K(A, n)$, $A \in \underline{\underline{Ab}}$, is isomorphic via the functor π_n to the category $\underline{\underline{Ab}}$ of abelian groups. The corresponding result, however, does not hold for the homotopy category $\underline{\underline{M}}^n$ of Moore spaces $M(A, n)$, $A \in \underline{\underline{Ab}}$. This creates the problem to find a suitable algebraic model of the category $\underline{\underline{M}}^n$. For $n \geq 3$ such a model category of $\underline{\underline{M}}^n$ is known (see (V.3a.8) in [4] and (I.§ 6) in [6]). The category $\underline{\underline{M}}^2$

is not completely understood. We shall use the cohomology of the category $\underline{\underline{Ab}}$ to describe various properties of the category $\underline{\underline{M}}^2$.

Let $\Gamma : \underline{\underline{Ab}} \to \underline{\underline{Ab}}$ be J.H.C. Whitehead's quadratic functor [15] with

$$\Gamma(A) = \pi_3 M(A,2) = H_4 K(A,2) \tag{2.1}$$

Then we obtain the $\underline{\underline{Ab}}$ -bimodule

$$Ext(-,\Gamma) : \underline{\underline{Ab}}^{op} \times \underline{\underline{Ab}} \to \underline{\underline{Ab}}$$

which carries (A, B) to the group $Ext(A, \Gamma(B))$.

(2.2) Proposition. *The category $\underline{\underline{M}}^2$ is part of a non split linear extension*

$$Ext(-,\Gamma) \overset{+}{\rightarrowtail} \underline{\underline{M}}^2 \overset{H_2}{\twoheadrightarrow} \underline{\underline{Ab}}$$

and hence $\underline{\underline{M}}^2$, up to equivalence, is characterized by a cohomology class

$$\{\underline{\underline{M}}^2\} \in H^2(\underline{\underline{Ab}}, Ext(-,\Gamma)).$$

Since the extension is non split we have $\{\underline{\underline{M}}^2\} \neq 0$.

Proof. For a free abelian group A_0 with basis Z let

$$M_{A_0} = \bigvee_Z S^1$$

be a one point union of 1-dimensional spheres S^1 such that $H_1 M_{A_0} = A_0$. For an abelian group A we choose a short exact sequence

$$0 \to A_1 \overset{d_A}{\to} A_0 \to A \to 0$$

where A_0, A_1 are free abelian. Let

$$d'_A : M_{A_1} \to M_{A_0}$$

be a map which induces d_A in homology and let M_A be the mapping cone of d'_A. Then

$$M(A,2) = \Sigma M_A$$

is the suspension of M_A. The homotopy type of M_A, however, depends on the choice of d'_A and is not determined by A. Using the cofiber sequence for d'_A we obtain the well known exact sequence of groups [11]

$$0 \to Ext(A, \pi_3 X) \overset{\Delta}{\to} [M(A,2), X] \overset{\mu}{\longrightarrow} Hom(A, \pi_2 X) \to 0$$

where $[Y, X]$ denotes the set of homotopy classes of pointed maps $Y \to X$. We now set $X = M(B,2)$. Then μ is given by the homology functor. We define the action of $\alpha \in Ext(A, \Gamma B)$ on $\xi \in [M(A,2), M(B,2)]$ by $\xi + \alpha = \xi + \Delta(\alpha)$ where we use the group structure in $[\Sigma M_A, M(B,2)]$. This action satisfies the linear distributivity law so that we obtain the linear extension in (2.2). Compare also (V.§ 3a) in [4] where we show $\{\underline{\underline{M}}^2\} \neq 0$.

(2.3) Remark. A Pontrjagin map τ_A for an abelian group A is a map

$$\tau_A : K(A,2) \to K(\Gamma(A),4)$$

which induces the identity of $\Gamma(A)$,

$$\Gamma(A) = H_4 K(A,2) \to H_4 K(\Gamma(A),4) = \Gamma(A)$$

Such Pontrjagin maps exist and are well defined up to homotopy. The map τ_A induces the Pontrjagin square which is the cohomology operation [15]

$$H^2(X,A) = [X, K(A,2)] \xrightarrow{(\tau_A)_*} [X, K(\Gamma(A),2)] = H^4(X, \Gamma(A))$$

The fiber of τ_A is the 3-type of $M(A,2)$. Therefore one gets isomorphisms of categories [9]

$$\underline{\underline{M}}^2 = \underline{\underline{P}}(\mathcal{X}) = \underline{\underline{Hopair}}(\mathcal{X})$$

where $\mathcal{X}$ is the class of all Pontrjagin maps τ_A, $A \in \underline{\underline{Ab}}$. Here $\underline{\underline{P}}(\mathcal{X})$ is the homotopy category of fibers $P(\tau_A)$, $\tau_A \in \mathcal{X}$, and $\underline{\underline{Hopair}}(\mathcal{X})$ is the category of homotopy pairs [10] between Pontrjagin maps. We have seen in [9] that via these isomorphisms the class $\{\underline{\underline{M}}^2\}$ is the image of the universal Toda bracket $\langle \underline{\underline{K}} \rangle_\Omega \in H^3(\underline{\underline{K}}, D_\Omega)$ where $\underline{\underline{K}}$ is the full subcategory of the homotopy category consisting of $K(A,2)$ and $K(\Gamma(A),4)$, $A \in \underline{\underline{Ab}}$. Hence we get by (2.2):

(2.4) Corollary. $\langle \underline{\underline{K}} \rangle_\Omega \neq 0$

§ 3 On the cohomology class $\{\underline{\underline{M}}^2\}$

The quadratic functor Γ can also be defined by the universal quadratic map $\gamma : A \to \Gamma(A)$. We have the natural exact sequence in $\underline{\underline{Ab}}$

$$\Gamma(A) \xrightarrow{H} A \otimes A \xrightarrow{q} \Lambda^2 A \to 0 \tag{3.1}$$

where H is defined by $H\gamma(a) = a \otimes a$, $a \in A \in \underline{\underline{Ab}}$, and where $\Lambda^2 A = A \otimes A / \{a \otimes a \sim 0\}$ is the exterior square with quotient map q. We also need the natural homomorphism

$$[1,1] = P : A \otimes A \to \Gamma(A) \tag{3.2}$$

with $P(a \otimes b) = \gamma(a+b) - \gamma(a) - \gamma(b) = [a,b]$. One readily checks that PH is multiplication by 2 on $\Gamma(A)$ and that $HP(a \otimes b) = a \otimes b + b \otimes a$. For $A \in \underline{\underline{Ab}}$ we obtain by P and H and q above the following natural short exact sequences of $\mathbb{Z}/2$ -vector spaces

$$(3.3) \qquad \begin{cases} S_1(A) : \Lambda^2(A) \otimes \mathbb{Z}/2 \overset{P}{\rightarrowtail} \Gamma(A) \otimes \mathbb{Z}/2 \overset{\sigma}{\twoheadrightarrow} A \otimes \mathbb{Z}/2 \\ S_2(A) : \Gamma(A) \otimes \mathbb{Z}/2 \overset{H}{\rightarrowtail} \otimes^2(A) \otimes \mathbb{Z}/2 \overset{q}{\twoheadrightarrow} \Lambda^2(A) \otimes \mathbb{Z}/2 \end{cases}$$

Here σ carries $\gamma(a) \otimes 1$ to $a \otimes 1$, $a \in A$. If we apply the functor $Hom(-, \Gamma(B) \otimes \mathbb{Z}/2)$ to the exact sequence $S_i(A)$, $i = 1,2$, we get the corresponding exact sequence of $\underline{\underline{Ab}}$ -bimodules denoted by $Hom(S_i(-), \Gamma(-) \otimes \mathbb{Z}/2)$. The associated Bockstein homomorphisms β_i yield thus homomorphisms

$$(3.4) \qquad \begin{array}{c} H^0(\underline{\underline{Ab}}, Hom(\Gamma(-) \otimes \mathbb{Z}/2, \Gamma(-) \otimes \mathbb{Z}/2)) \\ \downarrow \beta_2 \\ H^1(\underline{\underline{Ab}}, Hom(\Lambda^2(-) \otimes \mathbb{Z}/2, \Gamma(-) \otimes \mathbb{Z}/2)) \\ \downarrow \beta_1 \\ H^2(\underline{\underline{Ab}}, Hom(- \otimes \mathbb{Z}/2, \Gamma(-) \otimes \mathbb{Z}/2)) \end{array}$$

