DEGENERACY LOCI FORMULAS FOR MORPHISMS WITH SYMMETRIES

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0. The goal of the present note is to give new explicit formulas for the fundamental classes of degeneracy loci associated with the following vector bundles homomorphisms.

For a given pair $B \subset A$ of vector bundles, we denote by $B \vee A$ (resp. $B \wedge A$) the image of the canonical composition $B \otimes A \to A \otimes A \twoheadrightarrow S^2 A$ (resp. $B \otimes A \to A \otimes A \twoheadrightarrow \Lambda^2(A)$).

Let now $F^\vee \subset E^\vee$ be two vector bundles of ranks f and e over a scheme X over a field K. Let $\varphi : F \to E^\vee$ be a morphism coming from a section of $F^\vee \vee E^\vee$ (resp. $F^\vee \wedge E^\vee$). Suppose that an integer $0 \leqslant r \leqslant f$ is given. In this note, we describe the fundamental classes of the loci $D_r(\varphi) = \{x \in X : \mathit{rank}\ \varphi(x) \leqslant r\}$ with the help of some explicitly given polynomials in the Chern classes of E and F.

When $E = F$, our formulas specialize to the ones given in [J-L-P], [H-T] and [P1].

When $F = \bigoplus_{i=1}^{f} \mathcal{O}(n_i)$, $E = \bigoplus_{i=1}^{f} \mathcal{O}(n_i) + \bigoplus_{j=1}^{e-f} \mathcal{O}(m_j)$ are two vector bundles over a projective space, some formulas for the degree of the above degeneracy loci were established by Bottaso in [Bo] by different tools. The present paper offers a modern version and a "compact" generalization of the results of [Bo].

The method used follows the second author's paper [P1] and relies on the technique of "constructions with a nontrivial generic fiber" invented in Section 2 of loc. cit. This method is recalled in Theorem 1, where, in fact, an improvement of [P1, Section 2] is presented.

1. The most popular method to compute the fundamental class of a subscheme $D \subset X$ tries to find a scheme X' mapping properly to X, on which one has a locus Z that maps birationally onto D and for which one can compute its class $[Z]$. Usually this is because $[Z]$ is the zero locus of a section of some bundle whose rank is equal to $\mathit{codim}_{X'} Z$ so the class $[Z]$ is evaluated to be the top Chern class of the bundle. For example, this pattern was used in [J-L-P] and many other papers (see [F]).

To compute the fundamental classes of subvarieties, one can also use appropriate geometric constructions with a nontrivial generic fibre. This method was invented in [P1] in order to give a short proof of the formulas from [J-L-P] and [H-T], and is summarized and improved in the following simple theorem. In this theorem, we may assume that the Chow groups have rational coefficients. We follow [F] for all needed notions and notation from intersection theory.

Theorem 1. *Let D be an irreducible (closed) subscheme of a scheme X. Let $\pi : \mathbf{G} \to X$ be a proper morphism of schemes and W be a (closed) subscheme of $\mathbf{G}$ such that $\pi(W) = D$. We have the following two instances:*
(i) Suppose that $\mathbf{G}$ is smooth. Assume that there exists

$$\mathbf{g} \in A_{dim\mathbf{G}+dimD-dimW}(\mathbf{G})$$

and a point x in the smooth locus of D such that in $A_*(\mathbf{G}_x)$, where $\mathbf{G}_x$ is the fibre of π over x, one has:

$$i_x^*(\mathbf{g}) \cdot [W_x] = [point].$$

Here, W_x is the fibre of W over x and $i_x : \mathbf{G}_x \hookrightarrow \mathbf{G}$ is the inclusion. Then the following equality holds in $A_*(X)$:

