On the tensor algebra of a non abelian group and applications

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For the classifying space $BG = K(G, 1)$ of a group G we consider the word length filtration J_nBG, $n \geq 0$, in the infinite reduced product $J(BG)$ of James [15] which is homotopy equivalent to the loop space $\Omega\Sigma(BG)$. We introduce the "crossed tensor algebra" $\mathcal{J}(G)$ which is a differential algebra in the monoidal category of crossed chain complexes and we describe $\mathcal{J}(G)$ together with its differential d explicitly in terms of the elements of G. Our main result shows that one has a natural isomorphism (see (1.2) and (1.9))

$$(1) \qquad \mathcal{J}(G)_n \quad \cong \quad \pi_n(J_nBG, J_{n-1}BG)$$

where the right hand side is a relative homotopy group. This isomorphism is compatible with the boundary maps and with the multiplications in $\mathcal{J}(G)$ and $J(BG)$ respectively. Let IG be the augmentation ideal in the group ring $\mathbb{Z}[G]$. We form the quotient

$$I_1G = \frac{IG}{I(G,G).IG}$$

where (G,G) is the commutator subgroup. Moreover let $\hat{J}_nBG$ be the subspace of the universal covering $\hat{J}BG$ of JBG which is determined by J_nBG. Using (1) one gets the natural isomorphism of differential algebras

$$(2) \qquad T_{\mathbb{Z}[G]}(I_1G) \quad \cong \quad \bigoplus_{n\geq 0} H_n(\hat{J}_nBG, \hat{J}_{n-1}BG)$$

where the left hand side is the graded tensor algebra with a canonical differental, see (2.6). The right hand side of (2) is given by relative homology groups with integral coefficients. The proof of the isomorphisms above is based on the fundamental theory of Brown-Higgins on crossed complexes. The tensor algebra in (1) and (2) satisfy the algebraic formula

$$(3) \qquad T_{\mathbb{Z}[G]}(I_1G) \quad \cong \quad I_*G \quad \cong \quad C(JG)$$

where C is the chain functor (studied by J.H.C. Whitehead [21] and Brown-Higgins [7]) and where the algebra I_*G is defined by universal "multicrossed homomorphisms" on G, see (2.1) and (2.4). The connection between the homology groups π_nJG and $H_n(I_*G)$ is clarified by theorem (2.16); examples of such homology groups are computed in section 3. We also obtain the natural isomorphisms

$$(4) \qquad G\tilde{\otimes}G \quad \cong \quad \frac{JG_2}{dJG_3} \cong \quad \pi_2(J(BG), BG)$$

which yield a new topological interpretation of the tensor square of Brown-Loday [10]. The isomorphisms (4) are actually isomorphisms of G–crossed modules. Therefore one has the formula

$$(5) \quad \pi_3\Sigma BG \quad \cong \quad \pi_2J(BG) \quad \cong \quad \pi_2J(G) \quad \cong \quad ker(G\tilde{\otimes}G \longrightarrow G)$$

for the third homotopy group $\pi_3\Sigma BG$ of the suspended classifying space ΣBG. Using different methods this result is due to Brown-Loday in [10]. In addition to (5) we obtain new exact sequences for the homotopy group $\pi_4\Sigma BG$, see (3.5); in particular, one has always the surjection

$$(6) \qquad \pi_4\Sigma BG \quad \twoheadrightarrow \quad \pi_3(JG)$$

The computation of the homotopy groups $\pi_n\Sigma BG$ is analogous to the computation of the homotopy groups $\pi_n(BG)^+$ of Quillen's (+)-construction which is a fundamental problem of algebraic K-theory. For a perfect group G we study these homotopy groups by use of the suspension homomorphism

$$(7) \qquad \Sigma : \pi_n(BG)^+ \longrightarrow \pi_{n+1}(\Sigma(BG)^+) \quad = \quad \pi_{n+1}\Sigma BG$$

compare (3.15). In this case, however, the tensor algebras (1) and (2) are degenerate, that is $I_1 G = 0$ and $J(G)_n = 0$ for $n \geq 3$. For an abelian group or for a free group G the tensor algebra JG and $I_* G$ are highly non trivial, compare the computation of $H_n I_* G$ in (3.9) and (3.11)[1] .

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§ 1 The crossed tensor algebra of a group

We consider the classical infinite reduced product JX of a pointed space X as a special case of a free monoid in a monoidal category. The "crossed tensor algebra" of a group G is such a free monoid in the category of crossed chain complexes.

For a filtered space $X_* = \{X_0 \subset X_1 \subset X_2 \subset ...\}$ with $X_0 = *$ we have the (fundamental) crossed chain complex $\pi(X_*)$:

$$(1.1) \qquad ... \longrightarrow \pi_3(X_3, X_2) \xrightarrow{\delta} \pi_2(X_2, X_1) \xrightarrow{\delta} \pi_1(X_1)$$

given by relative homotopy groups. For example let $J(X)$ be the James construction of a pointed space $(X, *)$ with the word length filtration

$$J_*(X) = \{* = J_0X \subset X = J_1X \subset J_2X \subset ...\}$$

Then the crossed chain complex $\pi J_*(X)$ is defined by (1.1). A crucial result for this paper is the following computation of $\pi J_*(X)$ in case $X = BG = K(G, 1)$ is a classifying space of a discrete group G.

(1.2) Theorem: *Let G be a group, then there is a natural isomorphism of crossed chain complexes*

$$\tau \colon J(G) \cong \pi J_* BG$$

where $J(G)$ is the crossed tensor algebra of G defined as follows.

For the definition of $J(G)$ we introduce first the following general notion of a free monoid in a monoidal category.

(1.3) Definition: Let $(\boldsymbol{C}, \otimes)$ be a monoidal category with initial object $*$ satisfying $X \otimes * = X = * \otimes X$ for $X \in \boldsymbol{C}$. We suppose that the colimits below exist in $\boldsymbol{C}$. Then we get for the n-fold tensor product $X^{\otimes n}$ the maps

(1) $\quad i_t: \quad X^{\otimes(n-1)} \longrightarrow X^{\otimes n} \quad (1 \leq t \leq n)$

given by $i_t = X^{\otimes(t-1)} \otimes 0 \otimes X^{\otimes(n-t)}$ where $0: * \longrightarrow X$. These maps define the diagram

(2) $$* \longrightarrow X \rightrightarrows X^{\otimes 2} \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \ X^{\otimes 3} \ \ldots$$

the limit of which is *the free monoid* $J(X)$. Here $J(X)$ is filtered by $J(X) = \varinjlim J_n(X)$ where $J_n(X)$ is the limit of the finite subdiagram of (2) given by $X^{\otimes i}$, $i \leq n$. Moreover $J(X)$ is a monoid in $\boldsymbol{C}$ with multiplication $J(X) \otimes J(X) \longrightarrow J(X)$ in case the bifunctor $\otimes$ preserves the corresponding limits. In this case $J(X)$ is actually the free monoid generated by X in $\boldsymbol{C}$. Clearly J yields a functor $\boldsymbol{C} \longrightarrow \boldsymbol{C}$.

We consider the following examples (A), ... , (E) of free monoids.

(A) Let $(\boldsymbol{Set^*}, \times)$ be the category of pointed sets $(X, *)$ with the cartesian product $\times$. Then $J(X, *) = Mon(X - *)$ is actually the free monoid generated by the set $X - *$.

(B) Let R be a commutative ring and let $(\boldsymbol{Mod}_R^R, \otimes))$ be the category of R-modules under and over R, which are given by diagrams $(R \xrightarrow{i} X \xrightarrow{0} R) = X$ with $0\, i = 1$. Then $J(X) = T_R(\bar{X})$ is the classical tensor algebra of the R-module $\bar{X} = kernel(X \longrightarrow R)$ with $X = \bar{X} \oplus R$. Similarly we obtain the tensor algebra $T_R(\bar{X})$ of a graded R-module $\bar{X}$ by $J(\bar{X} \oplus R)$ where R is commutative in degree 0. Clearly here the monoidal structure is given by the (graded) tensor product $\otimes$.