Moreover we use the natural homomorphism

$$\chi : Hom(A \otimes \mathbb{Z}/2, \Gamma(B) \otimes \mathbb{Z}/2) \overset{g}{=} Ext(A \otimes \mathbb{Z}/2, \Gamma B) \overset{p^*}{\longrightarrow} Ext(A, \Gamma B)$$

where g is the natural isomorphism and where $p : A \to A \otimes \mathbb{Z}/2$ is the projection. Let

$$1_\Gamma \in H^0(\underline{\underline{Ab}}, Hom(\Gamma(-) \otimes \mathbb{Z}/2, \Gamma(-) \otimes \mathbb{Z}/2))$$

be the canonical class which carries the abelian group A to the identity of $\Gamma(A) \otimes \mathbb{Z}/2$. Then one gets the element

$$\chi_* \beta_1 \beta_2(1_\Gamma) \in H^2(\underline{\underline{Ab}}, Ext(-, \Gamma))$$

determined by 1_Γ and the homomorphisms above.

(3.5) Conjecture.

$$\{\underline{\underline{M}}^2\} = \chi_* \beta_1 \beta_2(1_\Gamma)$$

We shall prove various results which support this conjecture.

(3.6) Theorem. *Let $\underline{\underline{A}}$ be the full subcategory of $\underline{\underline{Ab}}$ consisting of direct sums of cyclic groups and let $i_{\underline{\underline{A}}} : \underline{\underline{A}} \to \underline{\underline{Ab}}$ be the inclusion functor. Then we have*

$$i_{\underline{\underline{A}}}^*\{\underline{\underline{M}}^2\} = i_{\underline{\underline{A}}}^*\chi_*\beta_1\beta_2(1_\gamma) \in H^2(\underline{\underline{A}}, Ext(-,\Gamma))$$

Proof. We write $C = (\mathbb{Z}/a)\alpha$ if C is a cyclic group isomorphic to $\mathbb{Z}/a$ with generator α, $a \geq 0$. A direct sum of cyclic groups

$$A = \bigoplus_i (\mathbb{Z}/a_i)\alpha_i$$

is indexed by an ordered set if the set of generators $\{\alpha_i, <\}$ is a well ordered set. The generator α_i also denotes the inclusion $\alpha_i : \mathbb{Z}/a_i \subset A$ and the corresponding inclusion

$$\alpha_i : \Sigma P_{a_i} \subset \bigvee_i \Sigma P_{a_i} = M(A,2) \tag{3.7}$$

Here $P_n = S^1 \cup_n e^2$ is the pseudo projective plane for $n > 0$ and $P_0 = S^1$ so that $\Sigma P_n = M(\mathbb{Z}/n, 2)$. Let $\alpha^i : A \to \mathbb{Z}/a_i$ be the canonical retraction of α_i with $\alpha^i\alpha_i = 1$ and $\alpha^j\alpha_i = 0$ for $j \neq i$. Let

$$\varphi : A = \bigoplus_i (\mathbb{Z}/\alpha_i)\alpha_i \to B = \bigoplus_j (\mathbb{Z}/b_j)\beta_j \tag{3.8}$$

be a homomorphism. The coordinates $\varphi_{ji} \in \mathbb{Z}$, $\varphi_{ji} : \mathbb{Z}/a_i \to \mathbb{Z}/b_j$, $\mathbf{1} \longmapsto \varphi_{ji}\mathbf{1}$, are given by the formula

$$\varphi\alpha_i = \sum \beta_j \varphi_{ji}.$$

Let B_2 be the splitting function

$$[\Sigma P_n, \Sigma P_m] \underset{B_2}{\rightleftarrows} Hom(\mathbb{Z}/n, \mathbb{Z}/m)$$

obtained in (III, Appendix D) of [5]. We define the map $s\varphi \in [M(A,2), M(B,2)]$ by the ordered sum

$$(s\varphi)\alpha_i = \sum_j^{<} \beta_j B_2(\varphi_{ji})$$

where we use the ordering $<$ of the generators in B. Hence we obtain a splitting function s

$$[M(A,2), M(B,2)] \underset{s}{\overset{H_2}{\rightleftarrows}} Hom(A,B) \tag{3.9}$$

with $H_2 s(\varphi) = \varphi$. Each element $\bar{\varphi} \in [M(A,2), M(B,2)]$ is of the form $\bar{\varphi} = s(\varphi) + \xi$ where $\xi \in Ext(A, \Gamma B)$. This way we can characterize all elements in $[M(A,2), M(B,2)]$ provided A and B are ordered direct sums of cyclic groups. We use s in (3.9) for the definition of the cocycle Δ_s representing $i^*\{\underline{\underline{M}}^2\}$ in (3.6), that is by (1.4):

$$s(\psi\varphi) = s(\psi)s(\varphi) + \Delta_s(\psi, \varphi)$$

Below we compute Δ_s. To this end we have to introduce the following groups.

q.e.d.

(3.10) Definition. Let A be an abelian group. We have the natural homomorphism between $\mathbb{Z}/2$ -vector spaces

$$(1) \qquad H : \Gamma(A) \otimes \mathbb{Z}/2 = \Gamma(A \otimes \mathbb{Z}/2) \otimes \mathbb{Z}/2 \to \otimes^2(A \otimes \mathbb{Z}/2)$$

with $H(\gamma(a) \otimes 1) = (a \otimes 1) \otimes (a \otimes 1)$. This homomorphism is injective and hence admits a retraction homomorphism

$$(2) \qquad r : \otimes^2(A \otimes \mathbb{Z}/2) \to \Gamma(A) \otimes \mathbb{Z}/2$$

with $rH = id$. For example, given a basis E of the $\mathbb{Z}/2$ -vector space $A \otimes \mathbb{Z}/2$ and a well ordering $<$ on E we can define a retraction $r^<$ on basis elements by the formula $(b, b' \in E)$

$$(3) \qquad r^<(b \otimes b') = \begin{cases} \gamma(b) \otimes 1 & \text{for} \quad b = b' \\ [b, b'] \otimes 1 & \text{for} \quad b > b' \\ 0 & \text{for} \quad b < b' \end{cases}$$

Now let $q \geq 1$ and let

$$(4) \qquad j_A : Hom(\mathbb{Z}/q, A) = A * \mathbb{Z}/q \subset A \overset{p}{\twoheadrightarrow} A \otimes \mathbb{Z}/2$$

be given by the projection p with $p(x) = x \otimes 1$. Also let

(5)

$$p_A : \Gamma(A) \otimes \mathbb{Z}/2 \overset{p}{\twoheadrightarrow} \Gamma(A) \otimes \mathbb{Z}/2 \otimes \mathbb{Z}/q = Ext(\mathbb{Z}/2 \otimes \mathbb{Z}/q, \Gamma(A)) \xrightarrow{p^*} Ext(\mathbb{Z}/q, \Gamma(A))$$

be defined by the indicated projections p. Then we obtain the homomorphism

$$(6) \qquad \begin{cases} \Delta_A : Hom(\mathbb{Z}/q, A) \otimes Hom(\mathbb{Z}/q, A) \to Ext(\mathbb{Z}/q, \Gamma A) \\ \Delta_A = p_A r(j_A \otimes j_A) \end{cases}$$

which depends on the choice of the retraction r in (2). Clearly Δ_A is not natural in A since r cannot be chosen to be natural. However one can easily check that Δ_A is natural for homomorphisms $\varphi : \mathbb{Z}/q \to \mathbb{Z}/t$ between cyclic groups that is

