VOLUMES, LATTICE POINTS, AND SINGULARITIES

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Introduction.

A classical result, cf. [La-1], for a positive definite quadratic form Q on $\mathbf{R}^n, n \geq 5$, states the following. For $t > 0$ set

$$\mathcal{N}_Q(t) = \#\{m \in \mathbf{Z}^n : Q(m) \leq t\}$$
$$\mathcal{V}_Q(t) = \int_{Q \leq t} dx_1 \cdots dx_n .$$

Then

$$\mathcal{N}_Q(t) - \mathcal{V}_Q(t) = O(\mathcal{V}_Q(t)/t^{\frac{n}{n+1}}) \quad as \ t \to \infty. \tag{0.1}$$

Furthermore, it is simple to see that $\mathcal{V}_Q(t) = At^{n/2}$, where

$$A = \int_{Q \leq 1} dx_1 \cdots dx_n.$$

This is to be understood as a significant improvement over the trivial error estimate of $O(\mathcal{V}_Q(t)/t^{\frac{1}{2}})$, obtained solely because of the homogeneity of Q and compactness of $\{Q \leq 1\}$.

The argument of Landau relies heavily upon the functional equation satisfied by the quadratic theta function. It is therefore not capable of generalization to polynomials of degree at least three. There are however quite reasonable problems to consider that require an extension of this type of result. For one example, a general question in numerical integration is the following.

Given an unbounded semialgebraic set $\mathcal{S} \subsetneq \mathbf{R}^n$ with positive Lebesgue measure, polynomial P that is proper on $\mathcal{S}$, and rational function φ, defined on $\mathcal{S}$, define

$$N_P(t, \varphi, \mathcal{S}) = \sum_{\{m \in \mathbf{Z}^n \cap \mathcal{S} : |P(m)| \leq t\}} \varphi(m)$$
$$V_P(t, \varphi, \mathcal{S}) = \int_{\{|P| \leq t\} \cap \mathcal{S}} \varphi dx_1 \cdots dx_n .$$

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Problem. Analyze the asymptotic of $N_P(t, \varphi, \mathcal{S}) - V_P(t, \varphi, \mathcal{S})$ as $t \to \infty$.

This general problem has both a geometric and arithmetic feature and is in general difficult. On the other hand, if $\mathcal{S}$ is $\mathbf{R}^n$ then some success has been achieved. For example, if P is homogenous and positive definite on $\mathbf{R}^n, n \leq 6$, the lower bound

$$\theta \geq \frac{2}{d} \cdot \frac{n}{n+1} \tag{0.2}$$

has been derived by using formulae for the indices of oscillation of the simple singularities, cf. [CdV-1], [Ra], [Va-1]. However, this method does not seem to extend so easily to inhomogeneous polynomials, nor to the inclusion of weights, as determined by φ. It is also not yet known how these results can be extended to arbitrary n.

An alternative and more functional analytic method exploits the existence of an integral representation for $N_P(t, \varphi, \mathbf{R}^n)$ as the Mellin transform of a meromorphic function whose first pole and growth at infinity in bounded vertical strips (of the complex plane) are understood reasonably well. The first significant result along these lines was given by Mahler-Bochner when P satisfied an ellipticity condition on $\mathbf{R}^n$ and φ was a quotient of relatively prime polynomials, each of which was elliptic on $\mathbf{R}^n$.

One says that $P \in \mathbf{R}[x_1, \ldots, x_n]$ is elliptic on $\mathbf{R}^n$ if the top degree term appearing in P is not zero on $\mathbf{R}^n - \{(0, \ldots, 0)\}$. Let d denote the degree of P and δ the difference of degrees of the numerator and denominator of φ. Mahler [Ma] and Bochner [Bo] investigated the behavior in t of $N_P(t, \varphi, \mathbf{R}^n) - V_P(t, \varphi, \mathbf{R}^n)$ in two different ways. One can combine their analysis into one result that says the following.

THEOREM. *There exist $A > 0, \theta \geq 1/nd$, and sequences of real numbers $\{A_i\}_i, \{\tau_i\}_i, \tau_i > 0$ and satisfying $\lim_{i \to \infty} \tau_i = +\infty$, such that*

$$V_P(t, \varphi, \mathbf{R}^n) = A t^{\frac{n+\delta}{d}} + \sum_{i=1}^{\infty} A_i t^{\frac{n+\delta-\tau_i}{d}}$$

$$N_P(t, \varphi, \mathbf{R}^n) - V_P(t, \varphi, \mathbf{R}^n) = O(V_P(t, \varphi)/t^{\theta}). \tag{0.3}$$

The reader will note that if $n \geq 2$, then the estimate obtained in (0.2) is considerably better than (0.3). Presumably, the methods of [CdV-2] apply to elliptic polynomials when $n \leq 6$.

Recently [Li-3,4,7], the author has extended (0.3) to a considerably larger class of P, Q, T where $\varphi = Q/T$. These consist of hypoelliptic polynomials on $\mathbf{R}^n$, cf. Section 1 for a definition. It is however, not yet known whether the estimate $1/nd$ can be improved by combining the geometric methods of [CdV-1, Ra] with the analytical ones developed in [ibid]. This appears to be a very interesting question to consider.

The set $\mathbf{R}^n$ being the largest possible, it is natural to search for smaller sets $\mathcal{S}$ for which some reasonable analogue of (0.2) is still true. The first goal of this paper is to describe some progress on this problem when S equals the box $[1,\infty)^n$ and the polynomials P, Q, T are hypoelliptic on $[1,\infty)^n$.

In the following let $N_P(t,\varphi), V_P(t,\varphi)$ denote the function $N_P(t,\varphi,[1,\infty)^n), V_P(t,\varphi,[1,\infty)^n)$.

If one is interested *only* in the standard lattice point problem, in which $\varphi \equiv 1$, then an elementary argument, given in Section 2, will show the following.

THEOREM A. *If P is hypoelliptic on $[1,\infty)^n$ then there exists $\theta > 0$ such that*

$$N_P(t,1) - V_P(t,1) = O(V_P(t,1)/t^\theta) \quad \text{as } \to \infty.$$

However the argument does not give so explicit an estimate for the error term's rate of growth. Indeed, the estimate for θ involves terms that are, so far, difficult to understand. Essentially, the behavior of P near the boundary of S can affect the estimate of θ, and this is difficult to understand simply in terms of P. This should be expected for any $S \subsetneq \mathbf{R}^n$ one might use.

More generally, when φ is a rational function whose numerator and denominator, in reduced form, are both hypoelliptic on $[1,\infty)^n$, it is natural to ask how well $N_P(t,\varphi)$ approximates $V_P(t,\varphi)$ (or vice versa) as $t \to \infty$. For the analysis of this weighted lattice point problem, the simple argument, used in the proof of Theorem A, no longer suffices, and one apparently needs to use a subtler analytical argument. To formulate the answer given here, one first recalls (cf. Theorem 2.1) the

THEOREM. *There exist $\rho_1(\varphi), \lambda_1(\varphi), \epsilon > 0$, nonzero polynomials $A_1(\varphi,u), B_1(\varphi,u) \in \mathbf{R}[u]$, as well as sequences of positive numbers τ_i and polynomials $A_i(\varphi,u), i \geq 2$, so that*

$$V_P(t,\varphi) = t^{\lambda_1(\varphi)} A_1(\varphi, logt) + \sum_{i=2}^{\infty} t^{\lambda_1(\varphi)-\tau_i} A_i(\varphi, logt) \qquad [Li-1, Ig]\,,$$
$$N_P(t,\varphi) = t^{\rho_1(\varphi)} B_1(\varphi, logt)\,(1 + O(t^{-\epsilon})) \qquad [Li-3]\,.$$

The first main result of this paper is proved in Section 4 and is a simple consequence of Theorem 4.1. Using the notation in this theorem, it states-

THEOREM B. *If P is hypoelliptic on $[1,\infty)^n$, and φ is a quotient of hypoelliptic polynomials on $[1,\infty)^n$, then*

$$\rho_1(\varphi) = \lambda_1(\varphi) \quad \text{and} \quad A_1(\varphi,u) = B_1(\varphi,u).$$

Set $\rho(\varphi)$ to be the number denoted $\rho_1(\varphi), \lambda_1(\varphi)$. From this result, one concludes

COROLLARY. *There exists $\theta > 0$ such that*

$$N_P(t,\varphi) - V_P(t,\varphi) = O(t^{\rho(\varphi)-\theta}) \text{ as } t \to \infty.$$

However, a reasonable expression or estimate for θ is not yet available. Obtaining such an estimate would be very useful.

Theorem B answers a question posed by Prof. Ehrenpreis to the author in the spring 1990. As noted below, this question has significance considerably beyond the context of lattice point problems involving one polynomial.

There are two immediate, but also interesting, applications of Theorems A, B. Theorem A provides an alternative "discrete" method for deriving the asymptotic for the number of eigenvalues at most t for a self-adjoint extension of a hypoelliptic PsDO. This is discussed in Section 5. Theorem B allows one to extend the results of [Li-2] to all hypoelliptic polynomials that are, in addition, "tame" on $\mathbf{C}^n$ in the sense of [Br]. This is discussed in Section 6. Thus, the main term in the asymptotic for $N_P(t, \varphi)$ is again shown to be a "cohomological invariant" whenever φ is a hypoelliptic polynomial on $[1, \infty)^n$. This gives further evidence of a topological "local-global" principle that describes the contribution to $N_P(t, \varphi)$ of the singularities of P in $\mathbf{C}^n$. This principle may be viewed as an analogue of an arithmetic local-global principle that determines when the singular series, arising in the Hardy-Littlewood-Vinogradov analysis (cf. [Dav-1]) of the counts $\#\{P = t\} \cap \mathbf{N}^n$ as $t \to \infty$, possesses a well-defined rate of growth as $t \to \infty$.

Because the methods used in this paper combine general analytic and geometric techniques, they provide a convenient framework to help analyze the precise asymptotics of "simultaneous" lattice point or volume problems, about which one can find very little in the literature. These are problems similar to that formulated at the beginning of the Introduction, but involving $k > 1$ polynomials P_i in place of the single polynomial P. The approach, taken to deal with such questions, attempts to use a multivariable inverse Mellin transform to extract asymptotic information in a manner analogous to the well known method in the complex plane (= one variable). An important theme will be the incorporation of geometric information obtained from a resolution of singularities at infinity to find the asymptotics, if they exist.

Understanding/determining precisely such asymptotics would be of interest in several subjects. A few of these will now be briefly described. There is an obvious connection to numerical integration and integer programming problems over semialgebraic subsets. Secondly, Gromov has pointed out how one can bound from below the first eigenvalue of a family of semialgebraic sets in terms of their volume. Thus, the ability to give precise asymptotics for volumes of multiparameter families of semialgebraic sets should find a use in Gromov's program, described in [Gro]. Thirdly, the asymptotics of the number of simultaneous eigenvalues of finitely many differential operators can sometimes be related to those of the asymptotics of the volume of regions like those considered in this paper [CdV-2]. Thus, one could give very precise descriptions of such asymptotics if this program proves successful. Fourthly, a method, successful in obtaining asymptotics of multivariable inverse Mellin transforms, should in principle be useful to analyse the convergence of the "singular integral" in the adelic Hardy-Littlewood method, an example of which is the Poisson formula of Igusa [Ig, ch. 4]. Fifthly, such a method should enable one to refine the recent results of Passare [Pas] who studied the behavior of residue currents, defined via limits of a certain integral over "admissible paths." That is, in place of limits, one should be able to describe the precise asymptotic behavior over such paths.

Section 7 is significant in light of these remarks. The most important results are contained in Theorems 7.8 and Theorem B_k, which are generalizations of Theorem 4.1. Theorem B_k is of basic importance and is the second main result of the paper. Essentially, Theorem B_k implies that if the dominant term(s) of the asymptotics of a simultaneous volume integral

$$\int_{\{P_1 \le t_1, \ldots, P_k \le t_k\} \cap [1,\infty)^n} \varphi \, dx_1 \cdots dx_n, \tag{0.4}$$

exist and are determined by a certain Newton polyhedron (cf. 7.7), then these are also the dominant terms of the corresponding simultaneous lattice point problem. Thus, one needs a criterion to determine when this polyhedron in fact does determine asymptotics of (0.4).

Theorem 7.14 gives one such criterion, which is expressed in geometric terms. It appears to be difficult, however, to show that the criterion holds in general.

A method of analysis, needed to determine if the criterion is satisfied, involves a natural (but not necessarily easy) extension of the techniques used to find the largest pole of the generalized functions $P_{\pm}^{s}$ in terms of data arising from a resolution of singularities, cf. [Ig], [Li-8], [Lo-1]. When the weight function φ is identically 1, [Li-6] will carry out this analysis for certain classes of *pairs* of nondegenerate polynomials, whose resolution data can be analyzed with sufficient precision in order to verify that the conditions of 7.14 are satisfied. Such results in fact enable one to sharpen the upper bound estimates of [Li-5, sec. 6] and therefore, to find the first nontrivial classes of examples of multivariable Tauberian theorems with precise asymptotics (cf. [Ga] for a much more general discussion of this type of problem). In this way, the reader will note the significant advantage of incorporating geometric reasoning into a fundamentally analytic problem.

