Non–commutative geometry, dynamics, and ∞–adic Arakelov geometry

Caterina Consani* and Matilde Marcolli †

We dedicate this work to Yuri Manin, with admiration and gratitude

Abstract

In Arakelov theory a completion of an arithmetic surface is achieved by enlarging the group of divisors by formal linear combinations of the "closed fibers at infinity". Manin described the dual graph of any such closed fiber in terms of an infinite tangle of bounded geodesics in a hyperbolic handlebody endowed with a Schottky uniformization. In this paper we consider arithmetic surfaces over the ring of integers in a number field, with fibers of genus $g \geq 2$. We use Connes' theory of spectral triples to relate the hyperbolic geometry of the handlebody to Deninger's Archimedean cohomology and the cohomology of the cone of the local monodromy N at arithmetic infinity as introduced by the first author of this paper. First, we construct a spectral triple, where the non–commutative space is given by the reduced C^*–algebra of the Schottky group acting on the cohomology of the cone via a representation induced by the presence of a polarized Lefschetz module structure. In this setting we recover the alternating product of the Archimedean factors from a zeta function of the spectral triple. Then, we introduce a second spectral triple, which is related to Manin's description of the dual graph of the fiber at infinity. Here the non–commutative space is a C^*–algebra representing the "reduction mod infinity" and acting on a "dynamical homology and cohomology" pair, defined in terms of the bounded geodesics in the handlebody and of a dynamical system T. Morita equivalence plays an important role in the definition of the action. The operator Φ, that represents the "logarithm of a Frobenius–type operator" on the Archimedean cohomology, gives the Dirac operator on these spectral triples. The Archimedean cohomology embeds in the dynamical cohomology, compatibly with the action of a real Frobenius $\bar{F}_\infty$, so that the duality isomorphism on the cohomology of the cone of N corresponds to the pairing of dynamical homology and cohomology. This suggests the existence of a duality between the monodromy N and the dynamical map $1 - T$.

*Partially supported by NSERC grant 72016789

†Partially supported by Humboldt Foundation Sofja Kovalevskaja Award

Золотое руно, где же ты, золотое руно?
Всю дорогу шумели морские тяжелые волны,
и покинув корабль, натрудивший в морях полотно,
Одиссей возвратился, пространством и временем полный
(Осип Мандельштам)

Contents

1 Introduction.

The aim of this paper is to show how non–commutative geometry provides a connection between two constructions in Arakelov theory concerning the Archimedean fibers of a one-dimensional arithmetic fibration. On the one hand, we consider the cohomological construction introduced by the first author in [14], that was inspired by the theory of limiting mixed Hodge structures on the limit fiber of a geometric degeneration over a disc and it is related to Deninger's archimedean cohomology and regularized determinants (*cf.* [18]). On the other hand, we re–interpret Manin's geometric realization of the dual graph of the fiber at infinity as an infinite tangle of bounded geodesics inside a real 3–dimensional hyperbolic handlebody (*cf.* [24]) in the context of dynamical systems. The problem of relating the results of [24] with the cohomological constructions of Deninger was already addressed by Manin in [24], but to our knowledge no further progress in this direction was made since then.

Let $\mathbb{K}$ be a number field and let $O_{\mathbb{K}}$ be the ring of integers. The choice of a model $X_{O_{\mathbb{K}}}$ of a smooth, algebraic curve X over $\mathbb{K}$ defines an arithmetic surface over $\mathrm{Spec}(O_{\mathbb{K}})$. A closed vertical fiber of $X_{O_{\mathbb{K}}}$ over a prime $\wp$ in $O_{\mathbb{K}}$ is given by $X_{\wp}$: the reduction mod $\wp$ of the model. It is well known that a completion of the fibered surface $X_{O_{\mathbb{K}}}$ is achieved by adding to $\mathrm{Spec}(O_{\mathbb{K}})$ the *Archimedean places* represented by the set of all embeddings $\alpha : \mathbb{K} \hookrightarrow \mathbb{C}$. The Arakelov divisors on the completion $\overline{X_{O_{\mathbb{K}}}}$ are defined by the divisors on $X_{O_{\mathbb{K}}}$ and by formal real combinations of the closed vertical fibers at infinity. Arakelov's geometry does not provide an explicit description of these fibers and it prescribes instead a Hermitian metric on each Riemann surface $X_{/\mathbb{C}}$, for each archimedean prime α. It is quite remarkable that the Hermitian geometry on each $X_{/\mathbb{C}}$ is sufficient to develop an intersection theory on the completed model, without an explicit knowledge of the closed fibers at infinity. For instance, Arakelov showed that intersection indices of divisors on the fibers at infinity are obtained via Green functions on the Riemann surfaces $X_{/\mathbb{C}}$.

Inspired by Mumford's p-adic uniformization of algebraic curves [33], Manin realized that one could enrich Arakelov's metric structure by a choice of a Schottky uniformization. In this way, the Riemann surface $X_{/\mathbb{C}}$ is the boundary at infinity of a 3-dimensional hyperbolic handlebody $\mathfrak{X}_\Gamma$, described as the quotient of the real hyperbolic 3–space $\mathbb{H}^3$ by the action of the Schottky group Γ. The handlebody contains in its interior an infinite link of bounded geodesics, which are interpreted as the dual graph of the closed fiber at infinity, thus providing a first geometric realization of that space.

A consequence of this innovative approach is a more concrete intuition of the idea that, in Arakelov geometry, the "reduction modulo infinity" of an arithmetic variety should be thought of as "maximally degenerate" (or totally split: all components are of genus zero). This is, in fact, the reduction type of the special fiber admitting a Schottky uniformization (*cf.* [33]).

In this paper we consider the case of an arithmetic surface over $\mathrm{Spec}(O_{\mathbb{K}})$ where the fibers are of genus $g \geq 2$. The paper is divided into two parts.

The first part consists of Sections 2 and 3. Here we consider the formal construction of a cohomological theory for the "maximally degenerate" fiber at arithmetic infinity, developed in [14]. Namely, the Riemann surface $X_{/\mathbb{C}}$ supports a double complex $(K^{\cdot,\cdot}, d', d'')$ endowed with an endomorphism N. This complex is made by real differential forms with certain "weight-type" conditions on the indices and was constructed as an Archimedean analogue of the one defined by Steenbrink on the semi-stable fiber of a degeneration over a disc [49]. The hyper-cohomologies of $(K^{\cdot}, d = d' + d'')$ and $(\mathrm{Cone(N)}^{\cdot}, \mathrm{d})$ are infinite dimensional, graded real vector spaces. We show that their summands are isomorphic twisted copies of a same real de-Rham cohomology group of X. The arithmetic meaning of $K^{\cdot}$ arises from the fact that the cohomology of $(\mathrm{Coker(N)}^{\cdot}, \mathrm{d})$ computes the real Deligne cohomology of $X_{/\mathbb{C}}$, and the regularized determinant of an operator Φ on the subspace $\mathbb{H}^{\cdot}(K^{\cdot}, d)^{N=0}$ of the hyper-cohomology of the complex $(K^{\cdot}, d)$ recovers the Archimedean factors of [18]. The complex $K^{\cdot}$ carries an important structure of bigraded polarized Lefschetz module à la Deligne and Saito (*cf.* [43]). In particular one obtains an induced inner product on the hyper-cohomology and a representation of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$.

The first part of this paper concentrates on the cohomology $H^{\cdot}(X^*)$ of $Cone(N)$. In the classical case of a semi–stable degeneration over a disc, the cohomology $H^{\cdot}(|\mathcal{G}|)$ of the dual graph of the special fiber can be described in terms of graded pieces, under the monodromy filtration on the cohomology of the geometric generic fiber, so that $H^{\cdot}(|\mathcal{G}|)$ provides at least a partial information on the mixed Hodge structure on $H^{\cdot}(X^*)$ (here X^* denotes the complement of the special fiber in the model and the cohomology of this space has a second possible description as hypercohomology of the complex $\mathrm{Cone(N)}^{\cdot}$). In the arithmetic case of a degeneration on the ring of integers of a local field, the cohomology group $H^{\cdot}(X^*)$ is still endowed with a graded structure, which is fundamental in arithmetic for determining the behavior of the local Euler factors at integer points on the left of the critical strip on the real line. In fact, the cohomology $H^{\cdot}(X^*)$ carries more arithmetical information than just the cohomology of the dual graph of the special fiber $H^{\cdot}(|\mathcal{G}|)$: think for instance of the relation between the torsion subgroup of $gr_2^{\mathrm{w}} H^2(X^*)(1)$ and the Tamagawa number.

Using non–commutative geometry, we link the cohomology $H^{\cdot}(X^*)$ at arithmetic infinity as defined in [14] to the hyperbolic handlebody considered by Manin.

More precisely, we prove that the bigraded polarized Lefschetz module structure on the complex $(K^{\cdot}, d = d' + d'')$, together with the data of the Schottky uniformization of the Riemann surface $X(\mathbb{C})$ and of its hyperbolic filling $\mathfrak{X}_\Gamma$, induces a representation of the real (reduced) C^*–algebra $\mathrm{C}^*(\Gamma)$ of the Schottky group on the Hilbert space completion of $H^{\cdot}(X^*)$ with respect to the inner product defined by the polarization on $K^{\cdot}$. This data, together with the operator Φ that determines the Archimedean factors of [18], define the structure of a *spectral triple* $(\mathrm{C}^*(\Gamma), H^*(X^*), \Phi)$ à la Connes. In this theory, Φ has the role of a Dirac operator, hence it defines a metric structure over the non-commutative space $\mathrm{C}^*(\Gamma)$.

The non-commutative space $\mathrm{C}^*(\Gamma)$ has the same non-commutative topology (K-theory) of the handlebody $\mathfrak{X}_\Gamma$. This can be seen through strong Morita equivalence and a result of Kasparov [22] relating $C^*(\Gamma)$ to the ring of functions on the handlebody $\mathfrak{X}_\Gamma$. The hyperbolic geometry of the handlebody is encoded in the Beltrami differentials of Bers' simultaneous uniformization, which we use in order to construct a representation of $C^*(\Gamma)$ on $H^\cdot(X^*)$ via the representation of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ determined by the Lefschetz structure on $K^\cdot$.

This result allows us to reinterpret Deninger's regularized determinants describing the Archimedean factors in terms of an integration theory on the "non-commutative manifold" $(C^*(\Gamma), H^*(X^*), \Phi)$. Theorem 3.17 shows that the alternating product of the Γ-factors $L_{\mathbb{C}}(H^q(X_{/\mathbb{C}}, \mathbb{C}), s)$ is recovered from a particular zeta function of the spectral triple. In §3.4 we interpret this alternating product as a Reidemeister torsion associated to the fiber at arithmetic infinity.

The second part of the paper (Section 4 and 5) concentrates on Manin's description of the dual graph $\mathcal{G}$ of the "fiber at infinity" of an arithmetic surface in terms of the infinite tangle of bounded geodesics in the hyperbolic handlebody $\mathfrak{X}_\Gamma$.

More precisely, the suspension flow $\mathcal{S}_T$ of a dynamical system T provides our model of the dual graph $\mathcal{G}$ of the fiber at infinity, which maps surjectively over the tangle of bounded geodesics considered in [24]. The map T is a subshift of finite type which partially captures the dynamical properties of the action of the Schottky group on its limit set Λ_Γ.

The first cohomology group of $\mathcal{S}_T$ is the ordered cohomology of the dynamical system T, in the sense of [8] [34] and it provides a model of the first cohomology of the dual graph of the fiber at infinity. The group $H^1(\mathcal{S}_T)$ carries a natural filtration, which is related to the periodic orbits of the subshift of finite type. We give an explicit combinatorial description of homology and cohomology of $\mathcal{S}_T$ and of their pairing.

We define a *dynamical cohomology* H^1_{dyn} of the fiber at infinity as the graded space associated to the filtration of $H^1(\mathcal{S}_T)$. Similarly, we introduce a *dynamical homology* H_1^{dyn} as the sum of the spaces in the filtration of $H_1(\mathcal{S}_T)$. These two graded spaces have an involution which plays a role analogous to the real Frobenius $\bar{F}_\infty$ on the cohomological theories of Section 2.

Theorem 5.5 relates the dynamical cohomology to the Archimedean cohomology by showing that the Archimedean cohomology sits as a particular subspace of the dynamical cohomology in a way that is compatible with the grading and with the action of the real Frobenius. The map that realizes this identification is obtained using the description of holomorphic differentials on the Riemann surface as Poincaré series over the Schottky group, which also plays a fundamental role in the description of the Green function in terms of geodesics in [24]. Similarly, in Theorem 5.10 we identify a subspace of the dynamical homology that is isomorphic to the image of the Archimedean cohomology under the duality isomorphism acting on $\mathbb{H}^\cdot(\mathrm{Cone}^\cdot(N))$. This way we reinterpret this arithmetic duality as induced by the pairing of dynamical homology and cohomology.

In Proposition 4.19 we show that the Cuntz–Krieger algebra $\mathcal{O}_A$ associated to the subshift of finite type T (*cf.* [15] [16]) acts on a completion of the cohomology $H^1(\mathcal{S}_T)$. This algebra carries a refined information on the action of the Schottky group on its limit set, and it induces an action by bounded operators on a Hilbert space completion $\mathcal{H}^1_{dyn}$ of the dynamical cohomology. In Proposition 5.13 we show that this representation extends to a representation of the C*-algebra $\mathcal{A} = (\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$. As a non-commutative space, this algebra provides an analog at arithmetic infinity of the p-adic reduction map considered in [24] and [33], which is realized in terms of certain finite graphs in a quotient of the Bruhat-Tits building.

Similarly, in Proposition 5.16 we construct a representation, on a Hilbert space completion $\mathcal{H}_1^{dyn}$ of the dynamical homology, of a C*-algebra strongly Morita equivalent to $\mathcal{A}$, which is the algebra of sections of a flat $\mathrm{C}(\Lambda_\Gamma)$-bundle over the handlebody $\mathfrak{X}_\Gamma$.

We introduce a Dirac operator D on the Hilbert space $\mathcal{H}^1_{dyn} \oplus \mathcal{H}_1^{dyn}$, whose restriction to the subspaces isomorphic to the Archimedean cohomology and its dual, recovers the Frobenius-type operator Φ of Section 2.

Using Connes' technique of transferring spectral geometry between Morita equivalent non-commutative spaces, we obtain a second spectral triple, which we call *dynamical spectral triple*, over the non-commutative space $\mathcal{A}$ acting on the Hilbert space $\mathcal{H}^1_{dyn} \oplus \mathcal{M} \otimes \mathcal{H}_1^{dyn}$, where $\mathcal{M}$ is the bimodule implementing the Morita equivalence.

In the last section of the paper, we outline some possible further questions and directions for future investigations.

Since this paper draws from the language and techniques of different fields (arithmetic geometry, non-commutative geometry, dynamical systems), we thought it necessary to include enough background material to make the paper sufficiently self contained and addressed to readers with different research interests.

Acknowledgments. In the course of this project we learned many things from different people to whom we are very grateful: to Paolo Aluffi for many useful suggestions for §4.3, to Alain Connes for beautiful lectures on spectral triples and noncommutative geometry, to Curt McMullen for very enlightening conversations on the dynamics of the shift T and on Schottky groups, to Victor Nistor for various remarks on cross product algebras, to Mika Seppälä for useful information on real algebraic curves, and of course to Yuri Manin for sharing his vision and enlightenment on many aspects of this project.

1.1 Preliminary notions and notation

The three-dimensional real hyperbolic space $\mathbb{H}^3$ is the quotient

$$\mathbb{H}^3 = \mathrm{PGL}(2, \mathbb{C})/\mathrm{SU}(2). \tag{1.1}$$

It can also be described as the upper half space $\mathbb{H}^3 \simeq \mathbb{C} \times \mathbb{R}^+$ endowed with the hyperbolic metric.

The group $\mathrm{PSL}(2, \mathbb{C})$ is the group of orientation preserving isometries of $\mathbb{H}^3$. The action is given by

$$\gamma : (z, y) \mapsto \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}y^2}{|cz+d|^2 + |c|^2 y^2}, \frac{y\,|ad-bc|}{|cz+d|^2 + |c|^2 y^2} \right), \tag{1.2}$$

for $(z, y) \in \mathbb{C} \times \mathbb{R}^+$ and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

The complex projective line $\mathbb{P}^1(\mathbb{C})$ can be identified with the conformal boundary at infinity of $\mathbb{H}^3$. The action (1.2) extends to an action on $\overline{\mathbb{H}}^3 := \mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$, where $\mathrm{PSL}(2, \mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations

$$\gamma : z \mapsto \frac{(az+b)}{(cz+d)}.$$

We begin by recalling some classical facts about Kleinian and Fuchsian groups (*cf.* [4] [7] [31]).

A *Fuchsian group* G is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$, the group of orientation preserving isometries of the hyperbolic plane $\mathbb{H}^2$. A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$, the group of orientation preserving isometries of three-dimensional real hyperbolic space $\mathbb{H}^3$.

For $g \geq 1$, a *Schottky group* of rank g is a discrete subgroup $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$, which is purely loxodromic and isomorphic to a free group of rank g. Schottky groups are particular examples of Kleinian groups.

A Schottky group that is specified by real parameters so that it lies in $\mathrm{PSL}(2, \mathbb{R})$ is called a *Fuchsian Schottky group*. Viewed as a group of isometries of the hyperbolic plane $\mathbb{H}^2$, or equivalently of the Poincaré disk, a Fuchsian Schottky group G produces a quotient $G\backslash\mathbb{H}^2$ which is topologically a Riemann surface with boundary.

In the case $g = 1$, the choice of a Schottky group $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ amounts to the choice of an element $q \in \mathbb{C}^*$, $|q| < 1$. This acts on $\mathbb{H}^3$ by

$$\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} (z, y) = (qz, |q|y).$$

One sees that $\mathfrak{X} = \mathbb{H}^3/(q^{\mathbb{Z}})$ is a solid torus with the elliptic curve $X_{/\mathbb{C}} = \mathbb{C}^*/(q^{\mathbb{Z}})$ as its boundary at infinity. This space is known in the theory of quantum gravity as Euclidean BTZ black hole [28].

In general, for $g \geq 1$, the quotient space

$$\mathfrak{X}_\Gamma := \Gamma\backslash\mathbb{H}^3 \tag{1.3}$$

is topologically a handlebody of genus g. These also form an interesting class of Euclidean black holes (*cf.* [28]).

We denote by Λ_Γ, the *limit set* of the action of Γ. This is the smallest non–empty closed Γ–invariant subset of $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$. Since Γ acts freely and properly discontinuously on $\mathbb{H}^3$, the set Λ_Γ is contained in the sphere at infinity $\mathbb{P}^1(\mathbb{C})$. This set can also be described as the closure of the set of the attractive and repelling fixed points $z^\pm(g)$ of the loxodromic elements $g \in \Gamma$. In the case $g = 1$ the limit set consists of two points, but for $g \geq 2$ the limit set is usually a fractal of some Hausdorff dimension $0 \leq \delta = \dim_H(\Lambda_\Gamma) < 2$.

We denote by Ω_Γ the *domain of discontinuity* of Γ, that is, the complement of Λ_Γ in $\mathbb{P}^1(\mathbb{C})$. The quotient

$$X_{/\mathbb{C}} = \Gamma \backslash \Omega_\Gamma \tag{1.4}$$

is a Riemann surface of genus g and the covering $\Omega_\Gamma \to X_{/\mathbb{C}}$ is called a *Schottky uniformization* of $X_{/\mathbb{C}}$. Every complex Riemann surface $X_{/\mathbb{C}}$ admits a Schottky uniformization.

The handlebody (1.3) can be compactified by adding the conformal boundary at infinity $X_{/\mathbb{C}}$ to obtain

$$\overline{\mathfrak{X}}_\Gamma := \mathfrak{X}_\Gamma \cup X_{/\mathbb{C}} = \Gamma \backslash (\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})). \tag{1.5}$$

Let $\{g_i\}_{i=1}^g$ be a set of generators of the Schottky group Γ. We write $g_{i+g} = g_i^{-1}$. There are $2g$ Jordan curves γ_k on the sphere at infinity $\mathbb{P}^1(\mathbb{C})$, with pairwise disjoint interiors D_k, such that the elements g_k are given by fractional linear transformations that map the interior of γ_k to the exterior of γ_j with $|k - j| = g$. The curves γ_k give a *marking* of the Schottky group.

The choice of a Schottky uniformization for the Riemann surface $X_{/\mathbb{C}}$ provides a choice of a set of generators a_i, $i = 1, \dots g$, for $Ker(I_*)$, where $I_* : H_1(X_{/\mathbb{C}}, \mathbb{Z}) \to H_1(\mathfrak{X}_\Gamma, \mathbb{Z})$ is the map induce by the inclusion of $X_{/\mathbb{C}}$ in $\overline{\mathfrak{X}}_\Gamma$ as the conformal boundary at infinity. The a_i are the images under the quotient map $\Omega_\Gamma \to X_{/\mathbb{C}}$ of the Jordan curves γ_i.

Recall that, if $\mathbb{K}$ is a number field with $n = [\mathbb{K} : \mathbb{Q}]$, there are n archimedean primes which correspond to the embeddings $\alpha : \mathbb{K} \hookrightarrow \mathbb{C}$. Among these n archimedean primes, there are r embeddings into $\mathbb{R}$, and s pairs of conjugate embeddings in $\mathbb{C}$ not contained in $\mathbb{R}$, so that $n = r + 2s$.

If X is an arithmetic surface over $\mathrm{Spec}(O_\mathbb{K})$, then at each archimedean prime we obtain a Riemann surface $X_{/\mathbb{C}}$. If the archimedean prime corresponds to a real embedding, the corresponding Riemann surface has a real structure, namely it is a smooth real algebraic curve $X_{/\mathbb{R}}$.

A smooth real algebraic curve $X_{/\mathbb{R}}$ is a Riemann surface $X_{/\mathbb{R}}$ together with an involution $\iota : X_{/\mathbb{R}} \to X_{/\mathbb{R}}$ induced by complex conjugation $z \mapsto \bar{z}$. The fixed point set X_ι of the involution is the set of real points $X_\iota = X_{/\mathbb{R}}(\mathbb{R})$ of $X_{/\mathbb{R}}$. If $X_\iota \neq \emptyset$, the components of X_ι are simple closed geodesics on $X_{/\mathbb{R}}$. A smooth real algebraic curve is called *orthosymmetric* if $X_\iota \neq \emptyset$ and the complement $X_{/\mathbb{R}} \backslash X_\iota$ consists of two connected components. If $X_\iota \neq \emptyset$, then $X_{/\mathbb{R}}$ can always be reduced to the orthosymmetric case upon passing to a double cover.

Even when not explicitly stated, all Hilbert spaces and algebras of operators we consider will be *separable*, *i.e.* they admit a dense (in the norm topology) countable subset.

An *involutive algebra* is an algebra over $\mathbb{C}$ with a conjugate linear involution $*$ (the adjoint) which is an anti-isomorphism. A C^**-algebra* is an involutive normed algebra, which is complete in the norm, and satisfies $\|ab\| \leq \|a\| \cdot \|b\|$ and $\|a^*a\| = \|a\|^2$. The analogous notions can be defined for algebras over $\mathbb{R}$.

The condition of strong Morita equivalence between two C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ means that there exists a bimodule $\mathcal{M}$, which is a right Hilbert module over $\mathcal{A}_1$ with an $\mathcal{A}_1$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}_1}$. and a left Hilbert module over $\mathcal{A}_2$ with an $\mathcal{A}_2$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}_2}$, such that the following properties hold:

1. The closure of the linear span of
$$\{\langle \xi_1, \xi_2 \rangle_{\mathcal{A}_i} : \xi_1, \xi_2 \in \mathcal{M}\}$$
gives all of $\mathcal{A}_i$, for both $i = 1, 2$.

2. The two inner products satisfy
$$\langle \xi_1, \xi_2 \rangle_{\mathcal{A}_1} \xi_3 = \xi_1 \langle \xi_2, \xi_3 \rangle_{\mathcal{A}_2}.$$
for all $\xi_1, \xi_2, \xi_3 \in \mathcal{M}$.

3. The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ act on $\mathcal{M}$ by bounded operators, namely
$$\langle a_2\xi, a_2\xi \rangle_{\mathcal{A}_1} \leq \|a_2\|^2 \langle \xi, \xi \rangle_{\mathcal{A}_1} \qquad \langle a_1\xi, a_1\xi \rangle_{\mathcal{A}_2} \leq \|a_1\|^2 \langle \xi, \xi \rangle_{\mathcal{A}_2}$$
for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$, $\xi \in \mathcal{M}$.

A Hilbert module $\mathcal{M}$ for a C^*-algebra $\mathcal{A}$ is finite projective if there exists an idempotent P in the matrix algebra $M_N(\mathcal{A})$, $P^2 = P$, such that $\mathcal{M} = P\mathcal{A}^N$.

2 Cohomological Constructions.

In this chapter we give an explicit description of a cohomological theory for the archimedean fiber of an Arakelov surface. The general theory, valid for any arithmetic variety, was defined in [14]. This construction provides an alternative definition and a refinement for the *archimedean cohomology* H^*_{ar} introduced by Deninger in [18]. The spaces $H^\cdot(\tilde{X}^*)$ (*cf.* Definition 2.8) are *infinite dimensional* real vector spaces endowed with a monodromy operator N and an endomorphism Φ (*cf.* Section 2.5). The groups $H^*_{\rm ar}$ can be identified with the subspace of the N-invariants (*i.e.* Ker(N)) over which (the restriction of) Φ acts in the following way. The monodromy operator determines an integer, even graduation on $H^\cdot(\tilde{X}^*) = \oplus_{p\in\mathbb{Z}} gr^{\rm w}_{2p} H^\cdot(\tilde{X}^*)$ where each graded piece is still *infinite dimensional*. We will refer to it as to the *weight graduation*. This graduation induces a corresponding one on the subspace $H^\cdot(\tilde{X}^*)^{N=0} := \oplus_{\cdot\geq 2p} gr^{\rm w}_{2p} H^\cdot(X^*)$. The summands $gr^{\rm w}_{2p} H^\cdot(X^*)$ are *finite dimensional* real vector spaces on which Φ acts as a multiplication by the weight p.

When $X_{/\kappa}$ is a non-singular, projective curve defined over $\kappa = \mathbb{C}$ or $\mathbb{R}$, the description of $gr^{\rm w}_{2p} H^\cdot(X^*)$ $(\cdot \geq 2p)$ is particularly easy. Proposition 2.23 shows that $H^\cdot(\tilde{X}^*)^{N=0}$ is isomorphic to an infinite direct sum of Hodge-Tate twisted copies of the same finite-dimensional vector space. For $\kappa = \mathbb{C}$, this space coincides with the de-Rham cohomology $H^*_{DR}(X_{/\mathbb{C}}, \mathbb{R})$ of the Riemann surface $X_{/\mathbb{C}}$.

For the reader acquainted with the classical theory of mixed Hodge structures for an algebraic degeneration over a disk (and its arithmetical counterpart theory of Frobenius weights), it will be immediately evident that the construction of the arithmetical cohomology defined in this chapter runs in parallel with the classical one defined by Steenbrink in [49] and refined by M. Saito in [43]. The notation: $H^\cdot(\tilde{X}^*)$, $H^\cdot(X^*)$, $H^\cdot(Y)$ followed in this section is purely formal. Namely, $\tilde{X}^*$, X^* and Y are only symbols although this choice is motivated by the analogy with Steenbrink's construction in which $\tilde{X}^*$, X^* and Y describe *resp.* the geometric generic fiber and the complement of the special fiber Y in the model. The space $H^\cdot(\tilde{X}^*)$ is the hypercohomology group of a double complex $K^{\cdot,\cdot}$ of real, differential twisted forms (*cf.* Section 2.1: (2.1)) on which one defines a structure of polarized Lefschetz module that descends to its hypercohomology (*cf.* Theorem 2.6 and Corollary 2.7).

The whole theory is inspired by the expectation that the fibers at infinity of an arithmetic variety should be thought to be semi-stable and more specifically to be 'maximally degenerate or totally split'. We like to think that the construction of the complex $K^{\cdot,\cdot}$ on the Riemann surface $X_{/\kappa}$, whose structure and behavior gives the arithmetical information related to the 'mysterious' fibers at infinity of an arithmetic surface, fits in with Arakelov's intuition that Hermitian geometry on $X_{/\kappa}$ is enough to recover the intersection geometry on the fibers at infinity.

2.1 A bigraded complex with monodromy and Lefschetz operators

Let $X_{/\kappa}$ be a smooth, projective curve defined over $\kappa = \mathbb{C}$ or $\mathbb{R}$. For $a, b \in \mathbb{N}$, we shall denote by $(A^{a,b} \oplus A^{b,a})_{\mathbb{R}}$ the abelian group of real differential forms (analytic or C^∞) on $X_{/\kappa}$ of type $(a,b) + (b,a)$.

For $p \in \mathbb{Z}$, the expression $(A^{a,b} \oplus A^{b,a})_{\mathbb{R}}(p)$ means the p-th Hodge-Tate twist of $(A^{a,b} \oplus A^{b,a})_{\mathbb{R}}$, *i.e.*

$$(A^{a,b} \oplus A^{b,a})_{\mathbb{R}}(p) := (2\pi\sqrt{-1})^p (A^{a,b} \oplus A^{b,a})_{\mathbb{R}}.$$

Let $i, j, k \in \mathbb{Z}$. We consider the following complex (*cf.* [14], §4 for the general construction)

$$K^{i,j,k} = \begin{cases} \bigoplus\limits_{\substack{a+b=j+1 \\ |a-b|\leq 2k-i}} (A^{a,b} \oplus A^{b,a})_{\mathbb{R}}(\frac{1+j-i}{2}) & \text{if } 1+j-i \equiv\ 0(2),\ k \geq \max(0,i) \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

On the complex $K^{i,j,k}$ one defines the following differentials

$$d' : K^{i,j,k} \to K^{i+1,j+1,k+1}; \qquad d'' : K^{i,j,k} \to K^{i+1,j+1,k}$$
$$d' = \partial + \overline{\partial} \qquad d'' = \sqrt{-1}(\overline{\partial} - \partial).$$

These maps satisfy the property that $d'^2 = 0 = d''^2$ (*cf.* [14] Lemma 4.2). Since $X_{/\kappa}$ is a projective variety (hence Kähler) one uses the existence of the fundamental real (closed) $(1,1)$-form ω to define the following *Lefschetz map* l. The operator N that is described in the next formula plays the role of the logarithm of the *local monodromy at infinity*

$$N : K^{i,j,k} \to K^{i+2,j,k+1}, \quad N(f) = (2\pi\sqrt{-1})^{-1} f \tag{2.2}$$

$$l : K^{i,j,k} \to K^{i,j+2,k}, \quad l(f) = (2\pi\sqrt{-1}) f \wedge \omega \tag{2.3}$$

These endomorphisms are known to commute with d' and d'' and satisfy $[l, N] = 0$ (*cf. op.cit.* Lemma 4.2). One sets $K^{i,j} = \oplus_k K^{i,j,k}$ and writes $K^* = \oplus_{i+j=*} K^{i,j}$ to denote the simple complex endowed with the total differential $d = d' + d''$ and with the action of the operators N and l.

Remark 2.1 In the complex (2.1) the second index j is subject to the constraint $a + b = j + 1$ (where $a + b$ is the total degree of the differential forms). This implies that j assumes only a finite number of values: $-1 \leq j \leq 1$, in fact $0 \leq a + b \leq 2$ (X is a Riemann surface).

2.2 A polarized Hodge–Lefschetz structure

In this paragraph we will review the theory of polarized bigraded Hodge-Lefschetz modules due to Deligne and Saito. The main result is Theorem 2.6 which states that the complex $K^{\cdot,\cdot}$ defined in (2.1) together with the maps N and l as in (2.2) and (2.3) determine a Lefschetz module. A detailed description of the structure of polarized Hodge-Lefschetz modules is contained in [43]; for a short and quite pleasant exposition we refer to [21].

Definition 2.2 *A bigraded Lefschetz module* $(K^{\cdot\cdot}, L_1, L_2)$ *is a bigraded real vector space* $K = \oplus_{i,j} K^{i,j}$ *with endomorphisms*

$$L_1 : K^{i,j} \to K^{i+2,j} \qquad L_2 : K^{i,j} \to K^{i,j+2} \tag{2.4}$$

satisfying $[L_1, L_2] = 0$. *Furthermore, the operators* L_i *are required to satisfy the following conditions*

1. $L_1^i : K^{-i,j} \to K^{i,j}$ *is an isomorphism for* $i > 0$
2. $L_2^j : K^{i,-j} \to K^{i,j}$ *is an isomorphism for* $j > 0$.

Bigraded Lefschetz modules correspond to representations of the Lie group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ (*cf.* [21] or [43]). Therefore, given a bigraded Lefschetz module $(K^{\cdot\cdot}, L_1, L_2)$ this corresponds to the representation

$$\sigma : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}(K^{\cdot\cdot})$$

satisfying

$$\sigma \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \right\} (x) = a^i b^j x \quad \text{for } x \in K^{i,j} \tag{2.5}$$

$$d\sigma \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0 \right\} = L_1 \tag{2.6}$$

$$d\sigma \left\{ 0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} = L_2. \tag{2.7}$$

The Weyl reflection

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

defines the elements $\tilde{w} = \{w, w\}$, $w_1 = \{w, 1\}$, $w_2 = \{1, w\} \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. They determine isomorphisms $\sigma(\tilde{w}) : K^{i,j} \to K^{-i,-j}$, $\sigma(w_1) : K^{i,j} \to K^{-i,j}$ and $\sigma(w_2) : K^{i,j} \to K^{i,-j}$.

Definition 2.3 *A bigraded Lefschetz module is a Hodge–Lefschetz module if each* $K^{i,j}$ *carries a pure real Hodge structure and the* L_i *(as in Definition 2.2) are morphisms of real Hodge structures.*

For convenience, we recall the definition of a *pure Hodge structure* over $\kappa = \mathbb{C}$ or $\mathbb{R}$. For a summary of mixed Hodge theory we refer to [50].

