REDUCIBILITY OF SIGNED CYCLIC SUMS OF MORDELL-TORNHEIM ZETA AND L-VALUES

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ABSTRACT. Matsumoto et al. define the Mordell-Tornheim L-functions of depth k by

$$L_{\rm MT}(s_1,\dots,s_{k+1};\chi_1,\dots,\chi_{k+1}) := \sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\frac{\chi_1(m_1)\cdots\chi_k(m_k)\chi_{k+1}(m_1+\cdots+m_k)}{m_1^{s_1}\cdots m_k^{s_k}(m_1+\cdots+m_k)^{s_{k+1}}}$$

for complex variables $s_1,\dots,s_{k+1}$ and primitive Dirichlet characters $\chi_1,\dots,\chi_{k+1}$. In this paper, we shall show that certain signed cyclic sums of Mordell-Tornheim L-values are rational linear combinations of products of multiple L-values of lower depths (i.e., reducible). This simultaneously generalizes some results of Subbarao and Sitaramachandrarao, and Matsumoto et al. As a direct corollary, we can prove that for any positive integer n and integer $k\ge 2$, the Mordell-Tornheim sums $\zeta_{\rm MT}(\{n\}_k,n)$ is reducible where $\{n\}_k$ denotes the string $(n,\dots,n)$ with n repeating k times.

1. Introduction

For $\mathbf{s}=(s_1,\dots,s_{k+1})\in\mathbb{C}^{k+1}$ and $\boldsymbol{\chi}=(\chi_1,\dots,\chi_{k+1})$ where χ_j's are primitive Dirichlet characters, Matsumoto et al. [16] define the *Mordell-Tornheim L-functions* by

$$L_{\rm MT}(\mathbf{s};\boldsymbol{\chi}) := \sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\frac{\chi_1(m_1)\dots\chi_k(m_k)\chi_{k+1}(m_1+\cdots+m_k)}{m_1^{s_1}\cdots m_k^{s_k}(m_1+\cdots+m_k)^{s_{k+1}}} \tag{1}$$

They show that when $k=3$ this function has analytic continuation to $\mathbb{C}^4$. As usual we call $|\mathbf{s}|:=s_1+\cdots+s_{k+1}$ the *weight* and k the *depth*. When all the characters are principle these are nothing but the traditional Mordell-Tornheim zeta functions

$$\zeta_{\rm MT}(s_1,\dots,s_{k+1}) := \sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\frac{1}{m_1^{s_1}\cdots m_k^{s_k}(m_1+\cdots+m_k)^{s_{k+1}}}. \tag{2}$$

Note that in the literature this function is also denoted by $\zeta_{{\rm MT},k}(s_1,\dots,s_k;s_{k+1})$. One can compare (1) to the classical multiple L-functions (here $\mathbf{s}=(s_1,\dots,s_k)$)

$$L(\mathbf{s};\chi_1,\dots,\chi_k) := \sum_{m_1>\cdots>m_k\ge 1}\frac{\chi_1(m_1)\dots\chi_k(m_k)}{m_1^{s_1}\cdots m_k^{s_k}} \tag{3}$$

and compare (2) to the classical multiple zeta functions

$$\zeta(\mathbf{s}) := \sum_{m_1>\cdots>m_k\ge 1}\frac{1}{m_1^{s_1}\cdots m_k^{s_k}} \tag{4}$$

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where $|\mathbf{s}|$ is called the *weight* and $\ell(\mathbf{s}) := k$ the *depth*. It is a little unfortunate, due to historical reasons, that the ordering of the indices in (3) and (4) is opposite to the one which naturally corresponds to that of (1) and (2). But this ordering of multiple zeta and L-functions has its advantages in many computations involving integral representations so we choose it to be consistent with our other recent works.

In the past several decades, relations among special values of the above functions at integers have gradually gained a lot interest among both mathematicians and physicists. In this paper, we slightly enlarge our scope our study. We call the number defined by (1) a *type 1 Mordell-Tornheim L-value* (1-MTLV for short) if at most one of the arguments is not a positive integer. We call it a *special 1-MTLV* if only the last variable s_{k+1} is allowed to be a complex number. We can define 1-MTZVs and special 1-MTZVs similarly by (2). Parallel to these, we can use (3) and (4) to define 1-MLVs and 1-MZVs (resp. their special versions) where only one variable (resp. only the leading variable s_1) is allowed to be a complex number. Note that MZVs are 1-MZVs and similarly for others. The following diagrams provide the relations between these numbers:

$$\begin{array}{ccc} \{\text{1-MTZVs}\} & \subset & \{\text{1-MTLVs}\} \\ \cup & & \cup \\ \{\text{MTZVs}\} & \subset & \{\text{MTLVs}\} \end{array} \qquad \begin{array}{ccc} \{\text{1-MZVs}\} & \subset & \{\text{1-MLVs}\} \\ \cup & & \cup \\ \{\text{MZVs}\} & \subset & \{\text{MLVs}\} \end{array} \tag{5}$$

Ordinary MTZVs were first investigated by Tornheim [24] in the case $k = 2$, and later by Mordell [17] and Hoffman [10] with $s_1 = \cdots = s_k = 1$. On the other hand, after the seminal work of Zagier [28] much more results concerning MZVs have been found (for a rather complete reference list please see Hoffman's webpage [11]). Our primary interest in this paper is to study the properties of the type 1 versions of these special values, especially their reducibility.

Definition 1.1. A linear combination of MTZVs (resp. MTLVs, MZVs, MLVs) is called *reducible* if it can be expressed as a $\mathbb{Q}$-linear combination of products of MTZVs (resp. MTLVs, MZV, MLVs) of lower depths. It is called *strongly reducible* (not defined for MZVs and MLVs) if we can further replace MTZVs (resp. MTLVs) by MZVs (resp. MVLs). One can similarly define the reducibility for corresponding type 1 values.

It is a well-known result that if the weight and length of a MZV have different parities then the MZV is reducible. This was proved by Zagier ([12, Cor. 8]), and later by Tsumura [25] independently. A similar result for MTZVs has been obtained by Bradley and the second author:

Theorem 1.2. *Every MTZV $\zeta_{\rm MT}(\mathbf{s})$ is a $\mathbb{Q}$-linearly combination of MVZs of the same weight and length (*[33, Theorem 5]*). Further, if $\ell(\mathbf{s}) = k + 1 \geq 3$ and $k + |\mathbf{s}|$ is odd then the MTZV $\zeta_{\rm MT}(\mathbf{s})$ is reducible and therefore strongly reducible (*[33, Theorem 2]*).*

In fact, MTZVs and MZVs are closely related so it is not surprising that similar results often hold for both. To illustrate this line of thought, in §3 we shall prove the reduction of special 1-MTLVs to special 1-MLVs, generalizing Theorem 1.2 of Bradley and the second authord. We also present a result relating colored 1-MTZVs to colored 1-MZVs (see Definition 2.1 and [4]).

Like in the classical case, when the weight and depth have the same parity the situation is more complicated. In 1985, Subbarao and Sitaramachandrarao [23] showed that $\zeta_{\rm MT}(2a, 2b, 2c) + \zeta_{\rm MT}(2b, 2c, 2a) + \zeta_{\rm MT}(2c, 2a, 2b)$ is reducible for positive integers a, b, c, which includes the special case $\zeta_{\rm MT}(2c,2c;2c)$ already known to Tornheim. In 2007, Tsumura [26] evaluated $\zeta_{\rm MT}(a,b,s)+(-1)^b\zeta_{\rm MT}(b,s,a)+(-1)^a\zeta_{\rm MT}(s,a,b)$ for positive integers a,b and complex number s. Nakamura [18] subsequently gave a simpler evaluation of the same quantity. More recently, a triple 1-MTLV analog is established by Matsumoto et al. [16, Theorem 3.5] (after slight reformation): for any positive integers a,b,c, and any primitive Dirichlet character χ

$$(-1)^{a+b+c}L_{\rm MT}(a,b,c,s;\mathbf{1},\mathbf{1},\mathbf{1},\chi)-(-1)^aL_{\rm MT}(b,c,s,a;\mathbf{1},\mathbf{1},\chi,\mathbf{1})\\ -(-1)^bL_{\rm MT}(c,s,a,b;\mathbf{1},\chi,\mathbf{1},\mathbf{1})-(-1)^cL_{\rm MT}(s,a,b,c;\chi,\mathbf{1},\mathbf{1},\mathbf{1}) \quad (6)$$

is reducible for all $s\in\mathbb{C}$ except at singular points, where $\mathbf{1}$ is the principal character.

In this paper, we shall generalize (6) to arbitrary depth k. For a letter v we denote by $\{v\}_n$ the string with letter v repeated n times. For a string $\mathbf{w}=(w_1,\dots,w_n)$, the operator $R_j(v,\mathbf{w})$ means to substitute v for w_j if $1\le j\le n$ and $R_j(v,\mathbf{w})=\mathbf{w}$ if $j>n$. Using some important properties of Bernoulli polynomials to be proved in §4 we shall show in §5 the following reducibility result.

Theorem 1.3. *Let k be a positive integer ≥ 2 and $\mathbf{s}=(s_1,\dots,s_k)\in\mathbb{N}^k$. Then*

$$(-1)^{k+|\mathbf{s}|}L_{\rm MT}(\mathbf{s},z;\{\mathbf{1}\}_k,\chi)+\sum_{j=1}^{k}(-1)^{s_j}L_{\rm MT}(R_j(z,\mathbf{s}),s_j;R_j(\chi,\{\mathbf{1}\}_{k+1})) \quad (7)$$

is reducible for all $z\in\mathbb{C}$ except at singular points. If z is also a positive integer then

$$\sum_{j=1}^{k+1}(-1)^{s_j}\zeta_{\rm MT}(R_j(z,\mathbf{s}),s_j) \quad (8)$$

is strongly *reducible, where $s_{k+1}=z$.*

We will in fact give a precise reduction formula in Theorem 5.3 which immediately implies Theorem 1.3. Unfortunately it is too complicated to state here. Note that (7) may not be strongly reducible. We call expressions like (7) or (8) *signed cyclic sums* of 1-MTLVs or MTZVs. The implication (7) $\Rightarrow$ (8) readily follows from Theorem 1.2. Theorem 1.3 has the following nice implication.

Corollary 1.4. *If $n\in\mathbb{N}$ and $k\ge 2$ then the MTZV $\zeta_{\rm MT}(\{n\}_{k+1})$ is reducible.*

Note the case $n=1$ of the corollary was already treated by Mordell [17, (5)]:

$$\zeta_{\rm MT}(\{1\}_{k+1})=k!\zeta(k+1).$$

It would be interesting to generalize this identity to arbitrary n.

The main idea in the proof of Theorem 1.3 comes from [16]. Both authors would like to thank Prof. Matsumoto and Tsumura for sending them many pre- and off-prints. The first author also wants to thank Max-Planck-Institut für Mathematik for providing financial support during his sabbatical leave when this work was done. The second author is supported by the National Natural Science Foundation of China, Project 10871169.

2. Analytic Continuation of Mordell-Tornheim Colored Zeta and L-functions

The main idea of this section is from [14, 16] and the result is perhaps known to the experts already. There are three reasons we want to include this section: first, the proof is relatively short so we can present it for completeness; second, this is the most natural place to introduce the term *colored Mordell-Tornheim functions* to be used later in the paper; and last, our main results Theorem 1.3 and Theorem 5.3 rely on the analytic continuation of Mordell-Tornheim colored zeta and L-functions.

