# Non-Normal Del Pezzo Surfaces


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## Introduction.

Let $Y$ be an irreducible reduced projective Gorenstein surface over $\mathbb{C}$. Then $Y$ is called a del Pezzo surface if the anti-canonical sheaf $\omega_Y^{-1}$ is ample. When $Y$ is normal, it is well-known by Brenton, Demazure and Hidaka–Watanabe that the minimal resolution $\widetilde{Y}$ is a rational surface or a ruled surface over an elliptic curve. Moreover the structure of $\widetilde{Y}$ is also investigated in detail (see [2] ,[3] ,[7]). In particular, putting $d := (\omega_Y^{-1})^2 > 0$, which is called the degree of $Y$, it was shown by Hidaka–Watanabe [7] :

(1) $\omega_Y^{-1}$ is very ample if $d \geq 3$.
(2) $\omega_Y^{-2}$ is very ample if $d = 2$.
(3) $\omega_Y^{-3}$ is very ample if $d = 1$.

When $Y$ is non-normal, the structure of such a surface was studied by Nagata [9], Mori [8] (see also Miyanishi [7]). Now, in this paper, we shall study the more detailed structure of non-normal Del Pezzo surfaces and their normalizations and give answers to the questions due to Miyanishi [7] :

**Question.** *Let $Y$ be a non-normal Del Pezzo surface. Then*

(1) *Does $Y$ have isolated singularities ?*
(2) *Is $\omega_Y^{\otimes -3}$ very ample ?*

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### Notation

$\omega_Y$ : dualizing sheaf of $Y$
$K_Y$ : canonical divisor on $Y$
$\mathbb{F}_t$ : Hirzebruch surface of degree $t \geq 0$
$\Sigma_t$ : minimal section of $\mathbf{F}_t$
$f_t$ : fiber of $\mathbf{F}_t$
$\mathbb{Q}_0^2$ : quadric cone
$\rho(Y)$ : Picard number of $Y$
$\sim$ : linear equivalence
$\mathrm{mult}_E Y$ : multiplicity of $Y$ at a generic point of $E$

## §1. The structure of non-normal Del Pezzo surfaces

**1**. Let $Y$ be an irreducible reduced projective Gorenstein surface over $\mathbb{C}$. The surface $Y$ is called a non-normal del Pezzo surface if $Y$ is non-normal and $\omega_Y^{-1}$ is ample. Let $Y$ be a non-normal del Pezzo surface and $\sigma : \overline{Y} \longrightarrow Y$ be the normalization, and $\mathcal{C} \subset \mathcal{O}_Y$ be the conductor of $\sigma$ defining closed subschemes $E := V_Y(\mathcal{C})$ in $Y$ and $\overline{E} := V_{\overline{Y}}(\mathcal{C})$ in $\overline{Y}$. Then Mori proved the following:

**(1.1) Lemma (cf.(3.35) in [8]).** *(i)* $h^0(\mathcal{O}_{\overline{E}}) = 1,\ h^1(\mathcal{O}_{\overline{E}}) = 0$.

*(ii)* $\chi(\mathcal{O}_{\overline{Y}}) = 1,\ (\sigma^*\omega_Y \cdot \overline{E}) = -2$.

*(iii)* $(\omega_Y \cdot E) = -1$ *and* $E$ *is irreducible reduced, in particular,* $E \cong \mathbb{P}^1$.

Now, let us consider an exact sequence (cf.(3.34.2) in [8]):

$$(1.2) \qquad 0 \longrightarrow \mathcal{O}_Y \longrightarrow \sigma_*\mathcal{O}_{\overline{Y}} \longrightarrow \omega_Y^{-1} \otimes \omega_E \longrightarrow 0$$

By operating $\otimes\omega_Y^{\otimes -n}$ $(n \geq 1,\ n \in \mathbb{Z})$ on (1.2), we obtain

$$(1.3) \qquad 0 \longrightarrow \omega_Y^{\otimes -n} \longrightarrow \sigma_*\mathcal{O}_{\overline{Y}} \otimes \omega_Y^{\otimes -n} \longrightarrow \omega_Y^{\otimes -n-1} \otimes \omega_E \longrightarrow 0$$

By the projection formula and the Serre duality theorem, we have:

$$\begin{aligned} H^0(Y; \sigma_*\mathcal{O}_{\overline{Y}} \otimes \omega_Y^{\otimes -n}) &\cong H^0(Y; \sigma_*\mathcal{O}_{\overline{Y}}(\sigma^*\omega_Y^{-1})) \\ &\cong H^0(\overline{Y}; \sigma^*\omega_Y^{\otimes -n}) \end{aligned}$$

and

$$\begin{aligned} H^0(Y; \omega_Y^{\otimes -n-1} \otimes \omega_E) &\cong H^1(E; \mathcal{O}_E \otimes \omega_Y^{\otimes n+1}) \\ &\cong H^1(E; \mathcal{O}_E(-n-1)) \\ &\cong H^0(E; \mathcal{O}_E(n-1)) \\ &\cong H^0(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(n-1)) \\ &\cong \mathbb{C}^n. \end{aligned}$$

Since $H^1(Y; \omega_Y^{\otimes -n}) = 0$ by Goto-Mori-Reid (cf. [7] ), we have an exact sequence

$$(1.4) \qquad 0 \longrightarrow H^0(Y; \omega_Y^{\otimes -n}) \longrightarrow H^0(\overline{Y}; \sigma^*\omega_Y^{\otimes -n}) \longrightarrow H^0(E; \mathcal{O}_E(n-1)) \longrightarrow 0$$

This implies

**(1.5) Lemma.** $h^0(\sigma^*\omega_Y^{\otimes -n}) = h^0(\omega_Y^{\otimes -n}) + n$ *for* $n \geq 1,\ n \in \mathbb{Z}$.

