MPIM-44/06
FIAN/TD-05/06
ITEP/TH-15/06

WDVV equations for 6d Seiberg-Witten theory and bi-elliptic curves

H.W.Braden, A.Marshakov, A.Mironov, A.Morozov

We present the generic derivation of the WDVV equations in 6d Seiberg-Witten theory, and extend it to the families of bi-elliptic spectral curves. Elliptization of naive 6d system requires approximate "doubling" of the number of moduli.

1 Introduction: WDVV equations and residue formulas

Complex geometry is playing an increasingly important role in non-perturbative physics. For example, modern (topological) string theory, including Seiberg-Witten (SW) theory, intensively exploits the prepotentials of complex manifolds, the (generalized) period matrices of which appear as couplings in the associated low-energy, effective field theory Lagrangians. In many cases these prepotentials satisfy some particularly nice non-linear differential equations, and these may be integrated using the basic properties of the underlying complex geometry. The general theory of these equations is still far from being complete. This paper will focus on one such class of equations, the WDVV equations, and provide some new solutions to these arising from six dimensional (6d) Seiberg-Witten theory. We begin by reviewing these equations.

In many known cases the nontrivial part of the complex geometry effectively reduces to families of one-dimensional complex manifolds, or curves, and both the equations and the curves are related to parts of well-known infinite-dimensional integrable hierarchies. The latter may be further effectively rewritten in group-theoretical terms, or even "linearized" in the form of the Virasoro/W-algebra constraints. Other ingredients of this picture suggest various universal properties, presumably generalizable beyond one complex dimension. This is particularly true of the relations satisfied by the generalized period matrices, including their first derivatives; equivalently the third derivatives of prepotentials. Such third derivatives have the sense of three-point functions in string theory and are very robust objects because of the large (three-dimensional) group of automorphisms of the world-sheet spheres. The basic relations satisfied by the third derivatives are known as the WDVV equations [1] and these were originally obtained from the crossing-symmetry of the four-point functions in topological string theory.

In their most general form [2] the WDVV equations can be presented as a system of algebraic relations

$$\mathcal{F}_I \mathcal{F}_J^{-1} \mathcal{F}_K = \mathcal{F}_K \mathcal{F}_J^{-1} \mathcal{F}_I, \qquad \forall\ I, J, K \tag{1.1}$$

for the matrices of third derivatives

$$\|\mathcal{F}_I\|_{JK} = \frac{\partial^3 \mathcal{F}}{\partial T_I\, \partial T_J\, \partial T_K} \equiv \mathcal{F}_{IJK} \tag{1.2}$$

of some function $\mathcal{F}(T_K)$. Originally one considered only a particular class of solutions to the system (1.1), where one of the matrices $\mathcal{F}_{I_0}$ was a constant (matrix) independent of $\{T_J\}$, see [1, 3]. This restriction corresponded to the existence of a distinguished vector (the vacuum) in the space of states of topological string theory with corresponding parameter T_{I_0}, the "cosmological constant". In the framework of SW theory however, there is no natural place for such a constraint, see [2, 4, 5, 6]; moreover, such a constraint violates the basic symmetries of the theory, like electric-magnetic duality [7]. In fact, it turns out that this constancy condition is inessential to the proof of solutions to the WDVV equations given by prepotentials of complex manifolds for which the third derivatives are expressed by a *residue formula*.

The residue formula relevant for the WDVV equations takes the following form [8]. One has a Riemann surface Σ endowed with a meromorphic generating one-form $dS = -z d\tilde{z}$. Then

$$\frac{\partial^3 \mathcal{F}}{\partial T_I \partial T_J \partial T_K} = \operatorname{res}_{d\tilde{z}=0} \left(\frac{d\Omega_I d\Omega_J d\Omega_K}{dz d\tilde{z}} \right) = \sum_\alpha \operatorname{res}_{z_\alpha} \left(\frac{\phi_I \phi_J \phi_K}{d\tilde{z}/dz} dz \right) = \sum_\alpha \frac{\phi_I(z_\alpha)\phi_J(z_\alpha)\phi_K(z_\alpha)}{\Psi_\alpha}, \tag{1.3}$$

where $d\tilde{z} = \Psi_\alpha \cdot (z - z_\alpha)dz + \ldots$ as $z \to z_\alpha$. The set of one-forms $\{d\Omega_I\}$ corresponds to the set of parameters $\{T_I\}$ via

$$\frac{\partial dS}{\partial T_I} = d\Omega_I \equiv \phi_I dz. \tag{1.4}$$

With such a residue formula the proof[1] of the WDVV equations (1.1) reduces to solving a system of linear equations [10, 11], the solution of which requires only two conditions to be fulfilled:

1. A matching condition between the number of deformation parameters (moduli) $n_m = \#(I)$ and the number of critical points $n_z = \#(\alpha)$ in the residue formula

$$n_z = n_m \tag{1.5}$$

2. Nondegeneracy of the matrix $\phi_I(z_\alpha)$,

$$\det_{I\alpha} \|\phi_I(z_\alpha)\| \neq 0 \tag{1.6}$$

This is believed to always be fulfilled in "general position".

Supposing these conditions to be satisfied the structure constants C_{IJ}^K of the associative algebra underlying the WDVV equations may be found from the system of linear equations (one for each z_α)

$$\phi_I(z_\alpha)\phi_J(z_\alpha) = \sum_K C_{IJ}^K(\xi)\phi_K(z_\alpha) \cdot \xi(z_\alpha). \tag{1.7}$$

This algebra is isomorphic to the more usual one considered in relation to the WDVV equations provided $\xi(z_\alpha) \neq 0$, which we henceforth assume. Utilising the matching and nondegeneracy assumptions we may solve (1.7) to give

$$C_{IJ}^K(\xi) = \sum_\alpha \frac{\phi_I(z_\alpha)\phi_J(z_\alpha)}{\xi(z_\alpha)} \left(\phi_K(z_\alpha)\right)^{-1}. \tag{1.8}$$

[1] An alternative way is to construct an associative algebra of forms, see [4, 5, 9].

The WDVV equations follow from the associativity of (1.7) once we establish the consistency of the relation

$$\mathcal{F}_{IJK} = \sum_L C_{IJ}^L(\xi)\eta_{KL}(\xi), \tag{1.9}$$

which expresses the structure constants in terms of the third derivatives. Here we have introduced a "metric"

$$\eta_{KL}(\xi) = \sum_M \xi_M \mathcal{F}_{KLM}, \tag{1.10}$$

where the parameters ξ_M are arbitrary, subject to η_{KL} being invertible. Upon defining the differential $\xi(z)dz$ with

$$\xi(z) = \sum_M \xi_M \phi_M(z),$$

one sees that the values of $\xi(z_\alpha)$ in (1.7) determine ξ_M and visa versa using

$$\xi_M = \sum_\alpha \phi_M(z_\alpha)^{-1}\xi(z_\alpha).$$

The consistency of (1.9) now follows simply if $\mathcal{F}_{KLM}$ are given by a residue formula (1.3):

$$\begin{aligned}\sum_K C_{IJ}^K(\xi)\eta_{KL}(\xi) &= \sum_{K,\alpha,\beta} \frac{\phi_I(z_\alpha)\phi_J(z_\alpha)}{\xi(z_\alpha)} \cdot (\phi_K(z_\alpha))^{-1} \cdot \phi_K(z_\beta)\phi_L(z_\beta)\xi(z_\beta)\Psi_\beta^{-1} \\ &= \sum_\alpha \frac{\phi_I(z_\alpha)\phi_J(z_\alpha)}{\xi(z_\alpha)}\phi_L(z_\alpha)\xi(z_\alpha)\Psi_\alpha^{-1} = \sum_\alpha \Psi_\alpha^{-1}\phi_I(z_\alpha)\phi_J(z_\alpha)\phi_L(z_\alpha) \\ &= \mathcal{F}_{IJL}.\end{aligned}$$

2 Seiberg-Witten prepotentials

We shall now construct a new class of SW prepotentials [12] and solutions to the WDVV equations using the general formalism described in the introduction. The focus of our attention will be on curves $\Sigma \subset \mathcal{E}_\tau \times \mathcal{E}_{\tilde{\tau}}$ lying in the product of two elliptic curves,

$$\Sigma: \quad \mathcal{H}(\tilde{z}, z) = 0, \quad dS = -z d\tilde{z}. \tag{2.1}$$

Here $\mathcal{H}$ is doubly-periodic in both z and $\tilde{z}$ with respective periods $(1,\tau)$ and $(1,\tilde{\tau})$. Such curves (2.1) are endowed with a "symmetric" generating differential: under exchange of the two tori this becomes a Legendre transform. In particular cases the $\tilde{z}$-torus will be taken to degenerate with $\tilde{\tau} \to +i\infty$. Then it is convenient to use the co-ordinate on a cylinder $\tilde{z} \to \log w$ instead of $\tilde{z}$ itself. In this case the generating differential becomes

$$dS = -z\frac{dw}{w} \tag{2.2}$$

and acquires the form more commonly appearing in SW theory (see e.g. [13, 14] and references therein). Our setting corresponds to a 6d gauge theory with two extra dimensions compactified onto the z-torus.

The variation of the generating differential (2.1) may be written as

$$\delta(dS) = \delta\mathcal{H}\frac{dz}{\mathcal{H}'_{\tilde{z}}} \;\in\; \bigoplus_{I=1}^{n_m} d\Omega_I, \tag{2.3}$$

where the right-hand side is understood as a linear combination of all *canonical* differentials. The most common basis of the space of all canonical differentials consists of the holomorphic or Abelian differentials of the first kind $\{d\omega_i\}$, and the meromorphic or Abelian differentials of the second and third kinds. If $\{A_i, B_i\} \subset H_1(\Sigma)$ are a canonical homology basis for our curve Σ the holomorphic differentials may be used to vary the A_i-periods of dS while the meromorphic differentials may be used to describe any poles of dS and any monodromy, or jumps, it may have. (Note that z and $\tilde{z}$ are non-single valued Abelian integrals and not functions on Σ.) Abelian differentials of the third kind, $d\Omega_{P_+,P_-}$, with residues ± 1 at $P_\pm$, also arise when allowing degenerations and a handle is shrunk to a pair of marked points ($P_\pm$). These various sorts of differentials will correspond in (2.3) to different variations of the parameters of $\mathcal{H}$.

