Convolutions of Hilbert modular forms, motives, and p-adic L-functions

by

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derivatives of order $\leq k-2$ vanish on $\mathbb{R}^{2m+d}$. When M is real–analytic, we may take Φ to be holomorphic. In particular, $D\Phi$ is $\mathbb{C}$–linear at each point of $\mathbb{R}^{2m+d}$ near the origin. We let $\Sigma \subset \mathbb{C}^n \times \mathbb{C}^{d\times m}$ be the local image of Φ near $\Phi(0) = (0, T_0^{\mathbb{C}}M)$. In the real–analytic case Σ is the usual complexification of $\hat{M}$.

Next we want to find a nonempty wedge

$$\mathscr{V}_0 = (\mathbb{R}^{2m+d} + i\Gamma_0) \cap V_0 \subset \mathbb{C}^{2m+d}$$

with edge $\mathbb{R}^{2m+d} \cap V_0$ (V_0 being a small neighborhood of the origin) such that

(16) $$\Phi(\mathscr{V}_0) \subset \mathscr{W} .$$

Let $\Gamma \subset \mathbb{R}^d$ be the cone determining the wedge $\mathscr{W}$. Choose an arbitrary finer cone $\Gamma' < \Gamma$ and let $\Gamma_0 \subset \mathbb{R}^{2m+d}$ be a cone contained in a small conical neighborhood of $\{0\}^{2m} \times \Gamma'$, satisfying $\Gamma_0 \cap (\{0\}^{2m} \times \mathbb{R}^d) = \Gamma'$. We claim that the inclusion (16) holds provided that Γ_0 and V_0 are chosen sufficiently small. To see this, notice that for each $t \in \mathbb{R}^d$ the vector

$$i\, D_t\Phi(0) = i \sum_{j=1}^{d} t_j\, \partial\Phi/\partial u_j(0)$$

belongs to $T_{\Phi(0)}\Sigma$. Since at the origin φ contains no quadratic terms except the Levi form, a simple calculation shows

$$i\, D_t\, \Phi(0) = (0, it, 0) .$$

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Introduction. Let p be a prime number. One of the purposes of this paper is to describe some p-adic properties of the special values of the Rankin convolution

$$L(s, \mathbf{f}, \mathbf{g}) = \sum_{\mathfrak{n}} C(\mathfrak{n}, \mathbf{f})\, C(\mathfrak{n}, \mathbf{g}) \mathcal{N}(\mathfrak{n})^{-s} \tag{0.1}$$

attached to two Hilbert modular forms $\mathbf{f}$, $\mathbf{g}$ over a totally real field F of degree $n = [F : \mathbf{Q}]$, where $C(\mathfrak{n}, \mathbf{f})$, $C(\mathfrak{n}, \mathbf{g})$ are normalized "Fourier coefficients" of $\mathbf{f}$ and $\mathbf{g}$, indexed by integral ideals $\mathfrak{n}$ of the maximal order $\mathcal{O}_F \subset F$. We suppose that $\mathbf{f}$ is a primitive cusp form of vector weight $k = (k_1, \cdots, k_n)$, and $\mathbf{g}$ a primitive cusp form of weight $l = (l_1, \cdots, l_n)$, such that

$$\text{for each } i \quad \text{either } k_i < l_i, \quad \text{or } l_i < k_i, \tag{0.2}$$

and the following parity conditions are satisfied:

$$k_1 \equiv k_2 \equiv \cdots \equiv k_n \bmod 2, \tag{0.3}$$

and

$$l_1 \equiv l_2 \equiv \cdots \equiv l_n \bmod 2. \tag{0.4}$$

Let $\mathfrak{c}(\mathbf{f}) \subset \mathcal{O}_F$ denote the conductor and ψ the character of $\mathbf{f}$ and $\mathfrak{c}(\mathbf{g})$, ω denote the conductor and the character of $\mathbf{g}$ ($\psi, \omega : \mathbf{A}_F^\times / F^\times \to \mathbf{C}^\times$ being Hecke characters of finite order, where $\mathbf{A}_F^\times / F^\times$ is the idele class group of F).

We formulate these p-adic properties in framework of the theory of motives over a number field F. Following a suggestion of A.A.Beilinson, we state first in 1.7 a refined form of the Deligne's conjecture on critical values of the corresponding L-functions twisted with Hecke characters $\chi : \mathbf{A}_F^\times / F^\times \to \mathbf{C}^\times$ of finite order. We formulate also in 1.8 a general conjecture on p-adic L-functions attached to motives over F. In case $F = \mathbf{Q}$ the above two conjectures were stated in [Co-PeRi]. Conditions (0.3) and (0.4) are essential for a geometric interpretation of $\mathbf{f}$ and $\mathbf{g}$ in terms of motives over F which is given below.

We give a description of the type of motives, which should correspond to a primitive Hilbert cusp form $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$, where the components k_i of the vector weight $k = (k_1, \cdots, k_n)$ of $\mathbf{f}$ are indexed by real embeddings $\sigma_i : F \to \mathbf{R}$. We assume that the L-function $L(s, \mathbf{f})$ of $\mathbf{f}$ is normalized so that

$$L(s, \mathbf{f}) = \sum_{\mathfrak{n}} C(\mathfrak{n}, \mathbf{f}) \mathcal{N}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} \left(1 - C(\mathfrak{p}, \mathbf{f}) \mathcal{N}(\mathfrak{p})^{-s} + \psi(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{k_0 - 1 - 2s}\right)^{-1},$$

where $k_0 = \max_i k_i$. If $\tau \in \mathrm{Aut}\mathbf{C}$ then we have $\mathbf{f}^\tau \in \mathcal{M}_{k^\tau}(\mathfrak{c}, \psi^\tau)$, so that the coefficient field T of $\mathbf{f}$ must contain the field F^k generated by products $\prod_{i=1}^n (x^{\sigma_i})^{k_i}, x \in F$, which can be characterized as the fixed field for all automorphisms $\tau \in \mathrm{Aut}\mathbf{C}$ such that $k^\tau = k$. Note that for each τ the vector k^τ is defined by means of the formula .

$$\prod_{i=1}^{n}(x^{\sigma_i})^{k_i^\tau} = \left(\prod_{i=1}^{n}(x^{\sigma_i})^{k_i}\right)^\tau .$$

In particular, if $k_1 = \cdots = k_n$ then $F^k = \mathbf{Q}$, and from the viewpoint of the theory of canonical models, the field F^k will coincide essentially with the reflex field of the Hodge structure of a motive $M(\mathbf{f})$ attached to $\mathbf{f}$.

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§1. Motives

1.1. By a motive M over a number field F with coefficients in (another) number field T we mean a collection of the following objects:

$$M_{B,\sigma} = M_\sigma,\ M_{DR},\ M_\lambda,\ I_{\infty,\sigma},\ I_{\lambda,\sigma},$$

where $\sigma : F \to \mathbf{C}$ runs over the set J_F of all complex embeddings of F, M_σ are vector spaces over T of dimension d (Betti realization of M relative to the complex embedding σ), which for real $\sigma \in J_F$ are endowed with a T-rational involutions ρ_σ; M_{DR} is a free module of rank d over the tensor product $T \otimes F$ (de Rham realization of M), which posesses a decreasing filtration $\{F^i_{DR}(M) \subset M_{DR} | i \in \mathbf{Z}\}$ of free $T \otimes F$-submodules; M_λ is a vector space of degree d over T_λ, a completion of T at an arbitrary finite place λ of the coefficient field T (λ-adic realization of M), which is a Galois module over $\mathrm{Gal}_F = \mathrm{Gal}(\overline{F}/F)$ so that we have a compatible system of λ-adic representations denoted by

$$r_{M,\lambda} = r_\lambda : \mathrm{Gal}(\overline{F}/F) \to GL(M_\lambda).$$

Also,

$$I_{\infty,\sigma} : M_\sigma \otimes \mathbf{C} \xrightarrow{\sim} M_{DR} \otimes_{F,\sigma} \mathbf{C}$$

is the (complex) comparison isomorfism of $T \otimes \mathbf{C}$-modules, and

$$I_{\lambda,\sigma} : M_\sigma \otimes T_\lambda \xrightarrow{\sim} M_\lambda$$

is the λ-adic comparison isomorphism of T_λ-vector spaces. It is assumed in the notation, that the complex vector space $M_\sigma \otimes \mathbf{C}$ is decomposed in the Hodge bigraduation

$$M_\sigma \otimes \mathbf{C} \cong \oplus_{i,j} M_\sigma^{i,j},$$

such that

$$\rho_\sigma(M_\sigma^{i,j}) = M_\sigma^{j,i},$$

provided σ is real, and

$$I_{\infty,\sigma}(\oplus_{i'\geq i} M_\sigma^{i',j}) = F_{DR}^i(M) \otimes_{F,\sigma} \mathbf{C}.$$

We assume that $i + j = w$ for some fixed integer w which is called in that case the weight of the motive M. The number d is called the rank of M.

The important propery of a motive M with coefficients in T is that T-structure of M is closely related to its Hodge structure since all of the terms $F_{DR}^i \otimes_{F,\sigma} \mathbf{C}$ of the Hodge filtration are free $T \otimes F \otimes \mathbf{C}$-modules. Recall that $T \otimes F$ is a product of fields F_i, which bijectively correspond to orbits under the action of $\mathrm{Gal}_{\mathbf{Q}}$ on the set

$$J_T \times J_F = \{(\tau,\sigma) | \tau \in J_T, \sigma \in J_F\}.$$

Let

$$M_{\sigma,\tau} \subset M_\sigma \otimes_{\mathbf{Q}} \mathbf{C}, \quad M_{DR,\sigma,\tau} \subset M_{DR} \otimes_{F,\sigma} \mathbf{C}$$

be the subspaces, on which T acts via $\tau \in J_T$, then the isomorphism $I_{\infty,\sigma}$ induces the isomorphism

$$I_{\infty,\sigma,\tau} : M_{\sigma,\tau} \xrightarrow{\sim} M_{DR,\sigma,\tau}.$$

Typical examples of motives come from algebraic varieties.

There are various definitions of motives (by means of correspondences, by means of absolute Hodge cycles and abelian varieties of CM-type etc., see [De3], [Bl1], [Bl2])

1.2. L-functions of motives. By definition, the L-function $L(M,s)$ of a motive M takes values in $T \otimes \mathbf{C} \cong \prod_\tau \mathbf{C}_\tau$, and it is defined as the Euler product

$$L(M,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(M, \mathcal{N}\mathfrak{p}^{-s})^{-1},$$

where

$$L_{\mathfrak{p}}(M,X)^{-1} = \det(1 - X \cdot r_\lambda(Fr_{\mathfrak{p}}^{-1}) | M_\lambda^{I_{\mathfrak{p}}}) = (1 - \alpha^{(1)}(\mathfrak{p})X) \cdots (1 - \alpha^{(d)}(\mathfrak{p})X),$$

and $Fr_{\mathfrak{p}}$ is the Frobenius element at $\mathfrak{p}$, defined modulo the inertia group $I_{\mathfrak{p}}$ (so that $Fr_{\mathfrak{p}}^{-1}$ is the "geometric Frobenius"), the upper index $I_{\mathfrak{p}}$ denotes the subspace of elements, pointwise fixed by the inertia group. We make here a standard hypothesis that coefficients of $L_{\mathfrak{p}}(M,X)^{-1} = 1 + A_1(\mathfrak{p})X + \cdots + A_d(\mathfrak{p})X^d$ belong to T, and we regard this polynomial over the ring $T \otimes \mathbf{C}$ so that

$$L_{\mathfrak{p}}(M,X)^{-1} = (L_{\mathfrak{p}}^{(\tau)}(M,X)^{-1}),\ L_{\mathfrak{p}}^{(\tau)}(M,X)^{-1} = 1 + A_1(\mathfrak{p})^\tau X + \cdots + A_d^\tau(\mathfrak{p})X^d.$$

According to the Deligne's theorem on Weil's conjectures, for a motive M of weight w and rank d, coming from algebraic varieties, all of the complex absolute values $|\alpha^{(i)}(\mathfrak{p})|$ are equal to $\mathcal{N}\mathfrak{p}^{w/d}$.

Properties of motives and their L-functions:

(a) *Dual motive* $\check{M}$ is defined, if we replace all realizations of M by their duals. In terms of L-functions,

$$L_{\mathfrak{p}}(\check{M},X)^{-1} = (1 - \alpha^{(1)}(\mathfrak{p})^{-1}X) \cdots (1 - \alpha^{(d)}(\mathfrak{p})^{-1}X),$$

where

$$L_{\mathfrak{p}}(M,X)^{-1} = (1-\alpha^{(1)}(\mathfrak{p})X)\cdots(1-\alpha^{(d)}(\mathfrak{p})X).$$

(b) *Direct sum* $M_1 \oplus M_2$. In terms of L-functions we have that

$$L(M_1 \oplus M_2, s) = L(M_1, s)L(M_2, s).$$

(c) *Tensor product* $M_1 \otimes M_2$. The corresponding L-function has the form of the convolution

$$L_{\mathfrak{p}}(M_1 \otimes M_2, X) = \prod_{i,j}(1-\alpha^{(i)}(\mathfrak{p})\beta^{(j)}(\mathfrak{p})X),$$

where $\alpha^{(i)}(\mathfrak{p})$, $\beta^{(j)}(\mathfrak{p})$ are the invers roots of the characteristic $\mathfrak{p}$-polynomials for M_1, M_2.