Clearly when $\Re(s_j) > 1$ all the functions in (1) to (4) converge. Denote the real part of s_j by $\Re(s_j) = \sigma_j$ for $1 \le j \le k+1$ and write $s = s_{k+1}$ and $\sigma = \sigma_{k+1}$. Since (2) (resp. (1)) remains unchanged if the arguments $s_1, \dots, s_k$ (resp. $(s_1, \chi_1), \dots, (s_k, \chi_k)$) are permuted, we may as well suppose that $s_1, \dots, s_k$ are arranged in the order of increasing real parts, i.e., $\sigma_1 \le \dots \le \sigma_k$. It follows from [33, Theorem 4] that the series (2) converges absolutely if

$$\sigma + \sum_{j=1}^{r} \sigma_j > r, \quad \forall r = 1, 2, \dots, k. \tag{9}$$

However, just like the Riemann zeta functions, all the functions in (1) to (4) should have analytic continuations to the whole complex space with clearly described singularities lying inside at most countably many hyperplanes. For multiple zeta and L-functions this has been worked out in [3] and [29] (independently [2]), respectively. Further, Matsumoto et al. have studied the Mordell-Tornheim zeta functions completely (see [15, Theorem 6.1]). On the other hand, in [16] Matsumoto et al. only treated depth three Mordell-Tornheim L-functions although it is possible to combine their ideas in [16] and [15, Theorem 6.1] to prove the general cases which we shall carry out in Theorem 2.3. To prepare for this we first introduce the "colored" version of the Mordell-Tornheim functions. In the depth 1 case, this is a special case of the more general Lerch series. In depth 2, Nakamura called these functions "double Lerch serires" (see [19]).

Definition 2.1. For any $x \in \mathbb{R}$ set $e(x) = e^{2\pi i x}$. For any given set of parameters $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_{k+1}) \in \mathbb{R}^{k+1}$, we define the function in complex variables $\mathbf{s} \in \mathbb{C}^{k+1}$

$$\zeta_{\rm MT}(\mathbf{s}; \boldsymbol{\alpha}) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \frac{e(\alpha_1 m_1 + \cdots + \alpha_k m_k + \alpha_{k+1}(m_1 + \cdots + m_k))}{m_1^{s_1} \cdots m_k^{s_k} (m_1 + \cdots + m_k)^{s_{k+1}}}, \tag{10}$$

where $\Re(s_j) \ge 1$ for all $j \le k+1$. This is called a *colored Mordell-Tornheim function* with variables s_j dressed with $e(\alpha_j)$.

This terminology is influenced by the name "colored MZVs" used in [4] in which similar generalizations of MZVs are considered. Of course colored MZVs can also be regarded as special values of multiple polylogarithms on the unit circle. To study L-functions we only need the "colors" to be roots of unity (i.e. $\alpha_j \in \mathbb{Q}$) in which case the colored MZVs have been investigated from different points of view in [9, 21, 30, 32].

For any $\boldsymbol{\beta} = (\beta_1, \dots, \beta_r) \in \mathbb{R}^r$ and a set A we let $A(\boldsymbol{\beta}) = A$ if $\boldsymbol{\beta} \in \mathbb{Z}^r$ and $A(\boldsymbol{\beta}) = \emptyset$ otherwise. Similar to the proof of [16, Prop. 2.1] we first have:

Proposition 2.2. *Let $\beta_j = \alpha_j + \alpha_{k+1}$ for all $j = 1, \dots, k$. Then the colored Mordell-Tornheim functions defined by* (10) *can be analytically continued to $\mathbb{C}^{k+1}$ with the singularities lying on* *the following hyperplanes:*

$$\{|\mathbf{s}|=k\}\cup\bigcup_{l=1}^{\infty}\bigcup_{r=1}^{k-1}\bigcup_{1\le j_1<\cdots<j_r\le k}\left\{\sum_{i=1}^{r}(s_{j_i}-1)+s_{k+1}=-l\right\}(\beta_{j_1},\dots,\beta_{j_r}). \tag{11}$$

When all $\beta_j=0$ *we recover the analytic continuation of Mordell-Tornheim functions defined by* (2) *(cf.* [15, Theorem 6.1]*).*

Proof. We proceed by induction on the depth. The case of depth two is given by [13, Theorem 1]. Assume in depth $k-1$ ($k\ge 3$) we already have the analytic continuations of $\zeta_{\rm MT}(\mathbf{s};\boldsymbol{\alpha})$ for every fixed $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_k)$ with the singularities given by the proposition. Let's consider the depth k situation in (10). First we assume $\Re(s_j)\ge 1$ for all $j=1,\dots,k+1$. By substitution $\alpha_j\to\alpha_j-\alpha_{k+1}$ for all $j=1,\dots,k$ we may also assume without loss of generality that $\alpha_{k+1}=0$. Set $n_r=\sum_{j=1}^r m_j$ for $r=1,\dots,k+1$. By Mellin-Barnes formula for all $b>0$ we have

$$\frac{1}{(1+b)^s}=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}b^z\,dz \tag{12}$$

where $s\in\mathbb{C}$, $\Re(s)>-c>0$ and (c) is the vertical line $\Re(z)=c$ pointing upward. Applying this with $b=m_{k+1}/n_k$ we get

$$\frac{1}{n_{k+1}^{s_{k+1}}}=\frac{1}{n_k^{s_{k+1}}}\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_{k+1}+z)\Gamma(-z)}{\Gamma(s_{k+1})}\left(\frac{m_{k+1}}{n_k}\right)^z\,dz$$

where $\Re(s_{k+1})>-c>0$. Setting $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k,0)$ we see that

$$\zeta_{\rm MT}(\mathbf{s};\boldsymbol{\alpha})\;=\;\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_{k+1}+z)\Gamma(-z)}{\Gamma(s_{k+1})}\zeta_{\rm MT}(\mathbf{s}',s_{k+1}+z;\boldsymbol{\alpha}',0)\phi(s_k-z,\alpha_k)\,dz \tag{13}$$

where $\mathbf{s}'=(s_1,\dots,s_{k-1})$, $\boldsymbol{\alpha}'=(\alpha_1,\dots,\alpha_{k-1})$, and $\phi(s,\alpha)=\sum_{j\ge 1}e(j\alpha)/j^s$. Note that $\zeta_{\rm MT}(\mathbf{s}',s_{k+1}+z;\boldsymbol{\alpha}',0)$ is well defined by our assumption $\Re(s_j)\ge 1$, $\Re(s_{k+1}+z)=\Re(s_{k+1})+c>0$ and (9). As $|e(\alpha m)|=1$ we still have the exponential decay of the integrand in (13) by Stirling's formula when $z\to c\pm i\infty$ and therefore the argument for [14, (3.2)] carries through without problem. When we shift the integration from (c) to $(M-\varepsilon)$ for large $M\in\mathbb{N}$ and very small $\varepsilon>0$ we need to consider the residues of the integrand of (13) between these two vertical lines. By the induction assumption, the singularities of $\zeta_{\rm MT}(\mathbf{s}',s_k+z;\boldsymbol{\alpha}',0)$ are given by:

$$\left\{\sum_{j=1}^{k-1}s_j+s_{k+1}+z=k-1\right\}\cup$$

$$\cup\bigcup_{l=1}^{\infty}\bigcup_{r=1}^{k-2}\bigcup_{1\le j_1<\cdots<j_r\le k-1}\left\{\sum_{i=1}^{r}(s_{j_i}-1)+s_{k+1}+z=-l\right\}(\alpha_{j_1},\dots,\alpha_{j_r}).$$

By assumption $\Re(s_j)\ge 1$ $(j\le k)$ and $\Re(s_{k+1}+z)>0$ none of these lies between the two vertical lines so the only relevant poles of the integrand of (13) are $z=j$ for $j=0,1,\dots,M-1$ given by $\Gamma(-z)$ and $\{z=s_k-1\}(\alpha_k)$ given by $\phi(s_k-z,\alpha_k)$. It is well-known that these poles are all simple poles. Thus by contour integration we get:

$$\begin{aligned}\zeta_{\rm MT}(\mathbf{s};\boldsymbol{\alpha}) =& \frac{1}{2\pi i}\int_{(M-\varepsilon)} \frac{\Gamma(s_{k+1}+z)\Gamma(-z)}{\Gamma(s_{k+1})}\zeta_{\rm MT}(\mathbf{s}', s_{k+1}+z;\boldsymbol{\alpha}',0)\phi(s_k-z,\alpha_k)\,dz\\ &-\left[\frac{\Gamma(s_{k+1}+s_k-1)\Gamma(1-s_k)}{\Gamma(s_{k+1})}\zeta_{\rm MT}(\mathbf{s}', s_{k+1}+s_k-1;\boldsymbol{\alpha}',0)\right]_{\alpha_k}\\ &+\sum_{j=0}^{M-1}\binom{-s_{k+1}}{j}\zeta_{\rm MT}(\mathbf{s}', s_{k+1}+j;\boldsymbol{\alpha}',0)\phi(s_k-j,\alpha_k)\end{aligned}$$

where $[x]_\alpha = x$ if $\alpha \in \mathbb{Z}$ and $[x]_\alpha = 0$ otherwise. As M can be arbitrarily large a careful computation now yields the correct set of poles for $\zeta_{\rm MT}(\mathbf{s};\boldsymbol{\alpha})$ as given in (11). This concludes the proof of the proposition. □

Theorem 2.3. *The Mordell-Tornheim L-function $L_{\rm MT}(\mathbf{s};\boldsymbol{\chi})$ defined by (1) can be analytically continued to a memorphic function over $\mathbb{C}^{|\mathbf{s}|}$ with explicitly computable singularities lying in at most countably many hyperplanes.*

Proof. Let f_j be the conductor of χ_j for $j = 1, \dots, k+1$. By [27, Lemma 4.7] we see that

$$L_{\rm MT}(\mathbf{s};\boldsymbol{\chi}) = \sum_{j_1=1}^{f_1}\cdots\sum_{j_{k+1}=1}^{f_{k+1}}\prod_{i=1}^{k+1}\frac{\chi_i(j_i)}{\tau(\overline{\chi_i})}\zeta_{\rm MT}(\mathbf{s}; j_1/f_1,\dots,j_{k+1}/f_{k+1}) \tag{14}$$

where $\tau(\overline{\chi_i})$ is the Gauss sum. Hence the theorem follows from Prop. 2.2 immediately. □

3. Reducing Mordell-Tornheim Type Values to Traditional Values

In this section we shall prove that the study of special 1-MTLVs and colored special 1-MTZVs can be reduced to that of special 1-MLVs and special 1-MZVs, respectively.

Theorem 3.1. *Fix a positive integer $k \geq 2$. Let $z \in \mathbb{C}$ and $\mathbf{s} = (s_1, \dots, s_k) \in \mathbb{N}^k$. Then for any primitive Dirichlet character χ the special 1-MTLV $L_{\rm MT}(\mathbf{s}, z; \{\mathbf{1}\}_k, \chi)$ is a $\mathbb{Q}$-linear combination of special 1-MLVs of the same weight and depth and of the same character type $(\{\mathbf{1}\}_k, \chi)$.*

Proof. Essentially the same proof of [33, Theorem 5] works here. For example, with their notation we can multiply $\chi(n_r)$ inside each sum appearing in their proof. Moreover, the variable s always appears in the last variable position of every function T_ℓ throughout the proof. This corresponds to the leading position of the 1-MLV so the values are always special. □

Due to the combinatorial nature of the proof it won't work for non-special 1-MTLVs or wrong character types. In fact, more generally, every special 1-MTLV $L_{\rm MT}(\mathbf{s}; \chi_1, \dots, \chi_{k+1})$ is a $\mathbb{Q}$-linear combination of the following values

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}\frac{\chi_1(m_1)\chi_2(m_2)\cdots\chi_k(m_k)\chi_{k+1}(m_1+\cdots+m_k)}{m_1^{r_1}(m_1+m_2)^{r_2}\cdots(m_1+\cdots+m_{k-1})^{r_{k-1}}(m_1+\cdots+m_k)^{s_{k+1}+r_k}}$$

where $\mathbf{r} \in \mathbb{Z}^k$. Notice that these are not 1-MLVs as Dirichlet characters are not additive in general. The situation for colored 1-MTZVs is little better, with no restriction on the type of the “colors”.