$$[D] = \pi_*\big(\mathbf{g} \cdot [W]\big).$$

(ii) Suppose that there exists a family of vector bundles $\{E^{(\alpha)}\}$ on $\mathbf{G}$ and $\mathbf{g} = P\big(\{c.(E^{(\alpha)})\}\big)$ - a homogeneous polynomial of degree $dimW - dimD$ in the Chern classes of $\{E^{(\alpha)}\}$ (deg $c_i(E^{(\alpha)}) = i$) with rational coefficients, such that in $A_*(\mathbf{G}_x)$,

$$P\big(\{c.(i_x^* E^{(\alpha)})\}\big) \cap [W_x] = [point],$$

where x, $\mathbf{G}_x$, W_x and i_x are as above. Then the following equality holds in $A_*(X)$:

$$[D] = \pi_*\big(\mathbf{g} \cap [W]\big).$$

Proof. (i) Using a standard dimension argument, we can replace, in the assertion, D by its smooth part, i.e., we can assume D is smooth. Write $\mathbf{G}_D = \mathbf{G} \times_X D$, $W_D = W \times_X D$, $\eta : \mathbf{G}_D \to D$ the projection induced by π, and $k : \mathbf{G}_D \to \mathbf{G}$ – the inclusion. Then, the assertion is a consequence of the following identity in $A_*(D)$:

$$\eta_*\big(k^*(\mathbf{g}) \cdot [W_D]\big) = [D].$$

To prove the latter equation, we first remark that the assumptions imply

$$\eta_*\big(k^*(\mathbf{g}) \cdot [W_D]\big) = m[D],$$

where $m \in \mathbb{Z}$. Let x be a point in D and consider the fibre square:

$$\begin{array}{ccc} \mathbf{G}_x & \xhookrightarrow{j} & \mathbf{G}_D \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \eta} \\ \{x\} & \xhookrightarrow{i} & D \end{array}$$

Using the assumptions on $\mathbf{g}$ and [F, Theorem 6.2], we have

$$\begin{aligned} i^*\eta_*\big(k^*(\mathbf{g}) \cdot [W_D]\big) &= p_*\Big(j^*\big(k^*(\mathbf{g}) \cdot [W_D]\big)\Big) \\ &= p_*\big(i_x^*(\mathbf{g}) \cdot [W_x]\big) = p_*\big([point]\big) = [point] \end{aligned}$$

This implies $m = 1$ and assertion (i) is proved.

The proof of (ii) is essentially the same. $\square$

Using this method, we now generalize the formulas from [J-L-P], [H-T] and [P1] to a wider class of degeneracy loci including, in the case of matrices of homogeneous forms, those studied in [Bo].

2. We follow the notation from Section 0. In the definition of the loci $D_r(\varphi)$ in the "$\wedge$-case", we assume r to be even. A proper scheme structure on $D_r(\varphi)$ is defined with the help of Schubert subschemes in Lagrangian (resp. orthogonal) Grassmannians. Let $V \subset U$ be vector spaces of dimensions f and e respectively. Let $X = \mathit{Spec}\ S^\bullet(V \vee U)$ (resp. $X = \mathit{Spec}\ S^\bullet(V \wedge U)$). In this situation, there exists a tautological morphism $\varphi : F = V_X \to (E = U_X)^\vee$. For such a φ, $D_r(\varphi)$ is the restriction to the "opposite big cell", of an appropriate Schubert variety in the Lagrangian (resp. orthogonal) Grassmannian of f-dimensional isotropic subspaces in K^{2e}. Hence, by results of [DC-L], $D_r(\varphi)$ is irreducible, normal and Cohen-Macaulay; moreover its codimension c equals

$$(e-f)(f-r)+(f-r)(f-r+1)/2 \text{ (resp. } (e-f)(f-r)+(f-r)(f-r-1)/2).$$

In general, $D_r(\varphi)$ can be obtained similarly as the scheme theoretic preimage of an open subset of a Schubert variety of a Lagrangian (resp. orthogonal) Grassmannian bundle. We omit the details of this fairly standard procedure. In the "$\vee$-case" the reduced scheme structure on $D_r(\varphi)$ is defined by the ideal generated locally by $(r+1)$-order minors of φ.