(C) Let $(\boldsymbol{Top^*}, \times)$ be the category of pointed topological spaces $X = (X, *)$ with the monoidal structure given by product of spaces. Then the free monoid $J(X)$ is the classical *James construction* or "infinite reduced product" of X, compare [15] and [12].

(D) Let $(\boldsymbol{CW}_o^*, \times)$ be the category of cellular maps between CW-complexes X with $X_0 = *$. Here we take the CW-topology for the product $X \times Y$ so that the monoid $J(X)$ is again a CW-complex. The cells of $J(X)$ are in 1-1 correspondance with the words in the free monoid $Mon(Z_*)$ where Z_* is the graded set of cells in $X - *$. Moreover we know by an old result of James [15] that there is a natural monoid homomorphism

$$JX \xrightarrow{\simeq} \Omega\Sigma X$$

which is a homotopy equivalence. Here $\Omega\Sigma X$ is the Moore loop space of the suspension ΣX.

(E) Crossed chain complexes $\rho = (\ldots \longrightarrow \rho_2 \xrightarrow{d} \rho_1)$ or "homotopy systems" as in (1.1) first appeared in the paper [21] of J.H.C. Whitehead, they are "reduced crossed complexes" in the sense of Brown-Higgins [7]. The category ***cross chain*** of crossed chain complexes for example is studied in chapter VI [1] and chapter III [2]. The monoidal structure for this category is given by the tensor product $\otimes$ of Brown-Higgins [8], see also (III §9) [2]. The initial object $*$ is the crossed chain complex which is trivial in each degree. We call the free monoid $J(\rho)$ the *crossed tensor algebra* of the crossed chain complex ρ. In fact, if $\rho_1 = *$ this is just the tensor algebra $T_{\mathbb{Z}}(\rho)$ of the chain complex ρ. In case $\rho = G$ is just a group (that is, ρ is concentrated in degree 1) we get the *crossed tensor algebra* $J(G)$ of the group G which is used in theorem (1.2). The tensor product preserves direct limits so that $J(\rho)$ and $J(G)$ are monoids in **cross chain** which we call "crossed chain algebra". Recall that for a crossed chain complex ρ we define

$$\pi_n(\rho) = \frac{kernel(d_n)}{image(d_{n+1})}$$

where $d_n = d\colon \rho_n \longrightarrow \rho_{n-1}$, $d_1 = 0$. An *n-equivalence* $f\colon \rho \longrightarrow \rho'$ is a map which induces isomorphisms $\pi_i(f)$ for $i \leq n$ and a *weak equivalence* f is a map for which $\pi_i(f)$ is an isomorphism, $i \geq 1$.

(1.4) Explicit description of $J(\rho)$. We now describe the crossed tensor algebra $J(\rho)$ in terms of the elements of the crossed chain complex $\rho = (\dots \longrightarrow \rho_2 \xrightarrow{d} \rho_1)$. We write $|x| = i$ if $x \in \rho_i$ and we set $|ab| = |a| + |b|$ if a, b are elements in the free monoid $Mon(\rho)$. Words of length one in $Mon(\rho)$ are denoted by $y = [y]$, $y \in \rho$. The action in a crossed complex ρ is written y^x where $y^x = y$ for $|x| \geq 2$. Moreover the group structure of ρ_i is denoted by $+$ for $i \geq 1$, clearly $x + x'$ is only defined if $|x| = |x'|$. We introduce the bracket

$$(1)\ \|x, y\| = \begin{cases} -x-y+x+y & \text{if} \quad |x| = |y| = 1 \\ -y^x+y & \text{if} \quad 1 = |x| < |y| \\ -x+x^y & \text{if} \quad |x| > |y| = 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in \rho$.

Now the crossed tensor algebra $J(\rho)$ is the crossed chain complex generated by the graded set $Mon(\rho)$ with the following relations where $x, x', y, y' \in \rho$ and $a, b \in Mon(\rho)$

$$(2)\quad [x]^y = [x^y] \qquad \text{and} \qquad [x+x'] = [x]+[x']$$
$$(3)\quad (a[x]b)^y = a[x^y]b \qquad \text{for} \qquad |x| \geq 2$$

$$(4)\quad a[x+x']b = \begin{cases} ax'b+(axb)^{x'} & \text{for} \quad |a|\geq 1 \\ (axb)^{x'}+ax'b & \text{for} \quad |b|\geq 1 \end{cases}$$

The boundary d of $J(\rho)$ is given by the formulas

$$(5)\qquad d[x] = [dx]$$
$$(6)\qquad d(ab) = (da)b + (-1)^{|a|+1}\|a, b\| + (-1)^{|a|}a(db)$$

where we set $da = 0$ for $|a| = 1$ and $0\,b = 0 = b\,0$.

The filtration of $J(\rho)$ is given by the subcomplex $J_n(\rho) \subset J(\rho)$ given by all words of length $\leq n$ in $Mon(\rho)$. Moreover the multiplication $J(\rho)\otimes J(\rho) \longrightarrow J(\rho)$ carries generators $a\otimes b$ to $a\ b$. We get $J(\rho)_1 = \rho_1$ by (2) and $J(\rho)_2$ is generated as a ρ_1-crossed module by $Mon(\rho)_2 = \rho_2 \cup \rho_1 \times \rho_1$. Moreover $J(\rho)_n$, $n \geq 3$, is generated as a $\pi_1(J(\rho))$-module by the set $Mon(\rho)_n$. Here $\pi_1(J(\rho)) = \pi_1(\rho)^{ab}$ is the abelianization of $\pi_1(\rho)$ by (6).

We now consider the functor

(1.5) $\quad \rho: \quad CW_0^* \longrightarrow$ ***cross chain***

which carries a CW-complex X, $X^0 = *$, to the crossed chain complex $\rho(X) = \pi(X^*)$ given by the CW-filtration $X^* = \{X^0 \subset X^1 \subset \ldots\}$. The crossed tensor algebra $J(\rho)$ above has the following important property.

(1.6) Proposition: *For X,Y in CW_0^* one has natural isomorphisms*

$$\tau: \rho(X) \otimes \rho(Y) \equiv \rho(X \times Y)$$
$$\tau: J\rho(X) \equiv \rho J(X)$$

The proposition is based on the following general property of the tensor product in ***cross chain***. Let X_*, Y_* be filtered topological spaces with $X_0 = * = Y_0$ and let $X_* \otimes Y_*$ be the filtered topological product for which $(X_* \otimes Y_*)_n$ is the union of all $X_p \times Y_q$, $p + q = n$ in $X \times Y$. Then one has by [9] or [2] theorem III2.3 a natural transformation

(1.7) $\quad \tau: \quad \pi(X_*) \otimes \pi(Y_*) \longrightarrow \pi(X_* \otimes Y_*)$

which is an isomorphism in case X_*, Y_* are CW-complexes in $CW_0{}^*$.

Since we assume $X^0 = *$ we see that the James filtration satisfies $J(X)^n = J_n(X)^n$. Whence the filtered map $i: J(X)^* \longrightarrow J_*(X)$ induces a natural surjective transformation

(1.8) $\quad i_*: \quad \rho J(X)^* = \pi J(X) \longrightarrow\!\!\!\!\rightarrow \pi J_*(X)$

In addition to theorem (1.2) we show the next result which characterises the isomorphism τ in (1.2).

(1.9) Theorem: *Let G be a group. Then the classifying space BG in CW_0^* admits a unique weak equivalence $\varepsilon : \rho(BG) \longrightarrow\!\!\!\!\rightarrow G$ in* ***cross chain*** *which induces the identity on π_1 and for which the following diagram commutes*

$$\begin{array}{ccc} \rho J(BG) & \xrightarrow{i_*} & \pi J_*(BG) \\ \tau \uparrow \cong & & \tau \uparrow \cong \\ J\rho(BG) & \xrightarrow{\varepsilon_*} & J(G) \end{array}$$

Moreover $\varepsilon_* = J(\varepsilon)$ *is a 2–equivalence but in general not a 3–equivalence.*

The theorem shows that J in general does not carry weak equivalence to weak equivalence. The functor J, however, carries weak equivalences between totally free objects in ***cross chain*** to homotopy equivalences.