Definition 2.4 *A pure Hodge structure H over κ is a finite dimensional $\mathbb{C}$-vector space $H = \oplus_{p,q} H^{p,q}$, together with a conjugate linear involution c and in case $\kappa = \mathbb{R}$ a $\mathbb{C}$-linear involution F_∞ such that*

1. *$c(H^{p,q}) = H^{q,p}$*
2. *the inclusion of $H_{\mathbb{R}} := H^{c=id}$ into H induces an isomorphism $H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$*
3. *in case $\kappa = \mathbb{R}$, F_∞ commutes with c and verifies $F_\infty(H^{p,q}) = H^{q,p}$. The action of F_∞ on the space $H^{p,p}$ decomposes it as $H^{p,p} = H^{p,+} \oplus H^{p,-}$. We denote the dimensions of the eigenspaces by $h^{p,\pm} := \dim_{\mathbb{C}} H^{p,\pm(-1)^p}$.*

   In the case $\kappa = \mathbb{R}$, H is called a real, pure Hodge structure.

Example 2.5 An example of pure Hodge structure is given by the singular (Betti) cohomology $H^*_B(X_{/\mathbb{C}}, \mathbb{C})$ on the Riemann surface $X_{/\mathbb{C}}$. The $\mathbb{C}$-linear involution F_∞ is induced by the complex conjugation on the Riemann surface.

On a bigraded Lefschetz module $(K^{\cdot,\cdot}, L_1, L_2)$ we consider the additional data of a differential d and a pairing ψ:

$$d : K^{i,j} \to K^{i+1,j+1}, \qquad \psi : K^{-i,-j} \otimes K^{i,j} \to \mathbb{R}(n), \qquad n \in \mathbb{N}$$

satisfying the following properties:

1. $d^2 = 0 = [L_i, d]$
2. $\psi(x, y) = (-1)^n \psi(y, x)$
3. $\psi(dx, y) = \psi(x, dy)$
4. $\psi(L_i x, y) + \psi(x, L_i y) = 0$
5. $\psi(\cdot, L_1^i L_2^j \cdot)$ is symmetric and positive definite on $K^{-i,-j} \cap \mathrm{Ker}(\mathrm{L}_1^{i+1}) \cap \mathrm{Ker}(\mathrm{L}_2^{j+1})$.

If (K, L_1, L_2, ψ) is a polarized bigraded Lefschetz module (*i.e.* (K, L_1, L_2) is a bigraded Lefschetz module satisfying the properties 1.-5.), then the bilinear form

$$\langle \cdot, \cdot \rangle : K \otimes K \to \mathbb{R}(n), \qquad \langle x, y \rangle := \psi\left(x, \sigma(\tilde{w}) y\right) \tag{2.8}$$

is symmetric and positive definite.

Theorem 2.6 *The differential complex $K^{\cdot,\cdot}$ defined in* (2.1) *endowed with the operators $L_1 = N$ (cf.* (2.2)*) and $L_2 = l$ (cf.* (2.3)*) is a polarized bigraded Lefschetz module. The polarization is given by*

$$\psi : K^{-i,-j,k} \otimes K^{i,j,k+i} \to \mathbb{R}(1)$$

$$\psi(x, y) := \left(\frac{1}{2\pi\sqrt{-1}} \right) \epsilon(n - j)(-1)^k \int_{X(\mathbb{C})} x \wedge Cy.$$

Here, for $m \in \mathbb{Z}$: $\epsilon(m) := (-1)^{\frac{m(m+1)}{2}}$ and $C(x) := (\sqrt{-1})^{a-b} x$ is the Weil operator, for x a differential form of type (a, b) (cf. [53] §V.1).

Proof. We refer to [14] Lemmas 4.2, 4.5, 4.6 and Proposition 4.7). ◇

Such elaborate construction on the complex $K^{\cdot,\cdot}$ allows one to set up a harmonic theory as in [14] pp. 350-1, so that the polarized bigraded Lefschetz module structure passes to the hypercohomology $\mathbb{H}^*(K^\cdot, d)$. More precisely, one defines a Laplace operator on $K^{\cdot,\cdot}$ as

$$\square := d({}^t d) + ({}^t d)d$$

where ${}^t d$ is the transpose of d relative to the bilinear form $\langle\cdot,\cdot\rangle$ defined in (2.8). Then, $\square$ commutes with the action of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ (*cf.* [53] Lemma at p. 153). Using the properties of the bilinear form $\langle\cdot,\cdot\rangle$ one gets

$$\mathbb{H}^*(K^\cdot, d) = \mathrm{Ker(d)} \cap \mathrm{Ker({}^t d)} = \mathrm{Ker}(\square)$$

and $\square$ is invariant for the action of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. The following result holds

Corollary 2.7 *The data $(\mathbb{H}^*(K^\cdot, d), N, l, \psi)$ define a polarized, bigraded Hodge-Lefschetz module.*

Proof. The statement follows from the isomorphism of complexes

$$K^\cdot \simeq \mathrm{Ker}(\square) \oplus \mathrm{Image}(\square)$$

and from the facts that $d = 0$ on $\mathrm{Ker}(\square)$ and that the complex $\mathrm{Image}(\square)$ is d-acyclic. These three statements taken together imply the existence of an induced action of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ on the hypercohomology of $K^\cdot$ (*cf.* [21] for details). ◇

2.3 Cohomology groups

It follows from the definition of the double complex $(K^{\cdot,\cdot}, d', d'')$ in (2.1) that the total differential $d = d' + d''$ satisfies $d^2 = 0$ and commutes with the operator N. In particular, d induces a differential on the graded groups

$$\mathrm{Ker(N)}^{\cdot,\cdot} = \mathrm{ker(N : K^{\cdot,\cdot} \to K^{\cdot+2,\cdot})}, \qquad \mathrm{Coker(N)}^{\cdot,\cdot} = \mathrm{coker(N : K^{\cdot,\cdot} \to K^{\cdot+2,\cdot})}$$

as well as on the mapping cone of N

$$\mathrm{Cone(N)}^{\cdot,\cdot} = \mathrm{Cone(N : K^{\cdot,\cdot} \to K^{\cdot+2,\cdot})} := \mathrm{K^{\cdot,\cdot}[1] \oplus K^{\cdot+2,\cdot}}, \quad \mathrm{D(a,b) = (-d(a), N(a) + d(b))}$$

Definition 2.8 *For any non-negative integer q and $p \in \mathbb{Z}$, define*

$$gr^w_{2p} H^q(\tilde{X}^*) = \frac{\mathrm{Ker(d : K^{q-2p,q-1} \to K^{q-2p+1,q})}}{\mathrm{Im(d : K^{q-2p-1,q-2} \to K^{q-2p,q-1})}}, \tag{2.9}$$

$$gr^w_{2p} H^q(Y) = \frac{\mathrm{Ker(d : Ker(N)^{q-2p,q-1} \to Ker(N)^{q-2p+1,q})}}{\mathrm{Im(d : Ker(N)^{q-2p-1,q-2} \to Ker(N)^{q-2p,q-1})}}, \tag{2.10}$$

$$gr^w_{2p} H^q_Y(X) = \frac{\mathrm{Ker(d : Coker(N)^{q-2p,q-3} \to Coker(N)^{q-2p+1,q-2})}}{\mathrm{Im(d : Coker(N)^{q-2p-1,q-4} \to Coker(N)^{q-2p,q-3})}}, \tag{2.11}$$

$$gr^w_{2p} H^q(X^*) = \frac{\mathrm{Ker(d : Cone(N)^{q-2p+1,q-2} \to Cone(N)^{q-2p+2,q-1})}}{\mathrm{Im(d : Cone(N)^{q-2p,q-3} \to Cone(N)^{q-2p+1,q-2})}}. \tag{2.12}$$

We define: $H^q(\tilde{X}^*) := \mathbb{H}^q(K^\cdot)$, $H^q(Y) := \mathbb{H}^q(\mathrm{Ker(N)}^\cdot)$, $H^q(X^*) := \mathbb{H}^q(\mathrm{Cone(N)}^\cdot)$ *and* $H^q_Y(X) := \mathbb{H}^q(\mathrm{Coker(N)}^\cdot)$. *These groups are identified with*

$$H^q(\tilde{X}^*) = \bigoplus\nolimits_{p\in\mathbb{Z}} gr^w_{2p} H^q(\tilde{X}^*), \quad H^q(Y) = \bigoplus\nolimits_{p\in\mathbb{Z}} gr^w_{2p} H^q(Y),$$

$$H^q(X^*) = \bigoplus\nolimits_{p\in\mathbb{Z}} gr^w_{2p} H^q(X^*), \quad H^q_Y(X) = \bigoplus\nolimits_{p\in\mathbb{Z}} gr^w_{2p} H^q_Y(X).$$

Remark 2.9 Note that the even graduation is a consequence of the parity condition $q + r \equiv\ 0\ (2)$ imposed on the indices of the complex (2.1).

Because $\dim X_{/\kappa} = 1$, it is easy to verify from the definition of $K^{\cdot,\cdot}$ that $H^q(\tilde{X}^*)$ and $H^q(Y)$ are $\neq 0$ only for $q = 0, 1, 2$. Furthermore, one easily finds that $H^q(X^*) \neq 0$ for $q = 0, 1, 2, 3$ and $H^q_Y(X) \neq 0$ only when $q = 2, 3, 4$.

The definition of these groups is inspired by the theory of degenerations of Hodge structures (*cf.* [49]) where the symbols $\tilde{X}^*$, Y and X^* have a precise geometric meaning: namely, they denote *resp.* the smooth fiber, the special fiber and the punctured space $\mathfrak{X} - Y$, where $\mathfrak{X}$ is the chosen model for a degeneration over a disk. In our set-up instead, $\tilde{X}^*$, Y, and X^* are only symbols but the general formalism associated to the hypercohomology of a double complex endowed with an operator commuting with the total differential can still be pursued and in fact it gives interesting arithmetical information. In the following we will show that the groups that we have just introduced enjoy similar properties as the graded quotients of the weight-filtration on the corresponding cohomology groups of *op.cit.* .

It is important to remark that the hypercohomology of the complex $\mathrm{Cone(N)}^\cdot$ contains both the information coming from the cohomologies of $\mathrm{Ker(N)}^\cdot$ and $\mathrm{Coker(N)}^\cdot$, as the following proposition shows

Proposition 2.10 *The following equality holds:*

$$\begin{aligned} H^q(X^*) &= \bigoplus_{p\in\mathbb{Z}} gr^w_{2p} H^q(X^*) = \\ &= \bigoplus_{2p\leq q-1} gr^w_{2p} H^q(Y) \oplus gr^w_q H^q(X^*) \oplus gr^w_{q+1} H^q(X^*) \oplus \bigoplus_{2p>q+1} gr^w_{2p} H^{q+1}_Y(X). \end{aligned}$$

The proof of proposition 2.10 as well as an explicit description of each addendum in the sum is a consequence of the following lemmas

Lemma 2.11 *For all* $p \in \mathbb{Z}$ *and for* $q \in \mathbb{N}$ *there are exact sequences*

$$\cdots \to gr^w_{2p} H^q(X^*) \to gr^w_{2p} H^q(\tilde{X}^*) \xrightarrow{N} gr^w_{2p-2} H^q(\tilde{X}^*) \to gr^w_{2p} H^{q+1}(X^*) \to \cdots \qquad (2.13)$$

$$\cdots \to gr^w_{2p} H^q(Y) \to gr^w_{2p} H^q(X^*) \to gr^w_{2p} H^{q+1}_Y(X) \to gr^w_{2p} H^{q+1}(Y) \to \cdots \qquad (2.14)$$

Furthermore, the maps $gr^w_{2p}H^q_Y(X)\to gr^w_{2p}H^q(Y)$ in (2.14) are zero unless $q=2p$, in which case they coincide with the morphism

$$\frac{(A^{p-1,p-1})_{\mathbb{R}}(p-1)}{\mathrm{Im(d'')}} \overset{2\pi\sqrt{-1}d'd''}{\longrightarrow} \mathrm{Ker(d' : (A^{p,p})_{\mathbb{R}}(p) \to (\bigoplus_{\substack{a+b=2p+1\\|a-b|\le 1}} A^{a,b})_{\mathbb{R}}(p)).}$$

Proof. We refer to [6]: Lemmas 3 and Lemma 4 and to [14]: Lemma 4.3. $\diamond$

Lemma 2.12 *For all $p\in\mathbb{Z}$ and for $q\in\mathbb{N}$:*

1. *The group $gr^w_{2p}H^q(Y)$ is zero unless $2p\le q$, in which case*

$$gr^w_{2p}H^q(Y) = \begin{cases} \mathrm{Ker(d' : (A^{p,p})_{\mathbb{R}}(p) \to \bigoplus_{\substack{a+b=2p+1\\|a-b|\le 1}} (A^{a,b})_{\mathbb{R}}(p))} & \text{if } q=2p \\ \dfrac{\mathrm{Ker(d' : (\bigoplus_{\substack{a+b=q\\|a-b|\le q-2p}} A^{a,b})_{\mathbb{R}}(p) \to (\bigoplus_{\substack{a+b=q+1\\|a-b|\le q-2p+1}} A^{a,b})_{\mathbb{R}}(p))}}{\mathrm{Im(d')}} & \text{if } q\ge 2p+1. \end{cases}$$

2. *The group $gr^w_{2p}H^q_Y(X)$ is zero unless $2p\ge q$, in which case*

$$gr^w_{2p}H^q_Y(X) = \begin{cases} \mathrm{Coker(d'' : (\bigoplus_{\substack{a+b=2p-3\\|a-b|\le 1}} A^{a,b})_{\mathbb{R}}(p-1) \to (A^{p-1,p-1})_{\mathbb{R}}(p-1))} & \text{if } q=2p \\ \dfrac{\mathrm{Ker(d'' : (\bigoplus_{\substack{a+b=q-2\\|a-b|\le 2p-q}} A^{a,b})_{\mathbb{R}}(p-1) \to (\bigoplus_{\substack{a+b=q-1\\|a-b|\le 2p-q-1}} A^{a,b})_{\mathbb{R}}(p-1))}}{\mathrm{Im(d'')}} & \text{if } q\le 2p-1. \end{cases}$$

Proof. We refer to [14] Lemmas 4.3. $\diamond$

Proof.(of Prop. 2.10) It is a straightforward consequence of Lemma 2.11. $\diamond$

Corollary 2.13 *The monodromy map*

$$N : gr^w_{2p}H^q(\tilde{X}^*)\to gr^w_{2(p-1)}H^q(\tilde{X}^*) \quad \text{is} \quad \begin{cases} \text{injective} & \text{if } q<2p-1 \\ \text{bijective} & \text{if } q=2p-1 \\ \text{surjective} & \text{if } q\ge 2p. \end{cases}$$

Therefore, the following sequences are exact

$$q\ge 2p: \quad 0\to gr^w_{2p}H^q(X^*)\to gr^w_{2p}H^q(\tilde{X}^*)\overset{N}{\to} gr^w_{2(p-1)}H^q(\tilde{X}^*)\to 0, \tag{2.15}$$

$$q\le 2(p-1): \quad 0\to gr^w_{2p}H^q(\tilde{X}^*)\overset{N}{\to} gr^w_{2(p-1)}H^q(\tilde{X}^*)\to gr^w_{2p}H^{q+1}(X^*)\to 0. \tag{2.16}$$

In particular, one obtains

$$\mathrm{Ker(N)} = \begin{cases} gr^w_{2p}H^q(X^*) & \text{if } q\ge 2p \\ 0 & \text{if } q\le 2p-1. \end{cases}$$

Proof. This follows from Lemma 2.11: (2.13) and from Lemma 2.12.$\diamond$

Remark 2.14 For future use, we explicitly remark that when $q \geq 2p$ one has the following decomposition:

$$gr^{\mathrm{w}}_{2p}H^q(\tilde{X}^*) = \underset{k\geq q-2p}{\oplus} \frac{\mathrm{Ker}(\mathrm{d}:\mathrm{K}^{\mathrm{q-2p,q-1,k}} \to \cdots)}{\mathrm{Im(d)}} = \tag{2.17}$$
$$= gr^{\mathrm{w}}_{2p}H^q(X^*) \underset{k\geq q-2p+1}{\oplus} \frac{\mathrm{Ker}(\mathrm{d}:\mathrm{K}^{\mathrm{q-2p,q-1,k}} \to \cdots)}{\mathrm{Im(d)}}.$$

Hence, when $q = 0, 1, 2$, the group $gr^{\mathrm{w}}_{2p}H^q(X^*)$ coincides with the homology of the complex $(\oplus_{k\geq q-2p}K^{q-2p,q-1,k}, d)$ at $k = q - 2p \geq 0$.

It is important to recall that the presence of a structure of polarized Hodge-Lefschetz module on the hypercohomology $\mathbb{H}^*(K^\cdot, d) = H^*(\tilde{X}^*)$ allows one to state the following results

Proposition 2.15 *For $q, p \in \mathbb{Z}$ satisfying the conditions $q - 2p > 0$, $q \geq 0$, the operator N induces isomorphisms*

$$N^{q-2p} : gr^w_{2(q-p)}H^q(\tilde{X}^*) \overset{\simeq}{\to} gr^w_{2p}H^q(\tilde{X}^*). \tag{2.18}$$

Furthermore, for $q \geq 2p$ the isomorphisms (2.18) *induce corresponding isomorphisms*

$$(gr^w_{2p}H^q(\tilde{X}^*)^{N=0} \simeq)\ gr^w_{2p}H^q(X^*) \underset{\simeq}{\overset{N^{2p-q}}{\to}} gr^w_{2(q-p+1)}H^{q+1}(X^*). \tag{2.19}$$

Proof. For the proof of (2.18) we refer to [14]: Proposition 4.8. For a proof of the isomorphisms (2.19) we refer to Corollary 2.13 and either the proof of Proposition 2.21 or to *op.cit.* .$\diamond$

2.4 Relations with Deligne's cohomology

The main feature of the complex (2.1) is its relation with the *real Deligne cohomology* of $X_{/\kappa}$. This cohomology (*cf.* Definition 2.16) measures how the natural real structure on the singular cohomology of a smooth projective variety is behaved with respect to the de-Rham filtration. One of the most interesting properties of Deligne's cohomology is its connection with arithmetics. Proposition 2.18 describes a precise relation between the ranks of some real Deligne's cohomology groups and the orders of pole, at non-positive integers, of the Γ-factors attached to a (real) Hodge structure $H = \oplus_{p,q}H^{p,q}$ over $\mathbb{C}$ (or $\mathbb{R}$). We recall that these factors are defined as (*cf.* [47])

$$L_{\mathbb{C}}(H, s) = \prod_{p,q} \Gamma_{\mathbb{C}}(s - \min(\mathrm{p,q}))^{\mathrm{h^{p,q}}} \tag{2.20}$$
$$L_{\mathbb{R}}(H, s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \prod_{p} \Gamma_{\mathbb{R}}(s-p)^{h^{p,+}}\Gamma_{\mathbb{R}}(s-p+1)^{h^{p,-}}$$

where $s \in \mathbb{C}$, $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ and $h^{p,\pm}$ is the dimension of the $\pm(-1)^p$-eigenspace of the $\mathbb{C}$-linear involution F_∞ on H (*cf.* Definition 2.4). One sets

$$\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s), \quad \Gamma_{\mathbb{R}}(s) = 2^{-\frac{1}{2}}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \quad \text{where} \quad \Gamma(\mathrm{s}) = \int_0^\infty \mathrm{e}^{-\mathrm{t}}\mathrm{t}^{\mathrm{s}}\frac{\mathrm{dt}}{\mathrm{t}}. \tag{2.21}$$

The following relation is known to hold (*cf. e.g.* [18]: Section 2)

$$\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1). \tag{2.22}$$

The Γ-function $\Gamma(s)$ is analytic in the whole complex plane except for simple poles at the non-positive integer points on the real axis.

Definition 2.16 *The real Deligne cohomology $H^*_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p))$ of a projective, smooth variety $X_{/\mathbb{C}}$ is the cohomology of the complex*

$$\mathbb{R}(p)_{\mathcal{D}}: \quad \mathbb{R}(p) \to \mathcal{O}_{X(\mathbb{C})} \to \Omega^1 \to \cdots \to \Omega^{p-1} \to 0.$$

The first arrow is the inclusion of the constant sheaf of Hodge-Tate twisted constants into the structural sheaf of the manifold: $\mathbb{R}(p) := (2\pi\sqrt{-1})^p\mathbb{R} \subseteq \mathbb{C} \subseteq \mathcal{O}_{X(\mathbb{C})}$. Ω^i denotes the sheaf of holomorphic i-th differential forms on the manifold.

The real Deligne cohomology of $X_{/\mathbb{R}}$ is defined as the subspace of elements invariant with respect to the de-Rham conjugation

$$H^*_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p)) := H^*_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p))^{\bar{F}_\infty}.$$

Precisely: $\bar{F}_\infty$ corresponds to the de-Rham conjugation under the canonical identification between de-Rham and Betti cohomology $H^*_{DR}(X_{/\mathbb{C}}, \mathbb{C}) = H^*_B(X_{/\mathbb{C}}, \mathbb{C})$ on the manifold $X_{/\mathbb{C}}$.

Remark 2.17 To understand correctly the meaning of $\bar{F}_\infty$ in the definition, it is worth to recall that both de-Rham and singular cohomology of the manifold $X_{/\mathbb{C}}$ have real structures. The real structure on the singular cohomology $H^*_B(X_{/\mathbb{C}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^*_B(X_{/\mathbb{C}}, \mathbb{C})$ is given by the $\mathbb{R}$-linear involution: $^-$ (on the right hand side) which is induced by the complex conjugation on the coefficients. On the other hand, by GAGA the algebraic de-Rham cohomology of $X_{/\mathbb{R}}$ defines a real structure $H^*_{DR}(X_{/\mathbb{R}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^*_{DR}(X_{/\mathbb{C}}, \mathbb{C})$ on the analytic de-Rham cohomology. The complex conjugation on the pair $(X_{/\mathbb{C}}, \Omega^\cdot)$ induces a $\mathbb{R}$-linear involution on the right hand side called de-Rham conjugation. Such de-Rham conjugation corresponds to the operator $\bar{F}_\infty$ on the Betti cohomology (*cf.* example 2.5) through the identification $H^*_{DR}(X_{/\mathbb{C}}, \mathbb{C}) = H^*_B(X_{/\mathbb{C}}, \mathbb{C})$ on the manifold $X_{/\mathbb{C}}$.

For example, let assume that $X_{/\mathbb{R}}$ is a smooth, real algebraic curve, that is, a symmetric Riemann surface with an involution $\iota : X_{/\mathbb{R}} \to X_{/\mathbb{R}}$ induced by the complex conjugation. Then, ι determines a corresponding involution ι^* in cohomology. On $H^0(X_{/\mathbb{R}}, \mathbb{R})$ this is the identity, whereas on $H^2(X_{/\mathbb{R}}, \mathbb{R})$ it reverses the orientation, hence is identified with $-\mathrm{id}$. On $H^1(X_{/\mathbb{R}}, \mathbb{R})$ ι^* acts as a non-trivial involution that exchanges $H^{1,0}_{\mathbb{R}}$ and $H^{0,1}_{\mathbb{R}}$, hence it separates the cohomology group into two eigenspaces corresponding to the eigenvalues of ± 1: $H^1(X_{/\mathbb{R}}, \mathbb{R}) = E_1 \oplus E_{-1}$, with $\dim E_1 = \dim E_{-1} = g =$ genus of X. On twisted cohomology groups such as $H^*(X_{/\mathbb{R}}, \mathbb{R}(p))$, $\bar{F}_\infty$ acts as the composition of ι^* with the involution that acts on the real Hodge structure $\mathbb{R}(p) = (2\pi i)^p\mathbb{R} \subset \mathbb{C}$ as $(-1)^p$.

From the short exact sequence of complexes

$$0 \to \Omega^{\cdot}_{<p}[-1] \to \mathbb{R}(p)_{\mathcal{D}} \to \mathbb{R}(p) \to 0$$

one gets the following long exact cohomology sequence that explains the statement we made at the beginning of this chapter ($F^p H^*_{DR}(X_{/\mathbb{C}}, \mathbb{C})$ = the Hodge filtration on de-Rham cohomology of the complex manifold)

$$\cdots \to H^*(X_{/\mathbb{C}}, \mathbb{R}(p)) \to H^*_{DR}(X_{/\mathbb{C}}, \mathbb{C})/F^p \to H^{*+1}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) \to H^{*+1}(X_{/\mathbb{C}}, \mathbb{R}(p)) \to \cdots$$

Proposition 2.18 *Let $X_{/\kappa}$ be a smooth, projective variety over $\kappa = \mathbb{C}$ or $\mathbb{R}$. Let i, m be two integers satisfying the conditions: $i \geq 0$, $2m \leq i$. Set $n = i + 1 - m$. Then*

$$-\dim_{\mathbb{R}} H^{i+1}_{\mathcal{D}}(X_{/\kappa}, \mathbb{R}(n)) = \operatorname*{ord}_{s=m} L_\kappa(H^i_B(X_{/\mathbb{C}}, \mathbb{C}), s)$$

Here $H^i_B(X_{/\mathbb{C}}, \mathbb{C})$ denotes Betti's cohomology of the manifold $X_{/\mathbb{C}}$ endowed with its pure Hodge structure over κ.

Proof. We refer to [44] Section 2.$\diamond$

The definition of the double complex $K^{\cdot,\cdot}$ in (2.1) was motivated by the expectation that the geometry on $X_{/\kappa}$ supports interesting arithmetical information on the archimedean fiber(s) of an arithmetic variety. Proposition 2.18 shows that the real Deligne cohomology of $X_{/\kappa}$ carries such information. The main goal was to construct a complex and an operator N acting on it which carries interesting arithmetic information so that the hypercohomology of $\mathrm{Cone(N)}^{\cdot}$ becomes isomorphic to the real Deligne cohomology of $X_{/\kappa}$. Proposition 4.1 of [14] shows that the complex $K^{\cdot,\cdot}$ has such property. When $X_{/\kappa}$ is a projective, non singular curve defined over $\kappa = \mathbb{C}$ or $\mathbb{R}$, this can be stated as follows

Proposition 2.19 *Let p, q be two non-negative integer. Then, for each fixed valued of p, the complex* $\mathrm{Cone(N : K^{q-2p,q-1} \to K^{q-2p+2,q-1})}$ *is quasi-isomorphic to the complex*

$$\ldots \xrightarrow{d''} \underbrace{(A^{p-2,p-1}(p-1) \oplus A^{p-1,p-2}(p-1))_{\mathbb{R}}}_{q=2p-2} \xrightarrow{d''} \underbrace{A^{p-1,p-1}_{\mathbb{R}}(p-1)}_{q=2p-1} \xrightarrow{2\pi\sqrt{-1}d'd''} \underbrace{A^{p,p}_{\mathbb{R}}(p)}_{q=2p} \xrightarrow{d'}$$
$$\xrightarrow{d'} \underbrace{(A^{p+1,p}(p) \oplus A^{p,p+1}(p))_{\mathbb{R}}}_{q=2p+1} \xrightarrow{d'} \ldots \quad (2.23)$$

whose homology, in each degree q, is isomorphic to the real Deligne cohomology of $X_{/\mathbb{C}}$.

Furthermore, taking the $\bar{F}_\infty$-invariants of such homology yields a description of $H^q_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p))$.

Proof. We refer to [14] Section 4, Proposition 4.1.$\diamond$

Remark 2.20 Notice that the left-side of (2.23), with respect to the central map $2\pi\sqrt{-1}d'd''$ is quasi-isomorphic to the complex obtained by exchanging the map d' with d''.

Proposition 2.21 *For $q \geq 2p \geq 0$, the following isomorphisms hold*

$$H^q_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) \simeq H^{q+1}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(q+1-p)) \tag{2.24}$$
$$H^q_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p)) \simeq H^{q+1}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(q+1-p))^{(-1)^q\bar{F}_\infty = id} \tag{2.25}$$

Proof. We consider in detail the case $q = 2p$. From Proposition 2.15, the following composite map (N^2) is an isomorphism

$$gr^{\mathrm{w}}_{2(p+1)}H^{2p}(\tilde{X}^*) \overset{N_{2(p+1)}}{\hookrightarrow} gr^{\mathrm{w}}_{2p}H^{2p}(\tilde{X}^*) \overset{N_{2p}}{\twoheadrightarrow} gr^{\mathrm{w}}_{2(p-1)}H^{2p}(\tilde{X}^*).$$

This implies, using the results of Corollary 2.13, that $gr^{\mathrm{w}}_{2p}H^{2p}(X^*) = \mathrm{KerN_{2p}}$ is mapped isomorphically to the group $gr^{\mathrm{w}}_{2(p+1)}H^{2p+1}(X^*) = \mathrm{Coker}(\mathrm{N_{2(p+1)}})$: this isomorphism is induced by the sequence of maps (2.13) in Lemma 2.11 (case $q = 2p$). It follows from Proposition 2.19 that $gr^{\mathrm{w}}_{2p}H^{2p}(X^*) \simeq H^{2p}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p))$ and that $gr^{\mathrm{w}}_{2p}H^{2p}(X^*)^{\bar{F}_\infty = id} \simeq H^{2p}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p))$. Similarly, one gets from the same proposition that $gr^{\mathrm{w}}_{2(p+1)}H^{2p+1}(X^*) \simeq H^{2p+1}_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p+1))$ and therefore we obtain (2.24). Taking the invariants for the action of $(-1)^q\bar{F}_\infty$ yields (2.25). The proof in the case $q \geq 2p+1$ is a generalization of the one just finished. For details on this part we refer to [14]: pp 352-3. $\diamond$

It is well known that the algebraic de-Rham cohomology $H^*_{DR}(X_{/\mathbb{C}}, \mathbb{R}(p))$, $(p \in \mathbb{Z})$ is the homology of the complex

$$0 \to \mathbb{R}(p) \to A^{0,0}_{\mathbb{R}}(p) \overset{d'}{\to} (A^{1,0} \oplus A^{0,1})_{\mathbb{R}}(p) \overset{d'}{\to} A^{1,1}_{\mathbb{R}}(p) \to 0; \quad d' = \partial + \bar{\partial}.$$

Using Lemma 2.12 together with Proposition 2.19 and Remark 2.20 we obtain the following description

Proposition 2.22 *Let $X = X_{/\kappa}$ be a smooth, projective curve over $\kappa = \mathbb{C}$ or $\mathbb{R}$.*
For $\kappa = \mathbb{C}$, the following descriptions hold

$$\begin{aligned} H^0(Y) &= \bigoplus_{p\leq 0} gr^w H^0(Y) = \bigoplus_{p\leq 0} H^0(X_{/\mathbb{C}}, \mathbb{R}(p)) \\ H^1(Y) &= \bigoplus_{p\leq 0} gr^w_{2p} H^1(Y) = \bigoplus_{p\leq 0} H^1(X_{/\mathbb{C}}, \mathbb{R}(p)) \\ H^2(Y) &= gr^w_2 H^2(Y) \oplus \bigoplus_{p\leq 0} gr^w_{2p} H^2(Y) = A^{1,1}_{\mathbb{R}}(1) \oplus \bigoplus_{p\leq 0} H^2(X_{/\mathbb{C}}, \mathbb{R}(p)). \end{aligned} \tag{2.26}$$

$$
\begin{aligned}
H^2_Y(X) =& gr^w_2 H^2_Y(X) \oplus \bigoplus_{p\geq 2} gr^w_{2p} H^2_Y(X) \simeq A^{0,0}_{\mathbb{R}} \oplus \bigoplus_{p\geq 2} H^1_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) \\
H^3_Y(X) =& \bigoplus_{p\geq 2} gr^w_{2p} H^3_Y(X) \simeq \bigoplus_{p\geq 2} H^2_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) \\
H^4_Y(X) =& \bigoplus_{p\geq 2} gr^w_{2p} H^4_Y(X) \simeq \bigoplus_{p\geq 2} H^3_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)).
\end{aligned}
\tag{2.27}
$$

For $X_{/\mathbb{R}}$ similar results hold by taking $\bar{F}_\infty$-invariants on both sides of the equalities.

Using Proposition 2.22, the description of $H^q(X^*)$ given in Proposition 2.10 can be made more explicit.

Proposition 2.23 *Let $X_{/\kappa}$ be a smooth, projective curve over $\kappa = \mathbb{C}$ or $\mathbb{R}$.*

1. *For $\kappa = \mathbb{C}$ and $\forall\ q \geq 0$ one has*

$$
H^q(\tilde{X}^*)^{N=0} = \underset{p\in\mathbb{Z}}{\oplus} gr^w_{2p} H^q(\tilde{X}^*)^{N=0} = \underset{q\geq 2p}{\oplus} gr^w_{2p} H^q(X^*)
$$

2. *In particular: $H^q(X^*) = 0$ for $q \notin [0,3]$ and the following description holds*

$$
\begin{aligned}
H^0(X^*) &= H^0(\tilde{X}^*)^{N=0} = \bigoplus_{p\leq -1} gr^w_{2p} H^0(Y) \oplus gr^w_0 H^0(X^*) = \bigoplus_{p\leq 0} H^0(X_{/\mathbb{C}}, \mathbb{R}(p)) \\
H^1(X^*) &= H^1(\tilde{X}^*)^{N=0} \oplus (gr^w_2 H^1(X^*) \oplus \bigoplus_{p\geq 2} gr^w_{2p} H^2_Y(X)) \simeq \\
&\simeq \bigoplus_{p\leq 0} H^1(X_{/\mathbb{C}}, \mathbb{R}(p)) \oplus \bigoplus_{p\geq 1} H^0(X_{/\mathbb{C}}, \mathbb{R}(p-1)) \\
H^2(X^*) &= H^2(\tilde{X}^*)^{N=0} \oplus \bigoplus_{p\geq 2} gr^w_{2p} H^3_Y(X) \simeq \\
&\simeq \bigoplus_{p\leq 1} H^2(X_{/\mathbb{C}}, \mathbb{R}(p)) \oplus \bigoplus_{p\geq 2} H^1(X_{/\mathbb{C}}, \mathbb{R}(p-1)) \\
H^3(X^*) &= (gr^w_4 H^3(X^*) \oplus \bigoplus_{p\geq 3} gr^w_{2p} H^4_Y(X)) \simeq \bigoplus_{p\geq 2} H^2(X_{/\mathbb{C}}, \mathbb{R}(p-1)).
\end{aligned}
$$

When $\kappa = \mathbb{R}$ similar results hold by taking $\bar{F}_\infty$-invariants on both sides.