(d) *Restriction of scalars.* For a subfield $k \subset F$ one can define a motive $M' = R_{F/k}(M)$ over k (with the same coefficient field) such that $L(M', s) = L(M, s)$, and the λ-adic Galois representation $r_{M',\lambda}$ of Gal_k is induced from the representation $r_{M,\lambda}$ of the subgroup $\mathrm{Gal}_F \subset \mathrm{Gal}_k$.

(e) *Extension of the ground field.* For an extension K/F one can define a motive $M'' = M \otimes_F K$ over K with coefficients in T such that the Galois representation $r_{M'',\lambda}$ Gal_K for M'' coincides with the restriction of $r_{\mathrm{M},\lambda}$ to the subgroup $\mathrm{Gal}_K \subset \mathrm{Gal}_F$.

Examples of motives and their L-functions.

(a) Cyclotomic (Tate) motive $F_{\mathrm{cyc}}(m)$ over F. Let us consider the cyclotomic character

$$\tilde{\psi}_{\mathrm{cyc}} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})^{\mathrm{ab}} \to \hat{\mathbf{Z}}^{\times},$$

then, for a number field F, define

$$\tilde{\psi}_{\mathrm{cyc}} : \mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}} \to \hat{\mathbf{Z}}^{\times}$$

by restriction. For any $m \in \mathbf{Z}$ define the cyclotomic motive $F_{\mathrm{cyc}}(m)$ over F with coefficients in $\mathbf{Q}$ in terms of its realizations:

(i) $F_{\mathrm{cyc}}(m)_{DR} = (2\pi i)^{-m}F$;

(ii) $F_{\mathrm{cyc}}(m)_\sigma = \mathbf{Q}; F_{\mathrm{cyc}}(m)_\sigma^{-n,-n} = F_{\mathrm{cyc}}(m)_{DR} \otimes \mathbf{C}$;

(iii) $F_{\mathrm{cyc}}(m)_p = \mathbf{Q}_p$ with the action of $\mathrm{Gal}(\overline{F}/F)$ given by $\tilde{\psi}_{\mathrm{cyc}}$;

(iv) $I_{\infty,\sigma}$ is the identification $(2\pi i)^{-m}F \otimes_{F,\sigma} \mathbf{C} = \mathbf{Q} \otimes \mathbf{C} = \mathbf{C}$;

(v) $I_{p,\sigma}$ is the identification $\mathbf{Q} \otimes \mathbf{Q}_p = \mathbf{Q}_p$.

In this case we have

$$L(F_{\mathrm{cyc}}(n), s) = \zeta_F(s+n)$$

(the Dedekind zeta function, shifted by n).

(b) Motive $M = H^1(A)$ of an abelian variety A over F has the Hodge structure of the type $M^{0,1} \oplus M^{1,0}$ and the corresponding Galois representation is (contragredient) to the representation of Gal_F on points of finite order.

(c) Motive $M(f)$ of a primitive cusp form (see [De3], [Ja], [Sch])

$$f = \sum_{n=1}^{\infty} a_n e(nz) \in \mathcal{S}_k(N,\psi) \quad (e(z) = \exp(2\pi i z))$$

of conductor N, weight k, and Dirichlet character ψ mod N is defined over $\mathbf{Q}$ and has coefficients in the field $\mathbf{Q}(f)$ generated by Fourier coefficients a_n. Its Hodge structure is of the type $M^{0,k-1} \oplus M^{k-1,0}$, $d = 2$, $w = k-1$, and

$$L(M(f), s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p \left(1 - a_p p^{-s} + \psi(p) p^{k-1-2s}\right)^{-1}.$$

1.3. Twist with a Hecke character of finite order. Let $\chi : \mathbf{A}_F^\times / F^\times \to \mathbf{C}^\times$ be a Hecke character of finite order, and

$$L(s, \chi) = \prod_{\mathfrak{p} \not| \mathfrak{c}} (1 - \chi(\mathfrak{p}) \mathcal{N}\mathfrak{p}^{-s})^{-1}$$

be its L-function, where $\chi(\mathfrak{p})$ denotes the value of χ at a uniformizing element for $\mathfrak{p}$, and the product is taken over all prime ideals $\mathfrak{p}$, not dividing the conductor $\mathfrak{c}$ of χ. Then there exists a motive $M = [\chi]$ of rank d, weight 0 over F with values in a field T, containing the field $\mathbf{Q}(\chi)$ generated by values of χ, such that $L(M, s) = L(s, \chi)$. In order to describe realizations of $[\chi]$, we denote by the same symbol $[\chi]$ a 1-dimensional vector space over T with the action of $\mathrm{Gal}(F^{ab}/F)$ given by the character $\tilde{\chi} : \mathrm{Gal}(\overline{F}/F)^{ab} \to T^\times$ which corresponds to χ by class field theory. The λ-adic representation $r_{M,\lambda} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}([\chi]_\lambda)$ is then determined by the character $\tilde{\chi}$. For each $\sigma \in J_F$, the Betti realization $[\chi]_\sigma$ is the 1-dimensional vector space over T such that for real σ the involution ρ_σ acts as $\varepsilon_\sigma = \chi(\rho_\sigma) = \pm 1$, where we denote by the same symbol ρ_σ the element of $\mathrm{Gal}(\overline{F}/F)$ given by the complex conjugation over the image $\sigma(F) \subset \mathbf{C}$. Also, we have that $[\chi]_\sigma \otimes \mathbf{C} = [\chi]_\sigma^{0,0}$.

Let F be totally real, and $\mathrm{Sgn}_F \subset F_\infty^\times$ be the *group of signs* of F (elements of order 2). If we denote by χ_∞ the restriction of χ to $F_\infty^\times = (F \otimes R)^\times$ then for a totally real F we have $\mathrm{sgn}(\chi_\infty) = (\varepsilon_\sigma)_\sigma \in \mathrm{Sgn}_F$. De Rham realization $[\chi]_{DR}$ is a one dimentional $T \otimes_{\mathbf{Q}} F$-module, and if we fix $\sigma \in J_F$ then $[\chi]_{DR}$ can be regarded as the submodule of

$$[\chi]_{DR} \otimes_{F,\sigma} \mathbf{C} \cong T \otimes_{\mathbf{Q}} \mathbf{C},$$

generated by the Gauss sum (see [De3])

$$G(\chi) = \sum_{x \in \mathfrak{d}^{-1}\mathfrak{c}^{-1}/\mathfrak{d}^{-1}} \chi_\infty(x) \chi((x)\mathfrak{c}\mathfrak{d})\, 1_{\sigma,\chi} \otimes_{\mathbf{Q}} e(\mathrm{Tr}(x)) \in T \otimes \mathbf{C};$$

here $n = [F : \mathbf{Q}]$, $\mathfrak{d}$ is the different of F, $\mathrm{Tr}(x) = x^{\sigma_1} + \cdots x^{\sigma_n}$, $e(z) = \exp(2\pi i z)$, and we denote by $1_{\sigma,\chi}$ a basis of the one dimentional T-vector space $[\chi]_\sigma$.

Moreover, the σ-comparison isomorphism (of $T \otimes F$-modules)

$$I_{\infty,\sigma} : [\chi]_\sigma \otimes \mathbf{C} \xrightarrow{\sim} [\chi]_{DR} \otimes_{F,\sigma} \mathbf{C}$$

is an isomorphism of $T \otimes F$-modules, such that F acts on the second factors via σ, and $I_{\infty,\sigma}$ becomes the identity map $T \otimes_{\mathbf{Q}} \mathbf{C} \to T \otimes_{\mathbf{Q}} \mathbf{C}$ if we use the above identification. We have that

$$I = \oplus_\sigma I_\sigma : [\chi] \otimes_{\mathbf{Q}} \mathbf{C} = \oplus_\sigma [\chi] \otimes_{F,\sigma} \mathbf{C} \xrightarrow{\sim} [\chi]_{DR} \otimes_{\mathbf{Q}} \mathbf{C}$$

is the isomorphism of free one dimentional $T \otimes F \otimes \mathbf{C}$-modules. Also, the λ-adic comparison isomorphism

$$I_{\lambda,\sigma} : [\chi]_\sigma \otimes T_\lambda \xrightarrow{\sim} [\chi]_\lambda$$

reduces to tensorization by T_λ over T.

Let $M[\chi] = M \otimes_F [\chi]$ denote the twist of a motive M over F with a Hecke character of finite order χ over F. Realizations of $M[\chi]$ are given by

$$M[\chi]_\sigma = M_\sigma \otimes_T [\chi]_\sigma,\ M[\chi]_{DR} = M_{DR} \otimes_{T\otimes F} [\chi]_{DR},\ M[\chi]_\lambda = M_\lambda \otimes_{T_\lambda} [\chi]_\lambda.$$

Without loss of generality we may regard M and $[\chi]$ as motives with the same coefficient field $T(\chi)$ generated over T by values of χ.

1.4. Periods. The following period construction is due to Deligne [De3] (see also [Bl1], [Bl2]. Consider the T-linear automorphism $\rho_\sigma : M_\sigma \xrightarrow{\sim} M_\sigma$ defined for real $\sigma \in J_F$ as above, and put

$$M_\sigma = M_\sigma^+ \oplus M_\sigma^-,\quad M_\sigma^\pm = \mathrm{Ker}(\rho_\sigma \mp 1 : M_\sigma \to M_\sigma).$$

Assume that ρ_σ acts on $M_\sigma^{w/2,w/2}$ by a scalar $\varepsilon(\sigma) = \pm 1$ if $M_\sigma^{w/2,w/2} \neq 0$. Define

$$\begin{aligned}
F_{DR}^- M &= F_{DR}^{w/2} M \quad (w \in 2\mathbf{Z}, \varepsilon = -1)\\
F_{DR}^+ M &= F_{DR}^{w/2} M \quad (w \in 2\mathbf{Z}, \varepsilon = 1)\\
F_{DR}^- M &= F_{DR}^{w/2+1} M \quad (w \in 2\mathbf{Z}, \varepsilon = 1)\\
F_{DR}^+ M &= F_{DR}^{w/2+1} M \quad (w \in 2\mathbf{Z}, \varepsilon = -1)\\
F_{DR}^+ M &= F_{DR}^- M \ = F_{DR}^{(w+1)/2} M \quad (w \text{ odd})
\end{aligned}$$

and set

$$M_{DR}^\pm \ = \ M_{DR}/F_{DR}^\pm M$$

(this is a free factormodule of the free $T \otimes F$-modules M_{DR} which is well defined only in case when all of the signes $\varepsilon(\sigma)$) are equal, say, to a fixed sign ε independent of σ, and in this case $M_{DR}^\pm$ depends only on ε). Put

$$d = \dim_T M_\sigma,\quad d_\sigma^\pm = \dim_T M_\sigma^\pm = \dim_{T\otimes F} M_{DR}^\pm.$$

Define

$$I_{\infty,\sigma}^\pm : M_\sigma^\pm \otimes \mathbf{C} \to M_{DR}^\pm \otimes_{F,\sigma} \mathbf{C},$$

to be the composition

$$M_\sigma^\pm \otimes \mathbf{C} \to M_\sigma \otimes \mathbf{C} \overset{I_{\infty,\sigma}}{\xrightarrow{\sim}} M_{DR} \otimes_{F,\sigma} \mathbf{C} \to M_{DR}^\pm \otimes_{F,\sigma} \mathbf{C},$$

and note that the $T \otimes F$ isomorphisms $I_{\infty,\sigma}$ and $I_{\infty,\sigma}^\pm$ give rise to the following isomorphisms of free $T \otimes F \otimes \mathbf{C}$-modules:

$$I_\infty = \oplus_\sigma I_{\infty,\sigma} :\ \oplus_\sigma M_\sigma \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{DR} \otimes_{\mathbf{Q}} \mathbf{C} = \oplus_\sigma M_{DR} \otimes_{F,\sigma} \mathbf{C},$$

and

$$I_\infty^\pm = \oplus_\sigma I_{\infty,\sigma}^\pm : \ \oplus_\sigma M_\sigma^\pm \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{DR}^\pm \otimes_{\mathbf{Q}} \mathbf{C} = \oplus_\sigma M_{DR}^\pm \otimes_{F,\sigma} \mathbf{C}.$$

Note that these isomorphisms describe the corresponding comparison isomorphisms for the motive $R_{F/\mathbf{Q}}M$ over $\mathbf{Q}$ obtained from M by restriction of scalars.

