Max-Planck-Institut für Mathematik
Bonn

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Max-Planck-Institut für Mathematik
Preprint Series 2020 (42)

Date of submission: August 2, 2020

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MPIM 20-42

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Abstract

We give a construction of Lagrangian torus fibrations with controlled discriminant locus on certain affine varieties. In particular, we apply our construction in the following ways:

- We find a Lagrangian torus fibration on the 3-fold negative vertex whose discriminant locus has codimension 2; this provides a local model for finding torus fibrations on compact Calabi-Yau 3-folds with codimension 2 discriminant locus.
- We find a Lagrangian torus fibration on a neighbourhood of the one-dimensional stratum of a simple normal crossing divisor (satisfying certain conditions) such that the base of the fibration is an open subset of the cone over the dual complex of the divisor. This can be used to construct an analogue of the non-archimedean SYZ fibration constructed by Nicaise, Xu and Yu.

1 Introduction

The Strominger-Yau-Zaslow conjecture [33, 14, 20] asserts that a Calabi-Yau manifold admitting a degeneration to a large complex structure limit also admits a Lagrangian torus fibration. Indeed, the fibres are special Lagrangian and the Calabi-Yau metric should undergo Gromov-Hausdorff collapse along the fibration as one approaches the limit. If we take a minimal semistable reduction of the degeneration, so that the singular fibre is a reduced simple normal crossing variety, the base of the SYZ torus fibration should be the dual complex[1] of the singular fibre [20, Section 3.3].

In this paper, we focus on the problem of finding Lagrangian torus fibrations (not necessarily special) on affine manifolds W arising as $X \setminus Y$ where X is a complex projective variety and Y is a simple normal crossing divisor. These should be thought of as local models for Lagrangian torus fibrations on compact Calabi-Yaus or other varieties (for example surfaces of general type). The idea

[1] called *Clemens complex* in [20].

is very simple: construct a Lagrangian torus fibration on the contact "link at infinity" of W, extend it to the interior of W using Liouville flow, and add a central fibre corresponding to the Lagrangian skeleton of W. If B is the base of the Lagrangian torus fibration on the link then we are careful to identify B with the dual complex of Y. The base of the Lagrangian torus fibration on W is then $\mathrm{Cone}(B)$.

The singularities of the Lagrangian torus fibrations constructed in this way are usually too severe to be of much use. However, in some cases, this construction yields Lagrangian fibrations with mild singularities which are difficult to construct any other way. We demonstrate this with two examples:

- Mikhalkin [23] introduced the *tropicalisation* map which, amongst other things, gives a topological torus fibration from a variety to its tropicalisation. In particular, one obtains a map from the 4-dimensional pair-of-pants to the cell complex which is the cone on the 1-skeleton of a tetrahedron. In Section 3.1, we give a version of this map with Lagrangian fibres. We anticipate that this will be useful for constructing Lagrangian torus fibrations on surfaces of general type.
- We give a Lagrangian torus fibration on the affine variety

$$W = \{(x, y, u_1, u_2) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 \ : \ xy = u_1 + u_2 + 1\} \tag{1}$$

  such that the base is a 3-ball and the discriminant locus (set of singular fibres) is a Y-graph. The existence of such a Lagrangian torus fibration was conjectured by Gross [13], who gave a topological torus fibration on this space with the same discriminant locus. The singular fibre over the vertex of the graph appears to be the same as the one Gross conjectured (a space with Euler characteristic -1 which we will call the *Gross fibre*, see [13, Example 2.6.(4)]).

This second example is of particular significance. In the early days of mirror symmetry, there was an expectation that the discriminant locus for a Lagrangian torus fibration on a Calabi-Yau 3-fold should be a trivalent graph. The only fibres with nontrivial Euler characteristic should live over the vertices of this graph, so there should be two local models where the discriminant is a Y-graph: one where the fibre over the vertex has Euler characteristic 1, and one where it has Euler characteristic -1 (as there are Calabi-Yau 3-folds with both positive and negative Euler characteristic). The local model with Euler characteristic 1 is easy to construct (see [12, Example 1.2]).

An explicit Lagrangian fibration on the variety given by Equation (1) was found by Castaño-Bernard and Matessi [3, 4], but the discriminant locus was a codimension 1 thickening of the Y-graph. Indeed, Joyce [16] had earlier argued that the property of having codimension 1 discriminant locus should be generic amongst singularities of special Lagrangian torus fibrations. For this reason, attention in recent years has focused on understanding the case where the discriminant locus has codimension 1. For example, the powerful method introduced by W.-D. Ruan [27, 28, 29] for producing Lagrangian torus fibrations on compact Calabi-Yaus produces fibrations with codimension 1 discriminant locus unless special care is taken.

In light of our construction, it seems reasonable to expect (continuous) Lagrangian torus fibrations on compact Calabi-Yau 3-folds with codimension 2 discriminant locus, as originally expected.

An alternative approach to the SYZ conjecture is to look for non-archimedean analogues of Lagrangian torus fibrations instead of special Lagrangian fibrations. This was first suggested in [20, §3.3] and [21], and details of this approach are explained in [25]. We give an overview in Section 6, which explains how it relates to the ideas in the current paper. According to [21, Conjecture 1], the smooth part of the Gromov-Hausdorff limit of maximally degenerating families of Calabi-Yau varieties should carry an integral affine structure, of which it should be possible to give a purely algebraic description; see [21, Conjecture 3] and [25, Theorem 6.1]. In Section 6, we show how the integral affine structure constructed in [25, Theorem 6.1] arises essentially from a Lagrangian torus fibration on an open subset of the total space of the degenerating family.

1.1 Outline

- In Sections 2.1–2.2, we give some basic definitions and setup; in particular, we define precisely what we mean by a Lagrangian torus fibration and a coisotropic fibration.
- In Sections 2.3–2.4, we explain how to construct a Lagrangian torus fibration on an affine variety $X \setminus Y$, and we discuss some obstructions to lift Lagrangian torus fibrations from subvarieties of Y.
- In Section 3, we apply the previous construction to find Lagrangian torus fibrations on the 4-dimensional pair-of-pants and on the negative vertex (the variety defined by Equation (1)).
- In Section 4, we discuss the dual complex of a simple normal crossing divisor and prove some results about the topology of the dual complex under the assumption that the complement of the simple normal crossing divisor is affine.
- In Section 5, we construct a coisotropic fibration on the contact boundary of $X \setminus Y$ such that the base of the fibration is the dual complex of the compactifying divisor Y.
- In Section 6, we explain a construction of Lagrangian torus fibrations with codimension 2 discriminant locus which is a symplectic analogue of the non-archimedean SYZ fibration constructed in [25].

1.2 Acknowledgements

We would like to acknowledge useful comments from the anonymous referees, which have helped us improve the exposition in this paper.

JE is supported by EPSRC Standard Grant EP/P02095X/1 and would like to acknowledge useful conversations with Ivan Smith, Paul Hacking, Peng Zhou, Renato Vianna, Dmitry Tonkonog, Mark McLean and Mohammed Tehrani.

MM is supported by the Max Planck Institute for Mathematics, and the Engineering and Physical Sciences Research Council [EP/L015234/1]. The EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London. MM expresses his gratitude to his advisor Paolo Cascini for his support and advice, and to Enrica Mazzon and Johannes Nicaise for helpful discussions on Section 6.

2 Definitions and setup

2.1 Stratifications

Definition 2.1. Recall that a stratification of a topological space B is a filtration

$$\emptyset = B_{-1} \subset B_0 \subset \cdots \subset B_d \subset B_{d+1} \subset \cdots \subset B,$$

where each B_d is a closed subset such that:

- the d-stratum $S_d(B) := B_d \setminus B_{d-1}$ is a smooth d-dimensional manifold for each d (possibly empty),
- $B = \bigcup_{d \geq 0} B_d$.

We say that B is finite-dimensional if the d-stratum is empty for sufficiently large d, and we say that B is n-dimensional if B is finite-dimensional and n is the maximal number for which $S_n(B)$ is nonempty, in which case we call $S_n(B)$ the *top stratum*, $S_{top}(B)$.

Definition 2.2 (Cones)**.** Given a stratified space B, the *cone over* B, denoted $\mathrm{Cone}(B)$, is the stratified space

$$\mathrm{Cone}(B) = ([-\infty, \infty) \times B)/(\{-\infty\} \times B),$$

whose strata are the open cones on the strata of B and the singleton stratum comprising the cone point $[\{-\infty\} \times B]$.

2.2 Lagrangian torus fibrations

We are interested in proper continuous maps $F\colon X \to B$ with connected fibres where B is a stratified space and, either:

- (X, ω) is a symplectic manifold of dimension $2n$, or
- (X, α) is a contact manifold of dimension $2n-1$ with a chosen contact 1-form α. In this case, we write $\omega = d\alpha$.

Definition 2.3 (Coisotropic fibration)**.** We call $F\colon X \to B$ a *coisotropic fibration* if it is a proper continuous map with connected fibres such that the fibres over the k-dimensional stratum are ω-coisotropic submanifolds (possibly with boundary and corners) of codimension k. We say a coisotropic fibration is *generically Lagrangian* if F is a smooth submersion over $S_{top}(B)$ and the fibres over $S_{top}(B)$ are Lagrangian. (Note that a Lagrangian submanifold of a contact manifold is one on which $d\alpha$ vanishes.)

Remark 2.4. If F is generically Lagrangian then $\dim B = n$, respectively $\dim B = n-1$, if X is symplectic, respectively contact.

Definition 2.5 (Lagrangian torus fibration)**.** We call $F\colon X \to B$ a *Lagrangian torus fibration* if

- F is a smooth submersion over the top stratum, with Lagrangian fibres.
- the fibres over other strata are themselves stratified spaces with isotropic strata.

We call the complement of the top stratum the *discriminant locus* $Discr(F) = B \setminus S_{top}(B)$.

Definition 2.6 (Integral affine structure)**.** Let M be a manifold of dimension n. The following definitions are equivalent.

1. An *integral affine structure* is a maximal atlas of M such that the transition functions belong to $\mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$.
2. An *integral affine structure* is a sheaf Aff_M of continuous functions on M such that (M, Aff_M) is locally isomorphic to $\mathbb{R}^n$ endowed with the sheaf of degree one polynomials with integral coefficients.
3. An *integral affine structure* is the datum of a torsion-free flat connection on the tangent bundle TM, and of a lattice of flat sections of maximal rank in TM.

See [21, §2.1] for a proof of the equivalence.

Remark 2.7. The Arnold-Liouville theorem tells us that the Lagrangian fibres of F are tori and, if X is symplectic, that the top stratum of B inherits an integral affine structure.

Remark 2.8. In the definition of coisotropic fibration, the fibres can be very large. For example, the fibres over the 0-strata of B are codimension zero submanifolds with boundary. One example of a coisotropic fibration would be taking a Lagrangian torus fibration $X \to B$ and postcomposing with a map which collapses an open set in B containing the discriminant locus. This motivates the following definition:

Definition 2.9. We will say that a Lagrangian torus fibration $F_1 \colon X \to B_1$ is a *refinement* of a coisotropic fibration $F_2 \colon X \to B_2$ if there is a continuous surjection $g \colon B_1 \to B_2$ such that $F_2 = g \circ F_1$.

We finish this section with some remarks on Lagrangian torus fibrations on contact manifolds.

Remark 2.10. Recall that the Reeb flow for a contact manifold (M, α) is always tangent to a $d\alpha$-Lagrangian submanifold of M.

Remark 2.11. Suppose that $f \colon X \to B$ is a Lagrangian torus fibration on a contact $(2n-1)$-manifold (X, α). The action integrals $\int_{C_1} \alpha, \ldots, \int_{C_n} \alpha$ of the contact 1-form around a basis for the homology of a general fibre define a map from the universal cover of the top stratum $S_{top}(B)$ to $\mathbb{R}^n$. This map necessarily avoids the origin. Otherwise the pullback of α to the fibre is a nullhomologous closed 1-form, which therefore vanishes somewhere on the fibre. However it must always evaluate positively on the Reeb vector, which is tangent to the fibre by Remark 2.10.

Remark 2.12. In the context of the previous remark, the action integrals $e^r \int_{C_1} \alpha$, $\ldots$, $e^r \int_{C_n} \alpha$ on the symplectisation give local action coordinates for the Lagrangian torus fibration $(r, f) \colon \mathbb{R} \times X \to \mathbb{R} \times B$ on the symplectisation $(\mathbb{R} \times X, d(e^r \alpha))$.

Remark 2.13. The following rigidity result for Lagrangian torus fibrations on contact manifolds with particular conditions on the Reeb dynamics will apply to some examples later (see Remark 5.8).