In this paper we will focus on the case when the variations (2.3) are *almost all* accounted for by the holomorphic differentials (or, when there is degeneration, Abelian differentials of the third kind). However, it turns out, that if we want to restrict ourselves to only this class of generalized holomorphic differentials, there is always a mismatch in unity with the matching condition (1.5) of validity of the WDVV equations. For us this mismatch is filled by a variation in τ of dS, producing in (2.3) a particular differential $d\Omega_\tau$ with jump along the B_i-cycles that nontrivially project to the B-cycle of the base z-torus. Thus we are considering variations $\delta(dS)$ of the form (for some point P_*)

$$\delta(dS) = \sum_{i=1}^{g_\Sigma} \delta \mathsf{a}_i d\omega_i + \sum_j \delta a_j \, d\Omega_{P_j,P_*} + \delta\tau d\Omega_\tau, \tag{2.4}$$

and corresponding "times" $\{T_K = \mathsf{a}_i, a_k, \tau\}$

$$\{T_K\}: \quad \mathsf{a}_i = \oint_{A_i} dS, \quad a_k = \mathrm{res}_{P_k} dS, \quad \tau = \oint_B dz. \tag{2.5}$$

Then we define (up to a constant) the function $\mathcal{F}(T_K)$ in terms of its derivatives $\{\frac{\partial \mathcal{F}}{\partial T_K}\}$ by

$$\frac{\partial \mathcal{F}}{\partial \mathsf{a}_i} = \oint_{B_i} \tilde{z} dz, \quad \frac{\partial \mathcal{F}}{\partial a_k} = \int_{P_k}^{P_*} \tilde{z} dz, \quad \frac{\partial \mathcal{F}}{\partial \tau} = \oint_A \tilde{z} dS. \tag{2.6}$$

The integrability of (2.6) and so the existence of a prepotential $\mathcal{F}(T_K)$ now follows from the Riemann bilinear identities satisfied by the differentials on a Riemann surface,

$$\int_\Sigma d\Omega_I \wedge d\Omega_J = 0. \tag{2.7}$$

Considering the canonical holomorphic differentials $\{d\omega_i\} \subset \{d\Omega_I\}$, for example, these identities ensure that the period matrix $T_{ij} = \partial^2 \mathcal{F}/\partial \mathsf{a}_i \partial \mathsf{a}_j$ of Σ is symmetric,

$$T_{ij} = \oint_{B_i} d\omega_j = \oint_{B_j} d\omega_i = T_{ji}, \qquad i, j = 1 \ldots g_\Sigma. \tag{2.8}$$

Likewise consideration of the whole set of differentials satisfying (2.7) similarly shows that the generalized period matrices (2.8) can be integrated to yield a (locally defined) function $\mathcal{F}(T_K)$ [8] of the periods, residues and jumps of the generating differentials. In addition to (2.8) one gets, for example, that

$$\frac{\partial^2 \mathcal{F}}{\partial a_k \partial \mathsf{a}_i} = \oint_{B_i} d\Omega_{P_k,P_*} = \int_{P_k}^{P_*} d\omega_i = \frac{\partial^2 \mathcal{F}}{\partial \mathsf{a}_i \partial a_k} \tag{2.9}$$

and

$$\frac{\partial^2\mathcal{F}}{\partial\tau\partial\mathsf{a}_i} = \oint_{B_i} d\Omega_\tau = \oint_A \tilde{z}\, d\omega_i = \frac{\partial^2\mathcal{F}}{\partial\mathsf{a}_i\partial\tau}. \tag{2.10}$$

At this juncture we simply note (to be elaborated upon below) that "contact terms", or the values of Abelian integrals at the intersection points $P_k = A_k \cap B_k$, appear in our setting as a consequence of the multi-valuedness of the generating differential dS.

In what follows we are going to apply these general formulae to particular examples of SW prepotentials. We start with the so called 6d supersymmetric QCD and then generalize it to the case of the curves (2.1).

2.1 The perturbative 6d case

We will now consider the perturbative prepotential of 6d supersymmetric QCD. By 6d supersymmetric QCD we follow [15, 16] and mean the $\mathcal{N} = 2$ SUSY four-dimensional gauge theory with $SU(N)$ gauge group and $N_f = 2N$ fundamental matter multiplets together with the extra Kaluza-Klein modes corresponding to adding *two* compact dimensions. All $N_f = 2N$ matter multiplets are taken throughout with vanishing masses. The perturbative prepotential of this hypothetical 6d theory compactified on the torus $\mathcal{E}_\tau = \mathbb{C}/(1,\tau)$ was calculated in [16] using the residue formula (1.3). Let

$$x = \wp(z|\tau), \quad y = -\tfrac{1}{2}\wp'(z|\tau), \qquad y^2 = \prod_{i=1}^{3}(x - e_i), \tag{2.11}$$

be standard (affine) coordinates for the torus $\mathcal{E}_\tau$. In terms of these the curve Σ is defined by

$$w = e^u \prod_{j=1}^{N} \frac{\theta_1(z - a_j)}{\theta_1(z)} = P(x) + y\,Q(x), \qquad \sum_{j=1}^{N} a_j = 0. \tag{2.12}$$

It is endowed with the generating differential (2.2). The perturbative curve (2.12) defines an elliptic function $w \in \mathbb{C}^*$ on the torus and so $g_\Sigma = 1$. The prepotential is computed as a function of the degenerate SW periods or residues

$$a_j = -\mathrm{res}_{P_j} dS = \mathrm{res}_{z=a_j} z\frac{dw}{w}, \qquad j = 1, \ldots, N. \tag{2.13}$$

Choosing a set of $N-1$ independent quantities from the N variables a_j subject to $\sum_{j=1}^{N} a_j = 0$ in a standard way, say $a_j \to a_j - a_N$, the resulting prepotential is easily written in terms of the quantum tri-logarithm function [16].

In addition to the variables (2.13) there are two more natural parameters in (2.12). There is the modulus τ of elliptic curve, the complexified ratio of the two compactification radii of the 6d theory, together with the coefficient of proportionality $\exp(u)$ which is "reminiscent" of the scale factor of the 6d theory and related to the coupling constant of the "micriscopic" gauge theory. (Here we are writing the coefficient of proportionality of [16] in the exponential form of [17].)

The residue formula (1.3), used in [16], can be extended to include these extra parameters. Let

$$\mathsf{a} = \oint_A dS, \qquad a_k = \mathrm{res}_{P_k} dS, \qquad \tau = \oint_B dz,$$

(recalling that we are in the genus one setting, so $A_1 = A$, $\mathsf{a}_1 = \mathsf{a}$ here). To apply general reasoning we need to relate u to the period a. The delicate point here is that dS is *not* single-valued. Unlike the usual SW setting we now need to specify a point $P_0 = A \cap B$ and fix some $z_0 = z(P_0)$ and $w_0 = w(P_0)$ [8]; the prepotential depends on the choice of homology cycles and various "contact terms" must be included. As

$$dS = -z\frac{dw}{w} = dz \log w - d\left(z \log w\right)$$

then

$$\mathsf{a} = -\int_{z_0}^{z_0+1} d\left(z\log w\right) + \oint_A \log w dz = -\log w_0 + \oint_A u dz + \sum_{j=1}^{N} \oint_A \log\frac{\theta_1(z-a_j)}{\theta_1(z)} dz \tag{2.14}$$

$$= -\log w_0 + u. \tag{2.15}$$

To see that the final term of (2.14) may be taken to vanish we use the identity

$$\log\frac{\theta_1\left(z-a_j\right)}{\theta_1\left(z\right)} = \log\frac{\sin\pi(z-a_j)}{\sin\pi z} - 4\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{2n}}{1-q^{2n}}\sin\pi n(2z-a_j)\sin\pi n a_j.$$

Then by periodicity the trigonometric terms vanish upon integration and

$$\sum_{j=1}^{N}\oint_A \log\frac{\theta_1(z-a_j)}{\theta_1(z)} dz = \sum_{j=1}^{N}\oint_A \log\frac{\sin\pi(z-a_j)}{\sin\pi z} dz.$$

The value of this last integral depends on the choice of contour A: if it is such that $\operatorname{Im} a_j < \operatorname{Im} z_0$ $(j = 1, \ldots, N)$ then the contour may be slid to infinity and the integrals vanish, while other choices of contour will differ by an integer multiple of $2\pi i$. Thus for our chosen homology basis we obtain (2.15) showing that (for constant w) variations in a and u are the same. Following from

$$\delta \log w = \delta u + \delta z\,\frac{d\log w}{dz} - \sum_i \delta a_i \frac{\theta_1'(z-a_i)}{\theta_1(z-a_i} + \delta\tau\left(\sum_{j=1}^{N}\frac{\theta_1''(z-a_j)}{\theta_1(z-a_j)} - N\frac{\theta_1''(z)}{\theta_1(z)}\right) \tag{2.16}$$

to each of the times $\{T_I = u, a_j, \tau\}$ (at constant w) we may associate the differentials $\{d\Omega_I = dz, d\Omega_j, d\Omega_\tau\}$ via

$$\frac{\partial dS}{\partial u} = dz, \tag{2.17}$$

$$\frac{\partial dS}{\partial a_j} = d\Omega_j = \left(\frac{\theta_1'(z-a_N)}{\theta_1(z-a_N)} - \frac{\theta_1'(z-a_j)}{\theta_1(z-a_j)}\right) dz, \quad j = 1, \ldots, N-1, \tag{2.18}$$

$$\frac{\partial dS}{\partial \tau} = d\Omega_\tau. \tag{2.19}$$

Thus (2.17) gives us the (unique) holomorphic differential on the torus. The Abelian differentials of the third-kind (2.18) used in [16] can also be expanded over the basis of

$$\frac{dz}{w}\left(1, \ldots, \wp(z)^{[N/2]}\right) = \frac{dx}{wy}\left(1, \ldots, x^{[N/2]}\right), \text{ and } \frac{dz}{w}\left(1, \ldots, \wp(z)^{[(N-3)/2]}\right)\wp'(z) = \frac{dx}{w}\left(1, \ldots, x^{[(N-3)/2]}\right), \tag{2.20}$$

where the coefficients of the expansion are such as to cancel all poles except for the two simple poles of (2.18). The differential (2.17) and the differentials (2.18) are single-valued on the torus. In contrast to this (2.19) is a *multi-valued* differential. Before turning to the residue formulae we first explain how the Riemann bilinear identity works for these latter differentials.

