VACCA-TYPE SERIES FOR VALUES OF THE GENERALIZED-EULER-CONSTANT FUNCTION AND ITS DERIVATIVE

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Abstract. We generalize well-known Catalan-type integrals for Euler's constant to values of the generalized-Euler-constant function and its derivatives. Using generating functions appeared in these integral representations we give new Vacca and Ramanujan-type series for values of the generalized-Euler-constant function and Addison-type series for values of the generalized-Euler-constant function and its derivative. As a consequence, we get base B rational series for $\log\frac{4}{\pi}$, $\frac{G}{\pi}$ (where G is Catalan's constant), $\frac{\zeta'(2)}{\pi^2}$ and also for logarithms of Somos's and Glaisher-Kinkelin's constants.

1. Introduction

In [11], J. Sondow proved the following two formulas:

$$\gamma=\sum_{n=1}^{\infty}\frac{N_{1,2}(n)+N_{0,2}(n)}{2n(2n+1)}, \tag{1}$$

$$\log\frac{4}{\pi}=\sum_{n=1}^{\infty}\frac{N_{1,2}(n)-N_{0,2}(n)}{2n(2n+1)}, \tag{2}$$

where γ is Euler's constant and $N_{i,2}(n)$ is the number of i's in the binary expansion of n. The series (1) is equivalent to the well-known Vacca series [13]

$$\gamma=\sum_{n=1}^{\infty}(-1)^n\frac{\lfloor\log_2 n\rfloor}{n}=\sum_{n=1}^{\infty}(-1)^n\frac{N_{1,2}\left(\lfloor\frac{n}{2}\rfloor\right)+N_{0,2}\left(\lfloor\frac{n}{2}\rfloor\right)}{n} \tag{3}$$

and both series (1) and (3) may be derived from Catalan's integral [6]

$$\gamma=\int_0^1\frac{1}{1+x}\sum_{n=1}^{\infty}x^{2^n-1}\,dx. \tag{4}$$

To see this it suffices to note that

$$G(x)=\frac{1}{1-x}\sum_{n=0}^{\infty}x^{2^n}=\sum_{n=1}^{\infty}(N_{1,2}(n)+N_{0,2}(n))x^n$$

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is a generating function of the sequence $N_{1,2}(n)+N_{0,2}(n)$, (see [10, sequence A070939]), which is the binary length of n, rewrite (4) as

$$\gamma = \int_0^1 (1-x)\frac{G(x^2)}{x}\,dx$$

and integrate the power series termwise. In view of the equality

$$1 = \int_0^1 \sum_{n=1}^{\infty} x^{2^n-1}\,dx,$$

which is easily verified by termwise integration, (4) is equivalent to the formula

$$\gamma = 1 - \int_0^1 \frac{1}{1+x}\sum_{n=1}^{\infty} x^{2^n}\,dx \tag{5}$$

obtained independently by Ramanujan (see [4, Cor. 2.3]). Catalan's integral (5) gives the following rational series for γ :

$$\gamma = 1 - \int_0^1 (1-x)G(x^2)\,dx = 1 - \sum_{n=1}^{\infty} \frac{N_{1,2}(n)+N_{0,2}(n)}{(2n+1)(2n+2)}. \tag{6}$$

Averaging (1), (6) and (4), (5), respectively, we get Addison's series for γ [1]

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{N_{1,2}(n)+N_{0,2}(n)}{2n(2n+1)(2n+2)}$$

and its corresponding integral

$$\gamma = \frac{1}{2} + \frac{1}{2}\int_0^1 \frac{1-x}{1+x}\sum_{n=1}^{\infty} x^{2^n-1}, \tag{7}$$

respectively. Integrals (5), (4) were generalized to an arbitrary integer base $B>1$ by S. Ramanujan and B. C. Berndt and D. C. Bowman (see [4])

$$\gamma = 1 - \int_0^1 \left(\frac{1}{1-x} - \frac{Bx^{B-1}}{1-x^B}\right)\sum_{n=1}^{\infty} x^{B^n}\,dx \qquad \text{(Ramanujan)}, \tag{8}$$

$$\gamma = \int_0^1 \left(\frac{B}{1-x^B} - \frac{1}{1-x}\right)\sum_{n=1}^{\infty} x^{B^n-1} \qquad \text{(Berndt-Bowman)}. \tag{9}$$

Formula (9) implies the generalized Vacca series for γ (see [4, Th. 2.6]) proposed by L. Carlitz [5]

$$\gamma = \sum_{n=1}^{\infty} \frac{\varepsilon(n)}{n}\lfloor \log_B n \rfloor, \tag{10}$$

where

$$\varepsilon(n) = \begin{cases} B-1 & \text{if} \quad B \quad \text{divides} \quad n \\ -1 & \text{otherwise,} \end{cases} \tag{11}$$

and the averaging integral of (8) and (9) produces the generalized Addison series for γ found by Sondow in [11]

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\lfloor \log_B Bn \rfloor P_B(n)}{Bn(Bn+1)\cdots(Bn+B)}, \tag{12}$$

where $P_B(x)$ is a polynomial of degree $B-2$ denoted by

$$P_B(x) = (Bx+1)(Bx+2)\dots(Bx+B-1)\sum_{m=1}^{B-1} \frac{m(B-m)}{Bx+m}. \tag{13}$$