Lemma 2.14. *Let (M, α) be a contact manifold with contact 1-form α and let $F_1 \colon M \to B_1$, $F_2 \colon M \to B_2$ be Lagrangian torus fibrations. Suppose that for an open set $U \subset (B_1)_{top}$ of regular fibres there is a dense set $V \subset U$ such that, for any $v \in V$, the fibre $F_1^{-1}(v)$ contains a dense Reeb orbit. Then F_2 is constant along the fibres of F_1. In other words, the fibration F_2 coincides with F_1 over $F_1^{-1}(U)$.*

Proof. The fibre of F_2 through a point $x \in F_1^{-1}(v)$, $v \in V$, necessarily contains the closure of the Reeb orbit through x, so it contains the whole fibre $F_1^{-1}(v)$ (recall that we have a standing assumption that fibres are connected). Since this is true for a dense set $V \subset U$, and since F_1 and F_2 are continuous, this means $F_2|_{F_1^{-1}(u)}$ is constant for all $u \in U$, as required. $\square$

2.3 Affine varieties

Let X be a smooth complex projective variety of complex dimension n. Let $Y \subset X$ be a simple normal crossing divisor with components Y_i, $i \in I$, for some indexing set I. Suppose that there is an ample divisor $\sum_{i\in I} m_i Y_i$, $m_i \geq 1$, fully supported on Y. Let L_i be the holomorphic line bundle with $c_1(L_i) = Y_i$, so that Y_i is the zero locus of a transversely vanishing section $s_i \in H^0(X, L_i)$. Let $|\cdot|_i$ be a Hermitian metric on L_i and let ∇_i be the corresponding Chern connection (uniquely determined by the condition of being a metric connection for $|\cdot|_i$ compatible with the holomorphic structure of E) with curvature F_{∇_i}. By [11, p.148], we can choose the Hermitian metrics in such a way that $\omega := -2\pi i \sum_{i\in I} m_i F_{\nabla_i}$ is a Kähler form on X with Kähler potential

$$\psi := -\sum_{i\in I} m_i \log |s_i|_i \colon W \to \mathbb{R}$$

defined on $W = X \setminus Y$, that is $\omega = 2\pi i \partial\bar{\partial}\psi$. In other words, ψ is a plurisubharmonic exhausting function on W.

We now have a standard package of geometric objects associated to ψ:

- its gradient (with respect to the Kähler metric on W) is a *Liouville vector field* $Z := \nabla\psi$; in other words $\mathcal{L}_Z\omega = \omega$. The 1-form $\lambda := \iota_Z\omega$ is a primitive for ω called the *Liouville form*. The flow ϕ_t for time t along Z is called the *Liouville flow*: it dilates the symplectic form in the sense that $\phi_t^*\omega = e^t\omega$.
- the critical locus of ψ is compact [32, Lemma 4.3].
- if ψ is Morse or Morse-Bott, the union of all downward manifolds of the Liouville flow is isotropic (called the *skeleton*, Σ); see [5, Proposition 11.9]. Since the critical locus of ψ is compact, this can be achieved by perturbing the Hermitian metrics on the bundles L_i (and hence the symplectic form) over a compact subset of W.
- if $M \subset W$ is a $(2n-1)$-dimensional submanifold transverse to the Liouville flow then M inherits a contact form α pulled back from the Liouville form λ. Moreover, if M is disjoint from the skeleton Σ then the complement $W \setminus \Sigma$ is symplectomorphic to a subset of the symplectisation SM of the form $\{(r, x) \in \mathbb{R} \times M \ :\ r < T(x)\}$, where $T(x) = \sup\{t \ :\ \phi_t(x) \text{ is defined}\}$. In particular, if Z is complete then $W \setminus \Sigma$ is symplectomorphic to the symplectisation of M.

Definition 2.15 (Link at infinity)**.** Let $M \subset W$ be a $(2n-1)$-dimensional submanifold transverse to the Liouville flow and disjoint from the skeleton. The contact manifold $(M, \lambda|_M)$ is called *link (at infinity) of* W, denoted by $\mathrm{Link}(W)$.

The contactomorphism type of $\mathrm{Link}(W)$ is independent of the choice of M by [5, Lemma 11.4].

Lemma 2.16. *Suppose there is a contact form* α *on* $\mathrm{Link}(W)$ *(not necessarily* $\lambda|_{\mathrm{Link}(W)}$*) for which there is a Lagrangian torus fibration* $f\colon \mathrm{Link}(W) \to B$*. We get a Lagrangian torus fibration* $F\colon \overline{W} \to \mathrm{Cone}(B)$ *where* $\overline{W}$ *is the symplectic completion of* W*.*

Proof. The complement of the skeleton $\overline{W} \setminus \Sigma$ is symplectomorphic to the symplectisation of the link: $\left(\mathbb{R}_t \times \mathrm{Link}(W), d\left(e^t \lambda|_{\mathrm{Link}(W)}\right)\right)$. Since $\alpha = e^f \lambda|_{\mathrm{Link}(W)}$ for some function f, this symplectisation is itself symplectomorphic to the symplectisation $(\mathbb{R}_{\mathfrak{t}} \times \mathrm{Link}(W), d(e^{\mathfrak{t}}\alpha))$, via the symplectomorphism $(\mathfrak{t}, x) \mapsto (t + f(x), x)$.

The fibration $F\colon \mathbb{R} \times \mathrm{Link}(W) \to \mathbb{R} \times B$, $F(\mathfrak{t}, x) = (\mathfrak{t}, f(x))$ defines a Lagrangian torus fibration on the complement $\bar{W} \setminus \Sigma$. This extends continuously to a map $\overline{W} \to \mathrm{Cone}(B)$ by sending Σ to the cone point. □

Remark 2.17. The discriminant locus of $F\colon W \to \mathrm{Cone}(B)$ is the cone on the discriminant locus of $f\colon \mathrm{Link}(W) \to B$.

Remark 2.18. Although the existence of a Lagrangian torus fibration on a contact manifold depends on the contact form α (the fibres must be $d\alpha$-Lagrangian), Lemma 2.16 only depends on the contactomorphism (not strict contactomorphism) type of the link.

Remark 2.19. The Lagrangian skeleton of W is not easy to find in general. Ruddat, Sibilla, Treumann and Zaslow [30] have given a conjectural description of the skeleton when W is a hypersurface in $(\mathbb{C}^*)^m \times \mathbb{C}^n$, and all the varieties we consider in this paper fall into this class. When W is hypersurface in $(\mathbb{C}^*)^m$, P. Zhou [35] has confirmed that there is a Liouville structure whose skeleton is precisely the RSTZ skeleton.

Remark 2.20. In the situation of the SYZ conjecture, where W is a local model for a Calabi-Yau variety, $\mathrm{Cone}(B)$ should be homeomorphic to a codimension zero submanifold of the n-sphere. One might wonder in that case if the map F is smooth *across* the different strata for some chosen smooth structure on $\mathrm{Cone}(B)$. That is not a question we will consider in this paper, as it does not appear natural from our perspective: in our more general setting, $\mathrm{Cone}(B)$ will not always be a topological manifold.

2.4 Symplectic neighbourhoods

Let $Y \subset X$ be a smooth codimension 2 symplectic submanifold with symplectic normal bundle $\pi\colon \nu \to Y$. Pick a Hermitian metric and a unitary connection ∇ on ν and let $\mu\colon \nu \to \mathbb{R}$ denote the function which generates the rotation of fibres. Let $\mathcal{H}$ denote the horizontal spaces of ∇ and define a 2-form Ω on ν which:

- equals $\pi^*\omega_Y + \langle \mu, F_\nabla \rangle$ on pairs of horizontal vectors (thinking of μ as $\mathfrak{u}(1)^*$-valued and F_∇ as $\mathfrak{u}(1)$-valued),

- equals the fibrewise Hermitian volume form on pairs of vertical vectors,
- makes $\mathcal{H}$ orthogonal to the fibres.

This 2-form Ω is closed and is nondegenerate on a neighbourhood of the zero-section; see for instance [7, Section 2.2].

Definition 2.21 ([7, Definition 2.9])**.** An ω-regularisation of Y in X is a symplectomorphism

$$\Psi\colon (\mathrm{nbhd}_\nu(Y), \Omega) \to (\mathrm{nbhd}_X(Y), \omega)$$

from a neighbourhood of the zero-section in ν to a neighbourhood of Y in X such that, along Y, $d\Psi$ induces the canonical isomorphism from the vertical distribution of ν to the normal bundle of Y.

The following lemma is immediate from the definition of Ω. In the special case when $F_\nabla = \omega$, it recovers Biran's circle bundle construction.

Lemma 2.22. *Suppose that $L \subset Y$ is a Lagrangian submanifold on whose tangent spaces the curvature F_∇ vanishes. Then, for all $c > 0$, the submanifold $\{x \in \nu \ : \ \pi(x) \in L,\ |x| = c\}$ is an Ω-Lagrangian circle-bundle over L.*

Remark 2.23. In an earlier version of this paper we implicitly assumed that Lemma 2.22 holds without the curvature hypothesis, and used this to lift Lagrangian torus fibrations from Y to the link. This does not work unless one is able to find connections for which the curvature vanishes along all Lagrangians in the torus fibration and which satisfy the compatibility conditions [7, Definitions 2.11 and 2.12] along the normal crossing locus. For example if Y is a complex curve then the curvature condition is empty because Lagrangians in Y are 1-dimensional.

All of this generalises to symplectic submanifolds of higher codimension whose normal bundle splits as a direct sum of Hermitian line bundles; the Lagrangian submanifolds from Lemma 2.22 are then torus-bundles.

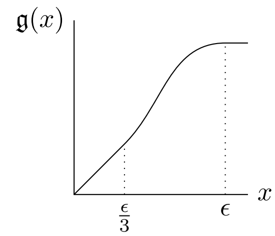
If we have a simple normal crossing divisor $Y = \bigcup_{i\in I} Y_i \subset X$ then an ω-regularisation of Y is a collection of ω-regularisations of the submanifolds $Y_J = \bigcap_{j\in J} Y_j$ satisfying compatibility conditions [7, Definitions 2.11 and 2.12].

By [7, Theorem 2.13], after an exact deformation of the symplectic form ω we can find an ω-regularisation of Y. By [22, Corollary 5.11], this deformation does not change the contactomorphism type of the link; though McLean's proof of this is written for "positively wrapped divisors" (like exceptional divisors of resolutions of singularities) it carries over with minor sign changes to the case of negatively wrapped divisors (like ample divisors).

Moreover, we can identify the link directly. Let r_i be a radial coordinate in the normal bundle ν_i of Y_i (in the sense of [22, Definition 5.7]). McLean calls a function $f\colon X \setminus Y \to \mathbb{R}$ compatible with Y if it has the form $\sum_i m_i \log r_i + \tau$ in a punctured neighbourhood of Y, for some constants m_i and a smooth function τ. The level sets $f^{-1}(\epsilon)$ of a compatible function are of contact type for sufficiently small ϵ [22, Proposition 5.8]. The plurisubharmonic function ψ on $X \setminus Y$ is compatible with Y [22, Lemma 5.25], and so is the function $\sum_{i \in I} \log(\mathfrak{g} \circ \mu_i)$ where $\mathfrak{g}$ is a cut-off function satisfying

$$\mathfrak{g}(x) = \begin{cases} x & \text{if } x \in [0, \epsilon/3] \\ 1 & \text{if } x \geq \epsilon, \end{cases} \tag{2}$$

and which is positive and strictly increasing on $[0, \epsilon]$.



The link of W (a level set of ψ near Y) is therefore contactomorphic to a plumbing of the circle bundles with fixed radii r_i in the normal bundles of the components Y_i (a level set of $\sum_{i \in I} \log(\mathfrak{g} \circ \mu_i)$).

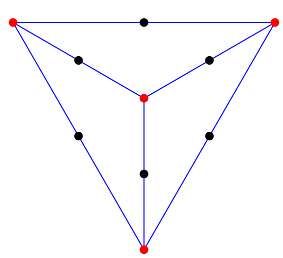
3 Examples

We now apply Lemma 2.16 in some examples to construct Lagrangian torus fibrations on affine varieties.

3.1 Example: pair-of-pants

Let Y be a union of four lines $Y_1, \dots, Y_4$ in general position in $\mathbb{CP}^2$. The complement $W := \mathbb{CP}^2 \setminus Y$ is called the *4-dimensional pair-of-pants*.

