Global Exponential Attractors for a Class of Almost–Periodic Parabolic Equations on $\mathbb{R}^N$

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S.34, Z. 10 v. u.: Lies » $g'(x_o)u$ « statt » $g'(x)u$ «
Lies » $g^{(r)}(x_o + \vartheta u)(u)^r$ « statt » $g^{(r)}(x_o + \vartheta tu)(u^1)$ «

S.35, Z. 15 v. u.: Lies » einer Bilinearform « statt » eine Bilinearform «
Z. 11 v. u.: Streiche » symmetrisch, also entstanden «
Z. 11/10 v. u.: Lies » Bilinearform entstanden voraussetzen « statt » Bilinearform voraussetzen «

S.36, Z. 11 v. o.: Füge hinter » $Q(x) \neq 0$ « ein » Ist m = 1, so gilt $Q(x) = ax^2$ mit $a \in \mathbb{R}$, und folglich nach Voraussetzung $a \neq 0$. Daher ist Q definit. Sei nun $m > 1$: «
Z. 13 v. o.: Lies » mit (wegen $m > 1$) $w(t) \neq 0$ « statt » mit $w(t) \neq 0$ «

S.37, Z. 9 v. o.: Lies » sämtlich « statt » sämtliche «
Z. 11 v. o.: Lies » sämtlich « statt » sämtliche «

S. 38, Z. 11 v. o.: Lies » $\frac{1}{a}(ax + by)^2$ « statt » $\frac{1}{2}(ax + by)^2$ «
Z. 13 v. u.: Lies » Ist das Gleichheitszeichen für $x \neq x_o$ ausgeschlossen, « statt » Ist das Gleichheitszeichen ausgeschlossen, «

S.39, Z. 10 v. u.: Streiche die Zeilen 10, 9, 8 v. u. und lies stattdessen » $g(x_o + tu)$ definiert und nach der Taylorformel gilt $g(x_o + tu) - g(x_o) = \frac{1}{2} g''(x_o + \vartheta tu)(tu,tu)$ mit $0 < \vartheta < 1$, also für $t \neq 0$:
$$\frac{1}{t^2}(g(x_o + tu) - g(x_o)) = g''(x_o + \vartheta tu)(u,u).$$ «
Z. 6 v. u.: Lies » der Stetigkeit von g'' in x_o « statt » der Stetigkeit von g in x_o «
Z. 4 v. u.: Lies » $g'(x_o)(x - x_o)$ « statt » $g'(x)(x - x_o)$ «

S.40, Z.16/17 v.o.: Lies » der "ersten partiellen Funktion von F im Punkte $(x,\bar{u})$" « statt » der "partiellen Funktion im Punkte (x,u)" «
Z. 15 v. u.: Lies » "zweite partielle Funktion h von F im Punkte $(\bar{x},\bar{u})$" « statt » partielle Funktion h im Punkte $(\bar{x},\bar{u})$ «
Z. 9 v. u.: Lies » $u_k \in \mathbb{R}^m$ mit $\|u_k\| = 1$ « statt » $u_k \in \mathbb{R}^m$ «
Z. 7/6 v. u.: Streiche Zeilen 7 und 6 v. u. und füge stattdessen ein » $\{u_k\}_{k=1,2,\dots}$ besitzt daher eine gegen $u_o \in A$ konvergente Teilfolge $\{u_{k_i}\}_{i=1,2,\dots}$. «
Z. 4 v. u.: Lies » (x_o,u_o) « statt » (x_o,u_u) «

S.45, Z. 11 v. o.: Lies » x_o « statt » x «
Z. 7 u. Z. 3 v. u.: Lies » mit der halben Kantenlänge « statt » mit der Norm «

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by

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Abstract. In this article, we construct global exponential attractors for a class of almost–periodic semilinear reaction–diffusion equations with Neumann boundary conditions on bounded regions of $\mathbb{R}^N$. The class of problems which we analyze here contains in particular Fisher's equations of population genetics.