**2.** Let $\mu : \widehat{Y} \longrightarrow \overline{Y}$ be the minimal resolution with the exceptional set $\cup_i A_i$. We put $\pi := \sigma \circ \mu : \widehat{Y} \longrightarrow \overline{Y}$

Since $\omega_{\overline{Y}} = \sigma^* \omega_Y \otimes \mathcal{C}$ ( namely, $K_{\overline{Y}} \sim \sigma^* K_Y - \overline{E}$ as a Weil divisor), we have

$$K_{\widehat{Y}} \sim \pi^* K_Y - \widehat{E} - A, \tag{1.6}$$

where $\widehat{E}$ is the proper transform of $\overline{E}$ in $\widehat{Y}$, and

$$A = \sum_i k_i A_i \;\; (k_i \in \mathbf{Z},\; k_i \geq 0).$$

Thus we have easily

**(1.7) Lemma.** *$P_m(\widehat{Y}) := dim\ H^0(\widehat{Y}; \mathcal{O}(mK_{\widehat{Y}})) = 0$ for every $m \in \mathbf{Z}$, $m > 0$. In particular, $\widehat{Y}$ is a rational or a ruled surface.*

**3.** We put $\mathcal{L} := -\pi^* K_Y$ . Then $\mathcal{L}$ is nef and big on $\widehat{Y}$ since $-K_Y$ is ample. On the other hand, since $-(K_{\widehat{Y}} + \mathcal{L}) = \widehat{E} + A$ is effective, the adjoint bundle $K_{\widehat{Y}} + \mathcal{L}$ is not nef. Hence, by the Cone theorem (cf. [6] [8]), there exists a contraction $\varphi : \widehat{Y} \longrightarrow Z$ of the extremal ray $R := \mathbf{R}_+[\ell]$, where $\ell \cong \mathbb{P}^1$ with

(i) $\rho(\widehat{Y}) = \rho(Z) + 1$,

(ii) $(K_{\widehat{Y}} + \mathcal{L}) \cdot R < 0$,

(iii) $\varphi(C)$ is a point for a curve $C$ iff $C \in R$.

**(1.8) Lemma.** *$dim\, Z \leq 1$ and $\rho(\widehat{Y}) \leq 2$.*

*Proof.* Assume that $\dim Z = 2$. Then $\varphi : \widehat{Y} \longrightarrow Z$ is birational and there exists an irreducible curve $C \in R$ such that $(C^2) < 0$ and $(K_{\widehat{Y}} + \mathcal{L}) \cdot C < 0$. Hence $C$ is a $(-1)$-curve and $(\mathcal{L} \cdot C) = 0$. This shows that $C$ is an exceptional curve of $\mu$. This is absurd because $\mu : \widehat{Y} \longrightarrow \overline{Y}$ is the minimal resolution. Thus we have $\dim Z \leq 1$, hence $\rho(\widehat{Y}) \leq 2$ by (i). □

By (1.8), we have two cases : $\rho(\widehat{Y}) = 1$ and $\rho(\widehat{Y}) = 2$.

**(1.9).** *The case of $\rho(\widehat{Y}) = 1$ :*

In this case, we obtain $\widehat{Y} = \overline{Y} \cong \mathbb{P}^2$ by (1.7). We put $d := (\omega_Y^2) = (K_Y^2) > 0$. Then we have $-\sigma^* K_{\widehat{Y}} \sim \sqrt{d} \cdot G$ for a line $G$ in $\mathbb{P}^2$. Since $-(K_{\widehat{Y}} - \sigma^* K_Y) \sim (3 - \sqrt{d}) \cdot G$ is ample on $\mathbb{P}^2$, we obtain $d = 1$ or $d = 4$.

**(1.9.1).** If $d = 1$, we have $\sigma^* \omega_Y^{-1} = \mathcal{O}_{\mathbb{P}^2}(1)$. Hence we have $h^0(\sigma^* \omega_Y^{-1}) = 3$ and $h^0(\omega_Y^{-1}) = 2$ by (1.4). Let $\Phi_{|\omega_Y^{-1}|} : Y - \!\!-\!\!- \succ \mathbb{P}^{|\omega_Y^{-1}|}$ be a rational map defined by the linear system $|\omega_Y^{-1}|$. Then the composition $\Phi_{|\omega_Y^{-1}|} \circ \sigma : \mathbb{P}^2 - \!\!-\!\!- \succ \mathbb{P}^1$ has a unique point of indeterminancy, that is, $Bs|\omega_Y^{-1}| \neq \emptyset$. Hence $\omega_Y^{-1}$ is not very ample. Moreover, since $(G \cdot \overline{E}) = (\sigma^*\omega_Y^{-1} \cdot \overline{E}) = 2$, $\overline{E}$ is a conic (not necessarily irreducible) in $\mathbf{P}^2$.

**(1.9.2).** If $d = 4$, we have $\sigma^*\omega_Y^{-1} = \mathcal{O}_{\mathbf{P}^2}(2)$. Hence we obtain $h^0(\sigma^*\omega_Y^{-1}) = 6$ and $h^0(\omega_Y^{-1}) = 5$ by (1.4). The linear system $|\sigma^*\omega_Y^{-1}|$ gives the Veronese embedding $\Phi_{|\omega_Y^{-1}|} : \overline{Y} \cong \mathbf{P}^2 \hookrightarrow \mathbf{P}^5$ and then $Y$ is obtained by the one point projection from the Veronese image $\Phi(\mathbf{P}^2) \hookrightarrow \mathbf{P}^5$ (see (1.4) with $n = 1$). One can see that $|\omega_Y^{-1}|$ is very ample. Moreover, since $(\sigma^*\omega_Y^{-1} \cdot \overline{E}) = 2(G \cdot \overline{E}) = 2$, $\overline{E}$ is a line on $\mathbf{P}^2$, and then the restriction $\sigma_{\overline{E}} : \overline{E} \longrightarrow E$ is a two-fold covering.