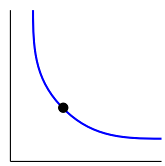
Lemma 3.1. *There is a Lagrangian torus fibration $f\colon \mathrm{Link}(W) \to B$, where B is the 1-skeleton of a tetrahedron:*



Proof. Using [7, Theorem 2.13], deform the symplectic form and choose a symplectic tubular neighbourhood of Y, that is a collection of symplectic embeddings $\Psi_i\colon \operatorname{nbhd}_{\mathcal{O}(1)}(Y_i) \to \mathbb{CP}^2$, where we have equipped $\mathcal{O}(1)$ with the symplectic form from Section 2.4. Write $N_i = \operatorname{Image}(\Psi_i)$ for the neighbourhood of Y_i, $N_{ij} = N_i \cap N_j$ for the overlaps.

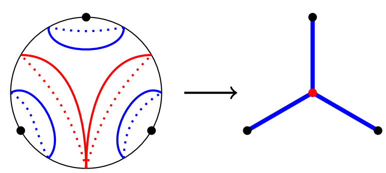
We now construct a Lagrangian torus fibration $f\colon \operatorname{Link}(\mathbb{CP}^2 \setminus Y) \to B$ as follows:

- on overlaps $(\mu_i, \mu_j)\colon N_{ij} \to [0,\infty)^2$ is a Lagrangian torus fibration. When restricted to $\operatorname{Link}(W) = \{x \ : \ \sum_{i\in I} \log(\mathfrak{g}(\mu_i(x))) = \epsilon'\}$, its image is the arc shown below:

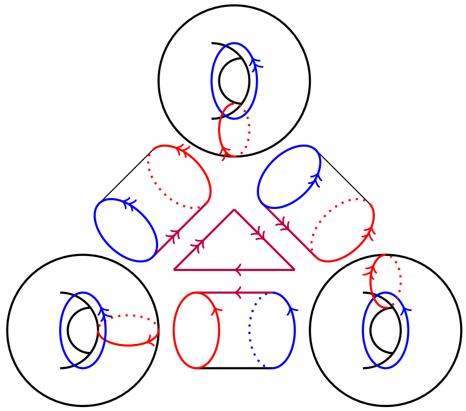
  (We have drawn the black dot to show how this arc sits inside the graph B). Note that (μ_i, μ_j) restricts to function $\varphi_i\colon Y_i \cap N_{ij} \to [0,\infty)$.

- on $N_i^\circ := N_i \setminus \bigcup_{j\neq i} N_{ij}$, the projection $\pi_i\colon \operatorname{Link}(W) \cap N_i^\circ \to Y_i$ is a circle bundle. Let B_i be a Y-graph and let $\varphi_i\colon Y_i \to B_i$ be a function extending the functions already constructed on $Y_i \cap N_{ij}$:

  The composition $\varphi_i \circ \pi_i\colon \operatorname{Link}(W) \cap N_i^\circ \to B_i$ is a Lagrangian torus fibration: by Lemma 2.22, the fibres are Lagrangian S^1-bundles over the 1-dimensional fibres of φ_i (the vanishing curvature condition is trivial). $\square$

The cone on the graph B can be visualised in $\mathbb{R}^3$ as the cone on the 1-skeleton of a tetrahedron. The result of applying Lemma 2.16 in this case is a Lagrangian torus fibration of $\mathbb{CP}^2 \setminus Y$ over this cone. The fibres over the cones on the blue edges are tori; the fibres over the cones on the red vertices are $S^1 \times 8$ (where 8 denotes the wedge of two circles). The fibre over the cone point is any Lagrangian skeleton for the pair-of-pants. In particular, since the pair-of-pants is a hypersurface in $(\mathbb{C}^*)^3$, Zhou's result [35] implies that we can take the RSTZ skeleton [30]. In this particular case, the RSTZ skeleton is obtained from three disjoint 2-tori by attaching three cylinders and a triangular 2-cell as indicated in the figure below. This is homotopy equivalent to the 2-skeleton of a 3-torus.



Remark 3.2. In this case, Mikhalkin gave a purely topological torus fibration of the pair-of-pants over the same base, namely the tropicalisation map [23]. See also the paper of Golla and Martelli [10] for a description of the Mikhalkin fibration. The fact that the 4-dimensional pair-of-pants is homotopy equivalent to the 2-skeleton of a 3-torus was proved much earlier by Salvetti [31].

3.2 Example: negative vertex

Take the affine variety

$$W = \{(x, y, u_1, u_2) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 \ : \ xy = u_1 + u_2 + 1\}.$$

We call this variety *the negative vertex*. It can be compactified to $\mathbb{CP}^3$ as follows. Rewrite the equation for W as $u_2 = xy - u_1 - 1$; W is the complement of $u_2 = 0$ in $\mathbb{C}^2_{xy} \times \mathbb{C}^*_{u_1}$, in other words it is the complement in $\mathbb{CP}^3_{[x:y:u_1:w]}$ of $(xy - u_1 w - w^2)u_1 w = 0$. Taking $L_1 = \mathcal{O}(2)$, $L_2 = L_3 = \mathcal{O}(1)$, $s_1 = xy - u_1 w - w^2$, $s_2 = u_1$, $s_3 = w$, we obtain a subvariety $Y = Y_1 \cup Y_2 \cup Y_3$ such that $W = \mathbb{CP}^3 \setminus Y$. However, this subvariety is not normal crossing: the components Y_1 and Y_3 intersect non-transversely at $[0 : 0 : 1 : 0]$ (Y_3 is the tangent plane to the quadric Y_1 at that point).

If we blow up along the line $[x : 0 : u_1 : 0]$ then the total transform of Y becomes simple normal crossing. We continue to write Y_j for the proper transform of Y_j, $j = 1, 2, 3$, and we write Y_4 for the exceptional divisor (a copy of $\mathbb{CP}^1 \times \mathbb{CP}^1$). We will consider the ample divisor $Y_1 + Y_2 + Y_3 + Y_4$ (i.e. giving all components multiplicity one). The symplectic form coming from this ample divisor is in the cohomology class $4H - E$, where H is the cohomology class Poincaré dual to the proper transform of a generic plane and E is the class of the exceptional divisor.

Without choosing very specific curvature forms on the normal bundles, we no longer have access to Lemma 2.22 for constructing Lagrangian torus fibrations

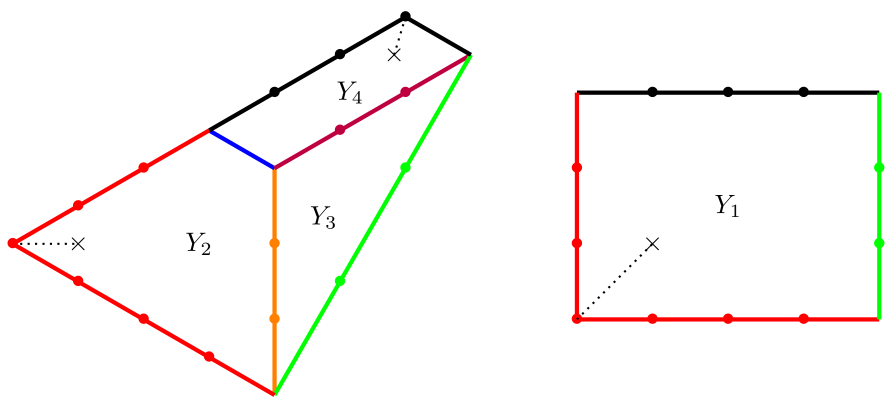


Figure 1: Almost toric base diagrams for a system of Lagrangian torus fibrations on the boundary divisor for the negative vertex.

on the link. Instead, we will proceed in a more *ad hoc* manner and construct the torus fibration directly using almost toric methods.

In Figure 1, we draw almost toric base diagrams for a system of Lagrangian torus fibrations $\varphi_j \colon Y_j \to B_j$. Recall that crosses indicate focus-focus singularities and dotted lines indicate branch cuts. Some of the edges are broken by a branch cut: they are nonetheless straight lines in the affine structure and the break point is not to be considered a vertex. The colours on edges indicate how these edges are to be identified in the pushout. Broken edges are decorated with the same colour on each segment, which is taken to mean that the edge continues beyond the break point, not that some kind of self-identification should be made. Dots on the edges are there to indicate interior integral points of the edges (for the integral affine structure). You can read the affine lengths of edges off from the integrals of $[\omega] = 4H - E$ over the corresponding spheres.

Proposition 3.3. *We can use this system of fibrations on Y to construct a Lagrangian torus fibration on the link. The base B is a topological 2-sphere and the discriminant locus consists of three points (the focus-focus singularities of the almost toric fibrations). We further obtain a Lagrangian torus fibration on W; the base is a 3-ball and the discriminant locus is the cone over three points, i.e. a Y-graph.*

Proof. We will prove this in Section 3.2.1 below. □

Theorem 3.4. *The negative vertex admits a Lagrangian torus fibration over the 3-ball such that the discriminant locus is a Y-graph. The fibre over the vertex of the Y-graph is a topological space obtained by attaching a solid torus to a wedge of two circles via an attaching map which is freely homotopic to a map $\phi\colon \partial(D^2 \times S^1) \to S^1 \vee S^1$ which induces the map $\phi_*\colon \mathbb{Z}^2 \to \mathbb{Z} \star \mathbb{Z} = \langle a, b\rangle$, $\phi_*(1,0) = aba^{-1}b^{-1}$, $\phi_*(0,1) = 1$ on fundamental groups.*

Proof. The existence of the Lagrangian torus fibration follows immediately from Lemma 2.16 and Proposition 3.3. It remains only to find the Lagrangian skeleton of W, which we do in Section 3.2.2 below. □

Remark 3.5. Note that since $S^1 \vee S^1$ is an Eilenberg-MacLane space, the induced map on fundamental groups determines the attaching map up to homotopy.

Remark 3.6. Note that the Gross fibre (cf [13, Example 2.6.(4)]) is also obtained by attaching a solid torus to a wedge of two circles by an attaching map in this homotopy class. It seems harder to see the relationship between the RSTZ skeleton in this case and the Gross fibre, though they are necessarily homotopy equivalent.

3.2.1 Proof of Proposition 3.3

We use [7, Theorem 2.13] to make an exact deformation of the symplectic structure so that Y admits an ω-regularisation. Now by Lemma A.1 and Remark A.2, a neighbourhood of Y in X can be obtained from the normal bundles $\nu_X Y_i$ by plumbing along the submanifolds $\nu_X Y_I$ ($|I| \geq 2$), and any two plumbing neighbourhoods are related by gauge transformations of $\nu_X Y_I$ preserving the stratification by subbundles $\nu_{Y_J} Y_I$. Isotopic gauge transformations yield symplectomorphic plumbings, so we can work with unitary gauge transformations without loss of generality (because $Sp(2n)$ retracts onto its maximal compact $U(n)$).

Momentarily, we will write down an almost toric base diagram for a 6-manifold $\mathcal{X}$ whose almost toric boundary $\mathcal{Y}$ is precisely the system of almost toric 4-manifolds shown in Figure 1, which is our divisor Y. The normal bundles to the components and strata of $\mathcal{Y}$ agree with those of Y.

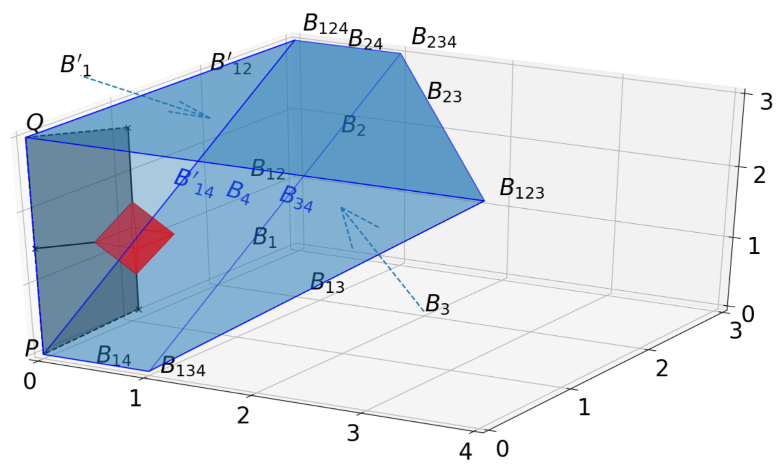
The submanifolds $\nu_{\mathcal{X}} \mathcal{Y}_I$ for $|I| \geq 2$ are toric: they are total spaces of $\mathcal{O}(m) \oplus \mathcal{O}(n) \to \mathbb{CP}^1$ for some m, n. If g is a unitary gauge transformation of this bundle preserving the stratification by subbundles $\mathcal{O}(m)$ and $\mathcal{O}(n)$, then it preserves the toric fibration: each toric fibre is invariant under the group of rotations $U(1) \times U(1)$ of the bundle fibre. Therefore all possible plumbings admit almost toric structures with the same base diagram[2]. In particular, this implies that X (which corresponds to some choice of plumbing) admits an almost toric fibration. Restricting this to the link of Y gives the desired Lagrangian torus fibration. For example, you can take the link to be the preimage of a hypersurface in the almost toric base diagram.