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Consider the class of real semilinear parabolic Neumann boundary value problems of the form

$$\left\{\begin{array}{lll} u_t(x,t) & = & \Delta u(x,t) + s(t)g(u(x,t)), \quad (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq & (u_0,u_1) & \\ \frac{\partial u}{\partial \nu}(x,t) & = & 0 \qquad\qquad , \quad (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \tag{1}$$

In (1), Ω denotes an open connected bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $N \in [2,\infty) \cap \mathbb{N}^+$, while Δ stands for Laplace's operator in the x–variable. Furthermore, $s : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is the restriction to $\mathbb{R}^+$ of a Bohr almost–periodic function on $\mathbb{R}$ which we shall also denote by s, while $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$ possesses at least two zeroes u_0 and u_1 such that $g(u) > 0$ for every $u \in (u_0,u_1)$, with the property that $g'(u_0) > 0$ and $g'(u_1) < 0$. Finally, $\mathrm{Ran}(u)$ denotes the range of u and ν stands for the normalized outer normal vector to $\partial\Omega$.

Problems of the form (1) occur in various fields of sciences, such as the theory of nerve pulse propagation and population genetics ([2]–[4]). It is then natural to ask whether there are conditions on the function s such that <u>every</u> classical solution $(x,t) \longrightarrow u(x,t)$ to Problem (1) which exists globally in time stabilizes around a stable attractor as $t \longrightarrow \infty$. It is the purpose of this article to show that this is indeed possible. We shall refer to the above properties of g as being the following hypothesis:

(G) We have $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$ and there exist $u_0,u_1 \in \mathbb{R}$ such that $g(u_0) = g(u_1) = 0$, $g'(u_0) > 0$, $g'(u_1) < 0$ and $g(u) > 0$ for every $u \in (u_0,u_1)$.

Now let $\mathbb{R}_B$ be the Bohr compactification of the real line ([7], [8], [9]). We shall identify the real Bohr almost–periodic function s of Problem (1) with its uniformly continuous extension on $\mathbb{R}_B$, namely $s \in \mathscr{C}(\mathbb{R}_B, \mathbb{R})$. We shall also write

$$\mu_B(s) = \lim_{\ell \to \infty} \ell^{-1} \int_0^{\ell} d\xi s(\xi) \tag{2}$$

for the time average of s. We shall moreover assume that the following two hypotheses hold:

(S_1) We have $\mu_B(s) \neq 0$ and $t \longrightarrow \int_0^t d\xi \hat{s}(\xi) = 0(1)$ as $|t| \longrightarrow \infty$, where $\hat{s} = s - \mu_B(s)$.

(S_2) The restriction of s to $\mathbb{R}^+$ is Hölder continuous.

Finally, we proceed to give the definition of classical solution which we shall use throughout this article. Let $[N/2]$ be the integer part of $N/2$; in the remaining part of this paper we shall assume that Ω has a $\mathscr{C}^{5+[N/2]}$–boundary in the sense of [1], in such a way that Ω lies on only one side of its boundary, and that it satisfies the interior ball condition for every $x \in \partial\Omega$ [5]. We denote by $\mathscr{C}^{2,1}(\Omega \times \mathbb{R}^+, \mathbb{R})$ the set consisting of all functions $z \in \mathscr{C}(\Omega \times \mathbb{R}^+, \mathbb{R})$ such that $(x,t) \longrightarrow \partial_t^\gamma D^\alpha z(x,t) \in \mathscr{C}(\Omega \times \mathbb{R}^+, \mathbb{R})$ for all $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, $\gamma \in \mathbb{N}$, satisfying $\sum_{j=1}^{N} \alpha_j + 2\gamma \leq 2$. In a similar way we define $\mathscr{C}^{1,0}(\bar{\Omega} \times \mathbb{R}^+, \mathbb{R})$. We then have the following

<u>Definition 1</u>. A function $u \in \mathscr{C}^{2,1}(\Omega \times \mathbb{R}^+, \mathbb{R}) \cap \mathscr{C}(\bar{\Omega} \times \mathbb{R}_0^+, \mathbb{R}) \cap \mathscr{C}^{1,0}(\bar{\Omega} \times \mathbb{R}^+, \mathbb{R})$ is said to be a classical solution to Problem (1) if the following conditions hold:

(C_1) The partial derivative $(x,t) \longrightarrow u_t(x,t)$ exists for every $t \in \mathbb{R}^+$ uniformly in $x \in \bar{\Omega}$.