In this short note, we generalize Catalan-type integrals (8), (9) to values of the generalized-Euler-constant function

$$\gamma_{a,b}(z) = \sum_{n=0}^{\infty} \left(\frac{1}{an+b} - \log\left(\frac{an+b+1}{an+b}\right)\right) z^n, \qquad a, b \in \mathbb{N}, \tag{14}$$

and its derivatives, which is related to constants (1), (2) as $\gamma_{1,1}(1) = \gamma$, $\gamma_{1,1}(-1) = \log\frac{4}{\pi}$. Using generating functions appeared in these integral representations we give new Vacca and Ramanujan-type series for values of $\gamma_{a,b}(z)$ and Addison-type series for values of $\gamma_{a,b}(z)$ and its derivative. As a consequence, we get base B rational series for $\log\frac{4}{\pi}$, $\frac{G}{\pi}$, (where G is Catalan's constant), $\frac{\zeta'(2)}{\pi^2}$ and also for logarithms of Somos's and Glaisher-Kinkelin's constants. We also mention on connection of our approach to summation of series of the form

$$\sum_{n=1}^{\infty} N_{\omega,B}(n) Q(n,B) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{N_{\omega,B}(n) P_B(n)}{Bn(Bn+1)\cdots(Bn+B)},$$

where $Q(n,B)$ is a rational function of B and n

$$Q(n,B) = \frac{1}{Bn(Bn+1)} + \frac{2}{Bn(Bn+2)} + \cdots + \frac{B-1}{Bn(Bn+B-1)}, \tag{15}$$

and $N_{\omega,B}(n)$ is the number of occurrences of a word ω over the alphabet $\{0,1,\dots,B-1\}$ in the B-ary expansion of n, considered in [2]. In this notation, the generalized Vacca series (10) can be written as follows:

$$\gamma = \sum_{k=1}^{\infty} L_B(k) Q(k,B), \tag{16}$$

where $L_B(k) := \lfloor \log_B Bk \rfloor = \sum_{\alpha=0}^{B-1} N_{\alpha,B}(k)$ is the B-ary length of k. Indeed, representing $n = Bk + r$, $0 \le r \le B-1$ and summing in (10) over $k \ge 1$ and $0 \le r \le B-1$ we get

$$\gamma = \sum_{k=1}^{\infty} \lfloor \log_B Bk \rfloor \left(\frac{B-1}{Bk} - \frac{1}{Bk+1} - \cdots - \frac{1}{Bk+B-1}\right) = \sum_{k=1}^{\infty} \lfloor \log_B Bk \rfloor Q(k,B).$$

By the same notation, the generalized Addison series (12) gives another base B expansion of Euler's constant

(17)

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{L_B(n)P_B(n)}{Bn(Bn+1)\cdots(Bn+B)} = \frac{1}{2} + \sum_{n=1}^{\infty} L_B(n)\left(Q(n,B) - \frac{B-1}{2Bn(n+1)}\right)$$

which converges faster than (16) to γ. Here we used the fact that

$$\sum_{n=1}^{\infty}\sum_{\alpha=0}^{B-1} \frac{N_{\alpha,B}(n)}{n(n+1)} = \frac{B}{B-1},$$

which can be easily checked by [3, Section 3]. On the other hand,

$$Q(n,B) - \frac{B-1}{2Bn(n+1)} = \frac{1}{2}\sum_{m=1}^{B-1}\left(\frac{1}{Bn} - \frac{2}{Bn+m} + \frac{1}{Bn+B}\right)$$

$$= \frac{1}{Bn(Bn+B)}\sum_{m=1}^{B-1}\left(2m - B + \frac{2m(B-m)}{Bn+m}\right) = \frac{P_B(n)}{Bn(Bn+1)\cdots(Bn+B)}.$$

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2. Analytic continuation

We consider the generalized-Euler-constant function $\gamma_{a,b}(z)$ defined in (14), where a, b are positive real numbers, $z \in \mathbb{C}$, and the series converges when $|z| \le 1$. We show that $\gamma_{a,b}(z)$ admits an analytic continuation to the domain $\mathbb{C} \setminus [1, +\infty)$. The following theorem is a slight modification of [12, Th.3].

Theorem 1. *Let a, b be positive real numbers, $z \in \mathbb{C}$, $|z| \le 1$. Then*

(18)

$$\gamma_{a,b}(z) = \int_0^1\int_0^1 \frac{(xy)^{b-1}(1-x)}{(1-zx^ay^a)(-\log xy)}\,dxdy = \int_0^1 \frac{x^{b-1}(1-x)}{1-zx^a}\left(\frac{1}{1-x} + \frac{1}{\log x}\right)dx.$$

The integrals converge for all $z \in \mathbb{C} \setminus (1, +\infty)$ and give the analytic continuation of the generalized-Euler-constant function $\gamma_{a,b}(z)$ for $z \in \mathbb{C} \setminus [1, +\infty)$.

Proof. Denoting the double integral in (18) by $I(z)$ and for $|z| \le 1$, expanding $(1 - zx^ay^a)^{-1}$ in a geometric series we have

$$I(z) = \sum_{k=0}^{\infty} z^k \int_0^1\int_0^1 \frac{(xy)^{ak+b-1}(1-x)}{(-\log xy)}\,dxdy$$

$$= \sum_{k=0}^{\infty} z^k \int_0^1\int_0^1\int_0^{+\infty} (xy)^{t+ak+b-1}(1-x)\,dxdydt$$

$$= \sum_{k=0}^{\infty} z^k \int_0^{+\infty}\left(\frac{1}{(t+ak+b)^2} - \left(\frac{1}{t+ak+b} - \frac{1}{t+ak+b+1}\right)\right)dt = \gamma_{a,b}(z).$$

On the other hand, making the change of variables $u = x^a$, $v = y^a$ in the double integral we get

$$I(z) = \frac{1}{a}\int_0^1\int_0^1 \frac{(uv)^{\frac{b}{a}-1}(1-u^{\frac{1}{a}})}{(1-zuv)(-\log uv)}\,dudv.$$

Now by [8, Corollary 3.3], for $z \in \mathbb{C}\setminus[1,+\infty)$ we have

$$I(z) = \frac{1}{a}\Phi\Big(z,1,\frac{b}{a}\Big) - \frac{\partial\Phi}{\partial s}\Big(z,0,\frac{b}{a}\Big) + \frac{\partial\Phi}{\partial s}\Big(z,0,\frac{b+1}{a}\Big),$$

where $\Phi(z,s,u)$ is the Lerch transcendent, a holomorphic function in z and s, for $z \in \mathbb{C}\backslash[1,+\infty)$ and all complex s (see [8, Lemma 2.2]), which is the analytic continuation of the series

$$\Phi(z,s,u) = \sum_{n=0}^{\infty}\frac{z^n}{(n+u)^s}, \qquad u > 0.$$

To prove the second equality in (18), make the change of variables $X = xy$, $Y = y$ and integrate with respect to Y. □