[2]Recall that an almost toric base diagram determines the almost toric manifold only once certain characteristic classes are fixed [36].

We now describe the almost toric base diagram for $\mathcal{X}$. Let

$$\begin{aligned} P &= (0,0,0), & Q &= (0,0,3) \\ B_{134} &= (1,0,0), & B_{123} &= (4,0,3) \\ B_{124} &= (0,3,3), & B_{234} &= (1,3,3), \end{aligned}$$

and consider the convex hull of these six points.



We name the facets:

- $B_1 = \langle P, B_{134}, B_{123}, Q\rangle$ (the quadrilateral at the front in the figure),
- $B'_1 = \langle P, Q, B_{124}\rangle$ (the triangle at the back on the left in the figure),
- $B_2 = \langle B_{124}, Q, B_{123}, B_{234}\rangle$ (the quadrilateral on the top in the figure),
- $B_3 = \langle B_{134}, B_{123}, B_{234}\rangle$ (the slanted triangle on the bottom right in the figure),
- $B_4 = \langle B_{124}, B_{234}, B_{134}, P\rangle$ (the slanted rectangle at the back underneath in the figure).

We label the edges $B_{ij} := B_i \cap B_j$ and $B'_{ij} := B'_i \cap B_j$.

We will excise a small neighbourhood of $(0.5, 0.5, 1.5)$ (red in the figure) and call the resulting polytope $\mathbf{B}$. We will decorate $\mathbf{B}$ with the data of a 3-dimensional almost toric base diagram:

- We make a branch cut using a half-plane emanating from the vertical line at $x = y = 0.5$ (the vertical black line in the figure) and containing the direction $(-1,-1,0)$. We call this the *branch plane*.
- We insert three rays of focus-focus singularities, emanating:
  - vertically downwards from $(0.5, 0.5, 3)$,

- vertically upwards from $(0.5, 0.5, 0.5)$,
- horizontally in the $(1, 1, 0)$-direction from $(0, 0, 1.5)$.

These three rays disappear into the excised neighbourhood. The horizontal ray cuts the branch plane into an upper and lower region.

- We reglue the affine structures across the branch plane as follows:
  - When crossing from $y < x$ to $y > x$ across the upper region of the branch plane, we use the matrix $M_1 = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
  - When crossing from $y < x$ to $y > x$ across the lower region of the branch plane, we use the matrix $M_2 = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

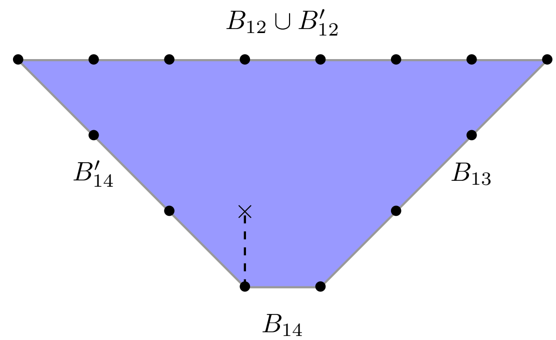
Definition 3.7. Let $\mathcal{X}$ be an almost toric 6-manifold associated to this almost toric base diagram, and $\mathcal{Y}_i$ the almost toric 4-manifold associated to the facet B_i. Note that the base for each $\mathcal{Y}_i$ is contractible, hence determines $\mathcal{Y}_i$ completely. On the other hand, the base for $\mathcal{X}$ has nontrivial second cohomology, so can arise as the base diagram for different almost toric 6-manifolds (distinguished by Zung's Chern class [36]).

Remark 3.8. The union $\mathcal{Y} = \bigcup_{i=1}^{4} \mathcal{Y}_i$ is a symplectic simple normal crossing divisor with pairwise symplectically orthogonal intersections.

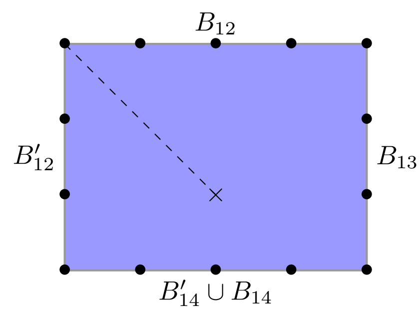
With this integral affine structure, the facets have the following properties:

- B_2 is the almost toric base diagram for an almost toric structure on the first Hirzebruch surface (obtained from the standard moment quadrilateral by a nodal trade). In particular, the edges B'_{12} and B_{12} form part of a straight line in the reglued integral affine structure (this is because the tangent vectors to these edges are related by $M_1 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$).
- B_3 is the standard toric moment polytope for $\mathbb{CP}^2$.
- B_4 is the almost toric base diagram for an almost toric structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ obtained from a standard moment rectangle by a nodal trade. In particular, the edges B'_{14} and B_{14} form part of a straight line in the reglued integral affine structure (this is because the tangent vectors to these edges are related by $M_2 \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$).
- $B_1 \cup B'_1$ forms a single facet in the reglued integral affine structure. This corresponds to an almost toric structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ with one nodal

fibre, shown in the diagram below:



By a mutation and a shear, this is equivalent to:



We now treat each $B'_{ij} \cup B_{ij}$ as a single edge and write it as B_{ij}. We write $\mathcal{Y}_{ij}$ for the corresponding symplectic 2-manifold in $\mathcal{X}$.

- $\mathcal{Y}_{13}$ is a copy of $\{pt\} \times \mathbb{CP}^1 \subset \mathcal{Y}_1 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ and a line in $\mathcal{Y}_3 \cong \mathbb{CP}^2$.
- $\mathcal{Y}_{14}$ is a copy of $\mathbb{CP}^1 \times \{pt\} \subset \mathcal{Y}_1$ and a $(1,1)$-curve in $\mathcal{Y}_4 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$.
- $\mathcal{Y}_{34}$ is a line in $\mathcal{Y}_3$ and a copy of $\{pt\} \times \mathbb{CP}^1$ in $\mathcal{Y}_4$.
- $\mathcal{Y}_{24}$ is a copy of $\mathbb{CP}^1 \times \{pt\}$ in $\mathcal{Y}_4$ and the -1-curve in $\mathcal{Y}_2 \cong \mathbb{F}_1$.
- $\mathcal{Y}_{23}$ is a line in $\mathcal{Y}_3$ and a fibre of $\mathbb{F}_1 \to \mathbb{CP}^1$ in $\mathcal{Y}_1$.
- $\mathcal{Y}_{12}$ is a $(1,1)$-curve in $\mathcal{Y}_1$ and a $(1,1)$ curve in $\mathcal{Y}_2$ (with respect to the basis given by a fibre and the section with square 1).

Note that from this you can read off the normal bundles of the components $\mathcal{Y}_i$; for example, $\nu\mathcal{Y}_3$ is determined by its restriction to $\nu_{\mathcal{Y}_1}\mathcal{Y}_{13}$ along $\mathcal{Y}_{13}$, which is trivial because $\{pt\} \times \mathbb{CP}^1$ has trivial normal bundle in $\mathbb{CP}^1 \times \mathbb{CP}^1$. The Poincaré duals of the normal bundles are:

$$\begin{aligned} c_1(\nu\mathcal{Y}_1) &= [\mathcal{Y}_{13}] + [\mathcal{Y}_{14}], & c_1(\nu\mathcal{Y}_2) &= [\mathcal{Y}_{24}] + [\mathcal{Y}_{23}] \\ c_1(\nu\mathcal{Y}_3) &= 0 & c_1(\nu\mathcal{Y}_4) &= [\mathcal{Y}_{24}] - [\mathcal{Y}_{34}]. \end{aligned}$$

This completes the construction of the base diagram for $\mathcal{X}$ and the identification of its almost toric boundary $\mathcal{Y}$ with Y.

3.2.2 Proof of Theorem 3.4

It remains to identify the Lagrangian skeleton of the negative vertex. The theorem then follows from Lemma 3.9 and Lemma 3.11 below:

- Lemma 3.9 identifies the critical locus of a suitably-chosen plurisubharmonic function ψ. It shows that there are three isolated critical points inside the locus $xy = 0$ (whose downward manifolds trace out a copy of $S^1 \vee S^1$) and a circle of critical points with $xy \neq 0$ (whose downward manifold is a solid torus).
- Lemma 3.11 identifies the attaching map for the solid torus to $S^1 \vee S^1$.

Lemma 3.9. *If $(x, y, u_1, u_2) \in W$, we will write $x = Xe^{i\alpha}$, $y = Ye^{i\beta}$, $u_j = e^{R_j + i\theta_j}$. Let $\Phi = \alpha + \beta$. Fix a real constant $c \in (-1, -\ln 2)$ and consider the plurisubharmonic function*

$$\psi = \frac{1}{2}\left(X^2 + Y^2 + \sum (R_j - c)^2\right).$$

The critical locus of $\psi|_W$ comprises:

- *the points*

  $$P_1 = \left(0, 0, -\frac{1}{2}, -\frac{1}{2}\right), \quad P_2 = (0, 0, e^{a(c)}, -e^{b(c)}), \quad P_3 = (0, 0, -e^{b(c)}, e^{a(c)}),$$

  where $(a(c), b(c))$ is the unique point of intersection between the curves $e^{R_1} + 1 = e^{R_2}$ and $(R_1 - c)e^{-(R_1 - c)} = -(R_2 - c)e^{-(R_2 - c)}$;
- *the circle of points*

  $$\left(e^{i\alpha}\sqrt{2e^{R(c)} + 1}, e^{-i\alpha}\sqrt{2e^{R(c)} + 1}, -e^{R(c)}, -e^{R(c)}\right),$$

  where $R(c) = c - \mathcal{W}(e^c)$, and $\mathcal{W}$ is the Lambert W-function[3].

The downward manifolds of the critical points P_2, P_3 trace out a figure 8 with vertex at P_1. The downward manifold of the circle of critical points is a solid torus.

Proof. We need to find the critical points of the constrained functional

$$\begin{aligned} &\frac{1}{2}\left(X^2 + Y^2 + \sum (R_j - c)^2\right) \\ &\quad -\lambda\left(XY\cos\Phi - 1 - \sum e^{R_j}\cos\theta_j\right) \\ &\quad\quad -\mu\left(XY\sin\Phi - \sum e^{R_j}\sin\theta_j\right). \end{aligned}$$

[3] The Lambert W-function is the inverse to the function $x \mapsto xe^x$ for $x > 0$.

where λ and μ are Lagrange multipliers imposing the constraint $xy = 1 + \sum u_j$. Differentiating, the critical point equations are

$$X = Y(\lambda \cos\Phi - \mu \sin\Phi) \tag{3}$$
$$Y = X(\lambda \cos\Phi - \mu \sin\Phi) \tag{4}$$
$$0 = XY(\lambda \sin\Phi - \mu \cos\Phi) \tag{5}$$
$$R_j - c = -e^{R_j}(\lambda \cos\theta_j + \mu \sin\theta_j) \tag{6}$$
$$\lambda \sin\theta_j = \mu \cos\theta_j. \tag{7}$$

We identify two cases: $XY = 0$ and $XY \neq 0$.

Case $XY \neq 0$. If X and Y are nonzero then Equation (5) implies $(\lambda, \mu) = k(\cos\Phi, \sin\Phi)$ for some $k \neq 0$. Equation (7) implies that, for all j, $(\lambda, \mu) = k_j(\cos\theta_j, \sin\theta_j)$ for some k_j. Overall, this means that the angles $\theta_1, \ldots, \theta_n$ and Φ agree modulo π and that $k_j e^{i\theta_j} = k e^{i\Phi}$, so $k/k_j = \pm 1$. The constraint equation becomes $\left(XY - \sum(ke^{R_i}/k_j)\right) e^{i\Phi} = 1$, which implies that Φ is either 0 or π. Equations (3) and (4) tell us that $X = kY$ and $Y = kX$, so $X = k^2 X$ and $k = \pm 1$. Indeed, since both X and Y are positive, this implies $k = 1$ and $X = Y$. Since $|k_j| = |k| = 1$, this implies that $k_j = \pm 1$.