(C_2) $x \longrightarrow u(x,t) \in \mathscr{C}^{(2)}(\overline{\Omega},\mathbb{R})$ for every $t \in \mathbb{R}^+$.

(C_3) $(x,t) \longrightarrow u_t(x,t) \in \mathscr{C}(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$ and $t \longrightarrow u_t(x,t) \in \mathscr{C}(\mathbb{R}^+,\mathbb{R})$ uniformly in $x \in \overline{\Omega}$.

(C_4) u satisfies relations (1) identically.

The main result of this article is then the following

Theorem 1. Consider Problem (1) where g satisfies hypothesis (G); assume moreover that s satisfies hypotheses (S_1) and (S_2). Set $r_{u_0} = g'(u_0)\mu_B(s)$, $r_{u_1} = g'(u_1)\mu_B(s)$ and let u be any classical solution to Problem (1). Then there exist $\epsilon_0 \in (0,\infty)$, $t_{\epsilon_0} \in (0,\infty)$, and a positive constant c_0 such that the following statements hold:

(1) If $\mu_B(s) < 0$, then the exponential decay estimates

$$\sup_{x\in\overline{\Omega}} |u(x,t)-u_0| \leq c_0\epsilon_0 \exp[r_{u_0}(t-t_{\epsilon_0})] \tag{3}$$

$$\sup_{x\in\overline{\Omega}} |\nabla u(x,t)| \leq c_0\epsilon_0 \exp[r_{u_0}(t-t_{\epsilon_0})] \tag{4}$$

hold for every $t \in [t_{\epsilon_0},\infty)$.

(2) If $\mu_B(s) > 0$, then a completely similar statement holds provided that we replace u_0 by u_1 everywhere in relations (3) and (4).

Remarks. (1) The above theorem asserts that if $\mu_B(s) < 0$, then u_0 is a global exponential attractor for Problem (1): every classical solution to Problem (1) stabilizes around u_0 exponentially rapidly, with a rate of decay depending solely on $g'(u_0)$ and $\mu_B(s)$, irrespective of the spectral properties of Laplace's operator. We can thereby also conclude that the global stabilization phenomenon described above is primarily governed by the reaction process in equation (1). Of course, a similar remark holds if $\mu_B(s) > 0$.

(2) Our result immediately implies that Problem (1) has no time almost–periodic classical solution. For if $t \longrightarrow u(x,t)$ were such a solution, relation (3) would immediately imply that $u(x,t) = u_0$ for every $(x,t) \in \bar{\Omega} \times \mathbb{R}^+$, a contradiction since u_0 is not a solution to Problem (1).

(3) If $\mu_B(s) = 0$ and if $t \longrightarrow \int_0^t d\xi s(\xi) = 0(1)$ as $|t| \longrightarrow \infty$, then it is possible to show that both u_0 and u_1 become unstable, but that the classical solutions to Problem (1) still stabilize around non constant almost–periodic attractors as $t \longrightarrow \infty$. However, the proof of these facts requires the elaboration of a suitable stable – and center – manifold theory, and is much more complex than that of the above theorem. We refer the reader to [11] and [12] for details.

(4) The simplest equation of the form (1) is the so–called Fisher's equation of population genetics, namely

$$\left\{\begin{array}{lll} u_t(x,t) = \Delta u(x,t) + s(t)u(x,t)(1-u(x,t)), & (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (0,1) & \\ \frac{\partial u}{\partial \underline{n}}(x,t) = 0 & , (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \quad (5)$$

Under the conditions of the above theorem, every classical solution to Problem (5) stabilizes exponentially rapidly toward $u_0 = 0$ or $u_1 = 1$, depending on whether $\mu_B(s) < 0$ or $\mu_B(s) > 0$. In the context of population genetics, equation (5) models for instance the fraction u of one of two alleles in the population of a migrating diploid species located in Ω, when the so–called selection function s takes almost–periodic seasonal variations into account. In this case, our result means that only one of the two alleles will eventually survive in the population.