Let

$$c^\pm(\sigma) = c^\pm(\sigma; M) = \det(I_{\infty,\sigma}^\pm) \in (T \otimes \mathbf{C})^\times,$$

and

$$\delta(\sigma) = \delta(\sigma; M) = \det(I_{\infty,\sigma}) \in (T \otimes \mathbf{C})^\times,$$

where the right-hand sides denote the determinants of matrices representing the maps relative to $T \otimes F$-rational bases of the sourse and target. Note that the quantities $c^\pm(\sigma)$, $\delta(\sigma) \in (T \otimes \mathbf{C})^\times$ are defined only modulo the multiplcative subgroup $(T \otimes \sigma(F))^\times$ of $(T \otimes \mathbf{C})^\times$.

However, the quantities

$$c^\pm = c^\pm(M) = \prod_\sigma c^\pm(\sigma), \quad \delta = \delta(M) = \prod_\sigma \delta(\sigma) \in (T \otimes \mathbf{C})^\times$$

are defined already modulo the multiplicative subgroup $T^\times \subset (T \otimes \mathbf{C})^\times$, and they are called *periods* of M. Using the above identification $T \otimes \mathbf{C} \cong \prod_\tau \mathbf{C}_\tau$ we may describe these periods as certain vectors

$$c^\pm(\sigma) = (c^\pm(\sigma; \tau, M))_\tau, \ \delta(\sigma) = (\delta(\sigma; \tau, M))_\tau \ \in (T \otimes \mathbf{C})^\times$$

in terms of τ-invariant subspaces.

Let

$$M_{\sigma,\tau}^\pm \subset M_\sigma^\pm \otimes_{\mathbf{Q}} \mathbf{C}, \quad M_{DR,\sigma,\tau}^\pm \subset M_{DR} \otimes_{F,\sigma}^\pm \mathbf{C}$$

be the subspaces, on which T acts via $\tau \in J_T$, then $I_{\infty,\sigma}^\pm$ and $I_{\infty,\sigma}$ induce the isomorphisms

$$I_{\infty,\sigma,\tau}^\pm : M_{\sigma,\tau}^\pm \xrightarrow{\sim} M_{DR,\sigma,\tau}^\pm$$

and

$$I_{\infty,\sigma,\tau} : M_{\sigma,\tau} \xrightarrow{\sim} M_{DR,\sigma,\tau}.$$

Let

$$c^\pm(\sigma; \tau, M) = \det(I_{\infty,\sigma,\tau}^\pm)$$

and

$$\delta(\sigma; \tau, M) = \det(I_{\infty,\sigma,\tau}),$$

where the right-hand sides also denote the determinants of matrices representing the maps relative to $T \otimes F$-rational bases (with F acting via σ on $\mathbf{C}$). Then

$$c^\pm(\sigma) = (c^\pm(\sigma; \tau, M))_\tau, \ \delta(\sigma) = (\delta(\sigma; \tau, M))_\tau \ \in (T \otimes \mathbf{C})^\times.$$

We shall often fix τ and M and allow us a convenient abuse of notation by writing

$$c^\pm(\sigma) = c^\pm(\sigma; \tau, M), \ \delta(\sigma) = \delta(\sigma; \tau, M) \in \mathbf{C}^\times \bmod (\sigma(F^\times)\tau(T^\times)).$$

Example. Let $[\chi]$ be the motive over a totally real field F attached to a Hecke character χ of finite order, then the periods $c^{\pm}([\chi])$ can be defined only in case when $\mathrm{sign}\chi$ is a scalar, i.e. $\varepsilon_\sigma(\chi) = \varepsilon(\chi)$ does not depend on σ. Then from the above description of realizations of $[\chi]$ we now deduce that that

$$c^{\varepsilon}([\chi]) = \delta([\chi]) = G(\chi)D_F^{1/2},\ c^{-\varepsilon}([\chi]) = 1.$$

Indeed, $[\chi]_\sigma^{-\varepsilon} = [\chi]_{DR}^{-\varepsilon} = \{0\}$ so that $I_{\infty,\sigma}^{-\varepsilon}$ are all trivial, proving the second equality. The determinant of the isomorphism

$$I_\infty^{\varepsilon} = I_\infty : \oplus_\sigma[\chi]_\sigma \otimes \mathbf{C} = \oplus_\sigma(T\otimes\mathbf{C})_\sigma \xrightarrow{\sim} [\chi]_{DR}\otimes\mathbf{C} = \oplus_\sigma[\chi]_{DR}\otimes_{F,\sigma}\mathbf{C},$$

of free $T\otimes\mathbf{C}$-modules is easy to compute using the standard T-rational base of $\oplus_\sigma(T\otimes\mathbf{C})_\sigma$, and the T-rational base of $[\chi]_{DR}\otimes\mathbf{C} = \oplus_\sigma[\chi]_{DR}\otimes_{F,\sigma}\mathbf{C}$ given by

$$\sum_\sigma G(\chi)\otimes(1\otimes x_i^\sigma) = \sum_{x\in\mathfrak{d}^{-1}\mathfrak{c}^{-1}/\mathfrak{d}^{-1}} \chi_\infty(x)\chi((x)\mathfrak{c}\mathfrak{d})\,1_{\sigma,\chi}\otimes_{\mathbf{Q}} x_i^\sigma e(\mathrm{Tr}(x)) \in (T\otimes\mathbf{C})^\times$$

$(i = 1,\cdots,n)$; here $\{x_i\}_{i=1}^n$ is a $\mathbf{Q}$-rational base of F over $\mathbf{Q}$ so that $\det((x_i^\sigma)_{i=1}^n) \sim D_F^{1/2}$, and the first equality follows.

Periods of a twisted motive. Let $M[\chi] = M\otimes_F[\chi]$ denote the twist of a motive M over a totally real field F with a Hecke character of finite order χ, and let $\varepsilon(\chi) = (\varepsilon_\sigma(\chi))_\sigma \in \mathrm{Sgn}_F$. Then

$$M[\chi]_\sigma = M_\sigma\otimes_T[\chi]_\sigma,\ M[\chi]_{DR} = M_{DR}\otimes_{T\otimes F}[\chi]_{DR},$$

and we can use the above description of the realizations of $[\chi]$ to compute the periods $c^{\pm}(M[\chi])$. First we suppose that $M_\sigma^{w/2,w/2} \neq \{0\}$ and ρ acts by a scalar $\varepsilon = \pm1$ which is independent of σ. Then the periods $c^{\pm}(M[\chi])$ may only be defined when $\mathrm{sign}\chi$ is a scalar, i.e. $\varepsilon_\sigma(\chi) = \varepsilon(\chi)$ does not depend on σ in which case we have that

$$c^{\pm}(M[\chi]) = G(\chi)^{-d^{-\varepsilon}} c^{\pm\varepsilon(\chi)}(M).$$

Next, assume that $M_\sigma^{w/2,w/2} = 0$, and χ has an arbitrary sign. Then the rank d of M is even, and we obtain from the above that

$$c^{\varepsilon}(M[\chi]) \sim G(\chi)^{-d/2}\prod_\sigma c(\sigma,M)^{\varepsilon_\sigma(\chi)\varepsilon} \bmod (T(\chi)\otimes F')^\times,$$

where $\varepsilon = \pm$, and $F' = \prod_\sigma \sigma(F)$ is a subfield of $\mathbf{C}$. However, we shall show using L-functions that the last equivalence also may make sense modulo $T(\chi)^\times$.

Example. For the cyclotomic Tate motive $F_{\mathrm{cyc}}(m)$ $(m\in\mathbf{Z})$, and a motive M over F we define $M(m) = M\otimes_F F(m)$. Then

$$c^{\pm}(\sigma;\tau,M(m)) \sim (2\pi i)^{d_\sigma^{\pm} m} c^{\pm(-1)^m}(\sigma;\tau,M)\ \ (\mathrm{mod}(\sigma(F^\times)\tau(T^\times))$$

and

$$\delta(\sigma;\tau,M(m)) \sim (2\pi i)^{dm}\delta(\sigma;\tau,M)\ \ (\mathrm{mod}(\sigma(F^\times)\tau(T^\times)).$$

Moreover, using the same arguments as in the previous exemple, we get

$$c^{\pm}(M(m)) \sim (2\pi i)^{d^{\pm}nm} D_F^{d^{\pm}m} c^{\pm(-1)^m}(M),\ \delta(M(m)) \sim (2\pi i)^{dnm} D_F^{dm} \delta(M) \pmod{T^{\times}}.$$

1.5. The gamma-factors of $L(M,s)$. We know that for a motive M over F with coefficients in T its L-function coincides with that of the motive $RM = R_{F/\mathbf{Q}}M$ over $\mathbf{Q}$ obtained from M by restriction of scalars, i.e.

$$L(M,s) = L(RM,s) \in T \otimes \mathbf{C} \text{ (for Re}(s) \text{ sufficiently large)}.$$

Therfore, we may restrict ourselves to the case of motives over $\mathbf{Q}$, and recall the conjectures about analytic prorerties and about the special values of $L(M,s)$.

Put $\Lambda(M,s) = L_\infty(M,s)L(M,s)$, where $L_\infty(M,s) = L_{\mathbf{C}}(M,s)L_{\mathbf{R}}(M,s)$ is the gamma-factor, defined by the Hodge structure of M:

$$M_B \otimes \mathbf{C} \cong \oplus_{i,j} M_B^{i,j},$$

as follows

$$L_{\mathbf{C}}(M,s) = \prod_{i<j} \Gamma_{\mathbf{C}}(s-i)^{h^{i,j}},\ L_{\mathbf{R}}(M,s) = \Gamma_{\mathbf{R}}(s-w/2)^{\gamma^+}\Gamma_{\mathbf{R}}(s+1-w/2)^{\gamma^-},$$

where $h^{i,j} = \dim M^{i,j}, \gamma^{\pm} = \dim(M^{w/2,w/2} \cap M^{(-1)^{w/2}\pm})$, and

$$\Gamma_{\mathbf{C}}(s) = (2\pi i)^{-s}\Gamma(s),\quad \Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

1.6. Conjecture on analytic properties of $\Lambda(M,s)$. *The function $L(M,s)$ can be analytically continued onto the entire complex plane and satisfies the functional equation*

$$\Lambda(M,s) = \varepsilon(M)c(M)^{(w+1)/2-s}\Lambda(\check{M},1-s),$$

where $\varepsilon(M) = (\varepsilon(M)^\tau)_\tau \in T \otimes \mathbf{C}$ $(\tau \in J_\tau)$ with $\varepsilon(M)^\tau$ being complex numbers of the absolute value equal to 1, which is independent of s and $c(M)$ is determined by conductors of the corresponding l-adic representations $r_{l,M}$. Moreover, $L(M,s)$ is holomorphic unless both w is even and the motive $\mathbf{Q}(-w/2)$ is a direct summand of M, in which case $L(M,s)$ may possibly have a pole at $s = 1 + (w/2)$.

Turning to critical values, we recall that $s = m$ is said to be *critical* for M if both Γ-factors $L_\infty(M,s)$ and $L_\infty(\check{M},1-s)$ are holomorphic at $s = m$ and eigher $\gamma^+ = 0$ or $\gamma^- = 0$, or $M^{w/2,w/2} = 0$.

1.7. Conjecture on critical values [De3]. *If $s = m$ is critical for M then*

$$\frac{L(M,m)}{c^+(M(m))} \in T$$

For example, if $M = [\chi]$ is the motive attached to a Hecke character of finite order χ, and F is totally real, then $s = m$ is critical if and only if all of the signes ε_σ are equal, say to $\varepsilon = \pm$, and either

(a) $m \equiv -\varepsilon$ for $m \leq 0$, when

$$c^{+}([\chi](m)) = 1, \quad L(m, \chi) \in \mathbf{Q}(\chi),$$

or

(b) $m \equiv \varepsilon$ for $m > 0$, when

$$c^{+}([\chi](m)) = G(\chi)(2\pi i)^{mn} D_F^{m/2}, \text{ and } \frac{L(M, \chi)}{G(\chi)(2\pi i)^{mn} D_F^{m/2}} \in \mathbf{Q}(\chi).$$

For motives M over F we set $\Lambda(M, s) = \Lambda(RM, s)$ and call $s = m$ critical for M iff it is critical for the corresponding motive RM over $\mathbf{Q}$.

We shall need the special case when

$$M^{w/2, w/2} = \{0\}. \tag{*}$$

One can deduce from the above formulae for $c^{\pm}(M(m))$ that under (*) for m critical the original conjecture takes the form [Co-PeRi]:

$$\frac{\Lambda(M, m)}{c^{(-1)^m}(M)} \in T.$$

Also, for the motive $M[\chi]$, obtained by twist with a Hecke character of finite order the above conjecture transforms in this case to the following:

$$\frac{\Lambda(M[\chi], m)}{c^{(-1)^m}(M[\chi])} \in T(\chi).$$

Following a suggestion of A.A.Beilinson, we formulate a refined form of the above conjecture assuming (*).