Corollary 1. *Let a, b be positive real numbers, $l \in \mathbb{N}$, $z \in \mathbb{C}\setminus[1,+\infty)$. Then for the l-th derivative we have*

$$\gamma_{a,b}^{(l)}(z) = \int_0^1\int_0^1 \frac{(xy)^{al+b-1}(x-1)}{(1-zx^ay^a)^{l+1}\log xy}\,dxdy = \int_0^1 \frac{x^{la+b-1}(1-x)}{(1-zx^a)^{l+1}}\left(\frac{1}{1-x}+\frac{1}{\log x}\right)dx.$$

From Corollary 1, [8, Cor.3.3, 3.8, 3.9] and [2, Lemma 4] we get

Corollary 2. *Let a, b be positive real numbers, $z \in \mathbb{C}\setminus[1,+\infty)$. Then the following equalities are valid:*

$$\gamma_{a,b}(1) = \log\Gamma\Big(\frac{b+1}{a}\Big) - \log\Gamma\Big(\frac{b}{a}\Big) - \frac{1}{a}\psi\Big(\frac{b}{a}\Big),$$

$$\gamma_{a,b}(z) = \frac{1}{a}\Phi\Big(z,1,\frac{b}{a}\Big) - \frac{\partial\Phi}{\partial s}\Big(z,0,\frac{b}{a}\Big) + \frac{\partial\Phi}{\partial s}\Big(z,0,\frac{b+1}{a}\Big),$$

$$\gamma'_{a,b}(z) = -\frac{b}{a^2}\Phi\Big(z,1,\frac{b}{a}+1\Big) + \frac{1}{a(1-z)} + \frac{b}{a}\frac{\partial\Phi}{\partial s}\Big(z,0,\frac{b}{a}+1\Big) - \frac{\partial\Phi}{\partial s}\Big(z,-1,\frac{b}{a}+1\Big) -$$
$$\frac{b+1}{a}\frac{\partial\Phi}{\partial s}\Big(z,0,\frac{b+1}{a}+1\Big) + \frac{\partial\Phi}{\partial s}\Big(z,-1,\frac{b+1}{a}+1\Big),$$

where $\Phi(z,s,u)$ is the Lerch transcendent and $\psi(x) = \frac{d}{dx}\log\Gamma(x)$ is the logarithmic derivative of the gamma function.

3. Catalan-type integrals for $\gamma^{(l)}_{a,b}(z)$.

In [4] it was demonstrated that for $x>0$ and any integer $B>1$, one has

$$\frac{1}{1-x}+\frac{1}{\log x}=\sum_{k=1}^{\infty}\frac{(B-1)+(B-2)x^{\frac{1}{B^k}}+(B-3)x^{\frac{2}{B^k}}+\cdots+x^{\frac{B-2}{B^k}}}{B^k(1+x^{\frac{1}{B^k}}+x^{\frac{2}{B^k}}+\cdots+x^{\frac{B-1}{B^k}})}. \tag{19}$$

The special cases $B=2,3$ of this equality can be found in Ramanujan's third note book [9, p.364]. Using this key formula we prove the following generalization of integral (9).

Theorem 2. *Let $a,b,B>1$ be positive integers, l a non-negative integer. If either $z\in\mathbb{C}\setminus[1,+\infty)$ and $l\ge 1$, or $z\in\mathbb{C}\setminus(1,+\infty)$ and $l=0$, then*

$$\gamma^{(l)}_{a,b}(z)=\int_0^1\left(\frac{B}{1-x^B}-\frac{1}{1-x}\right)F_l(z,x)\,dx \tag{20}$$

where

$$F_l(z,x)=\sum_{k=1}^{\infty}\frac{x^{(b+al)B^k-1}(1-x^{B^k})}{(1-zx^{aB^k})^{l+1}}. \tag{21}$$

Proof. First we note that the series of variable x on the right-hand side of (19) uniformly converges on $[0,1]$, since the absolute value of its general term does not exceed $\frac{B-1}{2B^{k-1}}$. Then for $l\ge 0$, multiplying both sides of (19) by $\frac{x^{la+b-1}(1-x)}{(1-zx^a)^{l+1}}$ and integrating over $0\le x\le 1$ we get

$$\gamma^{(l)}_{a,b}(z)=\sum_{k=1}^{\infty}\int_0^1\frac{x^{la+b-1}(1-x)}{(1-zx^a)^{l+1}}\cdot\frac{(B-1)+(B-2)x^{\frac{1}{B^k}}+\cdots+x^{\frac{B-2}{B^k}}}{B^k(1+x^{\frac{1}{B^k}}+x^{\frac{2}{B^k}}+\cdots+x^{\frac{B-1}{B^k}})}\,dx.$$

Replacing x by x^{B^k} in each integral we find

$$\begin{aligned}\gamma^{(l)}_{a,b}(z)&=\sum_{k=1}^{\infty}\int_0^1\frac{x^{(la+b)B^k-1}(1-x^{B^k})}{(1-zx^{aB^k})^{l+1}}\cdot\frac{(B-1)+(B-2)x+\cdots+x^{B-2}}{1+x+x^2+\cdots+x^{B-1}}\,dx\\&=\int_0^1\left(\frac{B}{1-x^B}-\frac{1}{1-x}\right)F_l(z,x)\,dx,\end{aligned}$$

as required. □

From Theorem 2 we readily get a generalization of Ramanujan's integral.