The general theory gives (cf. with [8])

$$\frac{\partial \mathcal{F}}{\partial \mathsf{a}} = \frac{\partial \mathcal{F}}{\partial u} = -\tau \log w_0 - \oint_B dS = -\int_{z_0}^{z_0+\tau} \log w \, dz, \tag{2.21}$$

$$\frac{\partial \mathcal{F}}{\partial a_k} = \int_{P_k}^{P_N} \tilde{z} dz, \tag{2.22}$$

$$\frac{\partial \mathcal{F}}{\partial \tau} = -\oint_A \tilde{z} dS = \frac{1}{2} (\log w_0)^2 - \frac{1}{2} \oint_A (\log w)^2 \, dz. \tag{2.23}$$

As a consequence we obtain

$$\frac{\partial^2 \mathcal{F}}{\partial \tau \partial u} = -\log w_0 - \oint_B d\Omega_\tau, \tag{2.24}$$

$$\frac{\partial^2 \mathcal{F}}{\partial u \partial \tau} = -\oint_A \log w \, dz = -\log w_0 + \oint_A z \frac{dw}{w} \tag{2.25}$$

and to show the integrability of $\mathcal{F}$ we must further investigate the *multi-valued* differential (2.19). Using the fact that theta functions satisfy the heat equation we may write

$$\begin{aligned} d\Omega_\tau &= dz \left(\sum_{j=1}^N \partial_\tau \log \theta_1(z - a_j) - N \partial_\tau \log \theta_1(z) \right) \\ &= \frac{dz}{4\pi i} \left(\sum_{j=1}^N (\log \theta_1(z-a_j))'' - N (\log \theta_1(z))'' \right) + \frac{dz}{4\pi i} \left(\sum_{j=1}^N ((\log \theta_1(z-a_j))')^2 - N \left((\log \theta_1(z))'\right)^2 \right). \end{aligned}$$

Here prime means derivative with respect to z and in the final equality we have separated the single-valued part of $d\Omega_\tau$ (the first term) from its multi-valued part. This gives rise to

$$\begin{aligned} \Delta_B d\Omega_\tau = d\Omega_\tau(z+\tau) - d\Omega_\tau(z) &= \frac{dz}{4\pi i} \left(\sum_{j=1}^N ((\log \theta_1(z-a_j))')^2 - N \left((\log \theta_1(z))'\right)^2 \right) \Bigg|_z^{z+\tau} \\ &= -dz \left(\sum_{j=1}^N (\log \theta_1(z-a_j))' - N (\log \theta_1(z))' \right) = -\frac{dw}{w}. \end{aligned} \tag{2.26}$$

In more invariant terms [8] this can be stated as the "jump" of the non-single valued differential (2.19) across the A-cycle, with jump $\Delta_B d\Omega_\tau = d\Omega_\tau^+ - d\Omega_\tau^- = -dw/w$. From (2.19) and the τ independence of (2.15) it also follows that

$$0 = \oint_A d\Omega_\tau. \tag{2.27}$$

The integrability condition for (2.24,2.25) now follows upon considering the integral over the boundary $\partial\Sigma$ of the cut z-torus

$$0=\int_{\partial\Sigma} z d\Omega_\tau = \oint_A \left(z d\Omega_\tau - (z+\tau)(d\Omega_\tau - \frac{dw}{w})\right) + \oint_B ((z+1)d\Omega_\tau - z d\Omega_\tau) = \oint_A z\frac{dw}{w} + \oint_B d\Omega_\tau . \tag{2.28}$$

Thus $\partial^2\mathcal{F}/\partial\tau\partial u = \partial^2\mathcal{F}/\partial u\partial\tau$. The equality (2.28) also follows upon differentiating

$$0=\int_{\partial\Sigma} z^2\frac{dw}{w} = \oint_A \left(z^2-(z+\tau)^2\right)\frac{dw}{w} + \oint_B \left((z+1)^2-z^2\right)\frac{dw}{w} = 2\tau\oint_A dS - 2\oint_B dS,$$

which also establishes that

$$\tau u = \oint_B dS. \tag{2.29}$$

which corresponds to the $\frac{1}{2}\tau u^2$ term in the prepotential.

Let us now turn to the residue formula (1.3). The addition of the extra variables now mean further terms to those calculated in [16],

$$\frac{\partial^3\mathcal{F}}{\partial a_j\partial a_j\partial a_k} = \operatorname{res}_{\frac{dw}{w}=0}\left(\frac{d\Omega_i d\Omega_j d\Omega_k}{dz\frac{dw}{w}}\right) \tag{2.30}$$

$$= \begin{cases} 2\sum_{l\neq N}\hat\zeta(a_{lN}) - \sum_{l\neq i,j,k,N}\hat\zeta(a_{lN}), & i\neq j\neq k,\\ -\hat\zeta(a_{ik}) + 4\hat\zeta(a_{iN}) + 2\hat\zeta(a_{kN}) + \sum_{l\neq i,k,N}\hat\zeta(a_{lN}), & i=j\neq k,\\ \sum_{l\neq i}\hat\zeta(a_{il}) + 6\hat\zeta(a_{iN}) + \sum_{l\neq N}\hat\zeta(a_{lN}), & i=j=k.\end{cases} \tag{2.31}$$

Here $a_{ij}=a_i-a_j$ and $\hat\zeta(z)\equiv\frac{d}{dz}\log\theta_1(z)$. (The function $\hat\zeta(z)$ differs from the usual Weierstrass ζ-function by a term linear in z: $\hat\zeta(z)=\zeta(z)-2\eta z$.) Additionally

$$\frac{\partial^3\mathcal{F}}{\partial u\partial u\partial\tau} = \operatorname{res}_{\frac{dw}{w}=0}\left(\frac{dz dz d\Omega_\tau}{dz\frac{dw}{w}}\right) = \operatorname{res}_{\frac{dw}{w}=0}\left(\frac{dz d\Omega_\tau}{\frac{dw}{w}}\right) = \oint_{\partial\Sigma_{\rm cut}}\left(\frac{dz d\Omega_\tau}{\frac{dw}{w}}\right)$$
$$= \oint_A dz\frac{d\Omega_\tau^- - d\Omega_\tau^+}{\frac{dw}{w}} = \oint_A dz = 1, \tag{2.32}$$

$$\frac{\partial^3\mathcal{F}}{\partial u^3} = \operatorname{res}_{\frac{dw}{w}=0}\left(\frac{dz dz dz}{dz\frac{dw}{w}}\right) = \operatorname{res}_{\frac{dw}{w}=0}\left(\frac{dz dz}{\frac{dw}{w}}\right) = 0, \tag{2.33}$$

$$\frac{\partial^3\mathcal{F}}{\partial u\partial\tau\partial\tau} = \operatorname{res}_{\frac{dw}{w}=0}\left(\frac{dz d\Omega_\tau d\Omega_\tau}{dz\frac{dw}{w}}\right) = \operatorname{res}_{\frac{dw}{w}=0}\left(\frac{d\Omega_\tau d\Omega_\tau}{\frac{dw}{w}}\right)$$
$$= \oint_A \frac{(d\Omega_\tau^+)^2 - (d\Omega_\tau^-)^2}{\frac{dw}{w}} = -\oint_A \left(d\Omega_\tau^+ + d\Omega_\tau^-\right) = 0. \tag{2.34}$$

Here we have used (2.26) and (2.27). Together these means that the tri-logarithmic expression of [16] should be corrected by adding the term $\frac{1}{2}\tau u^2$, consistent with (2.29). Finally adding the term $\partial^3\mathcal{F}/\partial u^3$, a function of τ, leads to the prepotential of [17].

2.2 WDVV for the perturbative 6d prepotential

From the formulas of the previous section it is obvious that the 6d perturbative prepotential satisfies the WDVV equations (1.1) as a function of the $n_m = N+1$ parameters $\{T_I\} = \{u, a_j\ (j = 1\ldots N-1), \tau\}$. Indeed, having the residue formulas, one has only to check the matching condition (1.5). If this holds the WDVV equations then simply follow from the associativity of the algebra of functions at the critical points $\{z_\alpha\}$, the solutions of

$$\frac{dw}{w} = \left(\sum_{j=1}^{N} \frac{\theta_1'(z-a_j)}{\theta_1(z-a_j)} - N\frac{\theta_1'(z)}{\theta_1(z)}\right) dz = 0. \tag{2.35}$$

Now since the differential (2.35) obviously has $N+1$ poles then it also has $\#(\alpha) = N+1$ zeroes. Thus $n_z = N+1 = n_m$ and the matching condition holds.