$$\Delta_A(\varphi^* \otimes \varphi^*) = \varphi^* \Delta_A. \tag{7}$$

We now define a group

$$G(q, A) = Hom(\mathbb{Z}/q, A) \times Ext(\mathbb{Z}/q, \Gamma(A)) \tag{8}$$

where the group law on the right hand side is given by the cocycle Δ_A, that is

$$(a, b) + (a', b') = (a + a', b + b' + \Delta_A(a \otimes a')). \tag{9}$$

For any abelian group A there is by (XII.1.6) [6] an isomorphism

$$\rho : G(q, A) \cong [\Sigma P_q, M(A, 2)] \tag{3.11}$$

which is natural in $\mathbb{Z}/q$, $q > 1$, and which is compatible with Δ and μ in the proof of (2.2). If A is a direct sum of cyclic groups as above we obtain maps

$$\bar{\alpha}_i : \Sigma P_{a_i} \to M(A, 2)$$

by $\bar{\alpha}_i = \rho(\alpha_i, 0)$ where $\alpha_i \in Hom(\mathbb{Z}/a_i, A)$ is the inclusion. These maps yield the homotopy equivalence

$$\bigvee_i \Sigma P_{a_i} \simeq M(A, 2)$$

which we use as in identification. Hence we may assume that ρ in (3.11) satisfies

$$\rho(\alpha_i, 0) = \alpha_i \tag{*}$$

where α_i is the inclusion in (3.7). We need the following function ∇_A, defined for an ordered direct sum A of cyclic groups,

$$\begin{aligned} &\nabla_A : Hom(\mathbb{Z}/q, A) \to Ext(\mathbb{Z}/q, \Gamma A) \\ &\nabla_A (x) = \sum_{i<j} \Delta_A(\alpha_i x_i \otimes \alpha_j x_j). \end{aligned} \tag{3.12}$$

Here $x_i \in Hom(\mathbb{Z}/q, \mathbb{Z}/a_i)$ is the coordinate of $x = \sum_i \alpha_i x_i$. We observe that $\nabla_A = 0$ is trivial if we define Δ_A by $r^<$ in (3.10) where the ordered basis E in $A \otimes \mathbb{Z}/2$ is given by the ordered set of generators in A. Clearly $2 \nabla_A (x) = 0$ since $2\Delta_A = 0$. The function ∇_A has the following crucial property:

(3.13) Lemma. *In the group $G(q,A)$ we have the formula*

$$\sum_i^{<} x_i^*(\alpha_i, 0) = (x, \nabla_A(x))$$

where the left hand side is the ordered sum of the elements $x_i^*(\alpha_i,0) = (\alpha_i x_i, 0)$ in the group $G(q,A)$.

The lemma is an immediate consequence of the group law (3.10) (9).

For $\varphi \in Hom(A,B)$ in (3.8) and $q \geq 1$ we define the function

$$\nabla(\varphi) : Hom(\mathbb{Z}/q, A) \to Ext(\mathbb{Z}/q, \Gamma(B)) \tag{3.14}$$

via the commutative diagram

$$\begin{array}{ccc}
\pi_2(\mathbb{Z}/q, M(A,2)) & \xrightarrow{(s\varphi)_*} & \pi_2(\mathbb{Z}/q, M(B,2)) \\
\| & & \| \\
G(q,A) & \xrightarrow{(s\varphi)_\sharp} & G(q,B) \\
\| & & \| \\
Hom(\mathbb{Z}/q, A) \times Ext(\mathbb{Z}/q, \Gamma A) & & Hom(\mathbb{Z}/q, B) \times Ext(\mathbb{Z}/q, \Gamma B)
\end{array}$$

where the isomorphisms are given as in (3.11). The homomorphism $(s\varphi)_\sharp$, induced by $s\varphi$ in (3.9), determines $\nabla(\varphi)$ by the formula

$$(s\varphi)_\sharp(x,\alpha) = (\varphi_* x, \Gamma(\varphi)_* \alpha + \nabla(\varphi)(x))$$

for $x \in Hom(\mathbb{Z}/q, A)$ and $\alpha \in Ext(\mathbb{Z}/q, \Gamma A)$. The function $\nabla(\varphi)$ is not a homomorphism.

(3.15) Lemma. *For $x \in Hom(\mathbb{Z}/q, A)$ we have*

$$\begin{aligned}
\nabla(\varphi)(x) = \Gamma(\varphi)_* \nabla_A(x) + \sum_i \nabla_B(\varphi\alpha_i x_i) \\
+ \sum_{i<t} \Delta_B(\varphi\alpha_i x_i \otimes \varphi\alpha_t x_t)
\end{aligned}$$

Since all summands are 2-torsion we have $\nabla(\varphi) = 0$ if q is odd.

Proof. For $(\alpha_i, 0) \in G(a_i, A)$ one has the formula

$$(s\varphi)_\sharp(\alpha_i, 0) = \sum_j^{<} (\beta_j \varphi_{ji}, 0)$$

as follows from property (3.11) (*) of the isomorphism χ. Hence we get by (3.13) the following equations

$$\begin{aligned}(s\varphi)_\sharp(x,0)+(0,\Gamma(\varphi)_*\nabla_A(x)) &= (s\varphi)_\sharp(x,\nabla_A(x))\\ &= (s\varphi)_\sharp(\sum_i^< x_i^*(\alpha_i,0))\\ &= \sum_i^< x_i^*(s\varphi)_\sharp(\alpha_i,0)\\ &= \sum_i^<(\sum_j^<(\beta_j\varphi_{ji}x_i,0))\\ &= \sum_i^<(\varphi\alpha_i x_i,\nabla_B(\varphi\alpha_i x_i))\end{aligned}$$

Here we have in $G(q,B)$ the equation

$$\sum_i^<(\varphi\alpha_i x_i,0)=(\varphi x,\sum_{i<t}\Delta_B(\varphi\alpha_i x_i\otimes\varphi\alpha_t x_t))$$

This yields the result in (3.15).

q.e.d.

We now describe cocycle δ in the class $\beta_1\beta_2(1_\Gamma)$. For this let A,B,C be ordered direct sums of cyclic groups and consider homomorphisms

(3.16) $$\psi\varphi : A\xrightarrow{\varphi}B\xrightarrow{\psi}C.$$

Let $r_A=r^<$ be the retraction of H in (3.10) (3)

$$\Gamma(A)\otimes\mathbb{Z}/2\underset{r_A}{\overset{H}{\rightleftarrows}}\otimes^2(A)\otimes\mathbb{Z}/2 \qquad \text{(see } S_2(A)\text{ in (3.3))}$$

Moreover let s_A be a splitting of σ

$$\Gamma(A)\otimes\mathbb{Z}/2\underset{s_A}{\overset{\sigma}{\rightleftarrows}}A\otimes\mathbb{Z}/2 \qquad \text{(see } S_1(A)\text{ in (3.3))}$$

defined by

$$s_A(\sum_i x_i\alpha_i\otimes 1)=\sum_i x_i\gamma(\alpha_i)\otimes 1.$$

Here the α_i are the generators of A as in (3.7). We now obtain derivations D_1,D_2 by setting

$$D_2(\psi)q = -\psi_* r_B + \psi^* r_C,$$
$$P\,D_1(\varphi) = -\varphi_* s_A + \varphi^* s_B.$$

For this we use the exact sequences $S_i(A)$ in (3.3). We define a 2-cocycle δ which carries (ψ, φ) to the composition

$$\delta(\psi,\varphi) : A \otimes \mathbb{Z}/2 \xrightarrow{D_1(\varphi)} \Lambda^2(B) \otimes \mathbb{Z}/2 \xrightarrow{D_2(\psi)} \Gamma(C) \otimes \mathbb{Z}/2$$

and we observe

(3.17) Lemma.

$$\beta_1 \beta_2 (1_\Gamma) = \{\delta\}$$

where β_1, β_2 are the Bockstein homomorphisms in (3.4). We leave the proof of the lemma as an exercise. The lemma yields a cocycle representing the right hand side in (3.6).