1.8. Modified conjecture on the critical values. *Assume that there exists an integer $s = m$ which is critical for a motive M over a totally real field F with coefficients in T satisfying $M_\sigma^{w/2, w/2} = 0$, where w is the weight of M. Then there exist constants*

$$\tilde{c}^{\pm}(\sigma; M) \in (T \otimes \mathbf{C})^{\times},$$

which are well defined modulo $T^{\times}$ such that

$$\tilde{c}^{\pm}(\sigma; M) \sim c^{\pm}(\sigma, M) \bmod (T \otimes \sigma(F))^{\times}$$

and if we put

$$\tilde{c}^{\pm}(M[\chi](m)) = G(\chi)^{-d/2} D_F^{m} \prod_{\sigma} (2\pi i)^{d^{\pm} nm} \tilde{c}^{\pm \varepsilon_\sigma(\chi)(-1)^m}(\sigma; M)$$

then

$$L^{*}(M[\chi], m) = \frac{L(M[\chi], m)}{\tilde{c}^{\pm}(M[\chi](m))} \in T(\chi)^{\times},$$

or equivalently

$$\frac{\Lambda(M[\chi],m)}{\tilde{c}^{\pm}(M[\chi])} \in T(\chi)^{\times}.$$

In other words, the L-function $L(M[\chi],s)$ determines a canonical choice of constants $\tilde{c}^{\pm}(\sigma;M) \in (T\otimes\mathbf{C})^{\times}$ modulo $T^{\times}$. The existence of such constants has been proven for a number of L-functions attached to Hilbert modular forms (see [Sh1] – [Sh3], [Ha3], (which conjecturally can be associated with certain motives).

Remark. Under the assumption (*) we can easily list all the critical points of the L-function $L(M[\chi],s)$. Let us set

$$h_* = \max_i \{i|h^{i,w-i} \neq 0, i < w/2\}, \quad h^* = \min_i \{i|h^{i,w-i} \neq 0, i > w/2\},$$

so that $h_* = w - h^*$. Then all the critical points of $L(M[\chi],s)$ are given by $s \in \mathbf{Z}, h_* < s \leq h^*$.

§2. p-adic L-functions

2.1. p-ordinary motives. We shall formulate a general conjecture on p-adic L-functions attached to motives over a totally real field F in terms of the existence of certain p-adic measures μ on the Galois group $\mathrm{Gal}_p = \mathrm{Gal}(F(p)/F)$, where $\mathrm{Gal}(F(p)/F)$ is the Galois group of the maximal abelian extension of F unramified outside p and ∞. For Hecke character χ of finite order whose conductor contains only prime divisors of p this conjecture interprets the critical values $\Lambda(M[\chi],r)$ in terms of certain integrals of χ over Gal_p, where we use the same symbol χ to denote a character of Gal_p attached to the Hecke character by class field theory. We start by recalling the notion of a p-ordinary motive [Co-PeRi] and suppose first that M is a motive over $\mathbf{Q}$ of weight w, and rank d. We fix an embedding $i_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$, where $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$ denotes the Tate field (completion of an algebraic closure of the p-adic field $\mathbf{Q}_p$).

Let

$$\Phi_p : D_p = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathbf{Z}_p^{\times}$$

be the cyclotomic character of D_p.

Definition. We say that M is ordinary at p if the following conditions are satisfied:

(i) The inertia group I_p acts trivially on M_λ for λ not dividing p in T.

(ii) There exists a filtration on M_λ

$$W_1(M) = M_\lambda(M) \underset{\neq}{\supset} W_2(M) \underset{\neq}{\supset} \cdots \underset{\neq}{\supset} W_{t+1}(M) = \{0\}$$

by T_λ-vector spaces which are stable under the action of D_p, and which are such that the inertia subgroup I_p operates on $W_i(M)/W_{i+1}(M)$ by some power of Φ_p, say $\Phi_p^{-e_i(M)}$ $(1 \leq i \leq t)$. Moreover, these integers satisfy

$$e_1(M) \geq e_2(M) \geq \cdots \geq e_t(M).$$

If M is ordinary at p there is a close connection between the Galois module M_λ (for λ dividing p) and the Hodge decomposition of $M_\sigma \otimes \mathbf{C}$. The precise conjecture on this connection is given below. Assuming M is ordinary at p condition (i) ensures that the p-Euler factor $L_p(M,X)$ has exact degree d. Hence factorizing it in $\overline{\mathbf{Q}}_p$ we have

$$L_p(M,X)^{-1} = (1-\alpha_1(p)X)\cdots(1-\alpha_d(p)X) \quad (\alpha_i(p) \in \overline{\mathbf{Q}_p})$$

where none of the $\alpha_i(p)$ is equal to zero. Here we fixed some embedding $\tau : T \to \overline{\mathbf{Q}}$ and assumed that

$$L_p(M,X)^{-1} = 1 + A_1X + \cdots + A_dX^d \in T[X] \subset \overline{\mathbf{Q}_p}[X]$$

Let ord_p denote the order valuation of $\overline{\mathbf{Q}_p}$, normalized so that $\mathrm{ord}_p(p) = 1$. We suppose that we have chosen our indices so that

$$\mathrm{ord}_p(\alpha_1(p)) \geq \mathrm{ord}_p(\alpha_2(p)) \geq \cdots \geq \mathrm{ord}_p(\alpha_d(p)).$$

2.2. Conjecture on p-ordinary motives. *Assume that M is ordinary at p. Then the above sequence of p-adic orders consists of e_1 repeated $h^{e_1,w-e_1}$ times, followed by e_2, repeated $h^{e_2,w-e_2}$ times, ..., followed finally by e_t repeated $h^{e_t,w-e_t}$ times.*

Actually, this conjecture has already been proven in a vast generality by the algebraic geometers [Blo-K].

2.3. Datum for the non-Archimedean construction. Let us consider the p-adic completion

$$\mathcal{O}_F \otimes \mathbf{Z}_p = \prod_{\mathfrak{p}|p} \mathcal{O}_\mathfrak{p}$$

of $\mathcal{O}_F$.

The domain of definition of our non-Archimedean L-functions is the p-adic analytic Lie group

$$\mathcal{X}_p = \mathrm{Hom}_{\mathrm{contin}}(\mathrm{Gal}_p, \mathbf{C}_p^\times)$$

of all continuouos p-adic characters of the Galois group Gal_p. Elements of finite order $\chi \in \mathcal{X}_p$ can be identified with those Hecke characters of finite order whose conductors contain only prime divisors of p. This identification uses the map

$$\chi : \mathbf{A}_F^\times \xrightarrow{\mathrm{CFT}} \mathrm{Gal}_p \to \overline{\mathbf{Q}}^\times \xrightarrow{i_p} \mathbf{C}_p^\times,$$

where CTF is the homomorphism of class field theory.

We shall use the same symbol χ to denote both Hecke character and the corresponding element of $\mathcal{X}_p$. Since $\mathbf{Q}(p) \subset F(p)$, the restriction of Galois automorphisms to $\mathbf{Q}(p)$ determines a natural homomorphism

$$\mathcal{N} : \mathrm{Gal}_p \to \mathrm{Gal}(\mathbf{Q}(p)/\mathbf{Q}) \cong \mathbf{Z}_p^\times.$$

We shall let $\mathcal{N}x_p$ denote the composition of this homomorphism with the inclusion $\mathbf{Z}_p^\times \subset \mathbf{C}_p^\times$.

Starting from the algebraicity property of the critical values in the modified conjecture 1.8, we shall describe the general form of p-adic L-function attached to a motive$M$

over a totally real field F. We call M ordinary at p if the motive RM of rank dn obtained from M by restriction of scalars to $\mathbf{Q}$ is p-ordinary. In this case we set

$$L_p(RM,X)^{-1} = \prod_{\mathfrak{p}|p} L_{\mathfrak{p}}(M,X)^{-1} =$$
$$(1-\alpha_1(p)X)\cdots(1-\alpha_{dn}(p)X) \quad (\alpha_i(p) \in \overline{\mathbf{Q}_p}),$$

where

$$L_{\mathfrak{p}_i}(M,X)^{-1} = (1-\alpha^{(1)}(\mathfrak{p}_i)X)\cdots(1-\alpha^{(d)}(\mathfrak{p}_i)X).$$

For the motive $\check{M}$ dual to M we set

$$L_p(R\check{M},X)^{-1} = \prod_{\mathfrak{p}|p} L_{\mathfrak{p}}(\check{M},X)^{-1} =$$
$$(1-\check{\alpha}_1(p)X)\cdots(1-\check{\alpha}_{dn}(p)X) \quad (\check{\alpha}_i(p) \in \overline{\mathbf{Q}_p}),$$

and

$$L_{\mathfrak{p}_i}(\check{M},X)^{-1} = (1-\check{\alpha}^{(1)}(\mathfrak{p}_i)X)\cdots(1-\check{\alpha}^{(d)}(\mathfrak{p}_i)^{-1}X),$$

so that we have

$$\check{\alpha}_i(p) = \alpha_{dn+1-i}(p)^{-1} \quad (1 \le i \le dn), \text{ and we set } \check{\alpha}^{(j)}(\mathfrak{p}_i) = \alpha^{(d-j)}(\mathfrak{p}_i)^{-1}.$$

In order to formulate precisely the conjecture we set

$$J_M = \left\{(i,j) \middle| \mathfrak{p}_i|p,\ \mathrm{ord}_p\alpha^{(j)}(\mathfrak{p}_i) \ge h^*\right\}, \quad \check{J}_M = \left\{(i,j) \middle| \mathfrak{p}_i|p,\ \mathrm{ord}_p\check{\alpha}^{(j)}(\mathfrak{p}_i) \ge 1-h^*\right\},$$

and

$$\Phi_p(M[\chi],s)^{-1} = \prod_{(i,j)\in J_M}(1-\chi(\mathfrak{p}_i)\alpha^{(j)}(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s}) \prod_{(i,j)\in \check{J}_M}(1-\chi^{-1}(\mathfrak{p}_i)\check{\alpha}^{(j)}(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{s-1}),$$

$$I_M = \left\{(i,j)|\mathrm{ord}_p\alpha^{(j)}(\mathfrak{p}_i) \le h_*\right\}.$$

Suppose that M is a motive over F with coefficients in $T \subset \overline{\mathbf{Q}} \subset \mathbf{C}$ which posesses a critical point. Let us also fix $\varepsilon_0 = (\varepsilon_{0,\sigma})_\sigma \in \mathrm{Sgn}_F$, where $\varepsilon_{0,\sigma} = \pm 1$, and define the following constant

$$\Omega(\varepsilon_0) = \prod_\sigma \tilde{c}^{\varepsilon_{0,\sigma}}(\sigma),$$

where $\tilde{c}^{\varepsilon_{0,\sigma}}(\sigma) \in \mathbf{C}^\times$ are the same as in the modified conjecture 1.8 on the critical values. Assume also that (*) is satisfied, i.e. $M_\sigma^{w/2,w/2} = 0$. Then d is even and all the critical points r are given by $h_* < r \le h^*$.

2.4. Conjecture on p-adic L-functions. *Under the conventions and notation as in 2.3 there exists a bounded $\mathbf{C}_p$-valued measure $\mu_{\varepsilon_0} = \mu_{\varepsilon_0}(M)$ on Gal_p which is uniquely determined by the following condition: for all Hecke characters $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ and all $r \in \mathbf{Z}$ satisfying*

$$(-1)^r\varepsilon_\sigma(\chi) = \varepsilon_{0,\sigma} \ (\textit{for all } \sigma), \quad h_* < r \le h^*$$

the following equality holds:

$$\int_{\mathrm{Gal}_p} \chi^{-1}\mathcal{N}x_p^r d\mu_{\varepsilon_0} =$$

$$i_p\left(\frac{D_F^{rd/2}(-1)^{[rd/4]}}{G(\chi)^{d/2}}\frac{\Lambda(M[\chi],r)}{\Omega(\varepsilon_0)\Phi_p(M[\chi],r)}\prod_{(i,j)\in I_M}\left(\frac{\mathcal{N}\mathfrak{p}_i^{r-1}}{\alpha^{(j)}(\mathfrak{p}_i)}\right)^{\mathrm{ord}_{\mathfrak{p}_i}c(\chi)}\right).$$

Note that the measure μ_{ε_0} defines a bounded $\mathbf{C}_p$-analytic function

$$L_{\varepsilon_0,M} : \mathcal{X}_p \to \mathbf{C}_p,\quad \mathcal{X}_p \ni x \mapsto \int_{\mathrm{Gal}_p} x\, d\mu_{\varepsilon_0}(M)$$

(the p-adic Mellin transform of $\mu_{\varepsilon_0}(M)$), which is uniquely determined by its values on the characters $x = \chi^{-1}\mathcal{N}x_p^r \in \mathcal{X}_p$.