Theorem 3.2. *Every colored special 1-MTZVs is a $\mathbb{Q}$-linear combination of colored special 1-MZVs of the same weight and same depth. More precisely, the colored special 1-MTZV $\zeta_{\mathrm{MT}}(s_1,\dots,s_{k+1};\alpha_1,\dots,\alpha_{k+1})$ is a $\mathbb{Q}$-linear combination of colored special 1-MZVs of the following form*

$$\zeta(s_{k+1}+r_k,r_{k-1},\dots,r_1;\alpha_k+\alpha_{k+1},\alpha_{k-1}-\alpha_k,\dots,\alpha_1-\alpha_2),$$

where $\mathbf{r}\in\mathbb{Z}^k$.

Proof. Modify the proof of [33, Theorem 5] by inserting "colors" into expressions of T_ℓ's there. □

Remark 3.3. Theorem 3.1 and Theorem 3.2 generalize [33, Theorem 5] about MTZVs in two different directions: the former to their L-function version while the latter to their colored version.

4. Preliminaries on Bernoulli Polynomials

By definition the Bernoulli polynomials $B_n(x)$ are periodic functions with period 1 defined by the generating function:

$$\frac{te^{xt}}{e^t-1}=\sum_{n\ge 0}B_n(x)\frac{t^n}{n!},\qquad x\in[0,1).$$

The values $B_n:=B_n(0)$ are the Bernoulli numbers which are linked to the Riemann zeta values :

$$B_1=\zeta(0)=-\frac{1}{2},\quad B_{2s}=-\frac{2(2s)!}{(2\pi i)^{2s}}\zeta(2s)\quad\forall s\in\mathbb{N}. \tag{15}$$

For any positive integer s and N we set

$$f^+_{s,N}(x)=\sum_{k=1}^{N}\frac{e(kx)}{k^s},\qquad f^-_{s,N}(x)=\sum_{k=-1}^{-N}\frac{e(kx)}{k^s}=(-1)^s f^+_{s,N}(-x),$$

where $e(x)=e^{2\pi i x}$, and

$$f_{s,N}(x)=f^+_{s,N}(x)+f^-_{s,N}(x).$$

Lemma 4.1. *For every positive integer s we have*

$$B_s(x)=-\frac{s!}{(2\pi i)^s}f_{s,\infty}(x). \tag{16}$$

Its derivative

$$B'_s(x)=sB_{s-1}(x) \tag{17}$$

and for $m\neq 0$ the integral

$$\int_0^1 B_n(x)e(mx)\,dx=-\gamma_{0,n}\frac{n!}{(-2\pi i m)^n} \tag{18}$$

where $\gamma_{0,n}=0$ if $n=0$ and $\gamma_{0,n}=1$ otherwise. For $\mathbf{s}=(s_1,\dots,s_t)\in\mathbb{N}^t$

$$C_{\mathbf{s}}:=\int_0^1\prod_{j=1}^t B_{s_j}(x)\,dx=\sum_{r_1=0}^{s_1}\cdots\sum_{r_t=0}^{s_t}\binom{s_1}{r_1}\cdots\binom{s_t}{r_t}\frac{B_{\mathbf{s}-\mathbf{r}}}{|\mathbf{r}|+1}, \tag{19}$$

where for any vector $\mathbf{v}=(v_1,\dots,v_t)$ we set $|\mathbf{v}|:=\sum_{j=1}^t v_j$ and $B_{\mathbf{v}}:=\prod_{j=1}^t B_j$.

Proof. All the statements are well-known (for e.g., see [1, pp. 804–805]) except perhaps (16) which is on [8, p.362]. □

Let $[t]:=\{1,\dots,t\}$ with the increasing order. By abuse of notation we let $\mathbf{i}=(i_1,\dots,i_\lambda)\subseteq[k]$ denote both a set and a vector such that $i_1<\cdots<i_\lambda$. No confusion should arise. We denote its length by $\ell(\mathbf{i})=\lambda$ and write $\mathbf{i}!=i_1!\cdots i_\lambda!$. If $\ell(\mathbf{j})=\ell(\mathbf{i})<t$ then we define the inflation of the vector $\mathbf{j}$ to length t with respect to $\mathbf{i}$ as

$$\mathrm{Inf}_{\mathbf{i}}^t(\mathbf{j})=(l_1,\dots,l_t),l_\beta=\begin{cases} j_\alpha & \text{if } \beta=i_\alpha\in\mathbf{i};\\ 1 & \text{if } \beta\notin\mathbf{i}.\end{cases}$$

Note that $|\mathrm{Inf}_{\mathbf{i}}^t(\mathbf{j})|=|\mathbf{j}|-\ell(\mathbf{i})+t$. This operation essentially stretches the vector $\mathbf{j}$ to a length t vector by redistributing its entries to $\mathbf{i}$-th positions while inserting 1's in other positions. Finally, for a vector $\mathbf{v}=(v_1,\dots,v_t)$ we write

$$\binom{|\mathbf{v}|}{\mathbf{v}}=\binom{|\mathbf{v}|}{v_1,\dots,v_t}.$$

The next proposition is not needed in the proof of the main results in the paper but it offers a simple and close expression of an arbitrary product of Bernoulli polynomials and therefore should have independent interest by itself. It generalizes the well-known result of Calitz [8].

Proposition 4.2. *Keep the same notation as in Lemma* 4.1*. Then*

$$B_{\mathbf{s}}(x)=C_{\mathbf{s}}+\sum_{\mathbf{i}\subsetneq[t]}\sum_{0\le j_{\mathbf{i}}\le s_{\mathbf{i}}}\binom{|\mathbf{s}|-|\mathbf{j}|+\ell(\mathbf{i})-t}{\mathbf{s}-\mathrm{Inf}_{\mathbf{i}}^t(\mathbf{j})}\frac{B_{\mathbf{j}}}{\mathbf{j}!}\cdot\frac{\mathbf{s}!B_{|\mathbf{s}|-|\mathbf{j}|+\ell(\mathbf{i})-t+1}(x)}{(|\mathbf{s}|-|\mathbf{j}|+\ell(\mathbf{i})-t+1)!}. \tag{20}$$

where for an vector $\mathbf{i}=(i_1,\dots,i_\ell)$ *we write the multiple sum* $\displaystyle\sum_{0\le j_{\mathbf{i}}\le s_{\mathbf{i}}}=\sum_{j_1=0}^{s_{i_1}}\cdots\sum_{j_\ell=0}^{s_{i_\ell}}$.

Proof. By induction it is easy to show that

$$\prod_{\tau=1}^t\frac{1}{e^{u_\tau}-1}=\frac{1}{e^{|\mathbf{u}|}-1}\sum_{\mathbf{i}\subsetneq[t]}\prod_{\tau=1}^{\ell(\mathbf{i})}\frac{1}{e^{u_{i_\tau}}-1}, \tag{21}$$

where the product on the right is 1 when $\mathbf{i}=\emptyset$. Indeed, if $t=1$ then $\mathbf{i}$ has to be $\emptyset$ and (21) is clear. Assume (21) holds for $t\ge1$. Then we have

$$\begin{aligned}\frac{1}{e^{u_{t+1}}-1}\prod_{\tau=1}^t\frac{1}{e^{u_\tau}-1}&=\frac{1}{(e^{u_{t+1}}-1)(e^{|\mathbf{u}|}-1)}\sum_{\mathbf{i}\subsetneq[t]}\prod_{i\in\mathbf{i}}\frac{1}{e^{u_i}-1}\\ &=\frac{1}{e^{|\mathbf{u}|+u_{t+1}}-1}\left(1+\frac{1}{e^{u_{t+1}}-1}+\frac{1}{e^{|\mathbf{u}|}-1}\right)\left(1+\sum_{\emptyset\ne\mathbf{i}\subsetneq[t]}\prod_{i\in\mathbf{i}}\frac{1}{e^{u_i}-1}\right)\\ &=\frac{1}{e^{|\mathbf{u}|+u_{t+1}}-1}\prod_{\tau=1}^{t+1}\left(1+\sum_{\emptyset\ne\mathbf{i}\subsetneq[t+1]}\prod_{i\in\mathbf{i}}\frac{1}{e^{u_i}-1}\right).\end{aligned}$$

Thus (21) is proved. Applying it we may transform the following power series ($\mathbf{u}=(u_1,\dots,u_t)$)

$$\begin{aligned}|\mathbf{u}|\sum_{\mathbf{s}\in(\mathbb{Z}_{\ge0})^t}\frac{B_{\mathbf{s}}(x)}{\mathbf{s}!}u_1^{s_1}\cdots u_t^{s_t}&=\frac{|\mathbf{u}|\prod_{\tau=1}^t u_\tau}{\prod_{\tau=1}^t(e^{u_\tau}-1)}e^{x|\mathbf{u}|}=\frac{|\mathbf{u}|e^{x|\mathbf{u}|}}{e^{|\mathbf{u}|}-1}\sum_{\mathbf{i}\subsetneq[t]}\prod_{i\in\mathbf{i}}\frac{u_i}{e^{u_i}-1}\prod_{i\notin\mathbf{i}}u_i\\ &=\sum_{n=0}^\infty B_n(x)\frac{|\mathbf{u}|^n}{n!}\sum_{\mathbf{i}\subsetneq[t]}\prod_{i\notin\mathbf{i}}u_i\sum_{0\le j_{\mathbf{i}}<\infty}\frac{B_{\mathbf{j}}}{\mathbf{j}!}\prod_{\tau=1}^{\ell(\mathbf{i})}u_{i_\tau}^{j_\tau}\end{aligned}$$

where $\boldsymbol{\infty}=(\infty,\dots,\infty)$. On the left, the coefficient for $u_1^{s_1}\cdots u_t^{s_t}/\mathbf{s}!$ is

$$\sum_{\tau=1}^{t} s_\tau B_{s_1}(x)\cdots B_{s_{\tau-1}}(x)B_{s_\tau-1}(x)B_{s_{\tau+1}}(x)\cdots B_{s_t}(x)=(B_{\mathbf{s}}(x))'$$

by (17). Integrating this we get

$$B_{\mathbf{s}}(x)=C+\sum_{\mathbf{i}\subsetneq[t]}\sum_{0\le j_{\mathbf{i}}\le s_{\mathbf{i}}}\binom{|\mathbf{s}|-|\mathbf{j}|+\ell(\mathbf{i})-t}{\mathbf{s}-\mathrm{Inf}_{\mathbf{i}}^t(\mathbf{j})}\frac{B_{\mathbf{j}}}{\mathbf{j}!}\cdot\frac{\mathbf{s}!B_{|\mathbf{s}|-|\mathbf{j}|+\ell(\mathbf{i})-t+1}(x)}{(|\mathbf{s}|-|\mathbf{j}|+\ell(\mathbf{i})-t+1)!},$$

for some constant C. Integrating again and noticing that $\int_0^1 B_n(x)\,dx=0$ whenever $n\ge1$ we get $C=C_{\mathbf{s}}$ by (19). This completes the proof of the lemma. □

The above lemma expresses the product of different Bernoulli polynomials explicitly as a linear combination of Bernoulli polynomials of different degrees. But it has the drawback that we cannot restrict the degrees in order to provide a general reduction formula in Theorem 1.3 since the terms on the right hand side of (20) do not always have the same weight where the weight of a product term in (20) is the sum of the indices of the all the Bernoulli numbers appearing in that product (this comes from relation (15)). However, Prop. 4.5 will enable us to quantify Theorem 1.3 even though it has much more complex structure than Prop. 4.2. To state it we need some more definitions and notations.