We now describe a certain geometric construction associated with φ. Let p be a natural number such that $2p \leqslant f$ and let $\pi_F : \mathbb{G}_F = G_{f-p}(F) \to X$, $\pi_E : \mathbb{G}_E = G_{e-p}(E) \to X$ be the Grassmannian bundles parametrizing $(f-p)$-subbundles of F and $(e-p)$-subbundles of E respectively. Consider the fibre product

$$\pi : \mathbb{G} = \mathbb{G}_F \times_X \mathbb{G}_E \to X\,.$$

Let $0 \to R_F \to F_{\mathbb{G}_F} \to Q_F \to 0$ and $0 \to R_E \to E_{\mathbb{G}_E} \to Q_E \to 0$ be two tautological sequences of vector bundles on $\mathbb{G}_F$ and $\mathbb{G}_E$. In $\mathbb{G}$, we have the "incidence" subvariety $\mathcal{I}$ parametrizing the points where $(R_F)_{\mathbb{G}} \subset (R_E)_{\mathbb{G}}$. We define a locus $W \subset \mathcal{I} \subset \mathbb{G}$ as the subscheme of zeros of the composite morphism:

$$(R_F)_{\mathcal{I}} \hookrightarrow F_{\mathcal{I}} \xrightarrow{\varphi_{\mathcal{I}}} E^\vee_{\mathcal{I}} \twoheadrightarrow (R^\vee_E)_{\mathcal{I}}.$$

Let $D = D_{2p}(\varphi)$. We have $\pi(W) = D$. Indeed, if $w \in W$ then the matrix of φ over $\pi(w)$ has the upper left $(e-p) \times (f-p)$ rectangle consisting of zeros and every $(2p+1)$-order minor of such a matrix vanishes (use the Laplace expansion w.r.t. the first $p+1$ columns).

We want now to get information about the generic fibre $W_x =: \mathcal{F}$ of $\pi|_W$ like that in Theorem 1. Let $V \subset U$ be vector spaces of dimensions f and e respectively. Let $\phi : V \to U^\vee$ be a morphism coming from a section of $V^\vee \vee U^\vee$ (resp. $V^\vee \wedge U^\vee$).

Since we are interested in a regular point ϕ in the space of homomorphisms of rank $\leqslant 2p$, we assume that rank $\phi = 2p$. Then the fibre $\mathcal{F}$ over ϕ is identified with

$$\{(L,M) \in I \mid \ p_{M^\vee} \circ \phi \circ i_L = 0\},$$

where $I = \{(L,M) \in G_{f-p}(V) \times G_{e-p}(U) \mid L \subset M\}$ and $i_L : L \hookrightarrow V$ and $p_{M^\vee} : U^\vee \twoheadrightarrow M^\vee$ are the canonical maps. We claim that the dimension of $\mathcal{F}$ is equal to $p(p-1)/2$ (resp. $p(p+1)/2$) and thus it does not depend on f and e). This can be calculated by applying our construction to X being the affine space $Spec\, S^\bullet(V \vee U)$ (resp. $Spec\, S^\bullet(V \wedge U)$), endowed with the tautological homomorphism. In this case, looking at local coordinates, one easily checks that $W \subset \mathcal{I}$ is a locally complete intersection of codimension equal to the rank of $R_F \vee R_E$ (resp. $R_F \wedge R_E$). Thus knowing the dimension of W, we get $dim\,\mathcal{F} = dim\,W - dim\,D_r(\varphi) = p(p-1)/2$ (resp. $dim\,\mathcal{F} = p(p+1)/2$).

The following very simple fact is helpful to find the class g satisfying the requirements of Theorem 1.

