Non-Archimedean L-Functions

Associated with Hilbert Modular Forms

by

A.A.Panchishkin

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

und

Moskauer Staatsuniversität
Mechanisch-Mathematische
Fakultät
119899 Moskau

MPI/89- 54

Non-Archimedean L-Functions
Associated with Hilbert Modular Forms

by

A.A.Panchishkin

Content

Introduction

Zeta- functions can be attached as certain Euler products to various objects such as diophantine equations,representations of Galois groups, modular forms etc. Deep interrelations between these objects discovered in last decades are based on identities for the corresponding zeta functions. They all presumably fit into a general concept of Langlands L- functions associated with automorphic representations of a reductive group G over a number field F. From this point of view the study of arithmetic properties of these zeta functions is becoming especially important.

The theory of non-Archimedean zeta-functions originates in the work of Kubota and Leopoldt [Ku-Le] containing p-adic interpolation of the special values of Riemann zeta-function $\zeta(s)$ at negative integers. Their construction turned out to by equivalent to classical Kummer congruences for the Bernoulli numbers and was used by Iwasawa [Iw] for the description of class groups of cyclotomic fields. Since then the class of functions admitting p-adic analogues has gradually extended. The theory of modular symbols (due to Mazur and Manin, see [Man1]-[Man5], [Maz-SD]) provided a non-Archimedean construction of functions, which correspond to the case of the group $G = \mathrm{GL}_2$ over $F = \mathbf{Q}$. Several authors (including Deligne, Ribet, N.M.Katz, Kurčanov and others, see [De-Ri], [Ka1]-[Ka3], [Kurč1] -[Kurč3], [Sho], [V1], [V2]) investigated this problem for the case $G = \mathrm{GL}_1$ and GL_2 over totally real fields and fields of CM-type. But the case of more general reductive groups remained unclear until the mid-eighties although important complex analytic properties of the Langlands L-functions had been proved. In recent years a general approach to consruction of non-Archimedean L-functions associated with various classes of automorphic forms was developed, in particular, for the case of symplectic groups of even degree over $F = \mathbf{Q}$ and the group $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ over a totally real field F.

In this paper we consider the case of the group $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ over a totally real field F, and we use the Rankin-Selberg method for obtaining both complex-valued and p-adic distributions as certain integrals involving cusp forms and Eisenstein series.

The first chapter contains an exposition of some basic properties of p-adic analytic functions, p-adic measures and their Mellin transforms (see [Ko2]).

In the second chapter we construct non-Archimedean convolutions of two Hilbert modular forms of different (scalar) weight. The exposition here is provided with some basic facts about Hilbert modular forms. Now let p be a prime number and S a finite set of primes containing p. We briefly discuss convolutions of Hilbert modular forms and their S -adic analogues; they correspond to the case of the group $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ over a totally real field F and have the following form

$$L(s, \mathbf{f}, \mathbf{g}) = \sum_{\mathfrak{n}} C(\mathfrak{n}, \mathbf{f})\, C(\mathfrak{n}, \mathbf{g}) \mathcal{N}(\mathfrak{n})^{-s},$$

where $\mathbf{f}$, $\mathbf{g}$ are Hilbert automorphic forms of 'holomorphic type' over F , $C(\mathfrak{n}, \mathbf{f})$, $C(\mathfrak{n}, \mathbf{g})$ their normalized Fourier coefficients (enumerated by integral ideals $\mathfrak{n}$ of the ring of integers $\mathcal{O}_F \subset F$). We consider the functions $\mathbf{f}$, $\mathbf{g}$ as being defined on the adelized group $G_{\mathbf{A}} = \mathrm{GL}_2(\mathbf{A}_F)$,where $\mathbf{A}_F$ the ring of adèles of F and we assume that $\mathbf{f}$ is a primitive cusp form of the scalar weight $k \geq 2$ of ĉonductor $\mathfrak{c}(\mathbf{f}) \subset \mathcal{O}_F$ with the charachter ψ and $\mathbf{g}$ a primitive cusp form of weight $l < k$ of conductor $\mathfrak{c}(\mathbf{g})$ with the character

ω, $(\psi, \omega : \mathbf{A}_F^\times \to \mathbf{C}^\times$ being Hecke characters of finite order). The non-Archimedean construction is based on the algebraicity properties of the special values of the function $L(s, \mathbf{f}, \mathbf{g})$ at the points $s = l, \cdots, k-1$ up to some constant involving the Petersson inner product $\langle \mathbf{f},\ \mathbf{f} \rangle$ of the automorphic form $\mathbf{f}$ [Shi6]. Our theorem about non-Archimedean interpolation is equivalent to certain generalized Kummer congruences for these special values. We need some more notations for the precise formulation of the result. Let ψ^*, ω^* be the ideal group characters of F associated with ψ, ω and let

$$L_{\mathfrak{c}}(s, \psi\omega) = \sum_{\mathfrak{n}+\mathfrak{c}=\mathcal{O}_F} \psi^*(\mathfrak{n})\omega^*(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}+\mathfrak{c}=\mathcal{O}_F} (1 - \psi^*(\mathfrak{p})\omega^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s})^{-1}$$

be the correspoding Hecke L-function with $\mathfrak{c} = \mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g})$. We now define the normalized zeta function

$$\Psi(s, \mathbf{f}, \mathbf{g}) = \gamma_n(s) L_{\mathfrak{c}}(2s + 2 - k - l, \psi\omega) L(s, \mathbf{f}, \mathbf{g})$$

where $n = [F : \mathbf{Q}]$ is the degree of F,

$$\gamma(s) = (2\pi)^{-2ns}\Gamma(s)^n\Gamma(s + 1 - l)^n$$

being the gamma-factor. Then the function $\Psi(s, \mathbf{f}, \mathbf{g})$ admits a holomorphic analytic continuation over the whole comlex plane and satisfies certain functional equation [Ja]. Put $\Omega(\mathbf{f}) = \langle \mathbf{f},\ \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}$, then the number

$$\frac{\Psi(l + r, \mathbf{f}, \mathbf{g})}{(2\pi i)^{n(1-l)}\Omega(\mathbf{f})}$$

is *algebraic* for all integers r with the condition $0 \le r \le k - l - 1$.

For the non-Archimedean construction we introduce the S-adic completion

$$\mathcal{O}_S = \prod_{q \in S} (\mathcal{O}_F \otimes \mathbf{Z}_q) = \prod_{\mathfrak{p} | q \in S} \mathcal{O}_{\mathfrak{p}}$$

of the ring $\mathcal{O}_F$. Put

$$S_F = \{\mathfrak{p} \mid \mathfrak{p} \ \ divides \ \ q \in S\}$$

and let $\mathrm{Gal}_S = \mathrm{Gal}(F(S)/F)$ be the Galois group of the maximal extension of F unramified outside S and ∞.

We take in this case the p-adic analytic Lie group

$$\mathcal{X}_S = \mathrm{Hom}_{\mathrm{contin}}(\mathrm{Gal}_S, \mathbf{C}_p^\times)$$

consisting of all continuouos p-adic characters of the Galois group Gal_S as a set on which our non-Archimedean L-functions will be defined. Elements of finite order $\chi \in \mathcal{X}_S$ can be obviously identified with such Hecke characters of finite order, whose conductors are divisible only by prime divisors belonging to S_F, via the decomposition

$$\chi : \mathbf{A}_F^\times \overset{\text{class field theory}}{\longrightarrow} \mathrm{Gal}_S \to \overline{\mathbf{Q}}^\times \xrightarrow{i_p} \mathbf{C}_p^\times.$$

Let us denote by the same letter χ both Hecke characters and the corresponding elements of $\mathcal{X}_S$. There is a natural homomorphism

$$\mathcal{N}: \mathrm{Gal}_S \to \mathrm{Gal}(\mathbf{Q}(S)/\mathbf{Q}) \cong \mathbf{Z}_S^\times = \prod_{q\in S} \mathbf{Z}_q^\times,$$

defined by the restriction of Galois automorphisms from $F(S)$ to $\mathbf{Q}(S)$, and we denote by $\mathcal{N} x_p$ composition of this homomorphism with the natural projection $\prod_q \mathbf{Z}_q^\times \to \mathbf{Z}_p^\times$ and the inclusion $\mathbf{Z}_p^\times \subset \mathbf{C}_p^\times$.

Again our essential assumption is that that the cusp form $\mathbf{f}$ is *p-ordinary*, i.e. for the fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ and for all $\mathfrak{p} \mid p$ there exists such a root $\alpha(\mathfrak{p})$ of Hecke $\mathfrak{p}$-polynomial of $\mathbf{f}$ that $\mid i_p(\alpha(\mathfrak{p})) \mid_p = 1$. We then fix such roots $\alpha(\mathfrak{p})$ and extend the definition of $\alpha(\mathfrak{m})$ to all integral ideals $\mathfrak{m} \subset \mathcal{O}_F$ by multiplicativity.

Theorem *(On non-Archimedean convolutions of Hilbert modular forms)*
Under the above notations and asumptions there exists a bounded $\mathbf{C}_p$-analytic function $\Psi_S : \mathcal{X}_S \to \mathbf{C}_p$ uniquely defined by the condition: for each Hecke character of finite order $\chi \in \mathcal{X}_S^{\mathrm{tors}}$ holds the following equality holds:

$$\mathcal{X}_S(\chi) = i_p\left[D_F^{2l}\omega(\mathfrak{m})\frac{\tau(\chi)^2\mathcal{N}\,\mathfrak{m}^{l-1}}{\alpha(\mathfrak{m})^2}\frac{\Psi(l, \mathbf{f}, \mathbf{g}^\rho(\bar{\chi}))}{(-2\pi i)^{n(1-l)}\langle \mathbf{f},\ \mathbf{f}\rangle}\right],$$

where D_F is the discriminant of F, $\tau(\chi)$ being the Gauss sum of χ, and $\mathbf{g}^\rho(\chi)$ the cusp form obtained from $\mathbf{g}$ by complex conjugation of its Fourier coefficients and by twisting it then with the character χ.

This result is also valid for the special values $\Psi(l+r, \mathbf{f}, \mathbf{g})$ with $r = 1, \cdots k-l$, if we replace $\chi \in \mathcal{X}_S$ by $\chi \mathcal{N} x_p^r \in \mathcal{X}_S$.

Recently this construction was extended by Mi Ving Quang (Moscow University) to the *non-p-ordinary*,i.e. *supersingular* case, when $\mid i_p(\alpha(\mathfrak{p}) \mid_p < 1$ for all $\mathfrak{p} \mid p$ (at least when $F = \mathbf{Q}$).In this situation the functions Ψ_S are also uniquely defined by the condition that they have only a prescribed logarithmic growth on $\mathcal{X}_S$.

To conclude with, we note that the Rankin-Selberg method was remarkably generalized by Rallis and Piatetskii-Shapiro [Ra-PSh] to other classical groups, and we hope that our p-adic construction can also be generalized to the corresponding automorphic L-functions.

Chapter 1. Non-Archimedean analytic functions, measures and distributions.

In this chapter we give an exposition of some standard facts from the theory of continuous and analytic functions over a non-archimeadian local field. We start by recalling the definitions and notations concerning p-adic and S-adic numbers. Then we discuss the theory of continuous p-adic functions and their p-adic interpolation, and also the basic properties of p-adic analytic functions. In §3 we introduce distributions and measures and give a general criterion for the existence of a non-Archimedean measure with given values of integrals of functions belonging to certain dense family ("generalized Kummer congruences"). The next §4 is devoted to a description of the algebra of bounded measures in terms of their non-Archimedean Mellin transforms (Iwasawa isomorphism). The chapter is completed with an exposition of a general construction of measures, attached to rather arbitrary Euler products. This construction provides a generalization of measures first introduced by Yu.I.Manin [Man4], B.Mazur and H.P.F.Swinnerton–Dyer [Maz-SD]. Our construction [Pa5], [Pa9] was already successesfully used in several problems concerning the p-adic analytic continuation of Dirichlet series [Ar], [Co-Sch], [Sch].

§1. p–adic numbers and the Tate field

1.1. Let p be a prime number, $\mathbf{Q}_p$ the field of p-adic numbers, i.e. the completion of the feld of rational numbers $\mathbf{Q}$ with respect to the p-adic metric, given by the p-adic valuation

$$|\cdot|_p : \mathbf{Q} \to \mathbf{R}_{\geq 0} = \{x \in \mathbf{R} | x \geq 0\}$$
$$|a/b|_p = p^{\mathrm{ord}_p - \mathrm{ord}_p a}, \quad |0|_p = 0,$$

where $\mathrm{ord}_p a$ is the highest power of p dividing the integer a. The function $|\cdot|_p$ is multiplicative, since

$$\mathrm{ord}_p(xy) = \mathrm{ord}_p x + \mathrm{ord}_p y, \tag{1.1}$$

and satisfies the non-Archimedean property

$$|x+y|_p \leq \max(|x|_p, |y|_p). \tag{1.2}$$

If K is a finite algebraic extension of $\mathbf{Q}_p$ then K is generated over $\mathbf{Q}_p$ by a primitive element $\alpha \in K$, so that α is a root of an irreducible polynimial of degree $d = [K : \mathbf{Q}_p]$,

$$f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbf{Q}_p[x].$$

The valuation $|\cdot|_p$ admits a unique extention to K defined by

$$|\beta|_p = (|\mathcal{N}_{K/\mathbf{Q}_p}(\beta)|_p)^{1/d}, \tag{1.3}$$

where $\mathcal{N}_{K/\mathbf{Q}_p}(\beta) \in \mathbf{Q}_p$ is the algebraic norm of an element $\beta \in K$. The formula (1.3) defines a unique extension of $|\cdot|_p$ to the algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, which satisfies (1.2) (see[Kol]). The fuction ord_p is then also extended to $\overline{\mathbf{Q}}_p$ by $\mathrm{ord}_p\alpha = \log_p |\alpha|_p$. The formula (1.3) implies that $\mathrm{ord}_p K^\times$ is an additive subgroup in $\frac{1}{d}\mathbf{Z}$, hence $\mathrm{ord}_p K^\times = \frac{1}{e}\mathbf{Z}$

for a certain positive integer e dividing d which is called *the index of ramification* of the extension $K/\mathbf{Q}_p$.

Put

$$\mathcal{O}_K = \{x \in K \,|\, |x|_p \le 1\} \quad M_K = \{x \in K \,|\, |x|_p < 1\} \tag{1.4}$$

then M_K is the maximal ideal in $\mathcal{O}_K$ and the residue field $\mathcal{O}_K/M_K$ is a finite extension of degree f of $\mathbf{F}_p$ and there is the relation $d = e \cdot f$, in which f is called *the inertial degree* of the extension. For each $x \in \mathcal{O}_K$ its *Teichmüller representative* is defined by

$$\omega(x) = \lim_{n\to\infty} x^{p^{fn}}, \quad \omega(x) \equiv x \,(\mathrm{mod}\, M_K), \tag{1.5}$$

and satisfies the equation

$$\omega(x)^{p^f} = \omega(x).$$

The map ω provides a homomorphism of the group of invertible elements

$$\mathcal{O}_K^\times = \mathcal{O}_K \backslash M_K = \{x \in K \,|\, |x|_p = 1\}$$

of $\mathcal{O}_K$ onto the group of roots of unity of degree $p^f - 1$ in K, denoted by μ_{p^f-1}, and the isomorphism

$$(O_K/M_K)^\times \xrightarrow{\sim} \mu_{p^f-1} \subset \mathcal{O}_K^\times. \tag{1.6}$$

For example, if $e = 1$ then the extension K is called *unramified*. In that case $f = d$ and the Teichmüller representatives generate K over $\mathbf{Q}_p$, therefore

$$K = \mathbf{Q}_p(1^{1/N}), \quad N = p^d - 1.$$

On the other hand, if $e = d$ then the extension K is called *totally ramified*. For example, if ζ is a primitive root of unity of degree p^n, then $\mathbf{Q}_p(\zeta)$ is totally ramified of degree $d = p^n - p^{n-1}$, and we have that

$$\mathrm{ord}_p(\zeta - 1) = \frac{1}{p^n - p^{n-1}}. \tag{1.7}$$

1.2. For a field K with the non-Archimedean valuation $|\cdot|_p$ let us define

$$D_a(r) = D_a(r; K) = \{x \in K \,|\, |x - a|_p \le r\}, \tag{1.8}$$

$$D_a(r^-) = D_a(r^-; K) = \{x \in K \,|\, |x - a|_p < r\} \tag{1.9}$$

("closed" and "open" discs of the radius r with the center at the point $a \in K$, $r \ge 0$). Then we have that $D_b(r) = D_a(r)$ for any $b \in D_a(r)$ and $D_b(r^-) = D_a(r^-)$ for any $b \in D_a(r^-)$. Note that in topological sense both discs (1.8) and (1.9) are open and closed subsets of the topological field K.

The important property of the field $\mathbf{Q}_p$ and of its finite extensions is that they are locally compact, so that the discs (1.8) and (1.9) are compact. The disc

$$\mathbf{Z}_p = D_0(1; \mathbf{Q}_p) = \{x \in \mathbf{Q}_p \,|\, |x| \le 1\}$$

is a compact topological ring (the ring of p-adic integers), which is isomorphic to the projective limit of residue rings $\mathbf{Z}/p^n\mathbf{Z}_p$ with respect to the homomorphisms of reduction modulo p^n:

$$\mathbf{Z}_p = \varprojlim_n \mathbf{Z}/p^n\mathbf{Z}_p,$$

Analogously, we have that

$$\mathcal{O}_K = \varprojlim_n \mathcal{O}_K/M_K^n,$$

and

$$\mathbf{Z}_p^\times = \varprojlim_n (\mathbf{Z}/p^n\mathbf{Z}_p)^\times, \mathcal{O}_K^\times = \varprojlim_n (\mathcal{O}_K/M_K^n)^\times$$

(the projective limit of groups).

1.3. The structure of the multiplicative group $\mathbf{Q}_p^\times$ and K. Put $\nu = 1$ for $p > 2$ and $\nu = 1$ for $p = 2$, and define

$$U = U_p = \{x \in |x \equiv 1 \bmod p^\nu\}. \tag{1.10}$$

Then there is an isomorphism $U \xrightarrow{\sim} \mathbf{Z}_p$ of the multiplicative group U_p and of the additive group $\mathbf{Z}_p$ which is provided by combining the natural homomorphism

$$U \xrightarrow{\sim} \varprojlim_n U/U^{p^n}$$

and by special isomorphisms

$$\alpha_{p^n} : U/U^{p^n} \xrightarrow{\sim} \mathbf{Z}/p^n\mathbf{Z},$$

given by

$$\alpha_{p^n}((1+p^n)^a) = a \bmod p^n \quad (a \in \mathbf{Z}). \tag{1.11}$$

one easily checks that (1.11) is well definied and gives the desired isomorphism. Therefore, the group U is a topological cyclic group, and $1+p^n$ can be taken as its generator. There are the following decompositions

$$\mathbf{Q}_p = p^{\mathbf{Z}} \times \mathbf{Z}_p^\times, \quad \mathbf{Z}_p^\times \cong (\mathbf{Z}/p^\nu\mathbf{Z})^\times \times U. \tag{1.12}$$

Analogously, if $[K : \mathbf{Q}_p] = d$, then

$$K^\times = \pi^{\mathbf{Z}} \times \mathcal{O}_K^\times, \quad \mathcal{O}_K^\times \cong (\mathcal{O}_K/M_K)^\times \times U_K,$$

where π is a generator of the principial ideal $M_K = \pi\mathcal{O}_K$ (i.e. any element $\pi \in K^\times$ with $\mathrm{ord}_p\pi = 1/e$),

$$U_K = \{x \in \mathcal{O}_K^\times | |x-1|_p < 1\} = D_1(1^-; K).$$

The structure of the group U_K is then described as a direct product of d copies of the additive group $\mathbf{Z}_p$ and a finite group consisting of all p-primary roots of unity contained in K (see [Ca-F], [Kol]).

1.4. The S-adic numbers. Let S be a finite set of prime numbers.For a positive integer M denote by $S(M)$ its support, i.e. the set of all prime numbers dividing M. Let us consider the projective limit

$$\mathbf{Z}_S = \varprojlim_{M,\, S(M)\subset S} \mathbf{Z}/M\mathbf{Z},$$

which is taken over all positive integers M with the condition $S(M) \subset S$ with respect to the homomorphisms of reduction. then it follows from the Chinese Remainder Theorem ([Se1], [Weil]) that

$$\mathbf{Z}_S \cong \prod_{q\in S} \mathbf{Z}_q.$$

Put

$$\mathbf{Q}_S = \prod_{q\in S} \mathbf{Q}_q, \quad M_0 = \prod_{q\in S} q.$$

1.5. The Tate field. For the purposes of analysis it is convenient to embed $\mathbf{Q}_p$ into a bigger field, which is already complete both in the topological and in algebraic sense. This field is constructed as completion $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$ of an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ with respect to its single valuation with the condition $|p|_p = \frac{1}{p}$. The proof of the fact $\mathbf{C}_p$ is algebraically closed is not difficult and is based on the Krasner's lemma (see [Ko1], [Ko2], [Wa]).

We will use the notations

$$\mathcal{O}_p = \{x \in \mathbf{C}_p \mid |x|_p \leq 1\}, \quad M_p = \{x \in \mathbf{C}_p \mid |x|_p < 1\}. \tag{1.13}$$

Note that the $\mathcal{O}_p$ and M_p are no longer compact , and therefore the field $\mathbf{C}_p$ is not locally compact. We also have that

$$\mathcal{O}_p/M_p = \overline{\mathbf{F}}_p \tag{1.14}$$

is the algebraic closure of $\mathbf{F}_p$.

§2. Continuous and analytic functions overe a non-Archimedean field

2.1. Let K be a closed subfield of the Tate field $\mathbf{C}_p$. For a subset $W \subset K$ we consider continuous functions $f : W \to \mathbf{C}_p$. The standard examples of continuous functions are provided by polynomials, by rational functions (at points, where they are finite), and also by locally constant functions. If W is compact then for any continuous function $f : W \to \mathbf{C}_p$ and for any $\varepsilon > 0$ there exists a polynomial $h(x) \in \mathbf{C}_p[x]$, such that

$$|f(x) - h(x)|_p < \varepsilon \ for \ all \ x \in W.$$

If $f(W) \subset L$ for a closed subfield L of $\mathbf{C}_p$ then $h(x)$ can be chosen so that $h(x) \in L[x]$ (see [Ko2], [Wa]).

Interesting examples of continuous p-adic functions are provided by interpolation of functions, defined on certain subsets , such as $W = \mathbf{Z}$ or $\mathbf{N}$ with $K = \mathbf{Q}_p$.

Let f be any function on non-positive integers with values in $\mathbf{Q}_p$ or in some (complete) $\mathbf{Q}_p$-Banach space. In order to continue $f(x)$ to all $x \in \mathbf{Z}_p$ we can use the interpolation polynomials

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}.$$

Then we have that $\binom{x}{n}$ is a polynomial of degree n of x, and for $x \in \mathbf{Z}, x \geq 0$ we get a binomial coefficient. If $x \in \mathbf{Z}_p$ then x is close (in p-adic topology) to a positive integer, hence the value of $\binom{x}{n}$ is also close to an integer, therefore $\binom{x}{n} \in \mathbf{Z}_p$.

The classical theorem of Mahler says that any continuous function $f : \mathbf{Z}_p \to \mathbf{Q}_p$ can be written in the form (see [Wa], p.99):

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \tag{2.1}$$

with $a_n \to 0$ (p-adically) for $n \to \infty$. For a function $f(x)$ defined for $x \in \mathbf{Z},\ x \geq 0$ one can write formally

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n},$$

where the coefficients can be found from the following system of linear equations

$$f(n) = \sum_{m=0}^{n} a_m \binom{n}{m},$$

that is

$$a_m = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(i).$$

The series for $f(x)$ is always reduced to a finite sum for all $x \in \mathbf{Z},\ x \geq 0$. If $a_n \to 0$ then this series is convergent for all $x \in \mathbf{Z}_p$. As was noticed above, the inverse statement is also valid ("Mahler's criterion"). If convergence of a_n to zero is so fast that the series defining the coefficients of the x-expansion of $f(x)$ also converge, then $f(x)$ can be continued to *an analytical function*, see 2.2 below. Unfortunately, for an arbitrary sequence a_n with $a_n \to 0$ the attempt to use (2.1) for continuation of $f(x)$ out of the subset $\mathbf{Z}_p$ of $\mathbf{C}_p$ may fail. However, in the sequel we mostly consider analytic functions, for which such a continuation is provided by summation of the series defining this function.