Next we determine the cocycle δ_s in (3.9). For this we use the injection

$$g : Ext(A, \Gamma C) \subset \underset{q>1}{\times} Hom(Hom(\mathbb{Z}/q, A), Ext(\mathbb{Z}/q, \Gamma C))$$

The element $g\Delta_s(\psi,\varphi)$ is given by the $\mathbb{Z}/q$ -natural homomorphism

$$(g\Delta_s(\psi,\varphi))_q : Hom(\mathbb{Z}/q, A) \to Ext(\mathbb{Z}/q, \Gamma C)$$

which satisfies

$$(g\Delta_s(\psi,\varphi))_q(x) = \Gamma(\psi)_* \nabla(\varphi)(x) + \nabla(\psi)(\varphi x) - \nabla(\psi\varphi)(x)$$

This equation is an easy consequence of (3.14). As in the remark following (3.12) we may assume that $\nabla_A = \nabla_B = \nabla_C = 0$ are trivial. Moreover we may assume that q is even since $(g\Delta_s(\psi,\varphi))_q$ is trivial if q is odd. We define a function

$$\rho_A : A \otimes \mathbb{Z}/2 \to \Lambda^2(A \otimes \mathbb{Z}/2)$$
$$\rho_A\Big(\sum_i x_i\alpha_i \otimes 1\Big) = \sum_{i<t} (x_i\alpha_i \otimes 1) \wedge (x_t\alpha_t \otimes 1)$$

(3.18) Lemma.

$$\nabla(\varphi)(x) = \chi_q\, D_2(\varphi)\rho_A(x \otimes \mathbb{Z}/2)$$

Here we have $x \in Hom(\mathbb{Z}/q, A)$ and

$$x \otimes \mathbb{Z}/2 \in Hom(\mathbb{Z}/q \otimes \mathbb{Z}/2, A \otimes \mathbb{Z}/2) = A \otimes \mathbb{Z}/2$$

since q is even. Moreover χ_q in lemma (3.18) is the composition

$$\chi_q : \Gamma(B) \otimes \mathbb{Z}/2 = Ext(\mathbb{Z}/2, \Gamma B) \to Ext(\mathbb{Z}/q, \Gamma B)$$

induced by $\mathbb{Z}/q \to \mathbb{Z}/q \otimes \mathbb{Z}/2 = \mathbb{Z}/2$. Lemma (3.18) is a consequence of the formula in (3.15) and the definition of $r_A = r^<$ in (3.10) (3). We apply Lemma (3.18) to the formula for $(g\Delta_s(\psi,\varphi))_q$ above and we get for $\bar{x} = x \otimes \mathbb{Z}/2$

(3.19) Lemma.

$$(g\Delta_s(\psi,\varphi))_q(x) = \chi_q D_2(\psi)(\rho_B(\varphi\bar{x}) - \varphi_*\rho_A(\bar{x}))$$

This follows easily from (3.18) since D_1 is a derivation. Finally we observe:

(3.20) Lemma.

$$\rho_B(\varphi\bar{x}) - \varphi_*\rho_A(\bar{x}) = D_1(\varphi)(\bar{x})$$

The proof of lemma (3.20) requires a lengthy computation with the definitions of ρ_B, ρ_A and $D_2(\varphi)$. By (3.19) and (3.20) we thus get

$$(3.21) \qquad (g\Delta_s(\psi,\varphi))_q(x) = \chi_q D_2(\psi) D_1(\varphi)(\bar{x})$$

and this yields the formula in (3.6). In fact (3.21) yields an easy algebraic description of the cocycle Δ_s in terms of the derivation D_1 and D_2 above since g is injective.

q.e.d.

§ 4 On the cohomology class $\{nil\}$ and James-Hopf invariants on $\underline{\underline{M}}^2$

In this section we prove a further formula for the class $\{\underline{\underline{M}}^2\}$ which, however, does not determine $\{\underline{\underline{M}}^2\}$ completely.

For the exterior square $\Lambda^2(B)$ of an abelian group B we have the exact sequence (3.1) which induces the exact sequence

$$Ext(A,\Gamma B) \xrightarrow{H_*} Ext(A, \otimes^2 B) \xrightarrow{q_*} Ext(A, \Lambda^2 B) \to 0$$

and hence we have the binatural short exact sequence

$$(4.1) \qquad H_* Ext(A,\Gamma B) \overset{i}{\rightarrowtail} Ext(A, \otimes^2 B) \overset{p_*}{\twoheadrightarrow} Ext(A, \Lambda^2 B)$$

together with the surjective map

$$H' : Ext(A, \Gamma B) \twoheadrightarrow H_* Ext(A, \Gamma B)$$

induced by H_*. The short exact sequence induces the Bockstein homomorphism

$$\beta : H^1(\underline{\underline{Ab}}, Ext(-,\Lambda^2)) \to H^2(\underline{\underline{Ab}}, H_* Ext(-,\Gamma))$$

(4.2) Theorem. *The algebraic class $\{nil\} \in H^1(\underline{\underline{Ab}}, Ext(-,\Lambda^2))$ defined below and the class $\{\underline{\underline{M}}^2\}$ of the homotopy category of Moore spaces in degree 2 satisfy the formula*

$$H'_*\{\underline{\underline{M}}^2\} = \beta\{nil\} \in H^2(\underline{\underline{Ab}}, H_*Ext(-,\Gamma))$$

This result is true in the cohomology of $\underline{\underline{Ab}}$. For the algebraic definition of the class $\{nil\}$ we need the following linear extension $\underline{\underline{nil}}$.

(4.3) Definition. Let $\langle Z\rangle$ be the free group generated by the set Z and let $\Gamma_n\langle Z\rangle$ be the subgroup generated by n-fold commutators. Then

$$A = \langle Z\rangle/\Gamma_2\langle Z\rangle = \bigoplus_Z \mathbb{Z} \tag{1}$$

is the free abelian group generated by Z and

$$E_A = \langle Z\rangle/\Gamma_3\langle Z\rangle \tag{2}$$

is the free $nil(2)$-group generated by Z. We have the classical central extension of groups

$$\Lambda^2 A \overset{w}{\rightarrowtail} E_A \overset{q}{\twoheadrightarrow} A \tag{3}$$

The map w is induced by the commutator map with

$$w(qx \wedge qy) = x^{-1}y^{-1}xy. \tag{4}$$

Here the right hand side denotes the commutator in the group E_A. Using (3) we get the linear extension of categories (compare also [3], [5])

$$Hom(-,\Lambda^2-) \overset{+}{\rightarrowtail} \underline{\underline{nil}} \overset{ab}{\twoheadrightarrow} \underline{\underline{ab}}. \tag{5}$$

Here $\underline{\underline{ab}}$ and $\underline{\underline{nil}}$ are the full subcategories of the category of groups consisting of free abelian groups and free $nil(2)$ -groups respectively. The functor $\underline{\underline{ab}}$ in (3) is abelianization and the action $+$ is given by

$$f + \alpha = f + w\alpha q \tag{6}$$

for $f : E_A \to E_B$, $\alpha \in Hom(A,\Lambda^2 B)$. The right hand side of (6) is a well defined homomorphism since (3) is central.