This question is also closely related to the work of Loeser in [Lo-2]. It should certainly be possible to show that the integrals in (0.4) are essentially functions of "regular singular type" ("de type singulier-régulier"), in the sense of [ibid, p. 458]. This would give a general and reasonably concrete representation of the volume integrals as $t_1, \ldots, t_k \to \infty$. It should then be possible, by the Mellin transform methods developed in Section 7, to connect such a description with the geometry of the Newton polyhedron, defined in 7.7. If this approach worked, it would lead to a very important generalization of the results, obtained in [Li-6]. Indeed, it would undoubtedly lead to a systematic calculus for multivariable asymptotic expansions "determined by geometry" that would prove valuable in many different subjects. Here the key point is the fact that the Newton polyhedron in 7.7 is effectively calculable given an explicit resolution of singularities of the type described in section 7. The recent work of Bierstone-Milman provides this, in principle.

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Notation. This paper will use the following notations.
(1) For $A = (A_1, \ldots, A_n) \in \mathbf{R}^n$, one sets $|A| = A_1 + \ldots + A_n$.
(2) $s = \sigma + it$ denotes a complex variable. $\mathbf{s} = (s_1, \ldots, s_k) \in \mathbf{C}^k$. It is sometimes useful to write $\mathbf{s} = \boldsymbol{\sigma} + i\mathbf{t}$ to emphasize the real and imaginary parts.
(3) If $x = (x_1, \ldots, x_n)$, then $\|x\| = max\,\{|x_i|\}$.
(4) Set $dx = dx_1 \cdots dx_n$.
(5) The interior of a set A is denoted $int(A)$.
(6) For a function $g : X \to \mathbf{R}$ define

$$g_+(x) = \begin{cases} g(x) \text{ if } g(x) > 0 \\ 0 \text{ if } g(x) \leq 0 \end{cases}$$
$$g_-(x) = \begin{cases} -g(x) \text{ if } g(x) < 0 \\ 0 \text{ if } g(x) \geq 0 \end{cases}$$

(7) If φ is a weight function and P is a polynomial,

$$N_P(t, \varphi) = \sum_{\substack{m \in \mathbf{N}^n \\ P(m) \leq t}} \varphi(m),$$
$$V_P(t, \varphi) = \int_{\{P \leq t\} \cap [1, \infty)^n} \varphi \, dx_1 \cdots dx_n .$$

When $\varphi \equiv 1$, one denotes these functions as $N_P(t), V_P(t)$.
(8) One writes $\mathbf{t}^{\mathbf{s}}$ for the product $t_1^{s_1} \cdots t_k^{s_k}$.

Section 1. *Some properties of hypoelliptic polynomials*

Recall that $P \in \mathbf{R}[x_1, \ldots, x_n]$ is hypoelliptic on $[1,\infty)^n$ if for all differential monomials D_x^A one has

$$\lim_{\substack{\|x\|\to\infty \\ x\in[1,\infty)^n}} \frac{D_x^A P(x)}{P(x)} = 0\,. \tag{1.1}$$

Hörmander [Hö, ch. 11] has shown that (1.1) is equivalent to

$$\text{There exist } c, C > 0 \text{ such that for all } x \in [1,\infty)^n \quad \left|\frac{D_x^A P(x)}{P(x)}\right| \le C\|x\|^{-c|A|}\,. \tag{1.2}$$

Further, he also observed that (1.2) is equivalent to the satisfaction of (1.1) for all A with $|A| = 1$.

Definition 1.3. The exponent of hypoellipticity for P is the largest c so that 1.2 holds. It will be denoted c_P in the following. ∎

The consequence of (1.2) of use in this paper is

PROPOSITION 1.4. *There exist $\alpha, C, D > 0$ such that*

$$|P(x_1,\ldots,x_n)| > C(x_1\cdots x_n)^\alpha \text{ if } (x_1,\ldots,x_n) \in [1,\infty)^n \cap \{\|(x_1,\ldots,x_n)\| \ge D\}. \tag{1.5}$$

The best α so that 1.5 holds is denoted α_P in the following.

Note. One will assume in the rest of the paper, without loss of generality, that P is positive on $[1,\infty)^n \cap \{\|(x_1,\ldots,x_n)\| \ge D\}$. Then

$$\text{there exists } b \ge 0 \text{ such that } P + b > 0 \quad \text{on } [1,\infty)^n\,. \tag{1.6}$$

Remark 1.7. If it is in fact necessary to use a positive b to insure (1.6), then one will replace P by $P+b$ below. However, for simplicity, this translation of P will continue to be denoted as P throughout the rest of the paper. As a result, one will assume throughout the rest of the article that P is positive on $[1,\infty)^n$.

The effect of the translation will not at all affect the conclusions in the Theorems. The reason for this follows from the properties stated in Corollaries 2.2, 2.4, which have been proved for arbitrary P hypoelliptic on $[1,\infty)^n$, and the easily established statements given below. In these, the notations from 2.2 and 2.4 are used.

$$N_P(t,\varphi) - N_{P+b}(t,\varphi) = N_P(t,\varphi) - N_P(t-b,\varphi) = O(t^{\rho_1(\varphi)-\theta})\,,$$
$$V_P(t,\varphi) - V_{P+b}(t,\varphi) = O(t^{\lambda_1(\varphi)-\theta})\,,$$

for some $\theta > 0$. ∎

A simple calculation will also show the following. Assume that $\varphi = Q/T$, with $gcd(Q,T) = 1$. If Q, T are both hypoelliptic on $[1,\infty)^n$, then for any nonzero index A one has

$$\lim_{\substack{\|x\|\to\infty \\ x\in[1,\infty)^n}} \frac{D_x^A \varphi(x)}{\varphi(x)} = 0\,. \tag{1.8}$$

A fraction $\varphi = Q/T$ with Q, T hypoelliptic polynomials will be called a "hypoelliptic fraction".

Remark 1.9. In order to avoid unnecessary complications in the discussion below, one will also assume in the rest of the article that $T(x) \neq 0$ for all $x \in [1,\infty)^n$. The industrious reader will easily be able to modify the arguments in the event that T is allowed to vanish on at most a compact subset of $[1,\infty)^n$. Since Q, T cannot, in general, vanish outside a compact subset of $[1,\infty)^n$, the sign of φ is constant outside such a set. Without loss of generality, one will assume that φ is positive. ∎

Section 2. *A Dirichlet series determined by φ and P*

Given P, φ satisfying the properties discussed in Section 1, define

$$D(s,\varphi) = \sum_{m\in\mathbf{N}^n} \frac{\varphi(m)}{P(m)^s}$$

$$I(s,\varphi) = \int_{[1,\infty)^n} \frac{\varphi}{P^s}\, dx\,.$$

Results proved in [Li-1,3,4] and [Ig, Ni] established the

THEOREM 2.1.
(i) There exists $B > 0$ such that if $\sigma > B$ then $D(s,\varphi), I(s,\varphi)$ are analytic.
(ii) $D(s,\varphi), I(s,\varphi)$ admit analytic continuations to $\mathbf{C}$ as meromorphic functions with polar locus contained in finitely many arithmetic progressions of rational numbers.

Let $\rho_1(\varphi)$ resp. $\lambda_1(\varphi)$ denote the largest pole of $D(s,\varphi)$ resp. $I(s,\varphi)$.

(iii) There exists $A > 0$ so that for each $\tau > 0$, and $\sigma_1 < \sigma_2 \leq \rho_1(\varphi)$ there exists $C = C(\tau, \sigma_1, \sigma_2)$ such that

$$|D(s,\varphi)| < C(1 + |t|^{A(\rho_1(\varphi)-\sigma)+\tau})$$

if $\sigma \in [\sigma_1, \sigma_2]$ and $|t| \geq 1$.
(iv) For any polynomial $B(s)$, and $\sigma_1 < \sigma_2$ there exists $C > 0$ such that

$$|B(s)\, I(s,\varphi)| < C, \text{ for all } \sigma \in [\sigma_1, \sigma_2] \text{ and } |t| \geq 1.$$

A tauberian argument, due to Landau [La-2], uses (i-iii) of (2.1) to prove:

COROLLARY 2.2. *Let $\rho_1(\varphi) > \rho_2(\varphi) > \cdots > \rho_\ell(\varphi) > \rho_1(\varphi) - \frac{1}{A} \geq \rho_{\ell+1}(\varphi) \cdots$ be the first $\ell + 1$ poles of $D(s,\varphi)$. Then there exist nonzero polynomials $A_1(\varphi, u), \ldots, A_\ell(\varphi, u) \in \mathbf{R}[u]$ such that*

$$N_P(t,\varphi) = \sum_{i=1}^{\ell} t^{\rho_i(\varphi)} A_i(\varphi, log t) + O_\epsilon(t^{\rho_1(\varphi) - \frac{1}{A} + \epsilon}) \quad \text{as } t \to \infty.$$

Remark 2.3. An estimate for the smallest possible A has been given in two cases. In [Sa-2], Sargos showed that if P has positive coefficients then one can choose $A = degP$. Moreover, this is an optimal (i.e. the smallest) estimate when taken over all polynomials with positive coefficients. If P is hypoelliptic, then [Li-4] showed that one can choose $A = n\, degP$. However, this is not an optimal estimate, as Sargos' example indicates. It is not clear what is an optimal estimate for A if the degree of P is at least two. ∎

A standard argument (cf. [Ig]) uses (i), (ii), (iv) of (2.1) to give a complete asymptotic expansion for $V_P(t, \varphi)$:

COROLLARY 2.4. *Let $\lambda_1(\varphi) > \lambda_2(\varphi) > \cdots$ be the poles of $I(s, \varphi)$. Then there exist nonzero polynomials $B_1(\varphi, u), B_2(\varphi, u), \ldots, \in \mathbf{R}[u]$ such that*

$$V_P(t,\varphi) = \sum_{i=1}^{\infty} t^{\lambda_i(\varphi)} B_i(\varphi, log t) \quad \text{as } t \to \infty.$$

One next observes the following identities. For b sufficiently large,

$$\begin{aligned} N_P(t,\varphi) &= \frac{1}{2\pi i} \int_{\sigma=b} t^s\, D(s,\varphi) \frac{ds}{s}\,, \\ V_P(t,\varphi) &= \frac{1}{2\pi i} \int_{\sigma=b} t^s\, I(s,\varphi) \frac{ds}{s}\,. \end{aligned} \tag{2.5}$$

Theorem B now follows by combining (2.5), (iii), (iv) of (2.1), a standard application of residue theory, and the following:

THEOREM B^*.
(i) $\rho_1(\varphi) = \lambda_1(\varphi)$.

Let $\rho(\varphi)$ denote the common first pole of $D(s, \varphi), I(s, \varphi)$.

(ii) $D(s, \varphi) - I(s, \varphi)$ is analytic at $s = \rho(\varphi)$.

Theorem B^*, in turn, follows immediately from the proof of Theorem 4.1.

Remark 2.6. Indeed, Theorem B^* implies that the principal parts of $D(s, \varphi)$, $I(s, \varphi)$ agree at the first pole $s = \rho(\varphi)$. The polynomials $A_1(\varphi, u), B_1(\varphi, u)$, appearing in the statement of Theorem B, are uniquely determined by the principal parts. Thus, Theorem B^* implies $A_1(\varphi, u) = B_1(\varphi, u)$. ∎

Of course, Theorem A will also follow from Theorem B^*. However, one can give an elementary argument that uses only (2.2), (2.4) when $\varphi \equiv 1$.

Proof of Theorem A: Let $\epsilon > 0$ and c_P the hypoellipticity exponent of (1.3). Define the sets

$$\begin{aligned}
\mathcal{U}_\epsilon(t) &= \{x \in [1,\infty)^n : P(x) \le t + t^{1-\epsilon c_P}\}, \\
\mathcal{L}_\epsilon(t) &= \{x \in [1,\infty)^n : \|x\| \ge t^\epsilon - \frac{1}{2} \text{ and } P(x) \le t - t^{1-\epsilon c_P}\}, \\
\ell_\epsilon(t) &= \{x \in [1,\infty)^n : \|x\| \le t^\epsilon - \frac{1}{2} \text{ and } P(x) \le t - t^{1-\epsilon c_P}\}, \\
\mathcal{C}_\epsilon(t) &= \bigcup_{\substack{m \in \mathbf{N}^n \\ \|m\| \ge t^\epsilon \\ P(m) \le t}} C(m),
\end{aligned}$$

where

$$C(m) = \{x : |x_i - m_i| < 1/2, \text{ for each i}\}.$$

The following is easily verified.

$$vol(\mathcal{L}_\epsilon(t)) + vol(\ell_\epsilon(t)) = vol\left(\{x \in [1,\infty)^n : P(x) \le t - t^{1-\epsilon c_P}\}\right), \tag{2.7.1}$$

and

$$vol(\ell_\epsilon(t)) \le vol(\{x \in [1,\infty)^n : \|x\| \le t^\epsilon - 1/2\}) \le C(t^\epsilon - 1/2)^n \text{ for some } C > 0. \tag{2.7.2}$$