Definition 4.3. For arbitrary $\mathbf{s}=(s_1,\dots,s_t)\in\mathbb{N}^t$ a *partition* of $\mathbf{s}$ is always an ordered partition $\mathbf{P}=(\mathbf{P}_1,\dots,\mathbf{P}_q)$ such that the concatenation of $\mathbf{P}$ is $\mathbf{s}$. A *pre-fat partition* of $\mathbf{s}$ is such a partition with $l_j:=\ell(\mathbf{P}_j)\ge2$ for all $j\le q-1$. Its *pre-associated index set* is the set $\mathrm{ind}'(\mathbf{P})$ of indices $\mathbf{r}=(\mathbf{r}_1,\dots,\mathbf{r}_q)$ where

$$\mathbf{r}_j=\begin{cases}(r_{j,1},r_{j,2},\dots,r_{j,l_j-2}) & \text{if } j\le q-1;\\ (r_{j,1},r_{j,2},\dots,r_{j,l_j-1}) & \text{if } j=q \text{ (vacuous if } l_q=1).\end{cases}$$

Setting $\mathbf{P}_j:=(s_{j,1},s_{j,2},\dots,s_{j,l_j})$ for $j=1,\dots,q$. For each j and each $i=1,2,\dots,\ell(\mathbf{r}_j)$, the range of the integer index $r_{j,i}$ goes from 0 to $\lfloor\max\{\sigma_i(\mathbf{P}_j)-2\sigma_{i-1}(\mathbf{r}_j),s_{j,i+1}\}/2\rfloor$ where for any vector $\mathbf{v}=(v_1,\dots,v_\ell)$ we denote its i-th partial sum by $\sigma_i(\mathbf{v}):=v_1+\cdots+v_i$ and $\sigma_0(\mathbf{v}):=0$.

Definition 4.4. A *fat partition* of $\mathbf{s}$ is a pre-fat partition $\mathbf{P}=(\mathbf{P}_1,\dots,\mathbf{P}_q)$ such that $l_q\ge2$, i.e., every part has length at least two. Its *associated index set* is the set $\mathrm{ind}(\mathbf{P})$ of $\mathbf{r}=(\mathbf{r}_1,\dots,\mathbf{r}_q)$ where for each $1\le j\le q$,

$$\mathbf{r}_j=(r_{j,1},r_{j,2},\dots,r_{j,l_j-2})$$

with each $r_{j,i}$ running over the same range as above in Definition 4.3.

It is an easy exercise to see that the number of fat partitions of $\mathbf{s}=(s_1,\dots,s_t)$ is given by the Fibonacci number F_{t-1} [22, p. 46, **14.b**], where $F_1=F_2=1$ and $F_{n+2}=F_{n+1}+F_n$ for all $n\ge1$. Obviously the number of pre-fat partitions of $\mathbf{s}$ is given by F_t.

Proposition 4.5. *Let $\mathcal{P}'(\mathbf{s})$ (resp. $\mathcal{P}(\mathbf{s})$) be the set of pre-fat (resp. fat) partitions of $\mathbf{s}=(s_1,\dots,s_t)\in\mathbb{N}^t$ with $t\ge2$. For each partition $\mathbf{P}$ let $q:=q(\mathbf{P})$ be the number of parts in $\mathbf{P}$.*

Then

$$B_{\mathbf{s}}(x) = \sum_{\mathbf{P}\in\mathcal{P}'(\mathbf{s})} \sum_{\mathbf{r}\in\mathrm{ind}'(\mathbf{P})} \prod_{j=1}^{q} \left\{ \left\{ \prod_{i=1}^{\ell(\mathbf{r}_j)} b_{j,i}(\mathbf{P},\mathbf{r}) \right\} B_j(\mathbf{P},\mathbf{r},x) \right\} + \sum_{\mathbf{P}\in\mathcal{P}(\mathbf{s})} \sum_{\mathbf{r}\in\mathrm{ind}(\mathbf{P})} \prod_{j=1}^{q} \left\{ \left\{ \prod_{i=1}^{\ell(\mathbf{r}_j)} b_{j,i}(\mathbf{P},\mathbf{r}) \right\} B_j(\mathbf{P},\mathbf{r}) \right\}, \tag{22}$$

where

$$B_j(\mathbf{P},\mathbf{r}) = (-1)^{1+s_{j,l_j}} \frac{(|\mathbf{P}_j| - s_{j,l_j} - 2|\mathbf{r}_j|)!(s_{j,l_j})!}{(|\mathbf{P}_j| - 2|\mathbf{r}_j|)!} B_{|\mathbf{P}_j|-2|\mathbf{r}_j|}, \tag{23}$$

where s_{j,l_j} is the last component of $\mathbf{P}_j$, and

$$B_j(\mathbf{P},\mathbf{r},x) = \begin{cases} B_j(\mathbf{P},\mathbf{r}), & \text{if } j < q; \\ B_{|\mathbf{P}_q|}(x), & \text{if } j = q \text{ and } l_q = 1; \\ B_{|\mathbf{P}_q|-2|\mathbf{r}_q|}(x) \quad , & \text{if } j = q \text{ and } l_q > 1. \end{cases} \tag{24}$$

By convention, if $l(\mathbf{r}_j) = 0$ then the innermost product is 1. If $l(\mathbf{r}_j) \ge 1$ then $b_{j,i}(\mathbf{P},\mathbf{r}) =$

$$\left[\binom{\sigma_i(\mathbf{P}_j) - 2\sigma_{i-1}(\mathbf{r}_j)}{2r_{j,i}} s_{j,i+1} + \binom{s_{j,i+1}}{2r_{j,i}} (\sigma_i(\mathbf{P}_j) - 2\sigma_{i-1}(\mathbf{r}_j)) \right] \frac{B_{2r_{j,i}}}{\sigma_{i+1}(\mathbf{P}_j) - 2\sigma_i(\mathbf{r}_j)}. \tag{25}$$

Proof. We prove the proposition by induction on t. If $t = 2$ then the proposition has a very explicit form given by [8, (3)] or [20]:

$$B_{s_1}(x)B_{s_2}(x) = \sum_{r=0}^{\lfloor \max\{s_1,s_2\}/2 \rfloor} \left[\binom{s_1}{2r} s_2 + \binom{s_2}{2r} s_1 \right] \frac{B_{2r}B_{|\mathbf{s}|-2r}(x)}{|\mathbf{s}| - 2r} - (-1)^{s_2} \frac{s_1!s_2!}{|\mathbf{s}|!} B_{|\mathbf{s}|}, \tag{26}$$

where $|\mathbf{s}| = s_1 + s_2$. Let's check formula (22) is correct. In this case the only pre-fat partition is the whole $\mathbf{s} = \mathbf{P}$ because in every pre-fat partition only the last part can have length equal to one. So $q = 1$, $\mathrm{ind}'(\mathbf{P}) = \{r : 1 \le r \le \lfloor \max\{s_1, s_2\}/2 \rfloor\}$ and $\mathrm{ind}(\mathbf{P}) = \emptyset$. Then

$$B_{s_1}(x)B_{s_2}(x) = \sum_{r\in\mathrm{ind}'(\mathbf{P})} b_{1,1}(\mathbf{P},r)B_1(\mathbf{P},r,x) + B_1(\mathbf{P},\emptyset)$$

where $B_1(\mathbf{P},r,x) = B_{|\mathbf{s}|-2r}(x)$, $B_1(\mathbf{P},\emptyset) = -(-1)^{s_2}s_1!s_2!B_{|\mathbf{s}|}/(|\mathbf{s}|)!$ and

$$b_{1,1}(\mathbf{P},r) = \left[\binom{s_1}{2r} s_2 + \binom{s_2}{2r} s_1 \right] \frac{B_{2r}}{|\mathbf{s}| - 2r}.$$

Thus the case $t = 2$ is verified.

Assume the proposition is true when $\ell(\mathbf{s}) = t \ge 2$. Then we can use (22) to compute $B_{\mathbf{s}}(x)B_n(x)$ for any positive integer n. Clearly when $B_n(x)$ is multiplied by the sums involving only Bernoulli numbers (the second line of (22)) we get exactly those terms corresponding to the pre-fat partitions $\mathbf{Q}$ of $(s_1,\dots,s_t,n)$ whose last part has length one. This can be readily explained by the map

$$\begin{aligned} \mathcal{P}(\mathbf{s}) &\longrightarrow \mathcal{P}'((\mathbf{s},n)) \\ \mathbf{P} &\longmapsto \mathbf{Q}' := (\mathbf{P},(n)). \end{aligned} \tag{27}$$

It is obvious that the pre-associated index set of $\mathbf{Q}'$ is exactly the same as the associated index set of $\mathbf{P}$. When $B_n(x)$ is multiplied on each of the terms in the first nested sum of (22) two kind of terms will appear according to (26). Let's consider the following two cases: (i) $l_q = 1$, and (ii) $l_q > 1$.

In case (i) we have

$$B_{|\mathbf{P}_q|}(x)B_n(x) = \sum_{r=0}^{\lfloor \max\{|\mathbf{P}_q|,n\}/2\rfloor} \left[\binom{|\mathbf{P}_q|}{2r}n + \binom{n}{2r}|\mathbf{P}_q|\right] \frac{B_{2r}B_{|\mathbf{P}_q|+n-2r}(x)}{|\mathbf{P}_q|+n-2r} + (-1)^{1+n}\frac{|\mathbf{P}_q|!n!}{(|\mathbf{P}_q|+n)!}B_{|\mathbf{P}_q|+n}.$$

The summation term in the above contribute exactly to those pre-fat partitions $\mathbf{Q}'$ of $(s_1,\dots,s_t,n)$ whose last part has length equal to two. The last term of the above corresponds to the fat partitions $\mathbf{Q}$ of $(s_1,\dots,s_t,n)$ whose last part has length equal to two. This can be summarized by the map

$$\begin{aligned} \mathcal{P}'(\mathbf{s}) &\longrightarrow \mathcal{P}'((\mathbf{s},n)) \times \mathcal{P}((\mathbf{s},n)) \\ \mathbf{P} &\longmapsto \mathbf{Q}' =: \big(\mathbf{P}_1,\dots,\mathbf{P}_{q-1},(\mathbf{P}_q,n)\big), \mathbf{Q} := \big(\mathbf{P}_1,\dots,\mathbf{P}_{q-1},(\mathbf{P}_q,n)\big). \end{aligned} \tag{28}$$

It's easy to check that the pre-associated index set of $\mathbf{Q}'$ is obtained from the pre-associated index set $\mathbf{r}$ of $\mathbf{P}$ by adding one more part at the end: (r) itself alone, which goes from 0 to $\lfloor \max\{|\mathbf{P}_q|,n\}/2\rfloor$. The corresponding term in (22) is thus determined by (25) and the third case of (24). It's also clear that the associated index set of $\mathbf{Q}$ is equal to $\mathbf{r}$ which is consistent with (23) since the last component of $\mathbf{Q}$ has length 2 which implies that the innermost product of the second sum in (22) is 1 by convention.