The corresponding associative algebra is simply realized as the algebra of functions at the points $\{z_\alpha\}$, with any appropriate choice of ξ. The corresponding basis can be chosen as

$$\begin{aligned}
\phi_j &= \frac{d\Omega_i}{dz} = \frac{\theta_1'(z-a_N)}{\theta_1(z-a_N)} - \frac{\theta_1'(z-a_j)}{\theta_1(z-a_j)}, \quad j = 1,\ldots,N-1,\\
\phi_\tau &= \frac{d\Omega_\tau}{dz} = \sum_{j=1}^{N} \frac{\theta_1''(z-a_j)}{\theta_1(z-a_j)} - N\frac{\theta_1''(z)}{\theta_1(z)},\\
\phi_z &= 1.
\end{aligned} \tag{2.36}$$

Our algebra of functions here 'accidentally' contains the "unity" $\phi_z = 1$, but we stress that this does not influence any of our statements made about the WDVV equations beyond the specific u-dependence of the prepotential via the $\frac{1}{2}\tau u^2$-term. This simple dependence almost certainly does not survive beyond the perturbative limit of the 6d theory.

The only delicate point to note here is that ϕ_τ (2.36) is *not* single valued on the torus Σ. However, this is not a problem when considering the associative algebra (1.7) since we restrict the values of the functions (2.36) to their *values* at critical points $\{z_\alpha\}$, where the ambiguity in the definition of ϕ_τ disappears,

$$\left(\phi_\tau^+ - \phi_\tau^-\right)\Big|_{z=z_\alpha} = \left(\frac{d\Omega_\tau^+}{dz} - \frac{d\Omega_\tau^-}{dz}\right)\Bigg|_{z=z_\alpha} = \left(\frac{d\log w}{dz}\right)\Bigg|_{z=z_\alpha} = 0. \tag{2.37}$$

Therefore the quantities $\phi_\tau(z_\alpha)$ are uniquely defined.

For later comparison it is instructive to write down the simplest $SU(2)$ case with $N+1=3$, when the WDVV equations (1.1) are already nontrivial. In this case (upon noting $\sigma(z) = e^{\eta z^2}\theta_1(z)$ and $\wp(z) = -\zeta'(z) = -2\eta - \frac{d^2}{dz^2}\log\theta_1(z)$)

$$w = e^u \frac{\theta_1(z-a)\theta_1(z+a)}{\theta_1(z)^2} = \theta_1^2(a) e^u \left(\wp(a) - \wp(z)\right) \tag{2.38}$$

and

$$\frac{dw}{w} = \frac{\wp'(z)dz}{\wp(z) - \wp(a)}. \tag{2.39}$$

The latter has three poles at $z = 0, \pm a$ and three zeroes at the half-periods of the z-torus

$$\{z_\alpha\} = \{\omega_\alpha\} = \frac{1}{2},\ \frac{\tau}{2},\ \frac{1+\tau}{2}. \tag{2.40}$$

The basis of functions (2.36) for this case is

$$\phi_a = \frac{d\Omega_a}{dz} = \frac{\theta_1'(z+a)}{\theta_1(z+a)} - \frac{\theta_1'(z-a)}{\theta_1(z-a)} = \hat{\zeta}(z+a) - \hat{\zeta}(z-a) = 2\hat{\zeta}(a) - \frac{\wp'(a)}{\wp(z)-\wp(a)}, \tag{2.41}$$

$$\phi_\tau = \frac{d\Omega_\tau}{dz} = \frac{\theta_1''(z+a)}{\theta_1(z+a)} + \frac{\theta_1''(z-a)}{\theta_1(z-a)} - 2\frac{\theta_1''(z)}{\theta_1(z)}, \tag{2.42}$$

$$\phi_z = 1. \tag{2.43}$$

Observe that the curve (2.38) in the $SU(2)$ case has an additional symmetry $z \leftrightarrow -z$. We will see below that the WDVV equations hold generally for a subfamily of bi-elliptic curves with this extra symmetry.

2.3 The non-perturbative 6d theory

Consider now the non-perturbative 6d $SU(N)$ theory associated to the curve

$$w + \frac{1}{w} = e^u \prod_{j=1}^{N} \frac{\theta_1(z-a_j)}{\theta_1(z)} \equiv e^u\, H(z), \qquad \sum_{j=1}^{N} a_j = 0, \tag{2.44}$$

and generating differential $dS = -z\frac{dw}{w}$. These curves have the $\mathbb{Z}_2$ symmetry $\sigma : w \leftrightarrow 1/w$ under which dS is odd. When $N = 2$, there is the additional symmetry $z \leftrightarrow -z$ noted above. The curves have genus $g = N+1$ for the $SU(N)$ gauge theory. We may picture this curve as two z-tori glued along N identical cuts on each. The ends of these are located at the zeroes of $H(z)^2 - 4$. One may choose a canonical basis of cycles with A_1 and A_{N+1} being A-cycles on the z-tori while the A_j-cycle ($j = 2,\dots,N$) surrounds the $(j-1)$-st handle joining them. (These handles degenerate to points in the perturbative limit.) The corresponding B-cycles are determined by $A_i \circ B_j = \delta_{ij}$. The cycle B_j may be taken as going from one torus through the $(j-1)$-st handle and returning through the N-th handle. This choice of cycles can be made so that $\sigma_\star(A_1) = A_{N+1}$, $\sigma_\star(B_1) = B_{N+1}$ and $\sigma_\star(A_j) = -A_j$, $\sigma_\star(B_j) = -B_j$ ($j = 2,\dots,N$).

For illustrative purposes we focus first on the $SU(2)$ case with the generalization to arbitrary N being briefly given later in the section. We will also restore the scale Λ. We then have

$$w + \frac{\Lambda^2}{w} = e^u H(z) = e^u \theta_1^2(a)\left(\wp(a) - \wp(z)\right). \tag{2.45}$$

Now we have two z-tori glued along 2 identical cuts, the ends which are located at the zeroes of

$$\left(w - \frac{\Lambda^2}{w}\right)^2 = e^{2u}H(z)^2 - 4\Lambda^2 = e^{2u}\theta_1^4(a)\left(\wp(z) - \wp(a) - 2\tilde{\Lambda}^2\right)\left(\wp(z) - \wp(a) + 2\tilde{\Lambda}^2\right),\ \tilde{\Lambda}^2 = \frac{\Lambda^2}{e^u\theta_1^2(a)}. \tag{2.46}$$

We may choose as a basis of holomorphic differentials

$$dz, \qquad \frac{w + \frac{\Lambda^2}{w}}{w - \frac{\Lambda^2}{w}}\left(\frac{\theta_1'(z+a)}{\theta_1(z+a)} - \frac{\theta_1'(z-a)}{\theta_1(z-a)}\right)dz, \qquad \frac{w + \frac{\Lambda^2}{w}}{w - \frac{\Lambda^2}{w}}\,dz, \tag{2.47}$$

or equivalently

$$dz, \qquad \frac{dz}{w - \frac{\Lambda^2}{w}}, \qquad \frac{w + \frac{\Lambda^2}{w}}{w - \frac{\Lambda^2}{w}}\,dz, \tag{2.48}$$

the two being related by

$$\frac{w+\frac{\Lambda^2}{w}}{w-\frac{\Lambda^2}{w}}\left(\frac{\theta_1'(z+a)}{\theta_1(z+a)}-\frac{\theta_1'(z-a)}{\theta_1(z-a)}\right)dz = 2\hat{\zeta}(a)\frac{w+\frac{\Lambda^2}{w}}{w-\frac{\Lambda^2}{w}}dz + e^u\theta_1^2(a)\wp'(a)\frac{dz}{w-\frac{\Lambda^2}{w}}. \tag{2.49}$$

Now consider the variation of the generating differential dS (at constant w). Up to total differentials

$$\begin{aligned}\delta\left(-z\frac{dw}{w}\right) &= \delta u\ dz\frac{w+\frac{\Lambda^2}{w}}{w-\frac{\Lambda^2}{w}} + \delta a\ \frac{w+\frac{\Lambda^2}{w}}{w-\frac{\Lambda^2}{w}}\left(\frac{\theta_1'(z+a)}{\theta_1(z+a)}-\frac{\theta_1'(z-a)}{\theta_1(z-a)}\right)dz \\ &\quad + \frac{\delta\tau}{4\pi i}\ \frac{w+\frac{\Lambda^2}{w}}{w-\frac{\Lambda^2}{w}}\left(\frac{\theta_1''(z+a)}{\theta_1(z+a)}+\frac{\theta_1''(z-a)}{\theta_1(z-a)}-2\frac{\theta_1''(z)}{\theta_1(z)}\right)dz\end{aligned} \tag{2.50}$$

The first two terms of this expansion are holomorphic Abelian differentials with the final term, corresponding to $\delta\tau$, being non-single valued. This is just as in the perturbative case. Note that the holomorphic differential dz does not appear in the expansion of $\delta(dS)$. Indeed this has a different behaviour under the $\mathbb{Z}_2$ symmetry (now $w \leftrightarrow \frac{\Lambda^2}{w}$) compared to the generating differential dS: dS itself and all the constituents of (2.50) are *odd* with respect to this symmetry, while dz is even. This latter differential is related with the deformation of dS with respect to the scale parameter

$$\partial_{\log\Lambda} dS\big|_{\frac{w}{\Lambda}} = dz.$$

Suppose now that $\{d\omega_i\}$ are canonical holomorphic differentials such that $\oint_{A_i} d\omega_j = \delta_{ij}$. Then using the symmetry under σ we have expansions

$$d\omega_- \equiv d\omega_1 - d\omega_3 = \alpha_-\frac{w+\frac{1}{w}}{w-\frac{1}{w}}\ dz + \beta_-\frac{w+\frac{1}{w}}{w-\frac{1}{w}}\left(\frac{\theta_1'(z+a)}{\theta_1(z+a)}-\frac{\theta_1'(z-a)}{\theta_1(z-a)}\right)dz \tag{2.51}$$

$$d\omega_2 = \alpha\frac{w+\frac{1}{w}}{w-\frac{1}{w}}\ dz + \beta\ \frac{w+\frac{1}{w}}{w-\frac{1}{w}}\left(\frac{\theta_1'(z+a)}{\theta_1(z+a)}-\frac{\theta_1'(z-a)}{\theta_1(z-a)}\right)dz \tag{2.52}$$

where the coefficients α_-, β_-, α, β are determined from

$$\oint_{A_2} d\omega_- = 0, \qquad \oint_{A_1} d\omega_- = 1 = -\oint_{A_3} d\omega_-, \qquad \oint_{A_2} d\omega_2 = 1, \qquad \oint_{A_1} d\omega_2 = 0 = -\oint_{A_3} d\omega_2. \tag{2.53}$$