(4.4) Definition. We define a derivation

$$nil : \underline{\underline{Ab}} \to Ext(-,\Lambda^2)$$

which carries a homomorphism $\varphi : A \to B$ in $\underline{\underline{Ab}}$ to an element $nil(\varphi) \in Ext(A,\Lambda^2 B)$. The cohomology class $\{nil\}$ represented by the derivation nil is the class used in (4.2). For the definition of nil we choose for each abelian group A a short exact sequence

$$0 \to A_1 \xrightarrow{d_A} A_0 \xrightarrow{q} A \to 0$$

where A_0, A_1 are free abelian groups. We also choose a homomorphism

$$\bar{d}_A : E_{A_1} \to E_{A_0}$$

between free $nil(2)$ -groups such that the abelianization of $\bar{d}_A$ is d_A. For the homomorphism $\varphi : A \to B$ we choose a commutative diagram in $\underline{\underline{Ab}}$

$$\begin{array}{ccccc} A_1 & \xrightarrow{d_A} & A_0 & \xrightarrow{q} & A \\ \varphi_1 \downarrow & & \downarrow \varphi_0 & & \downarrow \varphi \\ B_1 & \xrightarrow{d_B} & B_0 & \xrightarrow{q} & B \end{array}$$

and we choose a diagram of homomorphisms

$$\begin{array}{ccc} E_{A_1} & \xrightarrow{\bar{d}_A} & E_{A_0} \\ \bar{\varphi}_1 \downarrow & & \downarrow \bar{\varphi}_0 \\ E_{B_1} & \xrightarrow{\bar{d}_B} & E_{B_0} \end{array}$$

which by abelianization induces (φ_0, φ_1). This diagram, in general, cannot be chosen to be commutative. Since, however, $\varphi_0 d_A = d_B \varphi_1$ there is a unique element

$$\alpha \in Hom(A_1, \Lambda^2 B_0) \qquad \text{with} \qquad \bar{\varphi}_0 \bar{d}_A + \alpha = \bar{d}_B \bar{\varphi}_1.$$

Here we use the action in (4.3) (6). Now let

$$nil(\varphi) \in Ext(A, \Lambda^2 B) = Hom(A_1, \Lambda^2 B)/d_A^* Hom(A_0, \Lambda^2 B)$$

be the element represented by the composition

$$(\Lambda^2 q)\alpha : A_1 \to \Lambda^2 B_0 \to \Lambda^2 B$$

One can check that $nil(\varphi)$ does not depend on the choice of (φ_0, φ_1) and $(\bar{\varphi}_0, \bar{\varphi}_1)$ respectively and that nil is a derivation, that is $nil(\varphi\psi) = \varphi_* nil(\psi) + \psi^* nil(\varphi)$. This completes the definition of the cohomology class $\{nil\}$.

Next we use the derivation D_1 on $\underline{\underline{Ab}}$ defined as in (3.16). The derivation D_1 carries $\varphi : A \to B$ to

$$D_1(\varphi) \in Hom(A \otimes \mathbb{Z}/2, \Lambda^2(B) \otimes \mathbb{Z}/2) = Ext(A \otimes \mathbb{Z}/2, \Lambda^2 B)$$

and hence represents a cohomology class

$$\{D_1\} \in H^1(\underline{\underline{Ab}}, Ext(- \otimes \mathbb{Z}/2, \Lambda^2)).$$

Let

$$p_2 : Ext(A \otimes \mathbb{Z}/2, \Lambda^2 B) \to Ext(A, \Lambda^2 B)$$

be induced by the projection $A \twoheadrightarrow A \otimes \mathbb{Z}/2$.

(4.5) <u>Proposition</u>. *Let $\underline{\underline{A}}$ be the full subcategory of $\underline{\underline{Ab}}$ consisting of direct sums of cyclic groups. Then we have*

$$i^*_{\underline{\underline{A}}}(p_2)_*\{D_1\} = i^*_{\underline{\underline{A}}}\{nil\}$$

in $H^1(\underline{\underline{A}}, Ext(-, \Lambda^2))$.

We do not know whether this formula also holds if we omit $i^*_{\underline{\underline{A}}}$. Proposition (4.5) implies that the formulas in (4.2) and (3.6) are compatible. For the proof of (4.5) we need the following properties of $nil(2)$ -groups. A group G is a <u>$nil(2)$-group</u> if all triple commutators vanish in G. The commutators in G yield the central homomorphism

$$w : \Lambda^2(G^{ab}) \to G \tag{4.6}$$

where $G \to G^{ab}$, $x \longmapsto \{x\}$, is the abelianization of G. We define w by the commutator

$$w(\{x\} \wedge \{g\}) = x^{-1}y^{-1}xy$$

for $x, y \in G$. Let M be a set and let $f : M \to G$ be a function such that only finitely many elements $f(m)$, $m \in M$, are non trivial and let $<$, $<<$ be two total orderings on the set M. Then we have in G the formula

$$\sum_{m\in M}^{<<} f(m) = \sum_{m\in M}^{<} f(m) + w\left(\sum_{\substack{m<<m' \\ m'<m}} \{fm\} \wedge \{fm'\}\right)$$

For $a \in G$ and $n \in \mathbb{Z}$ let $na = a + \ldots + a$ be the n-fold sum in G in case $n \geq 0$, and let $na = -|n|a$ for $n < 0$. Then one gets in G the formula

$$n\sum_{m\in M}^{<} f(m) = \sum_{m\in M}^{<} nf(m) - w\left(\binom{n}{2}\sum_{m<m'} \{fm\} \wedge \{fm'\}\right)$$

where $\binom{n}{2} = n(n-1)/2$.

<u>Proof of</u> (4.5). Let A and B be direct sums of cyclic groups and let $\varphi : A \to B$ be given by $\varphi_{ji} \in \mathbb{Z}$ as in (3.8). Let A_0 be the free group generated by the set of generators $\{\alpha_i\}$ of A and let A_1 be the free group generated by the $\{\alpha_i,\, a_i \neq 0\}$. Then we choose, see (4.4),

$$\begin{cases} \bar{d}_A : E_{A_1} \to E_{A_0} \\ \bar{d}_A(\alpha_i) = a_i\alpha_i \end{cases}$$

Similarly we define $\bar{d}_B$. Moreover we define $\bar{\varphi}_1$ and $\bar{\varphi}_0$ by the ordered sum

$$\bar{\varphi}_0(\alpha_i) = \sum_j^{<} \varphi_{ji}\beta_j \in E_{B_0}$$

$$\bar{\varphi}_1(\alpha_i) = \sum_j^{<} (a_i\varphi_{ji}/b_j)\beta_j \in E_{B_1}$$

Hence we get α in (4.4) by the formula, see (4.6),

$$\begin{aligned} \bar{d}_B\bar{\varphi}_1(\alpha_i) - \bar{\varphi}_0\bar{d}_A(\alpha_i) &= \sum_j^{<} a_i\varphi_{ji}\beta_j - a_i\sum_j^{<} \varphi_{ji}\beta_j \\ &= w\binom{a_i}{2}\sum_{j<t}\{\varphi_{ji}\beta_j\} \wedge \{\varphi_{ti}\beta_t\} \end{aligned}$$

Hence $nil(\varphi) \in Ext(A, \Lambda^2 B)$ is given by the formula ($\alpha_i : \mathbb{Z}/a_i \subset A$ as in (3.7))

$$(\alpha_i)^* nil(\varphi) = \binom{a_i}{2}\sum_{j<t}\varphi_{ji}\varphi_{ti}(1 \otimes \beta_j \wedge \beta_t)$$

where $1 \otimes \beta_j \wedge \beta_t \in \mathbb{Z}/a_i \otimes \Lambda^2 B = Ext(\mathbb{Z}/a_i, \Lambda^2 B)$. Using the definition of D_1 in the proof of (3.16) it is easy to check that $(\alpha_i)^* p_2 D_1(\varphi)$ coincides with the right hand side of the formula so that we actually have

$$nil(\varphi) = p_2 D_1(\varphi).$$

This proves the proposition in (4.5).

q.e.d.

We will need the following element which projects to $nil(\varphi)$ above.

(4.7) ***Definition.*** For φ in the proof above let

$$\overline{nil}(\varphi) \in Ext(A, \otimes^2 B)$$

be given by the formula

$$(\alpha_2)^*\overline{nil}(\varphi) = \binom{a_i}{2}\sum_{j<t}\varphi_{ji}\varphi_{ti}(1 \otimes \beta_j \otimes \beta_t)$$

We clearly have $Ext(A, p)\overline{nil}(\varphi) = nil(\varphi)$ where $p : \otimes^2 B \twoheadrightarrow \Lambda^2 B$ is the projection.