2.2. Analytic functions and power series. (see [Ko2], p.13). The well known criterion of convergence of a series $\sum_{n=0}^{\infty} a_n$ is that the partial sums $\sum_{N \leq n \leq M} a_n$ are small for large N, M with $M > N$. In view of the non-Archimedean property (1.2) in $\mathbf{C}_p$ this occures if and only if $|a_n| \to 0$ or $\mathrm{ord}_p a_n \to \infty$ for $n \to \infty$. Therefore convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ depends only on $|x|_p$ but not on the precise value of x, hence there is no "conditional convergence" in this case. Thus, for any power series

$\sum_{n\geq 0} a_n x^n$ we can define its radius of convergence r such that only one of the following holds:

$$\sum_{n=0}^{\infty} a_n x^n \ \ converges \iff x \in D_0(r^-) \tag{2.2}$$

$$\sum_{n=0}^{\infty} a_n x^n \ \ converges \iff x \in D_0(r) \tag{2.3}$$

An example of the first alternative is $\sum_{n=0}^{\infty} x^n$, where (2.2) is satisfied with $r = 1$, and an example for the second is

$$\sum_{n=0}^{\infty} p^n x^{p^n - 1};$$

here the condition (2.3) is satisfied also with $r = 1$.

The important examples of analytic functions are $\exp(x)$ and $\log(x)$, which are given as the power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n,$$

and we have that

$$\exp(x) \ converges \ on \ D_0(p^{-1/(p-1)-}), \ r = p^{-1/(p-1)}, \tag{2.5}$$

$$\log(1+x) \ converges \ on \ D_0(1^-), \ r = 1. \tag{2.6}$$

(see [Ko2], [B-Šaf]), so that $\exp(x)$ converges in a disc smaller than the unit disc, and $\log(1+x)$ has better convergence that $\exp(x)$.

Since the identity

$$\log(xy) = \log(x) + \log(y) \tag{2.7}$$

holds as a formal power series identity, it follows that (2.7) holds in $\mathbf{C}_p$ as long as

$$|x-1|_p < 1, \quad and \quad |y-1|_p < 1$$

. in particular, since $|\zeta - 1|_p < 1$ for ζ any p^n-root of unity, we can obviously to apply (2.7) to conclude that $\log \zeta = 0$. Also we have that for all $x \in D_0(p^{-1/(p-1)-})$ the following identities hold

$$\exp(\log(1+x)) = 1 + x, \quad \log(\exp(x)) = x,$$

which are deduced from the corresponding properties of the formal series and can be used for establishing isomorphisms beween certain additive and multiplicative subgroups in $\mathbf{C}_p$ and $\mathbf{C}_p^\times$: for $U = D_1(p^{\nu-}; \mathbf{Q}_p)$ with ν as in 1.6 there are the isomorphisms

$$\exp : p^{\nu+n}\mathbf{Z}_p \xrightarrow{\sim} U^{p^n} \tag{2.8}$$

where $n \geq 0$.

2.3. Theorem *(on analyticity of interpolation series). Let $r < p^{-1/(p-1)} < 1$ and*

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \tag{2.1}$$

be a series with the condition $|a_n|_p \le Mp^n$ for some $M > 0$. Then $f(x)$ is expresible as a certain power series whose radius of convergence is not less than $R = (rp^{1/(p-1)})^{-1} > 1$ (see [Wa], p.53).

As an example consider the function $\langle a\rangle^x$ which is defined for $a \in \mathbf{Z}_p$ by means of the decomposition $a = \omega(a)\langle a\rangle$ where $\omega(a)$ is the Teichmüller representative of a for $p > 2$ and $\omega(a) = \pm 1$ for $p = 2$ with $\omega(a) \equiv a(\mathrm{mod}\,4)$, and the exponentiation is given by the binomial formula

$$\langle a\rangle^x = (1 + \langle a\rangle - 1)^x = \sum_{n=0}^{\infty} \binom{x}{n}(\langle a\rangle - 1)^n. \tag{2.9}$$

Since $|\langle a\rangle - 1|_p \le p^\nu$ we may put in the above theorem $r = p^{-\nu}$ and get that the function $\langle a\rangle^x$ is a power series in x with the radius of convergence not less than $p^{\nu-1/(p-1)} > 1$, and the following equality holds:

$$\exp(x\log\langle a\rangle) = \sum_{n=0}^{\infty} \binom{x}{n}(\langle a\rangle - 1)^n, \tag{2.10}$$

because both parts of of (2.10) are analytic in x and coincide for $x \in \mathbf{N}$.

2.4. Newton polygons (see [Ko2], p.21). The Newton polygon M_f for a power series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \in \mathbf{C}_p[[x]]$$

is defined as the convex hull of the points $(n, \mathrm{ord}_p a_n)$ (where we agree to take $\mathrm{ord}_p 0 = \infty$).

It is not hard to prove the following

Proposition. *If a segment of M_f has slope λ and horizontal length N (i.e., it extends from $(n, \mathrm{ord}_p a_n)$ to $(n+N, \lambda N + \mathrm{ord}_p a_n)$ then f has precisely N roots r_n with $\mathrm{ord}_p a_n = -\lambda$ (counting multiplicity).*

The following thorem is the p-adic analog of the Weierstrass Preparation Theorem (see [Ko2], p.21)

2.5. Theorem. *Let*

$$f(x) = a_m x^m + \cdots \in \mathbf{C}_p[[x]],\ a_m \ne 0$$

be a power series which converges on $D_0(p^\lambda; \mathbf{C}_p)$. let $(N, \mathrm{ord}_p a_N)$ be the right endpoint of the last segment of M_f with slope $\le \lambda$, if this N is infinite. Otherwise, there will be a last infinitely long segment of slope λ and only finitely many points $(n, \mathrm{ord}_p a_n)$ on that segment. In that case let N be the last such n. Then there exists a unique polynomial $h(x)$ of the form

$$b_m x^m + b_{m+1}x^{m+1} + \cdots + b_N x^N$$

with $b_m = a_m$ *and a unique power series* $g(x)$ *which converges and is non zero on* $D_0(p^\lambda; \mathbf{C}_p)$ *such that*

$$f(x) = \frac{h(x)}{g(x)} \quad on\ D_0(p^\lambda; \mathbf{C}_p).$$

In addition M_h *coincides with* M_f *as far as the point* $(N, \mathrm{ord}_p a_N)$.

Corollary. *A power series which converges everywhere and has no zeroes is a constant.*

A simple proof of the Weierstrass Preparation Theorem for power series of the type

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \in \mathcal{O}_p[[x]]$$

is based on a generalization of the Euclid algorithm (see [Man1]).

2.6. There exits another definition of the Newton polygon (see [Ko2], [V1]) of a series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \in \mathbf{C}_p[[x]]$$

Instead of the points $(n, \mathrm{ord}_p a_n)$ let us look at the lines $l_n : y = nx + \mathrm{ord}_p a_n$ with slope n and y-intercept l_n . Then $\tilde{M}_f$ is defined as the graph of the function $\min_n l_n(x)$.The x-coordinate of the points of intersection of the l_n which appear in $\tilde{M}_f$ give ord_p of the zeroes, and the difference between the slopes n of the successive l_n which appear in $\tilde{M}_f$ give the number of zeroes with given ord_p.

The first obvious consequence of consideration for this type of Newton polygon is that a function

$$f(x) = \sum_{n=1}^{\infty} a_n x^n \in \mathbf{C}_p[[x]]$$

is bounded in the open disc $D_0(1^-; \mathbf{C}_p)$ if and only if all the coefficients a_n are uniformly bounded.

§3. Distributions, measures, and the abstract Kummer congruences

3.1. Distributions. Let us consider a commutative associative ring R , an R-module $\mathcal{A}$ and a profinite (i.e. compact and totally disconnected) topological space Y. Then Y is a projective limit of finite sets:

$$Y = \lim_{\substack{\leftarrow \\ I}} Y_i \quad (\pi_{ij} : Y_i \to Y_j, \quad i, j \in I, \ i \geq j) \tag{3.1}$$

where I is a (partially ordered) inductive set and for $i \geq j, \ i, j \in I$ there are surjective homomorphisms $\pi_{i,j} : Y_i \to Y_j$ with the condition $\pi_{i,j} \circ \pi_{j,k} = \pi_{i,k}$ for $i \geq j \geq k$. The inductivity of I means that for any $i, j \in I$ there exist $k \in I$ with the condition $k \geq i, k \geq j$. By the universal property we have that for each $i \in I$ a unique map $\pi_i : Y \to Y_i$ is defined, which satisfies the property $\pi_{ij} \circ \pi_i = \pi_j$ (for each $i, j \in I$)

Let $\mathrm{Step}(Y, R)$ be the R-module consisting of all R-valued locally constant functions $\phi : Y \to R$.

Definition. *A distribution on Y with values in a R-module $\mathcal{A}$ is a R-linear homomorphism*

$$\mu : \mathrm{Step}(Y, R) \to \mathcal{A}. \tag{3.2}$$

For $\varphi \in \mathrm{Step}(Y, R)$ we use the notations

$$\mu(\varphi) = \int_Y \varphi d\mu = \int_Y \varphi(y) d\mu(y).$$

each distribution μ can be defined by a system of functions $\mu_i; Y_i \to \mathcal{A}$, satisfying to the following finite-additivity condition

$$\mu^{(j)}(y) = \sum_{x \in \pi_{i,j}^{-1}(y)} \mu^{(i)}(x) \quad (y \in Y_j,\ x \in Y_i). \tag{3.3}$$

In order to get such a system it suffices to put

$$\mu^{(i)}(x) = \mu(\delta_{i,x}) \in \mathcal{A} \quad (x \in Y_i),$$

where $\delta_{i,x}$ is the characteristic function of the inverse image $\pi_i^{-1}(x) \subset Y$ with respect to the natural projection $Y \to Y_i$. For an arbitrary function $\varphi_j : Y_j \to R$ and $i \geq j$ we define the functions

$$\varphi_i = \varphi_j \circ \pi_{ij}, \quad \varphi = \varphi_j \circ \pi_j, \varphi \in \mathrm{Step}(Y, R), \quad \varphi_i : Y_i \stackrel{\pi_{ij}}{\to} Y_j \to R$$

A convenient criterion for the fact that the system of functions $\mu^{(i)} : Y_i \to \mathcal{A}$ satisfies the finite additivity condition (3.3) (and hence is associated to a distribution) is given by the following condition: for all $j \in I$, and $\varphi_j : Y_j \to R$ the values of the sums

$$\mu(\varphi) = \mu^{(i)}(\varphi_i) = \sum_{y_i \in Y_i} \varphi_i(y) \mu^{(i)}(y) \quad \textit{is independent of } i \quad \textit{for all large enough } i \geq j, \tag{3.4}$$

When using (3.4), it suffices to check the condition (3.4) for some "basic" systems of functions. For example, if $Y = G = \lim\limits_{\leftarrow i} G_i$ is a profinite abelian group, and R is a domain containing all roots of unity of the order dividing the order of Y ("supernatural number") then it is sufficient to check the condition (3.4) for all characters of finite order $\chi : G \to R$, since their $R \otimes \mathbf{Q}$ -linear span coincides with the whole ring $\mathrm{Step}(Y, R \otimes \mathbf{Q})$ by the orthogonality properties for characters of a finite group [Ka3], [Maz–SD].

Example: Bernoulli distributions (see [La3]). Let M is a positive integer , $f : \mathbf{Z} \to \mathbf{C}$ is a periodic function with the period M (i.e. $f(x + M) = f(x)$, $f : \mathbf{Z}/M\mathbf{Z} \to \mathbf{C}$). The generalized Bernoulli number (see [Le1], [Šaf1]) $B_{k,f}$ is defined as $k!$ times the coefficient by t^k in the expansion in t of the rational quotient

$$\sum_{a=0}^{M-1} \frac{f(a) t e^{at}}{e^{Mt} - 1},$$

that is,

$$\sum_{k=0}^{\infty} \frac{B_{k,f}}{k!} = \sum_{a=0}^{M-1} \frac{f(a)te^{at}}{e^{Mt}-1} \tag{3.5}$$

consider the profinite ring

$$Y = \mathbf{Z}_S = \varprojlim_M \mathbf{Z}/M\mathbf{Z}, \quad (S(M) \subset S)$$

the projective limit being taken over the set of all positive integers M with support $S(M)$ in a fixed finite set S of prime numbers. The the periodic function $f : \mathbf{Z}/M\mathbf{Z} \to \mathbf{C}$ with $S(M) \subset S$ can be considered as an element of $\mathrm{Step}(Y, \mathbf{C})$. We claim that there exists a distribution $E_k : \mathrm{Step}(Y, \mathbf{C}) \to \mathbf{C}$ which is uniquely defined by the condition

$$E_k(f) = B_{k,f} \quad \textit{for} \quad \textit{all} \quad f \in \mathrm{Step}(Y, \mathbf{C}). \tag{3.6}$$

In order to prove the existence of this distribution we use the above criterion (3.4) and check that for every $f \in \mathrm{Step}(Y, \mathbf{C})$ the right hand side in (3.6) (i.e. $B_{k,f}$) is independent of the choice of a period M of the function f. It folows easily directly from the definition (3.5) ; however we give here a different proof which is based on an interpretation of the numbers $B_{k.f}$ as certain special values of L-functions.

For the function $f : \mathbf{Z}/M\mathbf{Z} \to \mathbf{C}$ let

$$L(s, f) = \sum_{n=1}^{\infty} f(n)n^{-s}$$

be the corresponding L- series which is absolutely convergent for all s with $\mathrm{Re}(s) > 1$ and admits an analytic continuation over all $s \in \mathbf{C}$. For this series we have that

$$L(1-k, f) = -\frac{B_{k,f}}{k}. \tag{3.7}$$

For example, if $f \equiv 1$ is the constant function with the period $M = 1$ then we have that

$$\zeta(1-k) = -\frac{B_k}{k}, \quad \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k = \frac{t}{e^t - 1},$$

B_k being the Bernoulli number. The formula (3.7) is established by means of the contour integral discovered by Riemann (see [La3], ch.XXI). This formula apparently implies the desired independence of $B_{k,f}$ on the choice of M. We note also that if $K \subset \mathbf{C}$ is an arbitrary subfield, and $f(Y) \subset K$ then we have from the formula (3.5) that $B_{k,f} \in K$ hence the distribution E_k is a K-valued distribution on Y.

3.2. Measures. Let R be a topological ring , and $\mathcal{C}(Y, R)$ be the topological module of all R-valued functions on Y.

Definition. *A measure on Y with values in the topological R-module $\mathcal{A}$ is a continuous homomorphism of R-modules*

$$\mu : \mathcal{C}(Y, R) \to \mathcal{A}.$$

The restriction of μ to the R-submodule $\mathrm{Step}(Y,R) \subset \mathcal{C}(Y,R)$ defines a distribution which we denote by the same letter μ, and the measure μ is uniquely determined by the corresponding distribution since the R-submodule $\mathrm{Step}(Y,R)$ is dense in $\mathcal{C}(Y,R)$. The last statement expresses the well known fact about the uniformal continuity of a continuous function on a compact topological space.

Now we consider any closed subring R of the Tate field $\mathbf{C}_p$, $R \subset \mathbf{C}_p$, and let $\mathcal{A}$ be a complete R- module with topology given by a norm $|\cdot|_{\mathcal{A}}$ on $\mathcal{A}$ compatible with the norm $|\cdot|_p$ on $\mathbf{C}_p$ so that the following conditions are satisfied:

for $x \in \mathcal{A}$ the equality $|x|_{\mathcal{A}} = 0$ is equivalent to $x = 0$,

for $a \in R$, $x \in \mathcal{A}$ $\quad |ax|_{\mathcal{A}} = |a|_p |x|_{\mathcal{A}}$,

for all $x, y \in \mathcal{A}$ $\quad |x+y|_{\mathcal{A}} \le \max(|x|_{\mathcal{A}}, |y|_{\mathcal{A}})$.

Then the fact that a distribution (a system of functions $\mu^{(i)} : Y_i \to \mathcal{A}$) gives rise to a $\mathcal{A}$-valued measure on Y is equivalent to the condition that the system $\mu^{(i)}$ is bounded, i.e. for some constant $B > 0$ and for all $i \in I$, $x \in Y_i$ the following uniform estimate holds:

$$|\mu^{(i)}(x)|_{\mathcal{A}} < B. \tag{3.8}$$

This criterion is an easy consequence of the non-Archimedean property

$$|x+y|_{\mathcal{A}} \le \max(|x|_{\mathcal{A}}, |y|_{\mathcal{A}})$$

of the norm $|\cdot|_{\mathcal{A}}$ (see [Man4], [V1]).

In particular if $\mathcal{A} = R = \mathcal{O}_p = \{x \in \mathbf{C}_p \mid |x|_p \le 1\}$ is the subring of integers in the Tate field $\mathbf{C}_p$ then the set of $\mathcal{O}_p$-valued distributions on Y coincides with $\mathcal{O}_p$-valued measures (in fact, both sets are R-algebras with multiplication defined by convolution, see §4).

Below we give some meaningful examples based on the following important criterion of existence of a measure with given properties.

3.3. Proposition (The abstract Kummer congruences) (see [Ka3], p.258). *Let $\{f_i\}$ be a family of continuous functions $f_i \in \mathcal{C}(Y, \mathcal{O}_p)$ in the ring $\mathcal{C}(Y, \mathcal{O}_p)$ of all continuous functions on the compact totally disconnected group Y with values in the ring of integers $\mathcal{O}_p$ of $\mathbf{C}_p$ such that $\mathbf{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(Y, \mathbf{C}_p)$. Let also $\{a_i\}$ be any family of elements $a_i \in \mathcal{O}_p$. Then the existence of an $\mathcal{O}_p$ -valued measure μ on Y with the property*

$$\int_Y f_i \, d\mu = a_i$$

is equivalent to the validity of the following congruences: for an arbitrary choice of elements $b_i \in \mathbf{C}_p$ almost all of which vanish

$$\sum_i b_i f_i(y) \in p^n \mathcal{O}_p \ \textit{for all}\ y \in Y \implies \sum_i b_i a_i \in p^n \mathcal{O}_p. \tag{3.9}$$

Remark. *Since $\mathbf{C}_p$ -measures are characterized as bounded $\mathbf{C}_p$-valued distributions, every $\mathbf{C}_p$-measure on Y becomes a $\mathcal{O}_p$-valued measure after multiplication by some non-zero constant.*

The proof of proposition 3.3. The nessecity is obvious since

$$\sum_i b_i a_i = \int_Y (p^n \mathcal{O}_p - valued\ function\ )d\mu =$$

$$p^n \int_Y (\mathcal{O}_p - valued\ function\ )d\mu \in p^n \mathcal{O}_p.$$

In order to prove the sufficiency we need to construct a measure μ from the numbers a_i. For a function $f \in \mathcal{C}(Y, \mathcal{O}_p)$ and a positive integer n there exist elements $b_i \in \mathbf{C}_p$ such that only a finite number of b_i does not vanish, and

$$f - \sum_i b_i f_i \in p^n \mathcal{C}(Y, \mathcal{O}_p)$$

according to the density of the $\mathbf{C}_p$ span of $\{f_i\}$in $\mathcal{C}(Y, \mathbf{C}_p)$. By the assumption (3.9) the value $\sum_i a_i b_i$ belongs to $\mathcal{O}_p$ and is well defined modulo p^n (i.e. does not depend on the choice of b_i). Following N.M.Katz [Ka3], we denote this value by "$\int_Y f\, d\mu \bmod p^n$". Then we have that the limit procedure

$$\int_Y f\, d\mu = \lim_{n\to\infty} \text{``}\int_Y f\, d\mu \bmod p^n\text{''} \in \varprojlim_n \mathcal{O}_p/p^n\mathcal{O}_p \ = \ \mathcal{O}_p,$$

provides the measure μ.

3.4. The S-adic Mazur measure. Let $c > 1$ be a positive integer coprime to

$$M_o = \prod_{q\in S} q$$

with S being a fixed set of prime numbers. Using the criterion 3.3 we show that the $\mathbf{Q}$-valued distribution defined by the formula

$$E_k^c(f) = E_k(f) - c^k E_k(f_c),\ f_c(x) = f(cx), \tag{3.10}$$

turns out to be a measure where $E_k(f)$ are defined in 3.1, $f \in \mathrm{Step}(Y, \mathbf{Q})$ and the field $\mathbf{Q}$ is considered as a subfield in $\mathbf{C}_p$. Define the generalized Bernoulli polynomials $B_{k,f}^{(M)}(X)$:

$$\sum_{k=0}^{\infty} B_{k,f}^{(M)}(X)\frac{t^k}{k!} = \sum_{a=0}^{M-1} f(a)\frac{t\, e^{(a+X)t}}{e^{Mt}-1} \tag{3.11}$$

and the generalized sums of powers

$$S_{k,f}(M) = \sum_{a=0}^{M-1} f(a)a^k.$$

Then the definition (3.11) formally implies that

$$\frac{1}{k}[B_{k,f}^{(M)}(M) - B_{k,f}^{(M)}(0)] = S_{k-1,f}(M), \tag{3.12}$$

and also we see that

$$B_{k,f}^{(M)}(X) = \sum_{i=0}^{k} \binom{k}{i} B_{i,f} X^{k-i} = B_{k,f} + kB_{k-1,f}X + \cdots + B_{0,f}X^k \tag{3.13}$$

The last identity can be rewritten symbolically as

$$B_{k,f}(X) = (B_f + X)^k.$$

The equality (3.12) enabels us to calculate the (generalized) sums of powers in terms of the (generalized) Bernoulli numbers. In particular this equality implies that the Bernoulli numbers $B_{k,f}$ can be obtained by the following p -adic limit procedure ([La3], Chapter XIII):

$$B_{k,f} = \lim_{n\to\infty} \frac{1}{Mp^n} S_{k,f}(Mp^n) \quad (p-adic \ \ limit), \tag{3.14}$$

where f is a $\mathbf{C}_p$-valued function on $Y = \mathbf{Z}_S$. Indeed, replacing M in (3.12) be Mp^n with growing n and let D be the common denominator of all coefficients of the polynomial $B_{k,f}^{(M)}(X)$. That we have from (3.13) that

$$\frac{1}{k}[B_{k,f}^{(Mp^n)}(M) - B_{k,f}^{(M)}(0)] \equiv B_{k-1,f}(Mp^n) \bmod \frac{1}{kD} p^2 n. \tag{3.15}$$

Then the proof of (3.14) finishes by division of (3.15) by Mp^n and by application of the formula (3.12).

Now we can directly show that the distributions E_k^c defined by (3.10) are in fact bounded measures. If we use (3.9) and take as a family $\{f_i\}$ the set of all functions from Step$(Y, \mathcal{O}_p)$. Let $\{b_i\}$ be such family of elements $b_i \in \mathbf{C}_p$ that for all $y \in Y$ the congruence

$$\sum_i b_i f_i(y) \equiv 0 (\bmod p^n). \tag{3.16}$$

holds. Put $f = \sum_i b_i f_i$ and without loss of generality assume that the number n is large enough so that for all i with with $b_i \neq 0$ the congruence

$$B_{k,f_i} \equiv \frac{1}{Mp^n} S_{k,f_i}(Mp^n) \bmod p^n \tag{3.17}$$

is valid in accordance with (3.14). Then we see that

$$B_{k,f} \equiv (Mp^n)^{-1} \sum_i \sum_{a=0}^{Mp^n-1} b_i f_i(a) a^k \bmod p^n \tag{3.18}$$

hence we get by the definition (3.10):

$$\begin{gathered} E_k^c(f) = B_{k,f} - c^k B_{k,f_c} \equiv \\ (Mp^n)^{-1} \sum_i \sum_{a=0}^{Mp^n-1} b_i [f_i(a) a^k - f_i(ac)(ac)^k] \bmod p^n. \end{gathered} \tag{3.19}$$

Let

$$a_c \in \{0, 1, \cdots, Mp^n\} \quad a_c \equiv ac \bmod Mp^n$$

then the map $a \mapsto a_c$ is well defined and it acts as a permutation of the set

$$\{0, 1, \cdots, Mp^n\}$$

hence (3.19) is equivalent to the congruence

$$E_k^c(f) = B_{k,f} - c^k B_{k,f_c} \equiv (Mp^n)^{-1} \sum_i \frac{a_c^k - (ac)^k}{Mp^n} \sum_{a=0}^{Mp^n-1} b_i f_i(a) a^k. \tag{3.20}$$

The assumption (3.16) now formally implies that $E_k^c(f) \equiv 0 \bmod p^n$ completing the proof of the abstract Kummer congruences and the construction of measures E_k^c.