§3. Hilbert modular forms

3.1. The group. We shall regard the group $\mathrm{GL}_2(F)$ as the group $G_{\mathbf{Q}}$ of all $\mathbf{Q}$-rational points of a certain $\mathbf{Q}$-subgroup $G \subset \mathrm{GL}_{2n}$. Then Hilbert modular forms will be regarded as complex functions on the adelic group $G_{\mathbf{A}} = G(\mathbf{A})$, which is apparently identified with the product

$$\mathrm{GL}_2(F_{\mathbf{A}}) \cong G_\infty \times G_{\widehat{\mathbf{Q}}},$$

where

$$G_\infty = \mathrm{GL}_2(F_\infty) \cong \mathrm{GL}_2(\mathbf{R})^n,\quad G_{\widehat{\mathbf{Q}}} = \mathrm{GL}_2(\widehat{F}).$$

The subgroup

$$G_\infty^+ \cong \mathrm{GL}_2^+(\mathbf{R})^n \subset G_\infty$$

consists of all elements

$$\alpha = (\alpha_1, ..., \alpha_n),\quad \alpha_\nu = \begin{pmatrix}\alpha_\nu & \beta_\nu\\ \gamma_\nu & \delta_\nu\end{pmatrix},$$

such that $\det\alpha_\nu > 0$, $\nu = 1, ..., n$. Every element $\alpha \in G_\infty^+$ acts on the product $\mathfrak{H}^n$ of the n copies of the upper half planes according to the formula

$$\alpha(z_1, \cdots, z_n) = (\alpha_1(z_1), \cdots, \alpha_n(z_n)),$$

where

$$\alpha_\nu(z_\nu) \;=\; (a_\nu z_\nu + b_\nu)/(c_\nu z_\nu + d_\nu).$$

For $z = (z_1, \cdots, z_n)$ we put $\{z\} = z_1 + \cdots + z_n$ and $e_F(z) = e(\{z\})$, with $e(x) = \exp(2\pi i x)$, and we use notation $\mathcal{N}(z) = z_1 \cdots z_n$. Let $\mathbf{i} = (i, \cdots, i) \in \mathfrak{H}^n$, then

$$\{\alpha \in G_\infty^+ | \alpha(\mathbf{i}) = \mathbf{i}\}/\mathbf{R}_+^\times \cong \mathrm{SO}(2)^n$$

is a maximal compact subgroup in $G_\infty^+/\mathbf{R}_+^\times$. For $\alpha \in G_\infty^+$, an integer n-tuple $k = (k_1, \cdots, k_n)$ and an arbitrary function $f : \mathfrak{H}^n \to \mathbf{C}$ we use the notation

$$(f|_k\alpha)(z) = \prod_{\nu=1}^{n}(c_\nu z + d_\nu)^{-k_\nu} f(\alpha(z))\det(\alpha_\nu)^{k_\nu/2}.$$

Let $\mathfrak{c} \subset \mathcal{O}_F$ be an integral ideal, $\mathfrak{c}_\mathfrak{p} = \mathfrak{c}\mathcal{O}_\mathfrak{p}$ its $\mathfrak{p}$-part, $\mathfrak{d}_\mathfrak{p} = \mathfrak{d}\mathcal{O}_\mathfrak{p}$ the local different. We shall need the open subgroups $W = W_\mathfrak{c} \subset G_\mathbf{A}$ defined by

$$\begin{aligned} W &= G_\infty^+ \times \prod_\mathfrak{p} W(\mathfrak{p}), \\ W(\mathfrak{p}) &= \\ &\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\mathfrak{p}) | b \in \mathfrak{d}_\mathfrak{p}^{-1}, c \in \mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p},\ a, d \in \mathcal{O}_\mathfrak{p}, ad - bc \in \mathcal{O}_\mathfrak{p}^\times \right\}. \end{aligned} \tag{3.1}$$

Let $h = |\widetilde{Cl_F}|$ be the number of ideal classes of F (in the narrow sense),

$$\widetilde{Cl_F} = I/\{(x) | x \in F_+^\times\},$$

and let us choose the ideles $t_1, \cdots, t_h$ so that $\tilde{t}_\lambda \subset \mathcal{O}_F$ form a complete system of representatives for $\widetilde{Cl_F}$, $(t_\lambda)_\infty = 1$ and $\tilde{t}_\lambda + \mathfrak{m}_0 = \mathcal{O}_F$ ($\lambda = 1, \cdots, h$, $\mathfrak{m}_0 = \prod_{\mathfrak{q} \in S_F} \mathfrak{q}$). If we put $x_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & t_\lambda \end{pmatrix}$ then there is the following decomposition into a disjoint union ("the approximation theorem"):

$$G_\mathbf{A} = \cup_\lambda G_\mathbf{Q} x_\lambda W = \cup_\lambda G_\mathbf{Q} x_\lambda^{-\iota} W, \tag{3.2}$$

where $x_\lambda^{-\iota} = \begin{pmatrix} t_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, ι denotes the involution given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(see [Sh1], p.647).

3.2. Hilbert automorphic forms. By a *Hilbert automorphic form* of weight $k = (k_1, \cdots, k_n)$, level $\mathfrak{c} \subset \mathcal{O}_F$, and Hecke character ψ we mean a function $f : G_\mathbf{A} \to \mathbf{C}^\times$ satisfying the following conditions (3.3) – (3.5):

$$\begin{aligned} &f(s\alpha x) = \psi(s)f(x) \text{ for all } x \in G_\mathbf{A} \\ &\text{for } s \in F_\mathbf{A}^\times \text{ (the center of } G_\mathbf{A}\text{), and } \alpha \in G_\mathbf{Q}. \end{aligned} \tag{3.3}$$

If we let $\psi_0 : (\mathcal{O}_F/\mathfrak{c})^\times \to \mathbf{C}^\times$ denote the $\mathfrak{c}$-part of the character ψ, and then extend the definition of ψ over W by the formula

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \psi_0(a_\mathfrak{c} \bmod \mathfrak{c}),$$

($a_\mathfrak{c}$ being the $\mathfrak{c}$-part of a) then for all $x \in G_\mathbf{A}$

$$f(x\,w) = \psi(w^\iota)f(x) \quad \text{for } w \in W_\mathfrak{c} \text{ with } \; w_\infty = 1. \tag{3.4}$$

If $w = w(\theta) = (w_1(\theta_1), ..., w_n(\theta_n))$ where

$$w_\nu(\theta_\nu) = \begin{pmatrix} \cos\theta_\nu & -\sin\theta_\nu \\ \sin\theta_\nu & \cos\theta_\nu \end{pmatrix},$$

then

$$f(x\,w(\theta)) = f(x)e^{-i(k_1\theta_1 + \cdots k_n\theta_n)} \qquad (x \in G_\mathbf{A}). \tag{3.5}$$

An automorphic form f is called a *cusp form* if

$$\int_{F_\mathbf{A}/F} f\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g\right) dt = 0 \text{ for all } g \in G_\mathbf{A}. \tag{3.6}$$

The vector space $\mathcal{M}_k(\mathfrak{c}, \psi)$ of Hilbert automorphic forms of *holomorphic type* is defined as the set of functions satisfying (3.3) – (3.5) and the following holomorphy condition (3.7): for any $x \in G_\mathbf{A}$ with $x_\infty = 1$ there exists a holmorphic function $g_x : \mathfrak{H}^n \to \mathbf{C}$, such that for all $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty^+$ we have

$$f(x\,y) \;\; = \;\; (g_x|_k y)(\mathbf{i}) \tag{3.7}$$

(in the case $F = \mathbf{Q}$ we must also require that the functions g_x be holomorphic at the cusps). The property (3.7) enables one to describe the automorphic forms $f \in \mathcal{M}_k(\mathfrak{c}, \psi)$ more explicitely in terms of Hilbert modular forms on $\mathfrak{H}^n$. For this purpose we put $f_\lambda = g_{x_\lambda^{-\iota}}$ where $x_\lambda^{-\iota} = \begin{pmatrix} t_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, then $f_\lambda(z) \in \mathcal{M}_k(\Gamma_\lambda, \psi_0)$ for the congruence subgroup

$$\begin{aligned} &\Gamma_\lambda = \Gamma_\lambda(\mathfrak{c}) \subset G_\mathbf{Q}^+, \\ &\Gamma_\lambda = x_\lambda W x_\lambda^{-1} \cap G_\mathbf{Q} = \\ &\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\mathbf{Q}^+ | b \in \tilde{t}_\lambda^{-1}\mathfrak{d}^{-1}, \;\; c \in \tilde{t}_\lambda \mathfrak{d}\mathfrak{c}, a, d \in \mathcal{O}_F, \;\; ad - bc \in \mathcal{O}_F^\times\right\}. \end{aligned}$$

This means that for all $\gamma \in \Gamma_\lambda(\mathfrak{c})$ the following condition (3.8) is satisfied:

$$f_\lambda|_k\gamma = \psi(\gamma)f_\lambda \quad \text{and} \quad f_\lambda(z) = \sum_\xi a_\lambda(\xi)e_F(\xi z), \tag{3.8}$$

where $0 \ll \xi \in \tilde{t}_\lambda$ or $\xi = 0$ in the sum over ξ (see [Sh1] for a more detailed discussion of Fourier expansions). The map $f \to (f_1, ...f_h)$ provides a vector space isomorophism

$$\mathcal{M}(\mathfrak{c}, \psi) \cong \oplus_\lambda \mathcal{M}_k(\Gamma_\lambda, \psi)$$

Put

$$C(\mathfrak{m}, f) = \begin{cases} a_\lambda(\xi)\mathcal{N}(\tilde{t}_\lambda)^{-k_0/2}, & \text{if the ideal } \mathfrak{m} = \xi\tilde{t}_\lambda^{-1} \text{ is integral;} \\ 0, & \text{if } \mathfrak{m} \text{ is not integral.} \end{cases} \tag{3.9}$$

We have the following Fourier expansion:

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{0 \ll \zeta \in F, \zeta = 0} \mathcal{N}(\widetilde{\zeta y})^{-(k_0/2)} C(\widetilde{\zeta y}, f)(\zeta y_\infty)^{k/2} e_F(\zeta i y_\infty)\chi(\zeta x), \tag{3.10}$$

where $\chi_F : F_{\mathbf{A}}/F \to \mathbf{C}^\times$ is a fixed additive character with the condition $\chi_F(x_\infty) = e_F(x_\infty)$ (see [Sh1], p. 650).

Let $\mathcal{S}_k(\mathfrak{c}, \psi) \subset \mathcal{M}(\mathfrak{c}, \psi)$ be the subspace of cusp forms and $f \in \mathcal{S}_k(\mathfrak{c}, \psi)$ then $a_\lambda(0) = 0$ for all $\lambda = 1, \cdots, h$.

3.3. Hecke operators are introduced by means of double cosets of the type WyW for y in the semigroup

$$Y_\mathfrak{c} = G_{\mathbf{A}} \cap (G_\infty^+ \times \prod Y_\mathfrak{c}(\mathfrak{p})),$$

where

$$Y_\mathfrak{c}(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\mathfrak{p}) \mid a\mathcal{O}_\mathfrak{p} + \mathfrak{c}_\mathfrak{p} = \mathcal{O}_\mathfrak{p},\ b \in \mathfrak{d}_\mathfrak{p}^{-1}, c \in \mathfrak{c}_\mathfrak{p}\mathfrak{d}_\mathfrak{p}, d \in \mathcal{O}_\mathfrak{p} \right\}. \tag{3.11}$$

The Hecke algebra $\mathcal{H}_\mathfrak{c}$ consists of all formal finite sums of the type $\sum_y c_y WyW$ with $y \in Y_\mathfrak{c}, c_y \in \mathbf{C}$ and with the standard multiplication law defined by means of decomposition of double cosets into a disjoint union of a finite number of left cosets. By definition, $T_\mathfrak{c}(\mathfrak{m})$ is an element of the ring $\mathcal{H}_\mathfrak{c}$ obtained by taking the sum of all different WyW with $y \in Y_\mathfrak{c}$ such that $\widetilde{\det(y)} = \mathfrak{m}$. Let

$$T'_\mathfrak{c} = \mathcal{N}(\mathfrak{m})^{(k_0-2)/2} T_\mathfrak{c}(\mathfrak{m}) \tag{3.12}$$

be the normalized Hecke operator, whose action on the Fourier coefficients of an automorphic form (of holomorphic type) $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ is given by the usual formula

$$C(\mathfrak{m}, \mathbf{f} | T'_\mathfrak{c}(\mathfrak{m}) = \sum_{\mathfrak{m}+\mathfrak{n}=\mathfrak{a}} \psi(\mathfrak{a}) \mathcal{N}(\mathfrak{a})^{k_0-1} C(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \mathbf{f}) \tag{3.13}$$

If $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ is an eigenfunction of all Hecke operators $T'_\mathfrak{c}(\mathfrak{m})$ with $\mathbf{f}|_k T'_\mathfrak{c}(\mathfrak{m}) = \lambda(\mathfrak{m})\mathbf{f}$ then we have that $C(\mathfrak{m}, \mathbf{f}) = \lambda(\mathfrak{m}) C(\mathcal{O}_F, \mathbf{f})$. If we normalize the form $\mathbf{f}$ by the condition $C(\mathcal{O}_F, \mathbf{f}) = 1$ then the L-function has the following Euler product expansion:

$$\begin{aligned} L(s, \mathbf{f}) &= \sum_\mathfrak{n} C(\mathfrak{n}, \mathbf{f}) \mathcal{N}(\mathfrak{n})^{-s} = \sum_\mathfrak{n} \lambda(\mathfrak{n}) \mathcal{N}(\mathfrak{n})^{-s} = \\ &\prod_\mathfrak{p} \left(1 - C(\mathfrak{p}, \mathbf{f}) \mathcal{N}(\mathfrak{p})^{-s} + \psi(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{k_0-1-2s}\right)^{-1}. \end{aligned} \tag{3.14}$$

For such a form $\mathbf{f}$ all of the numbers $C(\mathfrak{n}, \mathbf{f})$ are algebraic integers.