Proof: For $X_* = (BG, *)$ we have $\pi X_* = G$, whence we get by (1.7) natural maps

$$\tau: \quad G\otimes \ldots \otimes G \longrightarrow \pi(X_*\otimes \ldots \otimes X_*)$$

for n-fold tensor product, $n \geq 1$. These maps induce τ in (1.9) since one has the filtered map $\otimes^n X_* \longrightarrow J_*BG$ for all $n \geq 1$. Moreover the naturality of τ in (1.7) shows that the diagram in (1.9) commutes. We now show that τ on the right hand side of (1.9) is an isomorphism. For this we consider the following commutative diagram where $J_n = J_nBG$ and $J = JBG$, $\rho = \rho(BG)$. We have surjections

$$q_n: (J_n\rho)_{n+1} = \pi_{n+1}(J^{n+1}_n, J^n_n) \twoheadrightarrow \pi_{n+1}(J_n, J^n)$$

Moreover we have exact sequences of triples:

$$\begin{array}{ccccc} \pi_{n+1}(J_n, J^n) \xrightarrow{\partial} \pi_n(J^n, J^{n-1}) & \xrightarrow{i} & \pi_n(J_n, J^{n-1}) & & \\ & & \| & & \\ \pi_n(J_{n-1}, J^{n-1}) & \xrightarrow{i'} & \pi_n(J_n, J^{n-1}) & \xrightarrow{j} & \pi_n(J_n, J_{n-1}) \end{array}$$

where $ji = i_*$ is the map in (1.9). This actually shows that i_* is surjective. We now have the commutative diagram

$$\begin{tikzcd}
(J_n\rho)_{n+1} \arrow[r, "\partial q_n"] & \pi_n(J^n, J^{n-1}) \arrow[r, two heads, "i"] & \pi_n(J_n, J^{n-1}) \arrow[r, two heads, "j"] \arrow[dr, dashed, "x"] & \pi_n(J_n, J_{n-1}) \\
& (J\rho)_n \arrow[u, "\tau"', "\cong"] \arrow[rr, "\varepsilon_*"] & & (JG)_n \arrow[u, "\tau_n"'] \\
& & (J_{n-1}\rho)_n \arrow[ul, "i_{n-1}"] \arrow[uu, "i'q_{n-1}"'] &
\end{tikzcd}$$

By definition of ε it is clear that the compositions $\varepsilon_* \tau^{-1} \partial q_n$ and $\varepsilon_* i_{n-1}$ are trivial maps. Here we use the fact that $X^0 = *$ so that $(J_n\rho)_{n+1}$ and $(J_{n-1}\rho)_n$ do not contain generators which are products only of 1–cells in X. We know see that $\varepsilon_* \tau^{-1} \partial q_n = 0$ implies that there is a broken arrow x with $xi = \varepsilon_* \tau^{-1}$. Moreover x induces the inverse of τ_n since $xi'q_{n-1} = \varepsilon_* i_{n-1} = 0$. This completes the proof that τ on the right hand side of (1.9) is an isomorphism. □

The properties of $\varepsilon_* = J(\varepsilon)$ are obtained by the next lemma.

(1.10) Lemma: ε_* *in* (1.9) *induces the map*

$$\pi_n \varepsilon_* : \pi_n J\rho BG \longrightarrow \pi_n J(G)$$

which is an isomorphism for $n \leq 2$ *and surjective for* $n = 3$. *For* $G = \mathbb{Z}/2$ *the map* $\pi_3 \varepsilon_*$ *is not an isomorphism. Moreover there is a perfect group* G *for which* $\pi_3 \varepsilon_*$ *is not an isomorphism.*

This lemma is a consequence of the exact sequences in §3, see (3.6), (3.9) and (3.14).

We now study the crossed module $JG_2 \longrightarrow JG_1 = G$ with the tensor square $G \bar{\otimes} G$ of Brown-Loday [10] .

(1.11) Definition: Let G and B be groups.We say that a function $\Theta : G \times G \longrightarrow B$ is a *2–crossed morphism* if the following equation are satisfied ($x, y, z \in G$)

$$\begin{array}{lll} i) & (x+y)\,\Theta\, z - y\,\Theta\, z = -x\,\Theta\, y + x\,\Theta\,(z+y) \\ ii) & 0\,\Theta\, 0 = 0 \end{array}$$

Observe that, first taking $x=y=0$ and then, taking $y=z=0$ in $i)$ we get, using $ii)$

$$0\,\Theta\, x = x\,\Theta\, 0 = 0$$

(1.12) Proposition: *The function* $G \times G \longrightarrow JG_2$ *which carries* $(x\ ,\ y)$ *to the product* $x\,y$ *is the universal 2–crossed morphism for G.*

In §2 we shall see that the composition $G \times G \longrightarrow JG_2 \longrightarrow (JG_2)^{ab}$ is the universal 2–crossed homomorphism in the sense of (2.1) with $\varphi = ab\colon G \longrightarrow G^{ab}$.

Proof of (1.12): For JG_2 in (1.4) only reminds relation (3), which, in this case, can be written

$$x\,[z+y] = x\,y + (x\,z)^y$$
$$[x+y]\,z = (x\,z)^y + y\,z$$

These equalities imply relations $i)$ and $ii)$.

Now we can equivalently see JG_2 as a group generated by elements xy with $x, y \in G$ with relations

$$[x+y]\,z - y\,z = -x\,y + x\,[z+y]$$
$$0\,0 = 0$$

and define the operation of G on generators of JG_2 by

$$(x\,z)^y = -x\,y + x\,[z+y]\,;$$

as the equalities

$$(x\,z)^{y+y'} = ((x\,z)^y)^{y'} \quad \text{and} \quad (x\,z)^0 = x\,z$$

are consequences of this definition, this yields no further relation. □

We have the following examples of 2-crossed morphisms on G. First the commutator map

$$(\,,\,):\; G \times G \longrightarrow G$$

is a 2-crossed morphism which induces the homomorphism $d : JG_2 \longrightarrow G$ in JG. Moreover for the tensor product of Brown-Loday the function

$$\bar{\otimes}:\quad G \times G \longrightarrow G \bar{\otimes} G$$

is a 2-crossed morphism which induces the natural homomorphism $\bar{\otimes} : JG_2 \longrightarrow G \bar{\otimes} G$.

(1.13) Proposition: *The sequence*

$$JG_3 \xrightarrow{\;d\;} JG_2 \xrightarrow{\;\bar{\otimes}\;} G \bar{\otimes} G \longrightarrow 0$$

is exact . Moreover dJG_3 $=$ *kernel* $\bar{\otimes}$ *is generated as a normal subgroup by the relations*

$$(xy)^z \approx [x^z]\,[y^z]\;,\quad (x\,,y\,,z \in G\,).$$

Proof: The isomorphism

$$\frac{JG_2}{(xy)^z \approx [x^z][y^z]} \quad\cong\quad G \bar{\otimes} G$$

is given in [13].