Remark. *The formula (3.20) also implies that for all $f \in \mathcal{C}(Y, \mathbf{C}_p)$ the following hold*

$$E_k^c(f) = kE_1^c(x_p^{k-1} f) \tag{3.21}$$

where $x_p : Y \to \mathbf{C}_p \in \mathcal{C}(Y, \mathbf{C}_p)$ is the composition of the projection $Y \to \mathbf{Z}_p$ and the embedding $\mathbf{Z}_p \hookrightarrow \mathbf{C}_p$.

Indeed if we put $a_c = ac + Mp^n t$ for some $t \in \mathbf{Z}$ then we see that

$$a_c^k - (ac)^k = (ac + Mp^n t)^k - (ac)^k \equiv kMp^n t (ac)^{k-1} \bmod (Mp^n)^2,$$

and we get that in (3.20)

$$\frac{a_c^k - (ac)^k}{Mp^n} \equiv k(ac)^{k-1} \frac{a_c - ac}{Mp^n} (\bmod Mp^n).$$

The last congruence is equivalent to saying that the abstract Kummer congruences (3.9) are satisfied for all functions of the type $x_p^{k-1} f_i$ and for the measure E_1^c with $f_i \in \mathrm{Step}(Y, \mathbf{C}_p)$ establishing the identity (3.21).

§4. Iwasawa algebra and the non-Archimedean Mellin transform

4.1. The set of arguments for the non-Archimedean zeta functions. In the classical case the set on which zeta functions are defined is the set of complex numbers $\mathbf{C}$ which can be equally interpreted as the set of all continuous characters (more precisely, quasicharacters) via the following isomorphism:

$$\mathbf{C} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{contin}}(\mathbf{R}_+^\times, \mathbf{C}^\times); \quad s \mapsto (y \mapsto y^s). \tag{4.1}$$

The construction which associates to a function $h(y)$ on $\mathbf{R}_+^\times$ with certain growth conditions for $y \to \infty$ and $y \to 0$ the integral

$$L_h(s) = \int_{\mathbf{R}_+^\times} h(y)\, y^s \, \frac{dy}{y}$$

(which probably converges not for all values of s) is called the Mellin transform. For example, if $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ be the Riemann zeta function, then the function $\zeta(s)\Gamma(s)$ is the Mellin transform of the function $h(y) = 1/(1-e^{-y})$:

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{1}{1-e^{-y}} y^s \frac{dy}{y}, \tag{4.2}$$

so that the integral and the series are absolutely convergent for $\mathrm{Re}(s) > 1$. For an arbitrary function of the type

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

with $z = x + iy \in \mathfrak{H}$ in the upper half plane $\mathfrak{H}$ and with the growth condition $a(n) = \mathcal{O}(n^c)$ $(c > 0)$ on its Fourier coefficients, the zeta function

$$L(s, f) = \sum_{n=1}^{\infty} a(n) n^{-s},$$

coincides essentially with the Mellin transform of $f(z)$, that is

$$\frac{\Gamma(s)}{(2\pi)^s} L(s, f) = \int_0^\infty f(iy) y^s \frac{dy}{y}. \tag{4.3}$$

Both sides of the equality (4.3) converge absolutely for $\mathrm{Re}(s) > 1 + c$. The identities (4.2) and (4.3) are immediately deduced from the well known integral representation for the gamma -function

$$\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}, \tag{4.4}$$

where $\frac{dy}{y}$ is a measure on the group $\mathbf{R}_+^\times$ which is invariant under the group translations (Haar measure). The integral (4.4) is absolutely convergent for $\mathrm{Re}(s) > 0$ and it can be interpreted as the integral of the product of an additive character $y \to e^{-y}$ of the group $\mathbf{R}^{(+)}$ restricted to $\mathbf{R}_+^\times$, and the multiplicative character $y \mapsto y^s$ with respect to the Haar measure on the group $\mathbf{R}_+^\times$.

In the theory of non-archimeadian functions the group $\mathbf{R}_+^\times$ is replaced by the group $\mathbf{Z}_S^\times$ (the group of units of the S- adic completion of the ring of integers $\mathbf{Z}$) and the field $\mathbf{C}$ is replaced by the Tate field $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$ (the completion of an algebraic closure of $\mathbf{Q}_p$). The set, on which p-adic zeta functions are defined, is the p-adic analytic Lie group

$$X_S = \mathrm{Hom}(\mathbf{Z}_S^\times, \mathbf{C}_p^\times) = X(\mathbf{Z}_S^\times), \tag{4.5}$$

where

$$\mathbf{Z}_S^\times \cong \oplus_{q \in S} \mathbf{Z}_q^\times,$$

and the symbol

$$X(G) = \mathrm{Hom}_{\mathrm{contin}}(G, \mathbf{C}_p^\times) \tag{4.6}$$

denote the functor of all p-adic characters of a topological group G (see[V1]).

4.2.The analytic structure on X_S. Let us consider in more detail the structure of the topological group X_S. Define

$$U_p = \{x \in \mathbf{Z}_p^\times | x \equiv 1 \,(\mathrm{mod}\, p^\nu)\}$$

where $\nu = 1$ or $\nu = 2$ according as $p > 2$ or $p = 2$. Then we have the natural decomposition

$$X_S = X((\mathbf{Z}/p^\nu \mathbf{Z})^\times \times \prod_{q \neq p} \mathbf{Z}_q^\times) \times X(U_p). \tag{4.7}$$

The analytic structure on $X(U_p)$ is defined by the following isomorphism (which is equivalent to a special choice of a local parameter):

$$\varphi : X(U_p) \tilde{\to} T = \{z \in \mathbf{C}_p^\times | \ |t-1|_p < 1\},$$

where $\varphi(x) = x(1+p^\nu)$, $1+p^\nu$ being a topological generator of the multiplicative group $U_p \cong \mathbf{Z}_p$. Arbitrary character $\chi \in X_S$ can be uniquely represented in the form $\chi = \chi_0 \cdot \chi_1$ where χ_0 is trivial on the component U_p, and χ_1 is trivial on the other component

$$(\mathbf{Z}/p^\nu \mathbf{Z})^\times \times \prod_{q \neq p} \mathbf{Z}_q^\times.$$

The character χ_0 is called *the tame component*, and χ_1 *the wild component* of the character χ. We denote by the symbol $\chi_{(t)}$ the wild character which is uniquely determined by the condition

$$\chi_{(t)}(1+p^\nu) = t$$

with $t \in \mathbf{C}_p$, $|t|_p < 1$.

In some cases it is convenient to use another local coordinate s which is analogous to the classical argument s of the Dirichlet series:

$$\mathcal{O}_P \ni s \mapsto \chi^{(s)}((1+p^\nu)^\alpha) = (1+p^\nu)^{\alpha s} = \exp(\alpha s \log(1+p^\nu)).$$

The character $\chi^{(s)}$ is defined only for such s for which the series exp is p-adically convergent. In this domain of values of the argument we have that $t = (1+p^\nu)^s - 1$. But, for example, for $(1+t)^{p^n} = 1$ there is certainly no such value of s, so that the s-coordinate parametrizes a smaller neighborhood of the trivial character then the t-coordinate (which cover all wild characters) (see [Man4], [Man6]).

Recall that an analytic function $F : T \to \mathbf{C}_p$ is defined as the sum of a series of the type $\sum_{i=0}^\infty a_i(t-1)^i$ ($a_i \in \mathbf{C}_p$), which is assumed to be absolutely convergent for all $t \in T$. The notion of an analytic function is then obviously extended to the whole group X_S by means of the group translations. The function

$$F(t) = \sum_{i=0}^\infty a_i(t-1)^i$$

is bounded on T iff all its coefficients a_i are universally bounded. This last fact can be easily deduced for example from the basic properties of the Newton polygon of the series $F(t)$ (see [Ko2], [V1], [V2]). If we apply to such series the Weierstrass preparation theorem ([Ko1], [La3], [Man1] and 2.5) we see that in this case the function F has only a finite number of zeroes on T (if it is not identicaly zero). In particular, consider the torsion subgroup $X_S^{\mathrm{tors}} \subset X_S$. This subgroup is discrete in X_S in its elements $\chi \in X_S^{\mathrm{tors}}$ can be obviously identified with primitive Dirichlet characters $\chi \bmod M$ such that the support $S(\chi) = S(M)$ of the conducter of χ is contained in the fixed finite set S. This identification is provided by a fixed embedding

$$i_p : \overline{\mathbf{Q}}^\times \hookrightarrow \mathbf{C}_p^\times$$

if we note that each such character $\chi \in X_S^{\mathrm{tors}}$ can be factorized through some finite factor group $(\mathbf{Z}/M\mathbf{Z})^\times$:

$$\chi : \mathbf{Z}_S^\times \to (\mathbf{Z}/M\mathbf{Z})^\times \to \overline{\mathbf{Q}}^\times \overset{i_p}{\hookrightarrow} \mathbf{C}_p^\times,$$

and the smallest such number M coincides with the conductor of $\chi \in X_S^{\mathrm{tors}}$.

The symbol x_p will denote the composition of the natural projection $\mathbf{Z}_S^\times \to \mathbf{Z}_p^\times$ and of the natural embedding $\mathbf{Z}_p^\times \to \mathbf{C}_p^\times$, so that $x_p \in X_S$ and all integers k can be considered as the characters $x_p^k \ : \ y \mapsto y^k$.

Consider a bounded $\mathbf{C}_p$-analytic function F on X_S. The above statement about zeroes of bounded $\mathbf{C}_p$-analytic functions implies now that the function F is uniquely defined by its values $F(\chi_0\chi)$, where χ_0 is a fixed character and χ runs through all elements $\chi \in X_S^{\mathrm{tors}}$ with possible exclusion of a finite number of such characters in each analyticity component of the decomposition (4.7). This condition is satisfied, for example, by the set of characters $\chi \in X_S^{\mathrm{tors}}$ with the S-complete conductor (i.e. such that $S(\chi) = S$), and even for the smaller set of characters which is obtained by imposing the additional assumption that the character χ^2 is not trivial (see [Man4], [Man6], [V1]).

4.3. The non-Archimedean Mellin transform. Let μ be a (bounded) $\mathbf{C}_p$-valuedmeasure on $\mathbf{Z}_S^\times$. Then *the non-archimediian Mellin transform* of the measure μ is defined by

$$L_\mu(x) = \mu(x) = \int_{\mathbf{Z}_S^\times} x \, d\mu \quad (x \in X_S), \tag{4.8}$$

and defines a bounded $\mathbf{C}_p$-analytic function

$$L_\mu : X_S \to \mathbf{C}_p. \tag{4.9}$$

Indeed, the boundeness of the function L_μ is obvious since all characters $x \in X_S$ take values in $\mathcal{O}_p$ and μ also is bounded. The analyticity of this function expresses a general property of the integral (4.8), namely, the analytic dependence of it on the parameter $x \in X_S$. However, we give below a pure algebraic proof of this fact which is based on a description of the Iwasawa algebra. This description will also imply that every bounded $\mathbf{C}_p$-analytic function on X_S is the Mellin transform of a certain measure μ.

4.4. The Iwasawa algebra (see [La3], chapter XII). Let $\mathcal{O}$ be a closed subring in $\mathcal{O}_p = \{z \in \mathbf{C}_p \,|\, |z|_p \le 1\}$, $G = \lim\limits_{\leftarrow i} G_i \ \ (i \in I)$ a profinite group. Then the canonical homomorphism $G_i \overset{\pi_{ij}}{\longleftarrow} G_j$ induces a homomorphism of the corresponding group rings

$$\mathcal{O}[G_i] \longleftarrow \mathcal{O}[G_j]$$

. Then the completed group ring $\mathcal{O}[[G]]$ is defined as the projective limit

$$\mathcal{O}[[G]] = \varprojlim_i \mathcal{O}[[G_i]] \quad (i \in I)$$

Let us consider also the set $\mathcal{D}istr\,(G, \mathcal{O})$ of all $\mathcal{O}$-valued distributions on G which itself is an $\mathcal{O}$-module and a ring with respect to multiplication given by the convolution, which defined in terms of families of functions

$$\mu_1^{(i)}, \mu_2^{(i)} : G_i \to \mathcal{O}$$

(see the previous section) as follows:

$$(\mu_1 * \mu_2)^{(i)}(y) \;=\; \sum_{y=y_1 y_2} \mu_1^{(i)}(y_1)\mu_2^{(i)}(y_2) \quad (y_1,\ y_2 \in G_i) \tag{4.10}$$

Recall also that the $\mathcal{O}$-valued distributions are identified with $\mathcal{O}$-valued measures. Now we are going identify the $\mathcal{O}$-algebras $\mathcal{O}[[G]]$ and $\mathcal{D}istr\,(G, \mathcal{O})$. In the case when $G = \mathbf{Z}_p$ the algebra $\mathcal{O}[[G]]$ is called the Iwasawa algebra.

4.5.Theorem. *(a) Under the notation of 4.4 there is the canonical isomorphism of $\mathcal{O}$-algebras*

$$\mathcal{D}istr\,(G, \mathcal{O}) \xrightarrow{\sim} \mathcal{O}[[G]]; \tag{4.11}$$

(b) If $G = \mathbf{Z}_p$ then there is an isomorphism

$$\mathcal{O}[[G]] \xrightarrow{\sim} \mathcal{O}[[X]], \tag{4.12}$$

where $\mathcal{O}[[X]]$ is the ring of formal power series in X over $\mathcal{O}$. The isomorphism (4.12) depends on a choice of the topological generator of the group $G = \mathbf{Z}_p$.

4.6. Formulas for coefficients of power series. We noticed above that the theorem 4.5 would imply a description of $\mathbf{C}_p$-analytic bounded functions on X_S in terms of measures. Indeed, such functions are given on analyticity components of the decomposition (4.7) as certain power series with p-adicaly bounded coefficients, that is , power series, whose coefficients belong to $\mathcal{O}_p$ after multiplication by some non-zero constant from $\mathbf{C}_p^\times$. Formulas for the coefficients of such series can be also deduced from the proof of the theorem. However, we give a more direct computation of these coefficients in terms of the corresponding measures. Let us consider the component aU_p of the set $\mathbf{Z}_S^\times$ where

$$a \in (\mathbf{Z}/p^\nu\mathbf{Z})^\times \times \prod_{q \neq p} \mathbf{Z}_q^\times,$$

and let $\mu_a(x) = \mu(ax)$ be the corresponding measure on U_p defined by the restriction of μ on the subset $aU_p \subset \mathbf{Z}_S^\times$. Consider the isomorphism $U_p \cong \mathbf{Z}_p$ given by

$$y = \gamma^x \ (x \in \mathbf{Z}_p,\ y \in U_p)$$

with some choice of the generator γ of U_p (for example, we can take $\gamma = 1 + p^\nu$). Let μ'_a be the corresponding measure on $\mathbf{Z}_p$. Then this measure is uniquely determined by values of the integrals

$$\int_{\mathbf{Z}_p} \binom{x}{i} d\mu'_a(x) = a_i, \tag{4.13}$$

with the interpolation polynomials $\binom{x}{i}$, since the $\mathbf{C}_p$-span of the family

$$\left\{ \binom{x}{i} \right\} \ (i \in \mathbf{Z}, i \geq 0)$$

is dense in $\mathcal{C}(\mathbf{Z}_p, \mathcal{O}_p)$ according to Mahler's interpolation theorem for continuous functions on $\mathbf{Z}_p$ (see [Mah]). Then it follows from the basic properties of the interpolation polynomials that

$$\sum_i b_i \binom{x}{i} \equiv 0(\operatorname{mod} p^n) \ (\mathit{for\ all}\ x \in \mathbf{Z}_p) \Longrightarrow b_i \equiv 0(\operatorname{mod} p^n).$$

We can now apply the abstract Kummer congruences (see 3.3), which imply that for arbitrary choice of numbers $a_i \in \mathcal{O}_p$ there exist a measure with the property (4.13).

On the other hand we state that the Mellin transform L_{μ_a} of the measure μ_a is given by the power series $F_a(t)$ with coefficients as in (4.13), that is

$$\int_{U_p} \chi_{(t)}(y)\, d\mu(ay) = \sum_{i=0}^{\infty} \Big(\int_{\mathbf{Z}_p} \binom{x}{i} d\mu'_a(x) \Big)(t-1)^i \tag{4.14}$$

for all wild characters of the form $\chi_{(t)}$, $\chi_{(t)}(\gamma) = t$, $|t-1|_p < 1$. It suffices to show that (4.14) is valid for all characters of the type $y \mapsto y^m$, where m is a positive integer. In order to do this we use the binomial expansion

$$\gamma^{mx} = (1 + (\gamma^m - 1))^x = \sum_{i=0}^{\infty} \binom{x}{i} (\gamma^m - 1)^i,$$

from which follows that

$$\int_{U_p} y^m \, d\mu(ay) = \int_{\mathbf{Z}_p} \gamma^{mx} d\mu'_a(x) =$$
$$\sum_{i=0}^{\infty} \Big(\int_{\mathbf{Z}_p} \binom{x}{i} d\mu'_a(x) \Big) (\gamma^m - 1)^i,$$

establishing (4.14).

4.7. Example. The S-adic Mazur measure and the non- Archimedean Kubota – Leopoldt zeta function (see [La3], [Ku-Le], [Le2], [Wa]). Let us consider first a positive integer $c \in \mathbf{Z}_S^\times \cap \mathbf{Z}$, $c > 1$ coprime to all prime numbers in S. Then for each complex number $s \in \mathbf{C}$ there exists a complex distribution μ_s^c on $G_S = \mathbf{Z}_S^\times$ which uniquely determined by the following

$$\mu_s^c(\chi) = (1 - \chi^{-1}(c) c^{-1-s}) L_{M_0}(-s, \chi), \tag{4.15}$$

where $M_0 = \prod_{q \in S} q$ (see 3.1). Moreover, the right hand side of (4.15) is holomorphic for all $s \in \mathbf{C}$ including $s = -1$. If s is an integer and $s \geq 0$ then according to criterion (3.4) the right-hand side of (4.15) belongs to the field

$$\mathbf{Q}(\chi) \subset \mathbf{Q}^{\mathrm{ab}} \subset \overline{\mathbf{Q}}$$

generated by the values of the character χ, and we get a distribution with values in $\mathbf{Q}^{\mathrm{ab}}$. If we now apply to (4.15) the fixed embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ we get a $\mathbf{C}_p$-valued distribution $\mu^{(c)} = i_p(\mu_0^c)$ which turns out to be an $\mathcal{O}_p$-measure in view of 3.3, and the following equality holds

$$\mu^{(c)}(\chi x_p^r) = i_p(\mu_r^c(\chi)),$$

which connects the special values of the Dirichlet L-functions at different non-positive points. The function

$$L(x) = (1 - c^{-1}x(c)^{-1})^{-1} L_{\mu^{(c)}}(x) \quad (x \in X_S) \tag{4.16}$$

is uniquely defined and is holomorphic on X_S with the exception of a simple pole at the point $x = x_p \in X_S$ and is called *the non-Archimedean zeta-function of Kubota–Leopoldt*. The corresponding measure $\mu^{(c)}$ will be called *the S-adic Mazur measure.*

§5. Complex valued distributions, associated with Euler products

5.1. In this section we give a general construction of distributions, attached to rather arbitrary Euler products. This construction provides a generalization of measures first introduced by Yu.I.Manin [Man4], B.Mazur and H.P.F.Swinnerton–Dyer [Maz-SD]. Our construction [Pa5], [Pa9] was already successfully used in several problems concerning the p-adic analytic continuation of Dirichlet series [Ar], [Co-Sch], [Sch].

5.2. Let S be a fixed finite set of prime numbers and

$$\mathcal{D}(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad (s,\ a_n \in \mathbf{C}) \tag{5.1}$$

be a Dirichlet series with the following multiplicativity property of its coefficients a_n:

$$\mathcal{D}(s) = \prod_{q \in S} F_q(q^{-s})^{-1} \sum_{\substack{n=1 \\ (n,S)=1}}^{\infty} a_n n^{-s}, \tag{5.2}$$

where the condition $(n, S) = 1$ means that n is not divisible by any prime number from the set S, and $F_q(X)$ are polynomials with the constant term equal to 1:

$$F_q(X) = 1 + \sum_{i=1}^{m_q} A_{q,i} X^i. \tag{5.3}$$

We assume also that the series (5.1) is absolutely convergent in some right half plane $\mathrm{Re}(s) > 1 + c$ $(c \in \mathbf{R})$. This assumption is satisfied in most cases, for example, when the coefficients a_n satisfy the estimate $|a_n| = \mathcal{O}(n^c)$. For a Dirichlet character $\chi : (\mathbf{Z}/M\mathbf{Z})^\times \to \mathbf{C}^\times$ modulo $M \geq 1$ the twisted Dirichlet series is defined by

$$\mathcal{D}(s,\chi) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}. \tag{5.4}$$

For all $s \in \mathbf{C}$ such that the series (5.1) is absolutely convergent and $x \in \mathbf{Q}$ let us define the function $P_s : \mathbf{Q} \to \mathbf{C}$ by the equality

$$P_s(x) = \sum_{n=1}^{\infty} e(nx) a_n n^{-s} \quad e(x) = \exp(2\pi i x) \tag{5.5}$$

.

Using the functions (5.5) we construct distributions μ_s on the compact group $\mathbf{Z}_S^\times$ such that for every primitive Dirichlet charactder χ viewed as a homomorphism $\chi : \mathbf{Z}_S^\times \to \mathbf{C}^\times$ the value of the series $\mathcal{D}(S,\overline{\chi})$ at s with $\mathrm{Re}(s) > 1 + c$ is expressed in a canonical way in terms of the integral

$$\int_{\mathbf{Z}_S^\times} \chi \, d\mu_s \stackrel{\text{def}}{=} \sum_{\substack{a \bmod M \\ (a,S)=1}} \chi(a) \, \mu_s(a + (M)).$$

Let for every $q \in S$, $\alpha(q)$ denote a fixed root of the inverse polynomial

$$X^{m_q} F_q(X^{-1}) = X^{m_q} + \sum_{i=1}^{m_q} A_{q,i} X^{m_q - i}$$

(that is, an inverse root of $F_q(X)$). Suppose that $\alpha(q) \neq 0$ for every $q \in S$, and extend by multiplicativity the definition of numbers $\alpha(n)$ to all positive integers whose support is contained in S:

$$\alpha(n) = \prod_{q \in S} \alpha(q)^{\mathrm{ord}_q n} \quad (S(q) \subset S).$$

Let us define an auxiliary polynomial

$$H_q(X) = 1 + \sum_{i=1}^{m_q - 1} B_{q,i} X^i \tag{5.6}$$

by means of the relation

$$F_q(X) = (1 - \alpha(q) X) H_q(X), \tag{5.7}$$

which imply the identities

$$B_{q,i} = - \sum_{j=i+1}^{m_q} A_{q,j} \alpha(q)^{i-j} \quad (i = 1, \cdots, m_q - 1) \tag{5.8}$$

for the coefficients of the polynomial (5.6). Let us also introduce the following finite Euler product

$$\sum_{S(n)\subset S} B^S(n)n^{-s} = \prod_{q\in S} H_q(q^{-s}), \tag{5.9}$$

in which the coefficients $B^S(n)$ are given by means of (5.8), namely,

$$B^S(n) = \prod_{q\in S} B^S(q^{\mathrm{ord}_q n}) \quad (S(n)\subset S),$$

with

$$B^S(q^i) = \begin{cases} B_{q,i}, & \text{for } i < m_q \\ 0, & \text{otherwise.} \end{cases} \tag{5.10}$$

Now we state the main result of the section.