**(1.10).** *The case of* $\rho(\widehat{Y}) = 2$ *:*

In this case, $\varphi : \widehat{Y} \longrightarrow Z$ is a $\mathbf{P}^1$-bundle over a smooth algebraic curve $Z$. Let $f$ be a fiber of $\varphi$. Then, by (ii), we obtain $(K_{\widehat{Y}} + \mathcal{L}) \cdot f = -2 + (\mathcal{L} \cdot f) < 0$, hence we have $(\mathcal{L} \cdot f) = (\sigma^*\omega_Y^{-1} \cdot f) = 1$. Since $-(K_{\widehat{Y}} + \mathcal{L}) = \widehat{E} + A$, we have two cases:

(a) $(A \cdot f) = 0$, $(\widehat{E} \cdot f) = 1$, or

(b) $(A \cdot f) = 1$, $(\widehat{E} \cdot f) = 0$.

**(1.10.a).** In the case (a), since $(A \cdot f) = 0$, $\overline{Y}$ is smooth, hence we have $\widehat{Y} = \overline{Y}$, $\widehat{E} = \overline{E}$, $\mathcal{L} = -\sigma^*K_Y$ and $K_{\overline{Y}} \sim \sigma^*K_Y - \overline{E}$. By (1.1)-(i), $\overline{E}$ is connected and each irreducible component of $\overline{E}$ is a smooth rational curve. On the other hand, since $(\overline{E} \cdot f) = 1$, $\overline{E}$ has an irreducible component $\overline{E}_1$ with $(\overline{E}_1 \cdot f) = 1$. This implies $Z$ is rational and hence $\overline{Y} \cong \mathbf{F}_t$ for some $t \in \mathbf{Z}$, $t \geq 0$. Since $(\overline{E} \cdot f_t) = (-\sigma^*K_Y \cdot f_t) = 1$, we have linear equivalences:

$$\begin{aligned} -\sigma^*K_Y &\sim \Sigma_t + mf_t \\ \overline{E} &\sim \Sigma_t + nf_t , \end{aligned}$$

for some $m, n \in \mathbf{Z}$.

Taking into consideration that $(-\sigma^*K_Y \cdot \Sigma_t) > 0$ and $d = (-\sigma^*K_Y)^2$, one obtains the following

$$\begin{aligned} \overline{Y} &\cong \mathbf{F}_{d-2} \quad (d \geq 2) \\ -\sigma^*K_Y &\sim \Sigma_{d-2} + (d-1)f_{d-2} \\ \overline{E} &\sim \Sigma_{d-2} + f_{d-2}, \end{aligned} \tag{1.10.1}$$

or

$$\begin{aligned} \overline{Y} &\cong \mathbf{F}_{d-4} \quad (d \geq 4) \\ -\sigma^*K_Y &\sim \Sigma_{d-4} + (d-2)f_{d-4} \\ \overline{E} &= \Sigma_{d-4} \ . \end{aligned} \tag{1.10.2}$$

In both cases, we have $h^0(\sigma^*\omega_Y^{-1}) = d+2$ and hence $h^0(\omega_Y^{-1}) = d+1$ by (1.5). Then the linear system $|\sigma^*\omega_Y^{-1}|$ gives an embedding $\Phi := \Phi_{|\sigma^*\omega_Y^{-1}|} : \overline{Y} \hookrightarrow \mathbf{P}^{d+1}$ if $d \geq 3$ for

(1.10.1) (resp. $d \geq 4$ for (1.10.2)) with a relation $d = \deg \Phi(\overline{Y}) = \operatorname{codim} \Phi(\overline{Y}) + 1$. Next, take a general irreducible member $C \in |\omega_Y^{-1}|$. Let $\overline{C} \in |\sigma^*\omega_Y^{-1}|$ be the proper transform of $C$ in $\overline{Y} = \mathbb{F}_t$, where $t = d-2$ $(d \geq 3)$ or $d-4$ $(d \geq 4)$. Then $\overline{C}$ is a smooth rational curve with the self-intersection number $(\overline{C}^2) = d$ in $\mathbb{F}_t$. Since $p_a(C) = 1$, $C$ is a rational curve with a cusp or a node. Let us consider an exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \omega_Y^{-1} \longrightarrow \mathcal{O}_C \otimes \omega_Y^{-1} \longrightarrow 0 .$$

Since $H^1(Y; \mathcal{O}_Y) = 0$, we obtain

$$(\bigstar) \qquad 0 \longrightarrow H^0(Y; \mathcal{O}_Y) \longrightarrow H^0(Y; \omega_Y^{-1}) \longrightarrow H^0(C; \mathcal{O}_C \otimes \omega_Y^{-1}) \longrightarrow 0 .$$

One can easily show that $\mathcal{O}_C(\omega_Y^{-1})$ is very ample on $C$ if $d \geq 3$. Hence $\omega_Y^{-1}$ is also very ample by $(\bigstar)$. Indeed, $C \subset \mathbb{P}^{d-1}$ is obtained from $\overline{C} \subset \mathbb{P}^d$ by a point projection, and $Y$ is obtained from $\Phi(\overline{Y}) \hookrightarrow \mathbb{P}^{d+1}$ by a point projection (see (1.4) with $n = 1$).