The proof of the inclusion $dJG_3 \subset$ *kernel* $\bar{\otimes}$ is given in [11]. We translate it from left to right operations and commutators. We use the equality

$$(x \otimes y)^z \quad=\quad (x^z) \otimes (y^z)$$

in $G \bar{\otimes} G$. As $\quad x \otimes (z + y - z + z) \;=\; x \otimes z + (x \otimes (z + y - z))^z$

and $\quad (y + x - y + y) \otimes z \;=\; ((y + x - y) \otimes z)^y + y \otimes z$

we have (a) $x \otimes (z+y) = x \otimes z + (x^z) \otimes y$ and

(b) $(y+x) \otimes z = x \otimes (z^y) + y \otimes z$

Then, by (a) $(x \otimes y)^z = -x \otimes z + x \otimes (y+z) = -x \otimes z + x \otimes y + (x^y) \otimes z$

$= ((-x) \otimes z)^x + x \otimes y + (x^y) \otimes z = (-x) \otimes (z^x) + x \otimes y + (x^y) \otimes z$

$= x \otimes y + ((-x) \otimes z)^{x+(x,y)} + (x^y) \otimes z$ (here we use the second crossed modules property)

$= x \otimes y + ((-x)^{(x,y)}) \otimes (z^{x+(x,y)}) + (x^y) \otimes z$

$= x \otimes y + ((-x)^{-y+x+y}) \otimes (z^{x+(x,y)}) + (x^y) \otimes z$

We can write relation b) $\alpha \otimes (\gamma^\beta) = (\beta+\alpha) \otimes \gamma - \beta \otimes \gamma$

and take $\alpha = (-x)^{-y+x+y}$, $\beta = -y+x+y$, $\gamma = z$

Then we have $((-x)^{-y+x+y}) \otimes (z^{-y+x+y}) = (x,y) \otimes z - (x^y) \otimes z$

and $(x \otimes y)^z = x \otimes y + (x,y) \otimes z - (x^y) \otimes z + (x^y) \otimes z$

so that $(x,y) \otimes z = -x \otimes y + (x \otimes y)^z$

and $\bar{\otimes} d(x\,y\,z) = (x,y) \otimes z - (x \otimes y) + x \otimes y = 0$

thus we have $dJG_3 \subset kernel\,\bar{\otimes}$

Conversely we show that $[x^z][y^z]$ is congruent to $(x\ y)^z$ modulo dJG_3.

We have $[x^z][y^z] = [x+(x,z)]\,[y+(y,z)]$

$= (x\,[y+(y,z)])^{(x,z)} + [(x,z)]\,[y+(y,z)]$

$= -xz + x[y+(y,z)] + xz + [(x,z)][y+(y,z)]$

$= -xz + x\,[(y,z)] + (x\,y)^{(y,z)} + xz + [(x,z)]\,[(y,z)] + ([(x,z)]\,y)^{(y,z)}$

$= -xz + x\,[(y,z)] - yz + xy + yz + xz + [(x,z)]\,[(y,z)] - yz + [(x,z)]\,y + yz$

We have $[(x,z)]\,[(y,z)] = -d([(x,z)]\,y\,z) - (y\,z)^{(x,z)} + y\,z$

$= -d([(x,z)]\,y\,z) + (x\,z, y\,z)$

and $[(x,z)]\,y = d(x\,z\,y) - x\,z + (x\,z)y$

so that, taking congruences modulo dJG_3 we get

$[x^z][y^z] \equiv -xz + x\,[(y,z)] - yz + xy + yz + (x\,z)^y + yz$

Otherwise, using relation (4) in (1.4), $x\,[(y,z)] = x\,[-y-z+y+z]$

$= xz + (x\,y)^z + (x\,[-z])^{y+z} + (x\,[-y])^{-z+y+z}$

but $x\,[-z] = -(x\,z)^{-z}$ so that $(x\,[-z])^{y+z} = -(x\,z)^{y+(y,z)}$

$= -y\,z - (x\,z)^y + y\,z$

and $(x\,[-y])^{-z+y+z} = -(x\,y)^{(y,z)} = -y\,z - x\,y + y\,z$

so $x\,[(y,z)] = xz + (x\,y)^z - yz - (x\,z)^y - xy + yz$

thus we have $[x^z]\,[y^z] \equiv (x\ y)^z$ and the sequence is exact. □

(1.14) Corollary: *There are natural isomorphisms*

$$\pi_3\Sigma BG \quad \cong \quad \pi_2 J(G) \quad \cong \quad ker(G\,\bar{\otimes}\, G \longrightarrow G)$$

where the right hand side isomorphism is the commutator map.

Proof: The isomorphism on the left hand side is given by (1.3)(D) and (1.10), see also (3.5). The isomorphism on the right hand side is given by (1.13). In fact one has the following commutative diagram

$$\begin{tikzcd}
\pi_2(J_2BG, BG) \arrow[r, two heads] & \pi_2(JBG, BG) \arrow[r, "\partial"] & \pi_1 BG \\
JG_2 \arrow[u, "\cong"'] \arrow[r, two heads] & G\,\bar{\otimes}\, G \arrow[u, "\cong"'] \arrow[r] & G \arrow[u, "\cong"']
\end{tikzcd}$$

in which the vertical arrows are isomorphisms. □

This corollary was proved by Brown-Loday using different methods, see [10] proposition 4.12. In section 3 we shall give also some informations on the homotopy group $\pi_4\Sigma BG$.

There is a further description of $JG_2 \longrightarrow G$ by the following result of Gilbert-Higgins [13]. Consider the commutative diagram of groups

$$\begin{tikzcd}
G\Box G \arrow[r, dashed, "\varphi"] \arrow[d, "i"'] & JG_2 \arrow[d, "d"] \\
G*G \arrow[r, "\nabla"'] & G
\end{tikzcd}$$

where $G*G$ is the free product and where ∇ is the folding map. Moreover $G\Box G$ is the kernel of the projection map (p_1,p_2): $G*G \longrightarrow G\times G$

and i is the inclusion. The map i is a crossed module since the injection of a normal subgroup is a classical example of crossed module.

(1.15) Proposition: *There is a natural map φ such that d: $JG_2 \longrightarrow G$ is the G–crossed module induced by ∇.*

Proof: Let i_1 and i_2 be the two canonical injections: $G \longrightarrow G*G$. Then, by [17] (see also [18]) $G\Box G$ is a free group with basis the elements

$$x\Box z \quad = \quad -i_1x - i_2z + i_1x + i_2z \quad = \quad (i_1x, i_2z), \qquad x, z \in G - \{0\};$$

the group $G*G$ acting by conjugaison on $G\Box G$ we have, by commutators identities

$$\begin{aligned}(x\Box z)^{i_1y} &= [x+y]\Box z - y\Box z\\ (x\Box z)^{i_2y} &= -x\Box y + x\Box[z+y]\end{aligned}$$

For any $G*G$–equivarient homomorphism ψ with domain $G\Box G$ we have

$$\psi((x\Box z)^{i_1y}) \quad = \quad (\psi(x\Box z))^y \quad = \quad \psi((x\Box z)^{i_2y})$$

otherwise $\quad \psi([x+y]\Box z - y\Box z) = \psi(-x\Box y + x\Box[z+y])$

whence, relation *ii)* being clear, the map ψ factors throught the group JG_2. □

In the next section we give a simple description of the higher dimensional part of JG, namely $(JG)_n = I_nG$ for $n \geq 3$.

§ 2 The multi crossed algebra of a group

We consider multi crossed homomorphisms which generalise the notion of a crossed homomorphism introduced by J.H.C. Whitehead in [21]. The universal multi crossed homomorphisms yield an algebra I_*G which is actually a tensor algebra and which is isomorphic to the algebra $C(JG)$ of chains on JG.