Corollary 3. *Let $a,b,B>1$ be positive integers, l a non-negative integer. If either $z\in\mathbb{C}\setminus[1,+\infty)$ and $l\ge 1$, or $z\in\mathbb{C}\setminus(1,+\infty)$ and $l=0$, then*

$$\gamma^{(l)}_{a,b}(z)=\int_0^1\frac{x^{b+al-1}(1-x)}{(1-zx^a)^{l+1}}\,dx+\int_0^1\left(\frac{Bx^B}{1-x^B}-\frac{x}{1-x}\right)F_l(z,x)\,dx. \tag{22}$$

Proof. First we note that the series (21) considered as a sum of functions of variable x uniformly converges on $[0,1-\varepsilon]$ for any $\varepsilon>0$. Then integrating termwise we have

$$\int_0^{1-\varepsilon}F_l(z,x)\,dx=\sum_{k=1}^{\infty}\int_0^{1-\varepsilon}\frac{x^{(b+al)B^k-1}(1-x^{B^k})}{(1-zx^{aB^k})^{l+1}}\,dx.$$

Making the change of variable $y = x^{B^k}$ in each integral we get

$$\int_0^{1-\varepsilon} F_l(z,x)\,dx = \sum_{k=1}^{\infty} \frac{1}{B^k} \int_0^{(1-\varepsilon)^{B^k}} \frac{y^{b+al-1}(1-y)}{(1-zy^a)^{l+1}}\,dy.$$

Since the last series of variable ε uniformly converges on $[0,1]$, letting ε tend to zero we get

$$\int_0^1 F_l(z,x)\,dx = \frac{1}{B-1}\int_0^1 \frac{y^{b+al-1}(1-y)}{(1-zy^a)^{l+1}}\,dy. \tag{23}$$

Now from (20) and (23) it follows that

$$\gamma_{a,b}^{(l)}(z) - \int_0^1 \frac{y^{b+al-1}(1-y)}{(1-zy^a)^{l+1}}\,dy = \int_0^1 \left(\frac{Bx^B}{1-x^B} - \frac{x}{1-x}\right) F_l(z,x)\,dx,$$

and the proof is complete. □

Averaging both formulas (20), (22) we get the following generalization of integral (7).

Corollary 4. *Let $a, b, B > 1$ be positive integers, l a non-negative integer. If either $z \in \mathbb{C} \setminus [1,+\infty)$ and $l \geq 1$, or $z \in \mathbb{C} \setminus (1,+\infty)$ and $l = 0$, then*

$$\gamma_{a,b}^{(l)}(z) = \frac{1}{2}\int_0^1 \frac{x^{b+al-1}(1-x)}{(1-zx^a)^{l+1}}\,dx + \frac{1}{2}\int_0^1 \left(\frac{B(1+x^B)}{1-x^B} - \frac{1+x}{1-x}\right) F_l(z,x)\,dx.$$

4. Vacca-type series for $\gamma_{a,b}(z)$ and $\gamma'_{a,b}(z)$.

Theorem 3. *Let $a, b, B > 1$ be positive integers, $z \in \mathbb{C}$, $|z| \leq 1$. Then for the generalized-Euler-constant function $\gamma_{a,b}(z)$, the following expansion is valid:*

$$\gamma_{a,b}(z) = \sum_{k=1}^{\infty} a_k Q(k,B) = \sum_{k=1}^{\infty} a_{\lfloor \frac{k}{B} \rfloor} \frac{\varepsilon(k)}{k},$$

where $Q(k,B)$ is a rational function given by (15), $\{a_k\}_{k=0}^{\infty}$ is a sequence defined by the generating function

$$G(z,x) = \frac{1}{1-x}\sum_{k=0}^{\infty} \frac{x^{bB^k}(1-x^{B^k})}{1-zx^{aB^k}} = \sum_{k=0}^{\infty} a_k x^k \tag{24}$$

and $\varepsilon(k)$ is denoted in (11).

Proof. For $l = 0$, rewrite (20) in the form

$$\gamma_{a,b}(z) = \int_0^1 \frac{1-x^B}{x}\left(\frac{B}{1-x^B} - \frac{1}{1-x}\right) G(z,x^B)\,dx$$

where $G(z,x)$ is defined in (24). Then, since $a_0 = 0$, we have

$$\gamma_{a,b}(z) = \int_0^1 (B-1-x-x^2-\cdots-x^{B-1}) \sum_{k=1}^{\infty} a_k x^{Bk-1}\,dx. \tag{25}$$

Expanding $G(z,x)$ in a power series of x

$$G(z,x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} z^m x^{(am+b)B^k}(1+x+\cdots+x^{B^k-1})$$

we see that $a_k = O(\ln_B k)$. Therefore, by termwise integration in (25), which can be easily justified by the same way as in the proof of Corollary 3, we get

$$\begin{aligned}\gamma_{a,b}(z) &= \sum_{k=1}^{\infty} a_k \int_0^1 [(x^{Bk-1}-x^{Bk}) + (x^{Bk-1}-x^{Bk+1}) + \cdots + (x^{Bk-1}-x^{Bk+B-2})]\,dx \\ &= \sum_{k=1}^{\infty} a_k Q(k,B). \qquad \square\end{aligned}$$

Theorem 4. *Let $a, b, B > 1$ be positive integers, $z \in \mathbb{C}$, $|z| \le 1$. Then for the generalized-Euler-constant function, the following expansion is valid:*

$$\gamma_{a,b}(z) = \int_0^1 \frac{x^{b-1}(1-x)}{1-zx^a}\,dx - \sum_{k=1}^{\infty} a_k \widetilde{Q}(k,B),$$

where

$$\begin{aligned}\widetilde{Q}(k,B) &= \frac{B-1}{Bk(k+1)} - Q(k,B) \\ &= \frac{B-1}{(Bk+B)(Bk+1)} + \frac{B-2}{(Bk+B)(Bk+2)} + \cdots + \frac{1}{(Bk+B)(Bk+B-1)}\end{aligned}$$

and the sequence $\{a_k\}_{k=1}^{\infty}$ is defined in Theorem 3.