Equation (6) becomes $R_j - c = -k_j e^{R_j}$, or

$$(R_j - c)e^{-(R_j - c)} = -k_j e^c.$$

If $k_j < 0$ then, provided $c > -1$, this has no solutions (as $xe^{-x} \leq e^{-1}$ for all x). So if we choose $c > -1$ we need to take $k_j = 1$. In this case, there is a unique solution[4] $R(c)$ to the equation $(R - c)e^{-(R-c)} = -e^c$. This gives a circle of solutions

$$X = Y = \sqrt{e^{i\Phi} + 2e^{R(c)}}, \quad R_j = R(c), \quad \theta_j = \Phi,$$

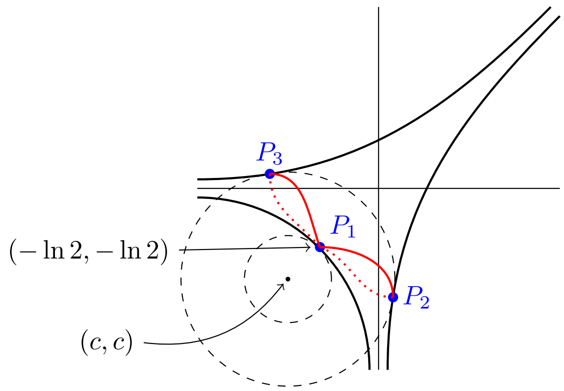
(as α and β vary subject to $\alpha + \beta = \Phi$) provided $e^{i\Phi} + 2e^{R(c)} \geq 0$.

If $e^{i\Phi} = 1$ then $e^{i\Phi} + 2e^{R(c)} > 0$, giving a circle of solutions. If $e^{i\Phi} = -1$ then we can ensure there are no solutions by taking c sufficiently small. Indeed, if $c < -0.3$ then $\ln 2 < \mathcal{W}(e^c) - c$ so $-1 + 2e^{R(c)} < 0$.

Case $XY = 0$: If one of X or Y vanishes, then so does the other (using Equations (3) and (4)). One could find the critical points from the statement of the lemma by a detailed computation as in the other case, but there is a nice "picture-proof" in this case. The subset of points in W with $x = y = 0$ is the curve $C = \{(u_1, u_2) \in (\mathbb{C}^*)^2 \ : \ u_1 + u_2 + 1 = 0\}$ in $(\mathbb{C}^*)^2$. If we draw the image of this curve under the map $(R_1, R_2)\colon W \to \mathbb{R}^2$ then we get the amoeba shown below. We also show the level sets of $\psi|_C$ as dotted circles, and it is easy to see these critical points (blue dots). In red, we have given an idea of how the gradient flowlines from these critical points look: the downward manifold from

[4] In fact, $R(c) = c - \mathcal{W}(e^c)$ where $\mathcal{W}$ is the Lambert W-function.

each of P_2 and P_3 is an interval, whose boundary points must tend to the index 0 critical point P_1 in the limit. Note that it is not true in general that one can find the critical points and flowlines by restricting to a submanifold in this way, but in this case, the Hessian is positive definite on the normal directions to C. $\square$



Lemma 3.10. *The fundamental group $\pi_1(W)$ is abelian.*

Proof. There is an affine conic fibration $\pi\colon W \to (\mathbb{C}^*)^2$, $\pi(x, y, u_1, u_2) = (u_1, u_2)$, with singular fibres over the curve $u_1 + u_2 + 1 = 0$. Let $U \subset (\mathbb{C}^*)^2$ be the complement of this curve and $V = \pi^{-1}(U)$. Since V is a Zariski open set in W, the inclusion map $i\colon V \to W$ induces a surjection $i_*\colon \pi_1(V) \to \pi_1(W)$.

Note that U is a 4-dimensional pair-of-pants; U therefore deformation retracts onto the 2-skeleton of a 3-torus, so $\pi_1(U) = \mathbb{Z}^3$. Let γ be a circle in a smooth fibre of π onto which the fibre deformation retracts. The fundamental group of V is a central extension of $\pi_1(U)$ by $\mathbb{Z}\langle\gamma\rangle$. We have $i_*(\gamma) = 0$ because the loop γ is a vanishing cycle for the conic fibration. Therefore $\pi_1(W)$ is a quotient of $\pi_1(U) = \mathbb{Z}^3$, hence abelian. $\square$

Lemma 3.11. *Let $\phi\colon \partial(D^2 \times S^1) \to S^1 \vee S^1$ be the attaching map for the solid torus to the 1-skeleton. Then, after possibly precomposing with a diffeomorphism of the solid torus and postcomposing with a conjugation, $\phi_*\colon \mathbb{Z}^2 \to \mathbb{Z} \star \mathbb{Z} = \langle a, b\rangle$ satisfies $\phi_*(1,0) = aba^{-1}b^{-1}$, $\phi_*(0,1) = 1$. This determines ϕ completely up to free homotopy (and precomposition by a diffeomorphism of the solid torus).*

Proof. The fact that ϕ is determined by ϕ_* follows from the fact that $S^1 \vee S^1$ is an Eilenberg-MacLane space.

The image of ϕ_* is a subgroup of a free group and hence free, however it is also abelian, so it is either trivial or has rank 1. In other words, $\phi_*(1,0) = c^m$ and $\phi_*(0,1) = c^n$ for some $c \in \langle a, b\rangle$ and $m, n \in \mathbb{Z}$. Suppose that $(1,0)$ is the loop in T^2 which bounds a disc in the solid torus.

Van Kampen's theorem tells us that $\pi_1(W) = \langle a, b \mid c^n \rangle$. By Lemma 3.10, $\pi_1(W)$ is abelian. Therefore $aba^{-1}b^{-1}$ is contained in the normal subgroup generated by c^n, so $aba^{-1}b^{-1} = hc^{nn'}h^{-1}$ for some $h \in \langle a, b \rangle$, $n' \in \mathbb{Z}$. However, $aba^{-1}b^{-1}$ is not a nontrivial power in the free group, so $n = n' = 1$ and $aba^{-1}b^{-1} = hch^{-1}$.

The Dehn twist around the loop $(0, 1)$ in T^2 extends to a diffeomorphism of the solid torus, and precomposing with a power of this diffeomorphism allow us to change m by a multiple of n. Since $n = 1$, we can achieve $m = 0$. This proves the lemma. $\square$

4 Dual complexes

4.1 Definition

Let Y be a pure-dimensional simple normal crossing variety of dimension $n-1$ (see Definition 1.8 of [18]), for example, a simple normal crossing divisor as in Section 2.3. Note that Y is stratified by the intersections of its irreducible components Y_i, i.e.

$$S_d(Y) = \bigcup_{J \subseteq I \,:\, |J| = n-d} Y_J^0,$$

where $Y_J^0 := \left(\bigcap_{j \in J} Y_j\right) \setminus \left(\bigcup_{j \notin J} Y_j\right)$.

Definition 4.1 (Dual complex)**.** The *dual complex* $\mathcal{D}(Y)$ of Y is a regular Δ-complex [15, Section 2.1] whose vertices are in correspondence with the irreducible components Y_i and whose d-cells correspond to connected components of $S_{n-d-1}(Y)$. Like any Δ-complex, the dual complex is stratified (the d-stratum is the union of its open d-cells).

Definition 4.2 (Maximal intersection)**.** We say that Y has *maximal intersection* if it admits a stratum of dimension zero. Equivalently, the corresponding cells of $\mathcal{D}(Y)$ have real dimension $\dim_{\mathbb{C}}(Y)$, and we say that $\mathcal{D}(Y)$ has maximal dimensional.

4.2 Dual boundary complexes of affine varieties

Let $Y \subset X$ be a simple normal crossing divisor of dimension $n-1$. In this section, we study the homotopy type of the dual complex $\mathcal{D}(Y)$, under the assumption that $X \setminus Y$ is affine.

First, recall the following result due to Danilov [6, Proposition 3].

Proposition 4.3. *If one of the irreducible components of Y is ample, then $\mathcal{D}(Y)$ has the homotopy type of a bouquet of spheres.*

The hypothesis of Proposition 4.3 is too restrictive for our purposes (for instance, it does not include all toric boundaries). Moreover, the statement does not provide any control on the number of spheres in the bouquet.

The following propositions can be regarded as generalizations and refinements of Proposition 4.3, and they are inspired by [26] and [19].

Proposition 4.4 (Rational cohomology)**.** *Let $Y \subset X$ be a simple normal crossing divisor of dimension $n-1$ and $W := X \setminus Y$ be an affine variety. Then,*

$$h^i(\mathcal{D}(Y), \mathbb{Q}) = 0 \qquad 0 < i < n-1. \tag{8}$$

If X has Hodge coniveau ≥ 1, i.e. $h^{0,i}(X) = 0$ for all $i > 0$, and $Y_J := \bigcap_{j \in J} Y_j$ does not admit global holomorphic canonical sections for any $J \subseteq I$ (e.g. if X and Y_J are rationally connected), then

$$h^{n-1}(\mathcal{D}(Y), \mathbb{Q}) = h^0(X, K_X + Y). \tag{9}$$

Proof. The vanishing (Equation (8)) of the rational cohomology is noted in [26, Section 6].

In order to prove Equation (9), we identify the $(n-1)$th cohomology group of $\mathcal{D}(Y)$ with the $(n-1)$th cohomology group of the structure sheaf $\mathcal{O}_Y$. Note indeed that the cohomology of $\mathcal{O}_Y$ is computed by a spectral sequence whose page E_1 is given by

$$E_1^{p,q} := \bigoplus_{J \subseteq I, |J| = q+1} H^p(Y_J, \mathcal{O}_{Y_J}),$$

and which degenerates at the page E_2; see [9, Proof of Proposition 1.5.3]. Since $H^{\dim Y_J}(Y_J, \mathcal{O}_{Y_J}) = 0$ for any $J \subseteq I$, and $\dim Y_J > 0$, we have that

$$H^{n-1}(Y, \mathcal{O}_Y) = E_2^{0,n-1} = H^{n-1}(\mathcal{D}(Y), \mathbb{Q}),$$

since the complex $(E_1^{0,*}, d_1)$ computes the cellular cohomology of $\mathcal{D}(Y)$.

Further, the short exact sequence

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

induces the following isomorphism in cohomology

$$H^{n-1}(Y, \mathcal{O}_Y) \simeq H^n(X, \mathcal{O}_X(-Y)) \simeq H^0(X, K_X + Y)^\vee,$$

since by hypothesis $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. We conclude that $H^{n-1}(\mathcal{D}(Y), \mathbb{Q}) \simeq H^0(X, K_X + Y)^\vee$, as desired. □

Proposition 4.5 (Fundamental group)**.** *Let $Y \subset X$ be a simple normal crossing divisor of dimension ≥ 2 and $W := X \setminus Y$ be an affine variety. If X is simply connected (e.g. if X is rationally connected), then $\mathcal{D}(Y)$ is so as well.*

Proof. By the Lefschetz hyperplane theorem, $\pi_1(X) \simeq \pi_1(Y)$. Note also that there is a natural surjective map $\pi_1(Y) \twoheadrightarrow \pi_1(\mathcal{D}(Y))$, induced for instance by the evaluation map in Definition 5.3; see also [19, Lemma 26]. We conclude that $\pi_1(\mathcal{D}(Y))$ is a quotient of $\pi_1(X)$. $\square$

Proposition 4.6 (Homotopy type)**.** *Let $Y = \bigcup_{i\in I} Y_i \subset X$ be a simple normal crossing divisor such that $X \setminus Y$ is an affine variety. Suppose that the full set of hypotheses of Proposition 4.4 hold.*

- *If* $\dim(X) = 2$*, then $\mathcal{D}(Y)$ is a graph with $h^0(X, K + X + Y)$ loops.*
- *If* $\dim(X) = 3$ *and X is simply-connected, then $\mathcal{D}(Y)$ has the homotopy type of a bouquet of $h^0(X, K_X + Y)$ spheres of dimension 2, unless it is contractible.*

Proof. In the two-dimensional case, the statement follows immediately from Proposition 4.4.(9). Suppose now that $\dim(X) = 3$. Since X is simply-connected, then $\mathcal{D}(Y)$ is simply-connected by Proposition 4.5. For dimensional reasons, $\mathcal{D}(Y)$ has torsion-free integral homology. Therefore Proposition 4.4 says that $\mathcal{D}(Y)$ has the integral homology of a bouquet of $h^0(X, K_X + Y)$ spheres of dimension 2, or of a point. In the former case, Hurewicz's theorem then gives a continuous map from such a bouquet to $\mathcal{D}(Y)$ inducing an isomorphism on homology. Since $\mathcal{D}(Y)$ is simply-connected, Whitehead's theorem implies that this map is a homotopy equivalence. $\square$

5 The evaluation map

In this section, we construct a map (the *evaluation map*) from the link of a simple normal crossing divisor to the dual complex and show that, with suitable choices, it is a coisotropic fibration.