Lemma 2. *Let $i : Y' \hookrightarrow Y$ be a closed embedding of smooth varieties, let $X \subset Y$ and $X' \subset Y'$ be two subvarieties such that $i(X') \subset X$ and $dim\,X' = dim\,X$. Assume that an element $z \in A^*(X)$ satisfies $[X'] \cdot i^*(z) = [point]$ in $A^*(Y')$. Then, $[X] \cdot z = [point]$ in $A^*(Y)$.*

Indeed, we have $i_*[X'] = [X]$, and by the projection formula we infer $[point] = i_*\big([X'] \cdot i^*(z)\big) = i_*[X'] \cdot z = [X] \cdot z$, as claimed.

In the next proposition and in the following, we use the notation "$s_I(E)$" for the Schur polynomial of a vector bundle E associated with a sequence of integers I, as defined in [P1,2]. In general, we refer the reader to [P2] for all unexplained here notions an notation concerning partitions and Schur polynomials. In particular, by ρ_p we understand the partition $(p, p-1, ..., 1)$.

Proposition 3. *The class $\mathbf{g} = 2^{-p} s_{\rho_{p-1}}\big((R_E^\vee)_{\mathcal{I}}\big)$ (resp. $\mathbf{g} = s_{\rho_p}\big((R_E^\vee)_{\mathcal{I}}\big)$) satisfies the assumption of Theorem 1(ii), with $\mathcal{I}$ playing the role of* $\mathbf{G}$.

Proof. We use the above description of the generic fibre $\mathcal{F}$ as well as the above notation. Moreover, let R denote the tautological rank $(e-p)$ bundle on $G_{e-p}(U)$.
1) Assume first that $e = f = 2p$ so $V = U$ and the corresponding bilinear form is nondegenerate. Then $[\mathcal{F}]$ is evaluated as the top Chern class of the bundle $S^2 R^\vee$ (resp. $\Lambda^2(R^\vee)$). We get by [L] (see also [M, p.48])

$$[\mathcal{F}] = 2^p s_{\rho_p}(R^\vee) \qquad (\text{resp. } [\mathcal{F}] = s_{\rho_{p-1}}(R^\vee)\ ).$$

The assertion now follows by taking the dual Schubert cycles in the Grassmannian $G_p(U)$ (see [F, Chap.14]).
2) Let now $2p < e = f$ (so again $V = U$), and let $U' \subset U$ be an inclusion of vector spaces of dimensions $2p$ and e, respectively. Assume that U is endowed with a symmetric (resp. antisymmetric) form ϕ of rank $2p$ such that the form $\phi|_{U'}$ is nondegenerate. We now use the lemma with the following data: $Y' = G_p(U')$

and $Y = G_{e-p}(U)$; $i : G_p(U') \hookrightarrow G_{e-p}(U)$ being defined by $L \mapsto L \oplus A$, where $U = U' \oplus A$. Moreover, X and X' are the generic fibres under consideration and $z = 2^{-p} s_{\rho_p}(R^\vee)$ (resp. $z = s_{\rho_{p-1}}(R^\vee)$). Then part 1) and the lemma yield the desired assertion.

3) Finally, suppose that $f < e$ and let $U = V \oplus B$, where $dim\ B = e - f$. We now apply the lemma to the following embedding:

$$i : \big(Y' = G_{f-p}(V)\big) \hookrightarrow \big(Y = I\big)$$

where $i(L) = (L, L \oplus B)$. Moreover, X and X' are the generic fibres under consideration and $z = 2^{-p} s_{\rho_p}(R_I^\vee)$ (resp. $z = s_{\rho_{p-1}}(R_I^\vee)$). Then part 2) and the lemma yield the desired result. □

3. We need the following algebraic identity, where $c_{top}(A)$ denotes the top Chern class of a bundle A.

Proposition 4. *If rank $E = e$ and rank $F = f$, then, with $n = e - f$,*

$$c_{top}(F \vee E) = 2^f s_{(e,e-1,\dots,n+2,n+1)}\big(F - z(E - F)\big)$$

$$\text{and} \qquad c_{top}(F \wedge E) = s_{(e-1,e-2,\dots,n+1,n)}\big(F - z(E - F)\big),$$

where z is a (formal) element of rank 1 in the corresponding λ-ring, specialized here with $z = -1$. More explicitly, one has

$$c_{top}(F \vee E) = 2^f \sum_I s_{(e,e-1,\dots,n+2,n+1)/I}(E - F)\ s_{I^\sim}(F),$$

where the sum runs over all $I \subset (e, e-1, \dots, n+2, n+1)$, and a similar expansion holds for $c_{top}(F \wedge E)$.