Proof. *1.* follows from Corollary 2.13. The first statement in *2.* is a consequence of $\dim X_{/\kappa} = 1$. For $q \in [0,3]$, the description of the graded groups $H^q(X^*)$ follows from Proposition 2.10, Proposition 2.19, Proposition 2.21. For $p, q \geq 2$, the isomorphisms $H^q_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(p)) \simeq H^{q-1}(X_{/\mathbb{C}}, \mathbb{R}(p-1))$ are a consequence of Remark 2.20, whereas the isomorphisms: $gr^{\mathrm{w}}_2 H^1(X^*) \simeq H^1_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(1)) \simeq H^0_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}) \simeq H^0(X_{/\mathbb{C}}, \mathbb{R})$ and $gr^{\mathrm{w}}_2 H^2(X^*) \simeq H^2_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(1)) \simeq H^3_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(2)) \simeq H^2(X_{/\mathbb{C}}, \mathbb{R}(1))$ follow from Proposition 2.21. In particular the last isomorphism holds because $\dim X = 1$. Finally, the case $\kappa = \mathbb{R}$ is a consequence of the fact that the $\bar{F}_\infty$-invariants of the homology of the complex (2.23) gives $H^*_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p))$. $\diamond$

2.5 Archimedean Frobenius and regularized determinants

On the *infinite dimensional* real vector space $gr^{\mathrm{w}}_{2p}H^*(\tilde{X}^*)$ (*cf.* (2.9)) one defines a linear operator

$$\Phi : gr^{\mathrm{w}}_{2p}H^*(\tilde{X}^*) \to gr^{\mathrm{w}}_{2p}H^*(\tilde{X}^*), \qquad \Phi(x) = p \cdot x \tag{2.28}$$

and then extend this definition to the whole group $H^*(\tilde{X}^*)$ according to the decomposition $H^q(\tilde{X}^*) = \oplus_p gr^{\mathrm{w}}_{2p}H^q(\tilde{X}^*)$.

In this section we will consider the operator Φ restricted to the subspace $H^*(\tilde{X}^*)^{N=0}$. Following the description of this space given in Proposition 2.23, we write $\Phi = \oplus_{q=0}^2 \Phi_q$, where

$$\Phi_q : H^q(\tilde{X}^*)^{N=0} \to H^q(\tilde{X}^*)^{N=0}.$$

Given a self-adjoint operator T with pure point spectrum, the *zeta-regularized determinant* is defined by (*cf.* [20])

$$\det_{\infty}(s - T) = \exp\left(-\frac{d}{dz}\,\zeta_T(s,z)_{|z=0}\right), \tag{2.29}$$

where

$$\zeta_T(s,z) = \sum_{\lambda \in \mathrm{Spec}(T)} m_\lambda (s-\lambda)^{-z}. \tag{2.30}$$

Here, $\mathrm{Spec}(T)$ denotes the spectrum of T and m_λ is the multiplicity of the eigenvalue λ. By the spectral theorem $m_\lambda = \dim E_\lambda(T)$, where $E_\lambda(T)$ is the related eigenspace.

Φ_q is a self-adjoint operator with respect to the inner product induced by (2.8), with spectrum

$$\mathrm{Spec}(\Phi_q) = \begin{cases} \{n \in \mathbb{Z}, n \le 0\} & q = 0,1 \\ \{n \in \mathbb{Z}, n \le 1\} & q = 2. \end{cases}$$

The eigenspaces $E_n(\Phi_q) = gr^{\mathrm{w}}_{2n}H^q(\tilde{X}^*)^{N=0}$ have dimensions $\dim E_n(\Phi_q) = b_q$, the q-th Betti number of $X_{/\mathbb{C}}$.

Proposition 2.24 *The zeta regularized determinant of Φ_q is given by*

$$\det_{\infty}\left(\frac{s}{2\pi} - \frac{\Phi_q}{2\pi}\right) = \Gamma_{\mathbb{C}}(s)^{-b_q}, \tag{2.31}$$

for $q = 0,1$ and

$$\det_{\infty}\left(\frac{s}{2\pi} - \frac{\Phi_2}{2\pi}\right) = \Gamma_{\mathbb{C}}(s-1)^{-b_2}, \tag{2.32}$$

with $\Gamma_{\mathbb{C}}(s)$ and $\Gamma_{\mathbb{R}}(s)$ as in (2.21).

Proof. We write explicitly the zeta function for the operator $\Phi_q/(2\pi)$. When $q = 0, 1$, this has spectrum $\{n/(2\pi)\}_{n\leq 0}$, hence we have

$$\zeta_{\Phi_q/(2\pi)}(s/(2\pi), z) = \sum_{n\leq 0} b_q(s/(2\pi) - n/(2\pi))^{-z} = b_q(2\pi)^z \zeta(s, z).$$

$\zeta(s, z)$ is the Hurwitz zeta function

$$\zeta(s, z) = \sum_{n\geq 0} \frac{1}{(s+n)^z}.$$

For $q = 2$, similarly we have

$$\zeta_{\Phi_2/(2\pi)}(s/(2\pi), z) = b_2(2\pi)^z(\zeta(s, z) + (s-1)^{-z}).$$

It is well known that the Hurwitz zeta function satisfies the following properties:

$$\zeta(s, 0) = \frac{1}{2} - s, \qquad \frac{d}{dz}\ \zeta(s, z)_{|z=0} = \log\Gamma(s) - \frac{1}{2}\log(2\pi). \tag{2.33}$$

When $q = 0, 1$, the computation of $\frac{d}{dz}\ \zeta_{\Phi_q/(2\pi)}(s/(2\pi), z)_{|z=0}$ yields

$$\frac{d}{dz}\ \zeta_{\Phi_q/(2\pi)}(s/(2\pi), z) = b_q\left(\log(2\pi)(2\pi)^z\zeta(s, z) + (2\pi)^z\frac{d}{dz}\zeta(s, z)\right).$$

At $z = 0$, this gives

$$\frac{d}{dz}\ \zeta_{\Phi_q/(2\pi)}(s/(2\pi), z)_{|z=0} = b_q\left(\log(2\pi)\zeta(s, 0) + \frac{d}{dz}\zeta(s, z)_{|z=0}\right)$$

$$= b_q\left(\log(2\pi)(\frac{1}{2} - s) + \log\Gamma(s) - \frac{1}{2}\log(2\pi)\right) = b_q(-s\log(2\pi) + \log\Gamma(s)).$$

Taking the exponential we get

$$\exp\left(-\frac{d}{dz}\ \zeta_{\Phi_q/(2\pi)}(s/(2\pi), z)_{|z=0}\right) = \exp(-b_q(-s\log(2\pi) + \log\Gamma(s))) =$$

$$= \left((2\pi)^{-s}\Gamma(s)\right)^{-b_q} = \Gamma_{\mathbb{C}}(s)^{-b_q}.$$

When $q = 2$, one has similarly

$$\frac{d}{dz}\ \zeta_{\Phi_2/(2\pi)}(s/(2\pi), z) = b_2\left((\log(2\pi)(2\pi)^z\zeta(s, z) + (2\pi)^z\frac{d}{dz}\zeta(s, z)\right) + b_2\frac{d}{dz}\ \frac{(2\pi)^z}{(s-1)^z}.$$

At $z = 0$ this gives

$$b_2\left(-s\log(2\pi) + \log\Gamma(s) + \log(2\pi) - \log(s-1)\right).$$

Thus, we have

$$\exp\left(-\frac{d}{dz}\,\zeta_{\Phi_2/(2\pi)}(s/(2\pi),z)_{|z=0}\right) = \left((2\pi)^{-s+1}\frac{\Gamma(s)}{(s-1)}\right)^{-b_2} = \left((2\pi)^{-(s-1)}\Gamma(s-1)\right)^{-b_2} =$$

$$= \Gamma_{\mathbb{C}}(s-1)^{-b_2}.$$

$\diamond$

Remark 2.25 When $X_{/\mathbb{C}}$ is a smooth complex algebraic curve, that is, when $X_{/\mathbb{C}} = X_{\alpha(K)}$ for an archimedean prime that corresponds to a complex (non-real) embedding $\alpha : K \hookrightarrow \mathbb{C}$, the description of the complex Euler factor is given by (*cf.* (2.20))

$$L_{\mathbb{C}}(H^q(X_{/\mathbb{C}},\mathbb{C}),s) = \begin{cases} \Gamma_{\mathbb{C}}(s)^{b_q} & q=0,1 \\ \Gamma_{\mathbb{C}}(s-1)^{b_2} & q=2, \end{cases}$$

where $H^q(X_{/\mathbb{C}},\mathbb{C})$ is the Betti cohomology. The relation to the determinants (2.31) (2.32) is then

$$\det_\infty\left(\frac{s}{2\pi}-\frac{\Phi_q}{2\pi}\right)^{-1} = L_{\mathbb{C}}(H^q(X_{/\mathbb{C}},\mathbb{C}),s). \tag{2.34}$$

This result was proved in [14]: §5, via comparison to Deninger's pair (H^*_{ar},Θ).

Assume now that $X_{/\mathbb{R}}$ is a smooth real algebraic curve; that is $X_{/\mathbb{R}} = X_{\alpha(K)}$ for an archimedean prime that corresponds to a real embedding $\alpha : K \hookrightarrow \mathbb{R}$. In this case $X_{/\mathbb{R}}$ is a symmetric Riemann surface, namely a compact Riemann surface with an involution $\iota : X_{/\mathbb{R}} \to X_{/\mathbb{R}}$ induced by complex conjugation. Such involution on the manifold induces an action of the real Frobenius $\bar{F}_\infty$ on $H^q(\tilde{X}^*)^{N=0}$: we refer to Remark 2.17 for the description of this operator.

For instance, following the decomposition given in Proposition 2.23, $H^1(\tilde{X}^*)^{N=0} = \oplus_{p\leq 0} gr^{\mathrm{w}}_{2p}H^1(X^*)$ splits as the sum of two eigenspaces for $\bar{F}_\infty$ with eigenvalues ± 1:

$$H^1(\tilde{X}^*)^{N=0} = E^+ \oplus E^-,$$

where

$$\begin{aligned} E^+ &:= H^1(\tilde{X}^*)^{N=0,\bar{F}_\infty=id} = \bigoplus_{p\leq 0} E_1(2p) \oplus \bigoplus_{p\leq -1} E_{-1}(2p+1), \\ E^- &:= H^1(\tilde{X}^*)^{N=0,\bar{F}_\infty=-id} = \bigoplus_{p\leq -1} E_1(2p+1) \oplus \bigoplus_{p\leq 0} E_{-1}(2p). \end{aligned} \tag{2.35}$$

We consider once more the operator Φ acting on $H^*(\tilde{X}^*)^{N=0}$, and we denote by $\hat{\Phi}_q$ the restriction of this operator to the subspace $H^q(\tilde{X}^*)^{N=0,\bar{F}_\infty=id}$.

Proposition 2.26 *The regularized determinant for the operator* $\hat{\Phi}_q = \Phi_{|_{H^q(\tilde{X}^*)^{N=0,\bar{F}_\infty=id}}}$ *is given by (g = genus of* $X_{/\kappa}$*)*

$$\det_\infty\left(\frac{s}{2\pi}-\frac{\hat{\Phi}_0}{2\pi}\right)=\Gamma_{\mathbb{R}}(s)^{-b_0} \tag{2.36}$$

$$\det_\infty\left(\frac{s}{2\pi}-\frac{\hat{\Phi}_1}{2\pi}\right)=\Gamma_{\mathbb{R}}(s)^{-b_1/2}\Gamma_{\mathbb{R}}(s+1)^{-b_1/2}=\Gamma_{\mathbb{C}}(s)^{-g} \tag{2.37}$$

$$\det_\infty\left(\frac{s}{2\pi}-\frac{\hat{\Phi}_2}{2\pi}\right)=\Gamma_{\mathbb{R}}(s-1)^{-b_2}. \tag{2.38}$$

Proof. We write explicitly the zeta function for the operators $\hat{\Phi}_q$ on $H^q(\tilde{X}^*)^{N=0,\bar{F}_\infty=id}$. The spectrum of $\hat{\Phi}_q$ is given by $\{n\in\mathbb{Z}, n\leq 0\}$ for $q=0,1$ and $\{n\in\mathbb{Z}, n\leq 1\}$ for $q=2$. Because complex conjugation is the identity on $H^0(X_{/\mathbb{R}},\mathbb{R}(2n))$, the eigenspaces where $\bar{F}_\infty$ acts as identity are $E_n(\hat{\Phi}_0)=gr^{\mathrm{w}}_{4n}H^0(X^*)$. On the other hand, $\bar{F}_\infty$ acts as the identity on $H^2(X_{/\mathbb{R}},\mathbb{R}(2n+1))$, hence $E_n(\hat{\Phi}_2)=gr^{\mathrm{w}}_{2(2n+1)}H^2(X^*)$. The action of $\bar{F}_\infty$ on $H^1(X_{/\mathbb{R}},\mathbb{R}(n))$ is the identity precisely on the eigenspaces $E_n(\hat{\Phi}_1)=E_1(2n)\oplus E_{-1}(2n+1)$ as in (2.35).

The zeta function of $\hat{\Phi}_0/(2\pi)$ is therefore of the form

$$\zeta_{\hat{\Phi}_0/(2\pi)}(s/(2\pi),z)=\sum_{n\geq 0}b_0\left(\frac{s+2n}{2\pi}\right)^{-z}=b_0(2\pi)^z\sum_{n\geq 0}\frac{1}{(s+2n)^z}=b_0(\pi)^z\zeta(s/2,z),$$

where $\zeta(s,z)$ is the Hurwitz zeta function. Using the identities (2.33), we obtain

$$\frac{d}{dz}\,\zeta_{\hat{\Phi}_0/(2\pi)}(s/(2\pi),z)_{|_{z=0}}=b_0(\log(\pi)(1/2-s/2)+\log\Gamma(s/2)-1/2\,\log(2\pi))$$

$$=b_0(-s/2\log(\pi)-1/2\log(2)+\log\Gamma(s/2)).$$

Hence, using the equalities (2.21), we obtain

$$\exp\left(-\frac{d}{dz}\,\zeta_{\hat{\Phi}_0/(2\pi)}(s/(2\pi),z)_{|_{z=0}}\right)=\exp(-b_0(-s/2\log(\pi)-1/2\log(2)+\log\Gamma(s/2)))$$

$$=\left(2^{-1/2}\pi^{-s/2}\Gamma(s/2)\right)^{-b_0}=\Gamma_{\mathbb{R}}(s)^{-b_0}.$$

The determinant for $\hat{\Phi}_1/(2\pi)$ is given by the product

$$\det_\infty\left(\frac{s}{2\pi}-\frac{\hat{\Phi}_1}{2\pi}\right)=$$

$$\det_\infty \left(\frac{s}{2\pi} - \frac{\hat{\Phi}_1}{2\pi}|_{\oplus_n E_1(2n)} \right) \cdot \det_\infty \left(\frac{s}{2\pi} - \frac{\hat{\Phi}_1}{2\pi}|_{\oplus_n E_{-1}(2n+1)} \right).$$

The zeta function for the first operator is given by

$$\zeta_{\frac{\hat{\Phi}_1}{2\pi}|_{\oplus_n E_1(2n)}}(s/(2\pi), z) = \frac{b_1}{2}\pi^z \zeta(s/2, z)$$

while the for the second operator is

$$\zeta_{\frac{\hat{\Phi}_1}{2\pi}|_{\oplus_n E_{-1}(2n+1)}}(s/(2\pi), z) = \frac{b_1}{2}(2\pi)^z \sum_{n\geq 0} \frac{1}{(s+1+2n)^z} = \frac{b_1}{2}\pi^z \zeta((s+1)/2, z).$$

Thus, we obtain

$$\det_\infty \left(\frac{s}{2\pi} - \frac{\hat{\Phi}_1}{2\pi}|_{\oplus_n E_1(2n)} \right) = \Gamma_{\mathbb{R}}(s)^{-b_1/2}$$

and

$$\det_\infty \left(\frac{s}{2\pi} - \frac{\hat{\Phi}_1}{2\pi}|_{\oplus_n E_{-1}(2n+1)} \right) = \Gamma_{\mathbb{R}}(s+1)^{-b_1/2}.$$

Then, (2.37) follows using the equality $\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s)$.

Finally, for $\hat{\Phi}_2/(2\pi)$, we have

$$\zeta_{\frac{\hat{\Phi}_2}{2\pi}}\left(\frac{s}{2\pi}, z\right) = b_2(2\pi)^z \sum_{n\geq 0} \frac{1}{(s-1+2n)^z} = b_2\pi^z\zeta((s-1)/2, z),$$

hence

$$\frac{d}{dz}\,\zeta_{\frac{\hat{\Phi}_2}{2\pi}}\left(\frac{s}{2\pi}, z\right)_{|z=0} = b_2\left(-(s-1)/2\log\pi - 1/2\,\log 2 + \log\Gamma((s-1)/2)\right).$$

Therefore

$$\det_\infty \left(\frac{s}{2\pi} - \frac{\hat{\Phi}_2}{2\pi} \right) = \left(2^{-1/2}\pi^{-(s-1)/2}\Gamma((s-1)/2)\right)^{-b_2} = \Gamma_{\mathbb{R}}(s-1)^{-1}.$$

$\diamond$

Remark 2.27 When $X_{/\mathbb{R}}$ is a smooth, real algebraic curve of genus g, that is, when $X_{/\mathbb{R}} = X_{\alpha(K)}$ for an archimedean prime that corresponds to a real embedding $\alpha : K \hookrightarrow \mathbb{R}$, the description of the real Euler factor is given by (*cf.* (2.20))

$$L_{\mathbb{R}}(H^q(X_{/\mathbb{R}}, \mathbb{R}), s) = \begin{cases} \Gamma_{\mathbb{R}}(s) & q = 0 \\ \Gamma_{\mathbb{C}}(s)^g & q = 1 \\ \Gamma_{\mathbb{R}}(s-1) & q = 2, \end{cases}$$

As for the complex case, this result was proved in [14]: §5, via comparison to Deninger's pair (H^*_{ar}, Θ)

3 Arithmetic spectral triple.

In this Section we show that the choice of a Schottky uniformization of a Riemann surface $X_{/\mathbb{C}}$ enriches the cohomological construction of Section 2, with the structure of a Connes *spectral triple* (*cf.* [10] §VI).

For a discrete group Γ, the (reduced) C*–algebra $\mathrm{C}^*(\Gamma)$ is the norm completion of the group ring $\mathbb{C}[\Gamma]$ in the algebra of bounded operators on the Hilbert space $\ell^2(\Gamma)$ of square summable functions on Γ.

If $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ is a Schottky group, the algebra $\mathrm{C}_0(\mathfrak{X}_\Gamma)$ of continuous functions vanishing at infinity on the handlebody $\mathfrak{X}_\Gamma = \Gamma\backslash\mathbb{H}^3$ is strongly Morita equivalent to a crossed product algebra (*cf.* Proposition 5.17), which by a result of [22] has the same K-theory of $\mathrm{C}^*(\Gamma)\otimes \mathrm{C}_0(\mathbb{H}^3)$. Since $\mathbb{H}^3$ is contractible, the non-commutative space $\mathrm{C}^*(\Gamma)$ carries the complete topological information on the handlebody: in the isomorphisms $K_i(\mathrm{C}_0(\mathfrak{X}_\Gamma)) \cong K_{i+1}(\mathrm{C}^*(\Gamma))$ the contribution of $\mathrm{C}_0(\mathbb{H}^3)$ is simply given by a degree shift (*cf.* [22]).

In this Section, we will only use *real coefficients*. In fact, in order to construct a spectral triple compatible with the arithmetic construction of Section 2, we need to preserve the structure of real vector spaces. For this reason, we consider the real (reduced) C*–algebra of a Schottky group Γ. This is a norm completion of $\mathbb{R}[\Gamma]$. The relation between the real $\mathrm{C}^*(\Gamma)$ and the ring of *real* functions on $\mathfrak{X}_\Gamma$ holds, with suitable modifications, as in the complex case (*cf.* [45], [3]).

Let $(H^\cdot(X^*),\Phi)$ be the cohomological theory of the fiber at the archimedean prime introduced in Section 2, endowed with the structure of polarized Hodge–Lefschetz module. Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ be a choice of Schottky uniformization of the Riemann surface $X_{/\mathbb{C}}$. In the following we often replace Γ with a lift to $\mathrm{SL}(2,\mathbb{C})$, which, with a slight abuse of notation, we still denote by Γ.

In Theorem 3.7 we show that there is a normal subgroup $\tilde\Gamma \subset \Gamma$ such that $\mathrm{C}^*(\tilde\Gamma)$ has a representation

$$\rho : \mathrm{C}^*(\tilde\Gamma) \to \mathcal{B}(H^\cdot(\tilde X^*)), \tag{3.1}$$

where $\mathcal{B}(H^\cdot(\tilde X^*))$ is the algebra of bounded operators on a real Hilbert space completion of $H^\cdot(\tilde X^*)$ (in the inner product determined by the polarization of $K^{\cdot,\cdot}$). The representation ρ is compatible with the structure of Hodge–Lefschetz module on $H^\cdot(\tilde X^*) = \mathbb{H}^\cdot(K,d)$.

The main result of this section is Theorem 3.11, where we prove that the inner product on $H^\cdot(\tilde X^*)$ and the representation (3.1) induce an inner product on $H^\cdot(X^*) = \mathbb{H}^\cdot(\mathrm{Cone}^\cdot(N))$ and a representation

$$\rho_N : \mathrm{C}^*(\tilde\Gamma) \to \mathcal{B}(H^\cdot(X^*)), \tag{3.2}$$

in such a way that the data $(\mathrm{C}^*(\tilde\Gamma), H^\cdot(X^*),\Phi)$ define a spectral triple for which Φ is the Dirac operator.

In non-commutative geometry, the notion of a spectral triple provides the correct generalization of the classical structure of a Riemannian manifold. The two notions agree on a commutative space. In the usual context of Riemannian geometry, the definition of the infinitesimal element ds on a smooth spin manifold can be expressed in terms of the inverse of the classical Dirac operator D. This is the key remark that motivates the theory of spectral triples. In particular, the geodesic distance between two points on the manifold is defined in terms of D^{-1} (*cf.* [10] §VI). The spectral triple that describes a classical Riemannian spin manifold is (A, H, D), where A is the algebra of complex valued smooth functions on the manifold, H is the Hilbert space of square integrable spinor sections, and D is the classical Dirac operator (a square root of the Laplacian). These data determine completely and uniquely the Riemannian geometry on the manifold. It turns out that, when expressed in this form, the notion of spectral triple extends to more general non-commutative spaces, where the data (A, H, D) consist of a C^*-algebra A (or more generally of some smooth subalgebra of a C^*-algebra) with a representation as bounded operators on a Hilbert space H, and an operator D on H that verifies the main properties of a Dirac operator. Under this perspective, our Theorem 3.11 states that the spectral triple $(\mathrm{C}^*(\tilde{\Gamma}), H^{\cdot}(X^*), \Phi)$ endows the non-commutative space $\mathrm{C}^*(\tilde{\Gamma})$ with the structure of a non-commutative Riemannian manifold, with the Riemannian metric determined by the Dirac operator Φ. We will refer to $(\mathrm{C}^*(\tilde{\Gamma}), H^{\cdot}(X^*), \Phi)$ as the *arithmetic spectral triple.*

The compatibility of the representation (3.1) with the polarized Hodge–Lefschetz module structure is obtained by applying Bers simultaneous uniformization (*cf.* [5] [7]) to the Schottky group Γ. Given the group Γ and the hyperbolic handlebody satisfying $\mathfrak{X}_\Gamma \cup X_{/\mathbb{C}} = \Gamma\backslash(\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C}))$, this process determines a pair of Fuchsian Schottky groups $G_1, G_2 \subset SL(2,\mathbb{R})$, which geometrically correspond to a decomposition of the Riemann surface $X_{/\mathbb{C}}$ as the union of two Riemann surfaces with boundary. The Fuchsian Schottky groups act on the complex $K^{\cdot,\cdot}$ and on the cohomology via the restriction of the representation of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ of the Hodge-Lefschetz module structure. The condition of compatibility with the arithmetic structure produces a natural geometric choice of a normal subgroup $\tilde{\Gamma} \subset \Gamma$, with $G_1 \simeq \tilde{\Gamma} \simeq G_2$. Geometrically, this group corresponds to the choice of a covering $\mathfrak{X}_{\tilde{\Gamma}} \to \mathfrak{X}_\Gamma$ of $\mathfrak{X}_\Gamma$ by a handlebody $\mathfrak{X}_{\tilde{\Gamma}}$. The non-commutative space $\mathrm{C}^*(\tilde{\Gamma})$ describes the topology of $\mathfrak{X}_{\tilde{\Gamma}}$.

In the interpretation of the tangle of bounded geodesics in the handlebody $\mathfrak{X}_{\tilde{\Gamma}}$ as the dual graph of the closed fiber at arithmetic infinity, the covering $\mathfrak{X}_{\tilde{\Gamma}} \to \mathfrak{X}_\Gamma$ produces a corresponding covering of the dual graph by geodesics in $\mathfrak{X}_{\tilde{\Gamma}}$. Passing to the covering $\mathfrak{X}_{\tilde{\Gamma}}$ may be regarded as an analog, at the archimedean primes, of the refinement of the dual graph of a Mumford curve that corresponds to a minimal resolution (*cf.* [33] §3).

When the archimedean prime corresponds to a real embedding $K \hookrightarrow \mathbb{R}$, so that the corresponding Riemann surface $X_{/\mathbb{R}}$ acquires a real structure, Proposition 3.13 shows that, if the set of real points $X_{/\mathbb{R}}(\mathbb{R})$ is non-empty, then we can construct the arithmetic spectral triple with the algebra $\mathrm{C}^*(\Gamma)$, with Γ the Fuchsian Schottky group determined by the real structure.

The theory of spectral triples encodes important arithmetic information on the underlying non-commutative space, expressed via an associated family of *zeta functions*. By studying the zeta functions attached to our triple $(C^*(\Gamma), H^{\cdot}(X^*), \Phi)$ we find a natural one whose associated Ray-Singer determinant is the alternating product of the Γ-factors for the real Hodge structure over $\mathbb{C}$ given by the Betti cohomology $H^q(X_{/\mathbb{C}}, \mathbb{C})$.

3.1 Spectral triples

We recall the basic setting of Connes theory of spectral triples. For a more complete treatment we refer to [12], [10], [13].

Definition 3.1 *a spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *consists of an involutive algebra* $\mathcal{A}$ *with a representation*

$$\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

as bounded operators on a Hilbert space $\mathcal{H}$*, and an operator* D *(called the Dirac operator) on* $\mathcal{H}$*, which satisfies the following properties:*

1. *D is self–adjoint.*
2. *For all $\lambda \notin \mathbb{R}$, the resolvent $(D-\lambda)^{-1}$ is a compact operator on $\mathcal{H}$.*
3. *For all $a \in \mathcal{A}$, the commutator $[D, a]$ is a bounded operator on $\mathcal{H}$.*

Remark 3.2 The property *2.* of Definition 3.1 generalizes ellipticity of the standard Dirac operator on a compact manifold. Usually, the involutive algebra $\mathcal{A}$ satisfying property *3.* can be chosen to be a dense subalgebra of a C*–algebra. This is the case, for instance, when we consider smooth functions on a manifold as a subalgebra of the commutative C*-algebra of continuous functions. In the classical case of Riemannian manifolds, property *3.* is equivalent the Lipschitz condition, hence it is satisfied by a larger class than that of smooth functions.

We review those aspects of the theory of spectral triples which are of direct interest to us. For a more general treatment we refer to [12], [10], [13].

Volume form. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be of dimension n, or *n–summable* if the operator $|D|^{-n}$ is an infinitesimal of order one, which means that the eigenvalues $\lambda_k(|D|^{-n})$ satisfy the estimate $\lambda_k(|D|^{-n}) = O(k^{-1})$.

For a positive compact operator T such that

$$\sum_{j=0}^{k-1} \lambda_j(T) = O(\log k),$$

the Dixmier trace $\mathrm{Tr}_\omega(T)$ is the coefficient of this logarithmic divergence, namely

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\log k} \sum_{j=1}^{k} \lambda_j(T). \tag{3.3}$$

Here the notation $\lim_\omega$ takes into account the fact that the sequence

$$S(k,T) := \frac{1}{\log k} \sum_{j=1}^{k} \lambda_j(T)$$

is bounded though possibly non-convergent. For this reason, the usual notion of limit is replaced by a choice of a linear form $\lim_\omega$ on the set of bounded sequences satisfying suitable conditions that extend analogous properties of the limit. When the sequence $S(k,T)$ converges (3.3) is just the ordinary limit $\mathrm{Tr}_\omega(T) = \lim_{k\to\infty} S(k,T)$. So defined, the Dixmier trace (3.3) extends to any compact operator that is an infinitesimal of order one, since any such operator is the difference of two positive ones. The operators for which the Dixmier trace does not depend on the choice of the linear form $\lim_\omega$ are called *measurable operators*.

On a non-commutative space the operator $|D|^{-n}$ generalizes the notion of a volume form. The volume is defined as

$$V = \mathrm{Tr}_\omega(|D|^{-n}). \tag{3.4}$$

More generally, consider the algebra $\tilde{\mathcal{A}}$ generated by $\mathcal{A}$ and $[D, \mathcal{A}]$. Then, for $a \in \tilde{\mathcal{A}}$, integration with respect to the volume form $|D|^{-n}$ is defined as

$$\int a := \frac{1}{V} \mathrm{Tr}_\omega(a|D|^{-n}). \tag{3.5}$$

The usual notion of integration on a Riemannian spin manifold M can be recovered in this context (*cf.* [10], [23]) through the formula (n even):

$$\int_M f dv = \left(2^{n-[n/2]-1}\pi^{n/2} n\Gamma(n/2)\right) \mathrm{Tr}_\omega(f|D|^{-n}).$$

Here D is the classical Dirac operator on M associated to the metric that determines the volume form dv, and f in the right hand side is regarded as the multiplication operator acting on the Hilbert space of square integrable spinors on M.

Zeta functions. An important function associated to the Dirac operator D of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is its zeta function

$$\zeta_D(z) := \mathrm{Tr}(|D|^{-z}) = \sum_\lambda \mathrm{Tr}(\Pi(\lambda, |D|))\lambda^{-z}, \tag{3.6}$$

where $\Pi(\lambda, |D|)$ denotes the orthogonal projection on the eigenspace $E(\lambda, |D|)$.

An important result in the theory of spectral triples ([10] §IV Proposition 4) relates the volume (3.4) with the residue of the zeta function (3.6) at $s = 1$ through the formula

$$V = \lim_{s\to 1+} (s-1)\zeta_D(s) = Res_{s=1}\mathrm{Tr}(|D|^{-s}). \tag{3.7}$$

There is a family of zeta functions associated to a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, to which (3.6) belongs. For an operator $a \in \tilde{\mathcal{A}}$, we can define the zeta functions

$$\zeta_{a,D}(z) := \mathrm{Tr}(a|D|^{-z}) = \sum_\lambda \mathrm{Tr}(a\,\Pi(\lambda, |D|))\lambda^{-z} \tag{3.8}$$

and

$$\zeta_{a,D}(s,z) := \sum_\lambda \mathrm{Tr}(a\,\Pi(\lambda, |D|))(s-\lambda)^{-z}. \tag{3.9}$$

These zeta functions are related to the heat kernel $e^{-t|D|}$ by Mellin transform

$$\zeta_{a,D}(z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}\mathrm{Tr}(a\, e^{-t|D|})\, dt \tag{3.10}$$

where

$$\mathrm{Tr}(a\, e^{-t|D|}) = \sum_\lambda \mathrm{Tr}(a\,\Pi(\lambda, |D|))e^{-t\lambda} =: \theta_{a,D}(t). \tag{3.11}$$

Similarly,

$$\zeta_{a,D}(s,z) = \frac{1}{\Gamma(z)} \int_0^\infty \theta_{a,D,s}(t)\, t^{z-1}\, dt \tag{3.12}$$

with

$$\theta_{a,D,s}(t) := \sum_\lambda \mathrm{Tr}(a\,\Pi(\lambda, |D|))e^{(s-\lambda)t}. \tag{3.13}$$

Under suitable hypothesis on the asymptotic expansion of (3.13) (*cf.* Theorem 2.7-2.8 of [25] §2), the functions (3.8) and (3.9) admit a unique analytic continuation (*cf.* [13]) and there is an associated regularized determinant in the sense of Ray–Singer (*cf.* [39]):

$$\det_{\infty\, a,D}(s) := \exp\left(-\frac{d}{dz}\zeta_{a,D}(s,z)|_{z=0}\right) \tag{3.14}$$

The family of zeta functions (3.8) also provides a refined notion of dimension for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, called the *dimension spectrum*. This is a subset $\Sigma = \Sigma(\mathcal{A}, \mathcal{H}, D)$ in $\mathbb{C}$ with the property that all the zeta functions (3.8), as a varies in $\tilde{\mathcal{A}}$, extend holomorphically to $\mathbb{C} \setminus \Sigma$.