In the perturbative limit $\frac{w+\frac{\Lambda^2}{w}}{w-\frac{\Lambda^2}{w}} \to \pm 1$, depending on which base torus solution we choose to (2.45). Similarly $\frac{w+\frac{\Lambda^2}{w}}{w-\frac{\Lambda^2}{w}}\ dz \to \pm dz$. The differential $\frac{w+\frac{\Lambda^2}{w}}{w-\frac{\Lambda^2}{w}}\left(\frac{\theta_1'(z+a)}{\theta_1(z+a)}-\frac{\theta_1'(z-a)}{\theta_1(z-a)}\right)dz$ becomes an Abelian differential of the third kind with poles at $\pm a$, the two cuts reducing to these two points. In this limit we have

$$\begin{pmatrix}\alpha_- & \beta_- \\ \alpha & \beta\end{pmatrix} \to \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}, \tag{2.54}$$

consistent with (2.53), and consequently

$$\oint_{B_2} d\omega_- \to 2\int_{-a}^{a} dz \tag{2.55}$$

In this way we recover our earlier perturbative results.

It remains to describe the matching conditions necessary for the WDVV equations. We see from (2.50) that we have $n_m = 3$ moduli here. On the nonperturbative curve (2.44) a holomorphic differential has $2(g-1) = 4$ zeroes, while the differential $\frac{dw}{w}$ has two simple poles and therefore $2 + 2(g-1) = 6$ zeroes. However these six zeros arise from $n_z = 3$ values $\{z_\alpha\}$ (with two w values for each). For the case at hand $z_\alpha = \omega_\alpha$, $\alpha = 1, \ldots, 3$ (2.40) are solutions to $H'(z) = 0$, The differentials in the residue formula only depend on z_α here, and so the symmetry of the curve enables us to get matching here.

Let us now consider the general N case which goes through in much the same way. Instead of (2.50) one now has[2]

$$\begin{aligned}\delta\left(-z\frac{dw}{w}\right) &= \delta u\; dz\frac{w+\frac{1}{w}}{w-\frac{1}{w}} + \sum_{j=1}^{N-1}\delta a_j\;\frac{w+\frac{1}{w}}{w-\frac{1}{w}}\left(\frac{\theta_1'(z-a_N)}{\theta_1(z-a_N)} - \frac{w+\frac{1}{w}}{w-\frac{1}{w}}\frac{\theta_1'(z-a_j)}{\theta_1(z-a_j)}\right)dz \\ &\quad + \frac{\delta\tau}{4\pi i}\;\frac{w+\frac{1}{w}}{w-\frac{1}{w}}\left(\sum_{j=1}^{N}\frac{\theta_1''(z-a_j)}{\theta_1(z-a_j)} - N\frac{\theta_1''(z)}{\theta_1(z)}\right)dz.\end{aligned} \tag{2.56}$$

The right hand side of (2.56) now consists of the N (out of the total $N+1$) holomorphic differentials that are odd with respect to the $\mathbb{Z}_2$ symmetry $w \leftrightarrow \frac{1}{w}$. In analogy with (2.20) these may be expanded over the non-perturbative basis

$$\frac{dz}{w-\frac{1}{w}}\left(1,\ldots,\wp(z)^{[N/2]}\right) = \frac{dx}{\left(w-\frac{1}{w}\right)y}\left(1,\ldots,x^{[N/2]}\right), \tag{2.57}$$

$$\frac{dz}{w-\frac{1}{w}}\left(1,\ldots,\wp(z)^{[(N-3)/2]}\right)\wp'(z) = \frac{dx}{w-\frac{1}{w}}\left(1,\ldots,x^{[(N-3)/2]}\right) \tag{2.58}$$

Now all of the differentials appearing in the residue formula are $\mathbb{Z}_2$-odd. Upon writing $d\Omega_I = h_I(z)\frac{w+\frac{1}{w}}{w-\frac{1}{w}}dz$ the residue formula (1.3) for the prepotential $\mathcal{F}$ becomes

$$\begin{aligned}\mathcal{F}_{IJK} &= \operatorname{res}_{\frac{dw}{w}=0}\;\left(\frac{w+\frac{1}{w}}{w-\frac{1}{w}}\right)^3\frac{h_I(z)h_J(z)h_K(z)dz^3}{dz\frac{dw}{w}} = \sum_\alpha \operatorname{res}_{z_\alpha}\frac{H(z)^3h_I(z)h_J(z)h_K(z)dz}{(H(z)^2-4)H'(z)} \\ &= \sum_\alpha\frac{H(z_\alpha)^3h_I(z_\alpha)h_J(z_\alpha)h_K(z_\alpha)}{(H(z_\alpha)^2-4)H''(z_\alpha)}\end{aligned} \tag{2.59}$$

where z_α are solutions of $0 = dw/w$. To describe what these are first observe that dz and $\frac{dz}{w-\frac{1}{w}}$ are holomorphic differentials and so have $2(g_\Sigma - 1) = 2N$ zeros. The divisor of dz are the $2N$ branch points of

$$\left(w-\frac{1}{w}\right)^2 = e^{2u}\left(H(z)^2-4\right),$$

the points at which $w - \frac{1}{w} = 0$. The divisor of $\frac{dz}{w-\frac{1}{w}}$ are the points $z = 0$ and $w = 0, \infty$. Now $H'(z)$ has a pole of order $N+1$ at $z = 0$. This pole cancels the zeros of $\frac{dz}{w-\frac{1}{w}}$ leaving a simple pole remaining (one at $(z = 0, w = 0)$ and one at $(z = 0, w = \infty)$) and the zeros of

$$\frac{dw}{w} = e^u H'(z)\,\frac{dz}{w-\frac{1}{w}}$$

[2]Below we shall put $\Lambda = 1$.

are the $(N+1)$ (as a function of z) zeros $\{z_\alpha\}$ of $H'(z)$, and so corresponds to $2(N+1)$ points on the curve Σ. Here we have $n_m = N+1$ moduli and, because of the symmetry of the differentials $d\Omega_I$, $n_z = \#\{z_\alpha\} = N+1$. This means that the non-perturbative $\mathcal{F}$, defined by (2.59), satisfies the WDVV equations as a function $\mathcal{F}(T_K) = \mathcal{F}(\mathsf{a}_1 - \mathsf{a}_{N+1}, \mathsf{a}_2, \ldots, \mathsf{a}_N, \tau)$ of N linear combinations of the $N+1$ SW periods $\mathsf{a}_i = \oint_{A_i} dS$ and the modular parameter τ of the base curve.

3 A bi-elliptic generalization

We now consider generalizing our discussion to the setting when the variable $\tilde{z}$ is also elliptic. First observe that the non-perturbative $SU(2)$ curve (2.45), (2.46) can be rewritten in "hyperelliptic" terms as

$$y^2 = \prod_{i=1}^{3}(x - e_i), \qquad Y^2 = (x - b_+)(x - b_-), \tag{3.1}$$

where $x = \wp(z)$ and $y = -\frac{1}{2}\wp'(z)$ are the affine coordinates of the torus (2.11) and $Y = e^{-u}\left(w - \frac{1}{w}\right)/\theta_1^2(a)$. By replacing the second equation in (3.1) with a polynomial of fourth degree on the right hand side,

$$y^2 = \prod_{i=1}^{3}(x - e_i), \qquad Y^2 = \prod_{j=1}^{4}(x - b_j), \tag{3.2}$$

one comes to a sort of a "bi-elliptic" system that we will further explore. (We reserve the term "double-elliptic" for the system considered in [18].) Equivalently via the fractional-linear transformation

$$\tilde{x} = \frac{ax+b}{cx+d}, \qquad x = \frac{d\tilde{x} - b}{-c\tilde{x} + a}, \qquad Y \propto (cx+d)^2\tilde{y}, \tag{3.3}$$

the curve (3.1) may be rewritten as two Weierstrass equations

$$y^2 = \prod_{i=1}^{3}(x - e_i), \qquad \tilde{y}^2 = \prod_{i=1}^{3}(\tilde{x} - \tilde{e}_i). \tag{3.4}$$

We remark that the curve (3.2) has the symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$: $z \leftrightarrow -z$ and $\tilde{z} \leftrightarrow -\tilde{z}$, where now $x = \wp(z;\tau)$, $y = -\frac{1}{2}\wp'(z;\tau)$ and $\tilde{x} = \wp(\tilde{z};\tilde{\tau})$, $\tilde{y} = -\frac{1}{2}\wp'(\tilde{z};\tilde{\tau})$ are the standard coordinates of the tori.