**(1.10.b).** In the case (b), since $(A \cdot f) = 1$, we have $\operatorname{Sing} \overline{Y} \neq \emptyset$, namely, $\widehat{Y} \neq \overline{Y}$. In particular, $A$ is the negative section of the $\mathbb{P}^1$-bundle $\overline{Y}$, hence $\overline{Y}$ is a cone over the curve $Z$. From the relation $(\widehat{E} \cdot f) = 0$, we obtain a linear equivalence $\widehat{E} \sim kf$ $(k \in \mathbb{Z})$. Since $(\sigma^*\omega_Y^{-1} \cdot \widehat{E}) = 2$ and $(\sigma^*\omega_Y^{-1} \cdot f) = 1$, we have easily $k = 2$, that is, $\widehat{E} \sim 2f$. This yields $(\widehat{E} \cdot A) = 2$. Thus $\overline{E}$ consists of two different generating lines or double generating lines. By the adjunction formula, one has $2p_a(A) - 2 = A \cdot (K_{\widehat{Y}} + A) = A(-\widehat{E} + \sigma^* K_Y) = -(A \cdot \widehat{E}) = -2$. This yields $p_a(A) = 0$, namely, the negative section $A$ is a smooth rational curve. From the relations

$$(\sigma^*\omega_Y^{-1} \cdot A) = 0, \ \ (\sigma^*\omega_Y^{-1} \cdot f) = 1, \ \ (\sigma^*\omega_Y^{-1})^2 = d,$$

we obtain

$$\text{(1.10.3)} \qquad \begin{aligned} \widehat{Y} &\cong \mathbb{F}_d \ \ (d \geq 2) \\ \sigma^*\omega_Y^{-1} &\sim \Sigma_d + \ df_d \\ A &= \Sigma_d \end{aligned}$$

Hence we have $h^0(\sigma^*\omega_Y^{-1}) = d + 2$ and $h^0(\omega_Y^{-1}) = d + 1$ by (1.5). If $d \geq 3$, then $\overline{Y} \hookrightarrow \mathbb{P}^{d+1}$ is a cone over a smooth rational curve of degree $d$ in $\mathbb{P}^d$ and $\overline{E}$ consists of two different generating lines or double generating lines. One can also see that $\omega_Y^{-1}$ is very ample and that $Y$ is a cone over a nodal or a cuspidal rational curve if $d \geq 3$.

Summarizing (1.9.1),(1.9.2),(1.10.1),(1.10.2) and (1.10.3), we have the following

**Theorem I (cf. [9], [7]).** *Let $Y$ be a non-normal del Pezzo surface and $\sigma : \overline{Y} \longrightarrow Y$ the normalization. Let $\mathcal{C} \subset \mathcal{O}_Y$ be the conductor of $\sigma$ defining closed subschemes $E := V_Y(\mathcal{C})$ in $Y$ and $\overline{E} := V_{\overline{Y}}(\mathcal{C})$ in $\overline{Y}$. Let $d := (\omega_Y^{-1})^2$ be the degree of $Y$. Then we have the following five cases:*

*(A)* $d = 1$ and

(1) $(\overline{Y}, \sigma^*\omega_Y^{-1}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.
(2) $\overline{E}$ *is a (not necessarily irreducible) conic in* $\overline{Y} \cong \mathbb{P}^2$. *In the case where* $\overline{E}$ *is a smooth conic,* $\sigma_{\overline{E}} : \overline{E} \longrightarrow E$ *is a two-fold covering.*
(3) $h^0(\omega_Y^{-1}) = 2$.
(4) $\omega_Y^{-1}$ *is not very ample.*
(5) $Bs|\omega_Y^{-1}| \neq \emptyset$.

*(B)* $d = 4$ and

(1) $(\overline{Y}, \sigma^*\omega_Y^{-1}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.
(2) $\overline{E}$ *is a line in* $\mathbb{P}^2$, *and* $\sigma_{\overline{E}} : \overline{E} \longrightarrow E$ *is a two-fold covering.*
(3) $h^0(\omega_Y^{-1}) = 5$.
(4) $\omega_Y^{-1}$ *is very ample and* $Y$ *is obtained by a point projection from the Veronese transform* $\Phi_{|\omega_Y^{-1}|} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ *of* $\mathbb{P}^2$ *in* $\mathbb{P}^5$.

*(C)* $d \geq 2$ and

(1) $(\overline{Y}, \sigma^*\omega_Y^{-1}) \cong (\mathbb{F}_{d-2}, \mathcal{O}(\Sigma_{d-2} + (d-1)f_{d-2}))$.
(2) $\overline{E} \sim \Sigma_d + f_d$ *is irreducible (it can occur only if* $d = 2, 3$*), or* $\overline{E} = \Sigma_{d-2} \cup f_{d-2}$ *(consisting of the negative section and a fiber). In the case where* $\overline{E}$ *is irreducible, the restriction* $\sigma|_{\overline{E}} : \overline{E} \longrightarrow E$ *is a two-fold covering.*
(3) $h^0(\omega_Y^{-1}) = d + 1$.
(4) *If* $d \geq 3$ *, then* $\overline{Y} \cong \mathbb{F}_{d-2}$ *is embedded into* $\mathbb{P}^{d+1}$ *with degree* $d$ *by the linear system* $|\Sigma_{d-2} + (d-1)f_{d-2}|$ *and then* $Y$ *is obtained by a point projection from* $\overline{Y} \hookrightarrow \mathbb{P}^{d+1}$, *in particular,* $\omega_Y^{-1}$ *is very ample.*