3.2 Hodge-Lefschetz modules and simultaneous uniformization

We begin by recalling the following elementary fact of hyperbolic geometry. Let Γ be a Kleinian group acting on $\mathbb{P}^1(\mathbb{C})$. Let $\Omega \subset \mathbb{P}^1(\mathbb{C})$ be a Γ-invariant domain. A subset $\Omega_0 \subset \Omega$ is Γ-stable if, for every $\gamma \in \Gamma$, either $\gamma(\Omega_0) = \Omega_0$ or $\gamma(\Omega_0) \cap \Omega_0 = \emptyset$. The Γ-stabilizer of Ω_0 is the subgroup Γ_0 of those $\gamma \in \Gamma$ such that $\gamma(\Omega_0) = \Omega_0$. Let π_Γ denote the quotient map $\pi_\Gamma : \Omega \to \Gamma\backslash\Omega$.

Claim 3.3 (*cf.* Theorem 6.3.3 of [4]). Let $\Omega_0 \subset \Omega$ be an open subdomain and let Γ_0 be the Γ-stabilizer of Ω_0. Then the quotient map π_Γ induces a conformal equivalence

$$\Gamma_0\backslash\Omega_0 \simeq \pi_\Gamma(\Omega_0).$$

If Γ is a Kleinian group, a *quasi circle* for Γ is a Jordan curve C in $\mathbb{P}^1(\mathbb{C})$ which is invariant under the action of Γ. In particular, such curve contains the limit set Λ_Γ.

In the case of Schottky groups, the following theorem shows that Bowen's construction of a quasi–circle for Γ (*cf.* [7]) determines a pair of Fuchsian Schottky groups $G_1, G_2 \subset \mathrm{PSL}(2,\mathbb{R})$ associated to $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$. The theorem describes the simultaneous uniformization by $\tilde{\Gamma}$ of the two Riemann surfaces with boundary $X_i = G_i\backslash\mathbb{H}^2$.

Theorem 3.4 *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ be a Schottky group of rank $g \geq 2$. Then the following properties are satisfied:*

1. *There exists a quasi–circle C for Γ.*
2. *There is a collection of curves $\hat{C}$ on the compact Riemann surface $X_{/\mathbb{C}} = \Gamma\backslash\Omega_\Gamma$ such that*
$$X_{/\mathbb{C}} = X_1 \cup_{\partial X_1 = \hat{C} = \partial X_2} X_2,$$
where $X_i = G_i\backslash\mathbb{H}^2$ are Riemann surfaces with boundary, and the $G_i \subset \mathrm{PSL}(2,\mathbb{R})$ are Fuchsian Schottky groups. The G_i are isomorphic to $\tilde{\Gamma} \subset \mathrm{PSL}(2,\mathbb{C})$, the Γ–stabilizer of the two connected components Ω_i of $\mathbb{P}^1(\mathbb{C})\backslash C$.

Proof. *1.* For the construction of a quasi–circle we proceed as in [7]. The choice of a set of generators $\{g_i\}_{i=1}^g$ for Γ determines $2g$ Jordan curves γ_i, $i = 1 \dots 2g$ in $\mathbb{P}^1(\mathbb{C})$ with pairwise disjoint interiors D_i such that, if we write $g_{i+g} = g_i^{-1}$ for $i = 1 \dots g$, the fractional linear transformation g_i maps the interior of γ_i to the exterior of $\gamma_{i+g \mod 2g}$. Now fix a choice of $2g$ pairs of points $\rho_i^\pm$ on the curves γ_i in such a way that g_i maps the two points $\rho_i^\pm$ to the two points $\rho_{i+g \mod 2g}^\mp$. Choose a collection C_0 of pairwise disjoint oriented arcs in $\mathbb{P}^1(\mathbb{C})$ with the property that they do not intersect the interior of the disks D_i. Also assume that the oriented boundary of C_0 as a 1-chain is given by $\partial C_0 = \sum_i \rho_i^+ - \sum_i \rho_i^-$. Then the curve

$$C := \Lambda_\Gamma \cup \bigcup_{\gamma\in\Gamma} \gamma C_0 \qquad (3.15)$$

is a quasi–circle for Γ.

2. The image of the curves γ_i in the quotient $X_{/\mathbb{C}} = \Gamma\backslash\Omega_\Gamma$ consists of g closed curves, whose homology classes a_i, $i = 1\ldots g$, span the kernel $Ker(I_*)$ of the map $I_* : H_1(X_{/\mathbb{C}}, \mathbb{Z}) \to H_1(\mathfrak{X}_\Gamma, \mathbb{Z})$ induced by the inclusion of $X_{/\mathbb{C}}$ as the boundary at infinity in the compactification of $\mathfrak{X}_\Gamma$. The image under the quotient map of the collection of points $\{\gamma\rho_i^\pm\}_{\gamma\in\Gamma, i=1\ldots 2g}$ consists of two points on each curve a_i, and the image of $C\cap\Omega_\Gamma$ consists of a collection $\hat{C}$ of pairwise disjoint arcs on $X_{/\mathbb{C}}$ connecting these $2g$ points. By cutting the surface $X_{/\mathbb{C}}$ along $\hat{C}$ we obtain two surfaces X_i, $i = 1, 2$, with boundary $\partial X_i = \hat{C}$.

Since C is Γ-invariant, the two connected components Ω_i, $i = 1, 2$, of $\mathbb{P}^1(\mathbb{C})\backslash C$ are Γ-stable. Let Γ_i denote the Γ-stabilizer of Ω_i. Notice that $\Gamma_1 = \Gamma_2$. In fact, suppose there is $\gamma \in \Gamma$ such that $\gamma \in \Gamma_1$ and $\gamma \notin \Gamma_2$. Then $\gamma(\mathbb{P}^1(\mathbb{C})) \subset \Omega_1 \cup C$, so that the image $\gamma(\mathbb{P}^1(\mathbb{C}))$ is contractible in $\mathbb{P}^1(\mathbb{C})$. This would imply that γ has topological degree zero, but an orientation preserving fractional linear transformation has topological degree one.

We denote by $\tilde{\Gamma}$ the Γ-stabilizer $\tilde{\Gamma} = \Gamma_1 = \Gamma_2$. Since the components Ω_i are open subdomains of the Γ-invariant domain Ω_Γ, Claim 3.3 implies that the quotients $\tilde{\Gamma}\backslash\Omega_i$ are conformally equivalent to the image $\pi_\Gamma(\Omega_i) \subset X_{/\mathbb{C}}$. By the explicit description of the surfaces with boundary X_i, it is easy to see that $\pi_\Gamma(\Omega_i) = X_i$.

The quasi-circle C is a Jordan curve in $\mathbb{P}^1(\mathbb{C})$, hence by the Riemann mapping theorem there exist conformal maps α_i of the two connected components Ω_i to the two hemispheres U_i of $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{P}^1(\mathbb{R})$,

$$\alpha_i : \Omega_i \xrightarrow{\simeq} U_i \qquad U_1 \cup U_2 = \mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{P}^1(\mathbb{R}). \qquad (3.16)$$

Consider the two groups

$$G_i := \{\alpha_i \gamma \alpha_i^{-1} : \gamma \in \tilde{\Gamma}\}.$$

These are isomorphic as groups to $\tilde{\Gamma}$, $G_i \simeq \tilde{\Gamma}$. Moreover, the G_i preserve the upper/lower hemisphere U_i, hence they are Fuchsian groups, $G_i \subset \mathrm{PSL}(2, \mathbb{R})$.

The conformal equivalence $\tilde{\Gamma}\backslash\Omega_i \simeq X_i$ implies that the G_i provide the Fuchsian uniformization of $X_i = G_i\backslash\mathbb{H}^2$, where $\mathbb{H}^2$ is identified with the upper/lower hemisphere U_i in $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{P}^1(\mathbb{R})$.

The group $\tilde{\Gamma} \subset \Gamma$ is itself a discrete purely loxodromic subgroup of $\mathrm{PSL}(2, \mathbb{C})$ isomorphic to free groups, hence a Schottky group, so that the G_i are Fuchsian Schottky groups.

$\diamond$

Let $X_{/\mathbb{R}}$ be an orthosymmetric smooth real algebraic curve. In this case, we can apply the following refinement of the result of Theorem 3.4. We refer to [1], [46] for a proof.

Proposition 3.5 *Let $X_{/\mathbb{R}}$ be a smooth real orthosymmetric algebraic curve of genus $g \geq 2$. Then the following holds.*

1. *$X_{/\mathbb{R}}$ has a Schottky uniformization such that the domain of discontinuity $\Omega_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ is symmetric with respect to $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$.*
2. *The reflection about $\mathbb{P}^1(\mathbb{R})$ gives an involution on Ω_Γ that induces the involution $\iota : X_{/\mathbb{R}} \to X_{/\mathbb{R}}$ of the real structure.*
3. *The circle $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$ is a quasi-circle for the Schottky group Γ, such that the image in $X_{/\mathbb{R}}$ of $\mathbb{P}^1(\mathbb{R}) \cap \Omega_\Gamma$ is the fixed point set X_ι of the involution.*
4. *The Schottky group Γ is a Fuchsian Schottky group.*

The choice of a lifting $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ of the Schottky group determines corresponding lifts of $\tilde{\Gamma} \subset \mathrm{SL}(2,\mathbb{C})$ and $G_i \subset \mathrm{SL}(2,\mathbb{R})$.

We consider the polarized bigraded Hodge-Lefschetz module $(K^{\cdot,\cdot}, N, \ell, \psi)$ associated to $X_{/\mathbb{C}}$ at the archimedean prime, as described in Section 2. We have the following result.

Lemma 3.6 *Let $\sigma : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}(K)$ be the representation associated to the bigraded Lefschetz module structure on the complex $K^{\cdot,\cdot}$. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ be a Schottky group that determines a Schottky uniformization of $X_{/\mathbb{C}}$. Let $\tilde{\Gamma}$ is the corresponding lift to $\mathrm{SL}(2,\mathbb{C})$ of the Γ-stabilizer of the components Ω_i in the complement of a quasi-circle C. Then $(K^{\cdot,\cdot}, N, \ell, \psi)$ carries a left and a right action of $\tilde{\Gamma}$,*

$$\sigma_1(\gamma) := \sigma\{\alpha_1 \gamma \alpha_1^{-1}, 1\} \tag{3.17}$$

$$\sigma_2(\gamma) := \sigma\{1, \alpha_2 \gamma \alpha_2^{-1}\}, \tag{3.18}$$

where α_i are the conformal maps (3.16) of Ω_i to the two hemispheres in $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{P}^1(\mathbb{R})$.

Proof. By Theorem 3.4 we obtain Fuchsian Schottky groups $G_i = \{\alpha_i \gamma \alpha_i^{-1}, \gamma \in \tilde{\Gamma}\}$ in $\mathrm{SL}(2,\mathbb{R})$. We consider the restriction of the representation $\sigma : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}(K)$ to $G_1 \times \{1\}$ and $\{1\} \times G_2$ as in (3.17) and (3.18).
$\diamond$

We set

$$\tilde{\Phi} : K^{i,j,k} \to K^{i,j,k} \qquad \tilde{\Phi}(x) = \frac{(1+j-i)}{2} x. \tag{3.19}$$

The operator $\tilde{\Phi}$ induces the operator Φ of (2.28) on the cohomology $H^\cdot(\tilde{X}^*)^{N=0}$.

We have the following result.

Theorem 3.7 *Let $(K^{\cdot\cdot}, d, N, \ell, \psi)$ be the polarized bigraded Lefschetz module associated to a Riemann surface $X_{/\mathbb{C}}$ of genus $g \geq 2$, and let $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ be a choice of Schottky uniformization for $X_{/\mathbb{C}}$. Let $\tilde{\Gamma}$ be a lift to $\mathrm{SL}(2, \mathbb{C})$ of the Γ–stabilizer of the two connected components of $\mathbb{P}^1(\mathbb{C}) \setminus C$ as in Theorem 3.4. Then the following holds.*

1. *The representation σ_2 of Lemma 3.6 extends to a representation of $\mathrm{C}^*(\tilde{\Gamma})$*
$$\rho : \mathrm{C}^*(\tilde{\Gamma}) \to \mathcal{B}(\mathbb{H}^{\cdot}(K, d)) \tag{3.20}$$
as bounded operators on a real Hilbert space completion of the cohomology $\mathbb{H}^{\cdot}(K, d)$ with the inner product defined by the polarization ψ.

2. *The operator $\tilde{\Phi}$ defined in* (3.19) *has bounded commutators with all the elements in $\rho(\mathrm{C}^*(\tilde{\Gamma}))$.*

Proof. *1.* The representations σ_1 and σ_2 of Lemma 3.6 extend by linearity to representations $\sigma_i : \mathbb{R}[\tilde{\Gamma}] \to \mathrm{Aut}(K)$.

By Theorem 2.6 and Corollary 2.7, the cohomology $\mathbb{H}^{\cdot}(K, d)$ has an induced Lefschetz module structure, thus we obtain induced representations of the real group ring $\mathbb{R}[\tilde{\Gamma}]$ on $\mathbb{H}^{\cdot}(K, d)$. We complete $H^{\cdot}(\tilde{X}^*) = \mathbb{H}^{\cdot}(K, d)$ to a real Hilbert space with respect to the inner product induced by the polarization ψ. Consider operators of the form (2.5) with $b = 1$,

$$U_a(x) := \sigma\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}(x) = a^i x \quad \text{for } x \in K^{i,j}.$$

A direct calculation shows that the U_a are in general *unbounded operators*: since the index i varies over a countable set, it is not hard to construct examples of infinite sums $x = \sum_i x_i$ that are in the Hilbert space completion of $H^{\cdot}(\tilde{X}^*)$ but such that $U_a(x)$ is no longer contained in this space.

On the other hand, the index j in the complex $K^{i,j}$ varies subject to the constraint $j + 1 = q$, where q is the degree of the differential forms (*cf.* Remark 2.1). Thus, expressions of the form

$$\sigma\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\right\}(x) = b^j x \quad \text{for } x \in K^{i,j} \tag{3.21}$$

give rise to *bounded operators.* Thus the representation σ_2 of $\mathbb{R}[\tilde{\Gamma}]$ extends to a representation of the real (reduced) C^*–algebra $\mathrm{C}^*(\tilde{\Gamma})$, by taking the norm closure of $\sigma_2(\mathbb{R}[\tilde{\Gamma}])$ in the algebra of bounded operators $\mathcal{B}(\mathbb{H}^{\cdot}(K, d))$.

(2) We can compute explicitly all the commutators with the operator $\tilde{\Phi}$. Elements of the form (2.5) commute with $\tilde{\Phi}$. Moreover, we have:

$$[N, \tilde{\Phi}]\,(x) = \frac{1}{(2\pi\sqrt{-1})}\left(\frac{(1 + j - i)}{2} - \frac{(1 + j - i + 2)}{2}\right) x = -N(x),$$

$$[\sigma_1(w_1),\tilde{\Phi}](x) = \left(\frac{(1+j-i)}{2} - \frac{(1+j+i)}{2}\right)\sigma_1(w_1)(x) = -i\sigma_1(w_1)(x).$$

$$[\ell,\tilde{\Phi}]\,(x) = \left(\frac{(1+j-i)}{2} - \frac{(1+j+2-i)}{2}\right)(2\pi\sqrt{-1})^{-1}x\wedge\omega = -\ell(x)$$

and

$$[\sigma_2(w_2),\tilde{\Phi}](x) = \left(\frac{(1+j-i)}{2} - \frac{(1-j-i)}{2}\right)\sigma_2(w)(x) = j\,\sigma_2(w)(x).$$

In particular, it follows that all the commutators that arise from the right representation are bounded operators (*cf.* Remark 2.1).
◇

Corollary 3.8 *The action of* $\mathrm{C}^*(\tilde{\Gamma})$ *on the odd cohomology* $\mathbb{H}^{2q+1}(K^\cdot,d)$ *is independent of the lift of* $\tilde{\Gamma}\subset\mathrm{PSL}(2,\mathbb{C})$ *to* $\mathrm{SL}(2,\mathbb{C})$.

Proof. It is sufficient to check that $-id$ acts trivially on the odd cohomology. For $x\in K^{i,j}$ we have

$$\sigma\left\{1,\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\}(x) = (-1)^j x.$$

Since $j+1=q$, where q is the degree of the differential forms, we obtain that the induced action is trivial on odd cohomology. Thus, the induced representation σ_2 of $\mathrm{SL}(2,\mathbb{R})$ on the odd cohomology descends to a representation of $\mathrm{PSL}(2,\mathbb{R})$.
◇

Remark 3.9 The data $(\mathrm{C}^*(\tilde{\Gamma}),\mathbb{H}^\cdot(K,d),\tilde{\Phi})$ of Theorem 3.7 do not define a spectral triple: the operator $\tilde{\Phi}$ does not satisfy all the properties of a Dirac operator. In fact, the eigenspaces of $\tilde{\Phi}$, which coincide with the graded pieces $gr^{\mathrm{w}}_{2p}H^q(\tilde{X}^*)$ of the cohomology, are not finite dimensional as the condition on the resolvent in Definition 3.1 would imply. Therefore, it is necessary to restrict the structure $(\mathrm{C}^*(\tilde{\Gamma}),\mathbb{H}^\cdot(K,d),\tilde{\Phi})$ to a suitable subspace of $\mathbb{H}^\cdot(K,d)$, which still carries all the arithmetic information.

Definition 3.10 *The operator* Φ *on* $\mathbb{H}^\cdot(\mathrm{Cone(N)}^\cdot)=\mathrm{H}^\cdot(\mathrm{X}^*)$ *is obtained by extending the action on its graded pieces*

$$\Phi|_{gr^{w}_{2p}H^q(X^*)} := \begin{cases} p & q\geq 2p\\ p-1 & q\leq 2p-1.\end{cases} \tag{3.22}$$

according to the decomposition $H^q(X^*)=\oplus_{p\in\mathbb{Z}}gr^{w}_{2p}H^q(X^*)$.

Notice that this definition is compatible with the operator $\tilde{\Phi}$ defined in (3.19), acting on the complex $K^{\cdot,\cdot}$ and with the induced operator Φ on $\mathbb{H}^\cdot(K,d)=H^\cdot(\tilde{X}^*)$. In fact, from the Wang exact sequence (2.13) and Corollary 2.13 we know that, for $q\geq 2p-1$ $gr^{\mathrm{w}}_{2p}H^q(X^*)$ is identified with a subspace of $gr^{\mathrm{w}}_{2p}H^q(\tilde{X}^*)$, hence the restriction of the operator Φ on $H^\cdot(\tilde{X}^*)$ acts on $gr^{\mathrm{w}}_{2p}H^q(X^*)$ as multiplication by p. In the case when

$q \leq 2(p-1)$, again using the exact sequence (2.13) (*cf.* (2.16) Corollary 2.13), we can define Φ of an element in $gr^{\rm w}_{2p}H^q(X^*)$ as Φ of a preimage in $gr^{\rm w}_{2(p-1)}H^{q-1}(\tilde{X}^*)$, hence as multiplication by $p-1$. This is obviously well defined, hence the definition (3.22) is compatible with the exact sequences and the duality isomorphisms. Moreover, the operator Φ of (3.22) agrees with the operator (2.28) on the subspace $H^{\cdot}(\tilde{X}^*)^{N=0}$ of $H^{\cdot}(X^*)$.

Theorem 3.11 *Let X be an arithmetic surface with 1-dimensional fibers of genus $g \geq 2$. Let Γ be a Schottky group uniformizing the Riemann surface $X_{/\mathbb{C}}$ at a given archimedean prime, and let $\tilde{\Gamma} \subset \Gamma$ be defined as in Theorem 3.4. Let $H^{\cdot}(X^*)$ be the hyper-cohomology of* Cone(N). *The inner product* (2.8) *defined by the polarization ψ induces an inner product on $H^{\cdot}(X^*)$. Moreover, the representation* (3.20) *induces a representation of* $\mathrm{C}^*(\tilde{\Gamma})$ *on the resulting real Hilbert space completion of $H^q(X^*)$. The data* $(\mathrm{C}^*(\tilde{\Gamma}), H^{\cdot}(X^*), \Phi)$, *with Φ as in* (3.22), *determine a 1–summable spectral triple.*

Proof. The Wang exact sequence (2.13) and Corollary 2.13 imply that the hyper-cohomology $H^{\cdot}(X^*)$ of Cone(N) injects or is mapped upon surjectively by the hyper-cohomology $\mathbb{H}^{\cdot}(K,d)$ of the complex, in a way which is compatible with the grading. Thus, we obtain an induced inner product and Hilbert space completion on $H^{\cdot}(X^*)$. Consider $Ker(N) \subset gr^{\rm w}_{2p}H^q(\tilde{X}^*)$. By Corollary 2.13, we know that this is non-trivial only if $q \geq 2p$, and in that case it is given by $gr^{\rm w}_{2p}H^q(X^*)$. Thus, we can show that there is an induced representation on $\oplus_{2p\leq q} gr^{\rm w}_{2p}H^q(X^*)$ by showing that the representation of $\mathrm{C}^*(\tilde{\Gamma})$ on $H^q(\tilde{X}^*)$ preserves $Ker(N)$.

In the definition of the complex $K^{i,j,k}$ in (2.1), the indices i,j,k and the integers p,q are related by $2p = j+1-i$ and $q = j+1$. Thus, the condition $q \geq 2p$ corresponds to $i \geq 0$. The representation σ_2 of $\mathrm{SL}(2,\mathbb{R})$ on $K^{i,j}$ preserves the subspace with $i \geq 0$. Similarly, by construction, the representation σ_2 preserves the subspaces $\oplus_j K^{i,j,k}$ of $\oplus_{j,k\geq i\geq 0} K^{i,j,k}$. This implies that the induced representation σ_2 on $\mathbb{H}^{\cdot}(K,d)$ preserves the summands of $gr^{\rm w}_{2p}H^q(\tilde{X}^*)$ as in Remark 2.14, and in particular it preserves $\mathrm{Ker}(N)$. Thus we obtain a representation

$$\rho_{Ker(N)} : \mathrm{C}^*(\tilde{\Gamma}) \to \mathcal{B}(\mathrm{Ker}(N)).$$

The duality isomorphisms N^{q-2p} of Proposition 2.15 determine duality isomorphisms between pieces of the hyper-cohomology $H^{\cdot}(X^*)$ of the cone:

$$\begin{aligned}
\delta_0 &: gr^{\rm w}_{2p}H^0(X^*) \stackrel{\simeq}{\to} gr^{\rm w}_{2r}H^1(X^*), \quad p \leq 0, r = -p+1 \geq 1 \\
\delta_1 &: gr^{\rm w}_{2p}H^1(X^*) \stackrel{\simeq}{\to} gr^{\rm w}_{2r}H^2(X^*), \quad p \leq 0, r = -p+2 \geq 2 \\
\delta_2 &: gr^{\rm w}_{2p}H^2(X^*) \stackrel{\simeq}{\to} gr^{\rm w}_{2r}H^3(X^*), \quad p \leq 1, r = -p+3 \geq 2.
\end{aligned} \tag{3.23}$$

We set $\delta = \oplus_{q=0}^2 \delta_q$ and we obtain an action of $\mathrm{C}^*(\tilde{\Gamma})$ on $H^{\cdot}(X^*)$ by extending the representation $\rho_{Ker(N)}$ by $\delta \circ \rho_{Ker(N)} \circ \delta^{-1}$ on the part of $H^{\cdot}(X^*)$ dual to $\mathrm{Ker}(N)$.

The operator $\tilde{\Phi}$ of (3.19) induces the operator Φ of (3.22) on $\mathbb{H}^{\cdot}(\mathrm{Cone}(N)) = H^{\cdot}(X^*)$. This has the properties of a Dirac operator: the eigenspaces are all finite dimensional by the result of Proposition 2.22, and the commutators are bounded by Theorem 3.7. The spectrum of Φ is given by $\mathbb{Z}$, so that Φ^{-1} on the complement of the zero modes is an infinitesimal of order one.
$\diamond$

Remark 3.12 Notice that the spectral triple we obtain is somewhat reminiscent of the spectral triple for the Hodge-Dirac operator (signature). In that case the Hilbert space of spinors H consists of L^2-differential forms, whereas in our setting we have a space of forms harmonic with respect to the harmonic theory defined by $\Box$ on the complex $K^{\cdot,\cdot}$ (*cf.* Theorem 2.6 and Corollary 2.7) that are "square integrable" with respect to the inner product (2.8) given by the polarization. Instead of the algebra of functions on a manifold, we have the algebra $\mathrm{C}^*(\tilde{\Gamma})$ which provides a noncommutative version of the manifold $\mathfrak{X}_\Gamma$.

If X is an arithmetic surface over $\mathrm{Spec}(O_{\mathbb{K}})$, where $O_{\mathbb{K}}$ is the ring of integers of a number field $\mathbb{K}$ with $n = [\mathbb{K} : \mathbb{Q}]$, the above result can be applied at each of the n archimedean primes, by choosing at each prime $\alpha : \mathbb{K} \hookrightarrow \mathbb{C}$ a Schottky uniformization of the corresponding Riemann surface $X_{\alpha(\mathbb{K})}$. At the primes that correspond to the r real embeddings, $X_{\alpha(\mathbb{K})}$ has a real structure.

We have the following refinement of Theorems 3.7 and 3.11. The proof is completely analogous, but we use Proposition 3.5 instead of Theorem 3.4.

Proposition 3.13 *Let X be an arithmetic surface over* $\mathrm{Spec}(O_{\mathbb{K}})$, *with the property that, at all the real archimedean primes, the Riemann surface $X_{\alpha(\mathbb{K})}$ is an orthosymmetric smooth real algebraic curve of genus $g \geq 2$. Let $(K^{\cdot,\cdot}, d, N, \ell, \psi)$ be the polarized bigraded Lefschetz module associated to $X_{/\mathbb{R}} = X_{\alpha(\mathbb{K})}$. Then the representation σ_2 extends to representations*

$$\rho : \mathrm{C}^*(\Gamma) \to \mathcal{B}(\mathbb{H}^{\cdot}(K, d)),$$

$$\rho_N : \mathrm{C}^*(\Gamma) \to \mathcal{B}(\mathbb{H}^{\cdot}(\mathrm{Cone}(N)))$$

with the properties as in Theorems 3.7 and 3.11, where $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ *is the Fuchsian Schottky uniformization for $X_{/\mathbb{R}}$.*

Remark 3.14 In [24], the condition $\dim_H(\Lambda_\Gamma) < 1$ on the limit set was necessary in order to ensure convergence of the Poincaré series that gives the abelian differentials on $X_{/\mathbb{C}}$, hence in order to express the Green function on $X_{/\mathbb{C}}$ in terms of geodesics in the handlebody $\mathfrak{X}_\Gamma$. Notice that this condition is *always satisfied* for an orthosymmetric smooth real algebraic curve $X_{/\mathbb{R}}$, with the choice of Schottky uniformization described above, since the limit set Λ_Γ is contained in the rectifiable circle $\mathbb{P}^1(\mathbb{R})$.

3.3 Some zeta functions and determinants

The duality isomorphisms N^{q-2p} of Proposition 2.15 and the induced isomorphisms δ_q of (3.23) give some further structure to the spectral triple.

Define subspaces $H^{\pm}(X^*)$ of $H^{\cdot}(X^*)$ in the following way:

$$\begin{aligned} H^{-}(X^*) &= \oplus_{p\le 0} gr^{\mathrm{w}}_{2p} H^0(X^*) \oplus \oplus_{p\le 0} gr^{\mathrm{w}}_{2p} H^1(X^*) \oplus \oplus_{p\le 1} gr^{\mathrm{w}}_{2p} H^2(X^*), \\ H^{+}(X^*) &:= \oplus_{p\ge 1} gr^{\mathrm{w}}_{2p} H^1(X^*) \oplus \oplus_{p\ge 2} gr^{\mathrm{w}}_{2p} H^2(X^*) \oplus \oplus_{p\ge 2} gr^{\mathrm{w}}_{2p} H^3(X^*), \end{aligned} \tag{3.24}$$

Let $\delta = \oplus_{q=0}^{2} \delta_q$ be the duality isomorphism of (3.23) and set

$$\omega = \begin{pmatrix} 0 & \delta^{-1} \\ \delta & 0 \end{pmatrix}.$$

The map ω interchanges the subspaces $H^{\pm}(X^*)$.

Lemma 3.15 *The map ω has the following properties:*

- $\omega^2 = id$, $\omega^* = \omega$.
- $[\omega, a] = 0$, *for all* $a \in \mathrm{C}^*(\tilde{\Gamma})$.
- $(\Phi\omega + \omega\Phi)|_{H^q(X^*)} = q \cdot id$.

Proof. By construction (*cf.* Theorem 3.11) the action of $\mathrm{C}^*(\tilde{\Gamma})$ commutes with ω. By (3.22), for $q \ge 2p$ the operator Φ on $gr^{\mathrm{w}}_{2p} H^2(X^*)$ acts as multiplication by p. The duality isomorphism, mapping $gr^{\mathrm{w}}_{2p} H^2(X^*)$ to $gr_{2(q-p+1)} H^{q+1}(X^*)$ (*cf.* Proposition 2.15), and Φ acts on $gr_{2(q-p+1)} H^{q+1}(X^*)$ as multiplication by $(q-p)$. Thus, we obtain

$$(\Phi\omega + \omega\Phi)|_{gr^{\mathrm{w}}_{2p} H^2(X^*)}(x) = (q-p)\cdot x + p \cdot x = q \cdot x.$$

$\diamond$

Remark 3.16 Recall that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called *even* if there is an operator ω such that $\omega^2 = id$ and $\omega^* = \omega$; the commutator $[\omega, a] = 0$, for all $a \in \mathcal{A}$ and the Dirac operator satisfies $D\omega + \omega D = 0$ (*cf.* [23]). The conditions of Lemma 3.15 provide a weaker version of this notion, depending on the degree of the cohomology $H^q(X^*)$.

Due to the presence of this further structure on the spectral triple, determined by the duality isomorphisms, in addition to the family of zeta functions (3.8), (3.9), we can consider zeta functions of the form

$$\zeta_{a, P_\pm\Phi}(s, z) := \sum_{\lambda \in \mathrm{Spec}(P_\pm\Phi)} \mathrm{Tr}(a\, \Pi(\lambda, P_\pm\Phi))(s-\lambda)^{-z}, \tag{3.25}$$

where $P_\pm$ are the projections on $H^{\pm}(X^*)$. In this setting we can now recover the Γ-factors.

Theorem 3.17 *Consider $a = \sigma_2(-id)$ as an element in $Aut(K^{\cdot,\cdot})$, acting on $H^{\cdot}(X^*)$ via the induced representation (cf. Theorem 3.11). Then the zeta function* (3.25)

$$\zeta_{a,P_-\Phi}(s,z) := \sum_{\lambda \in \mathrm{Spec}(P_-\Phi)} \mathrm{Tr}(a\Pi(\lambda, P_-\Phi))(s-\lambda)^{-z}$$

satisfies

$$\exp\left(-\frac{d}{dz}\zeta_{a,P_-\Phi/(2\pi)}(s/(2\pi),z)|_{z=0}\right)^{-1} = \frac{L_{\mathbb{C}}(H^1(X_{/\mathbb{C}},\mathbb{C}),s)}{L_{\mathbb{C}}(H^0(X_{/\mathbb{C}},\mathbb{C}),s)\cdot L_{\mathbb{C}}(H^2(X_{/\mathbb{C}},\mathbb{C}),s)}. \tag{3.26}$$

Proof. Notice that we have $P_-\Phi = \Phi|_{H^{\cdot}(\tilde{X}^*)^{N=0}}$. Moreover, recall that the element $a = \sigma_2(-id)$ acts as $(-1)^{q-1}$ on differential forms of degree q. We have

$$\exp\left(-\frac{d}{dz}\zeta_{a,P_-\Phi/(2\pi)}(s/(2\pi),z)|_{z=0}\right) = \prod_{q=0}^{2} \exp\left(-\frac{d}{dz}\zeta_{a,P_-\frac{\Phi}{2\pi}|_{H^q(X^*)}}(s/(2\pi),z)|_{z=0}\right)$$

$$= \prod_{q=0}^{2} \exp\left((-1)^q \frac{d}{dz}\zeta_{\Phi_q}(s/(2\pi),z)|_{z=0}\right),$$

where $\Phi_q = \Phi|_{H^q(\tilde{X}^*)^{N=0}}$. The result then follows by Proposition 2.24.
$\diamond$

We give a few more examples of computations with zeta functions related to the arithmetic spectral triple.

Example 3.18 *For $Re(s) >> 0$, the zeta function* (3.6) *of the Dirac operator Φ is given by*

$$\zeta_\Phi(s) = \mathrm{Tr}(|\Phi|^{-s}) = (4g+4)\zeta(s) + 1 + \frac{1}{2^s}, \tag{3.27}$$

where $\zeta(s)$ is the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

Proof. We compute explicitly $\mathrm{Tr}(|\Phi|^{-s})$. On the complement of the zero modes, the operator $|\Phi|$ has eigenvalues the positive integers, and the corresponding eigenspaces $E_n(|\Phi|)$ are described as follows:

$$\begin{array}{rlll} E_1(|\Phi|) = & gr_{-2}H^0(X^*)\oplus & & \\ & gr_{-2}H^1(X^*)\oplus & & gr_4H^1(X^*)\oplus \\ & gr_{-2}H^2(X^*)\oplus & gr_2H^2(X^*)\oplus & gr_4H^2(X^*)\oplus \\ & & gr_4H^3(X^*), & \end{array} \tag{3.28}$$

$$
\begin{array}{rlll}
E_2(|\Phi|) = & gr_{-4}H^0(X^*)\oplus & & \\
& gr_{-4}H^1(X^*)\oplus & & gr_6 H^1(X^*)\oplus \\
& gr_{-4}H^2(X^*)\oplus & gr_4 H^2(X^*)\oplus & gr_6 H^2(X^*)\oplus \\
& & & gr_6 H^3(X^*),
\end{array} \tag{3.29}
$$

$$
\begin{array}{rll}
E_n(|\Phi|) = & gr_{-2n}H^0(X^*)\oplus & \\
& gr_{-2n}H^1(X^*)\oplus & gr_{2(n+1)}H^1(X^*)\oplus \\
& gr_{-2n}H^2(X^*)\oplus & gr_{2(n+1)}H^2(X^*)\oplus \\
& & gr_{2(n+1)}H^3(X^*),
\end{array} \tag{3.30}
$$

for $n \geq 3$.