In (2.4), when $\varphi \equiv 1$, drop the "φ" as an argument of the exponents λ_i and polynomials $B_i(u)$. Then, setting $\beta = \lambda_1 - \lambda_2$, one has

$$\begin{aligned}
vol(\mathcal{U}_\epsilon(t)) &= \sum_{i=1}^{\infty} (t + t^{1-\epsilon c_P})^{\lambda_i} B_i\left(log\,(t + t^{1-\epsilon c_P})\right) \\
&= (t + t^{1-\epsilon c_P})^{\lambda_1} B_1\left(log\,(t + t^{1-\epsilon c_P})\right) + O_\kappa((t + t^{1-\epsilon c_P})^{\lambda_1 - \beta + \kappa}) \\
&= t^{\lambda_1} B_1(log\, t) + O_\kappa(t^{\lambda_1 - \epsilon c_P + \kappa}) + O_\kappa(t^{\lambda_1 - \beta + \kappa})
\end{aligned} \tag{2.8.1}$$

and similarly

$$vol(\mathcal{L}_\epsilon(t)) = t^{\lambda_1} B_1(log\, t) + O_\kappa(t^{\lambda_1 - \epsilon c_P + \kappa}) + O_\kappa(t^{\lambda_1 - \beta + \kappa}). \tag{2.8.2}$$

Property (1.2) of hypoellipticity implies, by means of the Taylor expansion of P around each point m, used in the definition of $\mathcal{C}_\epsilon(t)$, that for all t sufficiently large,

$$\mathcal{L}_\epsilon(t) \subset \mathcal{C}_\epsilon(t) \subset \mathcal{U}_\epsilon(t). \tag{2.9}$$

Moreover,

$$vol(C_\epsilon(t)) = N_P(t) - \nu_\epsilon(t),$$

where

$$\nu_\epsilon(t) = \#\{m \in [1,\infty)^n \cap \mathbf{N}^n : \|m\| \le t^\epsilon - \frac{1}{2}, \text{ and } P(m) \le t\}.$$

Clearly,

$$\nu_\epsilon(t) = O(t^{\epsilon n}).$$

Thus, combining this estimate with (2.7)-(2.9) implies

$$N_P(t) = V_P(t) + O(t^{\epsilon n}) + O_\kappa(t^{\lambda_1 - \epsilon c_P + \kappa}) + O_\kappa(t^{\lambda_1 - \beta + \kappa}).$$

Choosing ϵ so that $\epsilon n < \lambda_1 - \epsilon c_P$ then proves Theorem A. ∎

Section 3. *Sketch of analytic continuation of $D(s,\varphi)$*

For the benefit of the reader a brief sketch is now given of the analytic continuation of $D(s,\varphi)$ that uses the Euler-Maclaurin summation formula. More details may be found by consulting [Li-4, Ma, Lnd]. The key point is that hypoellipticity of P,φ allows this summation method to be used to give an analytic continuation to the entire s plane. For certain purposes, such as proving Theorem 4.1, this seems to give a simpler type of integral representation of the series than that provided by Cauchy residues, as has been used in [L-1,3,5].

To begin, one must introduce some convenient notations.

Notation (3.1).
(1) For any $\ell \in \mathbf{N}$ and $\mathbf{C} = (C_1,\ldots,C_\ell) \in (\mathbf{Z}_+^n)^\ell$ set

$$D_x^{\mathbf{C}}(P)/P^\ell = \prod_{i=1}^{\ell} D_x^{C_i}(P)/P.$$

The reader should note that implicit in this notation is the fact that the exponent of P equals the number of $n-$tuples comprising the components of $\mathbf{C}$.

(2) For each positive integer k set

$$\mathcal{I}_k' = \{I = (i_1,\ldots,i_n) : i_j < k \text{ for all } j\}$$
$$\mathcal{I}_k'' = \{I = (i_1,\ldots,i_n) : i_j \le k \text{ for all } j \text{ and } i_j = k \text{ for some j}\}.$$

(3) For any $I \in \mathcal{I}_k'$ and $\ell \in \mathbf{N}$, define

$$\mathcal{M}_\ell(I) = \{(B,\mathbf{C}) \in \mathbf{Z}_+^n \times \mathbf{Z}_+^{n\ell} : \text{ if } \mathbf{C} = (C_1,\ldots,C_\ell), \text{ then } |C_i| \ge 1 \text{ for each } i, \text{ and } B + C_1 + \ldots + C_\ell = I\},$$

and $\quad \mathcal{M}(I) = \cup_\ell \mathcal{M}_\ell(I)$.

It is convenient below to use the notation $\mathbf{C} \in \mathcal{M}(I)$. This means that $C_1 + \cdots + C_\ell = I$ when $\mathbf{C} = (C_1, \ldots, C_\ell)$.

(4) For $i = 0, 1, 2, \ldots$, set $[-s]_i = (-s)(-s-1)\cdots(-s-i)$. ∎

A simple calculation, left to the reader, shows

LEMMA 3.2. *For each positive integer k, index $I \in \mathcal{I}_k^I$ and pair $(B, \mathbf{C}) \in \mathcal{M}(I)$ there exist integers $n_{B,\mathbf{C}}$ such that*

$$D_x^I(\varphi/P^s) = \sum_{(B,\mathbf{C})\in\mathcal{M}(I)} n_{B,\mathbf{C}} \sum_{\ell=1}^{|\mathbf{C}|} [-s]_{\ell-1}\, D_x^B(\varphi) \cdot \frac{D_x^{\mathbf{C}}(P)}{P^\ell} \cdot \frac{1}{P^s}. \quad ∎$$

Using c_P, c_T from (1.2), α_P from (1.4), and Lemma 3.2, one next sees that the conditions satisfied by P and $\varphi = Q/T$ imply

PROPOSITION 3.3. *Set $N = n\, deg\, Q - \alpha_T$. Then*
(a) For each compact subset K of the halfplane $\sigma > (N+2)/\alpha_P$,

$$|\frac{\varphi}{P^s}| = O_K(\|x\|^{-2}), \quad x \in [1, \infty)^n.$$

(b) For each index I and each compact subset K of the halfplane $\sigma > (N + 2 - c_P|I|)/\alpha_P$

$$|D_x^I(\frac{\varphi}{P^s})| = O_K(\|x\|^{-2}), \quad x \in [1, \infty)^n.$$

PROOF: The straightforward verification is given in [Li-4]. ∎

The Euler-Maclaurin summation formula constructs for each $k = 1, 2, \ldots$, numbers $c_\ell(k)$, $i = 0, 1, \ldots, k-1$, and a periodic C^∞ function $\sigma_k(u)$, where u denotes a coordinate on $\mathbf{R}$ so that if $f(u + iv)$ is any holomorphic function satisfying the property

$$\lim_{u\to+\infty} f^{(i)}(u) = 0, \text{ for each } i = 0, 1, \ldots, \tag{3.4}$$

then

$$\sum_{\nu=1}^{\infty} f(\nu) = \int_1^\infty f(u)du + \sum_{i=0}^{k-1} c_i(k) f^{(i)}(1) + \int_1^\infty \sigma_k(u) f^{(k)}(u)\, du.$$

The precise values of the $c_i(k)$ and expressions of σ_k are not needed for this paper. The reader can work out their values by consulting [Lnd, pgs. 75-83].

Set, for each $k = 1, 2, \ldots$ and $i = 0, 1, \ldots, k-1$

$$h_i^{(k)}(u) = c_i(k)$$
$$h_k^{(k)}(u) = \sigma_k(u).$$

Proposition 3.3 implies that (3.4) is satisfied in the interval $[1, \infty)$ of the x_i coordinate plane, for each i, for any function of the form $D_x^I(\varphi/P^s)$, whenever σ is sufficiently large. One can therefore first set $k = 1$ and iterate the Euler-Maclaurin summation formula n-times to show

THEOREM 3.5. *If $\sigma > (N+2)/\alpha_P$, and $I = (i_1, \ldots, i_n)$, then*

$$D(s,\varphi) = \sum_{i_1=0}^{1} \cdots \sum_{i_n=0}^{1} \int_{[1,\infty)^n} h_{i_1}^{(1)}(x_1) \cdots h_{i_n}^{(n)}(x_n) D_x^I(\frac{\varphi}{P^s})\, dx$$

$$= I(s,\varphi) + \sum_{\substack{i_1,\ldots,i_n=0 \\ I \neq 0}}^{1} \int_{[1,\infty)^n} h_{i_1}^{(1)}(x_1) \cdots h_{i_n}^{(1)}(x_n) D_x^I(\frac{\varphi}{P^s})\, dx. \quad \blacksquare$$

Moreover, it is clear by Proposition 3.3 that for each $I \neq (0, \ldots, 0)$,

$$(3.6) \qquad \int_{[1,\infty)^n} h_{i_1}^{(1)}(x_1) \cdots h_{i_n}^{(1)}(x_n) D_x^I(\frac{\varphi}{P^s})\, dx \quad \textit{is analytic if} \quad \sigma > \frac{N+2-c_P}{\alpha_P}.$$

One can then repeat this procedure $k > 1$ times. In this way one proves

THEOREM 3.7. *If $\sigma > (N+2)/\alpha_P$ then*

$$(1) \qquad D(s,\varphi) = \sum_{i_1=0}^{k} \cdots \sum_{i_n=0}^{k} \int_{[1,\infty)^n} h_{i_1}^{(k)}(x_1) \cdots h_{i_n}^{(k)}(x_n) D_x^I(\frac{\varphi}{P^s})\, dx.$$

Thus, there exist constants $c(I)$ for each $I \neq (0, \ldots, 0) \in \mathcal{I}'_k$ so that

$$(2) \qquad D(s,\varphi) = I(s,\varphi) + \sum_{I \neq (0,\ldots,0) \in \mathcal{I}'_k} c(I) \int_{[1,\infty)^n} D_x^I(\frac{\varphi}{P^s})\, dx$$

$$+ \sum_{I \in \mathcal{I}''_k} \int_{[1,\infty)^n} h_{i_1}^{(k)}(x_1) \cdots h_{i_n}^{(k)}(x_n) D_x^I(\frac{\varphi}{P^s})\, dx.$$

As in (3.6) one observes that for any $I \in \mathcal{I}''_k$

$$(3.8) \qquad \int_{[1,\infty)^n} h_{i_1}^{(k)}(x_1) \cdots h_{i_n}^{(k)}(x_n) D_x^I(\frac{\varphi}{P^s})\, dx \quad \textit{is analytic if} \quad \sigma > \frac{N+2-kc_P}{\alpha_P}.$$

Shown in [Li-4] was the

THEOREM 3.9. *For each $I \in \mathcal{I}'_k$ the function*

$$s \to \int_{[1,\infty)^n} D_x^I(\varphi/P^s)\, dx$$

admits an analytic continuation to $\mathbf{C}$ as a meromorphic function with polar locus contained in finitely many arithmetic progressions of rational numbers.

Thus, (3.8), (3.9) imply

(3.10) *Any pole of $D(s,\varphi)$ in the strip*

$$\frac{N+2-kc_P}{\alpha_P} < \sigma \leq \frac{N+2-(k-1)c_P}{\alpha_P}$$

must be a pole of the analytic continuation of $\int_{[1,\infty)^n} D_x^I(\varphi/P^s)\, dx$ for some $I \in \mathcal{I}'_k$. $\blacksquare$

Notation. Given $I \in \mathcal{I}_k^l$ for some k and any $(B, \mathbf{C}) \in \mathcal{M}(I)$ one sets

$$I(s, B, \mathbf{C}, \varphi) = \int_{[1,\infty)^n} D_x^B(\varphi) \cdot \frac{D_x^{\mathbf{C}} P}{P^\ell} \cdot \frac{1}{P^s}\, dx\,. \quad \blacksquare$$

Remark 3.11. It is useful to write the integrand of this function as

$$\frac{D_x^B(\varphi)}{\varphi} \cdot \frac{D_x^{\mathbf{C}} P}{P^\ell} \cdot \frac{\varphi}{P^s},$$

since (1.2) allows one to interpret the factor of φ/P^s as a function that vanishes at infinity in $[1, \infty)^n$. $\blacksquare$

Section 4. *Proof of first main result*

From the description of the analytic continuation of $D(s, \varphi)$ sketched in Section 3, it is clear that Theorem B^* is a corollary of the following theorem. Using the notations from Section 3 this is

THEOREM 4.1. *Assume P, Q, T are polynomials hypoelliptic on $[1, \infty)^n$. Set $\varphi = Q/T$. Let k, ℓ be positive integers. Assume $I \in \mathcal{I}_k^l$ and $(B, \mathbf{C}) \in \mathcal{M}_\ell(I)$. Then the first pole of $I(s, B, \mathbf{C}, \varphi)$ is strictly smaller than the first pole of $I(s, \varphi)$.*

Preliminary remarks. Since the analysis needed to prove Theorem 4.1 is carried out at infinity, it is first necessary to define the following objects.

Definitions/Notations.

1) The chart at infinity in $(P^1\mathbf{R})^n$ will be denoted $(\mathbf{R}^n, (w_1, \ldots, w_n))$. The hyperplane at infinity $\{w_1 \cdots w_n = 0\}$ is denoted H_∞. The notations $1/w$ resp. dw are used to denote the point $(1/w_1, \ldots, 1/w_n)$ resp. the differential $dw_1 \cdots dw_n$.

2) Define the polynomials $G(w), \psi_1, \psi_2$ by these conditions.