The Petersson inner product is defined for $\mathbf{f} = (f_1, ..., f_h) \in \mathcal{S}_k(\mathfrak{c}, \psi)$ and $\mathbf{g} = (g_1, ..., g_h) \in \mathcal{M}_k(\mathfrak{c}, \psi)$ by setting

$$\langle \mathbf{f}, \mathbf{g} \rangle_\mathfrak{c} = \sum_{\lambda=1}^{h} \int_{\Gamma_\lambda(\mathfrak{c}) \backslash \mathfrak{H}^n} \overline{f_\lambda(z)} g_\lambda(z)\, y^k \, d\mu(z), \tag{3.15}$$

where

$$y^k = y_1^{k_1} \cdots y_n^{k_n}, \quad d\mu(z) = \prod_{\nu=1}^{n} y_\nu^{-2}\, dx_\nu\, dy_\nu$$

is a G_∞^+ -invariant measure on $\mathfrak{H}^n$.

§4. Motives and p-adic L-functions of Hilbert modular forms

4.1. Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ be a primitive Hilbert cusp eigenform of weight k, ($k = (k_1, \cdots, k_n)$), and character ψ. Then the important analytic property of the corresponding L-function (see [Sh1], p. 655)

$$L(s, \mathbf{f}) = \sum_{\mathfrak{n}} C(\mathfrak{n}, \mathbf{f}) \mathcal{N}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} \left(1 - C(\mathfrak{p}, \mathbf{f}) \mathcal{N}(\mathfrak{p})^{-s} + \psi(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{k_0 - 1 - 2s}\right)^{-1}$$

is (see [Sh1], p. 655) that it admits a holomorphic analytic continuation onto the entire complex plane, and if we set

$$\Lambda(s, \mathbf{f}) = \prod_{i=1}^{n} \Gamma_{\mathbf{C}}(s - (k_0 - k_i)/2) L(s, \mathbf{f}),$$

then $\Lambda(s, \mathbf{f})$ satisfies a certain functional equation expressing $\Lambda(s, \mathbf{f})$ in terms of the function $\Lambda(k_0 - s, \mathbf{f}^\rho)$. Also, for any $\phi \in \mathrm{Aut}\mathbf{C}$ we have that $\mathbf{f}^\phi \in \mathcal{S}_{k^\phi}(\mathfrak{c}, \psi^\phi)$, where

$$L(s, \mathbf{f}^\phi) = \sum_{\mathfrak{n}} C(\mathfrak{n}, \mathbf{f})^\phi \mathcal{N}(\mathfrak{n})^{-s},$$

and the action of ϕ on weights k is defined by the formula

$$(x^{k^\phi}) = (x^k)^\phi \text{ for all } x \in F^\times,$$

where $x^k = x_1^{k_1} \cdots x_n^{k_n}$, $x_i = \sigma_i(x)$. According to the above conjecture on analytic properties of $L(M, s)$ we may suggest a conjecture that $\mathbf{f}$ should correspond to a motive $M = M(\mathbf{f})$ over F of rank $d = 2$, and weight $w = k_0 - 1$ with coefficients in a field T containing $C(\mathfrak{n}, \mathbf{f})$ and x^k for all $\mathfrak{n} \subset \mathcal{O}$ and $x \in F$ such that

$$L(M, s) = L(s, \mathbf{f}), \quad \Lambda(M, s) = \Lambda(s, \mathbf{f}) \in T \otimes \mathbf{C},$$

and for a fixed $\sigma_i : F \to \mathbf{R}$ the Hodge decomposition of M_{σ_i} is given by

$$M_{\sigma_i} \otimes \mathbf{C} \xrightarrow{\sim} \oplus_{\tau \in J_T} \left(M_{\sigma_i}^{(k_0 - k_i^\tau)/2, (k_0 + k_i^\tau)/2 - 1} \oplus M_{\sigma_i}^{(k_0 + k_i^\tau)/2 - 1, (k_0 - k_i^\tau)/2} \right),$$

where $k^\tau = (k_1^\tau, \cdots, k_n^\tau)$ is the weight of the modular form $\mathbf{f}^\tau$ with coefficients $\tau(C(\mathfrak{n}, \mathbf{f}))$, which is obtained from $\mathbf{f}$ by action of a certain complex automorphism (note that the Hodge type of this decomposition does not depend on σ_i).

There are several confirmation of this conjecture. First of all, this is already proven in the elliptic modular case (F=$\mathbf{Q}$) by U.Jannssen and A.T.Scholl [Ja], [Sch]; the existence of Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ corresponding to λ-adic realizations of these motives was discovered earlier by P.Deligne [De1]. By the restriction of such representation to the subgroup $\mathrm{Gal}(\overline{F}/F) = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ we obtain the L-function of certain

Hilbert modular form, which is the Doi-Naganuma (or "base change") lift of the original elliptic cusp form.

In the general case the existence of Galois representations attached to Hilbert modular forms was established by R.Taylor (n odd) [Ta] and H.Carayol (n even)[Ca]. Also, a number of results on special values of $L(s,\mathbf{f})$ were proven, which match the above conjectures on the critical values and on p-adic L-function ([Sh1], [Man], [Ka3]).

As in the elliptic modular case there is a conjectural link of motives of the type $M(\mathbf{f})$ with Kuga – Shimura varieties, namely, that for the decomposition $R_{F/\mathbf{Q}}M = \oplus_{i=1}^{n} M^{\sigma_i}$ the tensor product $\otimes_{i=1}^{n} M^{\sigma_i}$, which is a motive over $\mathbf{Q}$ of rank 2^n, should be contained in the cohomology of certain Kuga – Shimura variety (fiber product of several copies of the universal Hilbert – Blumenthal abelian variety with a level structure, see the interesting discussion of this link in [Ha3], [O]).

If $k_1 = \cdots = k_n = 2$ then M^{σ_i} should have the Hodge type $H^{0,1} \oplus H^{1,0}$. In some cases (e.g. when $n = [F : \mathbf{Q}]$ is odd) the motives M^{σ_i} can be realized as factors of Jacobians of Shimura curves corresponding to quaternion algebras, which split at σ_i, ramified at σ_j $(i \neq j)$ (M.Harris, T.Oda) [Oda], [Ha3].

4.2. Periods of the Hilbert cusp forms. Let $\mathbf{f} \in S_k(\mathfrak{c}, \psi)$ be a primitive Hilbert cusp eigenform with coefficients in a field T as above, and

$$L(s, \mathbf{f}(\chi)) = \sum_{\mathfrak{n}} \chi(\mathfrak{n}) C(\mathfrak{n}, \mathbf{f}) \mathcal{N}(\mathfrak{n})^{-s} =$$

$$\prod_{\mathfrak{p}} \left(1 - \chi(\mathfrak{p}) C(\mathfrak{p}, \mathbf{f}) \mathcal{N}(\mathfrak{p})^{-s} + \chi(\mathfrak{p})^2 \psi(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{k_0 - 1 - 2s}\right)^{-1}$$

be its L-function, twisted with a Hecke character $\chi(\mathfrak{p})$ of finite order. Then it was established by G.Shimura [Sh1] using Rankin – Selberg method that there exist constants

$$c^{\pm}(\sigma, \mathbf{f}),\ \delta(\sigma, \mathbf{f}) \in (T \otimes \mathbf{C})^{\times}$$

defined modulo $T^{\times}$ such that

$$\prod_{\sigma} c^{+}(\sigma, \mathbf{f}) c^{-}(\sigma, \mathbf{f}) = \langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}}, \quad \prod_{\sigma} \delta(\sigma) = G(\psi)^{-1} (2\pi i)^{n(k_0 - 1)},$$

and if we put

$$h_* = \max_i \{(k_0 - k_i)/2\}, \quad h^* = \min_i \{(k_0 + k_i)/2 - 1\},$$

(so that $h_* = k_0 - 1 - h^*$), and

$$c^{\pm}(\chi, \mathbf{f}) = G(\chi)^{-1} \prod_{\sigma} c^{\pm \varepsilon_\sigma(\chi)}(\sigma, \mathbf{f}),$$

then for all the points $r \in \mathbf{Z}$, $h_* < r \leq h^*$ we have that

$$\frac{\Lambda(r, \mathbf{f}(\chi))}{c^{(-1)^m}(\chi, \mathbf{f})} \in T(\chi).$$

Note that from the point of view of the modified conjecture 1.8 on the critical values, the quantities $c^{\pm}(\sigma, \mathbf{f})$ determine the corresponding constants $\tilde{c}^{\pm}(\sigma, M(\mathbf{f}))$ under the assumption that the motive $M = M(\mathbf{f})$ exists. Earlier the analogous algebraicity result was established by Yu. I. Manin [Man] using the theory of generalized modular symbols on Hilbert – Blumenthal modular varieties.

4.3. p-ordinary Hilbert modular forms. Let us now analyze the condition that $\mathbf{f}$ is p-ordinary and assume for simplicity that p splits in F, $p\mathcal{O}_F = \mathfrak{p}_1 \cdots \mathfrak{p}_n$. We let in this case the ideals $\mathfrak{p}_i$ be indexed by the embeddings $\sigma_i : F \to \overline{\mathbf{Q}}$ so that $\mathrm{ord}_p \sigma_i(\mathfrak{p}_j) = \delta_{i,j}$ (we regard $\overline{\mathbf{Q}}$ as a subfield of $\mathbf{C}_p$ via i_p). Consider the numbers $\alpha(\mathfrak{p}_i)$, $\alpha'(\mathfrak{p}_i)$ (the roots of the Hecke polynomials

$$X^2 - C(\mathfrak{p}_i, \mathbf{f})X + \psi(\mathfrak{p}_i)\mathcal{N}(\mathfrak{p}_i)^{k_0-1} = (X - \alpha(\mathfrak{p}_i))(X - \alpha'(\mathfrak{p}_i)) \in \mathbf{C}_p[X]).$$

Then the condition that the motive $M = M(\mathbf{f})$ attached to $\mathbf{f}$ is p-ordinary takes the form:

$$\mathrm{ord}_p \alpha(\mathfrak{p}_i) = (k_0 - k_i)/2, \quad \mathrm{ord}_p \alpha'(\mathfrak{p}_i) = (k_0 + k_i)/2 - 1,$$

or equivalently, $\mathrm{ord}_p C(\mathfrak{p}_i, \mathbf{f}) = (k_0 - k_i)/2$.

An example of a p-ordinary motive. Let $K \supset F$ be a totally imaginary quadratic extension, and $\eta : \mathbf{A}_K^\times / K^\times \to \mathbf{C}^\times$ be an algebraic Hecke's Grössencharakter such that

$$\eta((\alpha)) = \left(\frac{\alpha^{\phi_1}}{|\alpha^{\phi_1}|}\right)^{w_1} \cdots \left(\frac{\alpha^{\phi_n}}{|\alpha^{\phi_n}|}\right)^{w_n} \cdot \mathcal{N}(\alpha)^{w_0/2-1}$$

for $\alpha \in K, \alpha \equiv 1(\mathrm{mod}\mathfrak{c}(\eta))$, where $\{\phi_i : K \to \mathbf{C}\}$ is the set of complex embeddings satisfying $\phi_i|F = \sigma_i$ (CM-type), w_i are positive integers, $w_0 = \max_i w_i$. Then there exists a Hilbert modular form $\mathbf{f}$ of weight $k = (w_1 + 1, \cdots, w_n + 1)$ such that $L(s, \mathbf{f}) = L(s, \eta)$, and $M(\mathbf{f})$ coincides with the motive $R_{K/F}[\eta]$ obtained by restriction of scalars from the motive $[\eta]$ (the last motive exists as an object of the category of motives of CM-type, see [Bl1]).

In order to give an example of a p-ordinary motive, let us assume that p totally splits in K:

$$p\mathcal{O}_K = \mathfrak{P}_1\mathfrak{P}_1' \cdots \mathfrak{P}_n\mathfrak{P}_n', \quad \mathfrak{p}_i\mathcal{O}_K = \mathfrak{P}_i\mathfrak{P}_i'.$$

Taking into account that $\mathrm{ord}_p \sigma_i(\mathfrak{p}_j) = \delta_{i,j}$ we shall assume that

$$\mathrm{ord}_p \phi_i(\mathfrak{P}_i) = 0, \ \mathrm{ord}_p \phi_i(\mathfrak{P}_i') = 1$$

then the roots $\alpha(\mathfrak{p}_i)$, $\alpha'(\mathfrak{p}_i)$ of the $\mathfrak{p}_i$-Hecke polynomial are equal to $\eta(\mathfrak{P}_i)$, $\eta(\mathfrak{P}_i')$, and from the above formula for $\eta((\alpha))$ it follows that

$$\mathrm{ord}_p \eta(\mathfrak{P}_i) = (w_i - w_0)/2 = (k_i - k_0)/2, \quad \mathrm{ord}_p \eta(\mathfrak{P}_i') = (w_i + w_0)/2 = (k_i + k_0)/2 - 1,$$

i.e. that M is p-ordinary.