Proof. We give here the proof of the proposition for the bundle $F \vee E$. By the splitting principle, a (more general) question concerning the total Chern class of $F \vee E$ leads to the calculation of the product:

$$\prod_{i \leqslant j}(1 + a_i + a_j) \prod_{i,j}(1 + a_i + b_j),$$

where, formally, $c(F) = \prod_i (1 + a_i)$ and $c(E/F) = \prod_j (1 + b_j)$. Using a well-known formula for the resultant (see, e.g., [M, p.59]) and the already quoted formula from [L], this, in turn, is equal to

$$2^{-N} \prod_{i \leqslant j} [(1 + 2a_i) + (1 + 2a_j)] \prod_{i,j} [(1 + 2a_i) + (1 + 2b_j)]$$
$$= 2^{-N+f} s_{\rho_f}(A^+)\ s_{(n)^f}(A^+ - zB^+)|_{z=-1}\ ,$$

where $N = rk\, F \vee E$, $A^+ = (1+2a_1, 1+2a_2, \dots, 1+2a_f)$, $B^+ = (1+2b_1, \dots, 1+2b_n)$ and z is a (formal) element of rank 1 in an appropriate λ-ring. By the factorization formula (see, e.g. [M, p.59], we can rewrite the latter expression as:

$$2^{-N+f} s_{(n+f,n+f-1,\dots,n+1)}(A^+ - zB^+)|_{z=-1},$$

the top-degree component of which gives the desired expression:

$$2^f s_{(e,e-1,\dots,n+1)}(A - zB)|_{z=-1}.$$

A computation for the bundle $F \wedge E$ is quite similar. □

4. We are now ready to perform the main computation.

Proposition 5. *The following equality holds in $A_*(X)$:*

$$\pi_*\Big(c_{top}(R_F^\vee \vee R_E^\vee) \cdot c_{top}(R_F^\vee \otimes Q_E) \cdot 2^{-p} s_{\rho_{p-1}}(R_E^\vee) \cap [\mathbb{G}]\Big)$$
$$= 2^{f-2p} s_{(e-2p,e-2p-1,\dots,n+2,n+1)}\big(F^\vee - z(E^\vee - F^\vee)\big)\big|_{z=-1} \cap [X],$$

and respectively

$$\pi_*\Big(c_{top}(R_F^\vee \wedge R_E^\vee) \cdot c_{top}(R_F^\vee \otimes Q_E) \cdot s_{\rho_p}(R_E^\vee) \cap [\mathbb{G}]\Big)$$
$$= s_{(e-2p-1,e-2p-2,\dots,n+1,n)}\big(F^\vee - z(E^\vee - F^\vee)\big)\big|_{z=-1} \cap [X],$$

where $n = e - f$ and z has the same meaning as in Proposition 4.

Proof. We treat the first case. We have by Proposition 4,

$$\pi_*\Big(c_{top}(R_F^\vee \vee R_E^\vee) \cdot c_{top}(R_F^\vee \otimes Q_E) \cdot 2^{-p} s_{\rho_{p-1}}(R_E^\vee) \cap [\mathbb{G}]\Big)$$
$$= 2^{f-2p} \pi_*\Big(\big[s_{(e-p,e-p-1,\dots,n+2,n+1)}\big(R_F^\vee - z(R_E^\vee - R_F^\vee)\big)$$
$$\cdot s_{(f-p)^p}(Q_E - R_F) \cdot s_{\rho_{p-1}}(R_E^\vee)\big]_{z=-1} \cap [\mathbb{G}]\Big).$$