Proof. From Corollary 3 with $l=0$, by the same way as in the proof of Theorem 3, we get

$$\begin{aligned}&\int_0^1 \left(\frac{Bx^B}{1-x^B} - \frac{x}{1-x}\right) F_0(z,x) = \int_0^1 \frac{1-x^B}{x}\left(\frac{Bx^B}{1-x^B} - \frac{x}{1-x}\right) G(z,x^B)\,dx \\ &= \int_0^1 (Bx^{B-1} - (1+x+\cdots+x^{B-1})) \sum_{k=1}^{\infty} a_k x^{Bk}\,dx \\ &= \sum_{k=1}^{\infty} a_k \int_0^1 [(x^{Bk+B-1}-x^{Bk+B-2}) + \cdots + (x^{Bk+B-1}-x^{Bk+1}) + (x^{Bk+B-1}-x^{Bk})]\,dx \\ &= -\sum_{k=1}^{\infty} a_k \widetilde{Q}(k,B). \qquad \square\end{aligned}$$

Theorem 5. *Let $a, b, B > 1$ be positive integers, $z \in \mathbb{C}$, $|z| \le 1$. Then for the generalized-Euler-constant function $\gamma_{a,b}(z)$ and its derivative, the following expansion is valid:*

$$\gamma_{a,b}^{(l)}(z) = \frac{1}{2}\int_0^1 \frac{x^{b+al-1}(1-x)}{(1-zx^a)^{l+1}}\,dx + \sum_{k=1}^{\infty} a_{k,l} \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}, \qquad l = 0,1,$$

where $P_B(k)$ is a polynomial of degree $B-2$ given by (13)*,* $(z-1)^2+(l-1)^2\neq 0$ *and the sequence* $\{a_{k,l}\}_{k=0}^{\infty}$ *is defined by the generating function*

$$G_l(z,x)=\frac{1}{1-x}\sum_{k=0}^{\infty}\frac{x^{(b+al)B^k}(1-x^{B^k})}{(1-zx^{aB^k})^{l+1}}=\sum_{k=0}^{\infty}a_{k,l}x^k,\qquad l=0,1. \tag{26}$$

Proof. Expanding $G_l(z,x)$ in a power series of x

$$G_l(z,x)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+l}{l}z^m x^{(b+al+am)B^k}(1+x+x^2+\cdots+x^{B^k-1})$$

we see that $a_{k,l}=O(k^l\ln_B k)$. Therefore, for $l=0,1$, by termwise integration we get

$$\int_0^1\left(\frac{B(1+x^B)}{1-x^B}-\frac{1+x}{1-x}\right)F_l(z,x)dx=\int_0^1\frac{1-x^B}{x}\left(\frac{B(1+x^B)}{1-x^B}-\frac{1+x}{1-x}\right)G_l(z,x^B)dx$$

$$=\int_0^1[(B-1)-2x-2x^2-\cdots-2x^{B-1}+(B-1)x^B]\sum_{k=1}^{\infty}a_{k,l}x^{Bk-1}\,dx$$

$$=\sum_{k=1}^{\infty}a_{k,l}\left(\frac{B-1}{Bk}-\frac{2}{Bk+1}-\frac{2}{Bk+2}-\cdots-\frac{2}{Bk+B-1}+\frac{B-1}{Bk+B}\right)$$

$$=2\sum_{k=1}^{\infty}a_{k,l}\frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where $P_B(k)$ is defined in (13) and the last series converges since $\frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}=O(k^{-3})$. Now our theorem easily follows from Corollary 4. □

5. Summation of series in terms of the Lerch transcendent

It is easily seen that the generating function (26) satisfies the following functional equation:

$$G_l(z,x)-\frac{1-x^B}{1-x}G_l(z,x^B)=\frac{x^{b+al}}{(1-zx^a)^{l+1}}, \tag{27}$$

which is equivalent to the identity for series:

$$\sum_{k=0}^{\infty}a_{k,l}x^k-(1+x+\cdots+x^{B-1})\sum_{k=0}^{\infty}a_{k,l}x^{Bk}=\sum_{k=l}^{\infty}\binom{k}{l}z^{k-l}x^{ak+b}.$$

Comparing coefficients of powers of x we get an alternative definition of the sequence $\{a_{k,l}\}_{k=0}^{\infty}$ by means of the recursion

$$a_{0,l}=a_{1,l}=\ldots=a_{al+b-1,l}=0$$

and for $k\geq al+b$,

$$a_{k,l}=\begin{cases}a_{\lfloor\frac{k}{B}\rfloor,l} & \text{if}\quad k\not\equiv b\pmod a,\\ a_{\lfloor\frac{k}{B}\rfloor,l}+\binom{(k-b)/a}{l}z^{\frac{k-b}{a}-l} & \text{if}\quad k\equiv b\pmod a.\end{cases} \tag{28}$$

On the other hand, in view of Corollary 2, $\gamma_{a,b}(z)$ and $\gamma'_{a,b}(z)$ can be explicitly expressed in terms of the Lerch transcendent, ψ-function and logarithm of the gamma function. This allows us to sum the series figured in Theorems 3-5 in terms of these functions.

6. Examples of rational series

Example 1. Suppose that ω is a non-empty word over the alphabet $\{0,1,\dots,B-1\}$. Then obviously ω is uniquely defined by its length $|\omega|$ and its size $v_B(\omega)$ which is the value of ω when interpreted as an integer in base B. Let $N_{\omega,B}(k)$ be the number of (possibly overlapping) occurrences of the block ω in the B-ary expansion of k. Note that for every B and ω, $N_{\omega,B}(0)=0$, since the B-ary expansion of zero is the empty word. If the word ω begins with 0, but $v_B(\omega)\neq 0$, then in computing $N_{\omega,B}(k)$ we assume that the B-ary expansion of k starts with an arbitrary long prefix of 0's. If $v_B(\omega)=0$ we take for k the usual shortest B-ary expansion of k.