(5) Our work concerning Problem (1) was primarily inspired by the recent results of [6]. In fact, the authors of [6] obtained results concerning the case where $t \longrightarrow s(x,t)$ is periodic and may also depend on $x \in \overline{\Omega}$. However, their method of proof seems to be strictly limited to the periodic case and does not provide the actual rates of decay.

The proof of the theorem rests upon the combination of a local geometric argument with a global one. We begin with the formulation of the local result. For $p \in (N,\infty)$, let $L^p(\mathbb{C}) = L^p(\Omega,\mathbb{C})$ be the usual Lebesgue space with respect to Lebesgue's measure on Ω; define

$$H^{2,p}_{\mathcal{N}}(\mathbb{R}) = \left\{ z \in H^{2,p}(\mathbb{R}) : \frac{\partial z}{\partial \underset{\sim}{n}}(x) = 0, \ x \in \partial\Omega \right\} \tag{6}$$

where $H^{2,p}(\mathbb{R}) = H^{2,p}(\Omega,\mathbb{R})$ is the usual real Sobolev space of functions on Ω. We may then assume that $H^{2,p}_{\mathcal{N}}(\mathbb{R})$ becomes a commutative Banach algebra with respect to the usual operations and the norm

$$\|z\|_{\lambda_0,2,p} = \|(\lambda_0 - \Delta_{p,\mathcal{N}})z\|_p \tag{7}$$

([1], [11]). In relation (7), $\Delta_{p,\mathcal{N}}$ is the $L^p(\mathbb{C})$–realization of Laplace's operator whose domain is given by the complexification of (6), while $\|\cdot\|_p$ denotes the usual L^p–norm and $\lambda_0 \in \rho(\Delta_{p,\mathcal{N}})$, the resolvent set of $\Delta_{p,\mathcal{N}}$. Endowed with the norm (7), the above Banach algebra will henceforth be denoted by $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. Our local result is then the following proposition, which we also believe to be new.

<u>Proposition 1</u>. Let s and g satisfy all of the hypotheses of Theorem 1, and fix $p \in (N,\infty)$. Then there exist $\epsilon_1 \in (0,\infty)$, $k_1 \in [1,\infty)$ and, for each $\epsilon \in (0,\epsilon_1]$, an open spherical neighborhood $\mathcal{N}_{(2k_1)^{-1}\epsilon}$ of radius $(2k_1)^{-1}\epsilon$ centered at the origin of $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, such that the following statements hold for every $t_0 \in \mathbb{R}$:

(1) If $\mu_B(s) < 0$, and if we define s_{t_0} by $s_{t_0}(t) = s(t+t_0)$ for every $t \in \mathbb{R}$, then for each $\eta \in \mathcal{N}^+_{(2k_1)^{-1}\epsilon} = \left\{\eta \in \mathcal{N}_{(2k_1)^{-1}\epsilon} : \eta > 0 \text{ on } \bar{\Omega}\right\}$, there exists a classical solution $(x,t) \longrightarrow \tilde{u}(x,t,\eta)$ to the problem

$$\left\{\begin{array}{lll} u_t(x,t) = \Delta u(x,t) + s_{t_0}(t)g(u(x,t)), & (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (u_0,u_1) & \\ \frac{\partial u}{\partial \mathcal{R}}(x,t) = 0 & , (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \tag{8}$$

which satisfies $\tilde{u}(x,0,\eta) = \eta(x) + u_0$ for every $x \in \bar{\Omega}$. Moreover, the exponential decay estimates

$$\sup_{x\in\bar{\Omega}} |\tilde{u}(x,t,\eta)-u_0| \leq c_1\epsilon \exp[r_{u_0}t] \tag{9}$$

$$\sup_{x\in\bar{\Omega}} |\nabla\tilde{u}(x,t,\eta)| \leq c_1\epsilon \exp[r_{u_0}t] \tag{10}$$

hold for every $t \in \mathbb{R}_0^+$ and for some positive constant c_1.