(2.1) Definition: Let $\varphi\colon G \longrightarrow A$ be a homomophism between groups and let M be a (right) A-module. A function

$$f\colon \quad G^n = G\times\ldots\times G \longrightarrow M$$

is a *multi φ-crossed homomorphism* if

(1) $f(g_1,\ldots,g_i+g_i',\ldots,g_n) = f(g_1,\ldots,g_i,\ldots,g_n)^{\varphi(g_i')} + f(g_1,\ldots,g_i',\ldots,g_n)$

for $g_1,\ldots,g_n, g_i' \in G$, $1\le i \le n$. Let

(2) $$h_n : \; G^n \longrightarrow I_n(G,\varphi)$$

be the universal multi φ-crossed homomorphism. For any multi φ-crossed homomorphism $f\colon G^{n+m} \longrightarrow M$ we define a $\mathbb{Z}$-bilinear map

(3) $\check{f}\colon \quad I_n(G,\varphi)\times I_m(G,\varphi)\longrightarrow M$

by the multi φ-crossed homomorphism

$$\bar{f}\colon \quad G^n \longrightarrow Hom_{\mathbb{Z}}(I_m(G,\varphi), M)$$

which carries $g = (g_1,\ldots,g_n)$ to the A-module map $\bar{f}(g)$ which is given by the multi φ-crossed homomorphism $(g_{n+1},\ldots,g_{n+m}) \to f(g_1,\ldots,g_{n+m})$. Thus, taking $M = I_{n+m}(G,\varphi)$ and $f = h_{n+m}$ we define a multiplication

(4) $\mu\colon \quad I_n(G,\varphi)\times I_m(G,\varphi) \longrightarrow I_{n+m}(G,\varphi)$

The map $\check{f}$ is even $\mathbb{Z}[A]$–bilinear if A is abelian. Whence μ yields a $\mathbb{Z}$-algebra for $I_*(G,\varphi)$ which is a $\mathbb{Z}[A]$-algebra if A is abelian. For the abelianization homomorphism $ab : G \longrightarrow G^{ab}$ we get the $\mathbb{Z}[G^{ab}]$-algebra

$$(5) \qquad I_*(G) \quad = \quad I_*(G, ab) \quad , \qquad * \geq 0 ,$$

which we call the *multi φ–crossed algebra of the group G.* Here we set $I_0(G) = \mathbb{Z}[G^{ab}]$.

As a well known special case we get the augmentation ideal $IG = kernel\ \varepsilon : \mathbb{Z}[G] \longrightarrow \mathbb{Z}$ of the group ring $\mathbb{Z}[G]$, that is

$$(2.2)\ I_1(G, 1_G) \quad = \quad IG \quad \text{with } h_1(g) \quad = \quad -1 + g$$

compare for example [14]. The algebra $I_*(G, \varphi)$ above, however, seems not to be considered in the litterature. For $n = 2$ there is a connection of $I_2(G, \varphi)$ with biderivations studied by Papakyriakopoulos in [19], p 266. This algebra has the following properties.

(2.3) Lemma: *For homomorphisms $G' \xrightarrow{\psi} G \xrightarrow{\varphi} A$ there is a natural isomorphism of A–modules*

$$I_n(G', \varphi\psi) \quad = \quad I_n(G', \psi) \otimes_{\mathbb{Z}[G]} \varphi^* \mathbb{Z}[A]$$

in particular for $\psi = 1$ we get

$$I_n(G, \varphi) \quad = \quad I_n(G, 1_G) \otimes_{\mathbb{Z}[G]} \varphi^* \mathbb{Z}[A]$$

Proof: The A-modules $I_n(G', \varphi\psi)$ and $I_n(G', \psi) \otimes_{\mathbb{Z}[G]} \varphi^*\mathbb{Z}[A]$ solve the same universal problem. More precisely any multi $\varphi\psi$–crossed homomorphism $f: G'^n \longrightarrow M$ is as well a ψ–crossed homomorphism, so it factors through a G-linear map: $I_n(G', \psi) \longrightarrow M$ which extends to a A–linear map. □

Using (VI 6.2) in [14] we thus get by (2.2) and (2.3) the formula

$$I_1(G) = I_1(G, ab) = IG \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G^{ab}] = IG \otimes_{\mathbb{Z}[(G,G)]} \mathbb{Z} = \frac{IG}{I(G,G).IG}$$

where (G,G) is the commutator subgroup of G.

(2.4) Proposition: *The multi crossed algebra $I_*(G)$ of a group G is the $\mathbb{Z}[G^{ab}]$–tensor algebra generated by I_1G, that is $I_*G = T_R(I_1G)$ with $R = \mathbb{Z}[G^{ab}]$, see* (1.3)(*B*).

Proof: When the group A is abelian the multiplication μ induces an isomorphism

$$I_n(G, \varphi) \otimes_{\mathbb{Z}[A]} I_m(G, \varphi) \cong I_{n+m}(G, \varphi)$$

since these two A-modules solve the same universal problem. Whence we have

$$I_n(G, \varphi) \cong \otimes^n_{\mathbb{Z}[A]} I_1(G, \varphi)$$

thus, in the case where $\varphi = ab$, we have $I_*G = T_R(I_1G)$. □

(2.5) Example: If G is perfect we have

$$I_1G = \frac{IG}{IG.IG} = H_1G = 0$$

so that in this case $I_*G = 0$, see also (3.12). In case G is abelian we get $I_1G = IG$ so that then $I_*G = T_{\mathbb{Z}[G]}(IG)$ is the tensor algebra of the augmentation ideal.

(2.6) Definition: The multi crossed algebra I_*G is a differential graded $\mathbb{Z}[G^{ab}]$–algebra by the differential

$$d : I_1G \longrightarrow I_0G = \mathbb{Z}[G^{ab}]$$

given by $dh_1(g) = 1 - ab(g)$. Here we use (2.5) and the fact that $g \to 1 - ab(g)$ is an ab-crossed homomorphism. On products we set $d(ab) = (da)b + (-1)^{|a|}a\, db$. Clearly (I_*G, d) is a functor in G.

(2.7) Proposition: *The differential in* (2.6) *as well is determined by the multi ab–crossed homomorphism*

$$\delta = dh_n : G^n \longrightarrow I_nG \longrightarrow I_{n-1}G$$

with

$$\delta(g_1,\ldots,g_n) = \sum_{i=1}^{n}(-1)^{i-1}\left(h_{n-1}(g_1,\ldots,g_{i-1},g_{i+1},\ldots,g_n) - h_{n-1}(g_1,\ldots,g_{i-1},g_{i+1},\ldots,g_n)^{ab(g_i)}\right)$$

Moreover $\delta(g_1, g_2) = h_1(-g_1 - g_2 + g_1 + g_2)$ *for* $n = 2$.

Proof: By (2.4) we have $h_n(g_1,\ldots,g_n) = h_1(g_1) \otimes h_1(g_2) \otimes \ldots \otimes h_1(g_n)$ thus, by induction on the differential of a product, we obtain

$$\delta(g_1,\ldots,g_n) = \sum_{i=1}^{n}(-1)^{i-1}(h_1(g_1)\otimes h_1(g_2)\otimes\ldots\otimes h_1(g_{i-1})\otimes dh_1(g_i)\otimes h_1(g_{i+1})\otimes \ldots\otimes h_1(g_n))$$

then the formula for $dh_1(g_i)$ and the $\mathbb{Z}[A]$-algebra structure gives the result.

Now, as the map h_1 is a crossed homomorphism, we have

$$h_1(-g) = -h_1(g)^{ab(-g)}$$

so that we can write

$$\begin{aligned} h_1(-g_1 - g_2 + g_1 + g_2) &= h_1(-g_1)^{ab(-g_2+g_1+g_2)} + h_1(-g_2)^{ab(g_1+g_2)} + h_1(g_1)^{ab(g_2)} + h_1(g_2) \\ &= -h_1(g_1)^{ab(-g_1-g_2+g_1+g_2)} - h_1(g_2)^{ab(-g_2+g_1+g_2)} + h_1(g_1)^{ab(g_2)} + h_1(g_2) \\ &= -h_1(g_1) - h_1(g_2)^{ab(g_1)} + h_1(g_1)^{ab(g_2)} + h_1(g_2) \quad = \quad \delta(g_1, g_2) \quad \square \end{aligned}$$

We now describe the topological signifiance of the multi crossed algebra I_*G of a group G. For this we consider the universal covering $q: \hat{X} \longrightarrow X$ of a filtered space X_* with $\varinjlim X_* = X$. We obtain the filtered space $\hat{X}_*$ by $\hat{X}_n = q^{-1}X_n$. We now replace the crossed chain complex $\pi(X_*)$ in (1.1) by the chain complex of $\pi_1(X)$-modules $\mathscr{H}(X_*)$:

$$(2.8) \quad \ldots \longrightarrow H_3(\hat{X}_3, \hat{X}_2) \xrightarrow{\delta} H_2(\hat{X}_2, \hat{X}_1) \xrightarrow{\delta} H_1(\hat{X}_1, \hat{X}_0) \xrightarrow{\delta} H_0(\hat{X}_0)$$

given by integral relative homology groups. Whence the filtered space $J_*X, X \in CW_0^*$, yields in the same way the chain complex $\mathscr{H}(\hat{J}_*X)$ which is actually a $\mathbb{Z}[G^{ab}]$–algebra with $G = \pi_1 X$ by using the multiplication of JX which induces a unique basepoint preserving covering map

$$\hat{J}_*X \times \hat{J}_*X \;=\; (J_*X \times J_*X)\hat{} \;\longrightarrow\; \hat{J}_*X$$

of filtered spaces.