In case (ii) we have

$$B_{|\mathbf{P}_q|-2|\mathbf{r}_q|}(x)B_n(x) = \sum_{r=0}^{\lfloor \max\{|\mathbf{P}_q|-2|\mathbf{r}_q|,n\}/2\rfloor} \left[\binom{|\mathbf{P}_q|-2|\mathbf{r}_q|}{2r}n + \binom{n}{2r}(|\mathbf{P}_q|-2|\mathbf{r}_q|)\right] \cdot \frac{B_{2r}B_{|\mathbf{P}_q|+n-2|\mathbf{r}_q|-2r}(x)}{|\mathbf{P}_q|+n-2|\mathbf{r}_q|-2r} + (-1)^{1+n}\frac{(|\mathbf{P}_q|-2|\mathbf{r}_q|)!n!}{(|\mathbf{P}_q|+n-2|\mathbf{r}_q|)!}B_{|\mathbf{P}_q|+n-2|\mathbf{r}_q|}.$$

Similarly to the above, this can be summarized by the map

$$\begin{aligned} \mathcal{P}'(\mathbf{s}) &\longrightarrow \mathcal{P}'((\mathbf{s},n)) \times \mathcal{P}((\mathbf{s},n)) \\ \mathbf{P} &\longmapsto \mathbf{Q}' =: \big(\mathbf{P}_1,\dots,\mathbf{P}_{q-1},(\mathbf{P}_q,n)\big), \mathbf{Q} =: \big(\mathbf{P}_1,\dots,\mathbf{P}_{q-1},(\mathbf{P}_q,n)\big) \end{aligned} \tag{29}$$

with the last component in both $\mathbf{Q}'$ and $\mathbf{Q}$ having length greater than 2. It is easy to check that index sets is consistent with (22) when $\mathbf{s}$ is replace by $(\mathbf{s},n)$ (let's call the equation after such a change $(22)'$) by inserting r into the end of the last component of $\mathbf{r}$.

The above argument shows that every term in the product expansion of $B_{|\mathbf{s}|}(x)B_n(x)$ appears in $(22)'$. Finally, one can check that in $(22)'$ every term is produced exactly once by the maps (27) to (29) combined as the number of terms produced in $\mathcal{P}'(\mathbf{s},n)$ and $\mathcal{P}((\mathbf{s},n))$ both follow the Fibonacci rule. This completes the proof of the proposition. □

5. Main Results

The notation in the proceeding section is still in force. Throughout this section we fix $\mathbf{s}' = (s_1,\dots,s_k) \in \mathbb{N}^k$, $\kappa := k+1$, $z = s_\kappa \in \mathbb{C}$ and $\mathbf{s} = (s_1,\dots,s_\kappa)$. For any subset $\mathbf{i} = (i_1,\dots,i_t) \subseteq [k]$ we write $\mathbf{s}(\mathbf{i}) = (s_{i_1},\dots,s_{i_t})$. For any real number α we define

$$S(\mathbf{s},\mathbf{i},\alpha) := \sum_{\substack{m_1,\dots,m_\kappa\in\mathbb{N}^\kappa \\ \sum_{j\in\mathbf{i}} m_j = \sum_{j\in[\kappa]\setminus\mathbf{i}} m_j}} \frac{e(m_\kappa\alpha)}{m_1^{s_1}\cdots m_\kappa^{s_\kappa}}.$$

Observe that if $i \le k$ then $S(\mathbf{s},\{i\},\alpha)$ is a colored 1-MTZV with only variable s_i dressed with $e(\alpha)$ while $S(\mathbf{s},[k],\alpha)$ is a colored *special* 1-MTZV with only variable z dressed with $e(\alpha)$. This observation and the next proposition is crucial to prove Theorem 1.3.

Proposition 5.1. *Let $\alpha \in \mathbb{R}$ and $\emptyset \ne \mathbf{i} \subseteq [k]$. Suppose $\Re(z) \ge 1$ then we have*

$$\lim_{N\to\infty} \int_0^1 \prod_{j\in\mathbf{i}} f_{s_j,N}(x) \prod_{j\in[k]\setminus\mathbf{i}} f^+_{s_j,N}(x) f^+_{z,N}(x+\alpha)\,dx = \sum_{\mathbf{j}\subseteq\mathbf{i}} (-1)^{|\mathbf{s}(\mathbf{j})|} S(\mathbf{s},\mathbf{j},\alpha). \tag{30}$$

If $2 \le \mathbf{i} \ne [k]$ then it is a $\mathbb{Q}$-linear combination of products of Riemann zeta values at a non-negative even integers and a colored 1-MTZV with only the complex variable z being dressed with $e(\alpha)$. This linear combination is explicitly given by

$$E(\mathbf{s},\mathbf{i},\alpha) = \sum_{\mathbf{P}\in\mathcal{P}'(\mathbf{s}(\mathbf{i}))} \sum_{\mathbf{r}\in\mathrm{ind}'(\mathbf{P})} (-1)^{|\mathbf{s}(\mathbf{i})|} 2^{\ell(\mathbf{i})-q} \prod_{j=1}^{q} \left\{ \prod_{i=2}^{\ell(\mathbf{r}_j)+1} c_{j,i}(\mathbf{P},\mathbf{r}) \right\} C_{j,\alpha}(\mathbf{P},\mathbf{r}) \tag{31}$$

where c's and $C_{j,\alpha}$'s are define as follows. For $\mathbf{s},\mathbf{i},\alpha$ as above and any positive integer n let $f_{\mathbf{s},\mathbf{i},\alpha}(n) := \zeta_{\mathrm{MT}}\big(\mathbf{s}([\kappa]\setminus\mathbf{i}), n; R_{k-\ell(\mathbf{i})+1}(\alpha,\{1\}_{k-\ell(\mathbf{i})+2})\big)$. Then

$$C_{j,\alpha}(\mathbf{P},\mathbf{r}) = \begin{cases} (-1)^{s_{j,l_j}} \tilde{\zeta}(|\mathbf{P}_j| - 2|\mathbf{r}_j|), & \text{if } j < q; \\ f_{\mathbf{s},\mathbf{i},\alpha}(|\mathbf{P}_q|), & \text{if } j = q \text{ and } l_q = 1; \\ f_{\mathbf{s},\mathbf{i},\alpha}(|\mathbf{P}_q| - 2|\mathbf{r}_q|), & \text{if } j = q \text{ and } l_q > 1. \end{cases} \tag{32}$$

where $\tilde{\zeta}(m) = \zeta(m)$ if m is even and $\tilde{\zeta}(m) = 0$ if m is odd. If $l(\mathbf{r}_j) = 0$ then the innermost product is 1; otherwise

$$c_{j,i}(\mathbf{P},\mathbf{r}) = \left[\binom{\sigma_i(\mathbf{P}_j) - 2\sigma_{i-1}(\mathbf{r}_j) - 1}{s_{j,i} - 1} + \binom{\sigma_i(\mathbf{P}_j) - 2\sigma_{i-1}(\mathbf{r}_j) - 1}{s_{j,i} - 2r_{j,i-1}} \right] \zeta(2r_{j,i-1}). \tag{33}$$

Proof. The equation (30) is straightforward. So we only need to prove the second part. In the following proof we often exchange limits without giving explicit justification. But they are easy to check by Lebesgue's Dominated Convergence Theorem because of the absolution convergence to be proved in Prop. 7.1.

Assume $\ell(\mathbf{i}) \ge 2$. First, by Lemma 4.1 we have

$$\lim_{N\to\infty} \prod_{j\in\mathbf{i}} f_{s_j,N}(x) = \frac{(2\pi i)^{|\mathbf{s}(\mathbf{i})|}}{(-1)^{\ell(\mathbf{i})} \cdot \mathbf{s}(\mathbf{i})!} B_{\mathbf{s}(\mathbf{i})}(x). \tag{34}$$

Prop. 4.5 now yields (with the same notation given there)

$$\text{LHS of (30)} = \frac{(2\pi i)^{|\mathbf{s}(\mathbf{i})|}}{(-1)^{\ell(\mathbf{i})} \cdot \mathbf{s}(\mathbf{i})!} \sum_{\mathbf{P}\in\mathcal{P}'(\mathbf{s}(\mathbf{i}))} \sum_{\mathbf{r}\in\mathrm{ind}'(\mathbf{P})} \left\{ \prod_{j=1}^{q-1} \left\{ \left\{ \prod_{i=1}^{\ell(\mathbf{r}_j)} b_{j,i}(\mathbf{P},\mathbf{r}) \right\} B_j(\mathbf{P},\mathbf{r}) \right\} \right.$$
$$\left. \cdot \prod_{i=1}^{\ell(\mathbf{r}_q)} b_{q,i}(\mathbf{P},\mathbf{r}) \cdot \int_0^1 B_q(\mathbf{P},\mathbf{r},x) \cdot \prod_{j\in[k]\setminus\mathbf{i}} f^+_{s_j,\infty}(x) f^+_{z,\infty}(x+\alpha)\,dx \right\}$$

This is equal to

$$E(\mathbf{s},\mathbf{i},\alpha) := \frac{(2\pi i)^{|\mathbf{s}(\mathbf{i})|}}{(-1)^{\ell(\mathbf{i})} \cdot \mathbf{s}(\mathbf{i})!} \sum_{\mathbf{P}\in\mathcal{P}'(\mathbf{s}(\mathbf{i}))} \sum_{\mathbf{r}\in\mathrm{ind}'(\mathbf{P})} \prod_{j=1}^{q} \left\{ \left\{ \prod_{i=1}^{\ell(\mathbf{r}_j)} b_{j,i}(\mathbf{P},\mathbf{r}) \right\} B_j(\mathbf{P},\mathbf{r}) \right\}. \tag{35}$$

Here by (25) and (15) if $l(\mathbf{r}_j)\ge 1$ then $b_{j,i}(\mathbf{P},\mathbf{r})=$

$$\left[\binom{\sigma_i(\mathbf{P}_j)-2\sigma_{i-1}(\mathbf{r}_j)}{2r_{j,i}}s_{j,i+1}+\binom{s_{j,i+1}}{2r_{j,i}}\big(\sigma_i(\mathbf{P}_j)-2\sigma_{i-1}(\mathbf{r}_j)\big)\right]\frac{-2(2r_{j,i})!\zeta(2r_{j,i})/(2\pi i)^{2r_{j,i}}}{\sigma_{i+1}(\mathbf{P}_j)-2\sigma_i(\mathbf{r}_j)}.$$

By (23) and (15) if $j<q$ then

$$B_j(\mathbf{P},\mathbf{r})=(-1)^{s_{j,l_j}}(|\mathbf{P}_j|-s_{j,l_j}-2|\mathbf{r}_j|)!(s_{j,l_j})!\frac{2\cdot\tilde{\zeta}(|\mathbf{P}_j|-2|\mathbf{r}_j|)}{(-2\pi i)^{|\mathbf{P}_j|-2|\mathbf{r}_j|}}$$

where s_{j,l_j} is the last component of $\mathbf{P}_j$. Finally , by (24) and (18)

$$B_q(\mathbf{P},\mathbf{r})=\begin{cases}-\dfrac{(|\mathbf{P}_q|)!}{(-2\pi i)^{|\mathbf{P}_q|}}\cdot\zeta_{\rm MT}(\mathbf{s}([\kappa]\setminus\mathbf{i}),|\mathbf{P}_q|;\{1\}_{k-\ell(\mathbf{i})},\alpha,1), & \text{if } l_q=1;\\ -\dfrac{(|\mathbf{P}_q|-2|\mathbf{r}_q|)!}{(-2\pi i)^{|\mathbf{P}_q-2|\mathbf{r}_q||}}\zeta_{\rm MT}(\mathbf{s}([\kappa]\setminus\mathbf{i}),|\mathbf{P}_q|-2|\mathbf{r}_q|;\{1\}_{k-\ell(\mathbf{i})},\alpha,1), & \text{if } l_q>1.\end{cases}$$