We use the following standard notation: if S, T are sets then S^T denotes the space of maps $T \to S$.

5.1 Evaluation map

Let $Y = \bigcup_{i\in I} Y_i \subset X$ be a simple normal crossing divisor. Here and in the following, we can assume that $Y_J = \bigcap_{j\in J} Y_j$ is connected for any $J \subset I$ (the assumption is not essential but it makes the notation lighter and allows us to realise $\mathcal{D}(Y)$ as the image of the evaluation map defined below. In fact, it can always be achieved via a sequence of blowups along connected components of the strata of Y).

Definition 5.1 (Full partition of unity)**.** Let $\{\chi_i\}_{i\in I}$ be a partition of unity subordinate to the open cover $\{U_i\}_{i\in I}$. Then $\{\chi_i\}_{i\in I}$ is called *full* if the following property holds: for any $J \subseteq I$ such that $U_J := \bigcap_{j\in J} U_j \neq \emptyset$, the map $\chi^J : U_J \to$

$[0,1]^J$, given by $\chi^J(j) = \chi_j(x)$, is surjective onto the open standard simplex in $(0,1]^J$ given by the equation $\sum_{j\in J} y_j = 1$.

Concretely, this means that the values $\chi_j(x)$ with $j \in K \subseteq J$ do not impose any constraints on the values that the other functions $\chi_{j'}(x)$ with $j' \in J \setminus K$ can attain, with the exception of the relation $\sum_{j\in J} \chi_j(x) = 1$.

Definition 5.2 (Plumbing neighbourhood)**.** A *plumbing neighbourhood* N *of* Y *is the union of tubular neighbourhoods* N_i *of* Y_i *for all* $i \in I$*.*

Definition 5.3 (Evaluation map)**.** Let $\{\chi_i\}_{i\in I}$ be a full partition of unity subordinate to the open cover $\{N_i\}_{i\in I}$ of the plumbing neighbourhood N. Then the *evaluation map*

$$ev : \mathrm{Link}(X \setminus Y) \subset N \to \mathbb{R}^I$$

is given by $ev(x)(i) = \chi_i(x)$.

Lemma 5.4. *The image of* ev *is homeomorphic to the dual complex* $\mathcal{D}(Y)$*.*

Proof. Let $N_J := \cap_{i\in J} N_i$ and $N_J^\circ := N_J \setminus \bigcup_{i\notin J} N_i$. By definition of fullness, the image of $N_J^\circ \cap \mathrm{Link}(Y)$ via ev is the convex hull of e_j with $j \in J$, where $\{e_i\}_{i\in I}$ is the standard basis of $\mathbb{R}^{|I|}$. In particular, it is a standard simplex of dimension $|J| - 1$, and it corresponds to the $(|J|-1)$-cell of $\mathcal{D}(Y)$ associated to Y_J. □

We will see that if (X, ω) is a symplectic manifold then the partition of unity can be chosen to make ev into a generically Lagrangian coisotropic fibration.

Remark 5.5 (Relation with other work)*.* The evaluation map ev appears in the geometric P=W conjecture [17, Conjecture 1.1]. Further, a similar map, called $\mathrm{Log}_{\mathcal{V}}$, is used to define the topology of hybrid spaces in [2, Definition 2.3]. The key point that we address in this paper is that ev can be *adapted* to a symplectic form ω, meaning that it can be turned into a generically Lagrangian fibration, unique up to homotopy.

5.2 Evaluation map as a coisotropic fibration

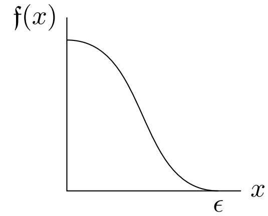
We now show that the evaluation map can be made into a coisotropic fibration on the contact hypersurface $\mathrm{Link}(X \setminus Y)$.

Let ν_i be the normal bundle of Y_i in X and pick an ω-regularisation $\{\Psi_i : \mathrm{nbhd}_{\nu_i}(Y_i) \to X\}_{i\in I}$ of Y as in Section 2.4; write $N_i = \mathrm{Image}(\Psi_i)$. Recall that $\mathrm{Link}(X \setminus Y)$ can be written as $f^{-1}(\epsilon)$ for the Y-compatible function $f(x) = \sum_{i\in I} \log(\mathfrak{g}(\mu_i(x)))$ (where $\mathfrak{g}$ is defined in Equation (2), μ_i is the moment map for the circle action which rotates the fibres of the normal bundle to Y_i, and $\epsilon > 0$ is a small parameter).

Definition 5.6. Let $\mathfrak{f}\colon \mathbb{R} \to [0,1]$ be a smooth cutoff function satisfying

$$\mathfrak{f}(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 0 & \text{if } x \ge \epsilon \end{cases} \tag{10}$$

and which is positive and strictly decreasing on $[0, \epsilon)$.



Proposition 5.7. *Assume that Y has maximal intersection. Then, there exists a partition of unity $\{\chi_i\}_{i\in I}$ subordinate to the open cover $\{N_i\}_{i\in I}$, such that the evaluation map $ev : \mathrm{Link}(X \setminus Y) \to \mathcal{D}(Y)$ is a generically Lagrangian coisotropic fibration (compatible with the stratification of $\mathcal{D}(Y)$).*

Proof. Let $\mathfrak{f}_i := \mathfrak{f} \circ \mu_i$ and choose the following partition of unity on N subordinate to $\{N_i\}_{i\in I}$:

$$\chi_i(x) := \frac{\mathfrak{f}_i(x)}{\sum_{j\in I} \mathfrak{f}_j(x)}.$$

Because the functions μ_i generate a Hamiltonian torus action, their Poisson brackets vanish. Since the functions χ_i depend only on the μ_j, they also Poisson commute with one another, and with $\prod_{i\in I} \mathfrak{g} \circ \mu_i$, so the restrictions of these functions to the link define a coisotropic fibration over the dual complex.

Note further that $\chi_i \equiv 0$ on $N_J^\circ := N_J \setminus \bigcup_{k\notin J} N_k$ if and only if $i \notin J$. Hence, the evaluation map projects $N_J^\circ \cap \mathrm{Link}(X \setminus Y)$ submersively onto the open $(|J|-1)$-cell in $\mathcal{D}(Y)$ corresponding to Y_J. The fibre $ev^{-1}(p)$ over a point p of this cell is coisotropic of codimension $|J| - 1$. The isotropic leaves of $ev^{-1}(p)$ are precisely the fibres of the projection $\pi_J \colon N_J^\circ \cap \mathrm{Link}(X \setminus Y) \to Y_J$, which are $|J|$-dimensional tori. In particular, ev is generically Lagrangian if Y has maximal intersection. □

Remark 5.8. Note that the functions $\{\mu_j\}_{j\in J}$ generate a Hamiltonian n-torus action on the regions N_J° with $|J| = n$. The Reeb flow on $N_J^\circ \cap \mathrm{Link}(X \setminus Y)$ is an $\mathbb{R}$-action inside that torus, and for a dense set of points in the corresponding n-cell of the dual complex, this $\mathbb{R}$-action has dense image in the n-torus. In particular, Lemma 2.14 applies to the evaluation map over these regions, and hence to any Lagrangian torus fibration which is a refinement of it.

6 Analogue of non-archimedean SYZ fibration

In Theorem 6.15 we construct a smooth Lagrangian torus fibration that can be regarded as a symplectic analogue (or dual) of the non-archimedean SYZ fibration constructed in [25, Theorem 6.1]. The base of this Lagrangian fibration has the homotopy type of the smooth (or affinoid) locus of the non-archimedean SYZ fibration. In particular, the integral affine structures induced on their bases have *dual* monodromy in the sense of Proposition 6.13.

We first give a biased introduction to Berkovich spaces and recall the notion of non-archimedean SYZ fibration. For further detail we refer the interested reader to [1], [34], [24] and [25].

6.1 A brief review of Berkovich spaces

Let X be a smooth connected variety over $\mathbb{C}$.

We denote by X^{an} the *Berkovich analytification* of X. As a set, X^{an} is the space of rank-one semi-valuations of the fraction field $\mathbb{C}(X)$ of X that extend the trivial valuation of $\mathbb{C}$, i.e. the set of functions $v : \mathbb{C}(X) \to \overline{\mathbb{R}}$ with the properties that:

1. $v(f \cdot g) = v(f) + v(g)$ for all $f, g \in \mathbb{C}(X)$;
2. $v(f + g) \leq \min\{v(f), v(g)\}$ for all $f, g \in \mathbb{C}(X)$;
3. $v(h) = 0$ for all $h \in \mathbb{C}^*$.

Example 6.1. Let D be a divisor on a birational modification of X. There is a semi-valuation which associates to any rational function on X its order of vanishing along D. These *divisorial valuations* form an important class of semi-valuations.

The Berkovich analytification is endowed with the coarsest topology for which the following maps are continuous:

- the *analytification morphism* $i : X^{an} \to X$, which sends any semi-valuation v to its kernel, i.e. the schematic point defined by the ideal of local regular functions g such that $v(g) = +\infty$. Note that here X is endowed with the Zariski topology.
- the *norm maps* $\|f\| : X^{an} \to \overline{\mathbb{R}}$, given by $\|f\|(v) = \exp(-v(f))$, for any $f \in \mathbb{C}(X)$.

In order to illustrate some aspects of the topology of Berkovich spaces, we describe in detail a fundamental example: the analytification of an algebraic torus $(\mathbb{C}^*)^n$.

Example 6.2 (Algebraic torus)**.** Let

$$T := (\mathbb{C}^*)^n \simeq \mathbb{C}^* \otimes N \simeq \operatorname{Spec}(\mathbb{C}[M])$$

be an algebraic torus with character lattice M and cocharacter lattice $N = M^\vee$ (i.e. the lattice of one-parameters subgroups).

Define the *tropicalization* as the map

$$\rho_T : T^{an} \to N_\mathbb{R} := N \otimes \mathbb{R} \qquad v \mapsto (M \to \mathbb{R} : m \mapsto v(m)).$$

It should be regarded as a non-archimedean analogue of the moment map for the standard Hamiltonian action of the real torus $(S^1)^n \subset T$

$$\mu_T : T \to M_\mathbb{R} \qquad x = (x_1, \ldots, x_n) \mapsto \left(\frac{1}{2}|x_1|^2, \ldots, \frac{1}{2}|x_n|^2\right).$$

Notice first that among the semi-valuations in T^{an} there are those which associate to any rational function f the value $-\log|f(x)|$ for $x \in T$. Hence, there is a copy of T which sits inside T^{an}; however, it is equipped with the discrete topology. The restriction of the tropicalization to T yields the following map

$$T \hookrightarrow T^{an} \xrightarrow{\rho_T} N_\mathbb{R} \qquad x = (x_1, \ldots, x_n) \mapsto (-\log|x_1|, \ldots, -\log|x_n|),$$

which coincides with the moment map μ_T up to a diffeomorphism of the image space.

The fibres of ρ_T should be viewed as non-archimedean Lagrangian tori, although at the moment it is not clear what a rigorous notion of non-archimedean Lagrangian analytic space should be in general.

A common feature of the maps ρ_T and μ_T is that they both induce an integral affine structure on their bases. The affine structure induced by ρ_T is constructed as follows: the norm maps $\|m\|$ of the invertible functions $m \in M$ are constant along the fibre of ρ_T, and so they descend to $N_\mathbb{R}$, and define a sheaf of affine functions; see also [21, Section 4.1, Lemma 1]. Via the identification of the tangent spaces of the points of $N_\mathbb{R}$ with $N_\mathbb{R}$ itself, the flat integer lattice of this affine structure can be identified with the cocharacter lattice N; see Definition 2.6.

Classically, the base of the moment map μ_T carries an integral affine structure given by the dual of the lattice of the 1-forms $\int_\gamma \omega$, where ω is the standard symplectic form on T, and γ is a loop of the fibre $\mu^{-1}(m)$, with $m \in M_\mathbb{R}$. According to the identification of the tangent spaces of the points of $M_\mathbb{R}$ with $M_\mathbb{R}$, the flat integer lattice of this affine structure can be identified with the character lattice M.

The natural pairing $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$ shows that the two affine structures are dual to each other.