Recall that we have for the bifunctor $Ext(-, \otimes^2)$ on $\underline{\underline{Ab}}$ the canonical split linear extension

$$Ext(-, \otimes^2) \rightarrowtail \underline{\underline{Ab}} \times Ext(-, \otimes^2) \twoheadrightarrow \underline{\underline{Ab}}$$

Objects in $\underline{\underline{Ab}} \times Ext(-, \otimes^2)$ are abelian groups and morphisms $(\varphi, \alpha) : A \to B$ are given by $\varphi \in Hom(A, B)$ and $\alpha \in Ext(A, \otimes^2 B)$ with composition $(\varphi, \alpha)(\psi, \beta) = (\varphi\psi, \varphi_*\beta + \psi^*\alpha)$. The derivation nil in (4.4) defines a subcategory

$$\underline{\underline{Ab}}(nil) \subset \underline{\underline{Ab}} \times Ext(-, \otimes^2) \tag{4.8}$$

consisting of all morphisms $(\varphi, \alpha) : A \to B$ which satisfy the condition

$$p_*(\alpha) = nil(\varphi) \in Ext(A, \Lambda^2 B).$$

Here $p : \otimes^2 B \twoheadrightarrow \Lambda^2 B$ induces $p_* = Ext(A, p)$. The exact sequence (4.1) shows that we have a commutative diagram of linear extensions of categories

$$\begin{array}{ccccc} H_* Ext(-, \Gamma) & \xrightarrow{+} & \underline{\underline{Ab}}(nil) & \longrightarrow & \underline{\underline{Ab}} \\ \cap & & \cap & & \| \\ Ext(-, \otimes^2) & \xrightarrow{+} & \underline{\underline{Ab}} \times Ext(-, \otimes^2) & \longrightarrow & \underline{\underline{Ab}} \end{array}$$

(4.9) Lemma. *The cohomology class represented by the linear extension for $\underline{\underline{Ab}}(nil)$ satisfies*

$$\{\underline{\underline{Ab}}(nil)\} = \beta\{nil\} \in H^2(\underline{\underline{Ab}}, H_* Ext(-, \Gamma))$$

where β is the Bockstein operator in (4.2).

Proof. Let $s : Ext(A, \Lambda^2 B) \to Ext(A, \otimes^2 B)$ be a set theoretic splitting of $Ext(A, p) = p_*$. Then $\beta\{nil\}$ is represented by the 2-cocycle $c = i^{-1}\delta(s\,nil)$ where i is the inclusion in (4.1) and where δ is the coboundary in (1.5). Hence c carries the 2-simplex (ψ, φ) in $\underline{\underline{Ab}}$ to

$$c(\psi, \varphi) = i^{-1}(\psi_* s\,nil(\varphi) - s\,nil(\psi\varphi) + \varphi^* s\,nil(\psi))$$

On the other hand we define a set theoretic section t for the linear extension $\underline{\underline{Ab}}(nil)$ by $t(\varphi) = (\varphi, s\,nil(\varphi))$. Then Δ_t in (1.4) is given by

$$s\,nil(\psi\varphi) = \psi_* s\,nil(\varphi) + \varphi^* s\,nil(\psi) + i\Delta_t(\psi, \varphi)$$

Hence $c = -\Delta_t$ yields the proposition. In fact, since the elements in (4.9) are of order 2 we can omit the sign.

q.e.d.

For Moore spaces $M(A, 2) = \Sigma M_A$ and $M(B, 2) = \Sigma M_B$ as in (2.2) we have the James-Hopf invariant [12], [7],

$$[\Sigma M_A, \Sigma M_B] \xrightarrow{\gamma_2} [\Sigma M_A, \Sigma M_B \wedge M_B] = Ext(A, B \otimes B) \tag{4.10}$$

which satisfies for $\alpha \in Ext(A, \Gamma B)$ the formula

$$(1) \qquad \lambda_2(\xi+\alpha) = \lambda_2(\xi) + H_*\alpha.$$

Hence γ_2 induces a well defined function

$$(2) \qquad \bar{\gamma}_2 : Hom(A,B) \to Ext(A, \Lambda^2 B)$$

defined by $\bar{\gamma}_2(\varphi) = q_*\gamma_2(\xi)$ where ξ induces $H_2(\xi) = \varphi : A \to B$. One can check that $\bar{\gamma}_2$ is a derivation which represents a cohomology class in $H^1(\underline{\underline{Ab}}, Ext(-,\Lambda^2 B))$. This cohomology class does not depend on the choice of M_A, M_B above.

(4.11) Theorem. *The cohomology class $\{\bar{\gamma}_2\}$ given by the James-Hopf invariant γ_2 coincides with*

$$\{\bar{\gamma}_2\} = \{nil\} \in H^1(\underline{\underline{Ab}}, Ext(-,\Lambda^2))$$

Moreover there is a full functor τ,

$$\underline{\underline{M}}^2 \xrightarrow{\tau} \underline{\underline{Ab}}(nil) \overset{i}{\subset} \underline{\underline{Ab}} \times Ext(-,\otimes^2)$$

which is the identity on objects and which is defined on morphisms by

$$\tau(\xi) = (H_2\xi, \gamma_2\xi)$$

The functor τ is part of the following commutative diagram of linear extensions

$$\begin{array}{ccccc} Ext(-,\Gamma) & \xrightarrow{+} & \underline{\underline{M}}^2 & \xrightarrow{H_2} & \underline{\underline{Ab}} \\ {\scriptstyle H'}\downarrow & & \downarrow{\scriptstyle \tau} & & \| \\ H_*Ext(-,\Gamma) & \xrightarrow{+} & \underline{\underline{Ab}}(nil) & \longrightarrow & \underline{\underline{Ab}} \end{array}$$

Proof of (4.2). The existence of the functor τ shows that $H'_*\{\underline{\underline{M}}^2\} = \{\underline{\underline{Ab}}(nil)\}$. Therefore we obtain (4.2) by (4.9).

q.e.d.

(4.12) Remark. We can give an alternative description of the functor τ in (4.11) by use of the singular chain complex of a loop space which yields the Adams-Hilton functor

$$C_*\Omega : Ho(\underline{\underline{Top}}^*) \to Ho(\underline{\underline{DA}})$$

between homotopy categories (compare [1] and also [4]). The functor $C_*\Omega$ restriced to $\underline{\underline{M}}^2$ leads to the following diagram where $\tilde{\underline{\underline{M}}}^2 \subset Ho(\underline{\underline{DA}})$ is the full subcategory consisting of $C_*\Omega M(A,2)$, $A \in \underline{\underline{Ab}}$,

$$\begin{array}{ccc} \underline{\underline{M}}^2 & \xrightarrow{C_*\Omega} & \tilde{\underline{\underline{M}}}^2 \subset Ho(\underline{\underline{DA}}) \\ {\scriptstyle \tau}\downarrow & & {\scriptstyle j}\uparrow{\scriptstyle \sim} \\ \underline{\underline{Ab}}(nil) & \xrightarrow{i} & \underline{\underline{Ab}} \times Ext(-,\otimes^2) \end{array}$$

where j is an equivalence of categories such that $ji\tau$ is naturally isomorphic to $C_*\Omega$.