Now we consider equation (27) with $l=0$, $z=1$

$$G(1,x)-\frac{1-x^B}{1-x}G(1,x^B)=\frac{x^b}{1-x^a} \tag{29}$$

and for a given non-empty word ω, set in (29) $a=B^{|\omega|}$ and

$$b=\begin{cases} B^{|\omega|} & \text{if} \quad v_B(\omega)=0 \\ v_B(\omega) & \text{if} \quad v_B(\omega)\neq 0. \end{cases}$$

Then by (28), it is easily seen that $a_k:=a_{k,0}=N_{\omega,B}(k)$, $k=1,2,\dots$, and by Theorem 3, we get one more proof of the following statement (see [2, Sections 3, 4.2]).

Corollary 5. *Let ω be a non-empty word over the alphabet $\{0,1,\dots,B-1\}$. Then*

$$\sum_{k=1}^{\infty} N_{\omega,B}(k)Q(k,B)=\begin{cases} \gamma_{B^{|\omega|},v_B(\omega)}(1) & \textit{if} \quad v_B(\omega)\neq 0 \\ \gamma_{B^{|\omega|},B^{|\omega|}}(1) & \textit{if} \quad v_B(\omega)=0. \end{cases}$$

By Corollary 2, the right-hand side of the last equality can be calculated explicitly and we have

(30)

$$\sum_{k=1}^{\infty} N_{\omega,B}(k)Q(k,B)=\begin{cases} \log\Gamma\left(\frac{v_B(\omega)+1}{B^{|\omega|}}\right)-\log\Gamma\left(\frac{v_B(\omega)}{B^{|\omega|}}\right)-\frac{1}{B^{|\omega|}}\psi\left(\frac{v_B(\omega)}{B^{|\omega|}}\right) & \text{if } v_B(\omega)\neq 0 \\ \log\Gamma\left(\frac{1}{B^{|\omega|}}\right)+\frac{\gamma}{B^{|\omega|}}-|\omega|\log B & \text{if } v_B(\omega)=0. \end{cases}$$

Corollary 6. *Let ω be a non-empty word over the alphabet $\{0,1,\dots,B-1\}$. Then*

$$\sum_{k=1}^{\infty}\frac{N_{\omega,B}(k)P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}$$

$$=\begin{cases} \gamma_{B^{|\omega|},v_B(\omega)}(1)-\frac{1}{2B^{|\omega|}}\left(\psi\left(\frac{v_B(\omega)+1}{B^{|\omega|}}\right)-\psi\left(\frac{v_B(\omega)}{B^{|\omega|}}\right)\right) & \textit{if} \quad v_B(\omega)\neq 0 \\ \gamma_{B^{|\omega|},B^{|\omega|}}(1)-\frac{1}{2B^{|\omega|}}\psi\left(\frac{1}{B^{|\omega|}}\right)-\frac{\gamma}{2B^{|\omega|}}-\frac{1}{2} & \textit{if} \quad v_B(\omega)=0. \end{cases}$$

Proof. The required statement easily follows from Theorem 5, Corollary 5 and the equality

$$\int_0^1 \frac{x^{b-1}(1-x)}{1-x^a}\,dx = \sum_{k=0}^{\infty}\left(\frac{1}{ak+b}-\frac{1}{ak+b+1}\right) = \frac{1}{a}\left(\psi\Big(\frac{b+1}{a}\Big)-\psi\Big(\frac{b}{a}\Big)\right). \quad \square$$

From Theorem 3, (27) and (28) with $a=1$, $l=0$ we have

Corollary 7. *Let $b, B > 1$ be positive integers, $z \in \mathbb{C}$, $|z| \le 1$. Then*

$$\gamma_{1,b}(z) = \sum_{k=1}^{\infty} a_k Q(k,B) = \sum_{k=1}^{\infty} a_{\lfloor \frac{k}{B}\rfloor}\frac{\varepsilon(k)}{k},$$

where $a_0 = a_1 = \ldots = a_{b-1} = 0$, $a_k = a_{\lfloor \frac{k}{B}\rfloor} + z^{k-b}$, $k \ge b$.

Similarly, from Theorem 5 we have

Corollary 8. *Let $b, B > 1$ be positive integers, $z \in \mathbb{C}$, $|z| \le 1$. Then*

$$\gamma_{1,b}(z) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{(k+b)(k+b+1)} + \sum_{k=1}^{\infty} a_k \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where $a_0 = a_1 = \ldots = a_{b-1} = 0$, $a_k = a_{\lfloor \frac{k}{B}\rfloor} + z^{k-b}$, $k \ge b$.

Example 2. If in Corollary 7 we take $z=1$, then we get that a_k is equal to the B-ary length of $\lfloor \frac{k}{b}\rfloor$, i. e.,

$$a_k = \sum_{\alpha=0}^{B-1} N_{\alpha,B}\left(\left\lfloor\frac{k}{b}\right\rfloor\right) = L_B\left(\left\lfloor\frac{k}{b}\right\rfloor\right).$$

On the other hand,

$$\gamma_{1,b}(1) = \log b - \psi(b) = \log b - \sum_{k=1}^{b-1}\frac{1}{k} + \gamma$$

and hence we get

$$\log b - \psi(b) = \sum_{k=1}^{\infty} L_B\left(\left\lfloor\frac{k}{b}\right\rfloor\right) Q(k,B). \tag{31}$$

If $b=1$, formula (31) gives (16). If $b>1$, then from (31) and (16) we get

$$\log b = \sum_{k=1}^{b-1}\frac{1}{k} + \sum_{k=1}^{\infty}\left(L_B\left(\left\lfloor\frac{k}{b}\right\rfloor\right) - L_B(k)\right) Q(k,B), \tag{32}$$

which is equivalent to [4, Theorem 2.8]. Similarly, from Corollary 8 we obtain (17) and

$$\log b = \sum_{k=1}^{b-1}\frac{1}{k} - \frac{b-1}{2b} + \sum_{k=1}^{\infty}\frac{\left(L_B(\lfloor\frac{k}{b}\rfloor) - L_B(k)\right)P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}. \tag{33}$$

Example 3. Using the fact that for any integer $B > 1$

$$L_B\left(\left\lfloor \frac{k}{B}\right\rfloor\right) - L_B(k) = -1,$$

from (30), (16) and (32) we get the following rational series for $\log\Gamma(1/B)$:

$$\log\Gamma\left(\frac{1}{B}\right) = \sum_{k=1}^{B-1}\frac{1}{k} + \sum_{k=1}^{\infty}\Big(N_{0,B}(k) - \frac{1}{B}L_B(k) - 1\Big)Q(k,B).$$

Example 4. Substituting $b = 1$, $z = -1$ in Corollary 7 we get the generalized Vacca series for $\log\frac{4}{\pi}$.