We readily see that all the powers of $2\pi i$ cancel out in (35) and (31) has the correct sign and 2-powers. Let's compute the jth term of the innermost product in (35) when $j<q$. The case $j=q$ is very similar and is left to the interested reader. For simplicity let us further assume $\mathbf{P}_j=(a_1,\dots,a_l)$, $\mathbf{r}_j=(r_1,\dots,r_{l-2})$, and $l\ge 3$. Without the signs, 2-powers and the zeta factors this product looks as follows:

$$\begin{aligned}&\left[\binom{a_1}{2r_1}a_2+\binom{a_2}{2r_1}a_1\right]\frac{(2r_1)!}{a_1+a_2-2r_1}\\ \times&\left[\binom{a_1+a_2-2r_1}{2r_2}a_3+\binom{a_3}{2r_2}(a_1+a_2-2r_1)\right]\frac{(2r_2)!}{a_1+a_2+a_3-2(r_1+r_2)}\\ &\qquad\vdots\\ \times&\left[\binom{\sum_{t=1}^{l-2}a_t-2\sum_{t=1}^{l-3}r_t}{2r_{l-2}}a_{l-1}+\binom{a_3}{2r_2}\Big(\sum_{t=1}^{l-2}a_t-2\sum_{t=1}^{l-3}r_t\Big)\right]\frac{(2r_{l-2})!}{\sum_{t=1}^{l-1}a_t-2\sum_{t=1}^{l-2}r_t}\\ \times&\frac{(\sum_{t=1}^{l-1}a_t-2\sum_{t=1}^{l-2}r_t)!}{a_1!a_2!\cdots a_{l-1}!}\end{aligned}$$

Now multiplying each fraction like $1/(a_1+a_2-2r_1)$ on the next $[\cdots]$, expanding the binomial coefficients, and canceling all $(2r_t)!$'s and $a_t!$'s we get:

$$\begin{aligned}&\left[\frac{1}{(a_1-2r_1)!(a_2-1)!}+\frac{1}{(a_2-2r_1)!(a_1-1)!}\right]\\ \times&\left[\frac{(a_1+a_2-2r_1-1)!}{(a_1+a_2-2(r_1+r_2))!(a_3-1)!}+\frac{1}{(a_3-2r_2)!}\right]\\ &\qquad\vdots\\ \times&\left[\frac{(\sum_{t=1}^{l-2}a_t-2\sum_{t=1}^{l-3})!}{(\sum_{t=1}^{l-2}a_t-2\sum_{t=1}^{l-2})!(a_{l-1}-1)!}+\frac{1}{(a_{l-1}-2r_{l-2})!}\right]\\ \times&\Big(\sum_{t=1}^{l-1}a_t-2\sum_{t=1}^{l-2}r_t-1\Big)!\end{aligned}$$

Dividing the first numerator appearing in each $[\cdots]$ (including the last line) and then multiplying it on the $[\cdots]$ immediately proceeding it we finally arrive at the displayed formula (31), as desired. This finishes the proof of the proposition. □

Remark 5.2. Clearly every $\zeta_{\rm MT}$ in (32) is a colored 1-MTZV with only variable $z=s_\kappa$ dressed with $e(\alpha)$. Further, the depth of colored 1-MTZV is $\kappa-\ell(\mathbf{i})\le k-1$ since we assumed $\ell(\mathbf{i})\ge 2$.

Also notice that in (32) $\kappa\in[\kappa]\setminus\mathbf{i}$ for all $\mathbf{i}\in[k]$ so the 1-MTZVs are not special and therefore we cannot use Theorem 3.2 to reduce (31) further to colored 1-MZVs.

Theorem 5.3. *Let k be a positive integer ≥ 2 and $\mathbf{s}=(s_1,\dots,s_k)\in\mathbb{N}^k$. Let χ be a primitive Dirichlet character. Then*

$$(-1)^{k+|\mathbf{s}|}L_{\rm MT}(\mathbf{s},z;\{\mathbf{1}\}_k,\chi)+\sum_{j=1}^{k}(-1)^{s_j}L_{\rm MT}(R_j(z,\mathbf{s}),s_j;R_j(\chi,\{\mathbf{1}\}_{k+1}))$$
$$=\sum_{\mathbf{i}\subseteq[k],\ell(\mathbf{i})\geq 2}(-1)^{\ell(\mathbf{i})}E(\mathbf{s},\mathbf{i},\chi)\quad(36)$$

for all $z\in\mathbb{C}$ except at singular points where with the notation in Prop. 4.5

$$E(\mathbf{s},\mathbf{i},\chi)=\sum_{\mathbf{P}\in\mathcal{P}'(\mathbf{s}(\mathbf{i}))}\sum_{\mathbf{r}\in{\rm ind}'(\mathbf{P})}(-1)^{|\mathbf{s}(\mathbf{i})|}2^{\ell(\mathbf{i})-q}\prod_{j=1}^{q}\left\{\prod_{i=2}^{\ell(\mathbf{r}_j)+1}c_{j,i}(\mathbf{P},\mathbf{r})\right\}C_{j,\chi}(\mathbf{P},\mathbf{r})\qquad(37)$$

where c's are defined by (33) and $C_{j,\chi}$'s are define as follows. For $\mathbf{s},\mathbf{i},\chi$ as above and any positive integer n let $f_{\mathbf{s},\mathbf{i},\chi}(n):=L_{\rm MT}\big(\mathbf{s}([\kappa]\setminus\mathbf{i}),n;R_{k-\ell(\mathbf{i})+1}(\chi,\{\mathbf{1}\}_{k-\ell(\mathbf{i})+2})\big)$. Then $C_{j,\chi}(\mathbf{P},\mathbf{r})$ can be obtained by replacing α by χ in (32).

Proof. First we assume that $\Re(z)\geq 1$. For any $\alpha\in\mathbb{R}$ we can take all possible subset $\mathbf{i}$ of $[k]$ of length t and add (30) together for all these $\mathbf{i}$'s. Within this sum we find that each $S(\mathbf{s},\mathbf{j},\alpha)$ with length $\ell(\mathbf{j})=r\leq t$ appears exactly $\binom{k-r}{t-r}$ times. Since for every fixed r, $1\leq r\leq k-1$

$$\sum_{t=r}^{k}\binom{k-r}{t-r}(-1)^t=0$$

we see that

$$\sum_{\mathbf{i}\subseteq[k]}(-1)^{\ell(\mathbf{i})}\sum_{\mathbf{j}\subseteq\mathbf{i}}(-1)^{|\mathbf{s}(\mathbf{j})|}S(\mathbf{s},\mathbf{j},\alpha)=(-1)^{k+s_1+\cdots+s_k}\zeta_{\rm MT}(\mathbf{s};\{1\},\alpha).\qquad(38)$$

because the only term with $r=k$ is when $\mathbf{i}=\mathbf{j}=[k]$. Further, if we take $\ell(\mathbf{i})=1$ in (38) then we get

$$-\sum_{j=1}^{k}(-1)^{s_j}S(\mathbf{s},\{j\},\alpha)=-\sum_{j=1}^{k}(-1)^{s_j}\zeta_{\rm MT}(R_j(z,\mathbf{s}),s_j;R_j(\alpha,\{1\}_{k+1})).$$

Moving these terms from the LHS of (38) to the RHS we have:

$$(-1)^{k+s_1+\cdots+s_k}\zeta_{\rm MT}(\mathbf{s};\{1\}_k,\alpha)+\sum_{j=1}^{k}(-1)^{s_j}\zeta_{\rm MT}(R_j(z,\mathbf{s}),s_j;R_j(\alpha,\{1\}_{k+1}))$$
$$=\sum_{\mathbf{i}\subseteq[k],\ell(\mathbf{i})\geq 2}(-1)^{\ell(\mathbf{i})}\sum_{\mathbf{j}\subseteq\mathbf{i}}(-1)^{|\mathbf{s}(\mathbf{j})|}S(\mathbf{s},\mathbf{j},\alpha)=\sum_{\mathbf{i}\subseteq[k],\ell(\mathbf{i})\geq 2}(-1)^{\ell(\mathbf{i})}E(\mathbf{s},\mathbf{i},\alpha)\quad(39)$$

by Prop. 5.1. Note that for any $j\leq k$ and any primitive Dirichlet character χ of conductor f by [27, Lemma 4.7] we have

$$L_{\rm MT}(R_j(z,\mathbf{s}),s_j;R_j(\chi,\{\mathbf{1}\}_\kappa))=\sum_{n=1}^{f}\frac{\chi(n)}{\tau(\overline{\chi})}\zeta_{\rm MT}(R_j(z,\mathbf{s}),s_j;R_j(f/n,\{1\}_{k+1})).\qquad(40)$$

Replacing α in (39) by n/f, multiplying by $\chi(n)/\tau(\overline{\chi})$, and summing over $n=1,\dots,f$ we finally arrive at (7). We notice that the theorem is now proved under the assumption that

$\Re(z) \geq 1$. But we can easily remove this restriction by analytic continuation using Theorem 2.3. This completes the proof of Theorem 5.3. □

Remark 5.4. We can compute the LHS of (30) explicitly in a not too complicated form by Prop. 4.2. In (18) taking $n = n(\mathbf{s}, \mathbf{i}, \mathbf{j}, \mathbf{P}) := |\mathbf{s}(\mathbf{i})| - |\mathbf{j}| + \ell(\mathbf{v}) - \ell(\mathbf{i}) + 1$ and using (34) we have

$$\lim_{N\to\infty} \int_0^1 \prod_{j\in\mathbf{i}} f_{s_j,N}(x) \prod_{j\in[\kappa]\setminus\mathbf{i}} f^+_{s_j,N}(x)\,dx = \frac{(2\pi i)^{|\mathbf{s}(\mathbf{i})|}}{(-1)^{\ell(\mathbf{i})}\mathbf{s}(\mathbf{i})!} \cdot \left\{ \sum_{\mathbf{v}\subsetneq[\ell(\mathbf{i})]} \sum_{0\le j_\mathbf{v}\le s_\mathbf{v}} \binom{n-1}{\mathbf{s}(\mathbf{i}) - \mathrm{Inf}^{\ell(\mathbf{i})}_\mathbf{v}(\mathbf{j})} \frac{B_\mathbf{j}}{\mathbf{j}!} \frac{\mathbf{s}(\mathbf{i})!}{n} \int_0^1 B_n(x) \prod_{j\in[\kappa]\setminus\mathbf{i}} f^+_{s_j,\infty}(x)\,dx \right\}$$
$$= -\sum_{\mathbf{v}\subsetneq[\ell(\mathbf{i})]} \sum_{0\le j_\mathbf{v}\le s_\mathbf{v}} \frac{(2\pi i)^{|\mathbf{s}(\mathbf{i})|-n} n!}{(-1)^{\ell(\mathbf{i})+n}\mathbf{s}(\mathbf{i})!} \binom{n-1}{\mathbf{s}(\mathbf{i}) - \mathrm{Inf}^{\ell(\mathbf{i})}_\mathbf{v}(\mathbf{j})} \frac{B_\mathbf{j}}{\mathbf{j}!} \frac{\mathbf{s}(\mathbf{i})!}{n} \zeta_{\mathrm{MT}}\big(\mathbf{s}([\kappa]\setminus\mathbf{i}), n\big).$$

Note however, this formula is not enough to prove Theorem 1.3.