Proof of (4.11). The image category of the functor

$$\tau : \underline{\underline{M}}^2 \to \underline{\underline{Ab}} \times Ext(-, \otimes^2)$$

is $\underline{\underline{Ab}}(nil)$ since we show

$$\bar{\gamma}_2 = nil \tag{1}$$

for compatible choices of $\bar{d}_A$, d'_A in (4.4) and (2.2). We use the equivalence of linear track extension described in (VI.4.7) of Baues [5]. This shows that a triple $(\bar{\varphi}_0, \bar{\varphi}_1, G)$ with $G \in Hom(A_1, \otimes^2 B_0)$ satisfying $p_* G = \alpha$ (see (4.4)) corresponds to a diagram

$$\begin{array}{ccc} \Sigma M_{A_1} & \xrightarrow{\Sigma d'_A} & \Sigma M_{A_0} \\ \Sigma\varphi'_1 \downarrow & \overset{G'}{\Longrightarrow} & \downarrow \Sigma\varphi'_0 \\ \Sigma M_{B_1} & \xrightarrow[\Sigma d'_B]{} & \Sigma M_{B_0} \end{array} \tag{2}$$

Here d'_A and d'_B induce $\bar{d}_A$ and $\bar{d}_B$ respectively and φ'_0, φ'_1 induces $\bar{\varphi}_0$, $\bar{\varphi}_1$ in (4.4). The track G' is determined by G. This track determines a principal map $\bar{\varphi} \in [\Sigma M_A, \Sigma M_B]$ such that $\tau(\bar{\varphi}) = (\varphi, (\otimes^2 q)_* \{G\})$ where $\{G\} \in Ext(A, \otimes^2 B_))$ is represented by G. This follows from the bijection (6) ... (11) in (VI.4.7) Baues [5]. Since $p_* G = \alpha$ we get $\bar{\gamma}_2 = nil$. q.e.d.

(4.13) Example. Let A and B be direct sums of cyclic groups as in (3.8) and let $s\varphi \in [M(A,2), M(B,2)]$ be defined as in (3.9). Then the functor τ in (4.11) satisfies

$$\tau(s\varphi) = (\varphi, \overline{nil}\,(\varphi))$$

where $\overline{nil}\,(\varphi)$ is defined in (4.7). We obtain this formula by the methods in the proof of (4.11) above. In this case we also can compute the James-Hopf invariant $\gamma_2(s\varphi)$ which actually is $\gamma_2(s\varphi) = \overline{nil}(\varphi)$.

As a corollary of (4.2) we get:

(4.14) Proposition. $\{\underline{\underline{M}}^2\}$ is a (non trivial) element of order 2.

Proof. We know that multiplication by 2 on $\Gamma(A)$ is the composition

$$2 = PH : \Gamma A \to \otimes^2 A \to \Gamma A$$

where $P = [1,1]$. Hence also the composition

$$\begin{array}{ccccc} Ext(A, \Gamma B) & \xrightarrow{H'} & H_* Ext(A, \Gamma B) & \xrightarrow{P'} & Ext(A, \Gamma B) \\ \| & & \cap & & \| \\ Ext(A, \Gamma B) & \xrightarrow{H_*} & Ext(A, \otimes^2 B) & \xrightarrow{P_*} & Ext(A, \Gamma B) \end{array}$$

is a multiplication by 2. Therefore we get by (4.2):

$$\begin{aligned} 2\{\underline{\underline{M}}^2\} &= (P'\,H')_*\{\underline{\underline{M}}^2\} \\ &= P'_*\,H'_*\{\underline{\underline{M}}^2\} \\ &= P'_*\,\beta\,\{nil\} \end{aligned}$$

Here the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_*Ext(A,\Gamma B) & \longrightarrow & Ext(A,\otimes^2 B) & \longrightarrow & Ext(A,\Lambda^2 B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle P'} & & \downarrow{\scriptstyle P_*} & & \downarrow & & \\ 0 & \longrightarrow & Ext(A,\Gamma B) & \longrightarrow & Ext(A,\Gamma B) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

shows that $P'_*\beta = 0$.

q.e.d.

(4.15) Proposition. Each element in $H^1(\underline{\underline{Ab}}, Ext(-,\Lambda^2))$ is of order 2, in particular, $2\{nil\} = 0$.

Proof. Let A, B be abelian groups and let $\varphi \in Hom(A,B)$. Let $2_A = 2\,id \in Hom(A,A)$ be multiplication by 2. Then we have

$$\varphi \circ 2_A = 2\varphi = 2_B \circ \varphi.$$

Now the derivation property of N with $\{N\} \in H^1(\underline{\underline{Ab}}, Ext(-,\Lambda^2))$ shows:

$$\begin{aligned} N(\varphi \circ 2_A) &= \varphi_*\,N(2_A) + (2_A)^*\,N(\varphi) \\ &= \varphi_*\,N(2_A) + 2N(\varphi) \\ N(2_B \circ \varphi) &= (2_B)_*N(\varphi) + \varphi^*\,N(2_B) \\ &= 4\,N(\varphi) + \varphi^*\,N(2_B) \end{aligned}$$

Hence we get

$$2N(\varphi) = \varphi_*\,N(2_A) - \varphi^*\,N(2_B)$$

so that $2N$ is an inner derivation.

q.e.d.

§ 5 A subcategory of $\underline{\underline{M}}^2$ given by diagonal elements

Let $\mathbb{Z}/2 * A$ be the 2-torsion of the abelian group A. We here construct a subcategory $\underline{\underline{H}}$ of the category of Moore spaces $\underline{\underline{M}}^2$ with the following property.

(5.1) <u>Theorem</u>. *There exists a subcategory $\underline{\underline{H}}$ of $\underline{\underline{M}}^2$ together with a commutative diagram of linear extensions*

$$\begin{array}{ccccc} \mathbb{Z}/2 * Ext(-,\Gamma) & \xrightarrow{+} & \underline{\underline{H}} & \longrightarrow & \underline{\underline{Ab}} \\ \cap & & \cap & & \| \\ Ext(-,\Gamma) & \xrightarrow{+} & \underline{\underline{M}}^2 & \longrightarrow & \underline{\underline{Ab}} \end{array}$$

The theorem shows that the class $\{\underline{\underline{M}}^2\}$ is in the image

$$i_* : H^2(\underline{\underline{Ab}}, \mathbb{Z}/2 * Ext(-,\Gamma)) \to H^2(\underline{\underline{Ab}}, Ext(-,\Gamma))$$

where i is the inclusion $\mathbb{Z}/2 * Ext(A,\Gamma(B)) \subset Ext(A,\Gamma(B))$.

(5.2) <u>Corollary</u>. *The extension $\underline{\underline{M}}^2 \to \underline{\underline{Ab}}$ is split on any full subcategory of $\underline{\underline{Ab}}$ consisting of objects A, B with $(\mathbb{Z}/2) * Ext(A,\Gamma B) = 0$.*

(5.3) <u>Corollary</u>. *Let A be any abelian group for which the 2-torsion of $Ext(A,\Gamma A)$ is trivial. Then the group of homotopy equivalences of $M(A,2)$ is given by the split extension*

$$Ext(A,\Gamma A) \rightarrowtail \mathfrak{E}(M(A,2)) \twoheadrightarrow Aut(A)$$

where $\varphi \in Aut(A)$ acts on $a \in Ext(A,\Gamma A)$ by $\varphi \cdot a = (\Gamma\varphi)_*(\varphi^{-1})^*(a)$.

<u>Proof of</u> (5.1). For a Moore space $M(A,2) = \Sigma M_A$ we have the <u>diagonal element</u>

(1) $$\Delta_A \in [\Sigma M_A, \Sigma M_A \wedge M_A] = Ext(A, A \otimes A)$$

which is given by the suspension of the reduced diagonal $M_A \to M_A \wedge M_A$. Let $[1_A, 1_A] : \Sigma M_A \wedge M_A \to \Sigma M_A$ be the Whitehead product for the identity 1_A of ΣM_A. Then