(2.9) Theorem: *Let G be a group with classifying space BG. Then there is a natural isomorphism of differential $\mathbb{Z}[G^{ab}]$–algebras*

$$\tau: \quad I_*G \;\cong\; \mathscr{H}\hat{J}_*BG$$

where I_*G is the multi crossed algebra of G.

We next compare the result with the corresponding result in (1.2). For this we observe first that the Hurewicz homomorphism yields a map $(n \geq 1)$

$$(2.10) \qquad h_n: \quad \pi_n(X_n, X_{n-1}) \cong \pi_n(\hat{X}_n, \hat{X}_{n-1}) \longrightarrow H_n(\hat{X}_n, \hat{X}_{n-1})$$

which gives a "chain map" $h: \pi\hat{X}_* \longrightarrow \mathscr{H}\hat{X}_*$. Moreover for any crossed chain complex ρ one has a natural map $h: \rho \longrightarrow C\rho$ which is the analogue of the map (2.10). For this we consider the commutative diagram

$$(2.11)\ldots\qquad \begin{array}{ccccccccc} \longrightarrow & \rho_3 & \longrightarrow & \rho_2 & \longrightarrow & \rho_1 & \overset{q}{\dashrightarrow} & \pi_1 \\ & \cong \Big\downarrow h_3 & & \Big\downarrow h_2 & & \Big\downarrow h_1 & & \\ \longrightarrow & (C\rho)_3 & \longrightarrow & (C\rho)_2 & \longrightarrow & (C\rho)_1 & \longrightarrow & (C\rho)_0 \end{array}$$

where q is the quotient map for $\pi_1 = \pi_1\rho$. We set $(C\rho)_0 = \mathbb{Z}[\pi_1]$, $(C\rho)_1 = I_1(\rho_1, q)$ with h_1 as in (2.1)(2) and $(C\rho)_2 = \rho_2^{ab}$ with $h_2 = ab$. For $n \geq 3$ the map h_n in (2.11) is the identity. The differential d in the bottom row is uniquely determined by the diagram and by the q–crossed homomorphism $\rho_1 \longrightarrow \mathbb{Z}[\pi_1]$, $x \to 1 - q(x)$. Let ***Chain***$_{\hat{\mathbb{Z}}}$ be the category of chain complexes over group rings, morphisms are pairs (φ, f) where φ is a homomorphism between groups and where f is a φ–equivariant homomorphism between modules, see [1]. The construction in (2.11) yields a functor

$$C: \quad \textbf{\textit{cross chain}} \longrightarrow \textbf{\textit{Chain}}_{\hat{\mathbb{Z}}}$$

which is studied in [21], [6] and in (VI.1.2) [1]. For a filtered space X with $X_0 = *$ one has a natural commutative diagram

$$(2.12)\qquad \begin{tikzcd} & \pi(X_*) \arrow[dl, "h"'] \arrow[dr, "h"] & \\ C\pi(X_*) \arrow[rr, "\lambda"] & & \mathscr{H}(X_*) \end{tikzcd}$$

where λ is induced by h in (2.10). It is a classical result of J.H.C. Whitehead [21] that λ is an isomorphism if $X_* = X^*$ is a CW-complex, moreover Brown-Higgins in (5.2) [6] show that λ is an isomorphism if X_* is a connected filtered space. For a CW-complex X with $X_0 = *$ for example the filtered space J_*X is connected so that we have an isomorphism

$$(2.13)\qquad \lambda : C\pi\, J_*X \cong \mathscr{H}\hat{J}_*X$$

We use this result in the following proof of (2.9).

(2.14) Proof of (2.9): We apply the functor C to the isomorphism τ in (1.2) and (1.9) so that we get isomorphisms

$$I_*G \overset{\alpha}{\cong} CJG \overset{C\tau}{\cong} C\pi J_*(BG) \overset{\lambda}{\cong} \mathcal{K}\hat{J}_*BG$$

The isomorphism α is a direct consequence of the definition of C and of the description of JG in (1.4). In fact, the multi crossed homomorphism

$$G^n \longrightarrow (CJG)_n\,, \qquad (g_1,\ldots,g_n) \to h(g_1,\ldots,g_n)$$

induces the isomorphism α. □

We now compare the homology of JG and I_*G, for this we use the natural map

$$(2.15) \qquad h\colon \quad JG \longrightarrow CJG \cong I_*G$$

given by (2.11) and (2.14).

(2.16) Theorem: *The natural map h in* (2.15) *induces an isomorphism $h:\pi_nJG \cong H_nI_*G$ for $n \geq 4$. Moreover one has the natural exact sequence*

$$0 \longrightarrow \pi_3JG \xrightarrow{h_*} H_3I_*G \longrightarrow \hat{\Gamma}_2J_*BG \longrightarrow \pi_2JG \xrightarrow{h_*} H_2I_*G \longrightarrow 0$$

where $\hat{\Gamma}_2J_*BG$ is the group defined in (3.2) *below.*

By definition of CJG it is clear that h_* is an isomorphism in degree ≥ 4 and injective in degree 3. The exact sequence of (2.16) is obtained in (3.7) below.

(2.17) Remark: Theorem (2.9) leads to a result of homological algebra which for example can be found in the book of Hilton-Stammbach

(VI, th6.3) [14]. For this we consider the homology sequence of the pair $(\hat{J}_1BG, \hat{J}_0BG) = (\hat{J}_1, \hat{J}_0)$. Since $J_0 = *$ we see $H_1\hat{J}_0 = 0$. Moreover since $\pi_1 JBG = G^{ab}$ we see that $\hat{J}_1$ is the G^{ab}–cover of $J_1 = BG$. Therefore we get $\pi_1\hat{J}_1 = (G, G)$ and whence $H_1\hat{J}_1 = (G, G)^{ab}$. Thus we have the following commutative diagram of exact sequences where the vertical arrows are isomorphisms

$$\begin{tikzcd}
0 \arrow[r] & (G,G)^{ab} \arrow[r] \arrow[d, "\cong"'] & IG\otimes_{\mathbb{Z}[G]}\mathbb{Z}[G^{ab}] \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z}[G^{ab}] \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & H_1\hat{J}_1 \arrow[r] & H_1(\hat{J}_1, \hat{J}_0) \arrow[r] & H_0\hat{J}_0 \arrow[r] & H_0\hat{J}_1 \arrow[r] & 0
\end{tikzcd}$$

Here we use (2.9) and (2.4).

§3 On the homology of the crossed tensor algebra $J(G)$.

In this section we embed the homology of the crossed tensor algebra $J(G)$ in an exact sequence which is an analogue of J.H.C. Whitehead 's certain exact sequence [20], see also [1]. This leads to applications concerning the homotopy groups of a suspended classifying space ΣBG.