6.2 The non-archimedean SYZ fibration

Just as the moment map μ_T is the local model for a Lagrangian submersion by the Arnold-Liouville theorem, the tropicalization should be taken as the local model of a non-archimedean Lagrangian submersion, also known as affinoid torus fibration.

Definition 6.3 (Affinoid torus fibration)**.** [25, Paragraph 3.3] A morphism $f : \mathfrak{X} \to B$ from an analytic space $\mathfrak{X}$ to a topological space B which is locally modelled on ρ_T is called *affinoid torus fibration*, i.e. B can be covered by open subsets U such that there exists an open $V \subseteq N_{\mathbb{R}}$ and a commutative diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r] \arrow[d, "f"'] & \rho_T^{-1}(V) \arrow[d, "\rho_T"] \\ U \arrow[r] & V \end{tikzcd}$$

such that the upper horizontal map is an isomorphism of analytic spaces and the lower horizontal map is a homeomorphism.

Example 6.2 is the local model for an affinoid torus fibration, and many of the local constructions described there can be globalised. A proof of these facts goes beyond the purposes of this paper, so we will give only statements and refer the interested reader to the literature.

Let $Y = \bigcup_{i \in I} Y_i$ be a simple normal crossing divisor of a smooth complex projective variety X.

Definition 6.4. The *centre* of a semi-valuation v is the schematic point defined by the ideal of local regular functions g such that $v(g) > 0$. The *analytic generic fibre* of the formal completion of X along Y, denoted by $\mathfrak{X}_\eta$, is the subset of X^{an} of valuations that admit center on Y (see [34, Paragraph 1.1.11]).

Theorem 6.5. [34, Theorem 3.26 and Corollary 3.27] *There exist strong deformation retracts*

$$\rho_{X,Y} : X^{an} \to \mathrm{Cone}(\mathcal{D}(Y))$$
$$\rho_{\mathfrak{X}_\eta} \coloneqq \rho_{X,Y}|_{\mathfrak{X}_\eta} : \mathfrak{X}_\eta \to \mathrm{Cone}(\mathcal{D}(Y)) \setminus \{\text{vertex}\}.$$

Assumption 6.6. Suppose that $Y = \bigcup_{i \in I} Y_i$ is a simple normal crossing divisor with maximal intersection and log Calabi-Yau along 1-dimensional components Y_K ($K \subset I$, $|K| = n-1$), i.e. each Y_K is a copy of $\mathbb{CP}^1$ which hits precisely two points Y_L ($L \subset I$, $|L| = n$) in the 0-stratum of Y.

Theorem 6.7. [25, Theorem 6.1][5] *Suppose that Assumption 6.6 holds and that*

[5] In non-archimedean mirror symmetry it is customary to normalise the points in X^{an} by the valuation of a local equation of Y. For this reason, in [25] the base of $\rho_{X,Y}$ is the dual

Z is the union of the strata of codimension ≥ 2 in $\mathrm{Cone}(\mathcal{D}(Y))$. *Then the retract $\rho_{\mathfrak{X}_\eta}$ is an affinoid torus fibration over* $\mathrm{Cone}(\mathcal{D}(Y)) \setminus Z$.

Remark 6.8. If Y is a singular fibre with maximal intersection of a minimal semistable degeneration of Calabi-Yau varieties, then $\rho_{X,Y}$ is the so-called *non-archimedean SYZ fibration.* The reason for this name is that conjecturally it should be possible to dualise these fibration on the affinoid locus, and construct a non-archimedean mirror. A GAGA principle for Berkovich spaces would give back the classical mirror, bypassing the difficult task of constructing a geometric special Lagrangian fibration in the first place. See [21].

Example 6.9 (Toric variety)**.** [25, Example 3.5] Let X be a smooth toric variety with fan $\Sigma \subset N_{\mathbb{R}}$, and Y its toric boundary. One can show that

$$\begin{aligned} \rho_{\mathfrak{X}_\eta} : \mathfrak{X}_\eta \to \mathrm{Cone}(\mathcal{D}(Y)) &\simeq \Sigma \subseteq N_{\mathbb{R}}, \\ v &\mapsto (M \to \mathbb{R} : m \mapsto v(m)). \end{aligned}$$

The base of $\rho_{\mathfrak{X}_\eta}$ is endowed with an integral affine structure, which can be identified with the lattice N as in Example 6.2 (in this case the integral affine structure actually extends through the codimension-two locus Z).

6.3 Non-archimedean monodromy

Let X be a smooth complex projective variety of complex dimension n, and $Y = \bigcup_{i \in I} Y_i \subset X$ be a simple normal crossing divisor satisfying Assumption 6.6.

The charts of the affinoid torus fibration $\rho_{\mathfrak{X}_\eta}$ in Theorem 6.5 are endowed with the integral affine structures defined in Example 6.2, which glue to a global affine structure on the whole $\mathrm{Cone}(\mathcal{D}(Y)) \setminus Z$; see [25, Section 6] for a detailed proof of the gluing. In this section, we show that the monodromy of this integral affine structure can be defined in purely topological terms.

Following [25], we present an atlas of $\mathrm{Cone}(\mathcal{D}(Y)) \setminus Z$, made of charts of the affinoid torus fibration of two different types.

1. An open set of first type consists of a top dimensional open cone $\mathrm{Cone}(\sigma_L)$. The $(n-1)$-cell $\sigma_L \subseteq \mathcal{D}(Y)$ corresponds to the 0-dimensional component Y_L, for some $L \subseteq I$ with $|L| = n$. In particular, the pair (X, Y) is locally modelled at Y_L on the affine space $\mathbb{C}^n$ with its coordinate hyperplanes Δ.

   By construction (cf [25, Section 2.4]), the fibration $\rho_{\mathfrak{X}_\eta}$ over $\mathrm{Cone}(\sigma_L)$ coincides with $\rho_{\mathbb{C}^n,\Delta}$, described in Example 6.9. In particular, the integral affine structure of the basis can be identified with the lattice of one-parameter subgroups of $\mathbb{C}^n$, and we denote it by $N(L)$.

complex instead of its cone. However, we prefer the non-normalized version because it makes the relation with the Lagrangian torus fibration Φ defined in Section 6.4 neater.

2. An open set of second type is the open star Star_K of the codimension-one open cells $\mathrm{Cone}(\sigma_K)$. The $(n-2)$-cell $\sigma_K \subseteq \mathcal{D}(Y)$ corresponds to the 1-dimensional component Y_K, for some $K \subseteq I$ with $|K| = n-1$.

   By [25, Proposition 5.4], Assumption 6.6 implies that there exists a (formal) toric tubular neighbourhood of Y_K. More precisely, the formal completion of X along Y_K is isomorphic to the formal completion of a toric vector bundle ν_K of rank $n-1$ on $Y_K \simeq \mathbb{CP}^1$ along its zero section.

   Again, the fibration $\rho_{\mathfrak{X}_\eta}$ over Star_K coincides with $\rho_{\nu_K,\Delta}$, where Δ is the toric boundary of ν_K. In particular, the integral affine structure on the basis can be identified with the cocharacter lattice of ν_K, and we denote it by $N(K)$.

Note that the inclusion $Y_L \subset Y_K$ induces the embedding $\mathrm{Cone}(\sigma_L) \subset \mathrm{Star}_K$, and an identification of the respective integral affine structures given by the linear map $\beta_{LK} : N(L) \to N(K)$.

Without loss of generality assume that $\mathrm{Cone}(\mathcal{D}(Y)) \setminus Z$ is connected, and fix b_0 a base point in Star_K. For any loop γ in $\mathrm{Cone}(\mathcal{D}(Y)) \setminus Z$ based at b_0, consider now an ordered finite sequence of open sets covering γ

$$\begin{tikzcd}[column sep=small]
\mathrm{Star}_K = \mathrm{Star}_{K_0} & & \mathrm{Star}_{K_2} & & \mathrm{Star}_{K_{n-2}} & & \mathrm{Star}_{K_n} = \mathrm{Star}_K \\
& \mathrm{Cone}(\sigma_{L_1}) \arrow[ul, hook] \arrow[ur, hook] & & \cdots \arrow[ul, hook] \arrow[ur, hook] & & \mathrm{Cone}(\sigma_{L_{n-1}}) \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Definition 6.10 (Non-archimedean monodromy)**.** The *monodromy representation* of the integral structure of the affinoid fibration $\rho_{\mathfrak{X}_\eta}$ is the group homomorphism

$$\rho_{\text{non-arch}} : \pi_1(\mathrm{Cone}(\mathcal{D}(Y)) \setminus Z) \to \mathrm{GL}(N(K)),$$

given by $\rho_{\text{non-arch}}([\gamma]) = \beta_{L_{n-1}K} \circ \beta^{-1}_{L_{n-1}K_{n-2}} \ldots \circ \beta_{L_1K_2} \circ \beta^{-1}_{L_1K}$.

Remark 6.11. In [25, Proposition 5.4], the algebraic tubular neighbourhood theorem for one-dimensional components Y_K is proved under the assumption that the conormal bundle of Y_K is ample. This condition can be always achieved by a sequence of stratum blow-ups at $Y_L \subset Y_K$ (or in the strict transform of Y_K)

$$X^m \xrightarrow{\pi^m} X^{m-1} \to \ldots \to X^0 = X.$$

Note that stratum blow-ups do not alter the integral affine structure on $\mathrm{Cone}(\sigma_L)$. The positivity assumption can be actually removed in the following way. The case of ample conormal bundle implies that the formal completion $\widehat{X}^m_{Y^m_K}$ of X^m along the strict transform Y^m_K of Y_K is isomorphic to the formal completion $\widehat{\nu}^m_K$ of the toric vector bundle ν^m_K along its zero section. The isomorphism, $f^m : \widehat{\nu}^m_K \to \widehat{X}^m_{Y^m_K}$, is constructed in [25, Proposition 5.4]. In particular, the stratum contracted by π^m is the image under f^m of a torus-invariant divisor in $\widehat{\nu}^m_K$. The corresponding toric blow-down is again the formal completion of a toric vector bundle ν_K^{m-1}. Now, the morphism $\pi^m \circ f^m$ factors through a morphism $f^{m-1} : \widehat{\nu}_K^{m-1} \to \widehat{X}^{m-1}_{Y_K^{m-1}}$ by the universal property of blow-ups, and we observe that f^{m-1} is an isomorphism, as required.

6.4 Topological monodromy

Let (X, ω) be a smooth complex projective variety of complex dimension n, and $Y = \bigcup_{i \in I} Y_i \subset X$ a simple normal crossing satisfying Assumption 6.6. We are particularly interested in the symplectic tubular neighbourhood of the complex 1-dimensional stratum: our goal is to construct a Lagrangian torus fibration over this neighbourhood, and compare its monodromy with the non-archimedean monodromy (Definition 6.10).

To this end, first make a perturbation of the symplectic form so that Y admits an ω-regularisation by [7]. For each $K \subset I$ of size $n-1$ and any $L \subset I$ of size n containing K with $Y_K \neq \emptyset$, a ω-regularisation gives us commutative squares

$$\begin{tikzcd} \mathrm{nbhd}_{\nu_K}(Y_K) \arrow[r, "\Psi_K"] & N_K \\ \mathrm{nbhd}_{\nu_L}(Y_L) \arrow[r, "\Psi_L"] \arrow[u] & N_L \arrow[u] \end{tikzcd}$$

where:

- ν_K is the normal bundle of $Y_K \subset X$; this splits as a direct sum of toric line bundles, due to Assumption 6.6.
- ν_L is the tangent space at Y_L.
- Ψ_K and Ψ_L are symplectic embeddings of $\mathrm{nbhd}_{\nu_K}(Y_K)$ and $\mathrm{nbhd}_{\nu_L}(Y_L)$ respectively into X, with $N_K = \mathrm{Image}(\Psi_K)$ and $N_L = \mathrm{Image}(\Psi_L)$.
- The vertical arrows are the natural inclusions.

By shrinking the domain of the regularisation, we can assume that N_K is invariant with respect to the torus action of ν_K, and that the inclusion $\mathrm{nbhd}_{\nu_L}(Y_L) \hookrightarrow \mathrm{nbhd}_{\nu_K}(Y_K)$ is toric. Therefore, we obtain other commutative squares

$$\begin{tikzcd} \mathrm{nbhd}_{\nu_K}(Y_K) \arrow[r, "\mu_K"] & B_K \subset M(K)_{\mathbb{R}} \\ \mathrm{nbhd}_{\nu_L}(Y_L) \arrow[r, "\mu_L"] \arrow[u] & B_L \subset M(L)_{\mathbb{R}} \arrow[u] \end{tikzcd}$$

where:

- $M(K) = N(K)^\vee$ and $M(L) = N(L)^\vee$ are the character lattices of the torus acting on ν_K and on ν_L respectively.
- μ_K and μ_L are toric moment maps.