Using then the result of Proposition 2.22 to compute the dimension of these eigenspaces, we have

$$
\begin{array}{llll}
\dim gr_{2p}H^0(X^*) = 1 & (p \leq 0) & & \\
\dim gr_{2p}H^1(X^*) = 2g & (p \leq 0) & \dim gr_{2p}H^1(X^*) = 1 & (p \geq 1) \\
\dim gr_{2p}H^2(X^*) = 1 & (p \leq 1) & \dim gr_{2p}H^2(X^*) = 2g & (p \geq 2) \\
& & \dim gr_2 H^2(X^*) = 1 & \\
& & \dim gr_{2p}H^3(X^*) = 1 & (p \geq 2)
\end{array}
$$

We obtain

$$
\dim E_n(|\Phi|) = \begin{cases} (4g+4) & n \geq 3 \\ (4g+5) & n = 1,2 \end{cases} \tag{3.31}
$$

This completes the calculation. Thus, we obtain

$$
\mathrm{Tr}(|\Phi|^{-s}) = \sum_{n\geq 1} \dim E_n(|\Phi|) n^{-s} = (4g+5)(1+\frac{1}{2^s}) + (4g+4)\sum_{n\geq 3} n^{-s}
$$

$$
= (4g+4)\zeta(s) + 1 + \frac{1}{2^s}.
$$

$\diamond$

As an immediate consequence of this calculation we obtain the volume of the non-commutative space $\mathrm{C}^*(\tilde{\Gamma})$.

Example 3.19 *The volume of the non-commutative space* $\mathrm{C}^*(\tilde{\Gamma})$*, in the metric determined by* Φ *is given by* $V = (4g+4)$*.*

Proof. We compute the volume using the residue formula (3.7). Recall that the Riemann zeta function has residue 1 at $s = 1$. In fact, the well known formula

$$
\lim_{s\to 1+} (s-1)\zeta_{\mathbb{K}}(s) = 2^{r_1}(2\pi)^{r_2}|d|^{-1/2} hRw^{-1},
$$

holds for an arbitrary number field $\mathbb{K}$, with class number h, w roots of unity, discriminant d and regulator R, with r_1 and r_2 counting the embeddings of $\mathbb{K}$ into $\mathbb{R}$ and $\mathbb{C}$. Applied to $\mathbb{K} = \mathbb{Q}$ this yields the result. This implies that, for the zeta function computed in (3.27), we obtain

$$V = \mathrm{Tr}_\omega(|\Phi|^{-1}) = \mathrm{Res}_{s=1}\mathrm{Tr}(|\Phi|^{-s}) = (4g+4)\mathrm{Res}_{s=1}\zeta(s) = (4g+4). \tag{3.32}$$

$\diamond$

Notice how, while the handlebody $\mathfrak{X}_{\tilde{\Gamma}}$ in its natural hyperbolic metric has infinite volume, the Dirac operator Φ induces on $C^*(\tilde{\Gamma})$, which is a non-commutative version of the handlebody, a metric of finite volume.

It is evident from the calculation of the eigenspaces $E_n(|\Phi|)$ in Example 3.18 that the Dirac operator Φ has a spectral asymmetry (*cf.* [2]). This corresponds to an eta invariant, which can be computed easily from the dimensions of the eigenspaces in Example 3.18, as follows.

Example 3.20 *The eta function of the Dirac operator Φ is given by*

$$\eta_\Phi(s) := \sum_{0\neq\lambda\in\mathrm{Spec}(\Phi)} \mathrm{sign}(\lambda)\frac{1}{|\lambda|^s} = 1 + \frac{1}{2^s}, \tag{3.33}$$

The eta invariant $\eta_\Phi(0) = 2$, measuring the spectral asymmetry, is independent of g.

3.4 Zeta function of the special fiber and Reidemeister torsion

In this paragraph we show that the expression (3.26) of Theorem 3.17 can be interpreted as a *Reidemeister torsion*, and it is related to a zeta function for the fiber at arithmetic infinity.

We begin by giving the definition of a zeta function of the special fiber of a semistable fibration, which motivates the analogous notion at arithmetic infinity.

Let X be a regular, proper and flat scheme over $\mathrm{Spec}(\Lambda)$, for Λ a discrete valuation ring with quotient field K and finite residue field k. Assume that X has geometrically reduced, connected and one-dimensional fibers. Let us denote by η and v *resp.* the generic and the closed point of $\mathrm{Spec}(\Lambda)$ and by $\bar{\eta}$ and $\bar{v}$ the corresponding geometric points. Assume that the special fiber X_v of X is a connected, effective Cartier divisor with reduced normal crossings defined over $k = k(v)$. This degeneration is sometime referred to as a *semistable* fibration over $\mathrm{Spec}(\Lambda)$.

Let N_v denote the cardinality of k. Then, define the zeta-function of the special fiber X_v as follows (u is an indeterminate)

$$Z_{X_v}(u) = \frac{P_1(u)}{P_0(u)P_2(u)}, \qquad P_i(u) = \det(1 - \mathrm{f}^*\mathrm{u} \mid \mathrm{H}^{\mathrm{i}}(\mathrm{X}_{\bar{\eta}}, \mathbb{Q}_\ell)^{\mathrm{I}_{\bar{v}}}), \tag{3.34}$$

where f^* is the geometric Frobenius *i.e.* the map induced by the Frobenius morphism $f : X_{\bar{v}} \to X_{\bar{v}}$ on the cohomological inertia-invariants at $\bar{v}$.

The polynomials $P_i(u)$ are closely related to the characteristic polynomials of the Frobenius $F_i(u) = \det(\mathrm{u}\cdot 1 - \mathrm{f}^* \mid \mathrm{H}^{\mathrm{i}}(\mathrm{X}_{\bar{\eta}}, \mathbb{Q}_\ell)^{\mathrm{I}_{\bar{v}}})$ through the formula

$$P_i(u) = u^{b_i} F_i(u^{-1}), \qquad b_i = \mathrm{degree}(\mathrm{F_i}). \tag{3.35}$$

The zeta function $Z_{X_v}(u)$ generalizes on a semistable fiber the description of the Hasse-Weil zeta function of a smooth, projective curve over a finite field.

Based on this construction we make the following definition for the fiber at an archimedean prime of an arithmetic surface:

$$Z_\Phi(u) := \frac{P_1(u)}{P_0(u)P_2(u)}, \tag{3.36}$$

where we set

$$P_q(u) := \det_\infty \left(\frac{1}{2\pi} - u\frac{\Phi_q}{2\pi} \right), \tag{3.37}$$

with $\Phi_q = \Phi|_{H^q(\tilde{X}^*)^{N=0}}$.

In order to see how this is related to the result of Theorem 3.17, we recall briefly a simple observation of Milnor (*cf.* §3 [32]). Suppose given a finite complex L and an infinite cyclic covering $\tilde{L}$, with $H_*(\tilde{L}, \kappa)$ finitely generated over the coefficient field κ. Let $h : \pi_1 L \to \kappa(s)$ be the composition of the homomorphism $\pi_1 L \to \Pi$ associated to the cover with the inclusion $\Pi \subset \mathrm{Units}(\kappa(s))$. The Reidemeister torsion for this covering is given (up to multiplication by a unit of $\kappa\Pi$) by the alternating product of the characteristic polynomials $F_q(s)$ of the κ–linear map

$$s_* : H_q(\tilde{L}, \kappa) \to H_q(\tilde{L}, \kappa),$$

$$\tau(s) \simeq F_0(s) F_1(s)^{-1} F_2(s) \cdots F_n(s)^{\pm 1}. \tag{3.38}$$

Moreover, for a map $T : L \to L$, let $\zeta_T(u)$ be the Weil zeta

$$\zeta_T(u) = P_0(u)^{-1} P_1(u) P_2(u)^{-1} \cdots P_n(u)^{\pm 1},$$

where the polynomials $P_q(u)$ of the map T_* are related to the characteristic polynomials $F_q(s)$ by (3.35) and b_q are the q-the Betti number of the complex L. By analogy with (3.38), Milnor writes the Reidemeister torsion $\tau_T(s)$ (up to multiplication by a unit) as

$$\tau_T(s) := F_0(s) F_1(s)^{-1} F_2(s) \cdots F_n(s)^{\mp 1},$$

where $F_q(s)$ are the characteristic polynomials of the map T_*. Then the relation between zeta function and Reidemeister torsion is given by:

$$\zeta_T(s^{-1}) \tau_T(s) = s^{\chi(L)}, \tag{3.39}$$

where $\chi(L)$ is the Euler characteristic of L.

Similarly, we can derive the relation between the zeta function of the fiber at infinity defined as in (3.36) and the alternating product of Gamma factors in (3.26).

First notice that the expression (3.26) is of the form (3.38). Namely, we write

$$\frac{L_{\mathbb{C}}(H^1(X_{/\mathbb{C}}, \mathbb{C}), s)}{L_{\mathbb{C}}(H^0(X_{/\mathbb{C}}, \mathbb{C}), s) \cdot L_{\mathbb{C}}(H^2(X_{/\mathbb{C}}, \mathbb{C}), s)} = \frac{F_0(s) \cdot F_2(s)}{F_1(s)}, \tag{3.40}$$

where we set

$$F_q(s) := \det_\infty \left(\frac{s}{2\pi} - \frac{\Phi_q}{2\pi} \right), \tag{3.41}$$

with $\Phi_q = \Phi|_{H^q(\tilde{X}^*)^{N=0}}$. For this reason we may regard (3.40) as the Reidemeister torsion of the fiber at arithmetic infinity:

$$\tau_\Phi(s) := \frac{F_0(s) \cdot F_2(s)}{F_1(s)}. \tag{3.42}$$

The relation between zeta function and Reidemeister torsion is then given as follows.

Proposition 3.21 *The zeta function Z_Φ of (3.36) and the Reidemeister torsion τ_Φ of (3.42) are related by*

$$Z_\Phi(s^{-1})\tau_\Phi(s) = s^{g-2} e^{\chi\, s \log s},$$

with g is the genus of the Riemann surface $X_{/\mathbb{C}}$ and $\chi = 2-2g$ its Euler characteristic.

Proof. The result follows by direct calculation of the regularized determinants as in Section 2.5. Namely, we compute (in the case $q = 0, 1$)

$$P_q(u) = \det_\infty \left(\frac{1}{2\pi} - u\frac{\Phi_q}{2\pi} \right) = \exp\left(-b_q \frac{d}{dz}((2\pi)^z \sum_{n\geq 0}(1+un)^{-z})|_{z=0} \right)$$

$$= \exp\left(b_q \left(\log\Gamma\left(\frac{1}{u}\right) + \frac{\log 2\pi}{u} + \frac{\log u}{2} + \frac{\log u}{u} \right) \right) = u^{b_q/2} e^{-b_q \frac{\log u}{u}} (2\pi)^{-1/u} \Gamma(1/u),$$

where b_q are the Betti numbers of $X_{/\mathbb{C}}$. The case $q = 2$ is analogous, but for the presence of the $+1$ eigenvalue in the spectrum of Φ_2, hence we obtain

$$P_2(u) = \exp\left(-b_2 \frac{d}{dz} \left((2\pi)^z u^{-z} \zeta(1/u, z) - \left(\frac{1}{u} - 1\right)^{-z} \right)_{z=0} \right)$$

$$= \Gamma_{\mathbb{C}}\left(\frac{1}{u} - 1\right)^{-1} u^{-3/2} e^{\frac{\log u}{u}}.$$

Thus, we obtain

$$Z_\Phi(s^{-1}) = \frac{L_{\mathbb{C}}(H^0(X_{/\mathbb{C}}, \mathbb{C}), s) \cdot L_{\mathbb{C}}(H^2(X_{/\mathbb{C}}, \mathbb{C}), s)}{L_{\mathbb{C}}(H^1(X_{/\mathbb{C}}, \mathbb{C}), s)} s^{g-2} e^{\chi\, s \log s}.$$

$\diamond$

4 Shift operator and dynamics.

In this section we consider an arithmetic surface X over $\mathrm{Spec}(O_{\mathbb{K}})$, with $\mathbb{K}$ a number field, and a fixed archimedean prime which corresponds to a *real* embedding $\alpha : \mathbb{K} \hookrightarrow \mathbb{R}$. We also assume that the corresponding Riemann surface $X_{/\mathbb{R}}$ is an orthosymmetric smooth real algebraic curve of genus $g \geq 2$.

We consider a Schottky uniformization of the Riemann surface $X_{/\mathbb{R}}$ and the hyperbolic filling given by the handlebody $\mathfrak{X}_\Gamma$ that has $X_{/\mathbb{R}}$ as the conformal boundary at infinity, $\mathfrak{X}_\Gamma \cup X_{/\mathbb{R}} = \Gamma\backslash(\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C}))$.

Geodesics in $\mathfrak{X}_\Gamma$ can be lifted to geodesics in $\mathbb{H}^3$ with ends on $\mathbb{P}^1(\mathbb{C})$. Among these, geodesics with one or both ends on $\Omega_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ correspond to geodesics in $\mathfrak{X}_\Gamma$ that reach the boundary at infinity $X_{/\mathbb{R}} = \Gamma\backslash\Omega_\Gamma$ in infinite time. The geodesics in $\mathbb{H}^3$ with both ends on $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ project in the quotient to geodesics contained in the *convex core* $\mathfrak{C}_\Gamma = \Gamma\backslash$ Hull (Λ_Γ) of $\mathfrak{X}_\Gamma$. Since the Schottky group Γ is geometrically finite, $\mathfrak{C}_\Gamma$ is a bounded region inside $\mathfrak{X}_\Gamma$ (*cf.* [31]). For this reason, geodesics in $\mathfrak{X}_\Gamma$ that lift to geodesics in $\mathbb{H}^3$ with both ends on Λ_Γ are called *bounded geodesics*.

We denote by $\Xi \subset \mathfrak{X}_\Gamma$ the image under the quotient map of all geodesics in $\mathbb{H}^3$ with endpoints on $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$, endowed with the induced topology. Similarly, we denote by $\Xi_c \subset \Xi$ the image in $\mathfrak{X}_\Gamma$ of all geodesics in $\mathbb{H}^3$ with endpoints of the form $\{z^-(h), z^+(h)\}$ for some primitive $h \in \Gamma$. Here $z^\pm(h)$ are the attractive and repelling fixed points of h, and the element h is primitive in Γ if it is not a power of some other element of Γ.

Definition 4.1 *We denote by $\tilde{\Xi}$ the orientation double cover of Ξ and we refer to it as the* infinite tangle of bounded geodesics. *Similarly, we define $\tilde{\Xi}_c$ to be the orientation double cover of Ξ_c and we refer to it as the* tangle of primitive closed geodesics.

Since Ξ is orientable, $\tilde{\Xi} \cong \Xi \times \mathbb{Z}/2$, where the second coordinate is the choice of an orientation on each geodesic. So, for instance, the geodesics in $\mathbb{H}^3$ with endpoints $\{z^-(h), z^+(h)\}$ and $\{z^+(h), z^-(h)\} = \{z^-(h^{-1}), z^+(h^{-1})\}$ correspond to the same closed geodesic in Ξ_c, while in the double cover $\tilde{\Xi}_c$ they give rise to the two different lifts of the geodesic in Ξ_c.

More precisely, let $L_{\{a,b\}}$ denote the geodesic in $\mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R}^+$ with endpoints $\{a, b\}$ in the complement of the diagonal in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$,

$$L_{\{a,b\}}(s) = \left(\frac{ae^s + be^{-s}}{e^s + e^{-s}}, \frac{|a-b|}{e^s + e^{-s}} \right) \qquad s \in \mathbb{R} \tag{4.1}$$

The parameter s in (4.1) becomes the parameterization by arc length on the corresponding geodesic $\pi_\Gamma(L_{\{a,b\}})$ in $\tilde{\Xi}$, where $\pi_\Gamma : \mathbb{H}^3 \to \mathfrak{X}_\Gamma$ is the quotient map. Then we have

$$\tilde{\Xi} = \{\pi_\Gamma(L_{\{a,b\}}(s)) : \ s \in \mathbb{R}, \ (a,b) \in (\Lambda_\Gamma \times \Lambda_\Gamma)^0\},$$

where

$$(\Lambda_\Gamma \times \Lambda_\Gamma)^0 := (\Lambda_\Gamma \times \Lambda_\Gamma) \smallsetminus \Delta$$

denotes the complement of the diagonal in $\Lambda_\Gamma \times \Lambda_\Gamma$.

Remark 4.2 The $\mathbb{Z}/2$ involution on $\tilde{\Xi}$ that has Ξ as quotient corresponds to the involution that exchanges the two factors in $\Lambda_\Gamma \times \Lambda_\Gamma$.

Remark 4.3 Notice that most constructions and results presented in this section are topological in nature, hence they do not require any assumption on the conformal structure of the Riemann surface $X_{/\mathbb{R}}$. In particular they are not sensitive to whether X is a complex or real smooth algebraic curve. However, when we refer to the results of the previous sections, by comparing the dynamical and archimedean cohomology in Theorem 5.5, we need to know that the $\mathbb{Z}/2$ cover $\tilde{\Xi}$ is compatible with the conformal structure on the Riemann surface. This requires the presence of the real structure, so that complex conjugation $z \mapsto \bar{z}$ on $\mathbb{P}^1(\mathbb{C})$, which induces the change of orientation on the geodesics $L_{\{a,b\}}$ in $\mathbb{H}^3$, determines an involution on the Γ–quotient.

According to [24], the tangle of bounded geodesics $\tilde{\Xi}$ provides a geometric realization of the *dual graph* $\mathcal{G}_\infty$ of the maximally degenerate closed fiber at arithmetic infinity.

In this section, we consider a resolution of $\tilde{\Xi}$ by the mapping torus (suspension flow) $\mathcal{S}_T$ of a dynamical system T (*cf.* Proposition 4.11), with a surjection $\mathcal{S}_T \to \tilde{\Xi}$.

In Theorem 4.12 we give an explicit description of the cohomology $H^1(\mathcal{S}_T)$. Such cohomology is endowed with a filtration whose graded pieces depend uniquely on the coding of geodesics in $\tilde{\Xi}$ (*cf.* Proposition 4.14 and Proposition 4.15). We also show that the C*-algebra describing the quotient Λ_Γ/Γ acts on the cohomology $H^1(\mathcal{S}_T)$

We recall the construction and basic properties of the Cuntz-Krieger algebra $\mathcal{O}_A$ associated to the shift of finite type $(\mathcal{S}, T)$. In Theorem 4.17 we show that this algebra describes, as a non-commutative space, the quotient of the limit set Λ_Γ by the action of the Schottky group Γ. Moreover, in Proposition 4.19, we show that the algebra $\mathcal{O}_A$ acts on a suitable Hilbert space completion of the cohomology $H^1(\mathcal{S}_T)$.

4.1 The limit set and the shift operator

Given a choice of a set of generators $\{g_i\}_{i=1}^g$ for the Schottky group Γ, there is a bijection between the elements of Γ and the set of all *reduced words* in the $\{g_i\}_{i=1}^{2g}$, where we use the notation $g_{i+g} := g_i^{-1}$, for $i = 1, \dots, g$. Here by reduced words we mean all finite sequences $w = a_0 \dots a_\ell$ in the g_i, for any $\ell \in \mathbb{N}$, satisfying $a_{i+1} \neq a_i$ for all $i = 0 \dots, \ell - 1$.

We also consider the set $\mathcal{S}^+$ of all right–infinite *reduced sequences* in the $\{g_i\}_{i=1}^{2g}$,

$$\mathcal{S}^+ = \{a_0 a_1 \dots a_\ell \dots \;|a_i \in \{g_i\}_{i=1}^{2g},\; a_{i+1} \neq a_i, \forall i \in \mathbb{N}\}, \tag{4.2}$$

and the set $\mathcal{S}$ of *doubly infinite reduced sequences* in the $\{g_i\}_{i=1}^{2g}$,

$$\mathcal{S} = \{\dots a_{-m} \dots a_{-1} a_0 a_1 \dots a_\ell \dots \ |a_i \in \{g_i\}_{i=1}^{2g},\ a_{i+1} \neq a_i, \forall i \in \mathbb{Z}\}. \tag{4.3}$$

On the space $\mathcal{S}$ we consider the topology generated by the sets $W^s(x,\ell) = \{y \in \mathcal{S} | x_k = y_k, k \geq \ell\}$, and the $W^u(x,\ell) = \{y \in \mathcal{S} | x_k = y_k, k \leq \ell\}$ for $x \in \mathcal{S}$ and $\ell \in \mathbb{Z}$. This induces a topology with analogous properties on $\mathcal{S}^+$ by realizing it as a subset of $\mathcal{S}$, for instance, by extending each sequence to the left as a constant sequence.

We define the *one-sided shift operator* T on $\mathcal{S}^+$ as the map

$$T(a_0 a_1 a_2 \dots a_\ell \dots) = a_1 a_2 \dots a_\ell \dots \tag{4.4}$$

We also define a *two-sided shift operator* T on $\mathcal{S}$ as the map

$$\begin{array}{cccccccccc} T(& \dots & a_{-m} & \dots & a_{-1} & a_0 & a_1 & \dots & a_\ell & \dots &) = \\ & \dots & a_{-m+1} & \dots & a_0 & a_1 & a_2 & \dots & a_{\ell+1} & \dots & \end{array} \tag{4.5}$$

Given a choice of a base point $x_0 \in \mathbb{H}^3 \cup \Omega_\Gamma$ we can define a map $Z : \mathcal{S}^+ \to \Lambda_\Gamma$ in the following way. For an eventually periodic sequence $w\overline{a_0 \dots a_N} \in \mathcal{S}^+$, with an initial word w, we set

$$Z(w\overline{a_0 \dots a_N}) = wz^+(a_0 \dots a_N). \tag{4.6}$$

Here we identify the finite reduced word w with an element in Γ, hence $wz^+(a_0 \dots a_N)$ is the image under $w \in \Gamma$ of the attractive fixed point of the element $a_0 \dots a_N$ of Γ. For sequences $a_0 \dots a_\ell \dots$ that are not eventually periodic, we set

$$Z(a_0 \dots a_\ell \dots) = \lim_{\ell \to \infty} (a_0 \dots a_\ell) x_0, \tag{4.7}$$

where again we identify a finite reduced word $a_0 \dots a_\ell$ with an element in Γ.

We also introduce the following notation. We denote by $\mathcal{S}^+(w) \subset \mathcal{S}^+$ the set of right infinite reduced sequences in the $\{g_i\}_{i=1}^{2g}$ that begin with an assigned word $w = a_0 \dots a_\ell$. If $g \in \Gamma$ is expressed as a reduced word w in the g_i, we write $\Lambda_\Gamma(g) := Z(\mathcal{S}^+(w))$.

In the following we denote by $\Lambda_\Gamma \times_\Gamma \Lambda_\Gamma$ the quotient by the diagonal action of Γ of the *complement of the diagonal* $(\Lambda_\Gamma \times \Lambda_\Gamma)^0$.

The group Γ acts on $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations, hence on Λ_Γ, which is a Γ-invariant subset of $\mathbb{P}^1(\mathbb{C})$. The group Γ also acts on $\mathcal{S}^+$: an element $\gamma \in \Gamma$, identified with a reduced word $\gamma = c_0 \dots c_k$ in the g_i, maps a sequence $a_0 \dots a_\ell \dots$ to the sequence obtained from $c_0 \dots c_k a_0 \dots a_\ell \dots$ by making the necessary cancellations that yield a reduced sequence.

Lemma 4.4 *The following properties are satisfied.*

1. *The spaces $\mathcal{S}^+$ and $\mathcal{S}$ are topologically Cantor sets. The one-sided shift T of* (4.4) *is a continuous surjective map on $\mathcal{S}^+$, while the two-sided shift T of* (4.5) *is a homeomorphism of $\mathcal{S}$.*

2. *The limit set Λ_Γ with the topology induced by the embedding in $\mathbb{P}^1(\mathbb{C})$ is also a Cantor set, and the map Z of* (4.6) *and* (4.7) *is a homeomorphism.*

   The shift operator T on $\mathcal{S}^+$ induces the map $ZTZ^{-1} : \Lambda_\Gamma \to \Lambda_\Gamma$ of the form

$$ZTZ^{-1}|_{\Lambda_\Gamma(g_i)}(z) = g_i^{-1}(z).$$

3. *The map Z is Γ-equivariant.*

Proof. *1.* The first claim can be verified easily.

2. It is not hard to see that the correspondence $a_0 \dots a_\ell \dots \mapsto \lim_{\ell\to\infty}(a_0 \dots a_\ell)x_0$ gives a bijection between the complement of eventually periodic sequences in $\mathcal{S}^+$ and the complement of the fixed points $\{z^-(h), z^+(h)\}_{h\in\Gamma}$ in Λ_Γ (*cf.* [17] Prop. 1.2). To see that the correspondence $w\overline{a_0 \dots a_N} \mapsto w\, z^+(a_0 \dots a_N) = z^+(wa_0 \dots a_N w^{-1})$ is a bijection between the set of eventually periodic sequences and the set of fixed points $\{z^-(h), z^+(h)\}_{h\in\Gamma}$ in Λ_Γ, we proceed as in [17]. For any $h \in \Gamma$, h can be written as a reduced word $h = a_0 \dots a_\ell$ in the g_i's. If this word satisfies $a_\ell \neq a_0^{-1}$, then $Z^{-1}(z^+(h)) = \overline{a_0 \dots a_\ell}$ and $Z^{-1}(z^-(h)) = \overline{a_\ell^{-1} \dots a_0^{-1}}$. If $a_\ell = a_0^{-1}$, then there is an element $\gamma \in \Gamma$, such that $h = \gamma a_{i_k} \dots a_{i_{k+N}} \gamma^{-1}$, with $a_{i_{k+N}} \neq a_{i_k}^{-1}$. In this case $Z^{-1}(z^+(h)) = \gamma\overline{a_{i_k} \dots a_{i_{k+N}}}$ and $Z^{-1}(z^-(h)) = \gamma\overline{a_{i_{k+N}}^{-1} \dots a_{i_k}^{-1}}$. As a continuous bijection from a compact to a Hausdorff space, Z is a homeomorphism. The expression for ZTZ^{-1} is then immediate.

3. By continuity it is sufficient to check that the map Z restricted to the dense subset $\{z^-(h), z^+(h)\}_{h\in\Gamma} \subset \Lambda_\Gamma$ is Γ-equivariant. This was proved already in *2.*, since $Z(\gamma\overline{a_0 \dots a_N}) = \gamma\, z^+(a_0 \dots a_N)$.
$\diamond$

We recall two notions that will be useful in the following.

Let $A = (A_{ij})$ be an $N \times N$ elementary matrix. The *subshift of finite type* with *transition matrix* A is the subset of the set of all doubly infinite sequences in the alphabet $\{1, \dots, N\}$ of the form

$$\mathcal{S}_A := \{\dots i_{-m} \dots i_{-1} i_0 i_1 \dots i_\ell \dots \, |1 \leq i_k \leq N,\, A_{i_k i_{k+1}} = 1, \forall k \in \mathbb{Z}\}. \tag{4.8}$$

A double sided shift operator of the form (4.5) can be defined on any subshift of finite type.

In the following we will consider the case where the elementary matrix A is the the symmetric $2g \times 2g$ matrix with $A_{ij} = 0$ for $|i - j| = g$ and $A_{ij} = 1$ otherwise.

Definition 4.5 *The pair $(\mathcal{S}, T)$ of a space and a homeomorphism is a* Smale space *if locally $\mathcal{S}$ can be decomposed as the product of expanding and contracting directions for T. Namely, the following properties are satisfied.*

1. *For every point $x \in \mathcal{S}$ there exist subsets $W^s(x)$ and $W^u(x)$ of $\mathcal{S}$, such that $W^s(x) \times W^u(x)$ is homeomorphic to a neighborhood of x.*

2. *The map T is contracting on $W^s(x)$ and expanding on $W^u(x)$, and $W^s(Tx)$ and $T(W^s(x))$ agree in some neighborhood of x, and so do $W^u(Tx)$ and $T(W^u(x))$.*

Lemma 4.6 *The following properties are satisfied.*

1. *The map $Q : \mathcal{S} \to \Lambda_\Gamma \times \Lambda_\Gamma$*

$$Q(\ldots a_{-m} \ldots a_{-1} a_0 a_1 \ldots a_\ell \ldots) = \big(Z(a_{-1}^{-1} a_{-2}^{-1} \ldots a_{-m}^{-1} \ldots)\,,\, Z(a_0 a_1 a_2 \ldots a_\ell \ldots)\big)$$

is an embedding of the space $\mathcal{S}$ in the Cartesian product $\Lambda_\Gamma \times \Lambda_\Gamma$. The image of the embedding Q is given by $\mathrm{Im}(Q) = \bigcup_{i \neq j} \Lambda_\Gamma(g_i) \times \Lambda_\Gamma(g_j)$. On $\mathrm{Im}(Q)$ the two-sided shift operator T of (4.5) induces the map QTQ^{-1}

$$\begin{aligned} QTQ^{-1}|_{\Lambda_\Gamma(g_i)\times\Lambda_\Gamma(g_j)}(\quad & Z(g_i b_1 b_2 \ldots b_m \ldots), & & Z(g_j a_1 \ldots a_\ell \ldots) \quad) = \\ (\quad & Z(g_j^{-1} g_i b_1 \ldots b_m \ldots), & g_j^{-1} g_j \quad & Z(a_1 a_2 \ldots a_\ell \ldots) \quad). \end{aligned}$$

2. *The map $Q : \mathcal{S} \to \Lambda_\Gamma \times \Lambda_\Gamma$ descends to a homeomorphism of the quotients*

$$\bar{Q} : \mathcal{S}/T \overset{\simeq}{\to} \Lambda_\Gamma \times_\Gamma \Lambda_\Gamma.$$

3. *The space $\mathcal{S}$ can be identified with the subshift of finite type $\mathcal{S}_A$ with the symmetric $2g \times 2g$ matrix $A = (A_{ij})$ with $A_{ij} = 0$ for $|i - j| = g$ and $A_{ij} = 1$ otherwise.*

   The two-sided shift operator T on $\mathcal{S}$ of (4.5) decomposes $\mathcal{S}$ in a product of expanding and contracting directions, so that $(\mathcal{S}, T)$ is a Smale space.

Proof. *1.* The first claim follows directly from the definitions and Lemma 4.4.

2. Notice that $\mathrm{Im}(Q)$ intersects each Γ orbit in the complement of the diagonal $(\Lambda_\Gamma \times \Lambda_\Gamma)^0$. Moreover, if $(a, b) = Q(x)$, then for $\gamma \in \Gamma$ the element $(\gamma a, \gamma b)$ is in $\mathrm{Im}(Q)$ iff $b \in \Lambda_\Gamma(\gamma^{-1})$, and in that case $(\gamma a, \gamma b) = Q(T^n x)$, for $n =$ length (γ) as a reduced word in the g_i. The statement on $\bar{Q}$ then follows easily.

3. We write $\mathcal{S}$ as the shift of finite type $\mathcal{S}_A$. Namely, we identify a reduced sequence $\ldots a_{-m} \ldots a_{-1} a_0 a_1 \ldots a_\ell \ldots$ where each $a_k = g_{i_k} \in \{g_i\}_{i=1}^{2g}$ with the sequence $\ldots i_{-m} \ldots i_{-1} i_0 i_1 \ldots i_\ell \ldots$ satisfying $A_{i_k i_{k+1}} = 1$, for all $k \in \mathbb{Z}$.

Then, for subshifts of finite type with the double sided shift (4.5), the sets

$$W^u(x) = \cup_{\ell \in \mathbb{Z}} W^u(x, \ell),$$

with

$$W^u(x, \ell) := \{y \in \mathcal{S} | x_k = y_k, k \leq \ell\}$$

and

$$W^s(x) = \cup_{\ell \in \mathbb{Z}} W^s(x, \ell).$$

with

$$W^s(x,\ell) := \{y \in \mathcal{S} | x_k = y_k, k \geq \ell\}$$

give the expanding and contracting directions, so that $(\mathcal{S}, T)$ satisfies the properties of a Smale space (*cf.* [37]).

$\diamond$

Remark 4.7 It is well known that one can associate different C*–algebras to Smale spaces (*cf.* [42], [37], [38]). For the Smale space $(\mathcal{S}, T)$ we consider, there are four possibilities: the crossed product algebra $C(\mathcal{S}) \rtimes_T \mathbb{Z}$ and the C*–algebras $\mathrm{C}^*(\mathcal{G}^s) \rtimes_T \mathbb{Z}$, $\mathrm{C}^*(\mathcal{G}^u) \rtimes_T \mathbb{Z}$, $\mathrm{C}^*(\mathcal{G}^a) \rtimes_T \mathbb{Z}$ obtained by considering the action of the shift T on the groupoid C*–algebra (*cf.* [40] for the definition of such algebra) associated to the groupoids $\mathcal{G}^s$, $\mathcal{G}^u$, $\mathcal{G}^a$ of the stable, unstable, and asymptotic equivalence relations on $(\mathcal{S}, T)$.

In the following we consider the algebras $C(\mathcal{S}) \rtimes_T \mathbb{Z}$ and $\mathrm{C}^*(\mathcal{G}^u) \rtimes_T \mathbb{Z}$ and show that the first is a non-commutative space describing the quotient $\Lambda_\Gamma \times_\Gamma \Lambda_\Gamma$ and the second is a non-commutative space describing the quotient Λ_Γ/Γ.

4.2 Coding of geodesics

Since the Schottky group Γ is a free group consisting of only loxodromic elements, the coding of geodesics in $\mathfrak{X}_\Gamma$ in terms of the dynamical system $(\mathcal{S}, T)$ is particularly simple. The following facts are well known. We recall them briefly for convenience.