For a filtered space X_* with $X_0 = *$ we define

$$\text{(3.1)} \qquad \Gamma_n X_* \;=\; image(\pi_n X_{n-1} \longrightarrow \pi_n X_n)$$

For the skeletal filtration X_* of a CW-complex X the group $\Gamma_n X = \Gamma_n X_*$ is the classical Γ-group of J.H.C. Whitehead in [20]. Moreover we define

$$\text{(3.2)} \qquad \hat{\Gamma}_n X_* \;=\; image(H_n \hat{X}_{n-1} \longrightarrow H_n \hat{X}_n)$$

where $\hat{X}_*$ as in (2.8). We now consider the filtered space J_*BG given by the James construction of the pointed classifying space BG. In this case we get the following natural commutative diagram; the top row in the diagram is the certain exact sequence of J.H.C. Whitehead applied to the space JBG.

$$\begin{tikzcd}
{} \arrow[r] & \Gamma_n(JBG) \arrow[r, "i"] \arrow[d, "i_*"] & \pi_{n+1}\Sigma BG \arrow[r, "h"] \arrow[d, "\cong"] & H_n(\hat{J}BG) \arrow[r, "b"] \arrow[d, "i_*"] & \Gamma_{n-1}(JBG) \arrow[d, "i_*"] \\
{} \arrow[r] & \Gamma_n(J_*BG) \arrow[r] \arrow[d, "h_*"] & \pi_n(JBG) \arrow[r] \arrow[d, "h_*"] & \pi_n(JG) \arrow[r] \arrow[d, "h_*"] & \Gamma_{n-1}(J_*BG) \arrow[d, "h_*"] \\
{} \arrow[r] & \hat{\Gamma}_n(J_*BG) \arrow[r] & H_n(\hat{J}BG) \arrow[r] & H_n(I_*G) \arrow[r] & \hat{\Gamma}_{n-1}(J_*BG)
\end{tikzcd}$$

(3.3) Proposition: *This diagram is well defined for $n \geq 2$ and the rows are long exact sequences.*

Proof: Let $BG \in CW_0^*$. Then $X = JBG$ is a CW-complex with the CW-filtration X^* and with the James filtration $X_* = J_*BG$. We have the filtered inclusion map $i: X^* \longrightarrow X_*$ since we assume $(BG)^0 = *$. Now we apply the certain exact sequence (III 10.7) [1] for $\mathcal{U} = X^*$ or $\mathcal{U} = X_*$ and we get the top row and the row in the middle respectively. Moreover we can apply (III 10.7) [1] in the category of chain complexes for $\mathcal{U} = C_*\hat{X}_*$ where C_* denotes the cellular chain complex. Then we get the bottom row of the diagram. The maps h_* and h are given by the Hurewicz map. □

We now consider the diagram above for low degrees, for this we first observe that one has the following formulas for the corresponding Γ-groups.

(3.4) Lemma:

$$
\begin{array}{lllllll}
\Gamma_1 JBG & = & \Gamma_1 J_*BG & = & \hat{\Gamma}_1 J_*BG & = & 0 \\
\Gamma_2 JBG & = & \Gamma_2 J_*BG & = & 0 & & \\
\Gamma_3 JBG & \cong & \Gamma \pi_3 \Sigma BG & & & &
\end{array}
$$

The natural isomorphism for $\Gamma_3 JBG$ follows by an old result of J.H.C. Whitehead which shows that for any CW-complex X one has $\Gamma_3 X = \Gamma \pi_2 X$, see [20]. The other equations in (3.4) are easily derived from the definitions. Using (3.4) we derive from (3.3) the following commutative diagram with exact rows in which $\twoheadrightarrow$ is a surjection and $\rightarrowtail$ an injection.

(3.5)

$$\begin{tikzcd}
{} \arrow[r] & \pi_4\Sigma BG \arrow[r, two heads] \arrow[d, "\cong"] & H_3\hat{J}BG \arrow[r] \arrow[d, "i_*"] & 0 \arrow[r] \arrow[d] & \pi_3\Sigma BG \arrow[r, "\cong"] \arrow[d, "\cong"] & H_2\hat{J}BG \arrow[d, "i_*"] \\
{} \arrow[r] & \pi_3 JBG \arrow[r, two heads] \arrow[d] & \pi_3 JG \arrow[r] \arrow[d, "h_*"] & 0 \arrow[r] \arrow[d] & \pi_2 JBG \arrow[r, "\cong"] \arrow[d, "\cong"] & \pi_2 JG \arrow[d, "h_*"] \\
{} \arrow[r] & H_3\hat{J}BG \arrow[r] & H_3 I_* G \arrow[r] & \hat{\Gamma}_2 J_* BG \arrow[r] & H_2\hat{J}BG \arrow[r, two heads] & H_2 I_* G
\end{tikzcd}$$

(3.6) **Proof of (1.10):** The map $\pi_n \varepsilon_*$, $n \geq 2$, can be identified with $i_*\colon H_n\hat{J}BG \longrightarrow \pi_n JG$ so that (1.10) follows immediately from (3.5) since i_* is an isomorphism for $n = 2$ and surjective for $n = 3$. □

(3.7) **Proof of (2.16):** The exact sequence (2.16) is a consequence of the bottom row of (3.5) since $\pi_3 JBG \longrightarrow \pi_3 JG$ is surjective. □

For $\hat{J}_n = \hat{J}_n BG$ we obtain the natural transformation ($n \geq 2$)

$$\alpha : H_n(G, \mathbb{Z}[G^{ab}]) \cong H_n \hat{J}_1 \longrightarrow H_n \hat{J}_{n-1} \twoheadrightarrow \hat{\Gamma}_n J_* BG$$

which is induced by the inclusion $\hat{J}_1 \subset \hat{J}_{n-1}$. One gets the isomorphism on the left hand side since $\hat{J}_1$ is the G^{ab}-cover of BG. Moreover one gets the

(3.8) **Lemma:** *There is an exact sequence*

$$H_3(G, \mathbb{Z}[G^{ab}]) \xrightarrow{\alpha} \hat{\Gamma}_3 J_* BG \longrightarrow P(G) \longrightarrow H_2(G, \mathbb{Z}[G^{ab}]) \xrightarrow{\alpha} \hat{\Gamma}_2 J_* BG \longrightarrow 0$$

where $P(G)$ *is the cokernel of the boundary* $\partial\colon H_4(\hat{J}_3, \hat{J}_2) \longrightarrow H_3(\hat{J}_2, \hat{J}_1)$.

The exact sequence is an easy consequence of the homology exact sequence for $(\hat{J}_3, \hat{J}_2, \hat{J}_1)$.

(3.9) Example: If G is abelian we see by (3.8) that $\hat{\Gamma}_2 J_* BG = 0$ and $\hat{\Gamma}_3 J_* BG = P(G)$. Therefore we get in this case by (3.3) and (3.5) the isomorphism

(1) $$H_2 \hat{J} BG \cong H_2 I_* G$$

and the exact sequence

(2) $$H_4 I_* G \longrightarrow P(G) \longrightarrow H_3 \hat{J} BG \longrightarrow H_3 I_* G \longrightarrow 0$$

Here $I_* G = (T_{\mathbb{Z}[G]}(IG), d)$ is a differential tensor algebra. If $G = \mathbb{Z}$ we have $P(\mathbb{Z}) = 0$. Moreover we have for $G = \mathbb{Z}$ the classifying space $BG = S^1$ so that by the well known homotopy equivalence $JS^1 \simeq S^1 \times J(S^2)$ one has the homotopy equivalence

$$\hat{J} B\mathbb{Z} = \hat{J} S^1 \simeq J(S^2)$$

Whence we get

(3) $$H_n(I_* \mathbb{Z}) \cong H_n \hat{J} B\mathbb{Z} = \begin{cases} \mathbb{Z} & \text{if n is even} \\ 0 & \text{otherwise} \end{cases}$$

Here the isomorphism on the left hand side is obtained by the bottom sequence in (3.3) since $\hat{\Gamma}_n J_* B\mathbb{Z} = 0$ by (3.2). It is not so easy to prove (3) directly by the formula for $(I_* G, d)$ in (2.6). For the group $G = \mathbb{Z}/2$ one has $IG \cong \mathbb{Z}$ generated by $x = 1 - [1]$ and with the action of G on $\mathbb{Z}$ by -1. One now can compute directly by the formula in (2.6)

(4) $$H_n(I_* \mathbb{Z}/2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2 & n \text{ even}, n > 0 \\ 0 & \text{n odd} \end{cases}$$

Since we know $\pi_4 \Sigma B\mathbb{Z}/2 = \mathbb{Z}/4$ by a result of Hennes [5] and $H_3\hat{J}B\mathbb{Z}/2 = \mathbb{Z}/2$ we see by (2) and (4) that $P(\mathbb{Z}/2) \neq 0$, see (3.8). This as well shows that for $G = \mathbb{Z}/2$ the map $\pi_3\varepsilon_*$ is not an isomorphism, see (1.10).