The genus of the curve (3.2) is $g = 5$. This coincides with the total number of holomorphic differentials, which are linear combinations of

$$\frac{dx}{yY}, \quad x\frac{dx}{yY}, \quad x^2\frac{dx}{yY}, \quad \frac{dx}{y} = dz, \quad \frac{dx}{Y} = \frac{d\tilde{x}}{\tilde{y}} = d\tilde{z}. \tag{3.5}$$

Taking into account the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry the number of zeroes of the holomorphic differentials (in particular of dz and $d\tilde{z}$) is seen to be $2(g-1) = 8$. Equally the genus may calculated using the Riemann-Hurwitz formula. The curve may be viewed as a 4-sheeted cover of the x-plane, with each sheet corresponding to particular choice of the sign for $(y, Y) = (\pm, \pm)$. There are a total of $B = 2 \cdot 4 + 2 \cdot 4 = 16$ branch points and so. Then

$$g - 1 = \#\ (\text{sheets})(g_0 - 1) + B/2 = 4(0-1) + 16/2 = 4, \tag{3.6}$$

or $g = 5$.

The example of (3.2) may be extended as follows. We consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric curve arising when the two elliptic equations (3.4) are rationally related via

$$\tilde{x} = \frac{P_m(x)}{R_m(x)} = \frac{p_m x^m + \cdots + p_0}{r_m x^m + \cdots + r_0}. \tag{3.7}$$

Upon introducing $Y \propto R_m(x)^2 \tilde{y}$ this may be rewritten in a form similar to (3.8),

$$y^2 = \prod_{i=1}^{3}(x - e_i), \qquad Y^2 = R_m(x) \prod_{j=1}^{3}(P_m(x) - \tilde{e}_j R_m(x)) \tag{3.8}$$

where on the right hand side of the last formula we now have a polynomial of degree $4m$. The trigonometric and rational degenerations of (3.7) then take the form

$$w + \frac{1}{w} = P_m(x) \tag{3.9}$$

and

$$w = P_m(x) \tag{3.10}$$

respectively. In contrast to (2.12) the righthand side here depends only on the Weierstrass function $x = \wp(z)$ which means that in this symmetric case one considers only the even differentials (2.57) and of (2.20) with $[N/2] = m$.

The genus of the curve (3.8) is $g = 4m + 1$. For a fixed x there are $4m$ Y-branch points, and as there are two z's for each x we obtain $8m$ branch points. The total number of branch points is then $B = 2 \cdot 4 + 2 \cdot 4m$ and application of the Riemann-Hurwitz formula (3.6) gives the stated genus. It is also clear that the curve defined by (3.7) together with (3.4), or by (3.8), can be visualized as a double-cover of the z-torus with $8m$ branch points. Taking derivative of (3.7), one gets

$$\tilde{y} d\tilde{z} = \frac{P_m'(x)R_m(x) - P_m(x)R_m'(x)}{R_m(x)^2} y dz \tag{3.11}$$

or

$$Y d\tilde{z} = (P_m'(x)R_m(x) - P_m(x)R_m'(x)) y dz \tag{3.12}$$

The number of zeroes of $d\tilde{z}$ is given by zeroes of the righthand side of (3.12) and equals

$$4 \cdot (2m - 2) + 2 \cdot 4 = 8m = 2(g - 1). \tag{3.13}$$

Here the first factor of 4 comes from taking into account the total $\mathbb{Z}_2 \times \mathbb{Z}_2$ multiplicity of the $2m-2$ zeros of the polynomial $P_m' R_m - P_m R_m' \sim x^{2m-2}$, while the second term in the left-hand side of (3.13) counts the 4 half-periods of the z-torus (including $z = 0$ if compared to (2.40)), with the factor 2 corresponding to the $\tilde{z} \leftrightarrow -\tilde{z}$ part of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

An alternative approach to obtain a generalization to that outlined above would be to consider the case with the elliptic cosine replacing the Weierstrass function $\tilde{x}$ in the left-hand side of (3.7). Naively, this would lead to an equation

$$\frac{\tilde{x} - \tilde{e}_1}{\tilde{x} - \tilde{e}_2} = \frac{P_m(x)^2}{R_m(x)^2} \tag{3.14}$$

with the polynomials on the righthand side being the squares of those appearing in (3.7). Analogous to our rewriting of (3.8) one now finds that

$$y^2 = \prod_{i=1}^{3}(x - e_i), \quad Y^2 = \left(P_m(x)^2 - R_m(x)^2\right)\left(\tilde{e}_{23}P_m(x)^2 - \tilde{e}_{13}R_m(x)^2\right), \tag{3.15}$$

where now

$$Y = \tilde{y}\frac{P_m^2 - R_m^2}{P_m R_m \tilde{e}_{12}} \tag{3.16}$$

and $\tilde{e}_{ij} \equiv \tilde{e}_i - \tilde{e}_j$. Again we find a polynomial of degree $4m$ on the righthand side of (3.15), and this case completely repeats our discussion of the curve (3.8).

One can also generalize to less symmetric curves where we only have a $\mathbb{Z}_2$ symmetry with no extra $z \leftrightarrow -z$ or $y \leftrightarrow -y$ symmetry. The analysis of this parallels what we have already presented and we simply will present the list of results in the next section.

4 Summary of general results

We now present a summary of the results for the separated case of the bi-elliptic curve (2.1) when the function $\mathcal{H}$ may be expressed as a sum of two terms; they being elliptic functions of z and $\tilde{z}$. Let us start with the case of an algebraic function $\mathcal{H}$ linear in the co-ordinate $\tilde{x} = \wp(\tilde{z}|\tilde{\tau})$,

$$\Sigma_{\rm ell}: \quad \tilde{x} = H_N(z) = \frac{P(x) + yQ(x)}{R(x) + yS(x)} = e^u \prod_{i=1}^{N} \frac{\theta(z - a_i|\tau)}{\theta(z - a_i'|\tau)}, \qquad dS = z d\tilde{z}, \tag{4.1}$$

with polynomials $P(x)$ and $R(x)$ of degree $\left[\frac{N}{2}\right]$ and the polynomials $Q(x)$ and $S(x)$ of degree $\left[\frac{N-3}{2}\right]$. Further $\sum_{i=1}^{N}(a_i - a_i') = 0$. We shall call this the elliptic case with $\mathbb{Z}_2$-symmetry; the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric subfamily (3.7) corresponds to the vanishing of Q and S in the righthand side of (4.1).

The non-perturbative 6d theory studied earlier corresponds to the trigonometric degeneration of the $\tilde{z}$-torus in (4.1),

$$\Sigma_{\rm trig}: \quad w + \frac{1}{w} = P(x) + yQ(x) = \prod_{i=1}^{N}\frac{\theta(z - a_i|\tau)}{\theta(z|\tau)}, \quad \sum_{i=1}^{N} a_i = 0, \qquad dS = z\frac{dw}{w}. \tag{4.2}$$

This may be viewed as an Inozemtsev limit (see [19, 20]), when $\tilde{\tau} \to +i\infty$ and $\tilde{z} = i\tilde{\tau}/2 - \log w$, so that $\tilde{x} \to {\rm const} + 2q^{1/4}(w + \frac{1}{w} + O(q^{9/4})$. After appropriate adjustment of the polynomials the bi-elliptic family (4.1) then turns then (4.2). This degeneration leads to the generic $SU(N)$ non-perturbative curve of genus $g = N + 1$ discussed earlier. Upon incorporating the scale Λ, redefining $w \to \Lambda^{-N}w$ and taking the perturbative limit $\Lambda \to 0$, the system (4.2) further reduces to (2.12).

4.1 Elliptic, trigonometric and rational cases: the case of enlarged $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry $\tilde{z} \leftrightarrow -\tilde{z}$ and $z \leftrightarrow -z$

	elliptic (3.7), (3.4	trigonometric (3.9)	rational (3.10)
spectral curve Σ	$\tilde{x} = \frac{P_m(x)}{R_m(x)}$	$w + \frac{1}{w} = P_m(x)$	$w = P_m(x)$
n'_m: # of parameters: the coefficients of polynomials	$2(m+1) - 1 = 2m + 1$	$m + 1$	$m + 1$
$n_m = n'_m + 1$ (inclusion of τ)	$2m + 2$	$m + 2$	$m + 2$
$\mathcal{H}'_{\tilde{z}}$	$\tilde{y}$	$w - \frac{1}{w}$	w
$Y^2 \sim (\mathcal{H}'_{\tilde{z}})^2$, expressed through x	$Y^2 = R_m^4(x)(\mathcal{H}'_{\tilde{z}})^2 =$ $R_m(x) \prod_{i=1}^{3} (P_m(x) - \tilde{e}_i R_m(x))$	$Y^2 = \left(w - \frac{1}{w}\right)^2 =$ $= P_m^2(x) - 4$	$Y = w = P_m(x)$
behaviour at large x	$Y \sim x^{2m}$	$Y \sim x^m$	$Y \sim x^m$
$\nu(\Sigma)$: # of branch points of Σ over z-torus, where $Y = 0$	$8m$	$4m$	$2m$ punctures: "contracted" $4m$ branch points (pairwise)
genus $g(\Sigma) = \frac{\nu(\Sigma)-2}{2} + 2$	$4m + 1$	$2m + 1$	$g = 1$ torus with $2m$ punctures