5.3. Theorem. *(a) For any choice of the inverse roots $\alpha(q)\neq 0$,($q\in S$ and for any s from the convergency region of the series (5.1) there exists a distribution $\mu_s = \mu_{s,\alpha}$ on $\mathbf{Z}_S^\times$ whose values on open compact subsets of the type $a+(M)\subset \mathbf{Z}_S^\times$ are given by the following*

$$\mu_s(a+(M)) = \frac{M^{s-1}}{\alpha(M)} \sum_{S(n)\subset S} B^S(n)P_s(\frac{an}{M})n^{-s}, \tag{5.11}$$

so that the sum in (5.11) is finite and the numbers $B^S(n)$ are defined by (5.10).

(b) For any primitive Dirichlet character χ viewed as a function $\chi : \mathbf{Z}_S^\times \to \mathbf{C}^\times$ the following equality holds

$$\begin{aligned}\int_{\mathbf{Z}_S^\times} \chi\, d\mu_s = &\prod_{q\in S\setminus S(\chi)} (1-\chi(q)\alpha(q)^{-1}q^{-s})H_q(\overline{\chi}(q)q^{-s})\times \\ &\times \frac{C_\chi^{s-1}}{\alpha(C_\chi)} G(\chi)\mathcal{D}(s,\overline{\chi}),\end{aligned} \tag{5.12}$$

with

$$G(\chi) = \sum_{a \bmod C_\chi} \chi(a)e(\frac{a}{C_\chi})$$

being the Gauss sum, C_χ the conductor, and $S(\chi)$ the support of the conductor of χ.

5.4. The following proof of this theorem differs from that given in [Pan5] and is based on the compatibility criterion (3.4). We check that the sum

$$\sum_{\substack{a \bmod M\\ (a,S)=1}} \chi(a)\mu_s(a+(M)), \tag{5.13}$$

does not depend on the choice of a positive integer M with the condition $C_\chi|M$,$S(M) = S$. This will be provided by a calculation which also implies that (5.13) coincides with the right hand side of (5.12) (and therefore is independent of M).

5.5.Lemma. *For an arbitrary positive integer n and $C_\chi|M$ put*

$$G_{n,M} = \sum_{\substack{a \bmod M \\ (a,S)=1}} \chi(a)e(an/M).$$

then the following holds

$$G_{n,M}(\chi) = \frac{M}{C_\chi} G(\chi) \sum_{d|(M/C_\chi)} \mu(d)d^{-1}\chi(d)\delta\left(\frac{dn}{(M/C_\chi)}\right)\overline{\chi}\left(\frac{dn}{(M/C_\chi)}\right),$$

in which μ denotes the Möbius function, $\delta(x) = 1$or 0 according as $x \in \mathbf{Z}$ or not, and we assume that the character χ is primitive modulo C_χ.

The proof of the lemma is deduced from the well known property of the Möbius function:

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if n¿1.} \end{cases}$$

Consequently, $G_{n,M}$ takes the following form:

$$\sum_{\substack{d|(a,M) \\ a \bmod M}} \mu(d)\chi(a)e(an/M) =$$

$$\sum_{d|M} \mu(d) \sum_{a_1 \bmod M/d} \chi(da_1)e(da_1n/M) =$$

$$\sum_{d|M} \mu(d)d^{-1}\chi(d) \sum_{a_1 \bmod M} \chi(a_1)e(da_1n/M) =$$

$$\sum_{d|(M/C_\chi)} \mu(d)d^{-1}\chi(d)\delta\left(\frac{dn}{(M/C_\chi)}\right) \sum_{a_1 \bmod M} \chi(a_1)e\left(da_1n, M, \right),$$

since $\chi(a_1)$ depend only on $a_1 \bmod C_\chi$, and

$$e(a_1dn/M) = e((a_1/C_\chi)(dn/(M/C_\chi))).$$

In the above equality we changed the order of summation , then we replaced the index of summation a by da_1 and extended the system of residue classes $a_1 \bmod M/d$ to $a_1 \bmod M$. Now we transform the summation into that one modulo C_χ. It remains to use the well known property of Gauss sums (see, for example, [Shi1], lemma 3.63):

$$G_{n,C_\chi}(\chi) = \overline{\chi}(n)\, G(\chi),$$

establishing the lemma.

5.6. In order to deduce the theorem, we now transform (5.13), taking into account the definition (5.11) and lemma 5.5:

$$\begin{aligned}
&\frac{M^{s-1}}{\alpha(M)} \sum_{\substack{a \bmod M \\ (a,M)=1}} \chi(a) \sum_n B^S(n) n^{-s} \sum_{n_1} a_{n_1} e(\frac{ann_1}{M}) n_1^{-s} = \\
&\frac{M^{s-1}}{\alpha(M)} \sum_n \sum_{n_1} B^S(n) n^{-s} a_{n_1} n_1^{-s} G_{nn_1,M}(\chi) = \\
&\frac{M^s}{\alpha(M) C_\chi} G(\chi) \sum_n \sum_{n_1} B^S(n) n^{-s} a_{n_1} n_1^{-s} \times \\
&\times \sum_{d|(M/C_\chi)} \frac{\mu(d)}{d} \chi(d)\, \delta\left(\frac{dnn_1}{(M/C_\chi)}\right) \overline{\chi}\left(\frac{dnn_1}{(M/C_\chi)}\right).
\end{aligned} \tag{5.14}$$

From the last formula we see that non-vanishing terms in the sum over n and n_1 must satisfy the condition $(M/C_\chi d)|nn_1$. Let us now split n_1 into two factors $n_1 = n_1' \cdot n_1''$ so that $S(n_1') \subset S$, and $(n_1'', S) = 1$. Then

$$\overline{\chi}\left(\frac{dnn_1}{(M/C_\chi)}\right) = \overline{\chi}(n_1'')\overline{\chi}\left(\frac{nn_1'}{(M/C_\chi d)}\right) \tag{5.15}$$

and

$$\delta\left(\frac{dnn_1}{(M/C_\chi)}\right) = \delta\left(\frac{nn_1'}{(M/C_\chi d)}\right), \tag{5.15a}$$

since $(n_1'', M) = 1$. According to (5.2) one has

$$\sum_{S(n_1')\subset S} a_{n_1'} n_1'^{-s} = \prod_{q\in S} F_q(q^{-s})^{-1} \tag{5.16}$$

. Now we use the definition of the finite Euler product (5.9) and of the polynomials $H_q(X)$ which we rewrite here in the form

$$\sum_n B^S(n) n^{-s} \prod_{q\in S} F_q(q^{-s})^{-1} = \prod_{q\in S} (1 - \alpha(q) q^{-s})^{-1}.$$

Consequently,

$$\sum_n B^S(n) n^{-s} \sum_{S(n_1')\subset S} a_{n_1'} n_1'^{-s} = \sum_{S(n_2)\subset S} a_{n_2} n_2^{-s}$$

and for $S(n_2) \subset S$ we have that

$$\alpha(n_2) = \sum_{n_2 = n\cdot n_1'} B^S(n) a_{n_1'}. \tag{5.17}$$

Keeping in mind (5.15) and (5.15a) we transform (5.14) to the following

$$
\begin{aligned}
&\frac{M^s}{\alpha(M)C_\chi}G(\chi)\sum_n \sum_{d|(M/C_\chi)} \frac{\mu(d)}{d}\chi(d) \sum_{(n_1'',S)=1} \overline{\chi}(n_1'')a_{n_1''}n_1''^{-s}\times \\
&\times \sum_{n,n_1'} B^S(n)a_{n_1'}(nn_1')^{-s}\delta\left(\frac{nn_1'}{(M/C_\chi d)}\right)\overline{\chi}\left(\frac{nn_1'}{(M/C_\chi d)}\right).
\end{aligned}
\tag{5.18}
$$

Now we transform (5.18) with help of the relation (5.17), taking into account that non-zero summands can only occur for such $n_2 = nn_1'$ which are divisible by $M/(C_\chi d)$, i.d. we put $n_2 = (M/C_\chi d)n_3$, $S(n_3) \subset S$. We also note that by the definition of our Dirichlet series we have

$$
\sum_{n,n_1'} B^S(n)a_{n_1'}(nn_1')^{-s} = \mathcal{D}(s,\overline{\chi}) \prod_{q\in S\setminus S(\chi)} F_q(\overline{\chi}(q)q^{-s}).
$$

Therefore (5.18) transforms to the following:

$$
\begin{aligned}
&\frac{M^s}{\alpha(M)C_\chi}G(\chi)\prod_{q\in S\setminus S(\chi)} F_q(\overline{\chi}(q)q^{-s})\times \\
&\qquad \sum_{d|(M/C_\chi)} \frac{\mu(d)}{d}\chi(d) \sum_{S(n_3)\subset S} \overline{\chi}(n_3)\alpha\left(\frac{n_3M}{C_\chi d}\right)\left(\frac{n_3M}{C_\chi d}\right)^{-s} = \\
&\frac{C_\chi^{s-1}}{\alpha(C_\chi)}G(\chi)\mathcal{D}(s,\overline{\chi})\prod_{q\in S\setminus S(\chi)} F_q(\overline{\chi}(q)q^{-s})\times \\
&\times \sum_{d|(M/C_\chi)} \mu(d)\, d^{s-1}\chi(d)\alpha(d)^{-1} \sum_{S(n_3)\subset S} \overline{\chi}(n_3)\alpha(n_3)n_3^{-s}.
\end{aligned}
$$

The proof of the theorem is accomplished by noting that

$$
\begin{aligned}
&\sum_{d|(M/C_\chi)} \mu(d)\, d^{s-1}\chi(d)\alpha(d)^{-1} = \prod_{q\in S\setminus S(\chi)} (1-\chi(q)\alpha(q)^{-1}q^{s-1}), \\
&\sum_{S(n_3)\subset S} \overline{\chi}(n_3)\alpha(n_3)n_3^{-s} = \prod_{q\in S\setminus S(\chi)} (1-\overline{\chi}(q)\alpha(q)q^{-s})^{-1} = \\
&\prod_{q\in S\setminus S(\chi)} F_q(\overline{\chi}(q)q^{-s})^{-1}H_q(\overline{\chi}(q)q^{-s}).
\end{aligned}
$$

5.7. Concluding remarks. This construction admits a generalization [Pa7] to the case of rather general Euler products over prime ideals in algebraic number fields. These Euler products have the form

$$
\mathcal{D}(s) = \sum_{\mathfrak{n}} a_{\mathfrak{n}}\mathcal{N}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} F_{\mathfrak{p}}(\mathcal{N}(\mathfrak{p})^{-s})^{-1},
$$

where $\mathfrak{n}$ runs over the set of integral ideals, and $\mathfrak{p}$ over the set of prime ideals of the ring of integers $\mathcal{O}_K$ of a number field K, with $\mathcal{N}(\mathfrak{n})$ denoting the adsolute norm of an ideal $\mathfrak{n}$, and $F_{\mathfrak{p}} \in \mathbf{C}[X]$ being polynomials with the condition $F_{\mathfrak{p}}(0) = 1$. In [Pa7] we constructed certain canonical distributions, which provide integral representations for special values of Dirichlet series of the type

$$\mathcal{D}(s,\chi) = \sum_{\mathfrak{n}} \chi(\mathfrak{n}) a_{\mathfrak{n}} \mathcal{N}(\mathfrak{n})^{-s},$$

where χ denote a Hecke character of finite order, whose conductor consists only of prime ideals beloning to a fixed finite set S of non-Archimedean places of K. The main result of [Pa7] provides a generalization of theorem 4.2 of the ealier work of Yu.I.Manin [Man6].

However, in the construction of non-Archimedean convolutions of Hilbert modular forms given below in chapter 2, we give another approach to local distributions, which is quite different of that given above and is applicable only to certain Dirichlet series (namely, to convolutions of Rankin type).

Chapter 2. Non-Archimedean convolutions of Hilbert modular forms

§0. Introduction

0.1. Now let p be a prime number and S a finite set of primes containing p. In this chapter we discuss convolutions of Hilbert modular forms and construct their S -adic analogues; they correspond to certain automorphic forms on the group $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ over a totally real field F and reduced to zeta functions of the form

$$L(s, \mathbf{f}, \mathbf{g}) = \sum_{\mathfrak{n}} C(\mathfrak{n}, \mathbf{f})\, C(\mathfrak{n}, \mathbf{g}) \mathcal{N}(\mathfrak{n})^{-s} \qquad (0.1),$$

where $\mathbf{f}$, $\mathbf{g}$ are Hilbert automorphic forms of "holomorphic type" over F , with $C(\mathfrak{n}, \mathbf{f})$, $C(\mathfrak{n}, \mathbf{g})$ being their normalized Fourier coefficients (enumerated by integral ideals $\mathfrak{n}$ of the ring of integers $\mathcal{O}_F \subset F$). We consider the functions $\mathbf{f}$, $\mathbf{g}$ as being defined on the adelized group $G_{\mathbf{A}} = \mathrm{GL}_2(\mathbf{A}_F)$, where $\mathbf{A}_F$ is the ring of adeles of F and we assume that $\mathbf{f}$ is a primitive cusp form of scalar weight $k \geq 2$ and of conductor $\mathfrak{c}(\mathbf{f}) \subset \mathcal{O}_F$ with the charachter ψ and $\mathbf{g}$ a primitive cusp form of weight $l < k$ and of conductor $\mathfrak{c}(\mathbf{g})$ with the character ω, $(\psi, \omega : \mathbf{A}_F^\times \to \mathbf{C}^\times$ being Hecke characters of finite order) . The non-Archimedean construction is based on the algebraicity properties of the special values of the function $L(s, \mathbf{f}, \mathbf{g})$ at the points $s = l, \cdots, k-1$ up to some constant involving the Petersson inner product $\langle \mathbf{f},\ \mathbf{f} \rangle$ of the automorphic form $\mathbf{f}$ [Shi6]. Our theorem about non-Archimedean interpolation is equivalent to certain generalized Kummer congruences for these special values.

0.2. We need some more notations for the precise formulation of the result (in a simplified form). Let ψ^*, ω^* be the ideal group characters of F associated with ψ, ω and let

$$L_{\mathfrak{c}}(s, \psi\omega) = \sum_{\mathfrak{n}+\mathfrak{c}=\mathcal{O}_F} \psi^*(\mathfrak{n})\omega^*(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}+\mathfrak{c}=\mathcal{O}_F} (1 - \psi^*(\mathfrak{p})\omega^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s})^{-1} \quad (0.2)$$

be the correspoding Hecke L-function with $\mathfrak{c} = \mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g})$. We now define the normalized zeta function

$$\Psi(s, \mathbf{f}, \mathbf{g}) = \gamma_n(s) L_{\mathfrak{c}}(2s + 2 - k - l, \psi\omega) L(s, \mathbf{f}, \mathbf{g})$$

where $n = [F : \mathbf{Q}]$ is the degree of F,

$$\gamma(s) = (2\pi)^{-2ns}\Gamma(s)^n \Gamma(s + 1 - l)^n$$

being the gamma-factor. Then the function $\Psi(s, \mathbf{f}, \mathbf{g})$ admits a holomorphic analytic continuation over the whole comlex plane and satisfies certain functional equation [Ja], [Shi6]. Put $\Omega(\mathbf{f}) = \langle \mathbf{f},\ \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}$, then the number

$$\frac{\Psi(l + r, \mathbf{f}, \mathbf{g})}{(2\pi i)^{n(1-l)}\Omega(\mathbf{f})} \ \textit{is algebraic for all integers } r \textit{ with } 0 \leq r \leq k - l - 1 \qquad (0.3)$$

.

For the non-Archimedean construction we introduce the S-adic completion

$$\mathcal{O}_S = \prod_{q \in S} (\mathcal{O}_F \otimes \mathbf{Z}_q) = \prod_{\mathfrak{p}|q \in S} \mathcal{O}_{\mathfrak{p}}$$

of the ring $\mathcal{O}_F$. Put
$$S_F = \{\mathfrak{p} \mid \mathfrak{p} \ \ divides \ \ q \in S\}$$
and let $\mathrm{Gal}_S = \mathrm{Gal}(F(S)/F)$ be the Galois group of the maximal abelian extension of F unramified outside S and ∞.

The non-Archimedean L-functions will be defined on the p-adic analytic Lie group

$$\mathcal{X}_S = \mathrm{Hom}_{\mathrm{contin}}(\mathrm{Gal}_S, \mathbf{C}_p^\times)$$

consisting of all continuouos p-adic characters of the Galois group Gal_S with $\mathbf{C}_p$ being the Tate field. Elements of finite order $\chi \in \mathcal{X}_S$ can be obviously identified with those Hecke characters of finite order whose conductors are divisible only by prime divisors belonging to S_F, via the decomposition

$$\chi : \mathbf{A}_F^\times \overset{\text{class field theory}}{\longrightarrow} \mathrm{Gal}_S \to \overline{\mathbf{Q}}^\times \overset{i_p}{\to} \mathbf{C}_p^\times.$$

Let us denote by the same letter χ both Hecke characters and the corresponding elements of $\mathcal{X}_S$. There is a natural homomorphism

$$\mathcal{N} : \mathrm{Gal}_S \to \mathrm{Gal}(\mathbf{Q}(S)/\mathbf{Q}) \cong \mathbf{Z}_S^\times = \prod_{q\in S} \mathbf{Z}_q^\times, \tag{0.4}$$

defined by the restriction of Galois automorphisms from $F(S)$ to $\mathbf{Q}(S)$, and we denote by $\mathcal{N}x_p$ the composition of this homomorphism with the natural projection $\prod_q \mathbf{Z}_q^\times \to \mathbf{Z}_p^\times$ and the inclusion $\mathbf{Z}_p^\times \subset \mathbf{C}_p^\times$

Our essential assumption is that that the cusp form $\mathbf{f}$ is *p-ordinary*, i.e. for the fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ and for all $\mathfrak{p} \mid p$ there exists such a root $\alpha(\mathfrak{p})$ of Hecke $\mathfrak{p}$-polynomial of $\mathbf{f}$ that $\mid i_p(\alpha(\mathfrak{p})) \mid_p = 1$. We then fix such roots $\alpha(\mathfrak{p})$ and extend the definition of $\alpha(\mathfrak{m})$ to all integral ideals $\mathfrak{m} \subset \mathcal{O}_F$ by multiplicativity.

0.3. Theorem *(On non-Archimedean convolutions of Hilbert modular forms) Under the above notations and assumptions there exists a bounded $\mathbf{C}_p$-analytic function $\Psi_S : \mathcal{X}_S \to \mathbf{C}_p$ uniquely defined by the condition: for each Hecke character of finite order $\chi \in \mathcal{X}_S^{\mathrm{tors}}$ the following equality holds*

$$\Psi_S(\chi) = i_p\Big[D_F^{2l}\omega^*(\mathfrak{m})\frac{\tau(\chi)^2\mathcal{N}\mathfrak{m}^{l-1}}{\alpha(\mathfrak{m})^2}\frac{\Psi(l,\mathbf{f},\mathbf{g}^\rho(\bar\chi))}{(-2\pi i)^{n(1-l)}\langle\mathbf{f},\ \mathbf{f}\rangle}\Big],$$

where D_F is the discriminant of F, $\tau(\chi)$ being the Gauss sum of χ, and $\mathbf{g}^\rho(\chi)$ the cusp form obtained from g *by complex conjugation of its Fourier coefficients and by twisting it then with the character χ.*

0.4. This result is also valid for the special values $\Psi(l+r,\mathbf{f},\mathbf{g})$ with $r = 1, \cdots k-l$, if we replace $\chi \in \mathcal{X}_S$ by $\chi\mathcal{N}x_p^r \in \mathcal{X}_S$ (see the Main theorem in §2).

Recently this construction was extended by Mi Ving Quang (Moscow University) to the *non-p-ordinary*, i.e. *supersingular* case, when $\mid i_p(\alpha(\mathfrak{p}) \mid_p < 1$ for all $\mathfrak{p} \mid p$ (at least when $F = \mathbf{Q}$). In this situation the functions Ψ_S are also uniquely defined by the condition that they have only a prescribed logarithmic growth on $\mathcal{X}_S$.

0.5. Content of the chapter. We recall in §1 some basic facts about Hilbert modular forms and their Fourier coefficients. The precise formulation of the main result of the chapter is given in §2. The most important element of our construction is the complex valued distributions $\tilde{\mu}_s$ on the Galois group Gal_S introduced in §3. We establish algebraicity properties of these distributions in §4 by means of the Rankin-Shimura integral representation for $s = l, \cdots, k-1$. After application of the fixed embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ these values become p-adic numbers and define p-adic measures on $\mathcal{X}_S$; then the function Ψ_S of the Main theorem is built as the non-Archimedean Mellin transform of the p-adic measure $i_p(\tilde{\mu}_l)$.

Notations

Let the symbols

$$\mathcal{O}_F,\ I,\ F_{\mathbf{A}},\ F_{\mathbf{A}}^{\times},\ \mathfrak{d} \subset \mathcal{O}_F,\ D_F = \mathcal{N}(\mathfrak{d})$$

denote, respectively, the maximal order (ring of integers), the group of fractional ideals, the ring of adeles, the group of ideles, the different and the discriminant of the totally real field F of degree n over $\mathbf{Q}$. Let $\Sigma = \Sigma_\infty \cup \Sigma_0$ denote the set of places (i.e. classes of normalized valuations) of F where $\Sigma_\infty = \{\infty_1, \cdots, \infty_n\}$ are archimedean places, $\Sigma_0 = \{\mathfrak{p} = \mathfrak{p}_v \subset \mathcal{O}_F\}$ finite (non-Archimedean) places , so that if $|\cdot|_v$ be the corresponding normalized valuations then the product formula hplds

$$\prod_v |x|_v \ = \ 1 \quad (x \in F^{\times},\ \ v \in \Sigma).$$

The archimedean places are induced by the real embeddings of $F : x \mapsto x^{(\nu)} \in \mathbf{R}$ $(\nu = 1, \cdots, n)$. An element $x \in F^{\times}$ is called totally positive ($x \gg 0$) if one has $x^{(\nu)} > 0$ for all ν and let $F_+^{\times}$ denote the multiplicative group of all totally positive elements of F. We put also $F_\infty = F \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathbf{R}^n \subset F_{\mathbf{A}}$, and let $\hat{F} \cong \hat{\mathcal{O}}_F \otimes_{\mathbf{Z}} \mathbf{Q} \subset F_{\mathbf{A}}$ be the subring of finite adeles where $\hat{\mathcal{O}}_F$ is the profinite completion of the ring $\mathcal{O}_F$ (with respect to all its ideals). Then $F_{\mathbf{A}} = F_\infty \oplus \hat{F}$, and for an adele $x = (x_v)_{v \in \Sigma}$ we write $x = x_\infty + x_0$ where $x_\infty \in F_\infty$, $x_0 \in \hat{F}$. On the other hand there is the decomposition $F_{\mathbf{A}}^{\times} = F_\infty^{\times} \times \hat{F}^{\times}$ and we often use the convenient though slitly ambiguous notation $y = y_\infty \cdot y_0$ with $y_\infty \in F_\infty^{\times}$, $y_0 \in \hat{F}^{\times}$. For the idele $y \in F_{\mathbf{A}}^{\times}$ let the symbol $\tilde{y} \in I$ denote the fractional ideal associated with y so that one has $\tilde{y}\widehat{\mathcal{O}}_F \ = \ y_0\widehat{\mathcal{O}}_F$.