(4.2)

$$R(w) =_{def} \frac{1}{P(1/w)} = \frac{w_1^{M_1} \cdots w_n^{M_n}}{G(w_1, \ldots, w_n)}, \quad w_1, \ldots, w_n \nmid G;$$
$$\Phi(w) =_{def} \varphi(1/w) = \frac{\psi_1(w)}{w_1^{m_1} \cdots w_n^{m_n} \psi_2(w)}, \quad gcd(w_1 \cdots w_n \cdot \psi_2, \psi_1) = 1\,.$$

3) For $\ell \in \mathbf{N}$ and $(B, \mathbf{C}) \in \mathbf{Z}_+^n \times \mathbf{Z}_+^{n\ell}$, define the rational function $\eta_{B,\mathbf{C}}(w)$

$$\eta_{B,\mathbf{C}}(w) = \frac{D_x^B(\varphi)\, D_x^{\mathbf{C}}(P)}{\varphi\, P^\ell}(1/w)\,.$$

Since T is hypoelliptic and assumed to be positive (for simplicity) on $[1,\infty)^n$, $\eta_{B,\mathbf{C}}$ and Φ are defined on $(0,1]^n$. This suffices for the proof of (4.1). ∎

By the assumptions of hypoellipticity for P, Q, T one concludes from (1.2) the existence of $c', C' > 0$ such that for all $w \in (0,1]^n$ and any $\eta_{B,\mathbf{C}}$ defined as in (4.1), one has

$$|\eta_{B,\mathbf{C}}(w_1,\ldots,w_n)| < C'|w_1\cdots w_n|^{c'}.$$

Thus, one concludes for each $p \in \partial[0,1]^n \cap H_\infty$

$$\lim_{\substack{w\to p\\ w\in(0,1]^n}} \eta_{B,\mathbf{C}}(w) = 0. \tag{4.3}$$

Note. It will be convenient in the following to fix a particular pair $B, \mathbf{C}$ and drop the subscript $B, \mathbf{C}$ from η whenever there is no possibility of confusion. ∎

The proof of (4.1) is based upon analyzing the integrands in the statement of the theorem, using a resolution of singularities "at infinity".

There exist a nonsingular real algebraic manifold Y and projective morphism $\pi : Y \to (\mathbf{R}^n, (w_1,\ldots,w_n))$ such that the following properties are satisfied.
(4.4)

i) There exists a divisor $\mathcal{D} \subset Y$ so that $\pi : Y - \mathcal{D} \to \mathbf{R}^n$ is an isomorphism onto its image;
ii) $\mathcal{D}$ is a normally crossing divisor. That is, $\mathcal{D} = \cup\mathcal{D}_\alpha$ where each $\mathcal{D}_\alpha$ is smooth and at each point $p \in \mathcal{D}$ the set of divisors containing p are mutually transverse;
iii) The divisor equal to the zero and polar locus defined by

$$[\prod_{i=1}^n (w_i - 1)\cdot \prod_{i=1}^n w_i \cdot R\cdot\Phi\cdot\eta]\circ\pi$$

has support in $\mathcal{D}$ (so that it too is locally normal crossing);
iv) $(0,1)^n \cap \pi(\mathcal{D}) = \emptyset$.

Thus, $(0,1)^n$ is disjoint from the locus of blowing up determined by π.

Next, one takes an open polydisc U containing $[0,1]^n$ in the chart at infinity and sets

$$\begin{aligned} X &= \pi^{-1}(U)\\ D &= \mathcal{D}\cap X\\ B &= \overline{\pi^{-1}(0,1)^n}\cap X. \end{aligned}$$

An elementary observation is the

LEMMA 4.5.

i) $\partial B \subset D$.
ii) $B \cap D = \partial B$.

PROOF: (i) follows from (4.4)(iii). To verify (ii), one notes that (4.4)(i,iv) imply

$$\pi^{-1}(0,1)^n \cap D = \emptyset.$$

Moreover, since π is continuous, $\pi^{-1}(0,1)^n$ is open in X and equals $int(B)$. Thus, $B \cap D = \partial B \cap D = \partial B$ by (i). ∎

A second elementary result will also be needed below. For each point $q \in \partial B$ there exists an open neighborhood $\mathcal{U}_q$ and coordinates $(z_1, \ldots, z_n)$, defined in $\mathcal{U}_q$ and centered at q, such that

$$\mathcal{U}_q \cap D \subset \cup_{i=1}^n \{z_i = 0\}. \tag{4.6}$$

A "sign distribution" is a function

$$\epsilon : \{1, \ldots, n\} \to \{+, -\}.$$

To each sign distribution one defines an open subset of any $\mathcal{U}_q$ by setting

$$\mathcal{O}_\epsilon = \{z \in \mathcal{U}_q : \epsilon(i) z_i > 0, \text{ for each } i = 1, \ldots, n\}.$$

One notes that the only geometric property of interest possessed by these sets is their disjointness from D.

LEMMA 4.7. *For each $q \in \partial B$ there exists a set $\mathcal{E}_q$ of sign distributions such that*

$$\cup_{\epsilon \in \mathcal{E}_q} \mathcal{O}_\epsilon = int(B) \cap \mathcal{U}_q\,.$$

PROOF: By (4.4)(i) and (4.5)(i), it is clear that

$$int(B) \cap \mathcal{U}_q \subset \cup_\epsilon \mathcal{O}_\epsilon.$$

Suppose for some ϵ_0 that $int(B) \cap \mathcal{U}_q \cap \mathcal{O}_{\epsilon_0} \neq \emptyset$. Further, suppose that $\mathcal{O}_{\epsilon_0} \not\subset int(B) \cap \mathcal{U}_q$. Then, Lemma (4.5) and (4.6) imply that

$$\begin{aligned} &\mathcal{O}_{\epsilon_0} \cap (int(B) \cap \mathcal{U}_q) \neq \emptyset \quad \text{and } \mathcal{O}_{\epsilon_0} \cap (B^c \cap \mathcal{U}_q) \neq \emptyset \\ \text{but} \quad &\mathcal{O}_{\epsilon_0} \cap (\partial B \cap \mathcal{U}_q) = \emptyset. \end{aligned}$$

Since $\mathcal{O}_{\epsilon_0}$ is connected this decomposition of $\mathcal{O}_{\epsilon_0}$ into two disjoint open subsets cannot occur. Thus, $\mathcal{O}_{\epsilon_0} \subset int(B) \cap \mathcal{U}_q$. This implies Lemma (4.7). ∎

To each irreducible component D_α of D one defines the following orders.

(4.8)

$$\begin{aligned} M_\alpha &= \mathrm{ord}_{D_\alpha} R \circ \pi =_{\mathrm{def}} \mathrm{ord}_{D_\alpha} (w_1^{M_1} \cdots w_n^{M_n}) \circ \pi - \mathrm{ord}_{D_\alpha} G \circ \pi \\ m_\alpha &= \mathrm{ord}_{D_\alpha} \Phi \circ \pi =_{\mathrm{def}} \mathrm{ord}_{D_\alpha} \psi_1 \circ \pi - \mathrm{ord}_{D_\alpha} (w_1^{m_1} \cdots w_n^{m_n}) \circ \pi \; - \mathrm{ord}_{D_\alpha} \psi_2 \circ \pi\,, \\ \kappa_\alpha &= \mathrm{ord}_{D_\alpha} \eta \circ \pi\,, \\ \gamma_\alpha &= \mathrm{ord}_{D_\alpha} Jac(\pi) - \mathrm{ord}_{D_\alpha} (w_1^2 \cdots w_n^2) \circ \pi\,, \end{aligned}$$

where $Jac(\pi)$ denotes the jacobian of π.

To each D_α for which $M_\alpha \neq 0$ define the ratios

$$\rho(D_\alpha) = \frac{-(1+m_\alpha+\gamma_\alpha)}{M_\alpha}$$
$$\beta(D_\alpha) = \frac{-(1+m_\alpha+\kappa_\alpha+\gamma_\alpha)}{M_\alpha}.$$

If $M_\alpha = 0$ one sets $\rho(D_\alpha) = \beta(D_\alpha) = -\infty$. The $\rho(D_\alpha)$ resp. $\beta(D_\alpha)$ are possible values for the first pole of $I(s,\varphi)$ resp. $I(s,B,\mathbf{C},\varphi)$.

Define

$$\rho(\pi) = sup_\alpha\{\rho(D_\alpha)\}, \qquad \beta(\pi) = sup_\alpha\{\beta(D_\alpha)\}. \tag{4.9}$$

Then any pole of $I(s,\varphi)$ is at most $\rho(\pi)$ and any pole of $I(s,B,\mathbf{C},\varphi)$ is at most $\beta(\pi)$. The key step in the proof of Theorem 4.1 is therefore the proof of the inequality

$$\rho(\pi) > \beta(\pi). \tag{4.10}$$

This will follow immediately from

LEMMA 4.11. *Suppose q is a point in ∂B such that $\pi(q) \in H_\infty$. Let D_α be any component of D containing q. Then $\kappa_\alpha > 0$.*

PROOF: Assume there exists a point $q \in \partial B$ with $\pi(q) \in H_\infty$ for which $\kappa_{\alpha'} \leq 0$ for some divisor $D_{\alpha'}$ containing q. Let $\mathcal{U}_q$ denote a neighborhood of the point so that (4.6) holds. Assume that coordinates are chosen so that the divisor $D_{\alpha'}$ satisfies the property $D_{\alpha'} \cap \mathcal{U}_q = \{z_1 = 0\}$. There exists at least one sign distribution ϵ so that $\mathcal{O}_\epsilon \subset int(B) \cap \mathcal{U}_q$. Given any point $p = (p_1, p') \in \mathcal{O}_\epsilon$ the path $\mu(t) = (1-t)p + t(0,p'), t \in [0,1)$ is entirely contained in $\mathcal{O}_\epsilon$. By definition, one has that

$$ord_t(\eta \circ \pi \circ \mu) = \kappa_{\alpha'}.$$

Thus, $\kappa_{\alpha'} \leq 0$ implies

$$\lim_{t\to 0} \eta \circ \pi \circ \mu(t) \neq 0.$$

On the other hand, $\mathcal{O}_\epsilon \subset int(B) \cap \mathcal{U}_q$ implies that for all $t > 0$, $\pi \circ \mu(t) \in (0,1)^n$. Moreover, as $t \to 0$, $\pi \circ \mu(t)$ approaches a point in H_∞. Thus, by (4.3) the limit of η along the path $\pi \circ \mu(t)$ must equal 0. So, the point q with the above properties must not exist. This proves the Lemma. ∎

An entirely similar argument that uses (1.4), as expressed in the $(w_1, \ldots, w_n)$ coordinates, shows

LEMMA 4.12. *Suppose q is a point of ∂B such that $\pi(q) \in H_\infty$. Let D_α be any component of D containing q. Then $M_\alpha > 0$. Moreover, if $q \in \partial B$ is such that $\pi(q) \notin H_\infty$ then $M_\alpha = 0$ for any component D_α containing q.*

Remark 4.13. Geometrically, Lemma 4.12 says that the strict transform of G is a component of D that is disjoint from B in X. That is, the polar divisor of $R \circ \pi\big|_X$ cannot intersect B. An entirely similar conclusion holds for the polar locus of $\varphi\big|_X$. This property is important in describing the polar part of $I(s, \varphi)$ at $s = \rho(\pi)$, as seen in (4.16)ff. ∎

Supplied with these preliminary observations, one can proceed to the

PROOF OF THEOREM 4.1: In light of (4.10), it evidently suffices to show, $\rho(\varphi) = \rho(\pi)$, (cf. Theorem 2.1), that is,

$$\rho(\pi) \text{ is the first pole of } I(s, \varphi). \tag{4.14}$$

It is clear that one can assume that the sign of φ is constant outside a compact subset of $[1, \infty)^n$. For simplicity, one may therefore assume that φ is positive on all but a compact subset of $[1, \infty)^n$.

One has for $\sigma \gg 1$,

$$\begin{aligned} I(s, \varphi) &= \int_{[0,1]^n} R^s \Phi \frac{dw}{w_1^2 \cdots w_n^2} \\ &=_{def} \lim_{\epsilon \to 0} \int_{[\epsilon,1]^n} R^s \Phi \frac{dw}{w_1^2 \cdots w_n^2} \\ &= \int_B (R \circ \pi)^s (\Phi \circ \pi) |\pi^*(\frac{dw}{w_1^2 \cdots w_n^2})| , \end{aligned}$$

where $|\pi^*(dw/w_1^2 \cdots w_n^2)|$ denotes a density on X.

Since π is proper and B is a closed subset of the compact set $\pi^{-1}[0,1]^n$, B is also compact. For each $q \in B$ there exists an open neigborhood $\mathcal{U}_q$ so that (4.6) holds iff $q \in \partial B$. The open cover $\{\mathcal{U}_q\}$ of B admits a finite open subcover $\{\mathcal{U}_i\}_{i=1}^N$, where $\mathcal{U}_i$ is centered at q_i. One now takes a finite partition of unity $\{v_c\}$ subordinate to the cover $\{\mathcal{U}_i\}$. Thus, for $\sigma \gg 1$

$$\int_B (R\circ\pi)^s (\Phi\circ\pi) |\pi^*(\frac{dw}{w_1^2 \cdots w_n^2})| = \sum_c \sum_i \int_{\mathcal{U}_i \cap B} (R\circ\pi)^s (\Phi\circ\pi) v_c |\pi^*(\frac{dw}{w_1^2 \cdots w_n^2})| . \tag{4.15}$$

One next fixes an arbitrary $\mathcal{U}_i$. One chooses the coordinates centered at q_i so that

$$\mathcal{U}_i \cap D = \cup_{j=1}^r \{z_j = 0\}.$$

Assume that $\{\epsilon_1, \ldots, \epsilon_{R(i)}\}$ are the sign distributions so that $\mathcal{O}_{\epsilon_k} \subset int(B) \cap \mathcal{U}_i, k = 1, \ldots, R(i)$. Define for each $j = 1, \ldots, r$

$$\begin{aligned} M_j(i) &= \mathrm{ord}_{D_j}\,(R \circ \pi) \\ m_j(i) &= \mathrm{ord}_{D_j}\,(\Phi \circ \pi) \\ \gamma_j(i) &= \mathrm{ord}_{D_j}\,|\pi^*(dw/w_1^2 \cdots w_n^2)|\,, \end{aligned}$$

and for each $i = 1, \ldots, N$

$$\nu(i, \rho(\pi)) = \#\{j : \frac{-(1 + m_j(i) + \gamma_j(i))}{M_j(i)} = \rho(\pi)\,\}\,.$$

Define

$$\mathcal{J}(\rho(\pi)) = \{i : \nu(i, \rho(\pi)) \geq 1\}\,.$$

By definition, $\mathcal{J}(\rho(\pi)) \neq \emptyset$. In this regard, one should also note that $r = 0$ is possible. This occurs iff $q_i \in int(B)$. In this case, each $M_j(i) = 0$ and $i \notin \mathcal{J}(\rho(\pi))$.