We denote by $\mathcal{S}^p \subset \mathcal{S}$ the set of periodic reduced sequences in the g_i, *i.e.* the set of periodic points of the shift T. We define

$$\hat{\tilde{\Xi}} := \left\{ \pi_\Gamma(L_{\{a,b\}}) : (a,b) \in (\Lambda_\Gamma \times \Lambda_\Gamma)^0 \right\},$$

and

$$\hat{\tilde{\Xi}}_c := \left\{ \pi_\Gamma(L_{\{z^+(h), z^-(h)\}}) : h \in \Gamma \setminus id \right\},$$

where $L_{\{a,b\}}$ denotes the geodesic as in (4.1).

The following Lemma gives a coding of the primitive closed geodesics in $\tilde{\Xi}_c$ by the quotient $\mathcal{S}^p/T$.

Lemma 4.8 *The correspondence* $\mathcal{L}_c : w\overline{a_0 \dots a_N} \mapsto \pi_\Gamma(L_{\{w\,z^+(a_0 \dots a_N), w\,z^-(a_0 \dots a_N)\}})$ *induces a bijection between* $\mathcal{S}^p/T$ *and* $\hat{\tilde{\Xi}}_c$

Proof. Arguing as in Lemma 4.4, we see that every closed geodesic in $\tilde{\Xi}_c$ is of the form $\pi_\Gamma(L_{\{z^+(a_0 \dots a_N), z^-(a_0 \dots a_N)\}})$ for some reduced sequence $a_0 \dots a_N$ with $a_N \neq a_0^{-1}$. If two elements $\overline{a_0, a_2, \dots, a_N}$ and $\overline{b_0, b_1, \dots, b_M}$ of $\mathcal{S}^p$ represent the same primitive closed geodesic, then the elements $h_a = a_0 a_2 \cdots a_N$ and $h_b = b_0 b_1 \cdots b_M$ are conjugate in Γ, $h_a = g h_b g^{-1}$, by an element $g = c_1 c_2 \cdots c_k$. It is easy to see that this implies $c_k = b_0^{-1}$, $c_{k-1} = b_1^{-1}$, etc. so that, for some $1 \leq N_0 \leq N$, we have

$$\overline{b_0, b_1, \dots, b_M} = T^{N_0}(\overline{a_0, a_1, \dots, a_N}),$$

that is, the two sequences are in the same equivalence class modulo the action of T. We refer to [17] for details.

$\diamond$

Via the map $\mathcal{L}_c$ of Lemma 4.8 we can define on $\hat{\Xi}_c$ a topology which makes it homeomorphic to the quotient $\mathcal{S}^p/T$.

Similarly, we obtain a coding of geodesics in $\tilde{\Xi}$ by the quotient space $\mathcal{S}/T$.

Lemma 4.9 *The map $\mathcal{L} : \mathcal{S} \to \hat{\Xi}$, $\mathcal{L} : x \mapsto \pi_\Gamma(L_{\{a,b\}})$, for $x \in \mathcal{S}$ and $(a,b) = Q(x)$ in $\Lambda_\Gamma \times \Lambda_\Gamma$, induces a bijection between $\mathcal{S}/T$ and $\hat{\Xi}$.*

Proof. Any geodesic in $\tilde{\Xi}$ lifts to a geodesic in $\mathbb{H}^3$ with ends (a,b) in the complement of the diagonal $(\Lambda_\Gamma \times \Lambda_\Gamma)^0$. Notice that in $\mathbb{H}^3$ we have $\gamma L_{\{a,b\}} = L_{\{\gamma a, \gamma b\}}$. The claim then follows easily.

$\diamond$

Via the map $\mathcal{L}$ of Lemma 4.9 we can define on $\hat{\Xi}$ a topology which makes it homeomorphic to the quotient $\mathcal{S}/T$.

We introduce a topological space defined in terms of the Smale space $(\mathcal{S}, T)$, which we consider as a graph associated to the fiber at arithmetic infinity. This maps onto the dual graph $\tilde{\Xi}$ considered in [24].

Definition 4.10 *The mapping torus (suspension flow) of the dynamical system $(\mathcal{S}, T)$ is defined as*

$$\mathcal{S}_T := \mathcal{S} \times [0,1]/(x,0) \sim (Tx,1) \tag{4.9}$$

Consider the map $\tilde{Q} : \mathcal{S}_T \to \tilde{\Xi}$, defined by

$$\tilde{Q}([x,t]) = \pi_\Gamma L_{\{a,b\}}(s(t)), \tag{4.10}$$

where $(a,b) = Q(x)$ in $(\Lambda_\Gamma \times \Lambda_\Gamma)^0$. The smooth change of parameterization $s(t)$ from (4.1) is introduced so that the geodesic line $L_{\{a,b\}}(s(t))$ in $\mathbb{H}^3$ crosses a fundamental domain for the action of Γ in time $t \in [0,1]$.

Proposition 4.11 *The map $\tilde{Q} : \mathcal{S}_T \to \tilde{\Xi}$ of* (4.10) *is a continuous surjection. It is a bijection away from the intersection points of different geodesics in $\tilde{\Xi}$.*

Proof. The parameterization $s(t)$ is chosen in such a way that the map (4.10) is well defined on equivalence classes. By *2.* of Lemma 4.6 and Lemma 4.9 the map is a continuous surjection.

$\diamond$

4.3 Cohomology and homology of $\mathcal{S}_T$

We give an explicit description of the cohomology $H^1(\mathcal{S}_T, \mathbb{Z})$.

Theorem 4.12 *The cohomology $H^1(\mathcal{S}_T, \mathbb{Z})$ satisfies the following properties.*

1. *There is an identification of $H^1(\mathcal{S}_T, \mathbb{Z})$ with the K_0-group of the crossed product* C^**-algebra for the action of T on $\mathcal{S}$,*

$$H^1(\mathcal{S}_T, \mathbb{Z}) \cong K_0(\mathrm{C}(\mathcal{S}) \rtimes_T \mathbb{Z}). \tag{4.11}$$

2. *The identification* (4.11) *endows $H^1(\mathcal{S}_T, \mathbb{Z})$ with a filtration by free abelian groups $F_0 \hookrightarrow F_1 \hookrightarrow \cdots F_n \hookrightarrow \cdots$, with* $\mathrm{rank} F_0 = 2g$ *and* $\mathrm{rank} F_n = 2g(2g-1)^{n-1}(2g-2)+1$*, for $n \geq 1$, so that*

$$H^1(\mathcal{S}_T, \mathbb{Z}) = \lim_n F_n.$$

Proof. *1.* The shift T acting on $\mathcal{S}$ induces an automorphism of the C^*–algebra of continuous functions $C(\mathcal{S})$. With an abuse of notation we still denote it by T. Consider the crossed product C^*–algebra $C(\mathcal{S}) \rtimes_T \mathbb{Z}$. This is a suitable norm completion of $C(\mathcal{S})[T, T^{-1}]$ with product $(V * W)_k = \sum_{r \in \mathbb{Z}} V_k \cdot (T^r W_{r+k})$, for $V = \sum_k V_k T^k$, $W = \sum_k W_k T^k$, and $V * W = \sum_k (V * W)_k T^k$.

The K–theory group $K_0(\mathrm{C}(\mathcal{S}) \rtimes_T \mathbb{Z})$ is described by the co–invariants of the action of T (*cf.* [8],[34]). Namely, let $\mathrm{C}(\mathcal{S}, \mathbb{Z})$ be the set of continuous functions from $\mathcal{S}$ to the integers. This is an abelian group generated by characteristic functions of clopen sets of $\mathcal{S}$. The *invariants* and *co–invariants* are given respectively by $\mathrm{C}(\mathcal{S}, \mathbb{Z})^T := \{f \in \mathrm{C}(\mathcal{S}, \mathbb{Z}) \,|\, f - f \circ T = 0\}$ and $\mathrm{C}(\mathcal{S}, \mathbb{Z})_T := \mathrm{C}(\mathcal{S}, \mathbb{Z})/\mathrm{B}(\mathcal{S}, \mathbb{Z})$, with $\mathrm{B}(\mathcal{S}, \mathbb{Z}) := \{f - f \circ T \,|\, f \in \mathrm{C}(\mathcal{S}, \mathbb{Z})\}$. We have the following result (*cf.* [8]):

- The C^*–algebra $\mathrm{C}(\mathcal{S})$ is a commutative AF–algebra (approximately finite dimensional), obtained as the direct limit of the finite dimensional commutative C^*–algebras generated by characteristic functions of a covering of $\mathcal{S}$. Thus, $K_0(\mathrm{C}(\mathcal{S})) \cong \mathrm{C}(\mathcal{S}, \mathbb{Z})$, being the direct limit of the K_0-groups of the finite dimensional commutative C^*–algebras, and $K_1(\mathrm{C}(\mathcal{S})) = 0$ for the same reason.

- The Pimsner–Voiculescu exact sequence (*cf.* [35]) then becomes of the form

$$0 \to K_1(\mathrm{C}(\mathcal{S}) \rtimes_T \mathbb{Z}) \to \mathrm{C}(\mathcal{S}, \mathbb{Z}) \stackrel{I-T_*}{\to} \mathrm{C}(\mathcal{S}, \mathbb{Z}) \to K_0(\mathrm{C}(\mathcal{S}) \rtimes_T \mathbb{Z}) \to 0, \tag{4.12}$$

  with $K_0(\mathrm{C}(\mathcal{S}) \rtimes_T \mathbb{Z}) \cong \mathrm{C}(\mathcal{S}, \mathbb{Z})_T$. Since the shift T is *topologically transitive*, i.e. it has a dense orbit, we also have $K_1(\mathrm{C}(\mathcal{S}) \rtimes_T \mathbb{Z}) \cong \mathrm{C}(\mathcal{S}, \mathbb{Z})^T \cong \mathbb{Z}$.

Now consider the cohomology group $H^1(\mathcal{S}_T, \mathbb{Z})$. Via the identification with Čech cohomology, we can identify $H^1(\mathcal{S}_T, \mathbb{Z})$ with the group of homotopy classes of continuous maps of $\mathcal{S}_T$ to the circle. The isomorphism

$$\mathrm{C}(\mathcal{S}, \mathbb{Z})_T \cong H^1(\mathcal{S}_T, \mathbb{Z}) \tag{4.13}$$

is then given explicitly by

$$f \mapsto [\exp(2\pi i t f(x))], \tag{4.14}$$

for $f \in \mathrm{C}(\mathcal{S},\mathbb{Z})$ and with $[\cdot]$ the homotopy class. The map is well defined on the equivalence class of f mod $B(\mathcal{S},\mathbb{Z})$ since, for an element $f - h + h\circ T$ the function

$$\exp(2\pi i t(f - h + h\circ T)(x)) = \exp(2\pi i t f(x)) \exp(2\pi i((1-t)h(x) + th(T(x)))),$$

since h is integer valued, but $\exp(2\pi i((1-t)h(x)+th(T(x))))$ is homotopic to the constant function equal to 1. It is not hard to see that (4.14) gives the desired isomorphism (4.13) (*cf.* §4-5 [8]). This proves the first statement.

2. There is a filtration on the set of coinvariants $\mathrm{C}(\mathcal{S},\mathbb{Z})_T$ (*cf.* Theorem 19 §4 of [34]). It is obtained in the following way. First, it is possible to identify

$$\mathrm{C}(\mathcal{S},\mathbb{Z})_T = \mathrm{C}(\mathcal{S},\mathbb{Z})/B(\mathcal{S},\mathbb{Z}) \cong \mathcal{P}/\delta\mathcal{P}, \tag{4.15}$$

where $\mathcal{P} \subset \mathrm{C}(\mathcal{S},\mathbb{Z})$ is the set of functions that depend only on future coordinates, and δ is the operator $\delta(f) = f - f\circ T$. In fact, since characteristic functions of clopen sets in $\mathcal{S}$ depend only on finitely many coordinates, any function in $\mathrm{C}(\mathcal{S},\mathbb{Z})$, when composed with a sufficiently high power of T becomes a function only of the future coordinates, *i.e.* of $\mathcal{S}^+$, the set of (right) infinite reduced sequences in the generators of Γ and their inverses, $\{g_i\}_{i=1}^{2g}$. Since all the $f\circ T^k$, $k \geq 0$, define the same equivalence class in $\mathrm{C}(\mathcal{S},\mathbb{Z})_T$, we have the identification of (4.15).

Then $\mathcal{P}$ can be identified with $\mathrm{C}(\mathcal{S}^+,\mathbb{Z})$ viewed as the submodule of the $\mathbb{Z}$-module $\mathrm{C}(\mathcal{S},\mathbb{Z})$ of functions that only depend on future coordinates. As such, it is generated by characteristic functions of clopen subsets of $\mathcal{S}^+$. A basis of clopen sets for the topology of $\mathcal{S}^+$ is given by the sets $\mathcal{S}^+(w) \subset \mathcal{S}^+$, where $w = a_0 \dots a_N$ is a reduced word in the g_i and $\mathcal{S}^+(w)$ is the set of reduced right infinite sequences $b_0 b_1 b_2 \dots b_n \dots$ such that $b_k = a_k$ for $k = 0\dots N$. Thus, $\mathcal{P}$ has a filtration $\mathcal{P} = \cup_{n=0}^{\infty}\mathcal{P}_n$, where $\mathcal{P}_n$ is generated by the characteristic functions of $\mathcal{S}^+(w)$ with w of length at most $n+1$. Taking into account the relations between these, we obtain that $\mathcal{P}_n$ is a free abelian group generated by the characteristic functions of $\mathcal{S}^+(w)$ with w of length exactly $n+1$. The number of such words is $2g(2g-1)^n$, hence $\mathrm{rank}\mathcal{P}_n = 2g(2g-1)^n$.

The map δ satisfies $\delta : \mathcal{P}_n \to \mathcal{P}_{n+1}$, with a 1-dimensional kernel given by the constant functions. More precisely, if we write $f(a_0 ... a_n)$ for a function in $\mathcal{P}_n$, then

$$(\delta f)(a_0 \dots a_n a_{n+1}) = f(a_0 \dots a_n) - f(a_1 \dots a_{n+1}).$$

The resulting quotients

$$F_n = \mathcal{P}_n/\delta\mathcal{P}_{n-1}$$

are torsion free (*cf.* Theorem 19 §4 of [34]) and have ranks

$$\mathrm{rank} F_n = 2g(2g-1)^{n-1}(2g-2) + 1$$

for $n \geq 1$, while $F_0 \cong \mathcal{P}_0$ is of rank $2g$. There is an injection $F_n \hookrightarrow F_{n+1}$ induced by the inclusion $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, and $\mathcal{P}/\delta\mathcal{P}$ is the direct limit of the F_n under these inclusions. Thus we obtain the filtration on $H^1(\mathcal{S}_T, \mathbb{Z})$:

$$H^1(\mathcal{S}_T, \mathbb{Z}) = \lim_n F_n.$$

This proves the second statement.
$\diamond$

Remark 4.13 There is an interesting degree shift between K–group $K_0(\mathrm{C}(\mathcal{S}) \rtimes_T \mathbb{Z})$ of the crossed product algebras associated to the Smale space $(\mathcal{S}, T)$ and the cohomology $H^1(\mathcal{S}_T, \mathbb{Z})$. There are other examples in non-commutative geometry where K_0 of a non-commutative space replaces the H^1 of its commutative counterpart. This happens, for instance, in the study of of non-commutative elliptic curves, where the Hodge filtration on H^1 has its non-commutative analog as a filtration on K_0, but not on K_1, [9] p. 132–139, [11] §XIII.

The following result computes the first homology of $\mathcal{S}_T$.

Proposition 4.14 *The homology group $H_1(\mathcal{S}_T, \mathbb{Z})$ has a filtration by free abelian groups $\mathcal{K}_N$,*

$$H_1(\mathcal{S}_T, \mathbb{Z}) = \lim_N \mathcal{K}_N, \tag{4.16}$$

with

$$K_N = \mathrm{rank}(\mathcal{K}_N) = \begin{cases} (2g-1)^N + 1 & N \text{ even} \\ (2g-1)^N + (2g-1) & N \text{ odd} \end{cases}$$

The group $H_1(\mathcal{S}_T, \mathbb{Z})$ can also be written as

$$H_1(\mathcal{S}_T, \mathbb{Z}) = \oplus_{N=0}^{\infty} \mathcal{R}_N$$

where $\mathcal{R}_n$ is a free abelian group of ranks $R_1 = 2g$ and

$$R_N = \mathrm{rank}(\mathcal{R}_N) = \frac{1}{N} \sum_{d|N} \mu(d)\, (2g-1)^{N/d},$$

for $N > 1$, with μ the Möbius function. This is isomorphic to the free abelian group on countably many generators, $\mathbb{Z}\langle \mathcal{S}^p/T \rangle$.

Proof. The lines $\{[x,t] : t \in \mathbb{R}\}$ are pairwise disjoint in the mapping torus $\mathcal{S}_T$, hence $H_1(\mathcal{S}_T, \mathbb{Z})$ is generated by *closed curves* $\{[x,t] : t \in \mathbb{R}\}$. By construction, a closed curve in $\mathcal{S}_T$ corresponds to a point $x \in \mathcal{S}$ such that $T^N x = x$ for some $N \geq 1$. More precisely, if $x = \overline{a_0 \ldots a_N}$ is a doubly infinite periodic sequence in $\mathcal{S}$, obtained by repeating the word $a_0 \ldots a_N$, so that $T^N x = x$, then the map $c_x : S^1 \to \mathcal{S}_T$ of the form

$$c_x : e^{2\pi i t} \mapsto [(x, Nt)], \quad T^N x = x. \tag{4.17}$$

defines a closed curve in $\mathcal{S}_T$, which is a non-trivial homology class in $H_1(\mathcal{S}_T,\mathbb{Z})$. Since all non-trivial homology classes can be obtained this way, the homology $H_1(\mathcal{S}_T,\mathbb{Z})$ is generated by all c_x as in (4.17), for x a reduced word $a_0\ldots a_N$ such that $a_N\neq a_0^{-1}$.

Let $\mathcal{K}_N$ be the free abelian groups generated by all the reduced words $a_0\ldots a_N$ of length $N+1$ satisfying $a_N\neq a_0^{-1}$. When we identify the elements of the $\mathcal{K}_N$ with homology classes via (4.17), we introduce relations between the $\mathcal{K}_N$ for different N, namely we have embeddings $k\mathcal{K}_N\hookrightarrow\mathcal{K}_{kN}$,

$$k\langle a_0\ldots a_N\rangle\mapsto\ \langle\underbrace{a_0\ldots a_N a_0\ldots a_N\ldots a_0\ldots a_N}_{k-times}\rangle. \qquad (4.18)$$

Thus, the homology $H_1(\mathcal{S}_T,\mathbb{Z})$ is computed as the limit $H_1(\mathcal{S}_T,\mathbb{Z})=\lim_N\mathcal{K}_N$ with respect to the maps $J_k:\mathcal{K}_N\longrightarrow\mathcal{K}_{kN}$ that send the c_x of (4.17) to the composite

$$S^1\stackrel{z\mapsto z^k}{\longrightarrow}S^1\stackrel{c_x}{\to}\mathcal{S}_T.$$

Thus, the homology $H_1(\mathcal{S}_T,\mathbb{Z})$ can be identified with the $\mathbb{Z}$-module generated by the elements of $\mathcal{S}^p/T$. This quotient can be written as a disjoint union $\mathcal{S}^p/T=\cup_{n=0}^\infty\mathcal{S}^p_n/T$, where $\mathcal{S}^p_n\subset\mathcal{S}^p$ is the subset of (primitive) periodic sequences with period of length $n+1$. This gives the description of $H_1(\mathcal{S}_T,\mathbb{Z})$ as direct sum of the $\mathcal{R}_n=\mathbb{Z}\langle\mathcal{S}^p_n/T\rangle$.

The computation of the ranks of the $\mathcal{K}_N$ and of the $\mathcal{R}_n$ is obtained as follows.

For a fixed a_0, denote by $p(N,k)$ the number of reduced sequences $a_0\ldots a_N$, such that the last k terms are all equal to a_0^{-1}. Then $2g\cdot p(N,0)=\mathrm{rank}\mathcal{K}_N$. Moreover, since the total number of all reduced sequences of length $N+1$ is $2g(2g-1)^N$, we have $\sum_{k=0}^N p(N,k)=(2g-1)^N$. It is not hard to see from the definition that the $p(N,k)$ satisfy $p(N,k)=p(N+1,k+1)$, $p(N,N)=0$, $p(1,0)=2g-1$, $p(N,0)=p(N+1,1)+p(N-1,0)$.

The calculation of $p(N,0)$ then follows inductively, using $p(N,1)=p(N-1,0)-p(N-2,0)$, and the sum $\sum_{k=0}^N p(N,k)=(2g-1)^N$, where $p(N,k)=p(N-k+1,1)$.

The rank of the $\mathcal{R}_N$ can be computed by first considering that $K_N=\sum_{d|N}d\,R_d$, since the total number of reduced sequences $a_0\ldots a_N$ with $a_N\neq a_0^{-1}$ is the sum of the cardinalities of the $\mathcal{S}^p_d$ over all d dividing N. These satisfy $\#\mathcal{S}^p_d=d\,R_d$. Then we obtain

$$R_N=\frac{1}{N}\sum_{d|N}\mu(d)K_{N/d}=\frac{1}{N}\sum_{d|N}\mu(d)(2g-1)^{N/d},$$

using the Möbius inversion formula in the first equality and the fact that $\sum_{d|N}\mu(d)=\delta_{N,1}$ in the second.

$\diamond$

Combining Theorem 4.12 with Proposition 4.14, we can compute explicitly the pairing of homology and cohomology for $\mathcal{S}_T$. This relates the filtration by the F_n on the cohomology of $\mathcal{S}_T$ to the coding of closed geodesics in $\tilde{\Xi}_c$.

Proposition 4.15 *Let F_n and $\mathcal{K}_N$ be the filtrations defined, respectively, in Theorem 4.12 and Proposition 4.14. There is a pairing*

$$\langle\cdot,\cdot\rangle:F_n\times\mathcal{K}_N\to\mathbb{Z}\qquad\langle[f],x\rangle=N\cdot f(\bar{x}), \qquad (4.19)$$

with $x = a_0 \dots a_N$. Here the representative $f \in [f]$ is a function that depends on the first $n+1$ terms $a_0 \dots a_n$ of sequences in $\mathcal{S}$, and $\bar{x}$ is the truncation of the periodic sequence $\overline{a_0 \dots a_N}$ after the first n terms. This pairing descends to the direct limits of the filtrations, where it agrees with the classical cohomology/homology pairing

$$\langle \cdot, \cdot \rangle : H^1(\mathcal{S}_T, \mathbb{Z}) \times H_1(\mathcal{S}_T, \mathbb{Z}) \to \mathbb{Z}. \tag{4.20}$$

Proof. First, it is not hard to check that the pairing (4.19) is compatible with the maps $F_n \hookrightarrow F_{n+1}$ and $J_k : \mathcal{K}_N \to \mathcal{K}_{kN}$. In fact, (4.19) is invariant under the maps $F_n \hookrightarrow F_{n+1}$, while under the map $J_k : \mathcal{K}_N \hookrightarrow \mathcal{K}_{kN}$ we have

$$\langle [f], J_k(x) \rangle = kNf(\bar{x}) = k \, \langle [f], x \rangle.$$

Thus, (4.19) induces a pairing of the direct limits

$$\langle \cdot, \cdot \rangle : \lim_n F_n \times \lim_N \mathcal{K}_N \to \mathbb{Z}. \tag{4.21}$$

In order to check that (4.21) agrees with the cohomology/homology pairing (4.20), notice that a class c in the homology $H_1(\mathcal{S}_T, \mathbb{Z})$ is realized as a finite linear combination of oriented circles in $\mathcal{S}_T$, where each such circle is described by a map $c_x : S^1 \to \mathcal{S}_T$ of the form (4.17).

The pairing $\langle u, c \rangle$ of an element u of the cohomology $H^1(\mathcal{S}_T, \mathbb{Z})$ with a generator c of the homology $H_1(\mathcal{S}_T, \mathbb{Z})$ is given by the homotopy class $[u \circ c]$ of

$$u \circ c : S^1 \longrightarrow S^1.$$

We write $u(x,t) = [2\pi i t f(x)]$, for a generator of $H^1(\mathcal{S}_T, \mathbb{Z})$ in F_n, where f is an element in $\mathcal{P}_n$, which depends only on the first $n+1$ terms in the sequences $a_0 \dots a_n \dots$ in $\mathcal{S}_T$ (*cf.* Theorem 4.12). If $y = a_0 \dots a_d$ is an element in $\mathcal{R}_d$ of period d, and c is the corresponding generator of $H_1(\mathcal{S}_T, \mathbb{Z})$ of the form $c_x(t) = [(x, d \cdot t) : x = \overline{a_0 \dots a_d}]$, then the homotopy class $[u \circ c_x] \in \pi_1(S^1) = \mathbb{Z}$ is equal to $d \cdot f(x)$ and this proves the claim. $\diamond$

4.4 Cuntz-Krieger algebra

A partial isometry is a linear operator S satisfying the relation $S = SS^*S$. The Cuntz–Krieger algebra $\mathcal{O}_A$ (*cf.* [15] [16]) is defined as the universal C*–algebra generated by partial isometries $S_1, \dots, S_{2g}$, satisfying the relations

$$\sum_j S_j S_j^* = I \tag{4.22}$$

$$S_i^* S_i = \sum_j A_{ij} \, S_j S_j^*, \tag{4.23}$$

where $A = (A_{ij})$ is the $2g \times 2g$ transition matrix of the subshift of finite type $(\mathcal{S}, T)$, namely the matrix whose entries are $A_{ij} = 1$ whenever $|i - j| \neq g$, and $A_{ij} = 0$ otherwise.

We give a more explicit description of the generators of the Cuntz–Krieger algebra $\mathcal{O}_A$.

Consider the following operators, acting on the Hilbert space $L^2(\Lambda_\Gamma, \mu)$, where μ is the Patterson–Sullivan measure on the limit set (*cf.* [51]):

$$(T_\gamma f)(x) := f(\gamma^{-1} x), \quad \text{and} \quad (P_\gamma f)(x) := \chi_\gamma(x) f(x), \tag{4.24}$$

where $\gamma \in \Gamma$ is identified with a reduced word in the generators $\{g_j\}_{j=1}^g$ and their inverses, and χ_γ is the characteristic function of the cylinder $\Lambda_\Gamma(\gamma)$ of all (right) infinite reduced words that begin with the word γ. Then, for all $\gamma \in \Gamma$, T_γ is a unitary operator with $T_\gamma^* = T_{\gamma^{-1}}$ and P_γ is a projector. In particular, with the usual notation $g_{j+g} = g_j^{-1}$, for $j = 1, \ldots, g$, we write

$$T_j := T_{g_j} \quad \text{and} \quad P_j := P_{g_j} \quad \text{for } j = 1, \ldots, 2g.$$

Proposition 4.16 *The operators*

$$S_i := \sum_j A_{ij} T_i^* P_j \tag{4.25}$$

are bounded operators on $L^2(\Lambda_\Gamma, \mu)$ satisfying the relations (4.22) *and* (4.23). *Thus, the Cuntz-Krieger algebra $\mathcal{O}_A$ can be identified with the subalgebra of bounded operators on the Hilbert space $L^2(\Lambda_\Gamma, \mu)$ generated by the S_i as in* (4.25).

Proof. The operators P_i are orthogonal projectors, *i.e.* $P_i P_j = \delta_{ij} P_j$. The composite $T_i^* P_j T_i$ satisfies

$$\sum_j A_{ij} (T_i^* P_j T_i\, f)(x) = \begin{cases} f(x) & \text{if } P_i(x) = x \\ 0 & \text{otherwise.} \end{cases}$$

In fact, for $x = g_i y$, we have

$$\begin{gathered} \textstyle\sum_j A_{ij} (T_i^* P_j T_i f)(x) = \sum_j A_{ij} T_i^* P_j f(g_i^{-1} g_i y) = \\ T_i^* (\textstyle\sum_j A_{ij} P_j) f(y) = T_i^* (I - P_{i+g}) f(y) = f(g_i y) = f(x). \end{gathered}$$

In the remaining cases with $x \neq g_i y$, we have

$$\sum_j A_{ij} (T_i^* P_j T_i f)(x) = \sum_j A_{ij} T_i^* P_j f(g_i^{-1} x) = T_i^* \sum_j A_{i\, i+g} P_{i+g} f(g_{i+g} x) = 0,$$

where the last equality follows from the fact that the transition matrix A satisfies $A_{i\ i+g} = 0$ (*cf.* Lemma 4.6.2). This implies that the S_i satisfy

$$S_i S_i^* = \sum_j A_{ij} T_i^* P_j T_i = P_i.$$

Since the projectors P_i satisfy $\sum_i P_i = I$, we obtain the relation (4.22). Moreover, since $T_i T_i^* = 1$, and the entries A_{ij} are all zeroes and ones, we also obtain

$$S_i^* S_i = \sum_{j,k} A_{ij} A_{ik} P_k T_i T_i^* P_j = \sum_j (A_{ij})^2 P_j = \sum_j A_{ij} P_j.$$

Replacing $P_j = S_j S_j^*$ from (4.22) we then obtain (4.23).
◇

The Cuntz–Krieger algebra $\mathcal{O}_A$ can be described in terms of the action of the free group Γ on its limit set Λ_Γ (*cf.* [41], [48]), so that we can regard $\mathcal{O}_A$ as a noncommutative space replacing the classical quotient Λ_Γ/Γ.

In fact (*cf.* Lemma 4.4), the action of Γ on $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ determines a unitary representation

$$\Gamma \to \mathrm{Aut}(\mathrm{C}(\Lambda_\Gamma)) \quad (T_\gamma f)(x) = f(\gamma^{-1}x), \tag{4.26}$$

where $\mathrm{C}(\Lambda_\Gamma)$ is the C^*–algebra of continuous functions on Λ_Γ. Thus, we can form the (reduced) crossed product C^*–algebra $\mathrm{C}(\Lambda_\Gamma) \rtimes \Gamma$.

In the following Theorem we construct an explicit identification between the algebra $\mathrm{C}(\Lambda_\Gamma) \rtimes \Gamma$ and the subalgebra of bounded operators on $L^2(\Lambda_\Gamma, \mu)$ generated by the S_i, which is isomorphic to $\mathcal{O}_A$.

Theorem 4.17 *The Cuntz–Krieger algebra $\mathcal{O}_A$ satisfies the following properties.*

1. *There is an injection $\mathrm{C}(\Lambda_\Gamma) \to \mathcal{O}_A$ which identifies $\mathrm{C}(\Lambda_\Gamma)$ with the maximal commutative subalgebra of $\mathcal{O}_A$ generated by the P_γ as in* (4.24)*.*
2. *The generators S_i of $\mathcal{O}_A$ given in* (4.25) *realize $\mathcal{O}_A$ as a subalgebra of $\mathrm{C}(\Lambda_\Gamma) \rtimes \Gamma$.*
3. *The operators T_γ of* (4.24) *are elements in $\mathcal{O}_A$, hence the injection of $\mathcal{O}_A$ inside $\mathrm{C}(\Lambda_\Gamma) \rtimes \Gamma$ is an isomorphism,*

$$\mathcal{O}_A \cong C(\Lambda_\Gamma) \rtimes \Gamma. \tag{4.27}$$

Proof. *1.* The algebra $\mathrm{C}(\Lambda_\Gamma)$ acts on $L^2(\Lambda_\Gamma, \mu)$ as multiplication operators. We identify the characteristic function $\chi_{\Lambda_\Gamma(\gamma)}$ of the subset $\Lambda_\Gamma(\gamma) \subset \Lambda_\Gamma$ with the projector P_γ defined in (4.24). A direct calculation shows that, for any $\gamma \in \Gamma$, the projector P_γ satisfies $P_\gamma = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*$, for $\gamma = g_{i_1} \cdots g_{i_k}$, hence it is in the algebra $\mathcal{O}_A$.

For a multi-index $\mu = \{i_1, \dots, i_k\}$, the *range projection* P_μ is the element $P_\mu = S_\mu S_\mu^*$ in $\mathcal{O}_A$. We have identified $\chi_{\Lambda_\Gamma(\gamma)}$ in $\mathrm{C}(\Lambda_\Gamma)$ with the range projection P_μ, for $\mu = \{i_1, \dots, i_k\}$ the multi-index of $\gamma = g_{i_1} \cdots g_{i_k}$. Thus, we have identified $\mathrm{C}(\Lambda_\Gamma)$ with the maximal commutative subalgebra of $\mathcal{O}_A$ generated by the range projections (*cf.* [15] [16]).

2. The operators S_i defined in (4.25) determine elements in $\mathrm{C}(\Lambda_\Gamma) \rtimes \Gamma$, by identifying the projectors P_j with $\chi_{\Lambda_\Gamma(g_j)}$ as in (1), and the operators T_i with the corresponding elements in the unitary representation (4.26). These elements still satisfy the relations (4.22) (4.23), hence the algebra generated by the S_i can be regarded as a subalgebra of $C(\Lambda_\Gamma) \rtimes \Gamma$.

3. Given *1.* and *2.*, in order to prove the isomorphism (4.27), it is enough to show that, for any $\gamma \in \Gamma$, the operators T_γ in the unitary representation (4.26) are in the subalgebra of $C(\Lambda_\Gamma) \rtimes \Gamma$ generated by the S_i. In fact, since this subalgebra contains $C(\Lambda_\Gamma)$, if we know it also contains the T_γ, it has to be the whole of $C(\Lambda_\Gamma) \rtimes \Gamma$. Again this follows by a direct calculation: $T_\gamma = T_{i_1} \cdots T_{i_k}$ and $T_i = P_i S_i^*$.
$\diamond$

Remark 4.18 It is shown in [15], [16], and [37] that the Cuntz–Krieger algebra $\mathcal{O}_A$ has a second description as a crossed product algebra. Namely, up to stabilization (*i.e.* tensoring with compact operators) we have

$$\mathcal{O}_A \simeq \mathcal{F}_A \rtimes_T \mathbb{Z}, \tag{4.28}$$

where $\mathcal{F}_A$ is an approximately finite dimensional (AF) algebra stably isomorphic to the groupoid C^*–algebra $C^*(\mathcal{G}^u)$ of Remark 4.7. It can be shown that the shift operator T induces an action by automorphisms on $\mathcal{F}_A$, so that the crossed product algebra (4.28) corresponds to $C^*(\mathcal{G}^u) \rtimes_T \mathbb{Z}$ associated to the Smale space $(\mathcal{S}, T)$. (*cf.* Remark 4.7).