*(D)* $d \geq 4$ and

(1) $(\overline{Y}, \sigma^*\omega_Y^{-1}) \cong (\mathbb{F}_{d-4}, \mathcal{O}(\Sigma_{d-4} + (d-2)f_{d-4}))$.
(2) $\overline{E} = \Sigma_{d-4}$ *is irreducible and* $\sigma|_{\overline{E}} : \overline{E} \longrightarrow E$ *is a two-fold covering.*
(3) $h^0(\omega_Y^{-1}) = d + 1$.
(4) $\overline{Y} \cong \mathbb{F}_{d-4}$ *is embedded into* $\mathbb{P}^{d+1}$ *by the linear system* $|\Sigma_{d-4} + (d-2)f_{d-4}|$ *with degree* $d$ *and* $Y$ *is obtained by a point projection from* $\overline{Y} \hookrightarrow \mathbb{P}^{d+1}$, *in particular,* $\omega_Y^{-1}$ *is very ample.*

*(E)* $d \geq 2$ and

(1) $(\overline{Y}, \sigma^*\omega_Y^{-1}) \cong (S_d, \mathcal{O}(1))$, *where* $S_d \hookrightarrow \mathbb{P}^{d+1}$ *is a cone over a smooth rational curve of degree* $d$ *in* $\mathbb{P}^d$ *if* $d \geq 3$.
(2) $\overline{E}$ *consists of two generating lines or double generating lines.*
(3) *If* $d \geq 3$, *then* $Y$ *is obtained by a point projection from* $S_d \hookrightarrow \mathbb{P}^{d+1}$, *hence* $Y$ *is a cone over a nodal or a cuspidal rational curve if* $d \geq 3$, *in particular,* $\omega_Y^{-1}$ *is very ample.*

**Corollary II.** *(1) The singular locus of $Y$ coincides with the non-normal locus of $Y$, that is, $Y$ doed not have isolated singularities.*

*(2) Let $Y^0$ be the smooth part of $Y$. Then $Y$ is a compactification of $\mathbb{C}^2$ if and only if the fundamental group $\pi_1(Y^0) = 1$.*

## §2. Very ampleness of $\omega_Y^{\otimes -3}$

**4.** Since $\omega_Y^{-1}$ is very ample if $d \geq 3$, we have only to consider the case of $d = 1$ and $d = 2$.

**(2.1) Proposition.** *$\omega_Y^{\otimes -3}$ is very ample if $d = 1$*

*Proof.* By Theorem I-(A), we have

$$\begin{cases} \overline{Y} & \cong \mathbb{P}^2 \\ \sigma^*\omega_Y^{-1} & = \mathcal{O}_{\mathbb{P}^2}(1) \\ \overline{E} & : \text{conic on } \mathbb{P}^2 \end{cases}$$

*(2.1.1).* In the case where $\overline{E} = \overline{E}_1 + \overline{E}_2$, where $\overline{E}_i$'s are two distinct lines on $\mathbb{P}^2$, we may assume

$$\begin{cases} \overline{E}_1 = \{z_1 = 0\} \\ \overline{E}_2 = \{z_0 = 0\}, \end{cases}$$

where $(z_0 : z_1 : z_2)$ is a system of homogeneous coordinates on $\mathbb{P}^2$.

Looking at the exact sequence (1.4) with $n = 3$, one can take a basis $\{h_0, ..., h_6\}$ of $H^0(Y, \omega_Y^{\otimes -3}) \cong \mathbb{C}^7$ as follows:

$$\begin{cases} \sigma^*h_0 & = z_0 z_1 z_2 \\ \sigma^*h_1 & = z_0^2 z_1 \\ \sigma^*h_2 & = z_0 z_1^2 \\ \sigma^*h_3 & = (z_0 + z_1)^3 \\ \sigma^*h_4 & = (z_0 + z_1)^2 z_2 \\ \sigma^*h_5 & = (z_0 + z_1) z_2^2 \\ \sigma^*h_6 & = z_2^3 \,. \end{cases}$$

Let

$$\overline{\Phi} := (\sigma^* f_0 : \sigma^* h_1 : .... : \sigma^* h_6) : \mathbb{P}^2 \longrightarrow \mathbb{P}^6$$
$$\Phi := (h_0 : h_1 : .... : h_6) : \mathbb{P}^2 \longrightarrow \mathbb{P}^6$$

be the associated morphisms. Since

$$\overline{\Phi}(\overline{E}_1) = (0 : 0 : 0 : z_0^3 : z_0^2 z_2 : z_0 z_2^2 : z_2^3) \cong \mathbb{P}^1 \hookrightarrow \mathbb{P}^3,$$

$$\overline{\Phi}(\overline{E}_2) = (0 : 0 : 0 : z_1^3 : z_1^2 z_2 : z_1 z_2^2 : z_2^3) \cong \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \,,$$

$\overline{\Phi}(\overline{E}_1) = \overline{\Phi}(\overline{E}_2) \cong \mathbb{P}^1 = \Phi(E)$ is a twisted cubic curve in $\mathbf{P}^3$.

On the affine part $\{z_2 \neq 0\}$, if we put $x := \dfrac{z_0}{z_2}$ and $y := \dfrac{z_1}{z_2}$, the morphism $\Phi : \mathbb{C}^2(x,y) \longrightarrow \mathbb{C}^6(X_0, \cdots, X_5)$ is given by:

$$\begin{cases} X_0 = xy \\ X_1 = x^2y \\ X_2 = xy^2 \\ X_3 = (x+y)^3 \\ X_4 = (x+y)^2 \\ X_5 = x+y\ . \end{cases}$$

One can easily verify that $\overline{\Phi}$ is one to one and that the Jacobian $J(\overline{\Phi})$ has the rank two on $\{x \neq 0,\ y \neq 0\}$. In particular, $\Phi$ is one to one on $Y$.