4.4. p-adic L-functions of Hilbert modular forms. We now state a general result on p-adic L-functions of Hilbert modular form, which was proven (in a weaker form) by Yu.I.Manin [Man]. In order to give a precise formulation we assume that $\mathbf{f}$ is p-ordinary, so that

$$\mathrm{ord}_p \alpha(\mathfrak{p}_i) = (k_0 - k_i)/2, \quad \mathrm{ord}_p \alpha'(\mathfrak{p}_i) = (k_0 + k_i)/2 - 1,$$

or equivalently, $\mathrm{ord}_p C(\mathfrak{p}_i, \mathbf{f}) = (k_0 - k_i)/2$. Then for a Hecke character χ of finite order whose conductor is divisible only by prime divisors of p in F we consider the L-function

$$L(s, \mathbf{f}(\chi)) = \sum_{\mathfrak{n}} \chi(\mathfrak{n}) C(\mathfrak{n}, \mathbf{f}) \mathcal{N}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} \left(1 - \chi(\mathfrak{p}) C(\mathfrak{p}, \mathbf{f}) \mathcal{N}(\mathfrak{p})^{-s} + \chi(\mathfrak{p})^2 \psi(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{k_0 - 1 - 2s}\right)^{-1}$$

of the Hilbert cusp eigenform $\mathbf{f}(\chi) \in \mathcal{S}(\mathfrak{c}\mathfrak{c}(\chi)^2, \psi\chi^2)$, obtained from $\mathbf{f}$ by twisting with χ, where $\mathfrak{c}(\chi)$ is the conductor of χ, and we assume that $\mathfrak{c}(\chi)$ is coprime with the level $\mathfrak{c}$ of $\mathbf{f}$. We mentioned above that there exist constants

$$c^{\pm}(\sigma, \mathbf{f}), \ \delta(\sigma, \mathbf{f}) \in (T \otimes \mathbf{C})^{\times}$$

defined modulo $T^{\times}$ such that

$$\prod_{\sigma} c^{+}(\sigma, \mathbf{f}) c^{-}(\sigma, \mathbf{f}) = \langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}}, \quad \prod_{\sigma} \delta(\sigma) = G(\psi)^{-1} (2\pi i)^{n(k_0 - 1)},$$

and if we put

$$h_* = \max_i \{(k_0 - k_i)/2\}, \quad h^* = \min_i \{(k_0 + k_i)/2 - 1\},$$
$$c^{\pm}(\chi, \mathbf{f}) = G(\chi)^{-1} \prod_{\sigma} c^{\pm\varepsilon_\sigma(\chi)}(\sigma, \mathbf{f}),$$

then for all of the points $r \in \mathbf{Z}, h_* < r \leq h^*$ we have that

$$\frac{\Lambda(r, \mathbf{f}(\chi))}{c^{(-1)^r}(\chi, \mathbf{f})} \in T(\chi).$$

In order to describe p-adic L-functions we set

$$\Phi_p(M[\chi], s)^{-1} = \prod_{\mathfrak{p} \in S(p) \setminus S(\chi)} (1 - \chi(\mathfrak{p}) \alpha'(\mathfrak{p}) \mathcal{N}\mathfrak{p}^{-s})(1 - \chi^{-1}(\mathfrak{p}) \alpha(\mathfrak{p})^{-1} \mathcal{N}\mathfrak{p}^{s-1}),$$

where $S(p)$ is the set of all prime divisors of p in F, and the product is extended over the subset of of those prime divisors which do not belong to the support $S(\chi)$ of χ. Let us also fix $\varepsilon_0 = (\varepsilon_{0,\sigma})_\sigma \in \mathrm{Sgn}_F$, where $\varepsilon_{0,\sigma} = \pm 1$, and define the constant

$$\Omega(\varepsilon_0, \mathbf{f}) = \prod_{\sigma} c^{\varepsilon_{0,\sigma}}(\sigma, \mathbf{f}).$$

Theorem *(on p-adic L-functions attached to Hilbert cusp forms). Under the conventions and notation as above there exists a bounded $\mathbf{C}_p$-valued measure $\mu_{\varepsilon_0} = \mu_{\varepsilon_0, \mathbf{f}}$ on Gal_p which is uniquely determined by the following condition: for all Hecke characters $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ and all $r \in \mathbf{Z}$ satisfying*

$$(-1)^r \varepsilon_\sigma(\chi) = \varepsilon_{0,\sigma} \ (\text{for all } \sigma), \quad h_* < r \leq h^*$$

the following equality holds

$$\int_{\mathrm{Gal}_p} \chi^{-1}\mathcal{N}x_p^r d\mu_{\varepsilon_0} =$$

$$i_p\left(\frac{D_F^r(-1)^{[r/2]}}{G(\chi)}\frac{\Lambda(\mathbf{f}(\chi),r)}{\Omega(\varepsilon_0,\mathbf{f})\Phi_p(\mathbf{f}(\chi),r)}\prod_{\mathfrak{p}}\left(\frac{\mathcal{N}\mathfrak{p}_i^{r-1}}{\alpha(\mathfrak{p})}\right)^{\mathrm{ord}_{\mathfrak{p}}\mathfrak{c}(\chi)}\right);$$

the measure μ_{ε_0} *defines a bounded* $\mathbf{C}_p$*-analytic function*

$$L_{\varepsilon_0,M}:\mathcal{X}_p\to\mathbf{C}_p,\quad \mathcal{X}_p\ni x\mapsto\int_{\mathrm{Gal}_p} x\,d\mu_{\varepsilon_0}(M)$$

(the p-adic Mellin transform of $\mu_{\varepsilon_0}(M)$*), which is uniquely determined by its values on the characters* $x=\chi^{-1}\mathcal{N}x_p^r\in\mathcal{X}_p$.

§5. Non-Archimedean convolutions of Hilbert modular forms

5.1. Let us consider the Rankin convolution

$$L(s,\mathbf{f},\mathbf{g})=\sum_{\mathfrak{n}} C(\mathfrak{n},\mathbf{f})\,C(\mathfrak{n},\mathbf{g})\mathcal{N}(\mathfrak{n})^{-s} \tag{5.1}$$

attached to two Hilbert modular forms $\mathbf{f}$, $\mathbf{g}$ over a totally real field F of degree $n=[F:\mathbf{Q}]$, where $C(\mathfrak{n},\mathbf{f})$, $C(\mathfrak{n},\mathbf{g})$ are normalized "Fourier coefficients" of $\mathbf{f}$ and $\mathbf{g}$, indexed by integral ideals $\mathfrak{n}$ of the maximal order $\mathcal{O}_F\subset F$ (see §3). We suppose that $\mathbf{f}$ is a primitive cusp form of vector weight $k=(k_1,\cdots,k_n)$, and $\mathbf{g}$ a primitive cusp form of weight $l=(l_1,\cdots,l_n)$ We assume that for a decomposition of J_F into a disjoint union $J_F=J\cup J'$ the following condition is satisfied

$$k_i>l_i\ (\text{ for }\sigma_i\in J),\quad\text{and}\quad l_i>k_i\ (\text{for }\sigma_i\in J'). \tag{5.2}$$

Also, assume that

$$k_1\equiv k_2\equiv\cdots\equiv k_n \bmod 2, \tag{5.3}$$

and

$$l_1\equiv l_2\equiv\cdots\equiv l_n \bmod 2. \tag{5.4}$$

Let $\mathfrak{c}(\mathbf{f})\subset\mathcal{O}_F$ denote the conductor and ψ the character of $\mathbf{f}$ and $\mathfrak{c}(\mathbf{g})$, ω denote the conductor and the character of $\mathbf{g}$ ($\psi,\omega:\mathbf{A}_F^\times/F^\times\to\mathbf{C}^\times$ being Hecke characters of finite order).

The essential property of the convolution

$$L(s,\mathbf{f},\mathbf{g}(\chi))=\sum_{\mathfrak{n}}\chi(\mathfrak{n})C(\mathfrak{n},\mathbf{f})\,C(\mathfrak{n},\mathbf{g})\mathcal{N}(\mathfrak{n})^{-s}$$

(twisted with a Hecke characer χ of finite order) is the following Euler product decomposition

$$
\begin{aligned}
&L_{\mathfrak{c}}(2s+2-k_0-l_0,\psi\omega\chi^2)L(s,\mathbf{f},\mathbf{g}(\chi)) = \\
&\prod_{\mathfrak{q}} \left((1-\chi(\mathfrak{q})\alpha(\mathfrak{q})\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1-\chi(\mathfrak{q})\alpha(\mathfrak{q})\beta'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})\times\right. \\
&\left.\times(1-\chi(\mathfrak{q})\alpha'(\mathfrak{q})\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1-\chi(\mathfrak{q})\alpha'(\mathfrak{q})\beta'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})\right)^{-1},
\end{aligned}
\tag{5.5}
$$

where the numbers $\alpha(\mathfrak{q})$, $\alpha'(\mathfrak{q})$, $\beta(\mathfrak{q})$, and $\beta'(\mathfrak{q})$ are roots of the Hecke polynomials

$$
X^2 - C(\mathfrak{q},\mathbf{f})X + \psi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{k_0-1} = (X-\alpha(\mathfrak{q}))(X-\alpha'(\mathfrak{q})),
$$

and

$$
X^2 - C(\mathfrak{q},\mathbf{g})X + \omega(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{l_0-1} = (X-\beta(\mathfrak{q}))(X-\beta'(\mathfrak{q})).
$$

The decomposition (5.5) is not difficult to deduce from the following elementary lemma on rational functions, applied to each of the Euler $\mathfrak{q}$-factors: if

$$
\sum_{i=0}^{\infty} A_i X^i = \frac{1}{(1-\alpha X)(1-\alpha' X)}, \qquad \sum_{i=0}^{\infty} B_i X^i = \frac{1}{(1-\beta X)(1-\beta' X)},
$$

then

$$
\sum_{i=0}^{\infty} A_i B_i X^i = \frac{1-\alpha\alpha'\beta\beta' X^2}{(1-\alpha\beta X)(1-\alpha\beta' X)(1-\alpha'\beta X)(1-\alpha'\beta' X)}. \tag{5.6}
$$

5.2. The Rankin convolution and the tensor product of motives. Assume that there exist motives $M(\mathbf{f})$ and $M(\mathbf{g})$ associated with $\mathbf{f}$ and $\mathbf{g}$. Then the identity (5.6) shows that

$$
L_{\mathfrak{c}}(2s+2-k-l,\psi\omega\chi^2)L(s,\mathbf{f},\mathbf{g}(\chi)) = L(M[\chi],s)
$$

where $M = M(\mathbf{f})\otimes_F M(\mathbf{g})$ is the tensor product of motives over F with coefficients in some common number field T. Using the Hogde decompositions for $M(\mathbf{f})$ and $M(\mathbf{g})$ (see §4) and the Künneth formula for $M = M(\mathbf{f})\otimes_F M(\mathbf{g})$ we see that under our assumption the motive M has $d=4$, $w = k_0+l_0-2$, and the following Hodge type:

$$
\begin{aligned}
&M_{\sigma_i}\otimes\mathbf{C}\cong \\
&\oplus_{\tau\in J_T}\left(M_{\sigma_i}^{(k_0+l_0-k_i^\tau-l_i^\tau)/2,(k_0+l_0+k_i^\tau+l_i^\tau)/2-2} \oplus M_{\sigma_i}^{(k_0+l_0+k_i^\tau+l_i^\tau)/2-2,(k_0+l_0-k_i^\tau-l_i^\tau)/2}\right. \\
&\left.\oplus M_{\sigma_i}^{(k_0+l_0-|k_i^\tau-l_i^\tau|)/2-1,(k_0+l_0+|k_i^\tau-l_i^\tau|)/2-1} \oplus M_{\sigma_i}^{(k_0+l_0+|k_i^\tau-l_i^\tau|)/2-1,(k_0+l_0-|k_i^\tau-l_i^\tau|)/2-1}\right).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\Lambda(M[\chi],s) = \Lambda(s,\mathbf{f},\mathbf{g}(\chi)) = \\
&\prod_{i=1}^{n}\left(\Gamma_{\mathbf{C}}(s-(k_0+l_0-k_i-l_i)/2)\Gamma_{\mathbf{C}}(s-(k_0+l_0-|k_i-l_i|)/2+1)\right)\times \\
&\times L_{\mathfrak{c}}(2s+2-k_0-l_0,\psi\omega\chi^2)L(s,\mathbf{f},\mathbf{g}(\chi)),
\end{aligned}
$$

and this function satisfies a functional equation of the type $s \mapsto k_0 + l_0 - 2 - s$.