Corollary 9. *Let $B \in \mathbb{N}$, $B > 1$. Then*

$$\log\frac{4}{\pi} = \sum_{k=1}^{\infty} a_k Q(k,B) = \sum_{k=1}^{\infty} a_{\lfloor\frac{k}{B}\rfloor}\frac{\varepsilon(k)}{k},$$

where

(34) $$a_0 = 0, \qquad a_k = a_{\lfloor\frac{k}{B}\rfloor} + (-1)^{k-1}, \quad k \ge 1.$$

In particular, if B is even, then

(35)
$$\log\frac{4}{\pi} = \sum_{k=1}^{\infty}(N_{odd,B}(k) - N_{even,B}(k))Q(k,B) = \sum_{k=1}^{\infty}\frac{\big(N_{odd,B}(\lfloor\frac{k}{B}\rfloor) - N_{even,B}(\lfloor\frac{k}{B}\rfloor)\big)}{k}\varepsilon(k),$$

where $N_{odd,B}(k)$ (respectively $N_{even,B}(k)$) is the number of occurrences of the odd (respectively even) digits in the B-ary expansion of k.

Proof. To prove (35), we notice that if B is even, then the sequence $\widetilde{a}_k := N_{odd,B}(k) - N_{even,B}(k)$ satisfies recurrence (34). □

Substituting $b = 1, z = -1$ in Corollary 8 with the help of (33) we get the generalized Addison series for $\log\frac{4}{\pi}$.

Corollary 10. *Let $B > 1$ be a positive integer. Then*

$$\log\frac{4}{\pi} = \frac{1}{4} + \sum_{k=1}^{\infty}\frac{\big(L_B(\lfloor\frac{k}{2}\rfloor) - L_B(k) + a_k\big)\,P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where the sequence a_k is defined in Corollary 9. In particular, if B is even, then

$$\log\frac{4}{\pi} = \frac{1}{4} + \sum_{k=1}^{\infty}\frac{\big(L_B(\lfloor\frac{k}{2}\rfloor) - 2N_{even,B}(k)\big)\,P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}.$$

Example 5. For $t > 1$, the generalized Somos constant σ_t is defined by

$$\sigma_t = \sqrt[t]{1\sqrt[t]{2\sqrt[t]{3\ldots}}} = 1^{1/t}2^{1/t^2}3^{1/t^3}\cdots = \prod_{n=1}^{\infty} n^{1/t^n}.$$

In view of the relation [12, Th.8]

(36) $$\gamma_{1,1}\left(\frac{1}{t}\right) = t\log\frac{t}{(t-1)\sigma_t^{t-1}},$$

by Corollary 7 and formula (32) we get

Corollary 11. *Let $B \in \mathbb{N}$, $B > 1$, $t \in \mathbb{R}$, $t > 1$. Then*

$$\log \sigma_t = \frac{1}{(t-1)^2} + \frac{1}{t-1} \sum_{k=1}^{\infty} \left(L_B\left(\left\lfloor \frac{k}{t} \right\rfloor\right) - L_B\left(\left\lfloor \frac{k}{t-1} \right\rfloor\right) - \frac{a_k}{t} \right) Q(k, B),$$

where $a_0 = 0$, $a_k = a_{\lfloor \frac{k}{B} \rfloor} + t^{1-k}$, $k \geq 1$.

In particular, setting $B = t = 2$ we get the following rational series for Somos's quadratic recurrence constant:

$$\log \sigma_2 = 1 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{a_k}{2k(2k+1)},$$

where $a_1 = 3$, $a_k = a_{\lfloor \frac{k}{2} \rfloor} + \frac{1}{2^{k-1}}$, $k \geq 2$.

From (36), (33) and Theorem 5 we find

Corollary 12. *Let $B \in \mathbb{N}$, $B > 1$, $t \in \mathbb{R}$, $t > 1$. Then*

$$\log \sigma_t = \frac{3t-1}{4t(t-1)^2}$$
$$+ \frac{t+1}{2(t-1)} \sum_{k=1}^{\infty} \left(L_B\left(\left\lfloor \frac{k}{t} \right\rfloor\right) - L_B\left(\left\lfloor \frac{k}{t-1} \right\rfloor\right) - \frac{2a_k}{t(t+1)} \right) \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where the sequence a_k is defined in Corollary 11.

In particular, if $B = t = 2$ we get

$$\log \sigma_2 = \frac{5}{8} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{a_k}{2k(2k+1)(2k+2)},$$

where $a_1 = 4$, $a_k = a_{\lfloor \frac{k}{2} \rfloor} + \frac{1}{2^{k-1}}$, $k \geq 2$.