Finally we show that the Cuntz-Krieger algebra acts on (a completion of) the dynamical cohomology.

Proposition 4.19 *1. The operators S_i defined in* (4.25) *determine endomorphisms of $C(\mathcal{S}^+, \mathbb{Z})$. These induce endomorphisms $\bar{S}_i$ of $H^1(\mathcal{S}_T, \mathbb{Z})$.*

2. Let $H^1(\mathcal{S}_T, \mathbb{C})$ be the dynamical cohomology with complex coefficients and denote by $\mathcal{O}_A^{alg}$ the $*$–algebra generated algebraically *by the S_i satisfying* (4.22) (4.23). *Then, the endomorphisms $\bar{S}_i$ of $H^1(\mathcal{S}_T, \mathbb{Z})$ determine a representation*

$$\rho : \mathcal{O}_A^{alg} \to End(H^1(\mathcal{S}_T, \mathbb{C})),$$

which induces an action of $\mathcal{O}_A$ on a suitable Hilbert space completion of $H^1(\mathcal{S}_T, \mathbb{C})$, in the inner product induced by $L^2(\Lambda_\Gamma, \mu)$, with μ the Patterson-Sullivan measure.

Proof. *1.* We write (4.25) equivalently as $S_i = \sum_j A_{ij} T_{i+g \mod 2g} P_j$, where

$$T_i f(a_0 \dots a_\ell \dots) = f(g_i^{-1} a_0 \dots a_\ell \dots)$$

and

$$P_i f(a_0 \dots a_\ell \dots) = \chi_{\mathcal{S}^+(g_i)}(a_0 \dots a_\ell \dots) \cdot f(a_0 \dots a_\ell \dots)$$

are well defined as endomorphisms of $C(\mathcal{S}^+, \mathbb{Z})$. It is then sufficient to show that the endomorphisms S_i commute with the endomorphism $\delta : f \mapsto f - f \circ T$. A direct calculation shows that this is the case. In fact, we have $S_i : \mathcal{P}_n \to \mathcal{P}_n$, and for any $f \in$

$\mathcal{P}_n$, we obtain $S_i(\delta f) = \delta(P_i f)$. Thus, there are induced endomorphisms $\bar{S}_i : F_n \mapsto F_n$ compatible with the inclusions $F_n \hookrightarrow F_{n+1}$, which in the limit induce endomorphisms of $H^1(\mathcal{S}_T, \mathbb{Z})$. The claim *2.* then follows directly by change of coefficients and norm completion (*cf.* Proposition 4.16).
$\diamond$

5 Dynamical spectral triple.

We define the *dynamical cohomology* H^1_{dyn} as the graded vector space given by the sum of the graded pieces of the filtration of $H^1(\mathcal{S}_T)$, introduced in Theorem 4.12. These graded pieces Gr_p are considered with coefficients in the p-th Hodge–Tate twist $\mathbb{R}(p)$. The resulting graded space carries an action of an involution, which we denote $\bar{F}_\infty$ by analogy with the real Frobenius acting on the cohomologies of §2. In Theorem 5.5 we show that there is an embedding of $H^1(\tilde{X}^*)^{N=0}$ in H^1_{dyn}, which is compatible with the graded structure. This embedding is equivariant with respect to the action of the real Frobenius.

Similarly, we define the *dynamical homology* H_1^{dyn} as the graded vector space given by the sum of the terms in the filtration of $H_1(\mathcal{S}_T)$, introduced in Proposition 4.14. These vector spaces are again considered with twisted $\mathbb{R}(p)$-coefficients. The resulting graded vector space also has an action of a real Frobenius $\bar{F}_\infty$. Theorem 5.10 shows that there is an identification of the dual of $H^1(\tilde{X}^*)^{N=0}$, under the duality isomorphisms in $\mathbb{H}^{\cdot}(X^*)$ of (2.18), with a subspace of H_1^{dyn}. The identification is compatible with the $\bar{F}_\infty$ action induced by the change of orientation $z \mapsto \bar{z}$ on $X_{/\mathbb{R}}$.

The pair

$$H^1_{dyn} \oplus H_1^{dyn} \tag{5.1}$$

provides a geometric setting, defined in terms of the dynamics of the shift operator T, which contains a copy of the archimedean cohomology $H^1(X^*)^{N=0}$ and of its dual under the duality isomorphisms acting on $\mathbb{H}^*(\mathrm{Cone}(N))$. In Theorem 5.10 we also prove that, under these identifications, the duality isomorphism corresponds to the homology/cohomology pairing between $H^1(\mathcal{S}_T)$ and $H_1(\mathcal{S}_T)$.

This construction also shows that the map $1-T$ plays, in this dynamical setting, a role dual to the monodromy map N of the arithmetic construction of Section 2 (*cf.* Remark 5.11).

We then construct a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{H}$ is a Hilbert space obtained from a completion of $H^1_{dyn} \oplus H_1^{dyn}$, the algebra $\mathcal{A}$ is a refinement of both $C_0(\mathfrak{X}_\Gamma)$ and $\mathcal{O}_A$, and the Dirac operator D, restricted to the subspace of $\mathcal{H}$ isomorphic to $H^1(\tilde{X}^*)^{N=0}$, agrees with the Frobenius-type operator Φ of Section 2. We refer to this as *dynamical spectral triple.*

The algebra $\mathcal{A}$ provides an analog, in the ∞-adic case, of the p-adic reduction map obtained by considering the reductions mod p^k (*cf.* [24] [33]).

In order to construct the dynamical spectral triple, we first introduce the Hilbert space completions. Then, in Propositions 5.13 and 5.16, we construct representations of two Morita equivalent C*-algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ acting as bounded operators on the completion of H^1_{dyn} and H_1^{dyn}, respectively. Finally, in Theorem 5.22 we obtain the spectral triple for the algebra $\mathcal{A}$ by tensoring H_1^{dyn} with the $\hat{\mathcal{A}} - \mathcal{A}$-bimodule implementing the Morita equivalence.

5.1 Dynamical cohomology of the fiber at infinity

In this paragraph we relate the archimedean cohomology of Section 2 to the sum of graded pieces of the filtration of Theorem 4.12. Notice that the terms F_n in this filtration can be regarded as real vector spaces of dimension $\dim F_n = 2g(2g-1)^{n-1}(2g-2)+1$ for $n \geq 1$ and $\dim F_0 = 2g$, since, as $\mathbb{Z}$-modules they are torsion free. We make the following definition.

Definition 5.1 *Let $H^1(\mathcal{S}_T, \mathbb{R}) = \lim_n F_n$, for a filtration F_n as in Theorem 4.12, with real coefficients. Let $\mathrm{Gr}_n = F_n/F_{n-1}$ be the corresponding graded pieces, with $\mathrm{Gr}_0 = F_0$. We define the* dynamical cohomology *as*

$$H^1_{dyn} := \oplus_{p\leq 0} gr^\Gamma_{2p} H^1_{dyn}, \tag{5.2}$$

where we set

$$gr^\Gamma_{2p} H^1_{dyn} := \mathrm{Gr}_{-p} \otimes_{\mathbb{R}} \mathbb{R}(p) \tag{5.3}$$

with $\mathbb{R}(p) = (2\pi\sqrt{-1})^p\mathbb{R}$.

Let $\iota_\Xi : \tilde{\Xi} \to \tilde{\Xi}$ be the involution on the orientation double cover $\tilde{\Xi}$ of Ξ given by the $\mathbb{Z}/2$-action. This corresponds to an involution $\iota_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ and an induced map $\iota^*_{\mathcal{S}} : H^1(\mathcal{S}_T, \mathbb{R}) \to H^1(\mathcal{S}_T, \mathbb{R})$ which preserves the F_n, hence it induces a map $\bar{\iota}^* : H^1_{dyn} \to H^1_{dyn}$.

Definition 5.2 *We define the action of the real Frobenius $\bar{F}_\infty$ on H^1_{dyn} as the composition of the involution $\bar{\iota}^*$ induced by the $\mathbb{Z}/2$-action on $\tilde{\Xi}$ and the action by $(-1)^p$ on $\mathbb{R}(p)$.*

We now proceed to identify a subspace of H^1_{dyn} which is isomorphic, as a graded vector space, to the archimedean cohomology $H^1_{ar} = H^1(\tilde{X}^*)^{N=0}$, in a way that is compatible with the action of the real Frobenius $\bar{F}_\infty$.

Let $\chi_{\mathcal{S}^+(w_{n,k})}$ be the characteristic functions of the sets $\mathcal{S}^+(w_{n,k}) \subset \mathcal{S}^+$, where $w_{n,k}$ is a word in the $\{g_j\}_{j=1}^{2g}$ of the form

$$w_{n,k} = \underbrace{g_k g_k \cdots g_k}_{n-times}.$$

Lemma 5.3 *The functions $\chi_{\mathcal{S}^+(w_{n,k})}$ define classes*

$$\chi_{n,k} := [\chi_{\mathcal{S}^+(w_{n,k})}] \in \mathrm{Gr}_{n-1}. \tag{5.4}$$

The $\chi_{n,k}$ are all linearly independent, hence they space a $2g$ dimensional subspace in each Gr_{n-1}.

Proof. It is not hard to see that the characteristic functions $\chi_{\mathcal{S}^+(w_{n,k})}$ for $n \geq 1$ and $k = 1, \dots, 2g$ are all linearly independent in $\mathcal{P}$ and no linear combination of the $\chi_{\mathcal{S}^+(w_{n,k})}$ in $\mathcal{P}_{n-1}$ lies in the image of $\delta = 1 - T$. Thus, passing to equivalence classes modulo the image of the map $\delta = 1 - T$, we obtain linearly independent elements

$$\big(\chi_{\mathcal{S}^+(w_{n,k})} \mod (1-T)\big) \in F_{n-1} \subset H^1(\mathcal{S}_T). \tag{5.5}$$

Again it is not difficult to check that, for any fixed n, no linear combination of the classes (5.5) lies in F_{n-2}. Thus, by further taking the equivalence classes of the (5.5) modulo F_{n-2}, we obtain $2g$ linearly independent elements (5.4) in each Gr_{n-1}.
$\diamond$

We obtain elements in H^1_{dyn} by considering

$$(2\pi\sqrt{-1})^p \chi_{-p+1,k} \in gr^\Gamma_{2p} H^1_{dyn}. \tag{5.6}$$

Definition 5.4 *We denote by $\mathcal{V} \subset H^1_{dyn}$ the graded subspace*

$$\mathcal{V} := \oplus_{p \leq 0} gr^\Gamma_{2p} \mathcal{V}$$

where $gr^\Gamma_{2p}\mathcal{V}$ is the subspace of $gr^\Gamma_{2p} H^1_{dyn}$ spanned by the elements of the form (5.6)*, for $k = 1, \dots, 2g$.*

For $k = 1, \dots, g$, let η_k denote a basis of holomorphic differentials on the Riemann surface $X_{/\mathbb{R}}$ satisfying the normalization

$$\int_{a_j} \eta_k = \delta_{jk}, \tag{5.7}$$

where the a_k are a basis of $Ker(I_*)$ for $I_* : H_1(X_{/\mathbb{R}}, \mathbb{Z}) \to H_1(\mathfrak{X}_\Gamma, \mathbb{Z})$ induced by the inclusion of $X_{/\mathbb{R}}$ as the boundary at infinity of the handlebody $\mathfrak{X}_\Gamma$. We refer to this basis as the *canonical basis* of holomorphic differentials.

Recall that, since we are considering an orthosymmetric smooth real algebraic curve, $X_{/\mathbb{R}}$ has a Schottky uniformization by a Fuchsian Schottky group as in Proposition 3.5. It is known then (*cf.* [24] [46]) that a holomorphic differential η on $X_{/\mathbb{R}}$ can be obtained as Poincaré series with exponent 1, $\eta = \Theta^1(f)$, where f is a meromorphic function on $\mathbb{P}^1(\mathbb{C})$ with divisor $D(f) \subset \Omega_\Gamma$, and

$$\Theta^m(f)(z) := \sum_{\gamma \in \Gamma} f(\gamma(z)) \left(\frac{\partial \gamma(z)}{\partial z} \right)^m \tag{5.8}$$

is the Poincaré series with exponent m. The fact that the Hausdorff dimension of the limit set satisfies $\dim_H(\Lambda_\Gamma) < 1$ ensures absolute convergence on compact sets in Ω_Γ (*cf.* Remark 3.14). In particular, consider the automorphic series

$$\omega_k = \sum_{h \in C(\cdot|g_k)} d_z \log\langle hz^+(g_k), hz^-(g_k), z, z_0\rangle, \tag{5.9}$$

where $\langle a, b, c, d\rangle$ is the cross ratio of points in $\mathbb{P}^1(\mathbb{C})$, and $C(\cdot|g_k)$ denotes a set of representatives of the coset classes $\Gamma/\mathbb{Z}\langle g_k\rangle$, for $\{g_k\}_{k=1}^g$ the generators of Γ, and z_0 a base point in Ω_Γ. By Lemma 8.2 of [24] (*cf.* Proposition 1.5.2 of [28]), the expression (5.9) gives the canonical basis of holomorphic differentials satisfying the normalization condition (5.7), in the form

$$\eta_k = \frac{1}{2\pi\sqrt{-1}}\omega_k, \qquad \text{so that} \qquad \int_{a_j} \omega_k = (2\pi\sqrt{-1})\delta_{jk}. \tag{5.10}$$

The formula (5.9) gives the explicit correspondence between the set of generators of Γ and the canonical basis of holomorphic differentials $g_k \mapsto \omega_k$, for $k = 1, \dots, g$, which we use in order to produce the following identification.

Theorem 5.5 *Consider the map*

$$U : gr^w_{2p} H^1(\tilde{X}^*)^{N=0} \longrightarrow gr^\Gamma_{2p}\mathcal{V}, \tag{5.11}$$

given by

$$U((2\pi\sqrt{-1})^{p-1}\varphi_k) := (2\pi\sqrt{-1})^p \ \frac{\chi_{-p+1,k} - \chi_{-p+1,k+g}}{2} \tag{5.12}$$

$$U((2\pi\sqrt{-1})^{p-1}\varphi_{k+g}) := (2\pi\sqrt{-1})^p \ \frac{\chi_{-p+1,k} + \chi_{-p+1,k+g}}{2}, \tag{5.13}$$

for $k = 1, \dots, g$ and $p \leq 0$, where we set

$$\varphi_k = (\omega_k + \bar{\omega}_k)/2 \qquad \text{and} \qquad \varphi_{g+k} = -i(\omega_k - \bar{\omega}_k)/2. \tag{5.14}$$

The map U is an isomorphism of $H^1(\tilde{X}^*)^{N=0}$ and $\mathcal{V}$ as graded vector spaces, which is equivariant with respect to the action of the real Frobenius $\bar{F}_\infty$.

Proof. The elements (5.14) give a basis of $H^1_{DR}(X_{/\mathbb{R}}, \mathbb{R}(1))$, where the twist is due to the choice of normalization (5.7) and the relation (5.10). A basis for $H^1(X_{/\mathbb{R}}, \mathbb{R}(p))$ is then given by the $(2\pi\sqrt{-1})^{p-1}\varphi_k$, $k = 1, \dots, 2g$.

By (2.26), we have $H^1(\tilde{X}^*)^{N=0} = \oplus_{p\leq 0} gr^{\mathrm{w}}_{2p} H^1(\tilde{X}^*)^{N=0}$, with $gr^{\mathrm{w}}_{2p} H^1(\tilde{X}^*)^{N=0} = H^1(X_{/\mathbb{R}}, \mathbb{R}(p))$. Thus, we obtain a basis for $H^1(\tilde{X}^*)^{N=0}$, of the form

$$\{(2\pi\sqrt{-1})^{p-1}\varphi_k : \ k = 1, \dots, 2g, \ p \leq 0\}.$$

It is then clear from the construction that the map (5.11) is an isomorphism of graded vector spaces. We need to check that it is equivariant with respect to the action of the real Frobenius.

In the case of a real $X_{/\mathbb{R}}$, the action of complex conjugation $z \mapsto \bar{z}$ corresponds geometrically to a change of orientation on $X_{/\mathbb{R}}$, which induces a change of orientation on the handlebody $\mathfrak{X}_\Gamma$. If $L_{\{a,b\}}$ is the geodesic in $\tilde{\Xi}$ such that the orientation of the geodesic at the endpoint $b \in \mathbb{P}^1(\mathbb{C})$ agrees with the outward pointing normal vector, then under the change of orientation induced by $z \mapsto \bar{z}$ the geodesic $L_{\{a,b\}}$ is exchanged with $L_{\{b,a\}}$, which is exactly the effect of the involution on $\tilde{\Xi}$. The induced involution on $\mathcal{P}$ exchanges $\chi_{\mathcal{S}^+(w_{n,k})}$ and $\chi_{\mathcal{S}^+(w_{n,g+k})}$, so that the corresponding involution on $\mathcal{V}$ is given by

$$\bar{F}_\infty((2\pi\sqrt{-1})^p \chi_{-p+1,k}) = (-1)^p (2\pi\sqrt{-1})^p \chi_{-p+1,g+k}.$$

On the other hand, under the action of the real Frobenius $\bar{F}_\infty$ we have $H^1(X_{/\mathbb{R}}, \mathbb{R}) = E_1 \oplus E_{-1}$ with $\dim E_{\pm 1} = g$ (*cf.* Remark 2.17), generated respectively by the $(2\pi\sqrt{-1})^{-1}\varphi_k$ and $(2\pi\sqrt{-1})^{-1}\varphi_{g+k}$, for $k = 1, \ldots, g$. This gives the corresponding splitting into eigenspaces $H^1(\tilde{X}^*)^{N=0} = E^+ \oplus E^-$ as in (2.35). Thus, we see that (5.12) and (5.13) are $\bar{F}_\infty$-equivariant, since we have

$$\begin{aligned}
\bar{F}_\infty\left((2\pi\sqrt{-1})^p \frac{1}{2}(\chi_{-p+1,k} \pm \chi_{-p+1,k+g})\right) &= \pm 1 \cdot (-1)^p \cdot \\
&\qquad (2\pi\sqrt{-1})^p \frac{1}{2}(\chi_{-p+1,k} \pm \chi_{-p+1,k+g}) \\
\bar{F}_\infty\left((2\pi\sqrt{-1})^{p-1}\varphi_k\right) &= +1 \cdot (-1)^{p-1}(2\pi\sqrt{-1})^{p-1}\varphi_k \\
\bar{F}_\infty\left((2\pi\sqrt{-1})^{p-1}\varphi_{k+g}\right) &= -1 \cdot (-1)^{p-1}(2\pi\sqrt{-1})^{p-1}\varphi_{k+g}.
\end{aligned}$$

$\diamond$

Remark 5.6 The characteristic function $\chi_{\mathcal{S}^+(w_{n,k})}$ can be regarded as the "best approximation" within $\mathcal{P}_n$ (*cf.* Theorem 4.12.*2.*) to a function (non-continuous) supported on the periodic sequence of period g_k,

$$f(g_k g_k g_k g_k g_k \ldots) = 1 \quad \text{and} \quad f(a_0 a_1 a_2 a_3 \ldots) = 0 \ \text{ otherwise.}$$

The periodic sequence $g_k g_k g_k g_k g_k \ldots$ represents under the correspondence of Lemma 4.8 the closed geodesic in $\tilde{\Xi}$ that is an oriented core handle of the handlebody $\mathfrak{X}_\Gamma$. Thus, the elements (5.6) that span the subspace $gr^\Gamma_{2p}\mathcal{V} \subset gr^\Gamma_{2p}H^1_{dyn}$ can be regarded as the "best approximations" within $gr^\Gamma_{2p}H^1_{dyn}$ to cohomology classes supported on the core handles of the handlebody. In other words, we may regard the index $p \leq 0$ in the graded structure $\mathcal{V} = \oplus_p gr^\Gamma_{2p}\mathcal{V}$ as measuring a way of "zooming in", with increasing precision for larger $|p|$, on the core handles of the handlebody $\mathfrak{X}_\Gamma$.

Notice that, while the archimedean cohomology $H^1(\tilde{X}^*)^{N=0}$ is identified with the *kernel* of the monodromy map, the dynamical cohomology is constructed by considering the *cokernel* of the map $\delta = 1 - T$. This suggests a duality between the monodromy N and the map $1 - T$. This will be made more precise in the next paragraph.

5.2 Dynamical homology and duality isomorphisms

Consider the $\mathbb{R}$-vector space $\mathcal{K}_N$ generated by all reduced sequences $a_0 \ldots a_N$ in the $\{g_i\}_{i=1}^{2g}$ with the condition $a_0 \neq a_N^{-1}$ (*cf.* Proposition 4.14).

Definition 5.7 *We define the* dynamical homology H_1^{dyn} *as*

$$H_1^{dyn} := \oplus_{p\geq 1} gr_{2p}^{\Gamma} H_1^{dyn}, \tag{5.15}$$

where we set

$$gr_{2p}^{\Gamma} H_1^{dyn} := \mathcal{K}_{p-1} \otimes_{\mathbb{R}} \mathbb{R}(p) \tag{5.16}$$

for $\mathbb{R}(p) = (2\pi\sqrt{-1})^p\mathbb{R}$. *The action of* $\bar{F}_\infty$ *on* H_1^{dyn} *is given by*

$$\bar{F}_\infty((2\pi\sqrt{-1})^p \ a_0 \ldots a_{p-1}) = (-1)^p (2\pi\sqrt{-1})^p \ a_{p-1}^{-1} \ldots a_0^{-1}. \tag{5.17}$$

We also define $\mathcal{W} \subset H_1^{dyn}$ *to be the graded subspace* $\mathcal{W} = \oplus_{p\geq 1} gr_{2p}^{\Gamma}\mathcal{W}$, *where* $gr_{2p}^{\Gamma}\mathcal{W}$ *is generated by the* $2g$ *elements*

$$(2\pi\sqrt{-1})^p \ \underbrace{g_k g_k \ldots g_k}_{p-times}.$$

Remark 5.8 Notice that the generators of H_1^{dyn} are periodic sequences $\overline{a_0 \ldots a_N}$, hence elements in $Ker(1 - T^d)$ for d the period length, $d|N$. Notice in particular that the subspace $\mathcal{W}$ can be identified with the part of H_1^{dyn} that is generated by elements in $Ker(1 - T)$, *i.e.* periodic sequences with period length $d = 1$.

The subspace $\mathcal{W}$ of the dynamical homology is related both to the subspace $\mathcal{V}$ of the dynamical cohomology, and to the space

$$\oplus_{r\geq 2} gr_{2r}^{w} H_Y^3(X) \cong \delta_1(\oplus_{p\leq 0} gr_{2p}^{w} H^1(\tilde{X}^*)^{N=0}),$$

for $r = -p + 2$, and with δ_1 the duality isomorphism of (3.23) (*cf.* Proposition 2.15).

Lemma 5.9 *The homology/cohomology pairing* (4.19) *induces an identification*

$$\mathcal{D} : gr_{2p}^{\Gamma}\mathcal{V} \xrightarrow{\simeq} gr_{2(-p+1)}^{\Gamma}\mathcal{W}$$

of the vector spaces $gr_{2p}^{\Gamma}\mathcal{W}$ *and* $gr_{2p}^{\Gamma}\mathcal{V}$, *for all* $p \leq 0$. *The map* $\mathcal{D}$ *satisfies* $\mathcal{D} \circ \bar{F}_\infty + \bar{F}_\infty \circ \mathcal{D} = 0$.

Proof. The pairing (4.19) of the class of the characteristic function $\chi_{\mathcal{S}^+(w_{-p+1,k})}$ with the element $\underbrace{g_j g_j \dots g_j}_{(-p+1)-times}$ is

$$\langle \chi_{\mathcal{S}^+(w_{-p+1,k})}, \underbrace{g_j g_j \dots g_j}_{(-p+1)-times} \rangle = (-p+1)\,\delta_{jk},$$

for $j,k = 1\dots 2g$. This induces a pairing

$$\langle \cdot,\cdot \rangle : gr^{\Gamma}_{2p}\mathcal{V} \times gr^{\Gamma}_{2(-p+1)}\mathcal{W} \to \mathbb{R}(1) \tag{5.18}$$

$$\langle (2\pi\sqrt{-1})^p \chi_{-p+1,k}, (2\pi\sqrt{-1})^{(-p+1)} \underbrace{g_j g_j \dots g_j}_{(-p+1)-times} \rangle = (2\pi\sqrt{-1})(-p+1)\delta_{jk}.$$

Via this pairing, we obtain an identification

$$\mathcal{D} : gr^{\Gamma}_{2p}\mathcal{V} \xrightarrow{\simeq} gr^{\Gamma}_{2(-p+1)}\mathcal{W}$$

of the form

$$\begin{aligned} \mathcal{D} :\ & (2\pi\sqrt{-1})^p \tfrac{1}{2}(\chi_{-p+1,k} \pm \chi_{-p+1,k+g}) \\ \mapsto\ & \tfrac{(2\pi\sqrt{-1})^{(-p+1)}}{(-p+1)} \tfrac{1}{2} \left(\underbrace{g_k g_k \dots g_k}_{(-p+1)-times} \pm \underbrace{g_k^{-1} g_k^{-1} \dots g_k^{-1}}_{(-p+1)-times} \right). \end{aligned} \tag{5.19}$$

The relation $\mathcal{D} \circ \bar{F}_\infty + \bar{F}_\infty \circ \mathcal{D} = 0$ follows by construction.
$\diamond$

We then obtain the following result.

Theorem 5.10 *For $p \leq 0$, consider the map*

$$\tilde{U} : gr^{w}_{2(-p+2)} H^2(X^*) \longrightarrow gr^{\Gamma}_{2(-p+1)}\mathcal{W}, \tag{5.20}$$

given by

$$\tilde{U}((2\pi\sqrt{-1})^{-p}\varphi_k) := (2\pi\sqrt{-1})^{-p+1} \frac{1}{2(-p+1)} \left(\underbrace{g_k \dots g_k}_{(-p+1)-times} - \underbrace{g_k^{-1} \dots g_k^{-1}}_{(-p+1)-times} \right) \tag{5.21}$$

$$U((2\pi\sqrt{-1})^{-p}\varphi_{k+g}) := (2\pi\sqrt{-1})^{-p+1} \frac{1}{2(-p+1)} \left(\underbrace{g_k \dots g_k}_{(-p+1)-times} + \underbrace{g_k^{-1} \dots g_k^{-1}}_{(-p+1)-times} \right), \tag{5.22}$$

for $k = 1, \dots, g$, with φ as in (5.14). The map $\tilde{U}$ is a $\bar{F}_\infty$-equivariant isomorphism of the graded vector spaces $\oplus_{r=-p+2\geq 2} gr^w_{2r} H^2(X^*)$ and $\mathcal{W}$. Moreover, the following diagram commutes and is compatible with the action of $\bar{F}_\infty$:

$$\begin{tikzcd} gr^w_{2p} H^1(\tilde{X}^*)^{N=0} \arrow[r, "\delta_1"] \arrow[d, "U"] & gr^w_{2(-p+2)} H^2(X^*) \arrow[d, "\tilde{U}"] \\ gr^\Gamma_{2p} \mathcal{V} \arrow[r, "\mathcal{D}"] & gr^\Gamma_{2(-p+1)} \mathcal{W} \end{tikzcd} \tag{5.23}$$

Proof. We have $gr^{\mathrm{w}}_{2(-p+2)} H^2(X^*) \cong H^1(X_{/\mathbb{R}}, \mathbb{R}(-p+1))$ by Propositions 2.22 and 2.23. The duality isomorphism (3.23) (*cf.* Proposition 2.15)

$$\delta_1 : gr^{\mathrm{w}}_{2p} H^1(\tilde{X}^*)^{N=0} \xrightarrow{\simeq} gr^{\mathrm{w}}_{2(-p+2)} H^2(X^*),$$

is given by

$$N^{2p-1} : H^1(X_{/\mathbb{R}}, \mathbb{R}(p)) \xrightarrow{\simeq} H^1(X_{/\mathbb{R}}, \mathbb{R}(-p+1)),$$
$$(2\pi\sqrt{-1})^{p-1}\varphi_k \mapsto (2\pi\sqrt{-1})^{-p}\varphi_k.$$

Notice that the duality isomorphism δ_1 also satisfies $\delta_1 \circ \bar{F}_\infty + \bar{F}_\infty \circ \delta_1 = 0$. The result then follows immediately.
$\diamond$

Remark 5.11 There is an intrinsic duality in the identifications of diagram (5.23). In fact, as discussed in the Remark 5.8, the space $H^1(\tilde{X}^*)^{N=0}$ corresponds to $\mathrm{Ker}(N)$ is identified with $\mathcal{V}$, which is obtained by considering the $\mathrm{Coker}(1-T)$, while the image of $H^1(\tilde{X}^*)^{N=0}$ under the duality isomorphism δ_1 is obtained by taking the $\mathrm{Coker}(N)$ in $\mathbb{H}^{\cdot}(X^*)$ and is identified with $\mathcal{W}$, which can be identified with $\mathrm{Ker}(1-T)$. Modulo this duality, we have a correspondence between the monodromy map N and the dynamical map $1-T$. The presence of this duality is not surprising, considering that the cohomological construction of Section 2 is a theory of the special fiber, while the dynamical construction of Section 4 and [24] is a theory of the *dual* graph.

5.3 Hilbert completions

In order to construct a spectral triple associated to the dynamical homology and cohomology, we first need to choose a suitable Hilbert space completions of the vector space $H^1_{dyn} \oplus H_1^{dyn}$.

Since we want to preserve the information given by the twisted coefficients $\mathbb{R}(p)$, we choose to work with real coefficients.

We consider the *real* Hilbert space $L^2(\Lambda_\Gamma, \mu)$, of real valued functions of Λ_Γ that are square integrable with respect to the Patterson–Sullivan measure μ. Since the operator $\delta = 1-T$ is a bounded operator on $L^2(\Lambda_\Gamma, \mu)$, its range is a closed subspace of $L^2(\Lambda_\Gamma, \mu)$, hence the quotient space $\mathrm{Coker}(1-T)$ inherits the structure of Hilbert space.

With a slight abuse of notation we denote by $\mathcal{P}_n$ the finite dimensional linear subspace of $L^2(\Lambda_\Gamma, \mu)$ spanned by functions that are constant on $\Lambda_\Gamma(\gamma) \subset \Lambda_\Gamma$, for all $\gamma \in \Gamma$ of word length $|\gamma| > n+1$. Since we have $\delta : \mathcal{P}_n \to \mathcal{P}_{n+1}$ as in the proof of Theorem (4.12), we realize the F_n of §5.1 as finite dimensional linear subspaces of the Hilbert space $\mathrm{Coker}(1-T)$. This induces an inner product on each Gr_n. We denote the corresponding norm by $\|\cdot\|_{Gr_n}$. This defines a real Hilbert space structure on H^1_{dyn}, with norm

$$\|\sum_p x_p\| := \left(\sum_p \|x_p\|^2_{Gr_{-p}}\right)^{1/2}. \tag{5.24}$$

We denote the Hilbert space completion of H^1_{dyn} in this norm by $\mathcal{H}^1_{dyn}$.

A real Hilbert space structure on H_1^{dyn} is obtained in the following way. Each summand $\mathcal{K}_N$ can be regarded as a finite dimensional linear subspace of the real Hilbert space $\ell^2(\Gamma)$, by identifying the generators $a_0 \dots a_N$ with $a_0 \neq a_N^{-1}$ with a subset of the set of all finite reduces words, which is a complete basis of $\ell^2(\Gamma)$. This determines the inner product on each $\mathcal{K}_N$. We denote the corresponding norm by $\|\cdot\|_{\mathcal{K}_N}$. We obtain a real Hilbert space structure on H_1^{dyn} with norm

$$\|\sum_p x_p\| := \left(\sum_p \|x_p\|^2_{\mathcal{K}_N}\right)^{1/2}. \tag{5.25}$$

We denote the Hilbert space completion of H_1^{dyn} in this norm by $\mathcal{H}_1^{dyn}$.

5.4 Representations

We now construct representations on the Hilbert spaces $\mathcal{H}^1_{dyn}$ and $\mathcal{H}_1^{dyn}$.

There is an action of the algebra $\mathcal{O}_A$ on H^1_{dyn}, which is described as follows.

Proposition 5.12 *Let $\mathcal{O}_A^{alg}$ be the $*$-algebra over $\mathbb{R}$ generated algebraically by the S_i satisfying* (4.22) (4.23)*. The action $S_i : F_n \mapsto F_n$ of Proposition 4.19 induces a representation*

$$\rho : \mathcal{O}_A^{alg} \to End(H^1_{dyn}).$$

This extends to a representation

$$\rho : \mathcal{O}_A \to \mathcal{B}(\mathcal{H}^1_{dyn}) \tag{5.26}$$

as bounded operators on the real Hilbert space completion $\mathcal{H}^1_{dyn}$, where $\mathcal{O}_A$ is considered here as a real C**-algebra.*

Proof. Recall that by Proposition 4.19 the generators S_i of $\mathcal{O}_A$ act on each F_n. In fact, we can write explicitly the action as

$$S_i : f(a_0 \dots a_n) \mapsto u(a_0 \dots a_{n-1}) = \begin{cases} f(g_i^{-1} a_0 \dots a_{n-1}) & a_0 \neq g_i \\ 0 & a_0 = g_i. \end{cases} \tag{5.27}$$

Thus, under the action of the S_i we have $S_i : F_0 \to F_0$ and $S_i : F_n \to F_{n-1}$ for each $n \geq 1$, hence we can consider the induced action

$$\bar{S}_i : Gr_n \to Gr_{n-1} \qquad \bar{S}_i : Gr_0 \to Gr_0.$$

The action of the S_i can be defined over real coefficients (*cf.* Proposition 4.19) hence we obtain an action of the algebra $\mathcal{O}_A^{alg}$ over $\mathbb{R}$ on the dynamical cohomology H^1_{dyn}. By Proposition 4.19 the elements of $\mathcal{O}_A^{alg}$ act as bounded operators with respect to the norm (5.24) on H^1_{dyn}, hence by taking the closure in the operator norm we obtain an action of the real $\mathcal{O}_A$ on $\mathcal{H}^1_{dyn}$.
$\diamond$

On the space $\mathcal{H}^1_{dyn}$ we can also define an action of the C*-algebra $(\mathrm{C}(\Lambda_\Gamma)\otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$. Here $\mathrm{C}_0(\mathbb{H}^3)$ denotes the C*-algebra of continuous functions that vanish on the sphere at infinity $\mathbb{P}^1(\mathbb{C})$ of $\mathbb{H}^3$.