The Gelfand-Shapiro-Shilov regularization method [G-S] applies to the integral over each open set $\mathcal{O}_{\epsilon_k}, k = 1, \ldots, R(i)$ and $i = 1, \ldots, N$, and thereby gives an analytic continuation to each summand on the right side of (4.15). In particular, if $i \in \mathcal{J}(\rho(\pi))$ then the principal part at $s = \rho(\pi)$ of

$$\int_{\mathcal{U}_i \cap B} (R \circ \pi)^s\,(\Phi \circ \pi)\,v_c\,|\pi^*(\frac{dw}{w_1^2 \cdots w_n^2})|$$

consists of at most $\nu(i, \rho(\pi))$ nonzero terms. The main point is to show that the term of order equal to $\nu(i, \rho(\pi))$ must be ***positive***.

Note. When one i is fixed, i and $\rho(\pi)$ are subsequently dropped as the arguments for ν. ∎

After reindexing, if necessary, one may assume that

$$\{j : \frac{-(1 + m_j(i) + \gamma_j(i))}{M_j(i)} = \rho(\pi)\} = \{1, 2, \ldots, \nu\} \subset \{1, \ldots, r\}\,.$$

One sets $z' = (z_{\nu+1}, \ldots, z_n)$.

Then the contribution from $\mathcal{U}_i$ to the term of order ν in the principal part has the form

$$\text{(4.16)} \qquad \sum_c \sum_{k=1}^{R(i)} \int_{\mathcal{U}_i \cap D_1 \ldots \cap D_\nu} (z_{\nu+1})_{\epsilon_k(\nu+1)}^{\zeta_{\nu+1}} \cdots (z_n)_{\epsilon_k(n)}^{\zeta_n}\, g_1(z')^{\rho(\pi)}\, g_2(z')\, v_c(z')\, g_3(z')\, dz'$$

where the following properties are satisfied:

(1) $\zeta_{\nu+1}, \ldots, \zeta_n > -1$, (cf. [Va-2] where this property was first used for a related problem);

(2) $g_1(z')$ is the restriction to $\cap_{j=1}^{\nu} D_j \cap \mathcal{U}_i$ of the quotient of the strict transforms of $w_1^{M_1} \cdots w_n^{M_n}$ and G;

(3) $g_2(z')$ is the restriction to $\cap_{j=1}^{\nu} D_j \cap \mathcal{U}_i$ of the quotient of the strict transforms of ψ_1 and $w_1^{m_1} \cdots w_n^{m_n} \cdot \psi_2$;

(4) $g_3(z')$ is the restriction to $\cap_{j=1}^{\nu} D_j \cap \mathcal{U}_i$ of the strict transform of $|Jac(\pi)|/(w_1^2 \cdots w_n^2) \circ \pi$.

From Lemma 4.12 and the assumed positivity of P, φ over $[1,\infty)^n$ (cf. section 1), one concludes that $g_1(z'), g_2(z'), g_3(z')$ are ***finite and positive*** over the domain of integration in (4.16). Moreover, since $\{v_c\}$ forms a partition of unity, one concludes that the double sum in (4.16) must be positive.

Thus, $\rho(\pi)$ ***must*** be a pole of $I(s,\varphi)$. Furthermore, any rational number larger than $\rho(\pi)$ could ***not*** be a pole of $I(s,\varphi)$ since it would be larger than any candidate pole ρ_α used to define $\rho(\pi)$. This proves (4.14) and therefore completes the proof of Theorem 4.1. ∎

Remark 4.17. At this point, it is also useful to discuss briefly the work of Sargos. In [Sa-1] the questions addressed in this section were considered for polynomials satisfying a nondegeneracy condition on $[1,\infty)^n$. This is formulated in terms of the Newton polyhedron of P at infinity. Recall that if S denotes the support of P then the Newton polyhedron at infinity is the boundary of the convex hull of $\bigcup_{I\in S}(I - \mathbf{R}_+^n)$. Given P, its polyhedron Γ_∞, and the set of vertices $\mathcal{V}$ of Γ_∞, one first defines the polynomial $P_{\Gamma\infty}$ by

$$P_{\Gamma\infty}(x_1,\ldots,x_n) = \sum_{(i_1,\ldots,i_n)\in\mathcal{V}} x_1^{i_1}\cdots x_n^{i_n}\,.$$

P is said to be nondegenerate with respect to Γ_∞, if there exists $C > 0$ such that

$$|P(x_1,\ldots,x_n)| > C P_{\Gamma\infty}(x_1,\ldots,x_n) \quad \text{for all } (x_1,\ldots,x_n) \in [1,\infty)^n.$$

Evidently, this condition, which generalizes the property that P has positive coefficients, is considerably weaker than ellipticity.

No elementary argument appears to be available to establish (0.3) when $\varphi \equiv 1$ and P is nondegenerate. Indeed, it is not in general true that $N_P(t,1)$ and $V_P(t,1)$ agree up to a strictly lower order in t. For example, if $a \neq b$ are positive integers, then the polynomial $P = x_1^a x_2^b$ is nondegenerate but $N_P(t,1)$ and $V_P(t,1)$ do not have the same dominant term as $t \to \infty$, as a simple calculation will verify.

The first result in [ibid] determines the precise rate of growth of $N_P(t,1), V_P(t,1)$, assuming nondegeneracy over $[1,\infty)^n$. Sargos showed:

THEOREM 1. *There are numbers ρ_0, ν, expressible in terms of the polyhedron of P at infinity, and positive numbers A, B such that*

$$N_P(t,1) = At^{\rho_0}log^{\nu}t\,(1 + O(1/logt))\,,$$
$$\text{and} \quad V_P(t,1) = Bt^{\rho_0}log^{\nu}t\,(1 + O(1/logt))\,.$$

The constants A, B are given by explicit expressions. Moreover, the values of ρ_0 and ν are expressed in terms of Γ_∞, in a manner analogous to that obtained in [Va-2, Vas].

The second major result of [ibid] showed:

THEOREM 2. *Assume that the diagonal intersects the Newton polyhedron at infinity of $P(x_1, \ldots, x_n)$ in compact faces only. Further, assume that P is nondegenerate with respect to this polyhedron. Then there exists $\theta > 0$ so that*

$$N_P(t,1) - V_P(t,1) = O(t^{\rho_0 - \theta}).$$

An extension of (0.3), involving weights determined by polynomials with positive coefficients, follows from the arguments given in [ibid]. Presumably, if φ is a rational function whose numerator and denominator are nondegenerate with respect to their polyhedra at infinity, Theorem 2 continues to hold. However, it is not yet known if one can give an "elementary" proof of this theorem, analogous to that given in Section 2, when $\varphi \equiv 1$. These would also be interesting questions to answer.

The precise relation between the nondegeneracy condition used in [Sa-1] and hypoellipticity is not yet completely understood. It would be interesting to characterize precisely the nondegenerate polynomials which must also be hypoelliptic on $[1,\infty)^n$, or on $[a,\infty)^n$ for some $a \in (0,1)$. To this end, it is useful to observe that Hörmander [Hö, ch. 11] found a class of polynomials that were hypoelliptic on $[1,\infty)^n$, degenerate with respect to their polyhedra at infinity, and for which the diagonal intersected each polyhedron in exactly one compact face. ∎

Section 5. *Distribution of eigenvalues for hypoelliptic PsDO's*

Let $p(x,\xi)$ denote the symbol of a pseudo differential operator $\mathcal{P}$ on $\mathbf{R}^n$. Let $\overline{\mathcal{P}}$ denote a self-adjoint extension to the Hilbert space $L^2([1,\infty)^n)$. Assume that the spectrum of $\overline{\mathcal{P}}$ is discrete. Denote the eigenvalues as $\lambda_1 \leq \lambda_2 \leq \ldots$. A standard problem is to understand the behavior as $t \to \infty$ of the spectral function

$$N(t) = \sum_{\lambda_n \leq t} 1.$$

This question has been studied by numerous authors under varying assumptions on $p(x,\xi)$. Of interest here is the behavior of $N(t)$ when $p(x,\xi)$ is hypoelliptic on $[1,\infty)^{2n}$. Robert [R] and Smagin [Sm] (among others, cf. their articles' bibliographies) have shown the following result. In the notation of the Introduction,

THEOREM.

$$N(t) \sim (\frac{1}{2\pi})^n V_p(t).$$

Theorem A shows that the asymptotic of $N(t)$ is determined by a discrete version of $V_p(t)$. That is, one sees immediately, again in the notation of the Introduction,

THEOREM 5.1.
$$N(t) \sim (\frac{1}{2\pi})^n N_p(t) \quad \text{as } t \to \infty.$$

Theorem 5.1 appears to be of interest because each $N_p(t)$ can be calculated in polynomial time as a function of t. That is, Proposition 1.3 implies that any lattice point $m \in \mathbf{N}^{2n}$ for which $|p(m)| \leq t$ must be contained inside the part of the hyperboloid given by

$$\{y_1 \cdots y_{2n} < t^{1/\alpha}\} \cap [1,\infty)^{2n}, \tag{5.2}$$

where α denotes an exponent for $p(x,\xi)$ so that (1.4) holds. Since the number of lattice points satisfying (5.2) is $O(t^{2n/\alpha})$, the complexity of determining $N_p(t)$ is clearly polynomial in t. One can therefore determine a reasonable approximation to $N(t)$ for any sufficiently large t in polynomial time by calculating $N_p(t)$, rather than the analytically more difficult function $V_p(t)$. The same properties hold if one replaces $[1,\infty)^{2n}$ by $\mathbf{R}^{2n}$ and insists that $p(x,\xi)$ be hypoelliptic on $\mathbf{R}^{2n}$.

A second interesting feature of Theorem 5.1 can be seen by comparing it with the results of Bochner [Bo], who studied $N(t)$ for a constant coefficient operator $P(D)$ on the n dimensional torus. He essentially showed that if the symbol $p(\xi)$ was hypoelliptic on $\mathbf{R}^n$, then the Dirichlet series
$$\sum_{\{m\in\mathbf{Z}^n:p(m)\neq 0\}} \frac{1}{p(m)^s}$$
determined the spectral function for P. The reason for this was that the *identity* $N(t) = N_p(t)$ follows easily from an explicit description of the eigenfunctions of $P(D)$. As a result, the eigenvalues of $P(D)$ are easily seen to be the values of p at the lattice points m.

Thus, Theorem 5.1 shows that a Dirichlet series of the form
$$\sum_n c_n\, e^{-s\log\eta_n}, \text{ where } 0 < \eta_1 < \eta_2 < \ldots,$$
exists *in general* so that
$$N(t) \sim \sum_{\eta_n \leq t} c_n\,.$$
In particular, this occurs even though the actual eigenvalues *are not* known to be expressible algebraically in terms of the values of p at any set of lattice points contained in $[1,\infty)^{2n}$. This intriguing property does not seem to have been observed earlier.

Section 6. *Cohomological invariance of the main term in $N_P(t,\varphi)$*

Suppose that φ_1, φ_2 are two polynomials that are positive outside a compact subset of $[1,\infty)^n$. By Theorem 2.1 (cf. [Li-3,4]), there exist nonzero real polynomials $A_1(u), A_2(u)$ and rational numbers $\rho(\varphi_1), \rho(\varphi_2)$ such that for some $\tau > 0$
$$N_P(t,\varphi_1) = t^{\rho(\varphi_1)} A_1(log t) + O(t^{\rho(\varphi_1)-\tau})$$
$$N_P(t,\varphi_2) = t^{\rho(\varphi_2)} A_2(log t) + O(t^{\rho(\varphi_2)-\tau}).$$

Denote the dominant term in $N_P(t,\varphi)$ by $\hat{N}_P(t,\varphi)$ below.

A natural question asks:
What conditions upon φ_1, φ_2 can be imposed that insures

$$\hat{N}_P(t,\varphi_1) = \hat{N}_P(t,\varphi_2)\,?$$

In [Li-2] a cohomological criterion was found that answered this question under conditions a bit more restrictive than were really necessary, in light of Theorem B of this paper. It will suffice here to state the extension of [Li-2, Theorem 4] that can now be made. To do so, first recall the standard constructions of the cohomology fiber bundle for a polynomial mapping on $\mathbf{C}^n$ and the section induced by the Leray residue operation.