(2) If $\mu_B(s) > 0$, a completely similar result holds, provided that we replace $\mathscr{N}^+_{(2k_1)^{-1}\epsilon}$ by $\mathscr{N}^-_{(2k_1)^{-1}\epsilon} = \left\{\eta \in \mathscr{N}_{(2k_1)^{-1}\epsilon} : \eta < 0 \text{ on } \bar{\Omega}\right\}$ and u_0 by u_1 everywhere in Statement (1).

Remark. Proposition 1 is in fact a result which amounts to constructing an infinite–dimensional stable manifold for Problem (8). For the sake of clarity, we postpone its proof until the end of this paper.

For $t_0 = 0$, Proposition 1 provides particular classical solutions of small norms to Problem (1) which satisfy exponential decay estimates similar to relations (3) and (4). In order to extend the validity of such estimates to all classical solutions of Problem (1), and thereby obtain a proof of Theorem 1, we do need the arbitrariness of t_0 in Proposition 1 (compare with the proof of Theorem 1 below), as well as the following global stabilization result.

Proposition 2. Let s and g satisfy all of the hypotheses of Theorem 1, and let $p \in (N,\infty)$. Let u be any classical solution to Problem (1) and define $t \longrightarrow u(t)$ by $u(t)(x) = u(x,t)$ for every $(x,t) \in \bar{\Omega}\times\mathbb{R}^+$. Then the following statements hold:

(1) If $\mu_B(s) < 0$, we have

$$\lim_{t\to\infty} \|u(t) - u_0\|_{\lambda_0,2,p} = 0 \tag{11}$$

(2) If $\mu_B(s) > 0$, we have

$$\lim_{t\to\infty} \|u(t) - u_1\|_{\lambda_0,2,p} = 0 \tag{12}$$

Proof. The fact that $u(t) \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ follows from condition (C_2) of Definition 1. The rest is a nearly verbatim adaptation of the proof of Theorem 3.1 of [10]. ∎

Besides Proposition 1, the other main contribution of this article is to show that we can prove Theorem 1 just by combining Propositions 1 and 2 in a suitable way. The main idea of the argument is very simple: since we already know that $u(t) \longrightarrow u_{0,1}$ in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ as $t \longrightarrow \infty$ by Proposition 2, and since the sets $\mathcal{N}^{\pm}_{(2k_1)^{-1}\epsilon}$ of Proposition 1 are in fact smooth stable manifolds in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, we just have to wait long enough until $u(t) - u_{0,1}$ hits the stable manifold $\mathcal{N}^{\pm}_{(2k_1)^{-1}\epsilon}$ at some time $t = t_\epsilon$. Using the parabolic maximum principle, we then proceed to identify $u(t)$ for $t \geq t_\epsilon$ with a small norm–solution of Proposition 2. The precise argument is given in the following

Proof of Theorem 1. Let ϵ_1 and k_1 be as in Proposition 1, and fix $\epsilon_0 \in (0,\epsilon_1)$. If $\mu_B(s) < 0$, then $u(t) \longrightarrow u_0$ in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ as $t \longrightarrow \infty$. Therefore, there exists $t_{\epsilon_0} \in (0,\infty)$ such that $\|u(t) - u_0\|_{\lambda_0,2,p} < (2k_1)^{-1}\epsilon_0$ for every $t \in [t_{\epsilon_0},\infty)$. With this in mind, define $\eta_0 = u(t_{\epsilon_0}) - u_0$; since u is a classical solution to Problem (1), we may conclude that $\eta_0 \in \mathcal{N}^{+}_{(2k_1)^{-1}\epsilon_0}$. It then follows from Proposition 1 with $t_0 = t_{\epsilon_0}$ that the boundary value problem

$$\left\{\begin{array}{lll} u_t(x,t) = \Delta u(x,t) + s_{t_{\epsilon_0}}(t)g(u(x,t)), & (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (u_0,u_1) & \\ \frac{\partial u}{\partial \underline{n}}(x,t) = 0 & , (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \tag{13}$$

possesses a classical solution $(x,t) \longrightarrow \tilde{u}(x,t,\eta_0)$ which satisfies