§1. Hilbert modular forms

1.1. The group. We consider here the group $\mathrm{GL}_2(F)$ as the group $G_{\mathbf{Q}}$ of all $\mathbf{Q}$-rational points of a $\mathbf{Q}$ -subgroup $G \subset \mathrm{GL}_{2n}$. Then the adelization $G_{\mathbf{A}} = G(\mathbf{A})$ can be identified with the product

$$\mathrm{GL}_2(F_{\mathbf{A}}) \cong G_\infty \times G_{\hat{\mathbf{Q}}},$$

where

$$G_\infty = \mathrm{GL}_2(F_\infty) \cong \mathrm{GL}_2(\mathbf{R})^n, \ \ G_{\hat{\mathbf{Q}}} = \mathrm{GL}_2(\hat{F}).$$

The subgroup

$$G_\infty^{+} \cong \mathrm{GL}_2^{+}(\mathbf{R})^n \subset G_\infty$$

consists of all elements

$$\alpha = (\alpha_1, \cdots, \alpha_n), \quad \alpha_\nu = \begin{pmatrix} \alpha_\nu & \beta_\nu \\ \gamma_\nu & \delta_\nu \end{pmatrix},$$

such that for all $\nu = 1, \cdots, n$ one has $\det\alpha_\nu > 0$. Every element $\alpha \in G_\infty^+$ acts on the product $\mathfrak{H}^n$ of the n copies of the upper half plane by the formula

$$\alpha(z_1, \cdots, z_n) = (\alpha_1(z_1), \cdots, \alpha_n(z_n)),$$

where

$$\alpha_\nu(z_\nu) \quad = \quad (a_\nu z_\nu + b_\nu)/(c_\nu z_\nu + d_\nu.$$

For $z = (z_1, \cdots, z_n)$ we put $\{z\} = z_1 + \cdots + z_n$ and $e_F(z) = e(\{z\})$, with $e(x) = \exp(2\pi i x)$. Let $\mathbf{i} = (i, \cdots, i) \in \mathfrak{H}^n$, then

$$\{\alpha \in G_\infty^+ | \alpha(\mathbf{i}) = \mathbf{i}\}/\mathbf{R}_+^\times \cong \mathrm{SO}(2)^n$$

is a maximal compact subgroup in $G_\infty^+/\mathbf{R}_+^\times$. For $\alpha \in G_\infty^+$, an integer k and an arbitrary function $f : \mathfrak{H}^n \to \mathbf{C}$ we use the notation

$$(f|_k\alpha)(z) = \mathcal{N}(cz+d)^{-k} f(\alpha(z)) \mathcal{N}\det(\alpha)^{k/2},$$

with $\mathcal{N}(z)^k = z_1^k \cdots z_n^k$. Let $\mathfrak{c} \subset \mathcal{O}_F$ be an integral ideal, $\mathfrak{c}_\mathfrak{p} = \mathfrak{c}\mathcal{O}_\mathfrak{p}$ its $\mathfrak{p}$- part,$\mathfrak{d}_\mathfrak{p} = \mathfrak{d}\mathcal{O}_\mathfrak{p}$ the local different. We will use the open subgroups $W = W_\mathfrak{c} \subset G_\mathbf{A}$ defined by

$$\begin{aligned} &W = G_\infty^+ \times \prod_\mathfrak{p} W(\mathfrak{p}), \\ &W(\mathfrak{p}) = \\ &\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\mathfrak{p}) | b \in \mathfrak{d}_\mathfrak{p}^{-1}, c \in \mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p},\ a, d \in \mathcal{O}_\mathfrak{p}, ad - bc \in \mathcal{O}_\mathfrak{p}^\times \right\}. \end{aligned} \tag{1.1}$$

Let $h = |\widetilde{Cl_F}|$ be the number of ideal classes of F (in the narrow sense),

$$\widetilde{Cl_F} = I/\{(x) | x \in F_+^\times\},$$

and let us choose the ideles $t_1, \cdots, t_h$ so that $\tilde{t}_\lambda \subset \mathcal{O}_F$ form a complete system of representatives for $\widetilde{Cl_F}$, $(t_\lambda)_\infty = 1$ and $\tilde{t}_\lambda + \mathfrak{m}_0 = \mathcal{O}_F$ ($\lambda = 1, \cdots, h$, $\mathfrak{m}_0 = \prod_{\mathfrak{q} \in S_F} \mathfrak{q}$). Put $x_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & t_\lambda \end{pmatrix}$ then there is the following decomposition into a disjoint union ("the approximation theorem"):

$$G_\mathbf{A} = \cup_\lambda G_\mathbf{Q} x_\lambda W = \cup_\lambda G_\mathbf{Q} x_\lambda^{-\iota} W, \tag{1.2}$$

where $x_\lambda^{-\iota} = \begin{pmatrix} t_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, ι denotes the involution given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota \quad = \quad \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(see [Shi6], p.647).

1.2. Definition of Hilbert automorphic forms of weight k and level $\mathfrak{c} \subset \mathcal{O}_F$ with a Hecke character ψ of finite order. We call a function $f : G_{\mathbf{A}} \to \mathbf{C}^\times$ a Hilbert automorphic form of weight k and level $\mathfrak{c} \subset \mathcal{O}_F$ with a Hecke character ψ if the following conditions (1.3) - (1.5) are satisfied:

$$\begin{aligned} &for\ all\ x \in G_{\mathbf{A}}\ one\ has\ f(s\alpha x) = \psi(s)f(x) \\ &for\quad s \in F_{\mathbf{A}}^\times(\ the\ center\ of\ G_{\mathbf{A}}), \alpha \in G_{\mathbf{Q}}. \end{aligned} \tag{1.3}$$

If we denote by $\psi_0 : (\mathcal{O}_F/\mathfrak{c})^\times \to \mathbf{C}^\times$ *the* $\mathfrak{c}$- *part of the character* ψ *and then extend the definition of* ψ *over* W *by*

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \psi_0(a_{\mathfrak{c}} \bmod \mathfrak{c}),$$

($a_{\mathfrak{c}}$ *being the* $\mathfrak{c}$*-part of* a*) then for all* $x \in G_{\mathbf{A}}$ *and for* $w \in W_{\mathfrak{c}}$ *with* $w_\infty = 1$ *we have that*

$$f(x\,w) = \psi(w^\iota)f(x). \tag{1.4}$$

If $w = w(\theta) = (w_1(\theta_1), \cdots, w_n(\theta_n))$ *with*

$$w_\nu(\theta_\nu) = \begin{pmatrix} \cos\theta_\nu & -\sin\theta_\nu \\ \sin\theta_\nu & \cos\theta_\nu \end{pmatrix},$$

then we have that

$$f(x\,w(\theta)) = f(x)e^{-ik\{\theta\}} \quad (x \in G_{\mathbf{A}}). \tag{1.5}$$

An automorphic form f is called *cusp form* if

$$\int_{F_{\mathbf{A}}/F} f\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g\right) dt = 0\ for\ all\ g \in G_{\mathbf{A}}. \tag{1.6}$$

The vector space $\mathcal{M}_k(\mathfrak{c}, \psi)$ *of Hilbert automorphic forms of holomorphic type* is then defined as the set of functions satisfying (1.3) - (1.5) and the following holomorphy condition (1.7): for each $x \in G_{\mathbf{A}}$ with $x_\infty = 1$ there exist a holmorphic function $g_x : \mathfrak{H}^n \to \mathbf{C}$ such that for all $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty^+$ we have that

$$f(x\,y) \quad = \quad (g_x|_k y)(\mathrm{i}) \tag{1.7}$$

(in case $F = \mathbf{Q}$ it is also assumed that the functions g_x are holomorphic at the cusps). The property (1.7) enables us to describe more explicitely the automorphic forms $f \in \mathcal{M}_k(\mathfrak{c}, \psi)$ in terms of Hilbert modular forms of $\mathfrak{H}^n$. For this purpose we put $f_\lambda = g_{x_\lambda^{-\iota}}$, where $x_\lambda^{-\iota} = \begin{pmatrix} t_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, then $f_\lambda(z) \in \mathcal{M}_k(\Gamma_\lambda, \psi_0)$ for the congruence subgroup

$$\begin{aligned} &\Gamma_\lambda = \Gamma_\lambda(\mathfrak{c}) \subset G_{\mathbf{Q}}^+, \\ &\Gamma_\lambda = x_\lambda W x_\lambda^{-1} \cap G_{\mathbf{Q}} = \\ &\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbf{Q}}^+ | b \in \tilde{t}_\lambda^{-1}\mathfrak{d}^{-1},\ c \in \tilde{t}_\lambda \mathfrak{d}\mathfrak{c}, a, d \in \mathcal{O}_F,\ ad - bc \in \mathcal{O}_F^\times\right\}. \end{aligned}$$

This means that for all $\gamma \in \Gamma_\lambda(\mathfrak{c})$ the following condition (1.8) is satisfied,

$$f_\lambda|_k\gamma = \psi(\gamma)f_\lambda \quad and \quad f_\lambda(z) = \sum_\xi a_\lambda(\xi)e_F(\xi z), \tag{1.8}$$

where $0 \ll \xi \in \tilde{t}_\lambda$ ore $\xi = 0$ in the sum over ξ (see [Shi6] for a more detailed discussion of such expansions). The map $f \to (f_1, \cdots f_h)$ provides an isomorophism of vector spaces

$$\mathcal{M}(\mathfrak{c}, \psi) \cong \oplus_\lambda \mathcal{M}_k(\Gamma_\lambda, \psi)$$

Put

$$C(\mathfrak{m}, f) = \begin{cases} a_\lambda(\xi)\mathcal{N}(\tilde{t}_\lambda)^{-k/2}, & \text{if the ideal } \mathfrak{m} = \xi\tilde{t}_\lambda^{-1} \text{ is integral;} \\ 0, & \text{otherwise.} \end{cases} \tag{1.9}$$

There is a Fourier expansion of the following type

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{0 \ll \zeta \in F, \zeta = 0} C(\zeta\tilde{y}, f)|y|^{k/2}e_F(\zeta i y_\infty)\chi(\zeta x), \tag{1.10}$$

where $\chi_F : F_{\mathbf{A}}/F \to \mathbf{C}^\times$ is a fixed additive character with the condition $\chi_F(x_\infty) = e_F(x_\infty)$ (see [Shi6], p. 650).

Let $\mathcal{S}_k(\mathfrak{c}, \psi) \subset \mathcal{M}(\mathfrak{c}, \psi)$ be the subspace of cusp forms and $f \in \mathcal{S}_k(\mathfrak{c}, \psi)$ then $a_\lambda(0) = 0$ for all $\lambda = 1, \cdots, h$.

1.3. Hecke operators (see [Miy], [Shi1], [Shi6]) are introduced by means of double cosets of the type WyW for y from the semigroup

$$Y_\mathfrak{c} = G_{\mathbf{A}} \cap (G_\infty^+ \times \prod Y_\mathfrak{c}(\mathfrak{p})),$$

where

$$Y_\mathfrak{c}(\mathfrak{p}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\mathfrak{p}) \mid a\mathcal{O}_\mathfrak{p} + \mathfrak{c}_\mathfrak{p} = \mathcal{O}_\mathfrak{p},\ b \in \mathfrak{d}_\mathfrak{p}^{-1}, c \in \mathfrak{c}_\mathfrak{p}\mathfrak{d}_\mathfrak{p}, d \in \mathcal{O}_\mathfrak{p}\right\}. \tag{1.11}$$

The Hecke algebra $\mathcal{H}_\mathfrak{c}$ consists of all formal finite sums of the type $\sum_y c_y WyW$ with $y \in Y_\mathfrak{c}, c_y \in \mathbf{C}$ and with the multiplication defined in the standard way by means of the decomposition of double cosets into a disjoint union of a finite number of left cosets. By definition $T_\mathfrak{c}(\mathfrak{m})$ is an element of the ring $\mathcal{H}_\mathfrak{c}$ which is the sum of all different cosets of the type WyW with $y \in Y_\mathfrak{c}$ such that $\det(y) = \mathfrak{m}$. Let

$$T'_\mathfrak{c} = \mathcal{N}(\mathfrak{m})^{(k-2)/2}T_\mathfrak{c}(\mathfrak{m}) \tag{1.12}$$

be the normalized Hecke operator, whose action on the Fourier coefficients of an automorphic form (of the holomorphic type) $\mathbf{f} \in \mathrm{M}_k(\mathfrak{c}, \psi)$ is given by the standard formula

$$C(\mathfrak{m}, \mathbf{f}|T'_\mathfrak{c}(\mathfrak{m}) = \sum_{\mathfrak{m}+\mathfrak{n}=\mathfrak{a}} \psi^*(\mathfrak{a})\mathcal{N}(\mathfrak{a})^{k-1}C(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \mathbf{f}) \tag{1.13}$$

If $\mathbf{f} \in \mathrm{M}_k(\mathfrak{c}, \psi)$ is an eigenfunction of all Hecke operators $T'_\mathfrak{c}(\mathfrak{m})$ with $\mathbf{f}|_k T'_\mathfrak{c}(\mathfrak{m}) = \lambda(\mathfrak{m})\mathbf{f}$ then we have that $C(\mathfrak{m}, \mathbf{f}) = \lambda(\mathfrak{m})C(\mathcal{O}_F, \mathbf{f})$. If we normalize the form $\mathbf{f}$ by the condition

$C(\mathcal{O}_F, f) = 1$ then there is the following expansion into an Euler product for the L-function of the form $\mathbf{f}$:

$$L(s,\mathbf{f}) = \sum_{\mathfrak{n}} C(\mathfrak{n},\mathbf{f})\mathcal{N}(\mathfrak{n})^{-s} = \sum_{\mathfrak{n}} \lambda(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{-s} = \\ \prod_{\mathfrak{p}}[1 - C(\mathfrak{p},\mathbf{f})\mathcal{N}(\mathfrak{p})^{-s} + \psi^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{k-1-2s}]^{-1}. \tag{1.14}$$

For such a form $\mathbf{f}$ all the numbers $C(\mathfrak{n},\mathbf{f})$ are algebraic integers.

1.4. The Petersson inner product is defined for $\mathbf{f} = (f_1,\cdots,f_h) \in \mathcal{S}_k(\mathfrak{c},\psi)$ and $\mathbf{g} = (g_1,\cdots,g_h) \in \mathcal{M}_k(\mathfrak{c},\psi)$ by the equality

$$\langle \mathbf{f},\mathbf{g}\rangle_{\mathfrak{c}} = \sum_{\lambda=1}^{h} \int_{\Gamma_\lambda(\mathfrak{c})\backslash\mathfrak{H}^n} \overline{f_\lambda(z)} g_\lambda(z)\mathcal{N}(y)^k \, d\mu(z), \tag{1.15}$$

where

$$d\mu(z) = \prod_{\nu=1}^{n} y_\nu^{-2}\, dx_\nu\, dy_\nu$$

is a G_∞^+ -invariant measure on $\mathfrak{H}^n$.

If $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c},\psi)$ and $\mathbf{f}|_k T'_{\mathfrak{c}}(\mathfrak{m}) = \lambda(\mathfrak{m})\mathbf{f}$ for all $\mathfrak{m}$ with $\mathfrak{m} + \mathfrak{c} = \mathbf{Q}_F$ then

$$\lambda(\mathfrak{m}) = \psi^*(\mathfrak{m})\overline{\lambda(\mathfrak{m})}, \\ \psi^*(\mathfrak{m})\langle \mathbf{f}|_k T'_\lambda(\mathfrak{m}),\mathbf{g}\rangle_{\mathfrak{c}} = \langle \mathbf{f},\mathbf{g}|_k T'_\lambda(\mathfrak{m})\rangle_{\mathfrak{c}}. \tag{1.16}$$

let $\mathfrak{q}$ be an integral ideal and $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c},\psi)$. Let us define the operators $\mathbf{f}|\mathfrak{q}, \mathbf{f}|U(\mathfrak{q})$ by their action on the Fourier coefficients:

$$C(\mathfrak{m},\mathbf{f}|\mathfrak{q}) = C(\mathfrak{q}^{-1}\mathfrak{m},\mathbf{f}), \quad C(\mathfrak{m},\mathbf{f}|U(\mathfrak{q})) = C(\mathfrak{q}\mathfrak{m},\mathbf{f}). \tag{1.17}$$

Here is given an explicit description of these operators in terms of the action of double cosets: for a finite idele $q \in F_{\mathbf{A}}^\times$ with $\tilde{q} = q$

$$(\mathbf{f}|\mathfrak{q})(x) \;=\; \mathcal{N}(\mathfrak{q})^{-k/2}\mathbf{f}(x\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}), \tag{1.18}$$

$$(\mathbf{f}|U(\mathfrak{q}))(x) = \mathcal{N}(\mathfrak{q})^{k/2-1} \sum_{v\in\mathcal{O}_F/\mathfrak{q}} f(x\begin{pmatrix} 1 & v \\ 0 & q\end{pmatrix}). \tag{1.19}$$

We define also an involution $J_{\mathfrak{c}}$ by the formula

$$(\mathbf{f}|J_{\mathfrak{c}})(x) = \psi(\det(x)^{-1})f(xb_0) \; with \; b_0 = \begin{pmatrix} 0 & 1 \\ c_0 & 0\end{pmatrix} \in G_{\hat{\mathbf{Q}}}, \tilde{c}_0 = \mathfrak{c}\mathfrak{d}^2, \tag{1.20}$$

then $\mathbf{f}|J_{\mathfrak{c}} \in \mathcal{M}_k(\mathfrak{c},\psi^{-1})$. If f is a primitive form (in the sense of Miyake [Miy]) of conductor $\mathfrak{c}$ then the following hold

$$\mathbf{f}|J_{\mathfrak{c}} = \Lambda(\mathbf{f})\mathbf{f}^\rho, \;\; |\Lambda(\mathbf{f})| = 1, \; C(\mathfrak{m},\mathbf{f}^\rho) = \overline{C(\mathfrak{m},f)}. \tag{1.21}$$

It follows from the definitions (1.18)–(1.20) that

$$\mathbf{f}|J_{\mathfrak{m}\mathfrak{c}} = \mathcal{N}(\mathfrak{m})^{k/2}(\mathbf{f}|J_{\mathfrak{c}})|\mathfrak{m}. \tag{1.22}$$

1.5. Gauss sums and the twist operator. Let χ be a Hecke character of finite order with a conductor $\mathfrak{m}$ and $\chi(x_\infty) = \operatorname{sign}(x_\infty)^r$ for $r = (r_1, \cdots, r_n) \in (\mathbf{Z}/2\mathbf{Z})^n$ (the parity of χ) and let χ^* be the corresponding character of the group of fractional ideals prime to $\mathfrak{m}$. Put $\chi^*(\mathfrak{a}) = 0$ for those $\mathfrak{a}$ which are not coprime to $\mathfrak{m}$ and define the Gauss sum by

$$\tau(\chi) = \sum_{x \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \operatorname{sign}(x)^r \chi^*((x)\mathfrak{m}\mathfrak{d}) e_F(x). \tag{1.23}$$

Then $|\tau(\chi)|^2 = \mathcal{N}(\mathfrak{m})$.

For an arbitrary element $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ there an exist an automorphic form $\mathbf{f}(\chi) \in \mathcal{M}_k(\mathfrak{c}\mathfrak{m}^2, \psi\chi^2)$ which is uniquely defined by the condition $C(\mathfrak{n}, \mathbf{f}(\chi)) = \chi^*(\mathfrak{n})C(\mathfrak{n}, \mathbf{f})$ for all $\mathfrak{n}$ with $\mathfrak{n} + \mathfrak{m} = \mathcal{O}_F$. If a cusp form $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ is primitive of conductor $\mathfrak{c}$ and $\mathfrak{c} + \mathfrak{m} = \mathcal{O}_F$, then the conductor of $\mathbf{f}(\chi)$ is equal to $\mathfrak{c}\mathfrak{m}^2$ and the "pseudo-eigenvalue" of the involution $J_{\mathfrak{c}\mathfrak{m}^2}$ on $\mathbf{f}$ is given by the well known formula

$$\Lambda(\mathbf{f}(\chi)) = \psi^*(\mathfrak{m})\chi^*(\mathfrak{c})\tau(\chi)^2\mathcal{N}(\mathfrak{m})^{-1}\Lambda(\mathbf{f}) \tag{1.24}$$

(see [Shi6], p.664)

§2. Description of the non-Archimedean Rankin convolution of Hilbert automorphic forms

2.1. As in the Introduction, let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}(\mathbf{f}), \psi)$, $g \in \mathcal{S}_l(\mathfrak{c}(\mathbf{g}), \omega)$ be the primitive cusp forms of (scalar) weights k and l of the conductors $\mathfrak{c}(\mathbf{f})$, $\mathfrak{c}(\mathbf{g})$ with the Hecke characters of finite order ψ and ω such that $k > l$ and $\psi(x) = \operatorname{sign}\mathcal{N}(x)^k$, $\omega(x) = \operatorname{sign}\mathcal{N}(x)^l$ for $x \in F_\infty^\times$. For an integral ideal $\mathfrak{a}$ denote by $S(\mathfrak{a})$ its support:

$$S(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{p} \ \ divides\ \mathfrak{a}\},$$

and put $S(\mathbf{f}) = S(\mathfrak{c}(\mathbf{f}))$, $S(\mathbf{g}) = S(\mathfrak{c}(\mathbf{g}))$ and $S(\chi) = S(\mathfrak{m})$ for a Hecke character of conductor $\mathfrak{m}$.

We assume that

$$S_F \cap S(\mathbf{f}) = S_F \cap S(\mathbf{g}) = S(\mathbf{f}) \cap S(\mathbf{g}) = \emptyset, \tag{2.1}$$

$$C(\mathfrak{c}(\mathbf{f}), \mathbf{f}) \cdot C(\mathfrak{c}(\mathbf{g}), \mathbf{g}) \neq 0, \tag{2.2}$$

$$|i_p(C(\mathfrak{q}, \mathbf{f}))|_p = 1 \ \ for\ all\ \mathfrak{q} \in S_F. \tag{2.3}$$

Put $\mathfrak{c} = \mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g})$. For $\mathfrak{q} \in S_F$ we denote by the symbol $\alpha(\mathfrak{q})$ such root of the Hecke polynomial

$$X^2 - C(\mathfrak{q}, \mathbf{f})X + \psi^*(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{k-1},$$

that $|i_p(\alpha(\mathfrak{q}))|_p = 1$ and let α' be the other root of the polynomial. Then it follows from the property (1.16) that the numbers

$$\hat{\alpha}(\mathfrak{q}) = \overline{\psi^*(\mathfrak{q})}\alpha(\mathfrak{q}), \quad \hat{\alpha}'(\mathfrak{q}) = \overline{\psi^*(\mathfrak{q})}\alpha'(\mathfrak{q})$$

coincide with the roots of the complex conjugate polynomial

$$X^2 - \overline{C(\mathfrak{q},\mathbf{f})}X + \psi^*(\mathfrak{q})^{-1}\mathcal{N}(\mathfrak{q})^{k-1}$$

for all $\mathfrak{q}$ such that $\mathfrak{q} + \mathfrak{c}(\mathbf{f}) = \mathcal{O}_F$.

Similarly, if

$$X^2 - C(\mathfrak{q},\mathbf{g})X + \omega^*(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{l-1} = (X - \beta(\mathfrak{q}))(X - \beta'(\mathfrak{q}))$$

then for $\mathfrak{q}$ with $\mathfrak{q} + \mathfrak{c}(\mathbf{g}) = \mathcal{O}_F$ the numbers

$$\hat{\beta}(\mathfrak{q}) = \overline{\omega^*(\mathfrak{q})}\beta(\mathfrak{q}), \quad \hat{\beta}'(\mathfrak{q}) = \overline{\omega^*(\mathfrak{q})}\beta'(\mathfrak{q})$$

coincide with the roots of the polynomial

$$X^2 - \overline{C(\mathfrak{q},\mathbf{g})}X + \omega^*(\mathfrak{q})^{-1}\mathcal{N}(\mathfrak{q})^{l-1}.$$

Let us extend the definition of the numbers $\alpha(\mathfrak{n}), \alpha'(\mathfrak{n}), \beta(\mathfrak{n}), \beta'(\mathfrak{n})$ to all integral ideals by multiplicativity. As in the introduction we denote by the same letter χ both Hecke characters and the corresponding elements of $\mathcal{X}_S$ and recall that

$$\mathcal{N} : \mathrm{Gal}_S \to \mathrm{Gal}(\mathbf{Q}(S)/\mathbf{Q}) \cong \mathbf{Z}_S^\times = \prod_{q \in S} \mathbf{Z}_q^\times$$

is a natural homomorphism defined by the restriction of Galois automorphisms from $F(S)$ to $\mathbf{Q}(S)$,so that $\mathcal{N}x_p$ is the composition of this homomorphism with the natural projection $\prod_q \mathbf{Z}_q^\times \to \mathbf{Z}_p^\times$ and the inclusion $\mathbf{Z}_p^\times \subset \mathbf{C}_p^\times$. Put

$$\Omega(\mathbf{f}) = \langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}$$

.