The *algebraic* crossed product $(\mathrm{C}(\Lambda_\Gamma)\otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes^{alg} \Gamma$ is the algebra whose elements consist of *finite sums* $\sum_\gamma f_\gamma T_\gamma$ with f_γ in $\mathrm{C}(\Lambda_\Gamma)\otimes \mathrm{C}_0(\mathbb{H}^3)$ and the product

$$(f_\gamma T_\gamma)(f_{\gamma'} T_{\gamma'}) = f_\gamma (T_\gamma f_{\gamma'}) T_{\gamma\gamma'}, \tag{5.28}$$

with $T_\gamma f_{\gamma'} = f_{\gamma'} \circ \gamma^{-1}$. The (reduced) crossed product $(\mathrm{C}(\Lambda_\Gamma)\otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$ is obtained by taking the closure of $(\mathrm{C}(\Lambda_\Gamma)\otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes^{alg} \Gamma$ in the algebra of bounded operators.

Proposition 5.13 *Let $x_0 \in \mathbb{H}^3$ be a fixed base point. For $f \in \mathrm{C}_0(\mathbb{H}^3)$, we define an operator $\tilde{\rho}(f) : F_n \to F_n$ as follows. For $h \in F_n$, we set*

$$(\tilde{\rho}(f)h)(a_0 \ldots a_n) = f((a_0 \ldots a_n)\cdot x_0)\cdot h(a_0 \ldots a_n), \tag{5.29}$$

where we identify the reduced word $a_0 \ldots a_n$ with an element of Γ. This extends the representation of $\mathcal{O}_A$ on $\mathcal{H}^1_{dyn}$ given in Proposition 5.12 to a representation

$$\tilde{\rho} : (\mathrm{C}(\Lambda_\Gamma)\otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma \to \mathcal{B}(\mathcal{H}^1_{dyn}). \tag{5.30}$$

Proof. We first show that, using the operators $\tilde{\rho}(f)$ of (5.29), we can construct a representation of the algebraic crossed product $(\mathrm{C}(\Lambda_\Gamma)\otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes^{alg} \Gamma$ as linear operators on $\mathcal{H}^1_{dyn}$. Consider an element $\sum_\gamma f_\gamma T_\gamma$ in the algebraic crossed product. This is a finite sum, hence in order to define $\tilde{\rho}(\sum_\gamma f_\gamma T_\gamma)$ it suffices to define $\tilde{\rho}(f_\gamma T_\gamma)$ for each term. The operators T_γ act on $\mathcal{H}^1_{dyn}$ as prescribed by Proposition 5.12, by viewing T_γ as an element of $\mathcal{O}_A$ (*cf.* Theorem 4.17). Thus, we set

$$\tilde{\rho}(f_\gamma T_\gamma) := \tilde{\rho}(f)\rho(T_\gamma),$$

where ρ is the representation of $\mathcal{O}_A$ as in (5.26). Notice that the composition rule (5.28) is satisfied by construction.

Next, we check that the operators $\tilde{\rho}(f)\rho(T_\gamma)$ acting on $\mathcal{H}^1_{dyn}$ are all bounded operators. It is sufficient to check that (5.29) are bounded operators. This follows by an estimate

$$\|\rho(f)h\| \leq (\sup_{x\in\mathbb{H}^3} |f(x)|)\, \|h\|,$$

where the supremum is finite, since f is continuous and vanishing at infinity. Thus, by taking the closure in the operator norm in $\mathcal{B}(\mathcal{H}^1_{dyn})$ we obtain a representation $\tilde{\rho}$ of the (reduced) crossed product $(\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$.

$\diamond$

On the space $\mathcal{H}_1^{dyn}$ we construct an action of the C*-algebra $\mathrm{C}(\Lambda_\Gamma)$ in the following way.

Lemma 5.14 *For f a real valued continuous functions on Λ_Γ, consider the operator*

$$\rho(f) : (2\pi\sqrt{-1})^p\, a_0 \dots a_p \mapsto f(\overline{a_0 \dots a_p})\ (2\pi\sqrt{-1})^p\, a_0 \dots a_p. \tag{5.31}$$

These operators define a representation of the algebra of real valued continuous functions $\mathrm{C}(\Lambda_\Gamma, \mathbb{R})$ as bounded operators on $\mathcal{H}_1^{dyn}$,

$$\rho : \mathrm{C}(\Lambda_\Gamma, \mathbb{R}) \to \mathcal{B}(\mathcal{H}_1^{dyn}).$$

Proof. We write an element in $\mathcal{H}_1^{dyn}$ as an infinite sum

$$X = \sum_p X_p \qquad X_p \in gr^\Gamma_{2p} H_1^{dyn},$$

with the condition that $\sum_p \|X_p\|^2_{\mathcal{K}_{p-1}} < \infty$. Each term X_p in turn can be written as a finite linear combination with real coefficients of basis elements $(2\pi\sqrt{-1})^p a_0 \dots a_p$, where $a_p \neq a_0^{-1}$. Given $f \in \mathrm{C}(\Lambda_\Gamma, \mathbb{R})$, we define the operator $\rho(f)$ as in (5.31), where the $(2\pi\sqrt{-1})^p\, a_0 \dots a_p$ are a complete basis of $\mathcal{H}_1^{dyn}$. The action extends by linearity. The elements $\rho(f)$ defined this way are bounded operators. In fact, the definition (5.31) implies that the following estimate holds:

$$\|\rho(f)\ (X)\| \leq (\sup_{x\in\Lambda_\Gamma} |f(x)|) \cdot \|X\|.$$

$\diamond$

For a C*-algebra $\mathcal{A}$, we denote by $\mathrm{C}_b(\mathbb{H}^3, \mathcal{A})$ the C*-algebra of bounded continuous functions $f : \mathbb{H}^3 \to \mathcal{A}$.

We extend the representation of $\mathrm{C}(\Lambda_\Gamma)$ on the space $\mathcal{H}_1^{dyn}$, given in Lemma 5.14, to a representation of the algebra $\mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma))$.

An element $f \in \mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma))$ is a collection f_u of elements in $\mathrm{C}(\Lambda_\Gamma)$, which vary continuously in the parameter $u \in \mathbb{H}^3$, and converge to zero in the norm of $\mathrm{C}(\Lambda_\Gamma)$ as u approaches the boundary at infinity $\mathbb{P}^1(\mathbb{C})$ in $\mathbb{H}^3$.

Lemma 5.15 *Let $x_0 \in \mathbb{H}^3$ be a fixed base point. For $f \in \mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma))$ we set:*

$$\tilde{\rho}(f)\, a_0 \ldots a_p = f_{(a_0 \ldots a_p)\, x_0}(\overline{a_0 \ldots a_p}) \cdot a_0 \ldots a_p. \tag{5.32}$$

This extends the representation (5.31) *to a representation*

$$\tilde{\rho} : \mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma)) \to \mathcal{B}(\mathcal{H}_1^{dyn}).$$

Proof. It is sufficient to check that the $\tilde{\rho}(f)$ are bounded operators. This again follows from the estimate

$$\|\tilde{\rho}(f)\, X\| \leq (\sup_{u \in \mathbb{H}^3} \sup_{x \in \Lambda_\Gamma} |f_u(x)|) \cdot \|X\|.$$

The first supremum is finite since we are considering bounded functions on $\mathbb{H}^3$.
$\diamond$

Let $\mathfrak{A}$ be a C*-algebra with an action of Γ by automorphisms. Let $\mathcal{E}$ be the flat $\mathfrak{A}$-bundle

$$\mathcal{E} = (\mathfrak{A} \times \mathbb{H}^3)/\Gamma \longrightarrow \mathfrak{X}_\Gamma, \tag{5.33}$$

where we consider the diagonal action of Γ on $\mathfrak{A} \times \mathbb{H}^3$. We consider the C*-algebra $\mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})$ of continuous sections of $\mathcal{E}$ that vanish on the boundary at infinity $X_{/\mathbb{R}}$ of the handlebody $\mathfrak{X}_\Gamma$.

Proposition 5.16 *The representation of* $\mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma))$ *defined by* (5.32) *induces a representation*

$$\tilde{\rho} : \mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E}) \to \mathcal{B}(\mathcal{H}_1^{dyn}),$$

where $\mathcal{E}$ is the bundle (5.33) *for* $\mathfrak{A} = \mathrm{C}(\Lambda_\Gamma)$.

Proof. The algebra $\mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})$ can be identified with a subalgebra of functions in $\mathrm{C}_b(\mathbb{H}^3, \mathrm{C}(\Lambda_\Gamma))$ satisfying $f_{\gamma x} = T_\gamma f_x$, where T_γ is the operator implementing the action of the element γ on $\mathrm{C}(\Lambda_\Gamma)$, as in (4.24). Thus, by restriction of the representation of Lemma 5.15, we obtain a representation of $\mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})$.
$\diamond$

We introduce the notation $\mathcal{A} := (\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$ and $\hat{\mathcal{A}} := \mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})$. In Propositions 5.13 and 5.16 we have obtained in representations

$$\begin{aligned} \tilde{\rho} : \mathcal{A} &\to \mathcal{B}(\mathcal{H}^1_{dyn}) \\ \tilde{\rho} : \hat{\mathcal{A}} &\to \mathcal{B}(\mathcal{H}_1^{dyn}). \end{aligned} \tag{5.34}$$

The important observation is that the algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ represent *the same* noncommutative space. To understand this point, we recall the following result (*cf.* [22]).

Proposition 5.17 *Let G be a connected Lie group and K a maximal compact subgroup. For $\Gamma \subset G$ a lattice and $\mathfrak{A}$ a C^*-algebra with an action of Γ by automorphisms, the algebras $(\mathfrak{A} \otimes C_0(G/K)) \rtimes \Gamma$ and $C_0(\Gamma\backslash G/K, \mathcal{E})$ are strongly Morita equivalent. Here $\mathcal{E}$ is the flat $\mathfrak{A}$-bundle over $\Gamma\backslash G/K$,*

$$\mathcal{E} = (\mathfrak{A} \times G/K)/\Gamma \longrightarrow \Gamma\backslash G/K,$$

where Γ acts diagonally on $\mathfrak{A} \times G/K$.

In our case, the result of Proposition 5.17 implies the following.

Corollary 5.18 *Let $\mathcal{K}$ be the algebra of compact operators on a separable Hilbert space. Then the algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ in (5.34) are strongly Morita equivalent.*

Proof. In the case of $\mathbb{H}^3$ and the Schottky group Γ the conditions of Proposition 5.17 are verified, by taking $G = SO_0(3,1)$, so that $\mathbb{H}^3 = G/K$, and $\mathfrak{A} = C(\Lambda_\Gamma)$.
$\diamond$

5.5 Reduction mod ∞

We argue that the non-commutative space considered in the previous paragraph, represented by the (Morita equivalence class of the) C^*-algebra $\mathcal{A}$ of (5.34), describes what Manin refers to in [24] as the "reduction mod ∞".

We recall briefly how the reduction map works in the non-archimedean setting of Mumford curves (*cf.* [24] [33]). Let $\mathfrak{K}$ be a finite extension of $\mathbb{Q}_p$ and let $\mathfrak{a}$ be its ring of integers. The correct analog for the archimedean case is obtained by "passing to a limit', replacing $\mathfrak{K}$ with its Tate closure in $\mathbb{C}_p$ (*cf.* [24] §3.1), however, for our purposes here it is sufficient to illustrate the case of a finite extension.

The role of the hyperbolic space $\mathbb{H}^3$ in the non-archimedean case is played by the Bruhat-Tits tree $\mathcal{T}_{BT}$ with vertices

$$\mathcal{T}^0_{BT} = \{\mathfrak{a} - \text{lattices of rank 2 in a 2-dim } \mathfrak{K}\text{-space}\}/\mathfrak{K}^*.$$

Vertices in $\mathcal{T}_{BT}$ have valence $|\mathbb{P}^1(\mathfrak{a}/\mathfrak{m})|$, where $\mathfrak{m}$ is the maximal ideal. Each edge in $\mathcal{T}_{BT}$ has length $\log|\mathfrak{a}/\mathfrak{m}|$. The set of ends of $\mathcal{T}_{BT}$ is identified with $X(\mathfrak{K}) = \mathbb{P}^1(\mathfrak{K})$. This is the analog of the conformal boundary $\mathbb{P}^1(\mathbb{C})$ of $\mathbb{H}^3$. Geodesics correspond to doubly infinite paths in $\mathcal{T}_{BT}$ without backtracking.

Fix a vertex v_0 on $\mathcal{T}_{BT}$. This corresponds to the closed fiber $X_\mathfrak{a} \otimes (\mathfrak{a}/\mathfrak{m})$ for the chosen $\mathfrak{a}$-structure $X_\mathfrak{a}$. Each $x \in \mathbb{P}^1(\mathfrak{K})$ determines a unique choice of a subgraph $e(v_0, x)$ in $\mathcal{T}_{BT}$ with vertices $(v_0, v_1, v_2, \dots)$ along the half infinite path without backtracking which has end x. The subgraphs $e(v_0, x)_k$ with vertices $(v_0, v_1, \dots v_k)$ correspond to the reduction mod $\mathfrak{m}^k$, namely

$$\{e(v_0, x)_k : x \in X(\mathfrak{K})\} \leftrightsquigarrow X_\mathfrak{a}(\mathfrak{a}/\mathfrak{m}^k).$$

Thus the finite graphs $e(v_0, x)_k$ represent $\mathfrak{a}/\mathfrak{m}^k$ points, and the infinite graph $e(v_0, x)$ represents the reduction of x.

A Schottky group Γ, in this non-archimedean setting, is a purely loxodromic free discrete subgroup of $\mathrm{PSL}(2, \mathfrak{K})$ in g generators. The doubly infinite paths in $\mathcal{T}_{BT}$ with ends at the pairs of fixed points $x^{\pm}(\gamma)$ of the elements $\gamma \in \Gamma$ produce a copy of the combinatorial tree $\mathcal{T}$ of the group Γ in $\mathcal{T}_{BT}$. This is the analog of regarding $\mathbb{H}^3$ as the union of the translates of a fundamental domain for the action of the Schottky group, which can be thought of as a 'tubular neighborhood' of a copy of the Cayley graph $\mathcal{T}$ of Γ embedded in $\mathbb{H}^3$. The ends of the tree $\mathcal{T} \subset \mathcal{T}_{BT}$ constitute the limit set $\Lambda_\Gamma \subset \mathbb{P}^1(\mathfrak{K})$. The complement $\Omega_\Gamma = \mathbb{P}^1(\mathfrak{K}) \smallsetminus \Lambda_\Gamma$ gives the uniformization of the Mumford curve $X(\mathfrak{K}) \simeq \Omega_\Gamma/\Gamma$. In turn, $X(\mathfrak{K})$ can be identified with the ends of the quotient graph $\mathcal{T}_{BT}/\Gamma$, just as in the archimedean case the Riemann surface is the conformal boundary at infinity of the handlebody $\mathfrak{X}_\Gamma$.

The reduction map is then obtained by considering the half infinite paths $e(v, x)$ in $\mathcal{T}_{BT}/\Gamma$ that start at a vertex v of the finite graph $\mathcal{T}/\Gamma$ and whose end x is a point of $X(\mathfrak{K})$, while the finite graphs $e(v, x)_k$ provide the $\mathfrak{a}/\mathfrak{m}^k$ points.

This suggests that the correct analog of the reduction map in the archimedean case is obtained by considering geodesics in $\mathbb{H}^3$ with an end on Ω_Γ and the other on Λ_Γ, as described in [24]. Arguing as in Lemma 4.9, we see that the set of such geodesics can be identified with the quotient $\Omega_\Gamma \times_\Gamma \Lambda_\Gamma$. The analog of the finite graphs $e(v, x)_k$ is then given by the quotient $\mathbb{H}^3 \times_\Gamma \Lambda_\Gamma$.

The C^*-algebra $\mathcal{A} = (\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$ corresponds to a way of describing this quotient as a non-commutative space.

In particular, in the definition of the action (5.30) and (5.32), we consider elements of the form $(a_0 \dots a_N) \cdot x_0$, for x_0 a fixed base point in $\mathbb{H}^3$, and the reduced word $a_0 \dots a_N$ defining an element $\gamma \in \Gamma$ of word length $|\gamma| = N + 1$. Notice that the choice of x_0 determines an embedding of a Cayley graph $\mathcal{T}$ of the group Γ in $\mathbb{H}^3$. This has set of vertices $\mathcal{T}^0 = \Gamma \cdot x_0$ and set of edges $\mathcal{T}^1$ given by the set of geodesic segments connecting each vertex $v \in \mathcal{T}^0$ to the vertices $g_k v$, for g_k the g generators of Γ. For a fixed $x = a_0 a_1 \dots a_N \dots$ in Λ_Γ, the points $(a_0 \dots a_N) \cdot x_0$, for $N \leq k$ correspond to the graph $e(v, x)_k$ of the non-archimedean setting. Thus, an archimedean analog of the reductions mod $\mathfrak{m}^k$ is built in with our choice of representations.

5.6 The Dirac operator

We consider the linear operator D on $H^1_{dyn} \oplus H_1^{dyn}$ defined by setting, for $p \leq 0$,

$$D : gr^\Gamma_{2p} H^1_{dyn} \to gr^\Gamma_{2p} H^1_{dyn} \qquad D : x \mapsto p \cdot x \tag{5.35}$$

and, for $p \geq 1$,

$$D : gr^\Gamma_{2p} H_1^{dyn} \to gr^\Gamma_{2p} H_1^{dyn} \qquad D : x \mapsto p \cdot x. \tag{5.36}$$

This defines an unbounded linear operator

$$D : \mathcal{H}^1_{dyn} \oplus \mathcal{H}^{dyn}_1 \to \mathcal{H}^1_{dyn} \oplus \mathcal{H}^{dyn}_1. \tag{5.37}$$

We show that this operator has the correct properties of a Dirac operator.

Proposition 5.19 *The operator D acting on the Hilbert space $\mathcal{H} = \mathcal{H}^1_{dyn} \oplus \mathcal{H}^{dyn}_1$ is self adjoint. For all $\lambda \notin \mathbb{R}$, the resolvent $R_\lambda(D) := (D-\lambda)^{-1}$ is a compact operator on $\mathcal{H}$. The restriction of the operator D to $\mathcal{V} \oplus \mathcal{W} \subset \mathcal{H}$ agrees with the Frobenius-type operator Φ of* (3.22)*:*

$$\begin{aligned} U\ \Phi|_{gr^w_{2p}H^1(\tilde{X}^*)^{N=0}}\, U^{-1} &= D|_{gr^\Gamma_{2p}\mathcal{V}} & p \le 0 \\ \tilde{U}\ \Phi|_{gr^w_{2p}H^2(X^*)}\, \tilde{U}^{-1} &= D|_{gr^\Gamma_{2(p-1)}\mathcal{W}} & p \ge 2. \end{aligned}$$

Proof. The operator D as in (5.37) is already given in diagonal form and is clearly symmetric with respect to the inner product of $\mathcal{H}$, hence it is self-adjoint on the domain $\mathrm{dom}(D) = \{X \in \mathcal{H} : DX \in \mathcal{H}\}$. For $\lambda \notin \mathrm{Spec}(D)$, the operators $R_\lambda(D)$ are bounded, and related by the resolvent equation

$$R_\lambda(D) - R_{\lambda'}(D) = (\lambda' - \lambda) R_\lambda(D) R_{\lambda'}(D).$$

This implies that, if $R_\lambda(D)$ is compact for one $\lambda \notin \mathrm{Spec}(D)$, then all the other $R_{\lambda'}(D)$, $\lambda' \notin \mathrm{Spec}(D)$ are also compact. In our case we have $\mathrm{Spec}(D) = \mathbb{Z}$, hence it is easy to check that, for instance, $R_{1/2}(D)$ is a compact operator with spectrum $\{(n+1/2)^{-1} : n \in \mathbb{Z}\} \cup \{0\}$. Furthermore, it is clear that the restriction of D to $\mathcal{V}$ agrees with the restriction of Φ to $H^1(\tilde{X}^*)^{N=0}$. On the other hand, recall that the operator Φ defined as in (3.22) acts on $gr^{\mathrm{w}}_{2p}H^2(X^*)$, for $p \ge 2$, as multiplication by $p-1$. The map $\tilde{U}$ of (5.20) identifies $gr^{\mathrm{w}}_{2p}H^2(X^*)$ with $gr^\Gamma_{2(p-1)}\mathcal{W}$, hence we obtain $\tilde{U}\Phi|_{gr^{\mathrm{w}}_{2p}H^2(X^*)}\tilde{U}^{-1} = D|_{gr^\Gamma_{2(p-1)}\mathcal{W}}$.
$\diamond$

We then consider the commutators of the Dirac operator D with the elements of the representations (5.30) and (5.32).

Proposition 5.20 *The operator D of* (5.37) *satisfies the following conditions:*

1. *The commutator $[D, \tilde{\rho}(a)]$ is a bounded operator on $\mathcal{H}^1_{dyn}$ for all $a \in (\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$, and $\tilde{\rho}$ as in* (5.30)*.*

2. *The operator D on $\mathcal{H}^{dyn}_1$ has trivial commutators $[D, \tilde{\rho}(a)] = 0$ with all the elements $a \in \mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})$, for $\tilde{\rho}$ as in* (5.32)*.*

Proof. *1.* First we check that the commutators $[D, \bar{S}_i]$ are bounded operators, where the $\bar{S}_i$ are the generators of the algebra $\mathcal{O}_A$ acting on $\mathcal{H}^1_{dyn}$ as in Lemma 5.12. This follows by the explicit description (5.27), which implies

$$[D, \bar{S}_i]h = -\bar{S}_i h \qquad h \in \mathrm{Gr}_n \quad n \ge 1.$$

This implies, in particular, that the commutators $[D, T_\gamma]$ are bounded operators on $\mathcal{H}^1_{dyn}$. Notice then that we have trivial commutators $[D, \tilde{\rho}(f)] = 0$ for $f \in C_0(\mathbb{H}^3)$, and $\tilde{\rho}$ as in (5.29). In fact, the operators $\tilde{\rho}(f)$ preserve each $gr^\Gamma_{2p}\mathcal{H}^1_{dyn}$, where D acts as a scalar multiple of the identity. This proves the result.

2. The argument is analogous to the case of (5.29) discussed above. The operators $\tilde{\rho}(f)$ with $f \in C_0(\mathfrak{X}_\Gamma, \mathcal{E})$ and $\tilde{\rho}$ as in (5.32) preserve each $gr^\Gamma_{2p}\mathcal{H}^1_{dyn}$, where D acts as a scalar multiple of the identity, hence $[D, \tilde{\rho}(f)] = 0$.

$\diamond$

5.7 Morita equivalence and spectral triples

In order to construct a spectral triple from the results stated in Propositions 5.13, 5.16, 5.19, and 5.20, we need to recall how to transfer the structure of spectral triple between strongly Morita equivalent algebras.

We have the following result (*cf.* [11], [10], [23]).

Proposition 5.21 *Let ρ be a representation $\rho_1 : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H})$, and let D be an operator on the Hilbert space $\mathcal{H}$, such that $(\mathcal{A}_1, \mathcal{H}, D)$ satisfies the properties of a spectral triple. Let $\mathcal{M}$ be a bimodule implementing a Morita equivalence between $\mathcal{A}_1$ and $\mathcal{A}_2$, which is a finite projective right Hilbert module over $\mathcal{A}_1$. Let $\Omega^1_D \subset \mathcal{B}(\mathcal{H})$ be the $\mathcal{A}_1$-bimodule generated by operators of the form $a_1[D, b_1]$, for $a_1, b_1 \in \mathcal{A}_1$. Let ∇ be a linear map*

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{A}_1} \Omega^1_D$$

such that, for all $\xi \in \mathcal{M}$ and $a_1 \in \mathcal{A}_1$,

$$\nabla(\xi a_1) = (\nabla\xi)a_1 + \xi \otimes [D, a_1]$$

and, for all $\xi_1, \xi_2 \in \mathcal{M}$,

$$\langle \xi_1, \nabla\xi_2 \rangle_{\mathcal{A}_1} - \langle \nabla\xi_1, \xi_2 \rangle_{\mathcal{A}_1} = [D, \langle \xi_1, \xi_2 \rangle_{\mathcal{A}_1}].$$

The choice of such ∇ determines a spectral triple $(\mathcal{A}_2, \tilde{\mathcal{H}}, \tilde{D})$. Here the Hilbert space is

$$\tilde{\mathcal{H}} = \mathcal{M} \otimes_{\mathcal{A}_1} \mathcal{H},$$

the representation $\rho_2 : \mathcal{A}_2 \to \mathcal{B}(\tilde{\mathcal{H}})$ is defined as

$$\rho_2(a_2)\ (\xi \otimes_{\mathcal{A}_1} x) := (a_2\xi) \otimes_{\mathcal{A}_1} x), \tag{5.38}$$

and the Dirac operator

$$\tilde{D}(\xi \otimes x) = \xi \otimes D(x) + (\nabla\xi)x.$$

Notice that the choice of the *Hermitian connection* ∇ is necessary, because in general the commutators $[D, \rho(a)]$ for $a \in \mathcal{A}_1$ are non-trivial, hence the operator $1 \otimes D$ would not be well defined on $\tilde{\mathcal{H}} = \mathcal{M} \otimes_{\mathcal{A}_1} \mathcal{H}$.

However, in our particular case, the condition of Proposition 5.20.*2.* implies that we are in a special case where we can simply consider the operator $\tilde{D} = 1 \otimes D$, with $\nabla = 0$. We obtain the following result.

Theorem 5.22 *Let* $\mathcal{H} = \mathcal{H}^1_{dyn} \oplus \mathcal{M} \otimes_{\hat{\mathcal{A}}} \mathcal{H}^{dyn}_1$*, with* $\mathcal{M}$ *a bimodule implementing the Morita equivalence between the* C*-*algebras* $\hat{\mathcal{A}}$ *and* $\mathcal{A}$ *of* (5.34)*. Then* $(\mathcal{A}, \mathcal{H}, \tilde{D})$ *is a spectral triple, where* $\tilde{D}|_{\mathcal{H}^1_{dyn}} = D$ *as defined in* (5.35)*, and* $\tilde{D}|_{\mathcal{M}\otimes\mathcal{H}^{dyn}_1} = 1 \otimes D$*, with* D *as defined in* (5.36)*. The representation* $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *is given by* $\rho = \tilde{\rho} \oplus \hat{\rho}$*, with* $\tilde{\rho}$ *as in* (5.30) *and* $\hat{\rho}$ *obtained as in* (5.38)*.*

6 Further structure at arithmetic infinity

We point out some other interesting aspects of the relation we have developed between the two approaches of Manin and Deninger to the theory of the closed fiber at arithmetic infinity. We hope to return to them in future work.

(i) Solenoids. The infinite tangle of bounded geodesics in $\mathfrak{X}_\Gamma$ can also be related to generalized solenoids, thus making another connection between Manin's theory of the closed finer at infinity and Deninger's theory of "arithmetic cohomology", for which the importance of solenoids was stressed in the remarks of §5.8 of [19].

There are various ways in which the classical p–adic solenoids can be generalized. One standard way of producing more general solenoids is by considering a map $\phi : C \to C$, which is a continuous surjection of a Cantor set C, and form the quotient space $(C \times [0,1])/\sim$ by the identification $(x, 0) \sim (\phi(x), 1)$. The resulting space is a generalized solenoid provided it is an *indecomposable* compact connected space, where the indecomposable condition means that the image under the quotient map of any $D \times [0,1]$ where D is a clopen subset of C is connected if and only if $D = C$. It is not hard to check that this condition is satisfied if ϕ is a homeomorphism of a Cantor set that has a dense orbit. Thus, our model for the dual graph of the fiber at arithmetic infinity, given by the mapping torus $\mathcal{S}_T$, is a generalized solenoid. This can also be related to generalized solenoids for subshifts of finite type [54].

(ii) Dynamical zeta function. The action of the shift operator T on the limit set Λ_Γ provides a dynamical zeta function on the special fiber at arithmetic infinity. In fact, there is a Perron–Frobenius theory for the shift T, with the Ruelle transfer operator

$$(\mathcal{L}_{-sf}\, g)(z) := \sum_{y:Ty=z} e^{-sf(y)} g(y), \tag{6.1}$$

depending on a parameter $s \in \mathbb{C}$ and with the function $f(z) = \log|T'(z)|$. This is the analog of the Gauss–Kuzmin operator studied in [29] [30] in the case of modular curves. For $\mathcal{B}_q$ a function space of q–forms on $\mathcal{U} \subset \mathbb{P}^1(\mathbb{C})$, with $\mathcal{U} = \cup_i D_i^\pm$, with analytic coefficients and uniformly bounded, we denote by $\mathcal{L}_{s,q}$ the operator (6.1) acting on $\mathcal{B}_q$. In the case $q = 0$, for $s = \delta$ the Hausdorff dimension of Λ_Γ, the operator $L_{\delta,0}$ has top eigenvalue 1. This is a simple eigenvalue, with a non–negative eigenfunction which is the density of the invariant measure. The analysis is similar to the case described in

[29]. For $\mathrm{Re}(s) >> 0$ the operators $\mathcal{L}_{s,q}$ are nuclear. If we denote by $\{\lambda_{i,q}(s)\}_i$ the eigenvalues of $\mathcal{L}_{s,q}$, we can define

$$P_q(s) = \det(1 - \mathcal{L}_{s,q}) = \prod_i (1 - \lambda_{i,q}(s)), \tag{6.2}$$

so that we have a Ruelle zeta function

$$Z_\Gamma(s) := \frac{P_1(s)}{P_0(s)P_2(s)}. \tag{6.3}$$

By the results of [36] §4, the Ruelle zeta function $Z_\Gamma(s)$ defined above is directly related to the Selberg zeta function of the hyperbolic handlebody $\mathfrak{X}_\Gamma$ of the form

$$Z(s) = \prod_{\gamma \in \Gamma} (1 - N(\gamma)^{-s}),$$

for $N(\gamma) = \exp(\ell_\gamma)$ the length of the corresponding primitive closed geodesic (equivalently $N(\gamma)$ is the norm of the derivative of γ at the repelling fixed point $z^+(\gamma) \in \mathbb{P}^1(\mathbb{C})$). If we consider the set of $s \in \mathbb{C}$ where 1 is an eigenvalue of $\mathcal{L}_{s,q}$, these produce possible zeroes and poles of the zeta function $Z_\Gamma(s)$: among these, the Hausdorff dimension $s = \delta$. The set of complex numbers where $Z_\Gamma(s)$ has a pole provides this way another notion of dimension spectrum of the fractal Λ_Γ extending the ordinary Hausdorff dimension.

It should be interesting to study the properties of $Z_\Gamma(s)$ in relation to the properties of the arithmetic zeta function (3.36) we defined in §3.4.

(iii) Missing Galois theory. The data of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ associated to the special fiber at arithmetic infinity may provide a different approach to what Connes refers to as a *missing Galois theory at archimedean places.* The idea is derived from the general setting of Hilbert's twelfth problem, for a local or global field K, and its maximal abelian (separable) extension K^{ab}. While class field theory provides a description of $Gal(K^{ab}/K)$, Hilbert's twelfth problem addresses the question of providing explicit generators of K^{ab} and an explicit action of the Galois group, much as in the Kronecker–Weber case over $\mathbb{Q}$. The approach to Hilbert's twelfth problem given by Stark's conjectures (see *e.g.* [52]) consists of considering a family of zeta functions (*e.g.* for the Kronecker–Weber case these consist of arithmetic progressions $\zeta_{n,m}(s) := \sum_{k\in m+n\mathbb{Z}} |k|^{-s}$) and generate numbers, which have the form of zeta–regularized determinants (*e.g.* $\exp(\zeta'_{n,m}(0))$ for the Kronecker–Weber case). For more general fields it is a conjecture that such numbers are algebraic and that they provide the desired generators with Galois group action. In [26] [27] Manin conjectured that, in the case of real quadratic fields, noncommutative geometry should produce Stark numbers and Galois action via the theory of noncommutative tori.

In our setting, whenever we associate the structure of a spectral triple to the archimedean places, we obtain, in particular, a family of zeta functions (3.8) determined by the spectral triple. These provide numbers of the form considered by Stark,

$$\exp \left(\frac{d}{ds} \zeta_{a,D}(s)|_{s=0} \right), \tag{6.4}$$

as regularized determinants. Of course, in general there is no reason to expect these numbers to be algebraic. However, it may be an interesting question whether there are spectral triples for which they are, and whether the original Stark numbers arise via a spectral triple. Notice that there is an explicit action of the algebra $\mathcal{A}$ on the collection of numbers (6.4), which may provide the right framework for a Galois action. We hope to return to this in a future work.

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Caterina Consani, Department of Mathematics, University of Toronto, Canada.
email: kc@math.toronto.edu

Matilde Marcolli, Max–Planck–Institut für Mathematik, Bonn Germany.
email: marcolli@mpim-bonn.mpg.de