holomorphic differentials on Σ (or with simple poles in rational case)	$\frac{(1,\dots,x^{2m})dx}{Yy}$ $\frac{(1,\dots,x^{2m-2})dx}{Y}$ $\frac{dx}{y}=dz$	$\frac{(1,\dots,x^{m})dx}{\left(w-\frac{1}{w}\right)y}$ $\frac{(1,\dots,x^{m-2})dx}{w-\frac{1}{w}}$ $\frac{dx}{y}=dz$	$\frac{(1,\dots,x^{m})dx}{wy}$ $\frac{(1,\dots,x^{m-2})dx}{w}$ $\frac{dx}{y}=dz$
total number of differentials $=g(\Sigma)+\#$marked points	$(2m+1)+(2m-1)+1$ $=4m+1$	$(m+1)+(m-1)+1$ $=2m+1$	$(m+1)+(m-1)+1$ $=2m+1$
symmetry of Σ	$\mathbb{Z}_2\times\mathbb{Z}_2:$ $\begin{cases} z\leftrightarrow -z, & y\leftrightarrow -y \\ \tilde{z}\leftrightarrow -\tilde{z}, & Y\leftrightarrow -Y \end{cases}$	$\mathbb{Z}_2\times\mathbb{Z}_2:$ $\begin{cases} z\leftrightarrow -z, & y\leftrightarrow -y \\ w\leftrightarrow \frac{1}{w}, & Y\leftrightarrow -Y \end{cases}$	$\mathbb{Z}_2:$ (different!) $z\leftrightarrow -z,\ y\leftrightarrow -y$
holomorphic differentials, *odd* under the symmetry	$\frac{(1,\dots,x^{2m})dx}{Yy}$	$\frac{(1,\dots,x^{m})dx}{\left(w-\frac{1}{w}\right)y}$	$\frac{(1,\dots,x^{m})dx}{wy}$ $\frac{dx}{y}=dz$
n_h'': # of such differentials	$2m+1$	$m+1$	$(m+1)+1=m+2$
n_h': # of hol. differentials, contributing to $\delta(d\tilde{S})$	$n_h'=n_h''=2m+1$	$n_h'=n_h''=m+1$	$n_h'=n_h''-1=m+1$ dz does not contribute
$n_h=n_h'+1$ due to addition of $d\Omega_\tau=\frac{\partial(dS)}{\partial\tau}$ with jump	$2m+2$	$m+2$	$m+2$
jump $\Delta_B(d\Omega_\tau)$	$d\tilde{z}$	$\frac{dw}{w}$	$\frac{dw}{w}$

$d\tilde{z}$ or $\frac{dw}{w}$	holomorphic	2 simple poles at $x=\infty$ $(z=0)$ related by $w \leftrightarrow \frac{1}{w}$	$2m+1$ simple poles at $x=\infty$ $(z=0)$ and zeroes of $P_m(x)$ doubled by $z \leftrightarrow -z$
zeroes of $d\tilde{z}$ (i.e. zeroes of H'_z)	$2(2m-2)+4=4m$ zeroes of $d\left(\frac{P_m(x)}{R_m(x)}\right) = \frac{P'_m R_m - P_m R'_m}{R^2_m} ydz$ produces two *different* zeroes related by $\tilde{z} \leftrightarrow -\tilde{z}$	$2(m-1)+3=2m+1$ zeroes of $dP_m(x)=P'_m(x)ydz$ produces two *different* zeroes related by $w \leftrightarrow \frac{1}{w}$	$2(m-1)+3=$ $=2m+1$ zeroes of $dP_m(x)=$ $=P'_m(x)ydz$
n'_z: number zeroes of $d\tilde{z}$ or $\frac{dw}{w}$	$8m = 2\cdot 4m = 2\,(g(\Sigma)-1)$	$4m+2 = 2\cdot(2m+1) =$ $= 2\,(g(\Sigma)-1) + \#\text{poles}$	$2m+1 =$ #poles
n_z: # of critical points in residue formula: (# of zeroes of $d\tilde{z}$ over symmetry)	$\frac{1}{4}(2\cdot 2(2m-2)) + \frac{1}{2}(2\cdot 3)$ $=2m+2$	$\frac{1}{4}(2\cdot 2(m-1)) + \frac{1}{2}(2\cdot 3)$ $=m+2$	$\frac{1}{2}2(m-1)+4$ $=m+2$
matching $n_m = n_h = n_z$ WDVV equations	+	+	+

This table completes our analysis for the most symmetric case. We see that the WDVV equations (1.1) hold in all three (perturbative, non-perturbative and bi-elliptic) cases.

4.2 Summary for elliptic, trigonometric and rational cases: the generic case of only $\mathbb{Z}_2$ symmetry $\tilde{z} \leftrightarrow -\tilde{z}$

	elliptic (4.1)	trigonometric (4.2)	rational (2.12)
spectral curve Σ	$\tilde{x} = \frac{P(x)+yQ(x)}{R(x)+yS(x)}$	$w + \frac{1}{w} = P(x) + yQ(x)$	$w = P(x) + yQ(x)$
n'_m: # of parameters: coefficients of polynomials	$2\left(\left(\left[\frac{N}{2}\right]+1\right)+\left(\left[\frac{N-3}{2}\right]+1\right)\right)-1 =$ $= 2N-1$	$\left(\left[\frac{N}{2}\right]+1\right)+\left(\left[\frac{N-3}{2}\right]+1\right)$ $= N$	$\left(\left[\frac{N}{2}\right]+1\right)+\left(\left[\frac{N-3}{2}\right]+1\right)$ $= N$
$n_m = n'_m + 1$ (inclusion of τ)	$2N$	$N+1$	$N+1$
$\mathcal{H}'_{\tilde{z}}$	$\tilde{y}$	$w - \frac{1}{w}$	w
$Y^2 \sim (\mathcal{H}'_{\tilde{z}})^2$, expressed through x	$Y^2 = (R+yS)^4(\mathcal{H}'_{\tilde{z}})^2 = (R+yS)\cdot$ $\cdot\prod_{i=1}^{3}\left((P-\tilde{e}_iR)+y(Q-\tilde{e}_iS)\right)$	$Y^2 = \left(w - \frac{1}{w}\right)^2 =$ $= (P+yQ)^2 - 4$	$Y = w = P + yQ$
behaviour at large x	$Y \sim x^N$	$Y \sim x^{N/2}$	$Y \sim x^{N/2}$
$\nu(\Sigma)$: # of branch points i.e. where $Y=0$	$4N$	$2N$	N punctures at pairwise contracted $2N$ branch points
genus $g(\Sigma) = \frac{\nu(\Sigma)-2}{2} + 2$	$2N+1$	$N+1$	$g=1$ torus with N punctures

holomorphic differentials on Σ (or with simple poles in rational case)	$\frac{(1,\dots,x^N)dx}{Yy}$ $\frac{(1,\dots,x^{N-2})dx}{Y}$ $\frac{dx}{y}=dz$	$\frac{(1,\dots,x^{[N/2]})dx}{\left(w-\frac{1}{w}\right)y}$ $\frac{(1,\dots,x^{[(N-3)/2]})dx}{w-\frac{1}{w}}$ $\frac{dx}{y}=dz$	$\frac{(1,\dots,x^{[N/2]})dx}{wy}$ $\frac{(1,\dots,x^{[(N-3)/2]})dx}{w}$ $\frac{dx}{y}=dz$
total number of differentials $=g(\Sigma)+\#$marked points	$(N+1)+(N-1)+1$ $=2N+1$	$\left(\left[\frac{N}{2}\right]+1\right)+$ $+\left(\left[\frac{N-3}{2}\right]+1\right)+1=$ $=N+1$	$\left(\left[\frac{N}{2}\right]+1\right)+$ $+\left(\left[\frac{N-3}{2}\right]+1\right)+1=$ $=N+1$
symmetry of Σ	$\mathbb{Z}_2:\tilde{z}\leftrightarrow-\tilde{z},\ Y\leftrightarrow-Y$	$\mathbb{Z}_2: w\leftrightarrow\frac{1}{w}$	none
holomorphic differentials, *odd* under the symmetry	$\frac{(1,\dots,x^N)dx}{Yy}$ $\frac{(1,\dots,x^{N-2})dx}{Y}$	$\frac{(1,\dots,x^{[N/2]})dx}{\left(w-\frac{1}{w}\right)y}$ $\frac{(1,\dots,x^{[(N-3)/2]})dx}{w-\frac{1}{w}}$	$\frac{(1,\dots,x^{[N/2]})dx}{wy}$ $\frac{(1,\dots,x^{[(N-3)/2]})dx}{w}$ $\frac{dx}{y}=dz???$
n_h'': # of such differentials	$2N$	N	$N+1$
n_h': # of hol. differentials, contributing to $\delta(dS)$	$n_h'=n_h''-1=2N-1$ one linear combination does not contribute	$n_h'=n_h''=N$	$n_h'=n_h''-1=N$ dz does not contribute
$n_h=n_h'+1$ due to addition of $d\Omega_\tau=\frac{\partial(dS)}{\partial\tau}$ with jump	$2N$	$N+1$	$N+1$
jump $\Delta_B(d\Omega_\tau)$	$d\tilde{z}$	$\frac{dw}{w}$	$\frac{dw}{w}$

$d\tilde{z}$	holomorphic	2 simple poles at $x = \infty$ $(z = 0)$ related by $w \leftrightarrow \frac{1}{w}$	$N+1$ simple poles: at $x = \infty$ $(z = 0)$ and N zeroes of $P + yQ$ or at $z = a_j$, $j = 1, \dots, N$
zeroes of $d\tilde{z}$ (or of H'_z)	$2N$ zeroes of $d\left(\frac{P+yQ}{R+yS}\right)$ doubled by $\tilde{z} \leftrightarrow -\tilde{z}$	$N+1$ zeroes of $d(P + yQ)$ doubled by $w \leftrightarrow \frac{1}{w}$	$N+1$ zeroes of $d(P + yQ)$
n'_z: number zeroes of $d\tilde{z}$	$4N = 2\,(g(\Sigma) - 1)$	$2N + 2 = 2\,(g(\Sigma) - 1) + \#\text{poles}$	$N + 1 = \#\text{poles}$
n_z: # of critical points (zeroes of $d\tilde{z}$ over symmetry)	$\frac{1}{2}n'_z = 2N$	$\frac{1}{2}n'_z = N + 1$	$n'_z = N + 1$
matching $n_m = n_h = n_z$ and validity of WDVV equations	+	+	+

5 Discussion

In this paper we have presented a generic check for the WDVV equations occuring for the 6d SW prepotentials and their bi-elliptic generalization. In particular, this is one of the rarely known cases, when it has been done for a non-hyperelliptic curve, though our argumentation intensively used the symmetry of the curves of bi-elliptic family, reminding the hyperelliptic involution. It is also basically the first example, where the differentials with jumps show up explicitly. Together with non single-valued generating differential (2.1) their appearence is a consequence of presence of compact part in the target-space and large gauge transformations in corresponding SW theory.