5.3. The critical values of the Rankin convolution. Let us now set

$$h_* = \max_i((k_0 + l_0 - |k_i - l_i|)/2 - 1),\ h^* = k_0 + l_0 - 2 - h_*.$$

The periods $c^{\pm}(\sigma, M)$ can be easily computed in terms of $c^{\pm}(\sigma, M)$ (as in the elliptic modular case; see a more general calculation in [Bl2]). As a result one gets that $c^{\pm}(\sigma, M) = c(\sigma, M)$ does not depend on the sign $\pm$, and is given by

$$c^{\pm}(\sigma, M) = \begin{cases} c^+(\sigma, \mathbf{f})c^-(\sigma, \mathbf{f})\delta(\sigma, \mathbf{g}), & \text{if } \sigma \in J \\ c^+(\sigma, \mathbf{g})c^-(\sigma, \mathbf{g})\delta(\sigma, \mathbf{f}), & \text{if } \sigma \in J'. \end{cases}$$

Moreover,

$$c^{\pm}(M[\chi]) = G(\chi)^{-2} \prod_{\sigma \in J} c^{\pm}(\sigma, M).$$

Let us apply the modified conjecture on special values 1.8 to the L-function

$$\Lambda(M[\chi], s) = \Lambda(s, \mathbf{f}, \mathbf{g}(\chi)),$$

and set $c(\mathbf{f}, \mathbf{g}) = \prod_\sigma c^+(\sigma, M)$,

$$c(J, \mathbf{f}) = \prod_{\sigma \in J} c^+(\sigma, \mathbf{f})c^-(\sigma, \mathbf{f}),\ c(J', \mathbf{g}) = \prod_{\sigma \in J'} c^+(\sigma, \mathbf{g})c^-(\sigma, \mathbf{g}),$$

and

$$\delta(J, \mathbf{f}) = \prod_{\sigma \in J} \delta(\sigma, \mathbf{f}),\ \delta(J', \mathbf{g}) = \prod_{\sigma \in J'} \delta(\sigma, \mathbf{g}).$$

Then we see in view of §4 that

$$c(J, \mathbf{f})c(J', \mathbf{f}) = \langle \mathbf{f}, \mathbf{f} \rangle,\quad \delta(J, \mathbf{f})\delta(J', \mathbf{f}) = G(\psi)^{-1}(2\pi i)^{n(k_0 - 1)},$$
$$c(J, \mathbf{g})c(J', \mathbf{g}) = \langle \mathbf{g}, \mathbf{g} \rangle,\quad \delta(J, \mathbf{f})\delta(J', \mathbf{g}) = G(\omega)^{-1}(2\pi i)^{n(l_0 - 1)},$$

and

$$c(M[\chi]) = c^{\pm}(M[\chi]) = G(\chi)^{-2} c(J, \mathbf{f})\delta(J, \mathbf{g})c(J', \mathbf{g})\delta(J', \mathbf{f}).$$

With this notation the conjecture 1.8 takes the following form: for all Hecke characters χ of finite order and $r \in \mathbf{Z}, h_* < r \le h^*$ we have that

$$\frac{\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))}{G(\chi)^{-2} c(J, \mathbf{f})\delta(J, \mathbf{g})c(J', \mathbf{g})\delta(J', \mathbf{f})} = \frac{\Lambda(M[\chi], r)}{G(\chi)^{-2} c(M)} \in \mathbf{Q}(\mathbf{f}, \mathbf{g}, \chi).$$

5.4. Let us consider the special case when $J' = \emptyset$, i.e. $k_i > l_i$ for all $\sigma_i \in J_F$. Then

$$c(J, \mathbf{f}) = c(J_F, \mathbf{f}) = \langle \mathbf{f}, \mathbf{f} \rangle,\quad \delta(J, \mathbf{g}) = \delta(J_F, \mathbf{g}) = G(\omega)^{-1}(2\pi i)^{n(l_0 - 1)},$$

and the above property transforms to the following:

$$\frac{\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))}{G(\chi)^{-2}\langle \mathbf{f}, \mathbf{f} \rangle, G(\omega)^{-1}(2\pi i)^{n(l_0 - 1)}} \in \mathbf{Q}(\mathbf{f}, \mathbf{g}, \chi),$$

where $\mathbf{Q}(\mathbf{f}, \mathbf{g}, \chi)$ denotes the subfield of $\mathbf{C}$ generated by the Fourier coefficients of $\mathbf{f}$ and $\mathbf{g}$, and the values of χ. This algebraicity property was established by G.Shimura [Sh1] by means of a version of the Rankin –Selberg method.

In the general case the above algebraicity property was also studies by G.Shimura [Sh2], [Sh3] (for some special Hilbert modular forms, coming from quaternion algebras) and by M.Harris [Ha3] using the theory of arithmetical vector bundles on Shimura varieties. The idea of the proof was to replace the original automorphic cusp form $\mathbf{f} : G(\mathbf{A}) \to \mathbf{C}$ of holomorphic type by another cusp form $\mathbf{f}^J : G(\mathbf{A}) \to \mathbf{C}$ such that

$$\mathbf{f}^J(g_1, \cdots, g_n) = \mathbf{f}(g_1 j_1, \cdots, g_n j_n),$$

where $g_i \in \mathrm{GL}_2(\mathbf{R})$,

$$j_i = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } i \in J \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \text{if } i \in J'. \end{cases}$$

Then $\mathbf{f}^J$ can be described by functions $\mathbf{f}_\lambda^J$ on $\mathfrak{H}^n$, which are holomorphic in z_i $(i \in J)$ and antiholomorphic in z_i $(i \in J')$. Then the differential forms

$$\mathbf{f}_\lambda^J \wedge_{i \in J} d\bar{z}_i$$

define a certain class $cl(\mathbf{f}^J)$ of the degree $|J|$ in the coherent cohomology of the Hilbert – Blumenthal modular variety, or rather its toroidal compactification ([Ha1], [Ha2]). This space of coherent cohomology has a natural rational structure over a certain number field F^J, defined in terms of canonical models. From the theory of new forms it follows that there exist a constant $\nu(J, \mathbf{f}) \in \mathbf{C}^\times$ such that the differential form attached to $\nu(J, \mathbf{f})^{-1}\mathbf{f}^J$ is rational over the extension of F^J obtained by adjoining the Hecke eigenvalues of $\mathbf{f}$. Then the critical values of the type $\Lambda(r, \mathbf{f}, \mathbf{g})$ can be expressed in terms of a cup product of the form

$$cl(\mathbf{f}^J) \cup cl(\mathbf{g}^{J'}) \cup E,$$

where E is a (nearly) holomorphic Eisenstein series. Then the above algebraicity property can be deduced from the fact that the cup product preserves the rational structure in the coherent cohomology. However, the technical details of the proof are quite difficult.

5.5. p-adic convolutions of Hilbert cusp forms. Now we give a precise description of the p-adic convolution of $\mathbf{f}$ and $\mathbf{g}$ assuming that both $\mathbf{f}$ and $\mathbf{g}$ are p-ordinary, i.e.

$$\mathrm{ord}_p\alpha(\mathfrak{p}_i) = (k_0 - k_i)/2, \quad \mathrm{ord}_p\alpha'(\mathfrak{p}_i) = (k_0 + k_i)/2 - 1,$$
$$\mathrm{ord}_p\beta(\mathfrak{p}_i) = (l_0 - l_i)/2, \quad \mathrm{ord}_p\beta'(\mathfrak{p}_i) = (l_0 + l_i)/2 - 1,$$

or equivalently, $\mathrm{ord}_p C(\mathfrak{p}_i, \mathbf{f}) = (k_0 - k_i)/2$, and $\mathrm{ord}_p C(\mathfrak{p}_i, \mathbf{g}) = (l_0 - l_i)/2$. We assume also that the conductors of $\mathbf{f}$ and $\mathbf{g}$ are coprime to p and we set

$$\begin{aligned}
&\Phi_p(s, \mathbf{f}, \mathbf{g}(\chi))^{-1} = \\
&\prod_{\sigma_i \in J \setminus S(\chi)} (1 - \chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s})(1 - \chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s}) \times \\
&\times (1 - \chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1})(1 - \chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta'(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1}) \times \\
&\times \prod_{\sigma_i \in J' \setminus S(\chi)} (1 - \chi(\mathfrak{p}_i)\alpha(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s})(1 - \chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s}) \times \\
&\times (1 - \chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1})(1 - \chi(\mathfrak{p}_i)^{-1}\alpha'(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1}).
\end{aligned}$$

Then we introduce the following constant:

$$\begin{aligned}
&\Omega(\mathbf{f}, \mathbf{g}) = c(J, \mathbf{f})\delta(J, \mathbf{g})c(J', \mathbf{g})\delta(J', \mathbf{f}) = \\
&\prod_{\sigma \in J} c^+(\sigma, \mathbf{f})c^-(\sigma, \mathbf{f})\delta(\sigma, \mathbf{g}) \prod_{\sigma \in J'} c^+(\sigma, \mathbf{g})c^-(\sigma, \mathbf{g})\delta(\sigma, \mathbf{f})
\end{aligned}$$

5.6. Description of the p-adic convolution. *Under the conventions and notation as above there exists a bounded $\mathbf{C}_p$-valued measure $\mu = \mu_{\mathbf{f},\mathbf{g}}$ on* Gal_p, *which is uniquely determined by the following condition: for all Hecke characters $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ and all $r \in \mathbf{Z}$ satisfying $h_* < r \leq h^*$ the following equality holds:*

$$\begin{aligned}
&\int_{\mathrm{Gal}_p} \chi^{-1}\mathcal{N}x_p^r d\mu_{\mathbf{f},\mathbf{g}} = \\
&\quad i_p\Big(\frac{D_F^{2r}(-1)^r}{G(\chi)^2}\frac{\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))}{\Omega(\mathbf{f}, \mathbf{g})\Phi_p(r, \mathbf{f}, \mathbf{g}(\chi))} \times \\
&\quad \times \prod_{\sigma_i \in J}\left(\frac{\mathcal{N}\mathfrak{p}_i^{r-1}}{\alpha(\mathfrak{p}_i)^2\beta(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)}\right)^{\mathrm{ord}_{\mathfrak{p}_i} c(\chi)} \prod_{\sigma_i \in J'}\left(\frac{\mathcal{N}\mathfrak{p}_i^{r-1}}{\beta(\mathfrak{p}_i)^2\alpha(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)}\right)^{\mathrm{ord}_{\mathfrak{p}_i} c(\chi)}\Big),
\end{aligned}$$

and the measure $\mu_{\mathbf{f},\mathbf{g}}$ defines a bounded $\mathbf{C}_p$-analytic function

$$L_{\mathbf{f},\mathbf{g}} : \mathcal{X}_p \to \mathbf{C}_p, \quad \mathcal{X}_p \ni x \mapsto \int_{\mathrm{Gal}_p} x \, d\mu_{\mathbf{f},\mathbf{g}}$$

(the p-adic Mellin transform of $\mu_{\mathbf{f},\mathbf{g}}$), which is uniquely determined by its values on the characters $x = \chi^{-1}\mathcal{N}x_p^r \in \mathcal{X}_p$.

(Note that the above expression could be written in a slightly simplier form if we take into account the equalities:

$$\alpha(\mathfrak{p})^2\beta(\mathfrak{p})\beta'(\mathfrak{p}) = \alpha(\mathfrak{p})^2\omega(\mathfrak{p})\mathcal{N}\mathfrak{p}^{l_0-1}, \quad \beta(\mathfrak{p})^2\alpha(\mathfrak{p})\alpha'(\mathfrak{p}) = \beta(\mathfrak{p})^2\psi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{k_0-1}.)$$

5.7. Concluding remarks. The existence of the p-adic measure in 5.6 is known in the special case, and $J = \emptyset$ (see [Pa2]), where $\mathbf{f}$ and $\mathbf{g}$ are assumed to be automorphic forms of scalar weights k and l, $k > l$. Also, this construction was recently extended by My Vinh Quang (Moscow University) to Hilbert automorphic forms $\mathbf{f}$ and $\mathbf{g}$ of arbitrary vector weights $k = (k_1, \cdots, k_n)$, and $l = (l_1, \cdots, l_n)$ such that $k_i > l_i$ for all $i = 1, \cdots, n$, and to the *non-p-ordinary*, i.e. *supersingular* case, when $| i_p(\alpha(\mathfrak{p}) |_p < 1$ for all $\mathfrak{p} \mid p$. In this situation the p-adic convolution of $L_{\mathbf{f},\mathbf{g}}$ is also uniquely determined by the above condition provided that it has the prescribed logarithmic growth on $\mathcal{X}_p$ (see [V1]).

In the general case the proof of the algebraic properties of the Rankin convolution in [Ha3] can be used also in order to carry out a p-adic construction. First of all, one obtains an expression for complex-valued distributions attached to $\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))$ in terms of the cup product of certain coherent cohomology classes, and one verifies that these distributions take algebraic values. Then, integrality properties of the arithmetic vector bundles can be used for proving some generalized Kummer congruences for the values of these distributions, which is equivalent to the existence of p-adic L-functions in 5.6. However, some essential technical difficulties remain in the general case, and 5.6 can not be regarded yet as a theorem proven in full generality.

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