We put

$$V := \Phi(\mathbb{P}^2).$$
$$\Delta := \Phi(\overline{E}).$$
$$V_0 := \Phi(z_2 \neq 0) = \Phi(\mathbb{C}^2) \hookrightarrow \mathbb{C}^6.$$

By an easy computation, we obtain the defining equation of $V_0$ :

$$V_0 = \{(X_0, X_1, X_5) \in \mathbb{C}^3 \mid X_0^3 - X_0X_1X_5 + X_1^2 = 0\}.$$

The non-normal locus of $V_0$ is the $X_5$-axis $\{X_0 = X_1 = 0\}$. Thus $V$ is a Gorenstein surface with $mult_\Delta V = 2$.

One sees that $\omega_V^{-1}$ is ample. In fact, take a non-vanishing holomorphic 2-form $\omega_{V_0}$ on $V_0$

$$\omega_{V_0} = \frac{dX_0 \wedge dX_1}{X_0X_1} (= \frac{dX_5 \wedge dX_1}{3X_0^2 - X_1X_5} = \frac{dX_5 \wedge dX_0}{X_0X_5 - 2X_1}).$$

Since

$$\Phi^*\omega_{V_0} = \frac{dx \wedge dy}{xy},$$

we have

$$\Phi^*\omega_V = \omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(\overline{E}_1 + \overline{E}_2) = \mathcal{O}_{\mathbf{P}^2}(-1).$$

Thus $\Phi^*\omega_V^{-1} = \mathcal{O}_{\mathbf{P}^2}(1)$ is ample. Since $\Phi$ is a finite morphism, $\omega_V^{-1}$ is also ample. Now since $(\omega_V^{-1})^2 = 1$ and since $H^2(V, \mathbf{Z}) \cong \mathbf{Z}$, we have $\operatorname{Pic} V \cong \mathbf{Z} \cdot \omega_V^{-1}$. Let $\mathcal{L}$ be a hyperplace section of $V \hookrightarrow \mathbb{P}^6$. Since $\Delta$ is a twisted cubic, we obtain $(\mathcal{L} \cdot \Delta) = 3$.

Since $\mathcal{L} = (\omega_V^{-1})^{\otimes k}$ for some $k \in \mathbf{Z}$, we obtain

$$3 = (\mathcal{L} \cdot \Delta) = k(\omega_V^{-1} \cdot \Delta) = k\ .$$

This yields $\mathcal{L} = \omega_V^{\otimes -3}$. By construction, one sees $Y$ is isomorphic to $V$ and hence $\omega_Y^{\otimes -3}$ is very ample.

*(2.1.2).* In the case where $\overline{E} = 2\overline{E}_0$, where $\overline{E}_0$ is a line on $\overline{Y} \cong \mathbb{P}^2$. We may assume

$$\overline{E}_0 = \{z_0 = 0\}.$$

In this case, we take a base $\{h_0, ..., h_6\}$ of $H^0(Y; \omega_Y^{\otimes -3}) \cong \mathbb{C}^7$ as follows:

$$\begin{cases} \sigma^* h_0 & = z_0^2 z_1 \\ \sigma^* h_1 & = z_0^2 z_2 \\ \sigma^* h_2 & = z_0^3 \\ \sigma^* h_3 & = z_1^3 \\ \sigma^* h_4 & = z_1^2 z_2 \\ \sigma^* h_5 & = z_1 z_2^2 \\ \sigma^* h_6 & = z_2^3 \, . \end{cases}$$

We put $\Phi := (\sigma^* h_0 : ... : \sigma^* h_6) : \mathbb{P}^2 \longrightarrow \mathbb{P}^6$. Then, by an argument similar to (2.1.1), one can verify that $V := \Phi(\mathbb{P}^2)$ is isomorphic to $Y$ and $\omega_Y^{\otimes -3}$ is very ample. In particular, the defining equation of $V_0 := \Phi(\{z_2 \neq 0\}) \subset \mathbb{C}^6(X_0, \cdots, X_5)$ is given by

$$\{(X_0, X_2, X_4) \in \mathbb{C}^3 \,| X_0^2 = X_2^3\}.$$

*(2.1.3).* In the case where $\overline{E}$ is an irreducible conic on $\overline{Y} \cong \mathbb{P}^2$, we may assume

$$\overline{E} = \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 \mid z_0^2 = z_1 z_2\}.$$

Then we can also take a basis $\{h_i\}$ $(0 \le i \le 6)$ of $H^0(Y; \omega_Y^{\otimes -3})$ as follows:

$$\begin{cases} \sigma^* h_0 & = z_0(z_0^2 - z_1 z_2) \\ \sigma^* h_1 & = z_1(z_0^2 - z_1 z_2) \\ \sigma^* h_2 & = z_2(z_0^2 - z_1 z_2) \\ \sigma^* h_3 & = z_1^3 \\ \sigma^* h_4 & = z_1^2 z_2 \\ \sigma^* h_5 & = z_1 z_2^2 \\ \sigma^* h_6 & = z_2^3 \, . \end{cases}$$

Let $\Phi := (\sigma^* h_0 : ... : \sigma^* h_6) : \mathbb{P}^2 \longrightarrow \mathbb{P}^6$ be the associated morphism of $\mathbb{P}^2$ to $\mathbb{P}^6$.

*Claim.* $\Phi(\overline{E}) =: \Delta \cong \mathbb{P}^1$ is a twisted cubic curve in $\mathbb{P}^3$.