According to Verdier [Ve]), for any polynomial mapping $P : \mathbf{C}^n \to \mathbf{C}$, there is a finite set $\Sigma_P \subset \mathbf{C}$ such that $P : \mathbf{C}^n - P^{-1}(\Sigma_P) \to \mathbf{C} - \Sigma_P$ is a locally trivial fibration. Set $\mathbf{C}^* = \mathbf{C} - \Sigma_P$, and define $P^* = P\big|_{P^{-1}(\mathbf{C}^*)}$. For $t \in \mathbf{C}^*$ set $X_t = P^{-1}(t)$. Let $\mathbf{H}^{n-1}$ denote the flat vector bundle on $\mathbf{C}^*$ with fiber at t equal to the finite dimensional vector space $H^{n-1}(X_t, \mathbf{C})$. Let $\mathcal{H}^{n-1} = \mathbf{H}^{n-1} \otimes \mathcal{O}_{\mathbf{C}^*}$ be the sheaf of germs of analytic sections of $\mathbf{H}^{n-1}$. Any rational differential n-form ω determines an analytic section of $\mathcal{H}^{n-1}$, defined as

$$\sigma(\omega) : t \to [\omega/dP|_{X_t}]$$

where $\omega/dP\,|_{X_t} = \text{Res}\,(\omega/(P-t))|_{X_t}$.

THEOREM 6.1. *Assume the following hypotheses.*

(1) *P is a tame polynomial (cf. [Br]).*
(2) *P, φ_1, φ_2 are hypoelliptic polynomials on $[1,\infty)^n$.*
(3) *For all $t \notin \Sigma_P$ one has*

$$\sigma(\varphi_1 dz)(t) = \sigma(\varphi_2 dz)(t),$$

where φdz is the $(n,0)$ form determined by φ.

Then $\hat{N}_P(t,\varphi_1) = \hat{N}_P(t,\varphi_2)$.

In this sense one can say that the dominant term is a "cohomological invariant". That such a result might be possible arose from studying the work of Cassou-Nogues [Ca-N]. On the other hand, one is still quite far from achieving a result with the precision found in [ibid], which was obtained under the assumption that P was a polynomial of the type studied by Dwork.

Section 7. *The case of $k > 1$ polynomials - proof of second main result*

The proof of Theorem B^* via Theorem 4.1 extends easily to a several variable setting. The main result of this section is Theorem B_k which is an immediate corollary of Theorem

7.8. Theorem B_k appears to be of particular interest because it is an essential ingredient, needed in an eventual sharpening of the results described in [Li-5]. This is discussed at the end of the section.

Let $P_1, \ldots, P_k \in \mathbf{R}[x_1, \ldots, x_n]$ be hypoelliptic on $[1,\infty)^n$. Let φ be a hypoelliptic fraction, in the sense of Section 1. One assumes, for simplicity, that each $P_i > 0$ and $\varphi > 0$ on $[1,\infty)^n$. Denote the best constants c, α so that (1.2), (1.5) hold for each P_i by c_i, α_i. Define, for $\delta > 0$,

$$\Omega(\delta) = \{\mathbf{s} : \sum_{i=1}^{k} \alpha_i \sigma_i > \delta, \text{ and } \sigma_i > 0, \text{ for each } i\}.$$

One also defines

$$D(\mathbf{s}, \varphi) = \sum_{m \in \mathbf{N}^n} \frac{\varphi(m)}{P_1^{s_1}(m) \cdots P_k^{s_k}(m)}.$$

An elementary argument shows

PROPOSITION 7.1. *There exists $\delta > 0$ so that $D(\mathbf{s}, \varphi)$ is absolutely convergent in $\Omega(\delta)$.*

In [Li-5] an analytic continuation of $D(\mathbf{s}, \varphi)$ to $\mathbf{C}^k$ was given, using a summatory formula (or integral representation) based upon an iteration of Cauchy residue theory, and a set of k functional equations in $s_1, \ldots, s_k$ deduced from the work of Sabbah [Sb]. In this way, one showed

THEOREM 7.2. *There exist $\mathbf{b}_i \in \mathbf{Z}_+^k, i = 1, \ldots, M, \beta_1, \ldots, \beta_M \in \mathbf{Z}$ such that the polar divisor of $D(\mathbf{s}, \varphi)$ is contained in*

$$\bigcup_{e=0}^{\infty} \bigcup_{i=1}^{M} \{\mathbf{s} : \mathbf{b}_i \cdot \mathbf{s} = \beta_i - e\}.$$

One denotes the polar divisor of $D(\mathbf{s}, \varphi)$ by Pol_D.

On the other hand, as observed in [Li-4], as well as Section 4 of this paper, a somewhat sharper set of results can be deduced for hypoelliptic polynomials if one uses the Euler-Maclaurin formula. A similar sharpening is also possible in the several variable setting. Define

$$I(\mathbf{s}, \varphi) = \int_{[1,\infty)^n} \frac{\varphi}{P_1^{s_1} \cdots P_k^{s_k}} \, dx.$$

To proceed as in Section 3, one must first give the natural extension of Lemma 3.2. Of course, the notations from Section 3 are adopted here.

For $J \in \mathbf{Z}_+^n$, set

$$\mathcal{P}(J) = \{(J_0, \ldots, J_k) \in (\mathbf{Z}_+^n)^{k+1} : J_0 + \cdots + J_k = J\}.$$

LEMMA 7.3. *For each index $J \in \mathbf{Z}_+^n$ and element $\mathbf{J} = (J_0, \dots, J_k) \in \mathcal{P}(J)$, there exist integers $a(\mathbf{J})$ such that*

$$D_x^J\left(\frac{\varphi}{P_1^{s_1} \cdots P_k^{s_k}}\right) = \sum_{\mathbf{J} \in \mathcal{P}(J)} a(\mathbf{J}) \cdot D_x^{J_0}(\varphi) \prod_{i=1}^{k} D_x^{J_i}\left(\frac{1}{P_i^{s_i}}\right)$$

$$= \sum_{\mathbf{J} \in \mathcal{P}(J)} a(\mathbf{J}) \cdot \frac{D_x^{J_0}(\varphi)}{\varphi} \prod_{i=1}^{k} \left\{ \sum_{\mathbf{C} \in \mathcal{M}(J_i)} \sum_{v=0}^{|\mathbf{C}|-1} [-s_i]_v \frac{D_x^{\mathbf{C}}(P_i)}{P_i^{v+1}} \right\} \frac{\varphi}{P_1^{s_1} \cdots P_k^{s_k}}.$$

The hypoellipticity of each P_i and φ shows clearly that

PROPOSITION 7.4. *For each J, the function*

$$\mathbf{s} \to \int_{[1,\infty)^n} D_x^J\left(\frac{\varphi}{P_1^{s_1} \cdots P_k^{s_k}}\right) dx$$

is analytic if $\mathbf{s} \in \Omega(\delta)$.

The arguments of [Li-4] extend easily to show

THEOREM 7.5. *For each J, the function defined in (7.4) admits an analytic continuation to $\mathbf{C}^k$ as a meromorphic function. Further, there exist $\mathbf{b}_i(J) \in \mathbf{Z}_+^k, i = 1, \dots, M(J), \beta_1(J), \dots, \beta_{M(J)}(J)$ $\mathbf{Z}$ such that its polar divisor is contained in*

$$\bigcup_{e=0}^{\infty} \bigcup_{i=1}^{M(J)} \{\mathbf{s} : \mathbf{b}_i(J) \cdot \mathbf{s} = \beta_i(J) - e\}.$$

Notation. One denotes the polar divisor of $I(\mathbf{s}, \varphi)$ by Pol_I. ∎

For given $r \in \mathbf{N}$ set

$$\mathcal{I}(r) = \{I = (i_1, \dots, i_n) : i_j \in [0, r], j = 1, \dots, n\}.$$

In the notation of (3.1) (with $k = r$), one sees the

THEOREM 7.6.

(1) *If $\mathbf{s} \in \Omega(\delta)$ and $r \geq 1$ then*

$$D(\mathbf{s}, \varphi) = \sum_{I \in \mathcal{I}(r)} \int_{[1,\infty)^n} h_{i_1}^{(r)}(x_1) \cdots h_{i_n}^{(r)}(x_n) D_x^I\left(\frac{\varphi}{P_1^{s_1} \cdots P_k^{s_k}}\right) dx.$$

(2) *There exist constants* $c(I)$ *for each* $I \neq (0,\ldots,0) \in \mathcal{I}'_r$ *so that*

$$D(\mathbf{s},\varphi) = I(\mathbf{s},\varphi) + \sum_{I\neq(0,\ldots,0)\in\mathcal{I}'_r} c(I)\int_{[1,\infty)^n} D^I_x\Big(\frac{\varphi}{P_1^{s_1}\cdots P_k^{s_k}}\Big)\,dx$$
$$+\sum_{I\in\mathcal{I}''_r}\int_{[1,\infty)^n} h^{(r)}_{i_1}(x_1)\cdots h^{(r)}_{i_n}(x_n)\,D^I_x\Big(\frac{\varphi}{P_1^{s_1}\cdots P_k^{s_k}}\Big)\,dx\,.$$

(3) *Set* $c = \min\{c_1,\ldots,c_k\}$. *For any* $I \in \mathcal{I}''_r$

$$\int_{[1,\infty)^n} h^{(r)}_{i_1}(x_1)\cdots h^{(r)}_{i_n}(x_n)\,D^I_x\Big(\frac{\varphi}{P_1^{s_1}\cdots P_k^{s_k}}\Big)\,dx \ \text{ is analytic if } \sum_i \alpha_i\sigma_i > \delta - cr\,.$$

Notation. For any $J_0, J_1, \ldots, J_k \in \mathbf{Z}^n_+$ and $(\mathbf{C}_1,\ldots,\mathbf{C}_k) \in \mathcal{M}(J_1)\times\ldots\times\mathcal{M}(J_k)$, set $\mathcal{C} = (J_0,\mathbf{C}_1,\ldots,\mathbf{C}_k)$. Define

$$I(\mathbf{s},\mathcal{C},\varphi) = \int_{[1,\infty)^n} \frac{D^{J_0}_x(\varphi)}{\varphi}\cdot\frac{D^{\mathbf{C}_1}_x(P_1)}{P_1^{v_1}}\cdots\frac{D^{\mathbf{C}_k}_x(P_k)}{P_k^{v_k}}\cdot\frac{\varphi}{P_1^{s_1}\cdots P_k^{s_k}}\,dx\,. \quad \blacksquare$$

To any meromorphic function on $\mathbf{C}^k$ analytic on a domain like $\Omega(\delta)$ and with polar divisor *pol* contained in a union of hyperplanes as described in (7.2), (7.4), one can associate a Newton polyhedron of *pol*, denoted $\Gamma(pol)$. This is an unbounded subset of $\mathbf{R}^k$ and defined as follows.

Definition 7.7. Suppose

$$pol \subset \bigcup_{e=0}^{\infty}\bigcup_{i=1}^{M}\{b^{(i)}\cdot\mathbf{s} = \beta^{(i)} - e\}\,.$$

Assume *each* hyperplane $\{b^{(i)}\cdot\mathbf{s} = \beta^{(i)}\}, i = 1,\ldots,M$, is a component of *pol*. Set

$$\mathcal{H} = \bigcap_{i=1}^{M}\{\sigma\in\mathbf{R}^k : b^{(i)}\cdot\boldsymbol{\sigma} \geq \beta^{(i)}, \sigma_1,\ldots,\sigma_k > 0\}$$

and $\qquad \Gamma(pol) = \partial\mathcal{H}\,. \quad \blacksquare$

Denote the Newton polyhedra of Pol_D resp. Pol_I by Γ_D resp. Γ_I. For simplicity these polyhedra will be called the Newton polyhedra of D resp. I.

The main result of this section is

THEOREM B_k. *Assume each* P_i *is hypoelliptic on* $[1,\infty)^n$, *and* φ *is a hypoelliptic fraction. Then*

(1) $\Gamma_I = \Gamma_D$.
Denote the common polyhedron in (1) by Γ.

(2) $D(\mathbf{s},\varphi) - I(\mathbf{s},\varphi)$ *is analytic at each point of* $\Gamma + i\mathbf{R}^k$.

Remark. One should interpret Theorem B_k as the several variable analogue of Theorem B^* of Section 3. In particular, one should think of Γ as the analogue of the "largest pole" for a series in one variable (with real poles). The reader should also note that if $\{b^{(i')} \cdot \boldsymbol{\sigma} = \beta^{(i')}\}$ contains a face of Γ, then the only $\mathbf{t} \in \mathbf{R}^k$ for which (2) is a nontrivial assertion are those that satisfy the equation $b^{(i')} \cdot \mathbf{t} = 0$. ∎

Theorem B_k will follow directly from Theorem 7.6 and the following result.

THEOREM 7.8. *For any $\mathcal{C} \neq (0, \ldots, 0)$ the Newton polyhedron of $I(\mathbf{s}, \mathcal{C}, \varphi)$ is strictly below Γ_I.*

Remark. By the phrase "strictly below" is meant that the Newton polyhedron of $I(\mathbf{s}, \mathcal{C}, \varphi)$ is contained in $\Gamma_I - (\epsilon, \infty)^k$, for some $\epsilon > 0$. ∎

PROOF: One can give a similar proof to that of Theorem 4.1. A sketch of the proof should therefore suffice here. The notation of (4.4) will be used below. In the following, an arbitrarily given $\mathcal{C} = (J_0, \mathbf{C}_1, \ldots, \mathbf{C}_k) \neq (0, \ldots, 0)$ is fixed throughout the discussion.