(2) $$[1_A, 1_A]\Delta_A = -1_A - 1_A + 1_A + 1_A = 0$$

is the trivial commutator. This implies that also

(3) $$\Delta_A \in Ker\{[1,1]_* : Ext(A, A \otimes A) \to Ext(A,\Gamma A)\}$$

with $[1,1]$ in (3.2). We have the short exact sequences (see (3.3))

$$\begin{array}{ccccccc} 0 \to Ext(A,\Gamma(A) \otimes \mathbb{Z}/2) & \xrightarrow{H_*} & Ext(A, \otimes^2(A) \otimes \mathbb{Z}/2) & \xrightarrow{q_*} & Ext(A, \Lambda^2(A) \otimes \mathbb{Z}/2) \to 0 \\ & & \downarrow{\scriptstyle [1,1]_*} & & \downarrow \\ & & & & Ext(A,\Gamma(A) \otimes \mathbb{Z}/2) \end{array}$$

which shows by (3) that for the projection $p : \otimes^2 A \to (\otimes^2 A) \otimes \mathbb{Z}/2$ there is a unique element $\Delta'_A \in Ext(A, \Gamma(A) \otimes \mathbb{Z}/2)$ with

$$(4) \qquad H_* \Delta'_A = p_* \Delta_A$$

We now choose by the surjection

$$p_* : Ext(A, \Gamma A) \to Ext(A, \Gamma(A) \otimes \mathbb{Z}/2)$$

an element $\Delta''_A \in Ext(A, \Gamma A)$ with

$$(5) \qquad p_* \Delta''_A = \Delta'_A$$

We call Δ''_A a diagonal structure for A. For the definition of the subcategory $\underline{\underline{H}}$ in $\underline{\underline{M}}^2$ we choose such a diagonal structure for each abelian group A in $\underline{\underline{Ab}}$. We define the set of morphisms in $\underline{\underline{H}}$ with

$$(6) \qquad \underline{\underline{H}}(A, B) \subset [\Sigma M_A, \Sigma M_B]$$

by the composition (compare (4.10))

$$[\Sigma M_A, \Sigma M_B] \xrightarrow{\gamma_2} Ext(A, B \otimes B) \xrightarrow{[1,1]_*} Ext(A, \Gamma B),$$

and by diagonal structures Δ''_A, Δ''_B, namely

$$(7) \qquad \bar{\varphi} \in \underline{\underline{H}}(A, B) \Leftrightarrow [1,1]_* \gamma_2 \bar{\varphi} = -\varphi_* \Delta''_A + \varphi^* \Delta''_B.$$

We show that for $\bar{\varphi} \in \underline{\underline{H}}(A, B)$ and $\bar{\psi} \in \underline{\underline{H}}(B, C)$ we actually have $\bar{\psi}\bar{\varphi} \in \underline{\underline{H}}(A, C)$ so that $\underline{\underline{H}}$ is a well defined subcategory of $\underline{\underline{M}}^2$. For this we need the fact that γ_2 is a derivation, namely

$$\gamma_2(\bar{\psi}\bar{\varphi}) = \psi_* \gamma_2(\bar{\varphi}) + \varphi^* \gamma_2(\bar{\varphi}).$$

Hence we get:

$$\begin{aligned}
[1,1]_* \gamma_2(\bar{\psi}\bar{\varphi}) &= [1,1]_*(\psi_* \gamma_2(\bar{\varphi}) + \varphi^* \gamma_2(\bar{\psi})) \\
&= \psi_*[1,1]_* \gamma_2(\bar{\varphi}) + \varphi^*[1,1]_* \gamma_2(\bar{\varphi}) \\
&= \psi_*(-\varphi_* \Delta''_A + \varphi^* \Delta''_B) + \varphi^*(-\psi_* \Delta''_B + \psi^* \Delta''_C) \\
&= -(\psi\varphi)_* \Delta''_A + (\psi\varphi)^* \Delta''_C.
\end{aligned}$$

The crucial observation needed for the proof of theorem (5.1) is the following equation where we use the interchange map $T : B \otimes B \to B \otimes B$ with $T(x \otimes y) = y \otimes x$,

$$(8) \qquad (1-T)_*\gamma_2(\bar\varphi) = \varphi_*\Delta_A - \varphi^*\Delta_B$$

This equation follows from the corresponding known property of James-Hopf invariants (Appendix A [6]) with respect to "cup products" which in our case has the form

$$\bar\varphi \cup \bar\varphi = \Delta_{1,1}\bar\varphi + (1+T_{2,1})\gamma_2(\bar\varphi).$$

This equation is equivalent to (10). We now consider the following commutative diagram.

$$\begin{array}{ccccc}
Ext(A,\Gamma B) & = & Ext(A,\Gamma B) & = & Ext(A,\Gamma B) \\
\downarrow + & & \downarrow H_* & & \downarrow \cdot 2 \\
[\Sigma M_A, \Sigma M_B] & \xrightarrow{\gamma_2} & Ext(A, B\otimes B) & \xrightarrow{[1,1]_*} & Ext(A,\Gamma B) \\
\downarrow \mu & & \downarrow & & \downarrow \\
Hom(A,B) & \xrightarrow{\bar\gamma_2} & Ext(A,\Lambda^2 B) & \xrightarrow{[1,1]_*} & Ext(A,\Gamma(B)\otimes \mathbb{Z}/2)
\end{array}$$

the columns are exact sequences. Here γ_2 is not a homomorphism; since however (4.10) (1) holds we see that the induced function $\bar\gamma_2$ is well defined. Moreover we use $[1,1]H = \cdot 2$ so that $[1,1]_*$ in the bottom row is well defined. We now claim that (8) implies the formula

$$(9) \qquad [1,1]_*\bar\gamma_2(\varphi) = -\varphi_*\Delta'_A + \varphi^*\Delta'_B$$

This shows by the diagram above that for any $\varphi \in Hom(A,B)$ there is an element $\bar\varphi$ which satisfies the condition in (7). Thus the functor $\underline{\underline{H}} \to \underline{\underline{Ab}}$ is full, moreover the diagram above shows that $\underline{\underline{H}}$ is part of a linear extension as described in the theorem. In fact for $\bar\varphi \in \underline{\underline{H}}(A,B)$ we have $\bar\varphi + \alpha \in \underline{\underline{H}}(A,B)$ if and only if $2\alpha = 0$.

It remains to prove (9). For this consider the commutative diagram

$$\begin{array}{ccccc}
Ext(A,B\otimes B) & & \longrightarrow & & Ext(A,\Gamma B) \\
\downarrow & & Ext(A,B\otimes B) & & \downarrow p_* \\
Ext(A,B\wedge B) & & \longrightarrow & & Ext(A,\Gamma(B)\otimes \mathbb{Z}/2) \\
 & & Ext(A,B\otimes B\otimes B) & &
\end{array}$$

The square in this diagram coincides with the corresponding square in the diagram above. Since for $x \otimes y \in B \otimes B$

$$H[1,1](x \otimes y) = x \otimes y + y \otimes x \equiv x \otimes y - y \otimes x \mod 2$$

we see that the diagram commutes. The homomorphism t is induced by $1 - T$. On the other hand H_* in the diagram is injective. This shows by the following equations that (9) holds.

$$\begin{aligned} H_*[1,1]_*\bar{\gamma}_2(\varphi) &= H_* p_*[1,1]_*\gamma_2\bar{\varphi} \\ &= p_*(1-T)_*\gamma_2\bar{\varphi} \\ &= p_*(\varphi_*\Delta_A - \varphi^*\Delta_B) \\ &= \varphi_*(p_*\Delta_A) - \varphi^*(p_*\Delta_B) \\ &= \varphi_*(H_*\Delta'_A) - \varphi^*(H_*\Delta'_B) \\ &= H_*(\varphi_*\Delta'_A - \varphi^*\Delta'_B). \end{aligned}$$

This completes the proof of theorem (5.1). q.e.d.

Formula (9) in the proof of (5.1) above and (1) in the proof of (4.11) show

$$\begin{aligned} [1,1]_*\, nil\,(\varphi) &= [1,1]_*\, \bar{\varphi}_2(\varphi) \\ &= -\varphi_*\, \Delta'_A + \varphi^*\, \Delta'_B \end{aligned}$$

Hence the composition $[1,1]_*\, nil$ with

$$[1,1]_* : Ext(A, \Lambda^2 B) \to Ext(A, \Gamma B \otimes \mathbb{Z}/2)$$

is an inner derivation. This implies

(5.4) Proposition.

$$[1,1]_*\{nil\} = 0$$

in $H^1(\underline{\underline{Ab}}, Ext(-, \mathbb{Z}/2 \otimes \Gamma))$.

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