In fact, we consicer an injection $\lambda : \mathbb{P}^1(s : t) \longrightarrow \mathbb{P}^2(z_0 : z_1 : z_2)$ with

$$\begin{cases} z_0 = st \\ z_1 = s^2 \\ z_2 = t^2. \end{cases}$$

Then we have easily $\overline{E} = \lambda(\mathbb{P}^1)$, hence

$$\Delta = \Phi \circ \lambda(\mathbb{P}^1) = (s^6 : s^4t^2 : s^2t^4 : t^6) \in \mathbb{P}^3.$$

Now we put

$$\begin{cases} u = s^2 \\ v = t^2, \end{cases}$$

then we obtain a two-fold covering $\mu : \mathbb{P}^1(s:t) \longrightarrow \mathbb{P}^1(u:v)$.
Then the morphism

$$(u:v) \longrightarrow (u^3 : u^2v : uv^2 : v^3)$$

yields an isomorphism $\Delta \cong \mathbb{P}^1(u:v)$, in particular, $\Delta$ is a twisted cubic curve in $\mathbb{P}^3$. Moreover, one can easily see that $\Phi|_{\overline{E}} : \overline{E} \longrightarrow \Delta \cong \mathbb{P}^1 \subset \mathbb{P}^3$ is a two-fold covering. Thus we have the claim.

Then one can also prove that $Y$ is isomorphic to $V := \Phi(\mathbb{P}^2)$ and that $\omega_Y^{\otimes -3}$ is very ample. The defining equation of $V_0 := \Phi(\{z_2 \neq 0\}) \subset \mathbb{C}^6(X_0, \cdots, X_5)$ is given by

$$\{(X_0, X_2, X_5) \in \mathbb{C}^3 \mid X_0^2 = X_2^2(X_2 + X_5)\}.$$

By (2.1.1),(2.1.2) and (2.1.3), we complete the proof of (2.1). □

**(2.2) Proposition.** *$\omega_Y^{\otimes -2}$ is very ample if $d = 2$.*

*Proof.* In this case, we have two cases by Theorem I-(C) and (E):

$$\begin{cases} \overline{Y} & \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ \sigma^*\omega_Y^{-1} & = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1) \\ \overline{E} & \sim \Sigma_0 + f_0, \end{cases}$$

or

$$\begin{cases} \overline{Y} & \cong \mathbb{Q}_0^2 \\ \sigma^*\omega_Y^{-1} & = \mathcal{O}_{\mathbb{Q}_0^2}(1) \\ \overline{E} & = g_1 + g_2 \text{ or } 2g_0, \end{cases}$$

where $g_i$ $(i = 0, 1, 2)$ is a generating line of $\mathbb{Q}_0^2$.

**—- The case of $\overline{Y} \cong \mathbb{P}^1 \times \mathbb{P}^1$ —-**

*(2.2.1).* In the case where $\overline{E} = \Sigma_0 + f_0$ (two irreducible components with the self-intersection number $\Sigma_0^2 = f_0^2 = 0$), let $(x:y) \times (u:v)$ be the homogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$, then we may assume

$$\begin{cases} \overline{E}_1 & := \Sigma_0 = \{x = 0\} \\ \overline{E}_2 & := f_0 = \{u = 0\} \end{cases}$$

By the exact sequence (1.4) with $n = 2$, one can take a basis $\{h_0, \cdots, h_6\}$ of $H^0(Y; \omega_Y^{\otimes -2}) \cong \mathbb{C}^7$ as follows:

$$\begin{cases} \sigma^* h_0 & = x^2u^2 \\ \sigma^* h_1 & = x^2uv \\ \sigma^* h_2 & = xyu^2 \\ \sigma^* h_3 & = xyuv \\ \sigma^* h_4 & = (xv+yu)^2 \\ \sigma^* h_5 & = (xv+yu)yv \\ \sigma^* h_6 & = y^2v^2 \,. \end{cases}$$

Let $\Phi := (\sigma^* h_0 : \cdots : \sigma^* h_6) : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^6$ be the associated morphism.

We put $V := \Phi(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^6$. Then one can prove that $Y$ is isomorphic to $V := \Phi(\mathbb{P}^1 \times \mathbb{P}^1)$ and that $\omega_Y^{\otimes -2}$ is very ample as before. Moreover the defining equation of $V_0 := \Phi(\{y \neq 0, v \neq 0\}) \subset \mathbb{C}^6(X_0, \cdots, X_5)$ is given by

$$\{(X_2, X_3, X_5) \in \mathbb{C}^3 \,|X_2^2 - X_2X_3X_5 + X_3^3 = 0\}.$$

*(2.2.2).* In the case where $\overline{E} \sim \Sigma_0 + f_0$ is irreducible, we may assume

$$\overline{E} = \{(x:y) \times (u:v) \in \mathbb{P}^1 \times \mathbb{P}^1 \,|xv = yu\}.$$

Then we take a basis of $H^0(Y; \omega_Y^{\otimes -2}) \cong \mathbb{C}^7$ as follows:

$$\begin{cases} \sigma^* h_0 & = (xv-yu)^2 \\ \sigma^* h_1 & = (xv-yu)xu \\ \sigma^* h_2 & = (xv-yu)uv \\ \sigma^* h_3 & = (xv-yu)yv \\ \sigma^* h_4 & = x^2y^2 \\ \sigma^* h_5 & = xyuv \\ \sigma^* h_6 & = u^2v^2 \,. \end{cases}$$

We put $\Phi := (\sigma^* h_0 : \cdots : \sigma^* h_6) : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^6$. Then one sees that $Y \cong V := \Phi(\mathbb{P}^1 \times \mathbb{P}^1) \hookrightarrow \mathbb{P}^6$ and $\omega_Y^{\otimes -2}$ is very ample as before. Moreover, the defining equation of $V_0 := \Phi(\{y \neq 0, v \neq 0\}) \subset \mathbb{C}^6(X_0, \cdots, X_5)$ is given by

$$\{(X_3, X_4, X_5) \in \mathbb{C}^3 \,|\, X_3^2 + X_3X_4 = X_4^2X_5\}.$$

**— The case of $\overline{\mathbf{Y}} \cong \mathbb{Q}_0^2$ —.**

In this case, the minimal resolution $\widehat{Y}$ of $\overline{Y}$ is isomorphic to the Hirzebruch surface $\mathbf{F}_2$. Let $\pi : \widehat{Y} \xrightarrow{\mu} \overline{Y} \xrightarrow{\sigma} Y$ be as before. Then the proper transform $\widehat{E} = \mu^* \overline{E} = \pi^*(E)$ consists of either two distinct fibers $\widehat{E}_1 + \widehat{E}_2$ or double fibers $2\widehat{E}_0$ of $\mathbf{F}_2$.