(3.10) Definition: Let $S(A,n)$ and $T(A,n)$ be the free graded *symmetric algebra* and *tensor algebra* respectively generated by the free abelian group A which is concentrated in degree n. Then $T(A,n)$ is a graded Lie algebra by setting

$$[x\,,y] = x\,y - (-1)^{|x||y|}\,y\,x$$

where the right hand side is defined via the multiplication for $x,y \in T(A,n)$. Let $L(A,\ n)$ be the *Lie subalgebra* generated by $A \subset T(A,\ n)$; this is a free graded Lie algebra.

(3.11) Theorem: *For a free group G with $A = G^{ab}$ one has natural isomorphisms*

$$H_n(I_*G) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ \Gamma A & n = 2 \\ L(A,1)_3 & n = 3 \end{cases}$$

Moreover there is a (non natural) isomorphism of graded free abelian groups

$$H_*(I_*G) \cong S(A,2) \otimes \bigotimes_{n \geq 2} S(L(A,2)_{2n}\,,n)$$

Proof: We can choose $BG = VS^1$ to be a one point union of 1-spheres. Therefore $J_*BG = (JBG)^*$ is the skeletal filtration and whence we get by (2.12) the isomorphism

$$(1) \quad \pi_n JG = H_n(I_*G) = H_n\hat{J}BG \quad , \quad n \geq 2.$$

Since $\pi_2 JG = \pi_3 \Sigma BG$ we get the isomorphism in (3.11) for $n = 2$, see also (3.4). Moreover we have the natural surjection (see (3.5))

$$(3) \qquad \pi_4\Sigma BG \quad = \quad \pi_3 JBG \quad \twoheadrightarrow \quad H_3\hat{J}BG$$

where $\pi_4\Sigma BG = \Gamma(A)\otimes\mathbb{Z}/2\oplus L(A,1)_3$, compare [3]. This surjection induces the isomorphism in (3.11) for $n = 3$. Finally we derive the formula for $H_*(I_*G)$ via (1) from the Hilton Milnor theorem. Let Q_n be a basis of $L(A,2)_{2n}$, $n \geq 1$. Then the Hilton Milnor theorem shows that one has a homotopy equivalence

$$(4) \qquad J(BG) \quad \approx \quad \Omega\Sigma BG \quad \approx \quad \mathcal{U}_1 \times \mathcal{U}_{\geq 2}$$

where $\quad \mathcal{U}_1 = \underset{x\in Q_1}{\times} J(S^1), \quad \mathcal{U}_{\geq 2} = \underset{n\geq 2}{\times}\ \underset{x\in Q_n}{\times} J(S^n)$

are products with the CW-topology. Since $\mathcal{U}_{\geq 2}$ is simply connected we see that

$$(5) \qquad \hat{J}(BG) \quad \approx \quad \hat{\mathcal{U}}_1 \times \mathcal{U}_{\geq 2}$$

where $\hat{\mathcal{U}}_1$ is the universal covering of $\mathcal{U}_1$ which by the argument in (3.9)(3) admits a homotopy equivalence

$$(6) \qquad \hat{\mathcal{U}}_1 \quad \approx \quad \underset{x\in Q_1}{\times} J(S^2)$$

Now one obtains the formula for H_*I_*G by (1), (5) and (6). $\square$

We finally consider perfect groups G for which we have the (+)–construction $BG \longrightarrow BG^+$ which induces an isomorphism in homology. Here BG^+ is 1-connected. Clearly the induced maps

$$(3.12) \qquad \Sigma BG \xrightarrow{\simeq} \Sigma BG^+ , \qquad JBG \xrightarrow{\simeq} JBG^+$$

are homotopy equivalences. This as well implies $I_nG = 0$ by use of (2.9), compare (2.5). Moreover one obtains by (3.12) easily the next result where $JBG = \hat{J}BG$ and where α is the map in (3.8).

(3.13) Proposition: *Let G be perfect. Then the map*

$$\alpha: \qquad H_n(G) \longrightarrow \hat{\Gamma}_n(J_*BG) \cong H_nJBG$$

is split injective for $n \geq 1$ and an isomorphism for $n \leq 3$. This implies $P(G) = 0$ by (3.8). *For $n = 4$ one has*

$$H_4JBG \cong H_4(G) \oplus H_2(G) \otimes H_2(G)$$

For the proof of (3.13) we use (3.12) and the well known homotopy equivalence $\Sigma JX \simeq \bigvee_{i\geq 1} \Sigma X^{(i)}$ where $X^{(i)}$ is the i-fold smash product.

(3.14) Example: Let G be a perfect group with $H_3G \neq 0$; such a perfect group exists since we can use the result in [16] on the existence of groups with prescribed homology. Since $I_*G = 0$ we get by (2.16) $\pi_3JG = 0$, but

$$\pi_3J\rho BG \cong H_3\hat{J}BG \cong \hat{\Gamma}_3J_*BG \cong H_3G \neq 0$$

Whence $\pi_3\varepsilon_*$ in (1.9) is not an isomorphism in this case.

Using (3.12) one obtains for a perfect group G the suspension operator

$$(3.15) \qquad \Sigma: \quad \pi_nBG^+ \longrightarrow \pi_{n+1}\Sigma BG^+ = \pi_{n+1}\Sigma BG$$

which by the Freudenthal suspension theorem is an isomorphism for $n = 2$ and surjective for $n = 3$. Moreover this operator is embedded in the following commutative diagram the rows of which are the certain exact sequences of J.H.C. Whitehead [20] for the spaces ΣBG, JBG^+ and JBG respectively. The bottom row coincides with the top row in (3.3) and (3.5).

$$
\begin{tikzcd}
{} \arrow[r] & \pi_5\Sigma BG \arrow[r] & H_4G \arrow[r] & (H_2G)\otimes\mathbb{Z}/2 \arrow[r] & \pi_4\Sigma BG \arrow[r, two heads] & H_3G \\
{} \arrow[r] & \pi_4BG^+ \arrow[r] \arrow[u, "\Sigma"'] \arrow[d, "\Sigma"] & H_4G \arrow[r] \arrow[u, equal] \arrow[d, "\alpha"] & \Gamma(H_2G) \arrow[r] \arrow[u, two heads, "\sigma"'] \arrow[d, equal] & \pi_3BG^+ \arrow[r, two heads] \arrow[u, two heads, "\Sigma"'] \arrow[d, two heads, "\Sigma"] & H_3G \arrow[u, equal] \arrow[d, equal] \\
{} \arrow[r] & \pi_5\Sigma BG \arrow[r] & H_4JBG \arrow[r] \arrow[d, equal] & \Gamma(H_2G) \arrow[r] & \pi_4\Sigma BG \arrow[r, two heads] & H_3G \\
& & H_4(G)\oplus H_2(G)\otimes H_2(G) & & &
\end{tikzcd}
$$

The map from the row for BG^+ to the top row is given by the functor Σ, compare the remark following (I.3.3) and (III.10.13) in [1]. Moreover the map to the bottom row is induced by the inclusion $BG^+ \subset J(BG)^+$.

The diagram for example shows that one has the exact sequence (G perfect)

$$H_2(G)\otimes H_2(G) \longrightarrow \pi_3(BG)^+ \xrightarrow{\Sigma} \pi_4\Sigma BG \longrightarrow 0$$

where $\pi_4\Sigma BG$ can be computed by the top row of the diagram. This seems to be a new estimation of $\pi_3(BG)^+$ in terms of the homology groups H_*G. The continuation of the diagram above uses the computation of the group Γ_4X which is described in [3], a further discussion of the operator Σ in (3.15) for $n = 4$ will appear elsewhere.

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