In the sequel we will denote the partition $(e-p, e-p-1, \dots, n+2, n+1)$ by T. Now using the addition/linearity formula, duality formula (see [M, p.72 and p.90]) and the following equality: $[R_E^\vee] - [R_F^\vee] = [E^\vee] - [F^\vee]$ in the Grothendieck group of $\mathcal{I}$ (where here, and in the following, we omit the pulback indices, for brevity), we rewrite the latter expression in the form:

$$2^{f-2p} \pi_*\Big(\big[\sum_I s_{T/I}(E^\vee - F^\vee)\, s_{I^\sim}(R_F^\vee)\, s_{(p)^{f-p}}(R_F^\vee - Q_E^\vee)\, s_{\rho_{p-1}}(R_E^\vee)\big] \cap [\mathbb{G}]\Big)$$

$$= 2^{f-2p} \sum_I s_{T/I}(E^\vee - F^\vee) \cap (\pi_E)_*\Big(\big[s_{\rho_{p-1}} R_E^\vee \cdot (\pi_F \times 1)_* s_{(p)^{f-p}+I^\sim}(R_F^\vee - Q_E^\vee)\big] \cap [\mathbb{G}_E]\Big)$$

$$= 2^{f-2p} \sum_I s_{T/I}(E^\vee - F^\vee) \cap (\pi_E)_*\Big(\big[s_{\rho_{p-1}}(R_E^\vee)\, s_{I^\sim}(F^\vee - Q_E^\vee)\big] \cap [\mathbb{G}_E]\Big),$$

where we have used the factorization formula quoted above and, e.g., [P2, Proposition 1.3] w.r.t. π_F. The latter expression can be rewritten with the help of the addition/linearity formula (quoted above) as follows:

$$2^{f-2p}\sum_I\sum_J s_{T/I}(E^\vee - F^\vee)\, s_{I^\sim/J}(F^\vee)\cap(\pi_E)_*\Big(\big[s_{\rho_{p-1}}(R_E^\vee)\cdot s_J(-Q_E^\vee)\big]\cap[\mathbb{G}_E]\Big).$$

Now, using [J-L-P, Proposition 1], we see that, in the above sum, there is only one partition J giving a non zero contribution while applying $(\pi_E)_*$: this is the partition $J=(p)^{e-p}/\rho_{p-1}=(p,\dots,p,p-1,\dots,1)$ ("p" occurs $e-2p+1$ times). Hence this sum equals:

$$2^{f-2p}\sum_I s_{(e-p,e-p-1,\dots,n+1)/I}(E^\vee-F^\vee)\, s_{\big(I/(e-p,e-p-1,\dots,e-2p+1)\big)^\sim}(F^\vee)\cap[X]$$
$$=2^{f-2p}\sum_K s_{(e-2p,e-2p-1,\dots,n+1)/K}(E^\vee-F^\vee)\, s_{K^\sim}(F^\vee)\cap[X],$$

where the latter sum runs over all $K\subset(e-2p,e-2p-1,\dots,n+1)$. (Partitions I indexing the former sum are related to partitions $K=(k_1,k_2,\dots)$ indexing the latter via the equality $I=(e-p,e-p-1,\dots,e-2p+1,k_1,k_2,\dots)$.) The latter expression is rewritten in the form:

$$2^{f-2p}s_{(e-2p,e-2p-1,\dots,n+2,n+1)}\big(F^\vee - z(E^\vee-F^\vee)\big)\big|_{z=-1}\cap[X],$$

as desired.