6. Some Corollaries and Examples

Because of the appearance of the odd powers of $2\pi i$ we don't get explicitly reduced form of (7) by using computations contained in Remark 5.4. We have to use the more involved Theorem 5.3. In Theorem 5.3 taking $k = 2$ we immediately get

Corollary 6.1. *Let $a, b \in \mathbb{N}$ and χ be any primitive Dirichlet character. Then*

$$L_{\mathrm{MT}}(a, b, z; \mathbf{1}, \mathbf{1}, \chi) + (-1)^b L_{\mathrm{MT}}(z, b, a; \chi, \mathbf{1}, \mathbf{1}) + (-1)^a L_{\mathrm{MT}}(a, z, b; \mathbf{1}, \chi, \mathbf{1})$$
$$= 2 \sum_{r=0}^{\lfloor \max\{a,b\}/2 \rfloor} \left[\binom{a+b-2r-1}{a-1} + \binom{a+b-2r-1}{a-2r} \right] \zeta(2r) L(a+b+z-2r; \chi) \quad (41)$$

for all complex number $z \in \mathbb{C}$ except at singular points.

This is in agreement with [16, Prop. 2.2] by Matsumoto et al. Note also that Tsumura's result [26, Theorem 4.5] should reduce to Cor. 6.1 with $\chi = \mathbf{1}$ (see [18, Theorem 1.2] and its remarks).

The depth $d = 3$ case is essentially the same as that of [16, Theorem 3.5]. We now look at depth $d = 4$. For any function $F(x_1, \dots, x_n)$ we define

$$\underset{x_1,\dots,x_n}{\boldsymbol{\pi}} F(x_1, \dots, x_n) = \sum_{j=1}^{n} F(x_1, \dots, \widehat{x_j}, \dots, x_n, x_j).$$

The following lemma will be used when we need to show strong reducibility result.

Lemma 6.2. *For any positive integers a, b, c and d we have*

$$\zeta_{\mathrm{MT}}(a, b, c) = \underset{a,b}{\boldsymbol{\pi}} \sum_{\nu=0}^{b-1} \binom{a+\nu-1}{\nu} \zeta(c+a+\nu, b-\nu).$$

and

$$\zeta_{\rm MT}(a,b,c,d) = \underset{a,b,c}{\boldsymbol{\pi}} \sum_{\nu_1=0}^{a-1}\sum_{\nu_2=0}^{b-1} \binom{\nu_1+\nu_2+c-1}{\nu_1,\nu_2,c-1}$$
$$\Bigg\{\sum_{\nu_3=0}^{a-\nu_1-1}\binom{b-\nu_2+\nu_3-1}{\nu_3}\zeta(c+d+\nu_1+\nu_2, b-\nu_2+\nu_3, a-\nu_1-\nu_3)$$
$$+\sum_{\nu_3=0}^{b-\nu_2-1}\binom{a-\nu_1+\nu_3-1}{\nu_3}\zeta(c+d+\nu_1+\nu_2, b-\nu_2-\nu_3, a-\nu_1+\nu_3).\Bigg\}$$

Proof. See the proof of [33, Theorem 5]. □

As the signed cyclic sum formula (7) has been derived in depth 3 in [16] we provide the depth 4 expression explicitly below.

Corollary 6.3. *Let* $\mathbf{s} = (s_1, s_2, s_3, s_4) = (a, b, c, d) \in \mathbb{N}^4$*. The signed cyclic sum of colored 1-MTZVs*

$$(-1)^{|\mathbf{s}|}\zeta_{\rm MT}(\mathbf{s}, z; \{1\}_4, \alpha) + \sum_{j=1}^{4}(-1)^{s_j}\zeta_{\rm MT}(R_j(z,\mathbf{s}), s_j; R_j(\alpha, \{1\}_4))$$
$$= \sum_{1\le i<j\le 4} E_2(\mathbf{s}, (i,j), \alpha) - \sum_{1\le i<j<k\le 4} E_3(\mathbf{s}, (i,j,k), \alpha) + E_4(\mathbf{s}, (1,2,3,4), \alpha)$$

is reducible for $z \in \mathbb{C}$ *except for singular points, where* E_ℓ $(\ell = 2, 3, 4)$ *are defined defined as follows:* $E_\ell(\mathbf{s}, (i_1, \dots, i_\ell), \alpha) = \pi_{\binom{1,\dots,t}{i_1,\dots,i_\ell}} E_\ell(\mathbf{s}, (1, \dots, \ell), \alpha)$ *(permuting the* s_j*'s) and*

$$\frac{E_2(\mathbf{s},(1,2),\alpha)}{2(-1)^{a+b}} = \underset{a,b}{\boldsymbol{\pi}} \sum_{r=0}^{\lfloor \max\{a,b\}/2\rfloor} \binom{a+b-2r-1}{b-1}\zeta(2r)\zeta_{\rm MT}(c,d,z,a+b-2r;1,1,\alpha,1),$$

$$\frac{E_3(\mathbf{s},(1,2,3),\alpha)}{2(-1)^{a+b+c}} = (-1)^b\tilde{\zeta}(a+b)\zeta_{\rm MT}(d,z,c;1,\alpha,1)$$
$$+2\cdot\underset{a,b}{\boldsymbol{\pi}} \sum_{\mu=0}^{\lfloor \max\{a,b\}/2\rfloor}\ \sum_{\nu=0}^{\lfloor \max\{a+b-2\mu,c\}/2\rfloor} \left[\binom{a+b-2\mu-1}{b-1}\binom{a+b+c-2\mu-2\nu-1}{c-1}\right.$$
$$\left.+\binom{a+b+c-2\mu-2\nu-1}{a-2\mu,\ b-1,\ c-2\nu}\right]\zeta(2\mu)\zeta(2\nu)\zeta_{\rm MT}(d,z,a+b+c-2\mu-2\nu;1,\alpha,1)$$

where $\tilde\zeta(n)=\zeta(n)$ if n is even and $\tilde\zeta(n)=0$ if n is odd. Finally, if $\mathbf{i}=[4]$ of length 4 then setting $\sigma=a+b+c+d$ we can get by (31)

$$\frac{E_4(\mathbf{s},(1,2,3,4),\alpha)}{4(-1)^{a+b+c+d}}=2\cdot\boldsymbol{\pi}_{a,b}\sum_{\mu=0}^{\lfloor\max\{a,b\}/2\rfloor}\sum_{\nu=0}^{\lfloor\max\{a+b-2\mu,c\}/2\rfloor}\sum_{\lambda=0}^{\lfloor\max\{a+b+c-2\mu-2\nu,d\}/2\rfloor}$$
$$\Bigg[\binom{a+b-2\mu-1}{b-1}\binom{a+b+c-2\mu-2\nu-1}{c-1}\binom{\sigma-2\mu-2\nu-2\lambda-1}{d-1}$$
$$+\binom{a+b+c-2\mu-2\nu-1}{a-2\mu,\ b-1,\ c-2\nu}\binom{\sigma-2\mu-2\nu-2\lambda-1}{d-1}$$
$$+\binom{a+b-2\mu-1}{b-1}\binom{\sigma-2\mu-2\nu-2\lambda-1}{a+b-2\mu-2\nu,\ c-1,\ d-2\lambda}$$
$$+\binom{\sigma-2\mu-2\nu-2\lambda-1}{a-2\mu,\ b-1,\ c-2\nu,\ d-2\lambda}\Bigg]\cdot\zeta(2\mu)\zeta(2\nu)\zeta(2\lambda)\zeta_{\mathrm{MT}}(z,\sigma-2(\mu+\nu+\lambda);\alpha,1)$$
$$+(-1)^b\tilde\zeta(a+b)\cdot\boldsymbol{\pi}_{c,d}\sum_{\mu=0}^{\lfloor\max\{c,d\}/2\rfloor}\binom{c+d-2\mu-1}{d-1}\zeta(2\mu)\zeta_{\mathrm{MT}}(z,c+d-2\mu;\alpha,1)$$
$$+(-1)^c\boldsymbol{\pi}_{a,b}\sum_{\mu=0}^{\lfloor\max\{a,b\}/2\rfloor}\binom{a+b-2\mu-1}{b-1}\zeta(2\mu)\tilde\zeta(a+b+c-2\mu)\zeta_{\mathrm{MT}}(z,d;\alpha,1)$$

Proof. This is follows from Theorem 5.3 easily. □

When $a=b=c=d=n$ setting $E_j=E(\mathbf{s},\{n\}_j,1)$ for $j=2,3,4$ we have

$$E_2=4\sum_{r=0}^{\lfloor n/2\rfloor}\binom{2n-2r-1}{n-1}\zeta(2r)\zeta_{\mathrm{MT}}(n,n,z,2n-2r;1,1,\alpha,1), \tag{42}$$

$$E_3=2\zeta(2n)\zeta_{\mathrm{MT}}(n,z,n)+8(-1)^n\cdot\sum_{\mu=0}^{\lfloor n/2\rfloor}\sum_{\nu=0}^{\lfloor\max\{2n-2\mu,n\}/2\rfloor}\Bigg[\binom{2n-2\mu-1}{n-1}\binom{3n-2\mu-2\nu-1}{n-1}$$
$$+\binom{3n-2\mu-2\nu-1}{n-2\mu,\ n-1,\ n-2\nu}\Bigg]\zeta(2\mu)\zeta(2\nu)\zeta_{\mathrm{MT}}(n,z,3n-2\mu-2\nu;1,\alpha,1) \tag{43}$$

$$E_4=16\cdot\sum_{\mu=0}^{\lfloor n/2\rfloor}\sum_{\nu=0}^{\lfloor\max\{2n-2\mu,n\}/2\rfloor}\sum_{\lambda=0}^{\lfloor\max\{3n-2\mu-2\nu,n\}/2\rfloor}$$
$$\Bigg[\binom{2n-2\mu-1}{n-1}\binom{3n-2\mu-2\nu-1}{n-1}\binom{4n-2\mu-2\nu-2\lambda-1}{n-1}$$
$$+\binom{3n-2\mu-2\nu-1}{n-2\mu,\ n-1,\ n-2\nu}\binom{4n-2\mu-2\nu-2\lambda-1}{n-1}$$
$$+\binom{2n-2\mu-1}{n-1}\binom{4n-2\mu-2\nu-2\lambda-1}{2n-2\mu-2\nu,\ n-1,\ n-2\lambda}$$
$$+\binom{4n-2\mu-2\nu-2\lambda-1}{n-2\mu,\ n-1,\ n-2\nu,\ n-2\lambda}\Bigg]\cdot\zeta(2\mu)\zeta(2\nu)\zeta(2\lambda)\zeta(z+4n-2(\mu+\nu+\lambda);\alpha)$$
$$+(-1)^n8\zeta(2n)\cdot\sum_{\mu=0}^{\lfloor n/2\rfloor}\binom{2n-2\mu-1}{n-1}\zeta(2\mu)\zeta(z+2n-2\mu;\alpha)$$
$$+8\sum_{\mu=0}^{\lfloor n/2\rfloor}\binom{2n-2\mu-1}{n-1}\zeta(2\mu)\tilde\zeta(3n-2\mu)\zeta(z+n;\alpha) \tag{44}$$

Thus we get

Corollary 6.4. *Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. Then signed sum of colored 1-MTZVs*

$$\zeta_{\rm MT}(\{n\}_4, z; \{1\}_4, \alpha) - 4\zeta_{\rm MT}(\{n\}_3, z, n; \{1\}_3, \alpha, 1) = 6E_2 - 4E_3 + E_4 \tag{45}$$

for all $z \in \mathbb{C}$ except at singular points, where E_2, E_3, and E_4 are defined by (42), (43), *and* (44), *respectively.*

Proof. This follow from Cor.6.3 since

$$\text{LHS of (45)} = \binom{4}{2}E_2 - \binom{4}{3}E_3 + E_4.$$

□

Corollary 6.5. *Let $\alpha \in \mathbb{R}$. For all $z \in \mathbb{C}$ the signed sum of 1-MTZVs*

$$\begin{aligned}4\zeta_{\rm MT}(\{1\}_3, z, 1; \{1\}_3, \alpha, 1) - \zeta_{\rm MT}(\{1\}_4, z; \{1\}_4, \alpha) = 12\zeta_{\rm MT}(1, 1, z, 2; \{1\}_3, \alpha, 1)\\ + 24\big[\zeta(2)\zeta_{\rm MT}(1, z, 1; 1, \alpha, 1) - \zeta_{\rm MT}(1, z, 3; 1, \alpha, 1) - \zeta(2)\zeta(z + 2; \alpha) + \zeta(z + 4; \alpha)\big],\end{aligned} \tag{46}$$

except at singular points.