2.2. Theorem. *Under the assumptions (2.1) - (2.3) and notations as above there exists a bounded $\mathbf{C}_p$-analytic function $\Psi_S : \mathcal{X}_S \to \mathbf{C}_p$ which is uniquely determined by the condition: for each Hecke character of finite order $\chi \in \mathcal{X}_S^{\mathrm{tors}}$ with $S(\chi) \subset S_F$ of conductor $\mathfrak{m} = \mathfrak{c}(\chi)$ and for each integers $r = 0, 1, \cdots k - l - 1$ the value $\Psi_S(\chi \mathcal{N} x_p^r)$is given by the image under the fixed embedding i_p of the algebraic number*

$$(-1)^{-r} D_F^{2r+2l} \omega^*(\mathfrak{m}) \frac{\tau(\chi)^2 \mathcal{N}\mathfrak{m}^{l+2r-1}}{\alpha(\mathfrak{m})^2} \frac{\Psi(l + r, \mathbf{f}, \mathbf{g}^\rho(\overline{\chi}))}{(-2\pi i)^{n(1-l)}\Omega(f)} A(r, \chi), \tag{2.4}$$

where

$$\begin{aligned} A(r,\chi) = & \\ \prod_{\mathfrak{q} \in S_F \setminus S(\chi)} & (1 - (\chi^* \alpha^{-1}\beta)(\mathfrak{q})\mathcal{N}(\mathfrak{q})^r)(1 - (\chi^* \alpha^{-1}\beta')(\mathfrak{q})\mathcal{N}(\mathfrak{q})^r) \times \\ & \times (1 - (\chi^{*-1}\alpha'\hat{\beta})(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-l-r})(1 - (\chi^{*-1}\alpha'\hat{\beta}')(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-l-r}). \end{aligned} \tag{2.5}$$

D_F is the discriminant of F, $\tau(\chi)$ being the Gauss sum of χ, and $\mathbf{g}^\rho(\chi)$ the cusp form obtained from $\mathbf{g}$ by complex conjugation of its Fourier coefficients and by twisting it then with the character χ (see §1).

§3. Distributions on the Galois group Gal_S

3.1. The function $\Psi_S(x, \mathbf{f}, \mathbf{g})$ of theorem 2.2 is constructed by means of the theory of non-Archimedean integration ([Man6], [Ko1], [Ko2], [V1]). Let $\mathcal{O}_p$ be the subring of integral elements in the Tate field $\mathbf{C}_p$, $\mathcal{O}_p = \{x \in \mathbf{C}_p \mid |x|_p \le 1\}$ and $\mathcal{C}(\mathrm{Gal}_S, \mathbf{C}_p)$ denote the ring of all continuous p-adic functions on Gal_S. Then $\mathcal{O}_p$ -measure on Gal_S is an arbitrary $\mathcal{O}_p$-linear map $\mu : \mathcal{C}(\mathrm{Gal}_S, \mathbf{C}_p) \to \mathcal{O}_p$ which is denoted also as

$$f \quad \mapsto \quad \int_{\mathrm{Gal}_S} f\, d\mu.$$

We note that

$$\mathcal{X}_S = \mathrm{Hom}_{\mathrm{contin}}(\mathrm{Gal}_S, \mathbf{C}_p^\times) \subset \mathcal{C}(\mathrm{Gal}_S, \mathbf{C}_p).$$

Therefore for an arbitrary $\mathcal{O}_p$ -measure μ we have that the function

$$L_\mu : \mathcal{X}_S \to \mathcal{O}_p \quad (L_\mu = \int_{\mathrm{Gal}_S} \chi d\mu \quad for \quad \chi \in \mathcal{X}_S)$$

which is called the non-Archimedean Mellin transform of the measure μ. The function L_μ is bounded and analytic on the p-adic analytic Lie group $\mathcal{X}_S$. If we fix a character $\chi_0 \in \mathcal{X}_S$ then we get that the values of the type $L_\mu(\chi_0\, \chi)$ with $\chi \in \mathcal{X}_S^{\mathrm{tors}}$ uniquely determine the function L_μ (see [V1], [Man6], [Ka3]). The analytic structure on $\mathcal{X}_S$ is defined by means of the class field theory. Namely, let $\mathfrak{m}$ be an integral ideal, $I(\mathfrak{m})$ be the group of fractional ideals coprime to $\mathfrak{m}$,

$$P(\mathfrak{m}) = \{(\alpha) \mid \alpha \in F_+^\times\} \ \alpha \equiv 1(\mathrm{mod}^\times \mathfrak{m}),$$
$$H(\mathfrak{m}) = I(\mathfrak{m}) / P(\mathfrak{m})$$

the ideal class group (in the narrow sense) of conductor $\mathfrak{m}$. Then if $F(\mathfrak{m})/F$ is the maximal abelian extension of F unramified outside primes dividing $\mathfrak{m}$ and infinity then the Artin symbol provides an isomorphism:

$$H(\mathfrak{m}) \xrightarrow{\sim} \mathrm{Gal}(F(\mathfrak{m})/F), \quad \mathfrak{a} \ \mapsto \ \left(\frac{\mathfrak{a}, F(\mathfrak{m})}{F}\right),$$

and we get

$$F(S) = \cup_{\mathfrak{m}} F(\mathfrak{m}), \quad \mathrm{Gal}_S = \varprojlim_{\mathfrak{m}} H(\mathfrak{m}) \quad (S(\mathfrak{m}) \subset S_F).$$

There is the following exact sequence

$$1 \ \to \ \mathrm{Gal}_S^0 \ \to \ \mathrm{Gal}_S \ \to \ \mathrm{Gal}_1^+ \ \to \ 1$$

where $\mathrm{Gal}_1^+ = \mathrm{Gal}(F(\emptyset))/F)) = \widetilde{Cl}_F$, $F(\emptyset)$ being the maximal abelian extension of F ramified only over ∞,

$$\mathrm{Gal}_S^0 = \mathrm{Gal}(F(S)/F(\emptyset)) = \mathcal{O}_S^\times / (\mathrm{clos}(\mathcal{O}_+^\times)),$$

where $\mathrm{clos}(\mathcal{O}_+^\times)$ denotes the closure of the group of all totally positive units $\mathcal{O}_+^\times$ in the group

$$\mathcal{O}_S^\times = \prod_{q\in S}(\mathcal{O}_F \otimes \mathbf{Z}_q)^\times.$$

There is the natural canonical $\mathbf{C}_p$-analytic structure on the group

$$\mathrm{Hom}_{\mathrm{contin}}((\mathcal{O}_F \otimes \mathbf{Z}_p)^\times, \mathbf{C}_p^\times)$$

so that this group is a $\mathbf{C}_p$-analytic Lie group which is n-dimensional over $\mathbf{C}_p$. We extend this structure onto the whole group $\mathcal{X}_S$ by translations (see [Man6], [V1]).

3.2. We construct a measure $\mu = \mu(\mathbf{f},\mathbf{g})$ on the group Gal_S for which the value $L_\mu(\chi\mathcal{N}x_p^r)$ is given by the formula (2.4) and set $\Psi_S(x) = L_\mu(x)$. In order to construct μ we use the theory of distributions. Recall that for an abelian group $\mathcal{A}$ then a distribution μ on Gal_S is a finite additive family of functions

$$\mu = \{\mu_\mathfrak{m}\} \quad \mu_\mathfrak{m} : \ H(\mathfrak{m}) \to \mathcal{A}$$

so that the compatibility condition

$$\sum_{x\equiv y \bmod^\times \mathfrak{m}_1} \mu_\mathfrak{m}(x) = \mu_{\mathfrak{m}_1}(y) \tag{3.1}$$

holds for all $\mathfrak{m}_1$ with $\mathfrak{m}_1|\mathfrak{m}$. If $\mathcal{A} = \mathcal{O}_p$ then μ defines an $\mathcal{O}_p$-measure. We call a $\mathbf{C}_p$-distribution a *bounded measure* if $a\,\mu$ is a $\mathcal{O}_p$ -measure for some $a \in \mathbf{C}_p^\times$. Now let $\mathcal{A}$ be a vector space over $\mathbf{Q}^{\mathrm{ab}}$ (the maximal abelian extension of $\mathbf{Q}$) and $\mu_\mathfrak{m} : H(\mathfrak{m}) \to \mathcal{A}$ be an arbitrary family of functions. For a character $\chi : H(\mathfrak{m}_1) \to \mathbf{Q}^{\mathrm{ab}\times}$ and for $\mathfrak{m}_1|\mathfrak{m}$ we define $\chi_\mathfrak{m}$ as the composition of the natural projection $H(\mathfrak{m}) \to H(\mathfrak{m}_1)$ $(\mathrm{mod}\,\mathfrak{m}_1)$ and χ and put

$$\mu_\mathfrak{m}(\chi_\mathfrak{m}) = \sum_{x\in H(\mathfrak{m})} \chi_\mathfrak{m}(x)\mu_\mathfrak{m}(x).$$

Then the compatibility condition (3.1) is equivalent to the fact that

$$\mu_\mathfrak{m}(\chi_\mathfrak{m}) \textit{ is independent on the choice of } \mathfrak{m} \textit{ with } \mathfrak{m}_1|\mathfrak{m}. \tag{3.2}$$

Using this compatibility criterion we construct $\mu(\mathbf{f},\mathbf{g})$ as a distribution defined by means of a certain family of functions

$$\tilde\mu_{s,\mathfrak{m}} : H(\mathfrak{m}) \to \overline{\mathbf{Q}}.$$

3.3. Proposition. *For every $s \in \mathbf{C}$ there exists a complex valued distribution $\{\tilde\mu_{s,\mathfrak{m}}\}_\mathfrak{m}$ which is uniquely determined by the condition*

$$\tilde\mu_{s,\mathfrak{m}}(\chi_\mathfrak{m}) = \frac{\mathcal{N}(\mathfrak{c}\mathfrak{d}^2)^s\mathcal{N}(\mathfrak{m}')^{s-(l/2)}}{\alpha(\mathfrak{m}')}\frac{\Psi(s,\mathrm{f}_0|\mathfrak{c}(\mathbf{g}),\mathbf{g}(\chi_\mathfrak{m})|J_{\mathfrak{c}\mathfrak{m}'})}{(-2\pi i)^{n(1-l)}\cdot\Omega(\mathbf{f})} \tag{3.3}$$

where $\mathfrak{m}, \mathfrak{m}'$ *are arbitrary ideals with the condition* $\mathfrak{m}_0\mathfrak{c}(\chi)|\mathfrak{m}$, $\mathfrak{c}(\chi)^2\mathfrak{m}_0^2|\mathfrak{m}'$,

$$\mathbf{f}_0 = \sum_{\mathfrak{a}|\mathfrak{m}_0} \mathrm{M}(\mathfrak{a})\alpha'(\mathfrak{a})\,\mathbf{f}|\mathfrak{a},$$

with $\mathrm{M}(\mathfrak{a})$ *being the Möbius function (of the ideals) so that this definition of* $\mathbf{f}_0$ *is equivalent to the identity*

$$L(s,\mathbf{f}_0) = \prod_{\mathfrak{q}|\mathfrak{m}_0}(1-\alpha'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})L(s,\mathbf{f}).$$

3.4. *The proof* of the proposition is carried out by means of the compatibility criterion (3.2) according to which it suffices to check that the right hand side of (3.3) is independent of $\mathfrak{m}$ and of $\mathfrak{m}'$. First of all $\mathbf{g}(\chi_{\mathfrak{m}}) = \mathbf{g}(\chi_{\mathfrak{m}_0})$ is independent of $\mathfrak{m}$ because of the property $S(\mathfrak{m}_0) = S(\mathfrak{m}) = S_F$ and

$$L(s,\mathbf{g}(\chi_{\mathfrak{m}})) = \sum_{\mathfrak{n}+\mathfrak{m}_0=\mathcal{O}_F} \chi(\mathfrak{n})C(\mathfrak{n},\mathbf{g})\mathcal{N}(\mathfrak{n})^{-s} = L(s,\mathbf{g}\mathfrak{m}_0(\chi)).$$

On the other hand, the formula (1.22) implies the identity: for $\mathfrak{m}' = \mathfrak{m}_0^2\mathfrak{c}(\chi)^2\mathfrak{m}_1$ we have that

$$\mathbf{g}(\chi_{\mathfrak{m}})|J_{\mathfrak{c}\mathfrak{m}'} = \mathcal{N}(\mathfrak{m}_1\mathfrak{c}(\mathbf{f}))^{l/2}(\mathbf{g}_{\mathfrak{m}_0}(\chi)|J_{\mathfrak{c}(\mathbf{g})\mathfrak{m}_0^2\mathfrak{c}(\chi)^2})|\mathfrak{m}_1\mathfrak{c}(\mathbf{f}). \tag{3.4}$$

Put

$$\sum_{\mathfrak{n}} A(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{-s}, \tag{3.5},$$

$$\sum_{\mathfrak{n}} B(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{-s} = L(s,\mathbf{g}_{\mathfrak{m}_0}(\chi)|J_{\mathfrak{c}(\mathbf{g})\mathfrak{m}_0^2\mathfrak{c}(\chi)^2}), \tag{3.6}$$

Then (1.17) and (1.21) imply the equalities

$$A(\mathfrak{n}\mathfrak{c}(\mathbf{f})\mathfrak{m}_1) = \alpha(\mathfrak{c}(\mathbf{f})\mathfrak{m}_1)A(\mathfrak{n}) = C(\mathfrak{c}(\mathbf{f}),\mathbf{f})\alpha(\mathfrak{m}_1)A(\mathfrak{n}), \tag{3.7}$$

$$B(\mathfrak{n}\mathfrak{c}(\mathbf{g})) = \overline{C(\mathfrak{c}(\mathbf{g}),\mathbf{g})}\chi^{*-1}(\mathfrak{c}(\mathbf{g}))B(\mathfrak{n}), \tag{3.8}$$

Taking into account that $\mathfrak{m}_1\mathfrak{c}(\mathbf{f}) + \mathfrak{c}(\mathbf{g}) = \mathcal{O}_F$ we obtain from (3.4), (3.7), (3.8) the following equality

$$\begin{aligned}
&L(s,\mathbf{f}_0|\mathfrak{c}(\mathbf{g}),g(\chi_{\mathfrak{m}_0})|J_{\mathfrak{c}\mathfrak{m}'}) = \\
&\mathcal{N}(\mathfrak{m}_1\mathfrak{c}(\mathbf{f}))^{l/2}\sum_{\mathfrak{n}} A(\mathfrak{n}\mathfrak{c}(\mathbf{g})^{-1})B(\mathfrak{n}\mathfrak{m}_1^{-1}\mathfrak{c}(\mathbf{f})^{-1})\mathcal{N}(\mathfrak{n})^{-s} = \\
&\mathcal{N}(\mathfrak{m}_1\mathfrak{c}(\mathbf{f}))^{l/2}\sum_{\mathfrak{n}} A(\mathfrak{n}\mathfrak{m}_1\mathfrak{c}(\mathbf{f}))B(\mathfrak{n}\mathfrak{m}_1\mathfrak{c}(\mathbf{g}))\mathcal{N}(\mathfrak{n}\mathfrak{m}_1\mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g}))^{-s} = \\
&\frac{\kappa(\mathbf{f},\mathbf{g})\alpha(\mathfrak{m}_1)\chi^{*-1}(\mathfrak{c}(\mathbf{g}))}{\mathcal{N}(\mathfrak{c})^s\mathcal{N}(\mathfrak{m}_1)^{s-(l/2)}}L(s,\mathbf{f}_0,g(\chi_{\mathfrak{m}_0})|J_{\mathfrak{c}(\mathbf{g})\mathfrak{c}(\chi)^2\mathfrak{m}_0^2})
\end{aligned} \tag{3.9}$$

with the constant

$$\kappa(\mathbf{f},\mathbf{g}) = \mathcal{N}(\mathfrak{c}(\mathbf{f}))^{l/2}C(\mathfrak{c}(\mathbf{f}),\mathbf{f})\overline{C(\mathfrak{c}(\mathbf{g}),\mathbf{g})}$$

which is not zero due to the assumption (2.2). It now follows from (3.9) and from the definition (0.2) that the following identity holds

$$\begin{gathered}\frac{\mathcal{N}(\mathfrak{c}\mathfrak{d}^2)^s\mathcal{N}(\mathfrak{m}')^{s-(l/2)}}{\alpha(\mathfrak{m}')}\Psi(s,\mathbf{f}_0|\mathfrak{c}(\mathbf{g}),g(\chi_{\mathfrak{m}})|J_{\mathfrak{c}\mathfrak{m}'}) = \\ \kappa(\mathbf{f},\mathbf{g})D_F^{2s}\frac{\mathcal{N}(\mathfrak{c}(\chi)\mathfrak{m}_0)^{2s-l}\chi^*(\mathfrak{c}(\mathbf{g}))^{-1}}{\alpha(\mathfrak{c}(\chi)\mathfrak{m}_0)^2}\cdot\Psi(s,\mathbf{f}_0,g(\chi_{\mathfrak{m}_0})|J_{\mathfrak{c}(\mathbf{g})\mathfrak{c}(\chi)^2\mathfrak{m}_0^2})\end{gathered}, \tag{3.10}$$

that is the terms with $\mathfrak{m}'$ cancel and we see that $\tilde{\mu}_{s,\mathfrak{m}}(\chi_{\mathfrak{m}})$ is independent of $\mathfrak{m}$ and of $\mathfrak{m}'$ proving the proposition.

Next we will use the property (1.24) of the twist operator in order to get a more explicit expression for the right hand side of (3.3) which is essential for the formula (2.4) in the theorem 2.2.

3.5. Proposition. *Under the notations and assumptions as above the following equality holds*

$$\begin{aligned}&\Psi(s,\mathbf{f}_0,g(\chi_{\mathfrak{m}_0})|J_{\mathfrak{c}(\mathbf{g})\mathfrak{c}(\chi)^2\mathfrak{m}_0^2}) = \\ &\qquad \alpha(\mathfrak{m}_0)^2\mathcal{N}(\mathfrak{m}_0)^{l-2s}A(s-l,\chi)\Lambda(\mathbf{g}(\chi))\Psi(s,\mathbf{f},\mathbf{g}^\rho(\chi^{-1})),\end{aligned} \tag{3.11}$$

with

$$\begin{aligned}A(s-l,\chi) = &\\ &\prod_{\mathfrak{q}\in S_F\setminus S(\chi)}(1-(\chi^*\alpha^{-1}\beta)(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{s-l})(1-(\chi^*\alpha^{-1}\beta')(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{s-l})\times\\ &\times(1-(\chi^{*-1}\alpha'\hat{\beta})(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1-(\chi^{*-1}\alpha'\hat{\beta}')(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})\end{aligned}$$

defined by (2.5) and with the twisted root number

$$\Lambda(\mathbf{g}(\chi)) = \omega^*(\mathfrak{c}(\chi))\chi^*(\mathfrak{c}(\mathbf{g})\tau(\chi)^2\mathcal{N}(\mathfrak{c}(\chi)))^{-1}\Lambda(\mathbf{g})$$

given by (1.24).

3.6. In order to prove the proposition we simply input (3.11) into the definition (3.3):

$$\begin{aligned}\tilde{\mu}_{s,\mathfrak{m}}(\chi_{\mathfrak{m}}) = &\kappa(\mathbf{f},\mathbf{g})\Lambda(\mathbf{g})A(s-l,\chi)D_F^{2s}\times\\ &\times\frac{\omega^*(\mathfrak{c}(\chi))\tau(\chi)^2\mathcal{N}(\mathfrak{c}(\chi))^{2s-l-1}}{\alpha(\mathfrak{c}(\chi))^2}\cdot\frac{\Psi(s,\mathbf{f},\mathbf{g}^\rho(\chi^{-1}))}{(-2\pi i)^{n(1-l)}\Omega(\mathbf{f})}.\end{aligned} \tag{3.12}$$

Now we see from the algebraicity property (0.3) that the values $\tilde{\mu}_{l+r,\mathfrak{m}}(y)$ belong to $\overline{\mathbf{Q}}$ for all $y\in H(\mathfrak{m})$. Put

$$\mu(y,\mathbf{f},\mathbf{g}) = i_p(\tilde{\mu}_l(y)/\Lambda(\mathbf{g})\kappa(\mathbf{f},\mathbf{g})), \tag{3.13}$$

In this way we have obtained a $\mathbf{C}_p$ -valued distribution on the Galois group Gal_S which satisfy the equality (2.4) with $r=0$. The p - adic boundness of the distribution, which is equivalent to certain generalized Kummer congruences for the numbers

$$\frac{\Psi(s,\mathbf{f},\mathbf{g}^\rho(\chi^{-1}))}{(-2\pi i)^{n(1-l)}\Omega(\mathbf{f})},$$

is established in §5 below.