As in (4.2), define each R_i by

$$R_i(w) =_{def} \frac{1}{P_i(1/w)} \quad \text{for each } i\,.$$

In addition, define the rational function $\Phi_{\mathcal{C}}$ as

$$\Phi_{\mathcal{C}}(w_1, \ldots, w_n) =_{def} \frac{D_x^{J_0}(\varphi)}{\varphi} \cdot \frac{D_x^{\mathbf{C}_1}(P_1)}{P_1^{v_1}} \cdots \frac{D_x^{\mathbf{C}_k}(P_k)}{P_k^{v_k}}(1/w)\,.$$

One constructs a nonsingular real algebraic manifold Y and proper birational map $\pi : Y \to (\mathbf{R}^n, (w_1, \ldots, w_n))$ such that properties (4.4)(i,ii,iv) are satisfied. On the other hand, one modifies (4.4)(iii) so that the divisor determined by

$$[\prod_{i=1}^{n} (w_i - 1) \cdot \prod_{i=1}^{n} w_i \cdot \prod_{i=1}^{k} R_i \cdot \Phi_{\mathcal{C}}] \circ \pi$$

has support in the normally crossing divisor $\mathcal{D}$. Restricting $\mathcal{D}$ to a divisor D over a polydisc U, as in the proof of (4.1), and setting $D = \cup_\alpha D_\alpha$ to denote its decomposition into irreducible components, one next defines the multiplicities

$$\begin{aligned} M_\alpha(i) &= ord_{D_\alpha}\, R_i \circ \pi \\ \kappa_\alpha(\mathcal{C}) &= ord_{D_\alpha}\, \Phi_{\mathcal{C}} \circ \pi \\ \gamma_\alpha &= ord_{D_\alpha}\, Jac(\pi) - ord_{D_\alpha}(w_1^2 \cdots w_n^2) \circ \pi\,. \end{aligned}$$

Additionally, for each component D_α, define the linear form and hyperplane

$$\begin{aligned} L_\alpha(\boldsymbol{\sigma}) &= \sum_{i=1}^{k} M_\alpha(i)\, \sigma_i + \gamma_\alpha\,, \\ \mathcal{H}(D_\alpha) &= \{\boldsymbol{\sigma} : L_\alpha(\boldsymbol{\sigma}) = -1\}\,. \end{aligned}$$

Define

$$\mathcal{G} = \bigcap_{\alpha} \{\boldsymbol{\sigma} : L_\alpha(\boldsymbol{\sigma}) \geq -1 \text{ and } \sigma_i \geq 0 \text{ for each } i\},$$

$$\mathcal{G}(\mathcal{C}) = \bigcap_{\alpha} \{\boldsymbol{\sigma} : L_\alpha(\boldsymbol{\sigma}) + \kappa_\alpha(\mathcal{C}) \geq -1 \text{ and } \sigma_i \geq 0 \text{ for each } i\}.$$

Define the polyhedra

$$\hat{\Gamma}_I = \partial\mathcal{G} \quad \text{and} \quad \hat{\Gamma}(\mathcal{C}) = \partial\mathcal{G}(\mathcal{C}).$$

$\hat{\Gamma}_I$ satisfies an important (and evident) "convexity" property that is used below:

(7.9) *Assume that the plane $\mathcal{H}(D_\alpha)$ is disjoint from $\hat{\Gamma}_I$. Then $L_\alpha(\boldsymbol{\sigma}) > -1$ whenever $\boldsymbol{\sigma} \in \hat{\Gamma}_I$.*

The hypoellipticity condition satisfied by each P_i implies by (1.2):

(7.10.1) *If q is a point of ∂B such that $\pi(q) \in H_\infty$ and D_α is any component of D containing q, then $\kappa_\alpha(\mathcal{C}) > 0$.*

(7.10.2) *Assume q is a point of ∂B such that $\pi(q) \in H_\infty$. Let D_α be any component of D containing q. Then, $M_\alpha(i) > 0$ for each i. In addition, if $q \in \partial B$ is such that $\pi(q) \notin H_\infty$, then $M_\alpha(i) = 0$ for each i and any component D_α containing q.*

It is clear that (7.10.1) implies:

$\hat{\Gamma}(\mathcal{C})$ must lie below $\hat{\Gamma}_I$ in the sense of the above Remark.

To complete the proof of Theorem 7.8, it suffices to show that $\hat{\Gamma}_I = \Gamma_I$. This is done by a straightforward adaptation of the analysis used to prove Theorem 4.1. In particular, one reduces to the local situation, as described in (4.16). The fact that $\hat{\Gamma}_I$ is the convex envelope of $\mathcal{G}$, which implies (7.9), is now used to insure that the four properties formulated below (4.16) are satisfied in this new setting. In particular, the exponents $\zeta_{\nu+1}, \ldots, \zeta_n$, appearing in (4.16), are now understood to be functions of $\mathbf{s} = \boldsymbol{\sigma} + i\mathbf{t}$. (7.9) implies that for any $\mathbf{s} \in \hat{\Gamma}_I + i\mathbf{R}^k$ one must have $Re(\zeta_{\nu+1}), \ldots, Re(\zeta_n) > -1$. The rest of the argument is entirely similar to that in the proof of Theorem 4.1. Details are left to the reader. ∎

There is an important application of Theorem B_k. To formulate this, some preliminary remarks are needed.

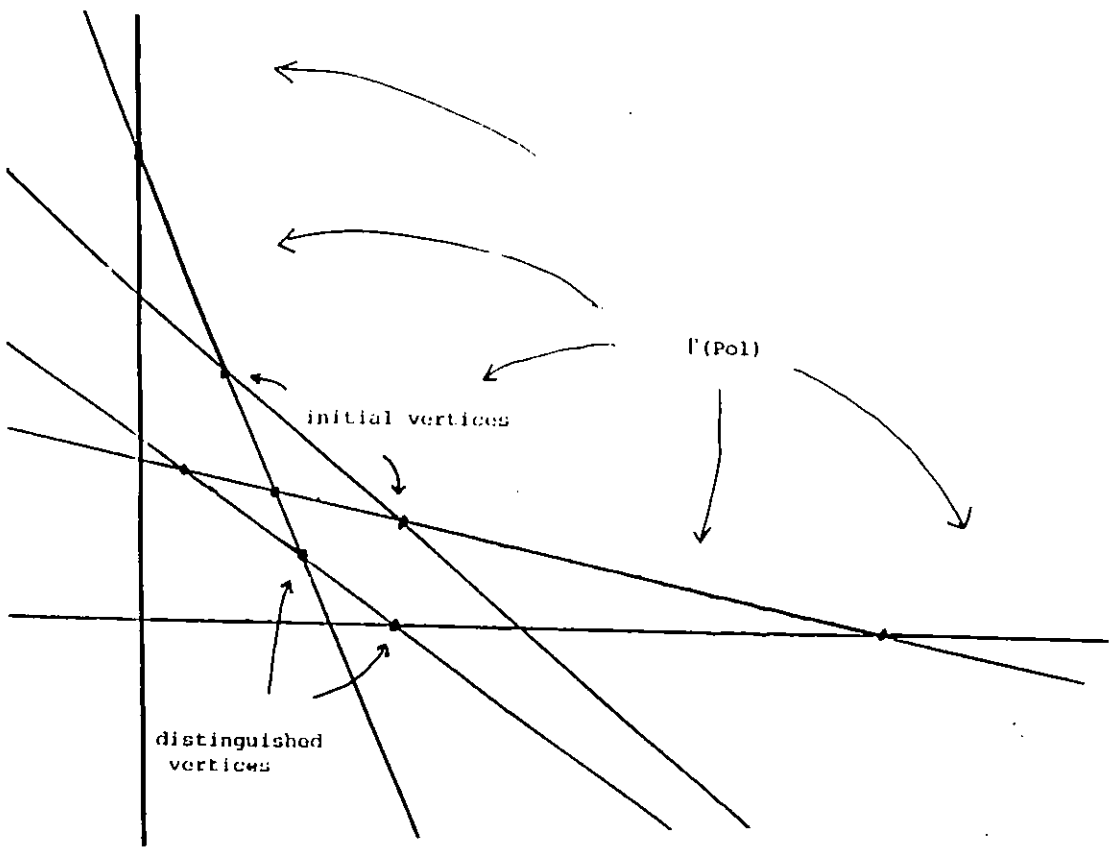
Definition 7.11. A component L of Pol_D satisfying the property that $L \cap \mathbf{R}^k$ contains a face of dimension $k-1$ of Γ is called an initial component of Pol_D. A divisor D_α in a resolution of singularities, as constructed in the proof of Theorem 7.8, is called an extremal divisor if the plane $\mathcal{H}(D_\alpha)$ is an initial component of Pol_D. Define

$$\hat{L} = L - \bigcup_{\{L' \neq L : L' \text{ a component of } Pol_D\}} (L \cap L'). \quad ∎$$

Definition 7.12. A vertex of the polyhedron Γ is called an initial vertex of Pol_D. More generally, set

$$\widehat{Pol_D} = Pol_D \cup \{s_1 \cdots s_k = 0\}.$$

The intersection of any k normally crossing components of $\widehat{Pol_D}$ is called a vertex of $\widehat{Pol_D}$. A distinguished vertex of $\widehat{Pol_D}$ is any vertex contained in an initial component of Pol_D. ∎



Notation. Define the differential forms

$$\omega_D(\mathbf{t},\mathbf{s}) = \mathbf{t}^{\mathbf{s}} D(\mathbf{s},\varphi)\frac{ds_1 \cdots ds_k}{s_1 \cdots s_k},$$
$$\omega_I(\mathbf{t},\mathbf{s}) = \mathbf{t}^{\mathbf{s}} I(\mathbf{s},\varphi)\frac{ds_1 \cdots ds_k}{s_1 \cdots s_k}.$$

Let $\mathbf{v}$ be a vertex of Pol_D. Let $q(\mathbf{v})$ equal the number of components of $\widehat{Pol_D}$ containing $\mathbf{v}$. Let $\mathcal{L}(\mathbf{v}) = \{L_1, \ldots, L_{q(\mathbf{v})}\}$ be the set of components.

In a neighborhood of $\mathbf{v}$ there exist meromorphic functions $\Phi_{i,j}$ which are polar along

$L_i \cup L_j$ for each $i \neq j \in \{1, \ldots, q(\mathbf{v})\}$ such that

$$\omega_D = \mathbf{t}^{\mathbf{s}} \sum_{i \neq j}^{q(\mathbf{v})} \Phi_{i,j}\, ds_1 \cdots ds_k.$$

Then, sufficiently near $\mathbf{v}$ one defines the residue form along the component L_e by the formula

$$\mathcal{R}_{L_e}(\omega_D) = \sum_{j \neq e}^{q(\mathbf{v})} Res_{L_e}\left(\mathbf{t}^{\mathbf{s}} \Phi_{e,j}\, ds_1 \cdots ds_k\right).$$

Here, Res_{L_e} is meant the standard $(k-1, 0)$ Leray residue form along the nonsingular variety $\hat{L}_e$.

Note. For simplicity in the following discussion, one now sets $k = 2$. When $k > 2$ one needs to iterate the discussion $k - 2$ additional times. This is left to the reader. ∎

As emphasized in [Li-5], the precise description of the polar locus of the $\mathcal{R}_L(\omega_D)$ (see below for the definition) is needed to determine asymptotics of certain simultaneous lattice point problems (cf. below).

Assume that $\mathbf{v}$ is an initial vertex. Let $L_e \in \mathcal{L}(\mathbf{v})$. *By definition*, $\mathbf{v}$ is a pole of the residue $(1,0)$ form $\mathcal{R}_{L_e}(\omega_D)$ if the iterated residue

$$Res_{\mathbf{v}} \mathcal{R}_{L_e}(\omega_D) =_{def} \sum_{j \neq e} Res_{L_j}\left(Res_{L_e} \mathbf{t}^{\mathbf{s}} \Phi_{e,j}\, ds_1 ds_2\right) \neq 0.$$

One also observes that the iterated residue with L_j in this sum is, up to a sign, the coefficient of the term $L_e^{-1} L_j^{-1}$ in the Laurent expansion of $\mathbf{t}^{\mathbf{s}} \Phi_{e,j}$.

Theorem B_k evidently implies the set of $\Phi_{i,j}$, used for $D(\mathbf{s}, \varphi)$, can be taken to be the same as those for $I(\mathbf{s}, \varphi)$ when $\mathbf{v}$ is an *initial* vertex. Thus, one concludes

THEOREM 7.13. *For any initial vertex* $\mathbf{v}$ *and component* L_e *of* Pol_D *containing* $\mathbf{v}$, *one has*

$$Res_{\mathbf{v}} \mathcal{R}_{L_e}(\omega_D) = Res_{\mathbf{v}} \mathcal{R}_{L_e}(\omega_I).$$

Theorem 7.13 is now seen to be helpful in detecting if an initial vertex is indeed a pole of the residue form taken along an initial component of Pol_D that contains this vertex. The reason for this is that the methods of Gelfand-Shapiro-Shilov, used to determine precisely poles of generalized functions $P_{\pm}^{s}$ for a polynomial P, can now also be applied to describe with some precision the polar locus of such residue forms.