$$\tilde{u}(x,0,\eta_0) = \eta_0(x) + u_0 = u(x,t_{\epsilon_0}) \tag{14}$$

for every $x \in \bar{\Omega}$. Moreover, the exponential decay estimates

$$\sup_{x\in\bar{\Omega}} |\tilde{u}(x,t-t_{\epsilon_0},\eta_0) - u_0| \le c_1\epsilon_0 \exp[r_{u_0}(t-t_{\epsilon_0})] \tag{15}$$

$$\sup_{x\in\bar{\Omega}} |\nabla\tilde{u}(x,t-t_{\epsilon_0},\eta_0)| \le c_1\epsilon_0 \exp[r_{u_0}(t-t_{\epsilon_0})] \tag{16}$$

hold for every $t \in [t_{\epsilon_0},\infty)$. We now define $(x,t) \longrightarrow w(x,t) = \tilde{u}(x,t-t_{\epsilon_0},\eta_0)$ for every $(x,t) \in \bar{\Omega} \times [t_{\epsilon_0},\infty)$. It then follows from relations (13), (14) and the definition of $s_{t_{\epsilon_0}}$ that w satisfies the initial boundary value problem

$$\left\{\begin{array}{lll} w_t(x,t) = \Delta w(x,t)+s(t)g(w(x,t)), & (x,t) \in \Omega\times(t_{\epsilon_0},\infty) \\ \mathrm{Ran}(w) \subseteq (u_0,u_1) & \\ w(x,t_{\epsilon_0}) = u(x,t_{\epsilon_0}) & , x \in \bar{\Omega} \\ \frac{\partial w}{\partial \underline{n}}(x,t) = 0 & , (x,t) \in \partial\Omega\times(t_{\epsilon_0},\infty) \end{array}\right\} \tag{17}$$

along with the exponential decay estimates (15) and (16). In order to complete the proof of Statement (1) of Theorem 1, it thus remains to prove that $w(x,t) = u(x,t)$ for every $(x,t) \in \bar{\Omega} \times [t_{\epsilon_0},\infty)$. We first notice that $\mathrm{Ran}(u) \subseteq (u_0,u_1)$, $\mathrm{Ran}(w) \subseteq (u_0,u_1)$; moreover s is uniformly bounded on $\mathbb{R}$ and g is smooth. The conclusion then follows from the third relation in (17) and the parabolic maximum principle. The proof of Statement (2) is of course similar. ■

We conclude this article in providing a

Proof of Proposition 1. If $t_0 = 0$, Proposition 1 is nothing but Corollary 3.1 to Theorem 3.1 of [11], which provides the existence of a local stable manifold around the trivial solution for initial value problems of the form

$$\left\{\begin{array}{ll} y'(t) = (\Delta_{\mathcal{N}}+s(t)g'(u_{0,1}))y(t)+s(t)\hat{g}_{u_{0,1}}(y(t)), & t \in \mathbb{R}^+ \\ y(0) = y_0 & \end{array}\right\} \tag{18}$$

in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. In expression (18), we have defined $\hat{g}_{u_{0,1}} : H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}) \longrightarrow H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ by

$$\hat{g}_{u_{0,1}}(z) = g\circ(u_{0,1}+z) - g'(u_{0,1})z \tag{19}$$

and, using the Banach algebra properties of $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, we can easily verify that the hypothesis $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$ implies that $\hat{g}_{u_{0,1}} \in \mathscr{C}^{(2)}(H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}), H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}))$. Moreover, $\Delta_{\mathcal{N}}$ stands for the $H^{2,p}_{\lambda_0,\mathcal{N}}$–realization of Laplace's operator on the domain

$$\mathrm{Dom}(\Delta_{\mathscr{N}}) = \left\{ z \in H^{4,p}_{\mathscr{N}}(\mathbb{C}) : \Delta_{p,\mathscr{N}} z \in H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{C}) \right\} \tag{20}$$

where $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{C})$ and $H^{4,p}_{\mathscr{N}}(\mathbb{C})$ are the complexifications of $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$ and $H^{4,p}_{\mathscr{N}}(\mathbb{R})$, respectively. In order to prove Proposition 1, we therefore fix $t_0 \in \mathbb{R}$ and consider initial value problems of the form