The second case is treated in an analogous way. $\square$

5. Here comes the main result.

Theorem 6. *If X is a pure-dimensional Cohen Macaulay scheme and $D_r(\varphi)$ is of expected pure codimension c or empty, then, in the "$\vee$-case", one has the equality:*

$$[D_r(\varphi)]=2^{f-r}s_{(e-r,e-r-1,\dots,n+2,n+1)}\big(F^\vee - z(E^\vee-F^\vee)\big)\big|_{z=-1}\cap[X],$$

and, in the "$\wedge$-case", the following equality holds:

$$[D_r(\varphi)]=s_{(e-r-1,e-r-2,\dots,n+1,n)}\big(F^\vee - z(E^\vee-F^\vee)\big)\big|_{z=-1}\cap[X].$$

Proof. We pass to the "generic case". For a given morphism $\varphi: F\to E^\vee$ of one of the two considered types, we define $\overline{X}=:$ *Spec* $S^\bullet(F\vee E)$ (resp. $\overline{X}=:$ *Spec* $S^\bullet(F\wedge E)$). Observe that φ induces a section $s: X\to\overline{X}$. On the other hand, there exists the tautological bundle homomorphism $\overline{\varphi}:\mathbb{F}\to\mathbb{E}^\vee$ where $\mathbb{F}=F_{\overline{X}}$, $\mathbb{E}=E_{\overline{X}}$ such that $s^*(\overline{\varphi})=\varphi$. If X is Cohen-Macaulay, then so is $D_r(\overline{\varphi})$ (cf. [DC-L]). Hence, if $D_r(\varphi)$ is of pure codimension c in X, then by [F, Sect.6 and 7], we get $[D_r(\varphi)]=s^*[D_r(\overline{\varphi})]$. Now for $D_r(\overline{\varphi})\subset\overline{X}$ and $r=2p$ we can apply Theorem 1 to the above W; the role of $\mathbf{G}$ is now played by $\mathcal{I}$ and the class $\mathbf{g}$ is given in Proposition 3. Applying the formula of Theorem 1 leads to the calculation performed in Proposition 5 where, however, we have "shifted" the computation from the bundle $\mathcal{I} \to X$ to the bundle $\pi : \mathbb{G} \to X$. This proves the theorem for even values of r.

We compute now the class of $D_r(\varphi)$ with an odd r (in the "$\vee$-case"). Namely, consider the following morphism $\varphi' = \varphi \oplus \mathbf{1} : F \oplus \mathbf{1} \to (E \oplus \mathbf{1})^\vee$ of vector bundles on X. Then the ideals defined by the minors of φ and φ' of respective orders $2p-1$ and $2p$ are equal. In particular, the codimension of $D_{2p}(\varphi')$ is expected. Hence

$$\begin{aligned}[D_{2p-1}(\varphi)] &= [D_{2p}(\varphi')] \\ &= 2^{f+1-2p} s_{(e+1-2p, e+1-2p-1, \ldots, n+1)}\Big((F^\vee \oplus \mathbf{1}) - z(E^\vee - F^\vee)\Big)\Big|_{z=-1} \cap [X] \\ &= 2^{f-(2p-1)} s_{(e-(2p-1), e-2p, \ldots, n+1)}\Big(F^\vee - z(E^\vee - F^\vee)\Big)\Big|_{z=-1} \cap [X]\end{aligned}$$

by the addition/linearity formula quoted above.

The proof of the theorem is complete. □

Remark 7. (Revision and corrigenda to [P1].) Theorem 1 gives an improvement of [P1, Sect.2]. The class "$\mathbf{g}$" in Theorem 1 corresponds to the class "$\mathbf{F}^d$" in [P1, Proposition 2.1]. The assumptions on $\mathbf{g}$ in Theorem 1 straighten an unprecise expression "the Poincaré dual of" from 5^0 in [P1, Proposition 2.1]. As a matter of fact, it was proved in [P1, Sect.3] that the classes $\mathbf{F}^d$ choosen in loc.cit., satisfy the assumptions imposed on the class $\mathbf{g}$ in Theorem 1; thus the computation in loc.cit. is complete. Moreover, the reference "Lemma 9 in [10]" on p.196 should be replaced by "[2, Sect.6 and 7]".

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