§4. The integral representation of Rankin - Shimura

4.1. We start by recalling the definition of Eisenstein series in the Hilbert modular case. Let $\mathfrak{a}, \mathfrak{b}$ be arbitrary fractional ideals , η a Hecke character of finite order modulo an integral ideal $\mathfrak{c} \subset \mathcal{O}_F$ such that $\eta^*((x)) = \mathrm{sign}\mathcal{N}(x)^m$ for $x \equiv 1 \bmod^\times \mathfrak{c}$, $x \in \mathcal{O}_F$. Put (for $\mathrm{Re}(s) > 2 - m$)

$$K_m(z, s; \mathfrak{a}, \mathfrak{b}; \eta) = \sum_{c,d} \mathrm{sign}\mathcal{N}(d)^m \, \eta^*(d\mathfrak{b}^{-1})\mathcal{N}(cz+d)^{-m}|\mathcal{N}(cz+d)|^{-2s}, \tag{4.1}$$

$$L_m(z, s; \mathfrak{a}, \mathfrak{b}; \eta) = \sum_{c,d} \mathrm{sign}\mathcal{N}(c)^m \, \eta^*(c\mathfrak{a}^{-1})\mathcal{N}(cz+d)^{-m}|\mathcal{N}(cz+d)|^{-2s}, \tag{4.2}$$

the summation in (4.1) and (4.2) being taken over a system of representatives (c, d) of equivalence classes with respect to the $\mathcal{O}_F^\times$ - equivalence relation for non-zero elements in $\mathfrak{a} \times \mathfrak{b}$ given by $(c, d) \sim (uc, ud)$ with $u \in \mathcal{O}_F^\times$. The series (4.1) and (4.2) can be extended to function on the adelized group $G_{\mathbf{A}}$ as in §1 so that

$$K_m(s; \mathfrak{a}, \mathfrak{b}; \eta)_\lambda = \mathcal{N}(\tilde{t}_\lambda)^{s+(m/2)}\mathcal{N}(y)^s K_m(z, s; \tilde{t}_\lambda \mathfrak{d}\mathfrak{a}, \mathfrak{b}; \eta), \tag{4.3}$$

$$L_m(s; \mathfrak{a}, \mathfrak{b}; \eta)_\lambda = \mathcal{N}(\tilde{t}_\lambda)^{-s-(m/2)}\mathcal{N}(y)^s L_m(z, s; \mathfrak{a}, \mathfrak{b}\tilde{t}_\lambda^{-1}\mathfrak{d}^{-1}; \eta). \tag{4.4}$$

The functions (4.1)–(4.4) admit analytic continuation over the whole complex plane with respect to the parameter $s \in \mathbf{C}$, and under the assumption of the primitivity of η modulo $\mathfrak{c}$ the following functional equation holds (see [Shi6], p.672):

$$\begin{aligned} \Delta_m(1-m-s)K_m(1-m-s; \mathfrak{a}, \mathfrak{b}; \eta) = \\ \tau(\eta)\mathcal{N}(\mathfrak{d}\mathfrak{a}\mathfrak{b}\mathfrak{c})^{m+2s-1}\Delta_m(s)L_m(s; \mathfrak{a}, \mathfrak{b}\mathfrak{c}; \overline{\eta}), \end{aligned} \tag{4.5}$$

with the Γ - factor $\Delta_m(s) = \pi^{-ns}\Gamma(s+m)^n$. If $\mathfrak{q}$ is an integral ideal then the action of the involution $J_\mathfrak{q}$ on (4.3) and (4.4) is easily calculated by the definition (1.20) and is given by the formula

$$K_m(s; \mathfrak{a}, \mathfrak{b}; \eta)|J_\mathfrak{q} = (-1)^{mn}\mathcal{N}(\mathfrak{q}\mathfrak{d}^2)^{-s-(m/2)}L_m(s; \mathfrak{b}, \mathfrak{a}\mathfrak{q}^{-1}; \eta). \tag{4.6}$$

4.2. The integral representation. Put

$$F = \mathrm{f}_0|\mathfrak{c}(\mathrm{g}) \in \mathcal{S}_k(\mathfrak{c}\mathfrak{m}_0, \psi), \quad G = g(\chi_{\mathfrak{m}_0})|J_{\mathfrak{c}\mathfrak{m}'} \in \mathcal{S}_l(\mathfrak{c}\mathfrak{m}', \omega^{-1}\chi^{-2}$$

Then the following integral representation of Rankin type holds (see [Shi6], (4.32)):

$$\Psi(s, F, G) = D_F^{1/2}\Gamma(s+1-l)^n\pi^{-ns}\langle F^\rho, V(s-k+1)\rangle_{\mathfrak{c}\mathfrak{m}'}, \tag{4.7}$$

where

$$V(s) = G \cdot K_{k-l}(s; \mathfrak{c}\mathfrak{m}', \mathcal{O}_F; \psi\omega^{-1}\chi^{-2}).$$

More precisely

$$\Psi(s,F,G) = D_F^{1/2}\Gamma(s+1-l)^n\pi^{-ns}\sum_{\lambda=1}^{h}\mathcal{N}(\tilde{t}_\lambda)^{s+1-(k+l)/2}\times$$
$$\times\int_{\Gamma_\lambda(\mathfrak{c}\mathfrak{m}')\backslash\mathfrak{H}^n}\overline{F_\lambda^\rho(z)}G_\lambda(z)\,K_{k-l}(z,s-k+1;\tilde{t}_\lambda\mathfrak{d},\mathcal{O}_F;\psi\omega^{-1}\chi^{-2})\mathcal{N}(y)^{s-1}\,dx\,dy. \tag{4.8}$$

4.2. Application of the trace operator. The method by which the right hand side of (4.7) can be explicitly calculated is based on an application of the trace operator (as in [Man-Pa], [Pa3], [Pa9], [Ar]) which is described below. Let R be a system of representatives of the right cosets $W_{\mathfrak{c}\mathfrak{m}'}\backslash W_{\mathfrak{c}\mathfrak{m}_0}$, then the trace $V|\mathrm{Tr}_{\mathfrak{c}\mathfrak{m}_0}^{\mathfrak{c}\mathfrak{m}'}$ is defined by the equality: for $x\in G_{\mathbf{A}}, h\in R$

$$(V|\mathrm{Tr}_{\mathfrak{c}\mathfrak{m}_0}^{\mathfrak{c}\mathfrak{m}'})(x) = \sum_{h\in R}\psi(h^{-\iota})V(x\,h). \tag{4.9}$$

Then the scalar product in the right hand side of (4.7) takes the form:

$$\langle F^\rho, V(s-k+1)\rangle_{\mathfrak{c}\mathfrak{m}'} = \langle F^\rho, V(s-k+1)|\mathrm{Tr}_{\mathfrak{c}\mathfrak{m}_0}^{\mathfrak{c}\mathfrak{m}'}\rangle_{\mathfrak{c}\mathfrak{m}_0}. \tag{4.10}$$

The explicit formula for the action of the trace on the Fourier expansions is provided by the equality:

$$V|\mathrm{Tr}_{\mathfrak{c}\mathfrak{m}_0}^{\mathfrak{c}\mathfrak{m}'} = \mathcal{N}(\mathfrak{m}'\mathfrak{m}_0^{-1})^{1-(k/2)}V|J_{\mathfrak{c}\mathfrak{m}'}U(\mathfrak{m}'\mathfrak{m}_0^{-1})J_{\mathfrak{c}\mathfrak{m}_0}, \tag{4.11}$$

which is deduced by the special choice of a system of representatives for $W_{\mathfrak{c}\mathfrak{m}'}\backslash W_{\mathfrak{c}\mathfrak{m}_0}$, namely

$$R=\left\{\begin{pmatrix}1 & 0\\ cm_0v & 1\end{pmatrix}\quad (v\in\mathcal{O}_F/\mathfrak{m}'\mathfrak{m}_0^{-1})\right\}$$

where cm_0 is an idele such that $\widetilde{cm_0}=\mathfrak{c}\mathfrak{m}_0$, and from the definitions (1.19), (1.20) of the operators $U(\mathfrak{m}'\mathfrak{m}_0^{-1})$ and $J_{\mathfrak{c}}$. If we now use (4.10) and (4.11) then the identity (4.7) transforms to

$$\Psi(s,F,G) = D_F^{1/2}\Gamma(s+1-l)^n\pi^{-ns}\mathcal{N}(\mathfrak{m}'\mathfrak{m}_0^{-1})^{1-(k/2)}\times$$
$$\times\langle F^\rho, V'(s-k+1)|U(\mathfrak{m}'\mathfrak{m}_0^{-1})J_{\mathfrak{c}\mathfrak{m}_0}\rangle_{\mathfrak{c}\mathfrak{m}_0}, \tag{4.12}$$

where

$$V'(s) = V(s)|J_{\mathfrak{c}\mathfrak{m}'} = (-1)^{nl}\mathrm{g}(\chi_{\mathfrak{m}_0})(K_{k-l}(s;\mathfrak{c}\mathfrak{m}',\mathcal{O}_F;\psi\omega^{-1}\chi^{-2}|J_{\mathfrak{c}\mathfrak{m}_0}) =$$

$$(-1)^{nk}\mathcal{N}(\mathfrak{c}\mathfrak{m}'\mathfrak{d}^2)^{-s-(k/2)}L_{k-l}(s;\mathcal{O}_F,\mathcal{O}_F;\psi\omega^{-1}\chi^{-2})$$

according to the formula (4.6).

4.3. Normalized Eisenstein series. Let us define the normalized Eisenstein series for all integers m and r with the condition $m \geq 0,\ m+r > 0$ by

$$E^*_{m,r}(\eta) = \frac{2^{-n} D_F^{1/2} \Gamma(m+r)^n}{(-4\pi)^{rn}(-2\pi i)^{mn}} L_{m+2r}(-r; \mathcal{O}_F, \mathcal{O}_F; \eta). \tag{4.13}$$

Then the integral representation takes the form

$$\begin{aligned}\Psi(s,F,G) = (-1)^{nk}\mathcal{N}(\mathrm{m}'\mathrm{m}_0^{-1})^{1-(k/2)}\mathcal{N}(\mathrm{c}\mathrm{m}'\mathfrak{d}^2)^{-(s-k+1)-(k-l)/2}\times \\ \times 2^n\pi^{n(k-l)}\langle F^\rho, [\mathrm{g}(\chi_{\mathrm{m}_0})E^*_{2-m,m+r-1}(\psi\omega^{-1}\chi^{-2})]|U(\mathrm{m}'\mathrm{m}_0^{-1})J_{\mathrm{cm}_0}\rangle_{\mathrm{cm}_0}\end{aligned} \tag{4.14}$$

with $s = l+r, m = k-l-2r, r = 0,\cdots,k-l-1$. This formula provides us also with the integral representation for the values of the distributions (3.3): for $s = l+r, m = k-l-2r\ \ (r = 0,\cdots,k-l-1)\ \ \mathrm{m}_0\ |\mathrm{m}\ |\ \mathrm{m}'$ we have that

$$\begin{aligned}&\tilde{\mu}_{s,\mathrm{m}}(\chi) = \\ &\gamma'(\mathrm{m}')\frac{\langle F^\rho, [\mathrm{g}(\chi_{\mathrm{m}_0})E^*_{2-m,m-r+1}(\psi\omega^{-1}\chi^{-2}]|U(\mathrm{m}'\mathrm{m}_0^{-1})J_{\mathrm{cm}_0}\rangle_{\mathrm{cm}_0}}{\Omega(\mathrm{f})},\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}\gamma(\mathrm{m}') &= 2^{nk}i^{k+1}\alpha(\mathrm{m}')^{-1}\mathcal{N}(\mathrm{m}_0)^{(k/2)-1}\mathcal{N}(\mathrm{c}\mathfrak{d}^2)^{(k+l)/2-1}, \\ \Omega(\mathrm{f}) &= \langle \mathrm{f},\mathrm{f}\rangle_{\mathrm{c}(\mathrm{f})}.\end{aligned} \tag{4.16}$$

§5. Integrality properties and congruences for the distributions

5.1. In order to prove the theorem 2.2 we first show that the distributions (3.3) for $s = 0,1,\cdots,k-l-1$ are bounded and then we prove a certain p-adic congruence involving their values on Dirichlet characters for different s ("generalized Kummer congruences") equivalent to the existence of the p-adic analytic function from the theorem 2.2. The proof is based on the integral representation (4.15).

5.2. Proposition. *(a) Under the assumotions and notation as in theorem 2.2 for each $r = 0,1,\cdots,k-l-1$ the $\mathbf{C}_p$-valued distribution $i_p(\tilde{\mu}_{l+r}$ on the group Gal_S is bounded.*
(b) the following p-adic equality holds

$$\int_{\mathrm{Gal}_S} \phi\mathcal{N}\, x_p^r\, di_p(\tilde{\mu}_l) = (-1)^{rn}\int_{\mathrm{Gal}_S}\phi\mathcal{N}\, di_p(\tilde{\mu}_{l+r}). \tag{5.1}$$

5.3. In the prove we use the function

$$V_r^*(\chi) = \mathcal{H}ol\,(\mathrm{g}(\chi_{\mathrm{m}_0})E^*_{2-k+l+2r,k-l-1-r}(\psi\omega^{-1}\chi^{-2})), \tag{5.2}$$

which is obtained by applying the holomorphic projection operator to the Hilbert automorphic type, which for $r \neq 0$ is not of the holomorphic type. This operator can be described in a very similar way as in the one-dimentional case (see [Shi6], [St1], [St2]); the necessary growth conditions are obviously satisfied due to the fact that $\mathrm{g}(\chi_{\mathrm{m}_0})$ is a cusp form so that the function (5.2) is a cusp form from $S_k(\mathfrak{c}\mathfrak{m}', \psi)$. Then the equality (4.15) takes the form

$$\tilde{\mu}_{l+r,\mathfrak{m}}(\chi) = \frac{\gamma'(\mathfrak{m}')}{\Omega(\mathfrak{f})} \langle F^\rho, V_r^*(\chi) | U(\mathfrak{m}'\mathfrak{m}_0^{-1}) J_{\mathfrak{c}\mathfrak{m}_0} \rangle_{\mathfrak{c}\mathfrak{m}_0}. \tag{5.3}$$

The functions (5.2) apparently satisfy the same compatibility condition as the values $\tilde{\mu}_{l+r,\mathfrak{m}}(\chi)$ and define a distribution on Gal_S with values in the spaces of Hilbert automorphic forms of holomorphic type of weight k with the character $\overline{\psi}$ whose level is growing with the growing conductor $\mathfrak{m}$ of the character χ.

5.4. Fourier coefficients of the function $V_r^*(\chi)$. Let us consider Fourier expansions of all functions $V_r^*(\chi)_\lambda(z)$ (for $\lambda = 1, \cdots, h$) on the Hilbert upper half plane $\mathfrak{H}^n$ which describe the automorphic form $V_r^*(\chi)$,

$$V_r^*(\chi)_\lambda(z) = \sum_{0 \ll \xi \in \tilde{t}_\lambda} v(\xi, r, \chi)_\lambda e_F(\xi z). \tag{5.4}$$

The coefficients $v(\xi, r, \chi)_\lambda$ obviously also define a distribution on Gal_S and they admit a fairly explicit calculation. If, for example, $r = k - l - 1$, then the function $V_{k-l-1}^*(\chi)$ is itself of holomorphic type and

$$\begin{aligned} V_{k-l-1}^*(\chi) &= \mathbf{g}_{\mathfrak{m}_0}(\chi) E_{k-l,0}^*(\psi\omega^{-1}\chi^{-2}), \\ \mathbf{g}_{\mathfrak{m}_0}(\chi)_\lambda(z) &= \sum_\xi b_\lambda(\xi) \chi^*(\xi \tilde{t}_\lambda^{-1}) e_F(\xi z), \end{aligned} \tag{5.5}$$

where the summation over ξ is taken with $0 \ll \xi \in \tilde{t}_\lambda$, $\tilde{\xi} + \mathfrak{m} = \mathcal{O}_F$, and

$$b_\lambda(\xi) = \mathcal{N}(\tilde{t}_\lambda)^{l/2} C(\xi \tilde{t}_\lambda^{-1}, \mathbf{g})$$

(see 1.9). Due to Klingen [Kl1], [Kl2] we know the Fourier expansion of the holomorphic Eisenstein series

$$\begin{aligned} &\mathcal{N}(\tilde{t}_\lambda)^{m/2} E_{m,0}(\eta)_\lambda(z) = \frac{2^{-n} D_F^{1/2} \Gamma(m)}{(-2\pi i)^m} L_m(z, 0; \mathcal{O}_F, \tilde{t}_\lambda^{-1}; \eta) = \\ &A(m;\eta)_\lambda + \sum_{0 \ll \xi \in \tilde{t}_\lambda} \Big(\sum_{\substack{\xi = b\tilde{c} \\ c \in \mathcal{O}_F, b \in \tilde{t}_\lambda}} \eta^*(\tilde{c}) \mathcal{N}(\tilde{b})^{m-1} \Big) e_F(\xi z) \end{aligned} \tag{5.6}$$

for $m \geq 1$, $\eta \bmod \mathfrak{c}$ is a Hecke character of finite order with the condition $\eta((x)) = \mathrm{sign}\mathcal{N}(x)^m$ for $x \equiv 1 \bmod^\times \mathfrak{c}, A(M;\eta)$ is some constant, which is explicitly given as a certain special value of, the Hecke L-function associated with η. In the general case the function $E_{m,r}^*(\eta)$ admits a certain Fourier expansion in $e_F(\xi z)$ in terms of the Whittaker functions (or confluent hypergeometric functions, [Fe], [Shi10]) and the Fourier expansion is then obtained by use of a certain integral formula as in [St1], [St2], or with help of the Shimura differential operators (as in [Shi6]). As a result we get for $r = 0, \cdots, k-l-1$ the following identity:

$$\mathcal{N}(\tilde{t}_\lambda)^r v(\xi, r, \chi)_\lambda = \chi^*_{\mathfrak{m}_0}(\xi\tilde{t}_\lambda^{-1}) b_\lambda(\xi) A'(r)(\eta) + \mathcal{N}(\xi) u_\lambda(\xi) +$$

$$\sum_{\xi=\xi_1+\xi_2} \chi^*_{\mathfrak{m}_0}(\xi_1 \tilde{t}_\lambda^{-1}) b_\lambda(\xi_1) \mathcal{N}(\xi_1)^{k-l-1-r} \sum_{\substack{\tilde{\xi}_2 = \tilde{b}\tilde{c} \\ c\in \mathcal{O}_F, b\in \tilde{t}_\lambda}} (\psi\omega^{-1}\chi^{-2})^*(\tilde{c}) \mathcal{N}(\tilde{b})^{1-k+l+2r}; \tag{5.7}$$

here $A'(0)(\eta) = A(k-l;\eta)$, $A'(r) = 0$for $r > 0$ with ξ_i is totally positive ($i = 1,2$) and the second sum is extended over all decompositions of $\tilde{\xi}_2$ into product of principial ideals $\tilde{c}\tilde{b}$ with the condition $c \in \mathcal{O}_F, b \in \tilde{t}_\lambda$ and $u_\lambda(\xi)$ is an integral linear combination of the numbers $\chi^*_{\mathfrak{m}_0}(\xi\tilde{t}_\lambda^{-1}) b_\lambda(\xi)$ and of values of the Hecke character $(\xi\omega^{-1}\chi^{-2})^*$ (which are certain roots of unity).

Let us consider the linear functional

$$\mathcal{L} : \Phi \quad \mapsto \quad \frac{\langle F^\rho, \Phi | J_{\mathfrak{c}\mathfrak{m}_0} \rangle_{\mathfrak{c}\mathfrak{m}_0}}{\Omega(f)} \tag{5.8}$$

on the linear space $\mathcal{S}_k(\mathfrak{c}\mathfrak{m}_0, \psi)$. It follows from the Atkin–Lehner theory (in the form of Miyake [Miy]) that $\mathcal{L}$ is defined over $\overline{\mathbf{Q}}$ i.e. for a finite number of ideals $\mathfrak{m}_i$ and for some fixed algebraic numbers $l(\mathfrak{m}_i) \in \overline{\mathbf{Q}}$ we have that

$$\mathcal{L}(\Phi) = \sum_i C(\mathfrak{m}_i, \Phi) l(\mathfrak{m}_i), \tag{5.9}$$

so that the distributions $\tilde{\mu}_{l+r}$ can be written in the form

$$\tilde{\mu}_{l+r,\mathfrak{m}}(\chi) = \gamma(\mathfrak{m}') \mathcal{L}(\Phi_{r,\mathfrak{m}'}(\chi)), \tag{5.10}$$

with

$$\Phi_{r,\mathfrak{m}'}(\chi) = V^*(\chi) | U(\mathfrak{m}'\mathfrak{m}_0^{-1}) \tag{5.11}$$

and the constant $\gamma(\mathfrak{m}')$ given by (4.16). Now we see from (5.9) that the properties (a) and (b) of the proposition 5.2 are equivalent to the corresponding statements about the Fourier coefficients $v(\xi, r, \chi)_\lambda$ for $\xi \equiv 0 \pmod{\mathfrak{m}'\mathfrak{m}_0^{-1}\tilde{t}_\lambda}$. Indeed, we see that in this case the Fourier coefficients in (5.9) are expressed according to (1.9) in terms of the numbers $v(\xi, r, \chi)_\lambda$ as follows:

$$\mathcal{N}(\tilde{t}_\lambda)^{k/2} C(\mathfrak{n}, \Phi_{l,\mathfrak{m}'}(\chi)) = \begin{cases} v(\xi, r, \chi)_\lambda, & \text{if the fractional ideal } \xi\tilde{t}_\lambda^{-1} = \mathfrak{n}\mathfrak{m}'\mathfrak{m}_0^{-1} \\ & \text{is not integral,} \\ 0, & \text{otherwise.} \end{cases} \tag{5.12}$$

Therefore the explicit expression (5.7) implies the following congruences

$$\begin{aligned} v(\xi, r, \chi)_\lambda \equiv & (-1)^{rn} \sum_{\xi=\xi_1+\xi_2} \sum_{\substack{\tilde{\xi}_2 = \tilde{b}\tilde{c} \\ c\in \mathcal{O}_F, b\in \tilde{t}_\lambda}} \chi^*(\tilde{c}^{-1}\tilde{b}\tilde{t}_\lambda^{-1}) \mathcal{N}(\tilde{c}^{-1}\tilde{b}\tilde{t}_\lambda^{-1})^r \times \\ & \times (\psi\omega^{-1})^*(\tilde{c}) \mathcal{N}(\tilde{c})^{k-l-1} \pmod{\mathcal{N}(\mathfrak{m}'\mathfrak{m}_0^{-1})}, \end{aligned} \tag{5.13}$$

for

$$0 \ll \xi_i \in \tilde{t}_\lambda \ (i = 1, 2), \quad c \in \mathcal{O}_F, \quad b \in \tilde{t}_\lambda, \quad \xi_2 + \mathfrak{m}_0 = \mathcal{O}_F.$$

Now proposition 5.2 and theorem 2.2 are directly deduced from the abstract Kummer congruences (see [Ka3], p.258 , or 3.3 of chapter 1) which provide a criterion of boundness of $\mathbf{C}_p$ -valued distributions.

Let $\{f_i\}$ be a family of continuous functions $f_i \in \mathcal{C}(Y, \mathcal{O}_p)$ in the ring $\mathcal{C}(Y, \mathcal{O}_p)$ of all continuous functions on the compact totally disconnected group $Y = \mathcal{X}_S^\times$ with values in the ring of integers $\mathcal{O}_p$ of $\mathbf{C}_p$ such that the $\mathbf{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(Y, \mathbf{C}_p)$. Let also $\{a_i\}$ be a family of elements $a_i \in \mathcal{O}_p$. Then the existence of an $\mathcal{O}_p$ -valued measure μ on Y with the property

$$\int_Y f_i \, d\mu = a_i$$

is equivalent to the validity of the following congruences: for an arbitrary choice of elements $b_i \in \mathbf{C}_p$ almost all of which vanish,

$$\sum_i b_i f_i(y) \in p^n \mathcal{O}_p \text{ for all } y \in Y \implies \sum_i b_i a_i \in p^n \mathcal{O}_p.$$

In our situation we take as a family $\{f_i\}$ the family of functions of the type $\chi \mathcal{N} x_p^s$ with s as in proposition 5.2 and with $\chi \in X_S^{\text{tors}}$ being Hecke characters; this family obviously has the dense $\mathbf{C}_p$-linear span. For any finite number of Hecke characters $\chi \in X_S^{\text{tors}}$ we choose such $\mathfrak{m}$ and a sufficiently large integer $\mathfrak{m}'$ that each of these characters is defined modulo $\mathfrak{m}$ and the formula (4.15) is valid for the values of the distributions $\tilde{\mu}_{l+r,\mathfrak{m}}(\chi)$. The proof is then completed by application of the congruences (5.13).

References

[An1] *Andrianov,A.N.*: Spherical functions for GL_n over local fields and summmation of Hecke series // Mat.Sbornik 83 (1970) 429–451 (in Russian)

[An2] *Andrianov,A.N.*: Euler products attached to Siegel modular forms of degree 2 // Uspekhi Mat. Nauk 29 (1974) 44–109 (in Russian)

[An3] *Andrianov,A.N.*: Symmetric squares of zeta functions of Siegel modular forms of degree 2 // Trudy Steklov Math.Institut USSR. 142 (1976) 22–45 (in Russian)

[An4] *Andrianov,A.N.*: On zeta functions of Rankin type associated with Siegel modular forms // Lecture Notes in Math. N 627 (1977) 325–338

[An5] *Andrianov,A.N.*: On the decomposition of Hecke polynomials for the symplectic group of degree n // Mat. Sbornic 104 (1978) 291–341 (in Russian)

[An6] *Andrianov A.N.*: Euler expansions of the theta transformations of Siegel modular forms of degree n // Mat. Sbornik 104 (1977) 390–427

[An7] *Andrianov,A.N.*: Multiplicative arithmetic of Siegel modular forms // Uspekhi Mat. Nauk 34 (1979) 67–135 (in Russian)

[An8] *Andrianov,A.N.*: Modular descent and the Saito-Kurokawa conjecture // Inv. Math. 53 (1979) 267–280

[An9] *Andrianov,A.N.*: On Siegel modular forms of genus n and weight $n/2$ // J. Fac. Sci. Univ. Tokyo. Ser.IA.28 (1981) 487–503

[An-K] *Andrianov,A.N., Kalinin, V.L.*: On analytic properties of standard zeta functions of Siegel modular forms // Mat. Sbornik 106 (1978) 323–339 (in Russian)

[An-M1] *Andrianov,A.N., Maloletkin, G.N.*: Behaviour of the theta series of degree n under modular substitutions // Izvestija Akad.Nauk. Ser.Matem.39 (1975) 243–258 (in Russian)

[An-M2] *Andrianov,A.N.,Maloletkin, G.N.*: Behaviour of the theta series of degree n of indefinite quadratic forms under modular substitutions // Trudy Steklov Mat. Institut USSR 148 (1978) 5–15 (in Russian)

[Ar] *Arnaud B.*: Interpolation p-adique d'un produit de Rankin // CRAS Paris. Ser.1, 299 (1984) 527–530

[A-Le] *Atkin A.,Lehner J.*: Hecke operators on $\Gamma_0(m)$ // Math.Ann. 185 (1970) 134–160

[A-Li] *Atkin A., Li W.-Ch. W*: Twists of newforms and pseudo - eigenvalues of W - opertors // Invent. Math. 48 (1978) 221–243

[Ba] *Baily W.L.*: Eisenstein series on tube domains // In: Problems in analysis. Princeton University Press, 1970

[Bö] *Böcherer S.*: ber die Functionalgleichung automorpher L- Funktionen zur Siegelscher Modulgruppe // J. reine angew. Math. 362 (1985) 146–168

[B-Š] *Borevič , S.I., Šafarevič , I.R.*: Number theory // 3d ed. Nauka: Moscow, 1985 (in Russian)

[Ca] *Cassels J.W.S.*: Rational quadratic forms // Acad. Press: London, N.Y., San Francisco, 1978, 413 p.