The observations above lead to a simple to state geometric criterion that locates precisely a pole of $\mathcal{R}_L(\omega_I)$ on the polygon Γ. This is formulated in terms of the geometry of the divisors D_α, appearing in a resolution of singularities, $\pi : Y \to \mathbf{R}^n$, that proves Theorem 7.8. Although this is, by no means, the most general version of the criterion, it will suffice, with some minor modifications, for [Li-6].

THEOREM 7.14. *Let L be an initial component of Pol_I (or equivalently of Pol_D) and D_α an extremal divisor such that $\mathcal{H}(D_\alpha) = L \cap \mathbf{R}^2$. Let $\mathbf{v}$ be an initial vertex contained in L. Assume that $q(\mathbf{v}) = 2$. Then $Res_{\mathbf{v}}\mathcal{R}_L(\omega_I) \neq 0$ if there exists an exceptional divisor $D_\beta \subset Y$ such that*

(7.14.1) $\qquad D_\beta \cap D_\alpha \cap B \neq \emptyset$

(7.14.2)

$$\mathcal{H}(D_\beta) \textit{ is the other initial component of } \widehat{Pol_I} \textit{ containing } \mathbf{v}\,.$$

Remark. It is clear that (7.14.2) must be true by hypothesis. The substance of the theorem lies in the validity of the geometric property (7.14.1), from which the analytical information is deduced. ▮

PROOF OF 7.14: This will follow from the regularization of the integral which represents the local contribution to ω_I in each open set $\mathcal{U}_i \cap B$, defined in (4.16). One first chooses and fixes one such pair of divisors D_α, D_β satisfying the conditions in (7.14).

In the following, the notation, employed in the proof of (7.8), will be used. (7.14.1) implies there exist open sets $\mathcal{U}_i(\alpha,\beta), i \in \{i_1,\dots,i_q\}$, (the set of indices apparently depend upon α,β) which intersect $D_\alpha \cap D_\beta$, such that in each connected component of $(\mathcal{U}_i(\alpha,\beta) - D) \cap B$, there are local coordinates $(z_1,\dots,z_n)$ so that the integrand

$$\pi^*(\varphi/P_1^{s_1}\cdots P_k^{s_k}\, dx_1\cdots dx_n)\big|_{(\mathcal{U}_i(\alpha,\beta)-D)\cap B}$$

has the form

$$\prod_{\ell=1}^{a} z_{\ell,\pm}^{L_\alpha(\mathbf{s})}\cdot \prod_{k=a+1}^{a+b} z_{k,\pm}^{L_\beta(\mathbf{s})}\cdot \Omega_{i,\alpha,\beta}(\mathbf{s},z)\,. \tag{7.15}$$

In this expression, one has grouped with the first product any z_i whose exponent is one less than a nonzero multiple of $L_\alpha(\mathbf{s})+1$. Similarly, in the second product, any z_j appears whose exponent is one less than a nonzero multiple of $L_\beta(\mathbf{s})+1$. Notation has been abused, for simplicity, by identifying the exponents by $L_\alpha(\mathbf{s}), L_\beta(\mathbf{s})$. This however will not affect in any significant manner the arguments below.

The choice of signs in (7.15) is made so that the product is positive on each connected component of $(\mathcal{U}_i(\alpha,\beta) - D) \cap B$. This is because one assumes, as indicated above, that each P_i and φ are positive outside a compact subset of $[1,\infty)^n$. Moreover, one has chosen, for simplicity, the indices so that $\{z_1 = 0\} = D_\alpha \cap \mathcal{U}_i(\alpha,\beta)$, $\{z_{a+1} = 0\} = D_\beta \cap \mathcal{U}_i(\alpha,\beta)$. In addition, $\Omega_{i,\alpha,\beta}$ is an $n-$ form which satisfies these properties for each i,α,β:

(1) it is real analytic in z whenever $z \in (\mathcal{U}_i(\alpha,\beta) - D) \cap B$;
(2) it is locally integrable if $\boldsymbol{\sigma}$ is contained in an *open* neighborhood of $\Gamma \cap L$;
(3) one has for each $i \in \{i_1,\dots,i_q\}$

$$\sum_{\substack{C_\epsilon \\ \text{a component of} \\ (\mathcal{U}_i(\alpha,\beta)-D)\cap B}} \int_{\overline{C_\epsilon}\cap\{z_1=\cdots=z_{a+b}=0\}} \Omega_{i,\alpha,\beta}(\mathbf{v},z)\,dz_{a+b+1}\dots dz_n > 0\,. \tag{7.16}$$

One calls the integral of $\pi^*(\varphi/P_1^{s_1}\cdots P_k^{s_k}\,dx_1\cdots dx_n)\big|_{(\mathcal{U}_i(\alpha,\beta)-D)\cap B}$, taken over the sum of the connected components of $(\mathcal{U}_i(\alpha,\beta)-D)\cap B$, the local contribution to ω_I from $\mathcal{U}_i(\alpha,\beta)$.

Adapting the method of regularization of Gelfand-Shilov-Shapiro, one now analytically continues

$$(7.17)\qquad \sum_{\substack{\alpha,\beta\\ \text{s.t. } D_\alpha, D_\beta \text{ satisfy}\\ 7.14.1, 7.14.2}} \sum_{i=i_1}^{i_q} \int_{(\mathcal{U}_i(\alpha,\beta)-D)\cap B} \pi^*(\varphi/P_1^{s_1}\cdots P_k^{s_k}\,dx_1\cdots dx_n).$$

Evidently, this is done by regularizing each summand in the local contribution to ω_I from $\mathcal{U}_i(\alpha,\beta)$ for each i,α,β. It is then straightforward to verify that the following crucial properties hold.

There exists a sufficiently small neighborhood $\mathcal{W}$ of $\mathbf{v}$ and an analytic function $A_{\mathbf{v}}(\mathbf{s})$ on $\mathcal{W}$ satisfying

$$A_{\mathbf{v}}(\mathbf{v}) > 0$$
$$\omega_I\big|_{\mathcal{W}} = \frac{A_{\mathbf{v}}(\mathbf{s})}{(L_\alpha+1)^a\,(L_\beta+1)^b} \text{ for some positive integers } a, b.$$

An elementary verification will now show that the positivity of $A_{\mathbf{v}}(\mathbf{v})$ implies that

$$Res_{\mathbf{v}}\,\mathcal{R}_L(\omega_I) \neq 0.$$

This completes the proof of Theorem 7.14. ∎

Remark 7.18. There is a different method of understanding the residue forms $\mathcal{R}_L(\omega_I)$ that allows one to weaken the condition (7.14.2), given a divisor D_β satisfying (7.14.1), to:

$$(\#)\qquad \mathcal{H}(D_\beta)\cap\mathcal{H}(D_\alpha) = \{\mathbf{v}\}.$$

The reason for this is that the sum in (7.17) also determines an integral representation of $\mathcal{R}_L(\omega_I)$ when $\mathbf{s}$ is contained in any open neighborhood $\mathcal{V}$ of a point in $\Gamma\cap\hat{L}$ that is disjoint from all other components of Pol_I. Now, fix one triple of i,α,β indexing the sum in (7.17). Let $(z_1,\ldots,z_n)$ be the coordinates on $\mathcal{U}_i(\alpha,\beta)$, used in (7.15).

The main observation to make is that inside $\mathcal{V}$ one has the following property. The convexity of Γ implies that the real part of the exponent of each coordinate $z_j, j\geq a+1$ is strictly larger than -1 whenever $\mathbf{s}\in\mathcal{V}$. This implies that if C is any connected component of $(\mathcal{U}_i(\alpha,\beta)-D)\cap D$ then the integral

$$\int_{\cap_{i=1}^a\{z_i=0\}\cap\overline{C}} \prod_{q=1}^{b} z_{a+q}^{L_\beta(\mathbf{s})}\cdot\Omega_{i,\alpha,\beta}(\mathbf{s},z)\,dz_{a+1}\cdots dz_n\Big|_{L\cap\mathcal{V}}$$

converges absolutely and represents an analytic function of a global parameter (say s_1 or s_2) on L.

Let $dz' = dz_{a+1} \cdots dz_n$. Now consider the following sum:

$$\sum_{\substack{\alpha,\beta \\ \text{s.t. } D_\alpha, D_\beta \text{ satisfy} \\ 7.14.1 \text{ and } \#}} \sum_{i=i_1}^{i_q} \sum_{\substack{C_a \\ \text{a component of} \\ (\mathcal{U}_i(\alpha,\beta)-D)\cap B}} \int_{\cap_{i=1}^{a}\{z_i=0\}\cap\overline{C_a}} \prod_{q=1}^{b} z_{a+q}^{L_\beta(\mathbf{s})} \cdot \Omega_{i,\alpha,\beta}(\mathbf{s},z)\, dz' \Big|_{L\cap\mathcal{V}}.$$

If s denotes a global parameter on L, then up to the factor $\mathbf{t}^{\mathbf{s}}$ this triple sum equals the factor of ds for $\mathcal{R}_L(\omega_I)$, when $\mathbf{s}$ is restricted to $\mathcal{V}\cap L$. So, one has an integral representation of the residue form inside some open subset of $\hat{L}$. Evidently, each summand can then be analytically continued to the entire s plane as a meromorphic function, using exactly the same regularization procedure of [G-S]. In a neighborhood of $\mathbf{v}$ it follows that the representation of $\mathcal{R}_L(\omega_I)$, given earlier in the section, must agree with that obtained by the process just described. One then observes that if (#) holds, there is at least one term in the triple sum which must have a pole at the value $s = s_{\mathbf{v}}$, corresponding to the point $\mathbf{v}$. By (7.16) the residue at $s_{\mathbf{v}}$ must also be positive. Summing over all such terms that can contribute to the pole at $s_{\mathbf{v}}$, one sees that the $(1,0)$ form $\mathcal{R}_L(\omega_I)$ also has a pole at $s_{\mathbf{v}}$. ∎

Concluding Remarks.

(1) To understand what is the relation of Theorem 7.14 to asymptotics of (weighted) lattice point counts, the following precis may be helpful. The reader can also consult [Li-5] for further details.

One can formulate a (simplified) "simultaneous" lattice point problem, whose asymptotic behavior can be precisely determined whenever the criteria in (7.14) are satisfied. Again, for simplicity only, $k=2$ is assumed. Let $\mathbf{a} = (a_1, a_2) \in (0,\infty)^2$, and $t > 0$. Define

$$N_{\mathbf{P}}(t, \mathbf{a}, \varphi) = \sum_{\{m\in\mathbf{N}^n : P_i(m)\le t^{a_i}, i=1,2\}} \varphi(m).$$

Evidently, one can interpret the family of counts $N_{\mathbf{P}}(t,\mathbf{a},\varphi)$ as a natural generalization of $N_P(t,\varphi)$ when given 2 polynomials. Thus, the analysis of the asymptotic behavior of $N_{\mathbf{P}}(t,\mathbf{a},\varphi)$ is an interesting problem to understand. Proceeding classically, one can now see why the polygon Γ should be defined so as to lie in $[0,\infty)^2$.

One starts with the following integral formula for $N_{\mathbf{P}}(t,\mathbf{a},\varphi)$. Assume that b is a sufficiently large positive number. Then, a 2-fold iteration of Perron's formula [Ti, ch. 9] shows:

$$N_{\mathbf{P}}(t,\mathbf{a},\varphi) = \frac{1}{(2\pi i)^2}\int_{\sigma_1=\sigma_2=b} t^{\mathbf{a}\cdot\mathbf{s}} D(\mathbf{s},\varphi)\frac{ds_1 ds_2}{s_1 s_2}. \tag{7.19}$$

Thus, ω_D is an evident $(2,0)$ form to introduce for purposes of studying $N_{\mathbf{P}}(t, \mathbf{a}, \varphi)$. The polygon Γ is evidently the polygon of the polar divisor of the meromorphic function appearing in ω_D.

[Li-5] gave a general description of an upper bound for any $N_{\mathbf{P}}(t, \mathbf{a}, \varphi)$ in terms of the geometry of the initial components of Pol_D. In order to sharpen such estimates, in particular, to determine a precise asymptotic for $N_{\mathbf{P}}(t, \mathbf{a}, \varphi)$, it is clear from the analysis in [ibid] that one needs to answer the following:

Question. For each initial component L, what is the polar locus of $\mathcal{R}_L(\omega_D)$?

The point of Theorem 7.14 is that it enables one to exploit geometric propertities implicit in the problem. [Li-6] will indicate one way of successfully doing so. Surely there are other ways, as indicated in the Introduction, that remain to be worked out.

(2) The relation between the number of lattice points in and volume of an expanding family (depending upon one or several parameters) of semialgebraic sets was (implicitly) the subject of a paper of Davenport [Dav-2], who actually worked in a considerably more general setting. It seems likely that if the semialgebraic sets in question are determined by the intersection of proper polynomial mappings on $\mathbf{R}^n$ then his analysis should apply and result in a pair of inequalities for the difference between the number of lattice points and the volume. However, this begs the question of how one actually determines the asymptotic behavior of the volume of the region.

Moreover, the proof of Theorem A can easily be seen to extend to analyze the difference of the number of lattice points in and volume of a family of intersections

$$\{P_1 \leq t_1\} \cap \ldots \cap \{P_k \leq t_k\} \quad \text{as each } t_i \to \infty,$$

when each P_i is hypoelliptic—once one has the asymptotics of the volume of these sets.

Neither method however extends to deal with the case of weights determined by reasonable classes of rational functions, which a general theory of multivariable asymptotic expansions should certainly be capable of incorporating. For this, ideas from the analytic theory of singularities appear essential.

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