Let $\{(u_i, v_i) \in U_i \times \mathbb{P}^1\}_{i=1,2}$ be a coordinate covering of $\widehat{Y} \cong \mathbf{F}_2$ with

$$\begin{cases} u_2 = \dfrac{1}{u_1} \\ v_2 = u_1^2 v_1, \end{cases}$$

where $U_i \cong \mathbb{C}$ and $v_i$ is a non-homogeneous coordinate of $\mathbb{P}^1$.

*(2.2.3).* In the case where $\widehat{E} = \widehat{E}_1 + \widehat{E}_2$, we may assume

$$\widehat{E}_i = \{u_i = 0\}\,(i = 1, 2).$$

Then one can take a basis $\{h_0, h_1, \cdots, h_5, h_6\}$ of $H^0(Y; \omega_Y^{\otimes -2}) \cong \mathbb{C}^7$ as follows:

$$\begin{cases} \pi^* h_0 = 1 \\ \pi^* h_1 = uv \\ \pi^* h_2 = u^2 v^2 \\ \pi^* h_3 = uv(1 + uv) \\ \pi^* h_4 = 1 + uv^2 \\ \pi^* h_5 = 1 + v + u^2 v \\ \pi^* h_6 = (1 + v + u^2 v)^2 \end{cases}$$

on $U_1 \times (\mathbb{P}^1 - \infty) \cong \mathbb{C}^2(u, v)$ , where we put $(u, v) := (u_1, v_1)$ for simplicity.

Now we put

$$\widehat{\Phi} := (\pi^* h_0 : \pi^* h_1 : \cdots : \pi^* h_5 : \pi^* h_6) : \widehat{Y} \longrightarrow \mathbb{P}^6$$
$$\Phi := (h_0 : h_1 : \cdots : h_5 : h_6) : Y \longrightarrow \mathbb{P}^6$$

Then one sees

(1) $\widehat{\Phi}(\widehat{E}_i) = \Phi(E) \cong \mathbb{P}^1$ is a conic for $i = 1, 2$.
(2) $\widehat{\Phi}$ is injective on $\widehat{Y} - (\widehat{E}_1 \cup \widehat{E}_2 \cup \Sigma_2)$.
(3) $\Phi$ is injective on $Y$.
(4) $\widehat{\Phi}(\Sigma_2)$ is a point.

We put $V := \widehat{\Phi}(\widehat{Y}) = \Phi(Y) \subset \mathbb{P}^6$. Then one can also see that $Y \cong V$ and that $\omega_Y^{\otimes -2}$ is very ample. The defining equation of $V_0 := \Phi(U_1 \times \mathbb{C}) \subset \mathbb{C}^6(X_1, \cdots, X_6)$ is given by

$$\{(X_1, X_4, X_5) \in \mathbb{C}^3 \mid (X_4 - 1)^2 - X_1(X_4 - 1)(X_5 - 1) + X_1^4 = 0\}.$$

*(2.2.4).* In the case where $\widehat{E} = 2\widehat{E}_0$, we may assume that

$$\widehat{E}_0 = \{u_1 = 0\}.$$

Then we take a basis $\{h_0, h_1, \cdots, h_5, h_6\}$ of $H^0(Y; \omega_Y^{\otimes -3}) \cong \mathbb{C}^7$ as follows:

$$\begin{cases} \pi^* h_0 = 1 \\ \pi^* h_1 = u^2 v \\ \pi^* h_2 = u^4 v^2 \\ \pi^* h_3 = u^2 v(1 + uv) \\ \pi^* h_4 = u^2 v(1 + v + u^2 v) \\ \pi^* h_5 = 1 + v + u^2 v \\ \pi^* h_6 = (1 + v + u^2 v)^2 \end{cases}$$

on $U_1 \times (\mathbb{P}^1 - \infty) \cong \mathbb{C}^2(u,v)$. Then we obtain $Y \cong V := \Phi(Y)$ and $\omega_Y^{\otimes -2}$ is very ample, where $\Phi := (h_0 : h_1 : \cdots : h_5 : h_6) : Y \longrightarrow \mathbb{P}^6$.

By (2.2.1),(2.2.2),(2.2.3) and (2.4), we complete the proof of (2.2). $\square$

Therefore we have finally

**Theorem II.** *Let $Y$ be a non-normal Del Pezzo surface and $d := (\omega_Y^{-1})^2$ the degree of $Y$. Then*

(1) $d = 1 \Longrightarrow \omega_Y^{\otimes -3}$ *is very ample.*
(2) $d = 2 \Longrightarrow \omega_Y^{\otimes -2}$ *is very ample.*
(3) $d \geq 3 \Longrightarrow \omega_Y^{-1}$ *is very ample.*

**Remark.**

(1) One can also prove the very ampleness of $\omega_Y^{-1}$ ($d \geq 3$) by the explicit way as above.
(2) If $Y$ is normal, then Theorem II is already known (cf. [Corollary 4.5 ;5]).

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