[Ca-F] *Cassels J.W.S.,Frölich A.(Eds)*: Algebraic number theory // Acad. Press: London, N.Y., San Francisco, 1967, 366 p.

[Co-Sch] *Coates J., Schmidt C.-G.*: Iwasawa thery for the symmmetric square of an elliptic curve // J. reine angew. Math. 375/376 (1987) 104–156

[De1] *Deligne P.*: Formes modulaires et reprèsentations l-adiques // Sém. Bourb. 1968/69 ,exp. no 335. Springer-Verlag, Lect. Notes in Math. N 179 (1971) 139–172

[De2] *Deligne P.*: La conjecture de Weil.I // Publ. math. IHES 43 (1974) 273–307

[De-R] *Deligne P.,Ribet ,K.A.*: Values of Abelian L-functions at negative integers over totally real fields // Invent. Math. 59 (1980) 227–286

[Ev1] *Evdokimov ,S.A.*: Euler products for congruence subgroups of the Siegel modular group of degree 2 // Mat.Sbornik 99 (1976) 483–513 (in Russian)

[Ev2] *Evdokimov ,S.A.*: Analytic properties of certain Euler products for congruence subgroups of $\mathrm{Sp}_2(\mathbf{Z})$ // Mat.Sbornik 110 (1979) 369–398 (in Russian)

[Fe] *Feit P.*: Poles and residues of Eisenstein series for symplectic and unitary groups // Memoir AMS 61 (1986) N 346, iv.+89p.

[Fo1] *Fomenko O.M.*: Applications of the theory of modular forms to number theory // In the series "Algebra. Topolology. Geometry." Vol. 15. VINITI Publ.: Moscow, 1977, 5–91 (in Russian)

[Fo2] *Fomenko O.M.*: On Fourier coefficients of Siegel cusp forms of degree n // Zap. Nauchn. Sem. LOMI 144 (1985) 155–166 (in Russian)

[Fr] *Freitag E.*: Siegelsche Modulfunktionen // Springer-Verlag: Berlin e.a.,1983, x+341p.

[Ga] *Garrett, P.B.*: Pullbacks of Eisenstein series: applications //In: Automorphic forms of several variables. Taniguchi symposium, 1983. Birkhaüser, Boston - Basel - Stuttgart, 1984

[Ge-PSh] *Gelbart S.,Piatetski-Shapiro I.I., Rallis S.*: Explicit constructions of automorphic L - functions // Springer-Verlag, Lect. Notes in Math. N 1254 (1987) 152p.

[G-G-PSh] *Gelfand ,I.M., Graev, M.I.,Piatgetskij-Shapiro ,I.I.*: Representation theory and automorphic functions // In the series: "Generalized functions" , Vol. 6. Nauka: Moscow ,1966 (English transl.: W.B.Saunders, Philadelphia,1969)

[Ger-Pa] *Gerzhoj, P.I., Panchishkin, A.A.*: A finiteness critereon for the number of rational points on twisted Weil elliptic curves // Zap.Nauchn. Sem. LOMI 160 (1987) 41–53

[Gr-Z] *Gross B., Zagier, Don B.*: Heegner points and derivatives of L- series // Invent. math. 84 (1986) 225–320

[HSh] *Haran Shai*: p- adic L- functions for modular forms —— Compos. Math. 62 (1986) 31–46

[Har1] *Harris M.*: Special values of zeta functions attached to Siegel modular forms // Ann. sci Ecole Norm. Sup. 14 (1981) 77–120

[Har2] *Harris M*: The rationality of holomorphic Eisenstein series series // Invent. Math. 63 (1981) 305–310

[Har3] *Harris M*: Maass operators and Eisenstein series // Math. Ann. 258 (1981) 135–144

[He] *Hecke E.*: Theorie der Eisensteinscher Reihen und ihre Anwendung auf Funktionentheorie und Arithmetik // Abh. Math. Sem. Hamburg 5 (1927) 199–224

[Hi1] *Hida H.*: A p-adic measure attached to the zeta functions associated with two elliptic cusp forms. I // Invent. Math. 79 (1985) 159–195

[Hi2] *Hida H*: Galois representations into $GL_2(\mathbf{Z}_p[[X]]$ attached to ordinary cusp forms // Invent. Math. 85 (1986) 545–613

[H-PSh1] *Howe R., Piatetski-Shapiro I. I.*: A counterexample to the "Generalized Ramanujan conjecture" for (quasi-) split groups // Automorphic Forms, Representations and L- functions. Proc. Symp. Pure Math. AMS. Vol.33. Part 1. Providence: R.I., 1979, 315–322

[H-PSh2] *Howe R., Piatetski-Shapiro I. I.*: Some examples of automorphic forms on Sp_4 //Duke Math. J. 50 (1983) 55–106

[HLK] *Hua Lo- Ken*: Harmonic analysis of functions of several complex variables in the classical domains // Providence: R.I., 1963

[Iw] *Iwasawa K.*: Lectures on p-adic L- functions // Ann. of Math.Studies , N 74. Princeton University Press, 1972

[Ja] *Jacquet H.*: Automorphic forms on GL(2). Part II // Springer-Verlag, Lect. Notes in Math. N 278 (1972)

[Ja-L] *Jacquet H., Langlands R.*: Automorphic Forms on GL(2) // Springer-Verlag, Lect. notes in Math. N 114 (1972) 548 p.

[K] *Kalinin V.L.*: Eisenstein series on the symplectic group // Mat. Sbornik 103 (1977) 519–549 (in Russian)

[Ka1] *Katz, N.M.*: p- adic interpolation of real analytic Eisenstein series // Ann. of Math. 104 (1976) 459–571

[Ka2] *Katz, N.M.*: The Eisenstein measure and p- adic interpolation // Amer. J. Math. 99 (1977) 238–311

[Ka3] *Katz, N.M*: p- adic L- functions for CM-fields // Invent. Math. 48 (1978) 199-297

[Ki1] *Kitaoka Y.*: Fourier coeficients of Siegel cusp forms of degree two // Proc. Jap. Acad. A58 (1982) 41–43

[Ki2] *Kitaoka Y.*: Lectures on Siegel modular forms and representations by quadratic forms // Springer - Verlag: Berlin e.a.,1986

[Kl1] *Klingen H.*: Über die Werte Dedekindschen Zetafunktionen // Math. Ann. 145 (1962) 265–272

[Kl2] *Klingen H.*: Über den arithmetischen Charakter der Fouriercoefficienten von Modulformen // Math. Ann. 147 (1962) 176–188

[Kl3] *Klingen H.*: Zum Darstellungssatz für Siegelsche Modulformen // Math. Z. 102 (1967) 30–43

[Kl4] *Klingen H.*: Über Poincarésche Reihen zur Siegelschen Modulgruppe // Math. Ann. 168 (1967) 157–170

[Kl5] *Klingen H.*: Über Poincarésche Reihen vom Exsponentialtyp // Math. Ann. 234 (1978) 145-157

[Ko1] *Koblitz Neal*: p-Adic Numbers , p-Adic Analysis and Zeta Functions // 2nd ed. Springer-Verlag, 1984

[Ko2] *Koblitz Neal*: *p*- Adic Analysis: A Short Course on Recent Work // London Math. Soc. Springer-Verlag, Lect. Notes Series, N 46. Cambridge University Press: London, Cambridge, 1980

[Ko3] *Koblitz Neal*: *p*- adic congruences and modular forms of half integral weight // Math. Ann. 274 (1986) 199–220

[Koe] *Koecher M.*: Zur Theorie der Modulformen n- ten Grades. I. II //Math. Z. 59 (1954) 399–416

[Koj] *Kojima H.*: On construction of Siegel modular forms of degree two // Math. Soc. Japan 34 (1982) 393–412

[Ku] *Kubota T.*: Elementary Theory of Eisenstein Series // Halsted Press: N.Y., 1973

[Ku-Le] *Kubota T., Leopoldt H.-W*: Eine p-adische Theorie der Zetawerte // J- reine angew. math. 214/215 (1964) 328–339

[Kurč1] *Kurčanov P.F.*: Zeta functions of elliptic curves over certain abelial extensions of imaginary quadratic fields // Mat.Sbornik 108 (1977) 56–70 (in Russian)

[Kurč2] *Kurčanov P.F.*: Cohomology of discrete groups and Dirichlet series associated with cusp forms of Jacquet – Langlands // Izvestija Akad. Nauk. Ser. Matem. 42 (1978) 588–601 (in Russian)

[Kurč3] *Kurčanov P.F.*: Local measures connected with cusp forms of Jacquet – Langlands over CM - fields // Mat. Sbornik 108 (1979) 483–503 (in Russian)

[Kur] *Kurokawa N.*: Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two // Invent. Math. 49 (1978) 149–165

[Kur-Miz] *Kurokawa N., Mizumoto S.*: On Eisenstein series of degree two // Proc. Jap. Acad. A57 (1981) 134–139

[La1] *Lang S.*: Algebraic Number Theory // Addison-Wesley, 1970

[La2] *Lang S.*: Elliptic functions // Addison-Wesley, 1970

[La3] *Lang S.*: Introduction to Modular Forms // Springer-Verlag, 1976

[La4] *Lang S.*: Cyclotomic fields //(Graduate Texts in Math.) Springer-Verlag, N.Y., 1980

[La5] *Lang S.*: Fundamentals of Diophantine Geometry // Springer-Verlag, 1983

[Ll1] *Langlands R.*: On the functional equations satisfied by Eisenstein series // Springer-Verlag, Lect.Notes in Math. N 544 (1976) 337 p.

[Ll2] *Langlands R.*: Problems in the theory of automorphic forms // Springer-Verlag, Lect. Notes in Math. N 170 (1970) 18–61

[Le1] *Leopoldt, H.-W.*: Eine Verallgemeinerung der Brnoullischen Zahlen // Abh. Math. Sem. Univ. Hamburg 22 (1958) 131–140

[Le2] *Leopoldt, H.-W.*: Eine p- adische Theorie der Zetawerte. II // J. reine angew. Math. 274/275 (1975) 224–239

[Li1] *Li W..-Ch. W*: Newforms and functional equations // Math. Annalen 212 (1975) 285–315

[Li2] *Li W.-Ch. W*: L- series of Rankin type and their functional equations // Math. Ann. 244 (1979) 135–166

[Maa] *Maass H.*: Siegel's modular forms and Dirichlet series // Springer-Verlag, Lect. Notes in Math. N 216 (1971)

[Mah] *Mahler K.*: An interpolation series for a continuous function of p- adic variable // J. reine angew. mth 199 (1958) 23–34

[Man1] *Manin Yu.I.*: Cyclotomic fields and modular curves // Uspekhi Mat. Nauk 26 (1971) 7–78 (in Russian)

[Man2] *Manin Yu.i.*: Cusp forms and zeta functions of modular curves // Izvestija Akad.Nauk. Ser.Matem. 36 (1972) 19–66 (in Russian)

[Man3] *Manin Yu.I.*: Explicit formulas for the eigenvalues of hecke operators // Acta arithm. 24 (1973) 239–249

[Man4] *Manin Yu.i.*: Periods of cusp forms and p-adic Hecke series // Mat. Sbornik 92 (1973) 378–401 (in Russian)

[Man5] *Manin Yu.i.*: The values of p-adic hecke series at integer points of the critical strip // Mat. Sbornik 93 (1974) 621–626 (in Russian)

[Man6] *Manin Yu.i.*: Non - archimedean integration and p -adic L- functions od Jacquet – Langlands // Uspekhi Mat. Nauk 31 (1976) 5–54 (in Russian)

[Man6] *Manin Yu.I.*: Modular forms and number theory // Proc. Int. Congr. Math. Helsinki, 1978, 177–186

[Man-Pa] *Manin Yu.I., Panchishkin A.A.*: Convolutions of Hecke series and their values at integral points // Mat. Sbornik, 104 (1977) 617 – 651 (in Russian)

[Maz1] *Mazur B.*: On the special values of L-functions // Invent. Math. 55 (1979) 207–240

[Maz2] *Mazur B.*: Modular curves and arithmetic // Proc. Int. Congr. Math. Warszawa,16–24 August 1982. North Holland: Amsterdam, 1984, 185–211

[Maz-SD] *Mazur B., Swinnerton- Dyer H.P.F.*: Arithmetic of Weil curves // Invent. Math. 25 (1974) 1-61

[Maz-W1] *Mazur B., Wiles A.*: Analogies between function fields and number fields // Am. J. Math. 105 (1983) 507-521

[Maz-W2] *Mazur B., Wiles A.*: Class fields of Abelian extensions of $\mathbf{Q}$ // Invent. math. 76 (1984) 179–330

[Maz-W3] *Mazur B., Wiles A.*: On p- adic analytic families of Galois representations // Compos. Math. 59 (1986) 231–264

[Mi-St] *Milnor J., Stasheff J.*: Characteristic classes // (Ann. of Math. Studies ,N 76) Princeton Univ. Press 1974, 330 p.

[Miy] *Miyake T.*: On automorphic forms on GL_2 and Hecke operators // Ann. of Math. 94 (1971) 174–189

[Miz1] *Mizumoto S.*: Fourier coefficients of generalized Eisenstein series of degree two // Proc. Jap. Acad. A59 (1983) 231–264

[Miz2] *Mizumoto S.*: Fourier coefficients of generalized Eisenstein series of degree two. II // Nagoya Math. J. 7 (1984) 86–110

[MF] : Modular functions of one variable. I–VI // Springer-Verlag, Lect. Notes in Math. NN 320, 349, 350 (1973); N 476 (1975); NN 601, 627 (1977)

[Ogg1] *Ogg, A.P.*: Modular forms and Dirichlet series // Benjamin, 1969

[Ogg2] *Ogg, A.P.*: On a convolution of L- series // Invent. Math. 7 (1969) 297–312

[Pa1] *Panchishkin A.A.*: There exist no Ramanujan congruences modulo 691^2 // Mat. Zametki 17 (1975) (in Russian)

[Pa2] -: On Dirichlet series attached to modular forms of integral and half integral weight // Izvestija Akad. Nauk.Ser.Mat.43 (1979) 1145-1157 (in Russian)

[Pa3] -: Symmetric squares of Hecke series and their values at integral points // Mat. Sbornik 108 (1979) 393-417(in Russian)

[Pa4] -: On p - adic Hecke series // In: "Algebra" (Ed.by A. I. Kostrikin), Moscow Univ. Press, 1980, pp. 68–71 (in Russian)

[Pa5] -: Complex valued measures attached to Euler products // Trudy Sem. Petrovskogo 7 (1981) 239–244 (in Russian)

[Pa6] -: Modular forms // In the series "Algebra. Topology. Geometry." Vol. 19. VINITI Publ.: Moscow , 1981, pp.135–180 (in Russian)

[Pa7] -: Local measures attached to Euler products in number fields // In "Algebra" (Ed. by A. I. Kostrikin),Moscow Univ. Press,1982, pp.119–138 (in Russian)

[Pa8] -: Automorphic forms and the functoriality principle // In: "Automorphic forms, representations and L-functions". Mir Publ.: Moscow, 1984, pp. 249–286 (in Russian)

[Pa9] *Pančiškin A.A.*: Le prolongement p - adique analytique de fonctions L de Rankin // CRAS Paris 294 (1982) 51–53; 227–230

[Pa10] *Panchishkin A.A.*: A functional equation of the non-archimedian Rankin convolution // Duke Math.J.54 (1987) 77-89

[Pa11] *Panchishkin A.A.*: Non-archimedian convolutions of Hilbert modular forms // Abstracts of the 19th USSR Algebraic Conference, September 1987, Lvov. Vol. 1, 211.

[Pa12] *Panchishkin A.A.*: Non-archimedian Rankin L-functions and their functional equations // Izvestija Akad.Nauk. Ser.Matem.52 (1988) 336–354(in Russian)

[Pal3] *Panchishkin A.A.*: Convolutions of Hilbert modular forms and their non-archimedian analogues // Mat. Sbornik 136 (1988) 574–587 (in Russian)

[Pal4] *Panchishkin A.A.*: Non-archimedian automorphic zeta-functions // Moscow University Press, 1988, 165p. (in Russian)

[PSh] *Piatetski-Shapiro I.I.*: Automorphic Functions and the Geometry of Classical Domains // Phismatgiz: Moscow, 1961 (English transl.: Gordon and Breach, 1969)

[PSh-R] *Piatetski-Shapiro I.I.* , *Rallis S.*: L-functions of automorphic forms on simple classical groups //In: Modul. Forms Symp., Durham, 30 June - 10 July 1983. Chichester, 1984

[Rag1] *Raghavan S.*: Modular forms of degree n and representations by quadratic forms // Ann. of Math. 70 (1959) 446–477

[Rag2] *Raghavan S.*: On an Eisenstein series of degree 3 // J. Indian Math. Soc. 39 (1975) 103–120

[Rag3] *Raghavan S.*: Singular modular forms of degree s //Papers Colloq. S. Ramanujan. - A Tribute. Berlin e.a.,1978, 263–272

[Rag4] *Raghavan S.*: Estimates of coefficients of modular forms and generalized modular relations // Automorphic Forms, Representation Theory and Arithmetic. Papers Colloq. ,Bombay, 8–15 January 1979. Berlin e.a. 1981, 247–254

[Ran1] *Rankin, R.A.*: Contribution to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions. I.II // Proc. Camb. Phil. Soc 35 (1939) 351–372

[Ran2] *Rankin, R.A.*: The scalar product of modular forms // Proc. London math. Soc. 35 (1939) 351–372

[Ran3] *Rankin, R.A.*: The construction of automorphic forms from the derivatives of a given form // J. Ind. Math. Soc. 20 (1956) 103–116

[Res] *Resnikoff H.L.*: Automorphic forms of singular weight are singular forms // Math. Ann. 215 (1975) 173–183

[Res-S] *Resnikoff H.L., Saldaña R.L.*: Some properties of Fourier coefficients of Eisenstein series of degree two // J.reine angew. Math. (1974) 90–109

[Šaf1] *Šafarevič I.R.*: Zeta Function // Moscow University Press, 1969

[Šaf2] *Šafarevič I.R.*: Basic concepts of algebra // In the series: "Modern Problems of Mathematics. Fundamental Directions", Vol. 11, Moscow: VINITI Publ., 1986

[Sat] *Satake I.*: Theory of spherical functions on reductive algenraic groups over p- adic fields // Publ. Math. IHES N 18 (1963)

[Sch] *Schmidt C.-G.*: The p- adic L-functions attached to Rankin convolutions of modular forms // J. reine angew. Math. 368 (1986) 201–220

[Sel] *Selberg A.*: Bemerkungen über Dirichletsche Reihe die der Theorie der Modulformen nahe verbunden ist // Arch. Math. Naturwid. 43 (1940) 47–50

[Se1] *Serre J.-P.*: Cours d'Arithmétique // Press Univ.France: Paris, 1970; 2nd ed. 1977, 188 p.

[Se2] *Serre J.-P.*: Abelian l-adic representations and elliptic curves // Benjamin Publ.: N.Y. 1968, 195 p.

[Shi1] *Shimura G.*: Introduction to the Arithmetic Theory of Automorphic Functions // Princeton Univ. Press, 1971

[Shi2] *Shimura G.*: On the holomorphy of certain Dirichlet series // Proc. Lond. Math. Soc. 31 (1975) 79–98

[Shi3] *Shimura G.*: The special values of the zeta functions associated with cusp forms // Comm. Pure Appl. Math. 29 (1976) 783–804

[Shi4] *Shimura G.*: On the periods of modular forms //Math. Annalen 229 (1977) 211–221

[Shi5] *Shimura G.*: On certain reciprocity laws for theta functions and modular forms //Acta Math. 141 (1978) 35–71

[Shi6] *Shimura G.*: The special values of zeta functions associated with Hilbert modular forms // Duke Math. J. 45 (1978) 637–679

[Shi7] *Shimura G.*: Arithmetic of differential operators on symmetric domains // Duke Math. J. 48 (1981) 813–843

[Shi8] *Shimura G.*: Confluent Hypergeometric functions on tube domains // Math. Ann. 260 (1982) 269–302

[Shi9] *Shimura G.*: Algebraic relations between critical values of zeta functions and inner products // Amer. J. Math. 105 (1983) 253–285

[Shi10] *Shimura G.*: On Eisenstein series // Duke Math. J. 50 (1983) 417–476

[Sho] *Shokurov V.V.*: The integrals of Shimura of cusp forms // Izvestija Akad. Nauk. Ser.Mat. 44 (1980) 670–718

[Sie1] *Siegel C.-L.*: Über die analytische theorie der quadratische Formen // Ann. Math. 36 (1935) 527–606

[Sie2] *Siegel C.-L.*: Über die Zetafunktionen indefiniter quadrqtische Formen // Math. Z. 43 (1938) 682–708

[Sie3] *Siegel C.-L.*: Einfürung in die Theorie der Modulfunktionen n- ten Grades // Math. Ann. 116 (1939) 617–657

[Sie4] *Siegel C.-L.*: On the theory of indefinite quadratic forms // Ann. Math. 45 (1944) 577–622

[Sie5] *Siegel C.-L.*: Die Funktionalgleichungen einiger Dirichletschen Reihen // Math. Z. 63 (1956) 363–373

[Sie6] *Siegel C.-L.*: Berechung von Zetafunktionen an ganzzahligen Stellen// Nachr. Acad. Wiss. Göttingen. II. Math.- Phys. Kl. 10 (1969) 1–16

[Sie7] *Siegel C.-L.*: Über die Fourierschen Koeffizienten von Modulformen // Nachr. Acad. Wiss. Göttingen. II. Math.- Phys. Kl. 3 (1970) 15–56

[St1] *Sturm J.*: Special values of zeta functions and Eisenstein series of half integral weght // Amer. J. Math. 102 (1980) 219–240

[St2] *Sturm J.*: The critical values of zeta-functions associated to the symplectic group // Duke Math. J. 48 (1981) 327–350

[V1] *Višik, M.M.*: Non archimedean measures associated with Dirichlet series // Mat.Sbornik 99 (1976) 248-260

[V2] *Višik, M.M.*: Non archimedean spectral theory // In the series: "Modern Problems of Mathematics" , Vol. 25,Moscow: VINITI Publ.,1984, 51–114

[Wa] *Washington L.*: Introduction to cyclotomic fields // Springer- Verlag: N.Y. e.a., 1982

[Weil] *Weil A.*: Basic Number Theory // (Die Grundlehren der mathematischen Wissenschaften,Bd.144) Springer-Verlag: Hdlbrg-N.Y.,1973, 312 p.

[We1] *Weissauer R.*: Über die construction von Spitzenformen mit Hilfe von Thetareihen // Abh. Math. Sem. Hamburg 56 (1986) 133–151

[We2] *Weissauer R.*: Stabile Modulformen und Eisensteinreihen // Springer-Verlag, Lect. Notes in Math. N 1219 (1987)

[We3] *Weissauer R.*: Divisors on the Siegel modular variety // Springer-Verlag, Lect. Notes in Math. N 1240 (1987)

[Za] *Zagier, Don B.*: Modular forms whose Fourier ccoefficients involve zeta-functions of quadratic fields // In: Modular functions.V, Springer-Verlag, Lect. Notes in Math. N 627 (1977) 106–168