In the most degenerate situation for the *perturbative* 6d prepotentials this fact has been already established in [17]. In this case the SW curve is basically equivalent to a meromorphic function (2.12) on complex torus of genus $g = 1$. Argumentation in the paper [17] was exploiting the parallels between this case and the Landau-Ginzburg (LG) model on torus [3], which is quite similar to the relation between the common LG model on on sphere with a single marked point (or just on complex plane) with a polynomial superpotential $W(X)$ and the *perturbative* 4d SW theory, see [2, 4, 11]. In the common LG model the WDVV equations follow from the polynomial ring modulo $W'(X)$, while in the perturbative SW theory the corresponding algebra is isomorphic to the completely equivalent ring of functions in the zeroes of $W'(X)$. The nontrivial fact for both these cases is existence of residue formulas for the *different* functions - the LG and (perturbative) SW prepotentials, which relate them to the structure constants of the isomorphic algebras.

Hence, it is not much surprising that similar parallels were noticed in [17] between the LG model on torus, where w from (2.12) can be considered as corresponding superpotential (i.e. one has to endow complex torus with a generating differential differential $dS^{\rm LG} = zdw$) and the perturbative 6d SW theory we have discussed above. Moreover, it is well-known, that the isomorphisms with LG *algebra* holds for the SW theory *beyond* the perturbative case, see [2, 11]. Moreover, it is easy to write down an explicit formula, expressing isomorphic structure constants through the third derivatives of the LG and *non-perturbative* SW prepotentials, and therefore relating two latter sets to each other; such formulas were essentially used in [21, 22, 11], and they completely cover the LG/SW duality discussed in [17] and even prolong it to the non-perturbative regime of gauge theory.

We have to stress also, that the fact of validity of WDVV equations itself does not depend on any extra requirements of [3] which are important for the class of solutions to WDVV equations related with the simplest topological string theories. In particular, we have already observed above, that appearence of "unity" in the basis of functions and corresponding constant-metric term in the prepotential is accidental: both these features do not survive in the non-perturbative and bi-elliptic cases, but the simple counting argument (1.5), based on residue formula (1.3) remains completely intact, as demonstrated in our tables.

Let us also point out some similarity between the families of curves we have considered in the paper with another distinguished family of the SW curves: the softly-broken $\mathcal{N} = 4$ theory described by the Calogero-Moser integrable systems [23] in spirit of the correspondence of [24]. The SW curves for this family also cover a complex torus, and as in the above case the prepotential, as function of only of the SW periods does not satisfy the WDVV equations [5, 11]. However, in the Calogero-Moser case the counting argument shows [11] that one has to add minimally $(N-2)$ parameters and differentials, contrarily to a single differential with jump and corresponding modulus of the torus, like it can be done for the whole bi-elliptic family.

Finally, it would be interesting to extend analysis of this paper to the curves of *double*-elliptic family, the simplest examples of which was considered in [18, 25]. Compare to bi-elliptic curves, in the double-elliptic case

$$\Sigma_{\rm dell}: \quad \mathcal{H}(z,\tilde{z}) = \alpha(z)\mathrm{cn}\left(\beta(z)\tilde{z}\,\middle|\,\frac{\alpha(z)}{\beta(z)}\tilde{k}\right) - E = 0, \tag{5.1}$$

with $\alpha^2 = 1 - c\wp(z) = 1 - cx$ and $\beta^2 = 1 - \tilde{k}^2 c\wp(z) = 1 - \tilde{k}^2 cx$, there is a non-trivial periodicity in $\tilde{z}$-variable, with the periods, depending themselves on z. For the double-elliptic curve (5.1) one gets

$$\delta(dS)|_{\delta\tilde{z}=0} = \delta\mathcal{H}\frac{d\tilde{z}}{\mathcal{H}'_z} \propto \frac{\delta\mathcal{H}dz}{\sqrt{x-U}} \propto \frac{\delta\mathcal{H}dx}{\sqrt{(x-U)\prod_{i=1}^3(x-e_i)}} \tag{5.2}$$

with $U = \frac{1-E^2}{c}$. Now, expression (5.2) can be thought of as a differential on a double cover of the (z,τ)-torus, ramified at two solutions of $x - U = 0$, or on a curve of genus $g\left(\Sigma_{\rm dell}\right) = 2$. A naive number of holomorphic differentials is then $n_z = 2\left(g\left(\Sigma_{dell}\right) - 1\right) = 2$, and this is not enough to analyze validity of the WDVV equations, even restricting number of moduli to $n_h = 2$. Note, however, that the derivation of (5.2) contains a lot of surprising cancellations, for example due to $\beta^2 - \tilde{k}^2\alpha^2 = \mathrm{const}$, and it means that one can look for the nontrivial double-elliptic solutions to the WDVV equations for the families of curves, similar to (5.1). We are going to return to this problem elsewhere.

Acknowledgements

We are grateful to I. Krichever and A. Rosly for discussions. The work was partially supported by RFBR grants 04-01-00642 (A.Mar.), 04-01-16646 (A.Mir.), 04-02-16880 (A.Mor.), the grants for support of Scientific Schools LSS-4401.2006.2 (A.Mar.) and 8004.2006.2 (A.Mir., A.Mor.), the NWO projects

047.017.015 (A.Mar.) and 047.011.2004.026 (A.Mir., A.Mor.) and the ANR-05-BLAN-0029-01 project "Geometry and Integrability in Mathematical Physics". H.W.B. and A.Mar. would like to thank the the Max Planck Institute for Mathematics, where this work was completed.

References

[1] E. Witten, Nucl. Phys. **B340** (1990) 281;
R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. **B352** (1991) 59.

[2] A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. **B389** (1996) 43, [arXiv:hep-th/9607109].

[3] B. Dubrovin, arXiv:hep-th/9407018.

[4] A. Marshakov, A. Mironov and A. Morozov, Mod. Phys. Lett. A **12** (1997) 773 [arXiv:hep-th/9701014].

[5] A. Marshakov, A. Mironov and A. Morozov, Int. J. Mod. Phys. A **15** (2000) 1157 [arXiv:hep-th/9701123].

[6] A. Morozov, Phys.Lett. **B427** (1998) 93-96 [arXiv:hep-th/9711194];
A. Mironov and A. Morozov, Phys.Lett. **B424** (1998) 48-52 [arXiv:hep-th/9712177].

[7] B. de Wit and A. Marshakov, Theor. Math. Phys. **129** (2001) 1504, [Teor. Mat. Fiz. **129** (2001) 230], [arXiv:hep-th/0105289].

[8] I. Krichever, Commun. Pure. Appl. Math. **47** (1992) 437, hep-th/9205110.

[9] A. Mironov, the review in the second book of [13], [arXiv:hep-th/9903088].

[10] A.Boyarsky, A.Marshakov, O.Ruchayskiy, P.Wiegmann and A.Zabrodin, Phys.Lett. **515B** (2001) 483-492, hep-th/0105260

[11] A. Marshakov, hep-th/0108023;
Theor.Math.Phys. **132** (2002) 895, [arXiv:hep-th/0201267].

[12] N.Seiberg and E.Witten, Nucl. Phys. **B426** (1994) 19; arXciv:hep-th/9407087.

[13] A. Marshakov, *Seiberg-Witten Theory and Integrable Systems*, (World Scientific, Singapore, 1999);
H.W. Braden and I. Krichever (Eds.), *Integrability: The Seiberg-Witten and Whitham Equations*, (Gordon and Breach, 2000)

[14] A. Gorsky and A. Mironov, Integrable Many-Body Systems and Gauge Theories, arXiv:hep-th/0011197

[15] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, arXiv:hep-th/9604078.

[16] A. Marshakov and A. Mironov, Nucl. Phys. B **518** (1998) 59 [arXiv:hep-th/9711156].

[17] A. Riley and I. Strachan, arXiv:math-ph/0511048.

[18] H.W. Braden, A. Marshakov, A. Mironov and A. Morozov, Nucl. Phys. **B573** (2000) 553-572, [arXiv:hep-th/9906240]

[19] V. Inozemtsev, Comm. Math. Phys. **121** (1989) 629.

[20] H. Itoyama and A. Morozov, Nucl. Phys. B **491** (1997) 529 [arXiv:hep-th/9512161].

[21] K. Ito and S.-K. Yang, Phys. Lett. **B433** (1998) 56-62, [arXiv:hep-th/9803126].

[22] L. Hoevenaars and R. Martini, Lett. Math. Phys. **57** (2001) 175-183, [arXiv:hep-th/0102190].

[23] R.Donagi and E.Witten, Nucl.Phys. **B460** (1996) 299; [arXiv:hep-th/9510101];
E.Martinec, Phys.Lett. **B367** (1996) 91-96; [arXiv:hep-th/9510204];
A.Gorsky and A.Marshakov, Phys.Lett. **B375** (1996) 127, [arXiv:hep-th/9510224];
H.Itoyama and A.Morozov, Nucl.Phys. **B477** (1996) 855; [arXiv:hep-th/9511126]; [arXiv:hep-th/9601168];
E. D'Hoker and D. H. Phong, Nucl. Phys. B **513** (1998) 405 [arXiv:hep-th/9709053].

[24] A.Gorsky, I.Krichever, A.Marshakov, A.Mironov and A.Morozov, Phys. Lett. **B355** (1995) 466; [arXiv:hep-th/9505035].

[25] A. Mironov and A. Morozov, Phys.Lett. **B475** (2000) 71-76, [arXiv:hep-th/9912088];
A. Mironov and A. Morozov, Double Elliptic Systems: Problems and Perspectives, arXiv:hep-th/0001168.