Proof. Specializing (42) to (44) further to $n = 1$ we get:

$$\begin{aligned}E_2 =& - 2\zeta_{\rm MT}(1, 1, z, 2; 1, 1, \alpha, 1),\\ E_3 =& 6(\zeta(2)\zeta_{\rm MT}(1, z, 1; 1, \alpha, 1) - \zeta_{\rm MT}(1, z, 3; 1, \alpha, 1)),\\ E_4 =& 24(\zeta(2)\zeta_{\rm MT}(z + 2; \alpha) - \zeta_{\rm MT}(z + 4; \alpha)).\end{aligned}$$

So the corollary follows from Cor. 6.4 at once. □

Corollary 6.6. *For all $n \in \mathbb{N}$ the signed cyclic sum of MTZVs*

$$\begin{aligned}4\zeta_{\rm MT}(\{1\}_3, n, 1) - \zeta_{\rm MT}(\{1\}_4, n) = 12\Bigg\{ 2\zeta(n + 4) - 2\zeta(n + 3, 1) + 2\zeta(n + 2, 1, 1)\\ + 2\zeta(2)(\zeta(n+1, 1) - \zeta(n+2)) + \sum_{\nu=0}^{n-1}\sum_{\mu=0}^{n-1-\nu}\big[\zeta(3+\nu, 1+\mu, n-\nu-\mu) + \zeta(3+\nu, n-\nu-\mu, 1+\mu)\big]\\ + \sum_{\nu=0}^{n-1}\big[2\zeta(2)\zeta(2 + \nu, n - \nu) - 2\zeta(4 + \nu, n - \nu) + \zeta(3 + \nu, 1, n - \nu) + \zeta(3 + \nu, n - \nu, 1)\big]\Bigg\}\end{aligned}$$

is strongly reducible.

Proof. The corollary follows from Lemma 6.2 after specializing $z = n$ and $\alpha = 1$ in (46). □

Remark 6.7. Corollary 6.6 is consistent with [10, Cor. 4.2] when we take $n = 1$. In fact both sides are then equal to $72\zeta(5)$ since one can show by double shuffle relations of MVZs that $\zeta(4, 1) = \zeta(3, 1, 1)$ and $\zeta(4, 1) + \zeta(2)\zeta(3) = 2\zeta(5)$ (see [31, §3]).

Maple computation by Cor. 6.6 also shows that

$$\begin{aligned}\zeta_{\rm MT}(\{2\}_5) = \frac{12}{5}\Big\{2\zeta(2)\zeta_{\rm MT}(2, 2, 2, 2) + 24\zeta(2)\zeta_{\rm MT}(2, 2, 4) - 10\zeta(4)\zeta_{\rm MT}(2, 2, 2)\\ - 30\zeta_{\rm MT}(2, 2, 6) - 3\zeta_{\rm MT}(2, 2, 2, 4)\Big\} + 2\zeta(10)\end{aligned}$$

is reducible, which, by Lemma 6.2, can be further reduced to

$$2\zeta(10)-\frac{216}{5}\Big\{\zeta(6,2,2)+2\zeta(6,3,1)+2\zeta(7,1,2)+4\zeta(7,2,1)+6\zeta(8,1,1)\Big\}$$
$$-144\Big\{\zeta(8,2)+2\zeta(9,1)\Big\}-48\zeta(4)\Big\{6\zeta(4,2)+2\zeta(5,1)\Big\}$$
$$+\frac{144}{5}\zeta(2)\Big\{\zeta(4,2,2)+2\zeta(4,3,1)+2\zeta(5,1,2)+4\zeta(5,2,1)+6\zeta(6,1,1)+4\zeta(6,2)+8\zeta(7,1)\Big\}.$$

By parity consideration [12, Cor. 8] we know that the above can be reduced further to products of MZVs of depth one or two. In fact, after using double shuffle relations we find

$$\begin{aligned}\zeta_{\rm MT}(\{2\}_5)=&\frac{79}{5}\zeta(10)+12\Big\{15\zeta(8,2)+30\zeta(9,1)-12\zeta(2)\zeta(6,2)-24\zeta(2)\zeta(7,1)\\&-4\zeta(4)\zeta(4,2)-8\zeta(4)\zeta(5,1))\Big\}\\=&7\zeta(10)+36\Big\{5\zeta(8,2)+10\zeta(9,1)-4\zeta(2)\zeta(6,2)-8\zeta(2)\zeta(7,1)\Big\}\end{aligned}$$

since $\zeta(4,2)+2\zeta(5,1)=\zeta(6)/6$. We now can look up the table [5] and get

$$\zeta_{\rm MT}(\{2\}_5)=\frac{1376}{385}\zeta(2)^5+180\Big\{\zeta(8,2)-\zeta(5)^2-2\zeta(3)\zeta(7)\Big\}+144\Big\{2\zeta(2)\zeta(3)\zeta(5)-\zeta(2)\zeta(6,2)\Big\}$$

Remark 6.8. Note that only one depth 2 weight 10 term, namely $\zeta(8,2)$, appears in the reduction of $\zeta_{\rm MT}(\{2\}_5)$. By the table in [5] we know it's the only depth two weight 10 MZV in the basis (there are only 7 $\mathbb{Q}$-linearly independent MVZs of weight 10 by Zagier's conjecture). Moreover, by Broadhurst conjecture [7, (3)] this depth two value cannot be reduced further, hence neither can $\zeta_{\rm MT}(\{2\}_5)$.

We also have calculated $\zeta_{\rm MT}(\{2\}_5)$ by the method in Theorem 1.2. Using EZ-face we find our two methods produce the same value $\zeta_{\rm MT}(\{2\}_5)=.163501600521337009\dots$. Similarly we have also verified by two methods that

$$\begin{aligned}\zeta_{\rm MT}(\{2\}_6)=&\,1200\Big\{21\zeta(2)\zeta(5)^2+33\zeta(2)\zeta(8,2)+30\zeta(2)\zeta(3)\zeta(7)+12\zeta(8,2,1,1)-\zeta(3)^4\Big\}\\&+60\Big\{\frac{1056}{7}\zeta(2)^3\zeta(3)^2-4264\zeta(3)\zeta(9)-1068\zeta(10,2)-6627\zeta(5)\zeta(7)\Big\}\\&+7488\Big\{\zeta(5)\zeta(3)\zeta(2)^2+2\zeta(2)^2\zeta(6,2)\Big\}+\frac{13944719168}{525525}\zeta(2)^6=.15311508886\dots\end{aligned}$$

and $\zeta_{\rm MT}(\{3\}_6)=.01255766232\dots$. Note that only one depth 4 term, namely $\zeta(8,2,1,1)$, appears in the reduction of $\zeta_{\rm MT}(\{2\}_6)$. By reasons similar to those explained in Remark 6.8 we see that $\zeta_{\rm MT}(\{2\}_6)$ cannot be reduced further.

7. Convergence Problem

In the proof of Prop. 5.1 we need the following results which guarantees our exchange of limits there.

Proposition 7.1. *Let k be a positive integer. For all $(s_1,\dots,s_k)\in\mathbb{N}^k$ and $s_{k+1}\in\mathbb{C}$ with $\Re(s_{k+1})\ge 1$, the following series converges:*

$$S_{\mathbf{i}}(\mathbf{s})=\sum_{\substack{m_1,\dots,m_{k+1}\in\mathbb{N}^{k+1}\\ \sum_{j\in\mathbf{i}}m_j=\sum_{j\in[\kappa]\setminus\mathbf{i}}m_j}}\frac{1}{m_1^{s_1}m_2^{s_2}\cdots m_{k+1}^{s_{k+1}}}.$$

Proof. If $\ell(\mathbf{i})=1$ then this is the well known MTZV $\zeta_{\rm MT}(\{1\}_{k+1})$ which certainly converges. So we assume $\ell(\mathbf{i})=t+1\ge 2$. Then we only need to show the following series converges:

$$S:=\sum_{\substack{m_1,\dots,m_{k+1}\in\mathbb{N}^{k+1}\\ m_1+\dots+m_{t+1}=m_{t+2}+\dots+m_{k+1}}} \frac{1}{m_1m_2\cdots m_{k+1}}$$

Let $n=m_1+\dots+m_{t+1}=m_{t+2}+\dots+m_{k+1}$, $a_i=\sum_{j=1}^i m_i$, $b_i:=\sum_{j=t+2}^i m_j$. Then

$$S:=\sum_{n=1}^\infty \sum_{\substack{m_1,\cdots,m_k\in\mathbb{N}^k,\\ m_1+\dots+m_t<n,\\ m_{t+2}+\dots+m_k<n}} \frac{1}{m_1\cdots m_t(n-a_t)m_{t+2}\cdots m_k(n-b_k)}.$$

Repeatedly using partial fractions we have

$$\frac{1}{m_1\cdots m_t(n-a_t)}=\frac{1}{m_1\cdots m_{t-1}(n-a_{t-1})}\left(\frac{1}{m_t}+\frac{1}{n-a_t}\right)$$
$$=\dots=\frac{1}{n}\prod_{j=1}^t\left(\frac{1}{m_j}+\frac{1}{n-a_j}\right). \quad (47)$$

Similarly

$$\frac{1}{m_{t+2}\cdots m_k(n-b_k)}=\frac{1}{n}\prod_{j=t+2}^k\left(\frac{1}{m_j}+\frac{1}{n-b_j}\right). \quad (48)$$

Note that the summation in S can be written as

$$\sum_{n=1}^\infty\sum_{m_1=1}^n\sum_{m_2=1}^{n-a_1-1}\cdots\sum_{m_t=1}^{n-a_{t-1}-1}\sum_{m_{t+2}=1}^{n}\sum_{m_{t+3}=1}^{n-b_{t+2}-1}\cdots\sum_{m_k=1}^{n-b_{k-1}-1} \quad (49)$$

After expanding (47) and taking the summation like in (49) we see that there are 2^t products each of which has factors $\sum 1/m_j$ or $\sum 1/(n-a_j)$ (but not both for each j) for $j=1,\dots,t$. Starting from $j=t$ down to $j=1$ we now make change of the index $m_j\to n-a_j$ if and only if $1/(n-a_j)$ appears in the product. Now if $j<t$ this substitution will affect the l-th summation if and only if two conditions are satisfied: (i) $l>j$ and (ii) such change of index was not carried out for l-th summation, namely, it is still of the form $\sum_{m_l=1}^{n-a_l}(1/m_j)$. The effect is to change the l-th summation to $\sum_{m_l=1}^{m_j+a_j-a_l-1}(1/m_j)$. If this happens the l-th summation will not change anymore under substitutions for indices $m_1,\dots,m_{j-1}$. The upshot is, we can bound each j-th summation by $2\sum_{m_j=1}^n 1/m_j$ for $j=1,\dots,t$. After similarly treating (48) we see immediately that

$$S\le\sum_{n=1}^\infty\frac{4^{k-1}}{n^2}\left(\sum_{m=1}^n\frac{1}{m}\right)^{k-1}\ll\sum_{n=1}^\infty\frac{4^k\log^k(n)}{n^2}<\infty.$$

This finishes the proof of the proposition. □

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