Let $\mathbf{B}$ be the colimit of the diagram of spaces whose vertices are the bases B_K with $|K| \in \{n-1, n\}$ and whose morphisms are the inclusions $B_L \to B_K$; that is, $\mathbf{B}$ is the quotient of $\bigsqcup_{K:|K|=n-1} B_K \sqcup \bigsqcup_{L:|L|=n} B_L$ by the equivalence relation which identifies each B_L with its image under the inclusion map $B_L \subset B_K$. Note that $\mathbf{B}$ is a strong deformation retract of $\mathrm{Cone}(\mathcal{D}(Y)) \setminus Z$.

Let $\mathcal{N} \subset X$ be the union of N_K for all $K \subset I$ of size $n-1$ with $Y_K \neq \emptyset$.

Proposition 6.12. *There exists a Lagrangian torus fibration*

$$\phi : \mathcal{N} \to \mathbf{B}.$$

with only toric singularities, along the boundary of $\mathbf{B}$*.*

Proof. Set $\phi(x) = \mu_K \circ \Psi_K^{-1}(x)$ for all $x \in N_K$. The commutative squares above ensure that the maps $\mu_K \circ \Psi_K^{-1}$ agree on plumbing regions, and so that ϕ is well-defined. Moreover, ϕ is a Lagrangian fibration with only toric singularity along the boundary of $\mathbf{B}$, as μ_K are so. ∎

By the Arnold-Liouville theorem, $\mathbf{B}^\circ$ inherits an integral affine structure, and up to fibred symplectomorphisms, the Lagrangian torus fibration ϕ can be locally identified with the fibration

$$T^\vee \mathbf{B}^\circ / \Lambda^\vee \to \mathbf{B}^\circ,$$

where Λ is the integral lattice of tangent vectors defining the integral affine structure on $\mathbf{B}^\circ$; see Definition 2.6(3). The integral affine structure on $\mathbf{B}^\circ$ actually extends to $\mathbf{B}$, since ϕ has only toric singularities.

We study now the monodromy of the integral affine structure on $\mathbf{B}$. The key observation is that the toric inclusions $N_L \to N_K$ are induced by the same linear maps $\beta_{LK} : N(L) \to N(K)$, defined in Section 6.3, or dually by $(\beta_{LK}^{-1})^t : M(L) \to M(K)$.

Fix a point $b_0 \in B_K$ of $\mathbf{B}$. As above, the integral affine structure of $\mathbf{B}$ at b_0 can be identified with the character lattice $M(K)$. Therefore, the monodromy of ϕ

$$\rho_{\mathrm{Lagr}} : \pi_1(\mathbf{B}) \simeq \pi_1(\mathrm{Cone}(\mathcal{D}(Y)) \setminus Z) \to \mathrm{GL}(M(K)),$$

is given by

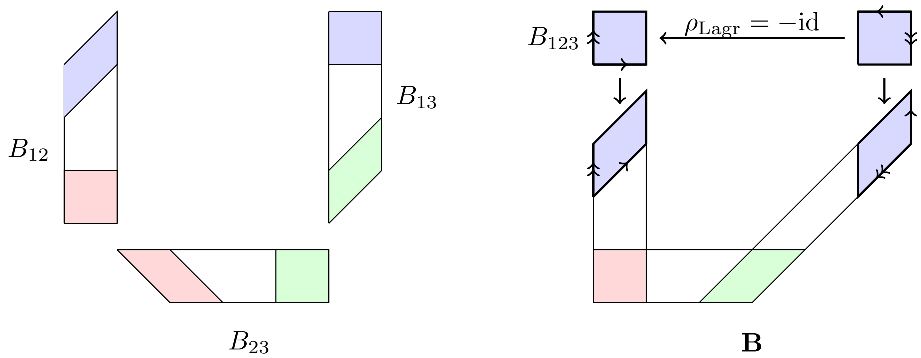
$$\rho_{\mathrm{Lagr}}([\gamma]) = (\beta_{L_{n-1}K}^{-1})^t \circ (\beta_{L_{n-1}K_{n-2}})^t \ldots \circ (\beta_{L_1 K_2}^{-1})^t \circ (\beta_{L_1 K})^t$$

for any loop $\gamma \in \pi_1(\mathbf{B}) \simeq \pi_1(\mathrm{Cone}(\mathcal{D}(Y)) \setminus Z)$. In particular, we have that

$$\rho_{\mathrm{Lagr}}([\gamma]) = (\rho_{\mathrm{Lagr}}([\gamma])^{-1})^t.$$

Proposition 6.13. *The natural pairing $\langle \cdot, \cdot \rangle : N(K) \times M(K) \to \mathbb{Z}$ induces the identification $\rho_{\textit{non-arch}} = (\rho_{Lagr}^{-1})^t$.*

Example 6.14. Let $X \subset \mathbb{P}^3$ be a smooth cubic surface and $Y = \bigcup_{i=1}^{3} Y_i$ be a hyperplane section which consists of three lines pairwise intersecting in three double points. The normal bundle of Y_i in X is isomorphic to the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$. In this case, B_L are squares, and B_{12}, B_{23}, B_{13} are right trapezia. In the picture, we draw also the colimit $\mathbf{B}$, up to an identification which consists of an affine transformation whose linear part is $-\mathrm{id}$. By definition, this linear map is the monodromy of the integral affine structure of $\mathbf{B}$ along a generator of its fundamental group $\pi_1(\mathbf{B}) \simeq \pi_1(\mathcal{D}(Y)) \simeq \pi_1(S^1) \simeq \mathbb{Z}$.



To conclude, we summarize the results of this section in the following theorem.

Theorem 6.15. *Suppose that (X, Y) is a simple normal crossing pair of maximal intersection and log Calabi-Yau along the irreducible components of the 1-dimensional stratum of Y. Let $\mathfrak{X}_\eta$ be the* analytic generic fibre *of the formal completion of X along Y, and Z be the union of the cells of codimension ≥ 2 in* $\mathrm{Cone}(\mathcal{D}(Y))$.

There exists an affinoid torus fibration

$$\rho_{\mathfrak{X}_\eta} : \mathfrak{X}_\eta \to \mathrm{Cone}(\mathcal{D}(Y)) \setminus Z,$$

a neighbourhood $\mathcal{N}$ of the 1-dimensional stratum of Y and a Lagrangian torus fibration

$$\phi : \mathcal{N} \to \mathbf{B} \subset \mathrm{Cone}(\mathcal{D}(Y)) \setminus Z,$$

where $\mathbf{B}$ is a retract of $\mathrm{Cone}(\mathcal{D}(Y)) \setminus Z$, such that the monodromy of the integral affine structure induced by $\rho_{\mathfrak{X}_\eta}$ and ϕ are dual in the sense of Proposition 6.13.

A Symplectic plumbing neighbourhoods

Lemma A.1. *Suppose you have two symplectic normal crossing divisors $Y \subset X$ and $Y' \subset X'$, each with pairwise symplectically orthogonal components, each* *indexed by the set I. Suppose that there is a homeomorphism $f\colon Y \to Y'$ which restricts to a symplectomorphism $f_i\colon Y_i \to Y_i'$ on each component. Suppose moreover that there are symplectic bundle isomorphisms $F_i\colon \nu_X Y_i \to \nu_{X'} Y_i'$ with the property that $F_i|_{Y_{ij}}$ agrees with $df_j\colon \nu_{Y_j} Y_{ij} \to \nu_{Y_j'} Y_{ij}'$. Then there are symplectomorphic neighbourhoods* $\mathrm{nbhd}_X(Y)$ *and* $\mathrm{nbhd}_{X'}(Y')$.

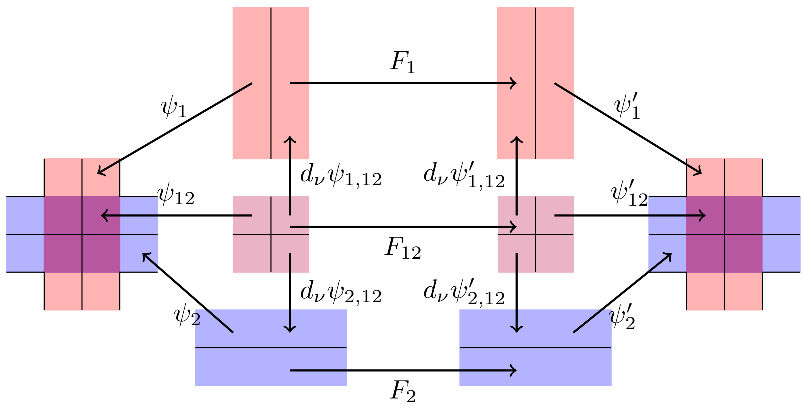
Proof. By [7, Proposition 4.2] we can construct smooth regularisations ψ_J of Y and ψ_J' of Y' with the property that the normal bundle of each stratum Y_J, $J \subset I$, is identified with the symplectic orthogonal complement TY_J^{ω} (both in X and within all intermediate strata). These regularisations are compatible in the sense that if $K \subset J$ then $\psi_J = \psi_K \circ d_\nu \psi_{K;J}$ where $d_\nu \psi_{K;J}$ denotes the linearisation of ψ_K in the normal directions to Y_J (see [8, Definition 4]).

In particular, this gives regularisations of each stratum Y_J within each Y_i with $i \in J$. Now, stratum by stratum starting at the deepest, we modify f_i by an isotopy so that the following diagrams commute:

$$\begin{tikzcd}
Y_i \arrow[r, "f_i"] & Y_i' \\
\nu_{Y_i} Y_J \arrow[u] \arrow[r, "d_\nu f_i"'] & \nu_{Y_i'} Y_J \arrow[u]
\end{tikzcd}$$

where the vertical maps are the regularisations, and d_ν denotes the linearisation of f_i along the normal directions to Y_J in Y_i. The idea here is that, in local coordinates, a diffeomorphism can be made to coincide locally with its derivative by an isotopy (by isotoping its graph to coincide with the graph of its linearisation).

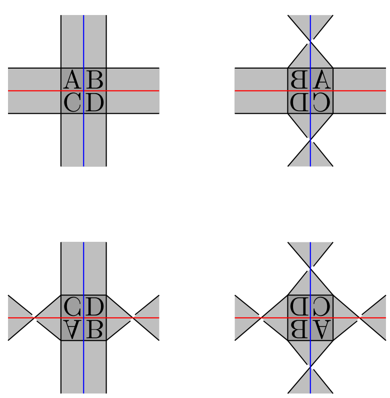
We now use our regularisations and the symplectic bundle isomorphisms F_i to construct a diffeomorphism from a neighbourhood of Y to a neighbourhood of Y' which induces an isomorphism of symplectic vector bundles $TX|_Y$ to $TX'|_{Y'}$. The idea is to use $\Psi_i := \psi_i' \circ F_i \circ \psi_i^{-1}$ on the codomain of the regularisation ψ_i. For this to make sense, we need $\Psi_i = \Psi_j$ on all intersections. For the sake of notational simplicity, we illustrate this only in the case when $Y = Y_1 \cup Y_2$.



Here, F_{12} is the restriction of F_1 to $\nu_{Y_2}Y_{12}$ plus the restriction of F_2 to $\nu_{Y_1}Y_{12}$. Because of the way we modified our homeomorphism f, all parts of the diagram commute, so we see that $\Psi_i = \Psi_j$.

Now write ω for the symplectic form on X and ω' for the pullback of the symplectic form on X' along this diffeomorphism. Since ω and ω' are symplectic and agree along Y, the 2-forms $\omega_t = \omega + t(\omega' - \omega)$ give an isotopy of symplectic forms in a (smaller) neighbourhood of Y. By Moser's isotopy theorem, this means that ω and ω' are symplectomorphic when restricted to this smaller neighbourhood. $\square$

Remark A.2. Another way of thinking about this is that the neighbourhood of Y is obtained from the normal bundles $\nu_X Y_i$ by plumbing along subbundles $\nu_X Y_I$. The different possible neighbourhoods can be obtained by twisting the plumbing identification using symplectic gauge transformations of these subbundles preserving the stratification by subbundles $\nu_{Y_J} Y_I$. This is analogous to plumbing 2-dimensional surfaces along square patches, where one has the freedom to twist neither, either or both of the square patches, corresponding to the $\mathbb{Z}/2 \times \mathbb{Z}/2$ of gauge transformations of $\mathbb{R}^2$ preserving the x- and y-axes.



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