$$\left\{ \begin{array}{l} y'(t) = (\Delta_{\mathscr{N}} + s_{t_0}(t) g'(u_{0,1})) y(t) + s_{t_0}(t) \hat{g}_{u_{0,1}}(y(t)), \ \ t \in \mathbb{R}^+ \\ y(0) = y_0 \end{array} \right\} \tag{21}$$

on $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$. Since $s \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$, and since s_{t_0} is just the translate of s, we have $s_{t_0} \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$ as well and $\mu_B(s_{t_0}) = \mu_B(s)$. In addition, s_{t_0} also satisfies hypotheses (S_1) and (S_2) since s does. The proof of Proposition 1 then becomes identical to that of Theorem 3.1 of [11], if we can show that the basic estimates of exponential dichotomy <u>remain valid uniformly in</u> t_0. In order to see this let $\{W_{\Delta_{\mathscr{N}}}(t)\}_{t \in \mathbb{R}_0^+}$ be the restriction to $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$ of the diffusion semigroup generated by $\Delta_{\mathscr{N}}$ on $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{C})$ and, for every $t_0 \in \mathbb{R}$, define the family of evolution operators $\{U_{u_{0,1},t_0}(t,r)\}_{t \geq r \geq 0}$ by

$$U_{u_{0,1,t_0}}(t,r) = \exp\left[g'(u_{0,1}) \int_r^t d\xi\, s_{t_0}(\xi) \right] W_{\Delta_{\mathscr{N}}}(t-r) \tag{22}$$

It is easily verified that relation (22) with $r = 0$ provides the evolution operators which solve the linear part of equation (21). Moreover, since $\mu_B(s_{t_0}) = \mu_B(s)$, we may write

$$U_{u_{0,1},t_0}(t,r) = \exp\left[r_{u_{0,1}}(t-r)\right]\exp\left[g'(u_{0,1})\int_r^t d\xi\, \widehat{s}_{t_0}(\xi)\right]W_{\Delta_{\mathcal{N}}}(t-r) \tag{23}$$

where $\widehat{s}_{t_0} = s_{t_0} - \mu_B(s)$. But since s satisfies hypothesis (S_1), we have

$$\int_r^t d\xi\, \widehat{s}_{t_0}(\xi) = \int_r^t d\xi(\hat{s})_{t_0}(\xi) = 0(1) \tag{24}$$

uniformly in r, t and t_0. Since $\{W_{\Delta_{\mathcal{N}}}(t)\}_{t\in\mathbb{R}_0^+}$ is a contraction semigroup on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, we conclude that there exists a positive constant c_2 , independent of t_0 , such that the exponential decay estimates

$$\left|\left|\left|U_{u_{0,1},t_0}(t,r)\right|\right|\right|_{\infty,\lambda_0,2,p} \leq c_2 \exp\left[r_{u_{0,1}}(t-r)\right] \tag{25}$$

hold for the corresponding operator norms. But estimates (25) are identical to the estimates (3.3) and (3.4) of [11], so that the remaining part of the proof of Proposition 1 is identical to that of Corollary 3.1 of [11]. In particular, the constants ϵ_1 and k_1 may be chosen uniformly in $t_0 \in \mathbb{R}$, an absolutely essential fact in the proof of Theorem 1. ■

Remarks. (1) It is not clear whether our main result remains true if g is not at least five times continuously differentiable, for then $\hat{g}_{u_{0,1}}$ is not any longer twice continuously Fréchet differentiable on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. However, Proposition 2 still holds if $g \in \mathscr{C}^{(1)}(\mathbb{R},\mathbb{R})$ [10].

(2) In the case where $\mu_B(s) = 0$ and $t \longrightarrow \int_0^t d\xi s(\xi) = 0(1)$ as $|t| \longrightarrow \infty$, the stabilization of the classical solutions of Problem (1) around the appropriate almost–periodic attractors is in general not any longer exponentially rapid, because of a typical center manifold behavior due to the fact that $\lambda = 0$ is an eigenvalue of $\Delta_{\mathscr{N}}$. We refer the reader to [12] for details.

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