Example 6. The Glaisher-Kinkelin constant is defined by the limit [7, p.135]

$$A := \lim_{n \to \infty} \frac{1^2 2^2 \cdots n^n}{n^{\frac{n^2+n}{2} + \frac{1}{12}} e^{-\frac{n^2}{4}}} = 1.28242712\ldots.$$

Its connection to the generalized-Euler-constant function $\gamma_{a,b}(z)$ is given by the formula [12, Cor.4]

$$\gamma'_{1,1}(-1) = \log \frac{2^{11/6} A^6}{\pi^{3/2} e}. \tag{37}$$

By Theorem 5, since

$$\int_0^1 \frac{x(1-x)}{(1+x)^2} \, dx = 3 \log 2 - 2,$$

we have

$$\log A = \frac{4}{9} \log 2 - \frac{1}{4} \log \frac{4}{\pi} + \frac{1}{6} \sum_{k=1}^{\infty} a_{k,1} \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where the sequence $a_{k,1}$ is defined by the generating function (26) with $a = b = l = 1$, $z = -1$, or using (28) by the recursion

$$a_{0,1} = a_{1,1} = 0, \qquad a_{k,1} = a_{\lfloor \frac{k}{B} \rfloor,1} + (-1)^k (k-1), \quad k \geq 2.$$

Now by Corollary 10 and (33) we get

Corollary 13. *Let $B > 1$ be a positive integer. Then*

$$\log A = \frac{13}{48} - \frac{1}{36} \sum_{k=1}^{\infty} \left(7L_B(k) - 7L_B\left(\left\lfloor \frac{k}{2} \right\rfloor \right) + b_k \right) \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where $b_0 = 0$, $b_k = b_{\lfloor \frac{k}{B} \rfloor} + (-1)^{k-1}(6k+3)$, $k \geq 1$.

In particular, if $B = 2$ we get

$$\log A = \frac{13}{48} - \frac{1}{36} \sum_{k=1}^{\infty} \frac{c_k}{2k(2k+1)(2k+2)},$$

where $c_1 = 16$, $c_k = c_{\lfloor \frac{k}{2} \rfloor} + (-1)^{k-1}(6k+3)$, $k \geq 2$.

Using the formula expressing $\frac{\zeta'(2)}{\pi^2}$ in terms of Glaisher-Kinkelin's constant [7, p.135]

$$\log A = -\frac{\zeta'(2)}{\pi^2} + \frac{\log 2\pi + \gamma}{12}$$

by Corollaries 8, 10 and 13, we get

Corollary 14. *Let $B > 1$ be a positive integer. Then*

$$\frac{\zeta'(2)}{\pi^2} = -\frac{1}{16} + \frac{1}{36} \sum_{k=1}^{\infty} \left(4L_B(k) - L_B\left(\left\lfloor \frac{k}{2} \right\rfloor \right) + c_k \right) \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where $c_0 = 0$, $c_k = c_{\lfloor \frac{k}{B} \rfloor} + (-1)^{k-1} 6k$, $k \geq 1$.

Example 7. First we evaluate $\gamma_{2,1}^{(l)}(-1)$ for $l = 0, 1$. From Corollaries 1, 2 and [12, Ex.3.12, 3.13] we have

$$\gamma_{2,1}(-1) = \int_0^1 \int_0^1 \frac{(x-1)\, dx dy}{(1+x^2y^2)\log xy} = \frac{\pi}{4} - 2\log\Gamma\left(\frac{1}{4}\right) + \log\sqrt{2\pi^3}$$

and

$$\begin{aligned}\gamma_{2,1}'(-1) = &-\frac{1}{4}\Phi(-1,1,3/2) + \frac{1}{2}\Phi(-1,0,3/2) + \frac{1}{2}\frac{\partial\Phi}{\partial s}(-1,0,3/2) \\ &- \frac{\partial\Phi}{\partial s}(-1,-1,3/2) - \frac{\partial\Phi}{\partial s}(-1,0,2) + \frac{\partial\Phi}{\partial s}(-1,-1,2).\end{aligned}$$

The last expression can be evaluated explicitly (see [12, Section 2]) and we get

$$\gamma_{2,1}'(-1) = \frac{G}{\pi} + \frac{\pi}{8} - \log\Gamma\left(\frac{1}{4}\right) - 3\log A + \log\pi + \frac{1}{3}\log 2,$$

or

$$\frac{G}{\pi} = \gamma_{2,1}'(-1) - \frac{1}{2}\gamma_{2,1}(-1) + \frac{1}{4}\log\frac{4}{\pi} + 3\log A - \frac{7}{12}\log 2. \tag{38}$$

On the other hand, by Theorem 5 and (28) we have

$$\gamma_{2,1}(-1) = \frac{\pi}{8} - \frac{1}{4}\log 2 + \sum_{k=1}^{\infty} a_{k,0} \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}, \tag{39}$$

where $a_{0,0} = 0$, $a_{2k,0} = a_{\lfloor \frac{2k}{B} \rfloor,0}$, $k \geq 1$, $a_{2k+1,0} = a_{\lfloor \frac{2k+1}{B} \rfloor,0} + (-1)^k$, $k \geq 0$, and

$$\gamma'_{2,1}(-1) = \frac{\pi}{16} - \frac{1}{4}\log 2 + \sum_{k=1}^{\infty} a_{k,1} \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}, \tag{40}$$

where $a_{0,1} = 0$, $a_{2k,1} = a_{\lfloor \frac{2k}{B} \rfloor,1}$, $k \geq 1$, $a_{2k+1,1} = a_{\lfloor \frac{2k+1}{B} \rfloor,1} + (-1)^{k-1}k$, $k \geq 0$. Now from (38) – (40), (33) and Corollary 10 we get the following expansion for G/π.

Corollary 15. *Let $B > 1$ be a positive integer. Then*

$$\frac{G}{\pi} = \frac{11}{32} + \sum_{k=1}^{\infty} \left(\frac{1}{8} L_B\left(\left\lfloor \frac{k}{2} \right\rfloor \right) - \frac{1}{8} L_B(k) + c_k \right) \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},$$

where $c_0 = 0$, $c_{2k} = c_{\lfloor \frac{2k}{B} \rfloor} + k$, $k \geq 1$, $c_{2k+1} = c_{\lfloor \frac{2k+1}{B} \rfloor} + \frac{(-1)^{k-1}-1}{2}(2k+1)$, $k \geq 0$.

In particular, if $B = 2$ we get

$$\frac{G}{\pi} = \frac{11}{32} + \sum_{k=1}^{\infty} \frac{c_k}{2k(2k+1)(2k+2)},$$

where $c_1 = -\frac{9}{8}$, $c_{2k} = c_k + k$, $c_{2k+1} = c_k + \frac{(-1)^{k-1}-1}{2}(2k+1)$, $k \geq 1$.

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