Nash resolution for binomial varieties as Euclidean division. Apriori termination bound, polynomial complexity in dim 2

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Abstract. We establish (novel for desingularization algorithms) apriori bound on the length of resolution of singularities by means of the composites of normalizations with Nash blowings up, albeit that only for affine binomial varieties of (essential) dimension 2 . Contrary to a common belief the latter algorithm turn out to be of a very small complexity (in fact polynomial).

To that end we prove a structure theorem for binomial varieties and, consequently, the equivalence of the Nash algorithm to a combinatorial algorithm that resembles Euclidean division in dimension ≥ 2 and, perhaps, makes Nash termination conjecture of the Nash algorithm particularly interesting.

An explicit bound on the length of normalized Nash resolution of a minimal surface singularity via the size of the dual graph of its minimal desingularization is in the Appendix (by M. Spivakovsky).

Contents

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1. Introduction.

1.1. **Summary.** We establish isomorphisms of irreducible components of affine binomial varieties $\hat{V}$ with their toric components V and of the étale germs of the latter with the products of their subtori and subvarieties $\hat{Y}_V$ defined by setting nonvanishing on V coordinates to 1 . Resulting structure theorem implies 'no change' in the singularities constraining the nonvanishing coordinates to any nonsingular variety, e. g. allowing linear combinations of two monomials as binomials.

Our proof of the equivalence of the Nash algorithm for desingularization of binomial varieties to a combinatorial algorithm that resembles, surprizingly, Euclidean division (in dim ≥ 2) is by following the changes in the exponents of a monomial parametrization of the torus of the toric component Y_V of $\hat{Y}_V$ under Nash blowings up. However, termination of the *Euclidean multidimensional algorithm* predicted by Nash conjecture seems very hard to establish even in dimension 2 .

On the other hand, when $\dim Y_V = 2$, a combinatorial version of the composites of normalizations with Nash blowings up unexpectedly yields a (first for desingularization algorithms and sharp) apriori bound $2 \cdot \log_2 D$ on the length of such sequences leading to a resolution of singularities, where D is the area of the parallelogram on the shortest integral generators of the cone spanned by the exponents of any monomial parametrization of the torus of Y_V . Moreover, every affine chart is covered after the normalized Nash blowing up by at most 5 affine charts with at most 3 among them being singular.

1.2. **Nash blow ups and normalizations: conjectures.** For a reduced equidimensional algebraic variety X , say of $\dim X = n$, over an algebraically closed field $\mathbb{K}$ of zero characteristic (this requirement is relaxed in Sections 2, 5) the Gauss map G_X is defined off singular points $\operatorname{Sing} X$ of X and sends every point $\mathcal{P} \in \operatorname{Reg} X := X \setminus \operatorname{Sing} X$ to the tangent space $T_{\mathcal{P}}X$ (to X at $\mathcal{P}$) as points of the respective Grassmanian bundle restricted over X . (Using embeddings of affine charts of X in $\mathbb{K}^N$ consider the Grassmanian variety of n-dimensional subspaces of $\mathbb{K}^N$. The latter naturally embeds into projective space $\mathbb{P}(\wedge^n \mathbb{K}^N)$ by means of Plücker coordinates, i. e. the homogeneous coordinates in $\wedge^n \mathbb{K}^N$.) The Nash blow up $N(X)$ of X is the closure of the graph of G_X with the natural projection $N_X : N(X) \to X$.

Nash conjecture. The sequence of Nash blowings up starting with any algebraic variety stabilizes resulting in a desingularization.

Over affine charts the ring of 'regular functions' $\mathbb{K}[\mathcal{N}(X)]$ on normalization $\mathcal{N}_X : \mathcal{N}(X) \to X$ of variety X is the integral closure of

$\mathbb{K}[X]$ in its field of fractions. When Z is nonsingular and $X \simeq Z \times Y$ (locally) it follows that $\mathcal{N}(X) \simeq Z \times \mathcal{N}(Y)$ (of course also only locally). Normalization separates all étale (in the completions of local rings) irreducible components. We refer to the composites of normalizations with Nash blowings up as *normalized Nash blowings up.*

Normalized Nash conjecture. Normalized Nash blowings up starting with any algebraic variety result in a desingularization.

So far though Nash and normalized Nash desingularizations remain elusive in dimensions larger than one and two, respectively. Moreover,

Remark 1.1. In dimension larger than one an apriori estimate for the length of normalized Nash desingularization is novel (as well as in any reasonable sense for other desingularizations).

(i) If Nash blow up $N_X : N(X) \to X$ is an isomorphism then X is nonsingular, see [10] and [11].

(ii) Nash conjecture is valid when $\dim X = 1$ and there is a simple estimate for the length of sequences by Nash blowings up leading to a desingularization (e. g. by means of Newton-Puiseux expansion).

(iii) M. Spivakovsky proved that the sequence of normalized Nash blowings up terminates when $\dim X = 2$, see [13] and [8]. Bound $1 + \log_2(\#\Gamma)$ on the length of normalized Nash desingularization of a minimal surface singularity, where $\#\Gamma$ is the number of vertices of the so called *dual graph* Γ of its *minimal desingularization*, appears below in the Appendix authored by M. Spivakovsky.

1.3. **Desingularization results briefly.** With *affine binomial* varieties (see e. g. [3]) defined as the closures in $\mathbb{K}^N$ of the vanishing sets off coordinate hyperplanes of collections of differences of pairs of monomials (called binomials) Theorem 2.7 on the structure of affine binomial varieties provides a reduction of Nash and of normalized Nash desingularizations to that of *essential* varieties, i. e. affine toric varieties containing the origin. (The singularities of the germs of an essential variety occur in every neighbourhood of its origin, see Remark 2.13, while the convex hulls of the exponents $\mathcal{E} \subset \mathbb{Z}^m$ of monomial parametrizations of its dense torus do not contain the origin, see Remark 2.2. Essential variety is nonsingular iff its 'parametrizing exponents' $\mathcal{E}$ are generated over positive integers by a subset of size m, see Remark 2.16.)

Following the process of changes in the set $\mathcal{E}$ of parametrizing exponents under successive Nash and normalized Nash algorithms for these toric varieties we establish in Section 4 their respective 'combinatorial' versions. The combinatorial version of Nash algorithm resembles Euclidean division (in dim ≥ 2) the termination of which so far remains elusive even in dimension 2 in spite of its combinatorial nature and simple formulation. But by means of the combinatorial version of the normalized Nash algorithm we establish in essential dimension 2 an explicit (sharp) apriori bound (of Section 1.1) in terms of the set $\mathcal{E}$ of parametrizing exponents with each branching of the algorithm being bounded by 5 and the complexity along a single branch being polynomial in the binary size of the input.

Remark 1.2. Of course, if X consists of several irreducible components $X = \cup_i X_i$ then $N(X) = \cup_i N(X_i)$ and $N(X_i)$ are the irreducible components of $N(X)$. Also, when locally (in Zariski or even in the étale topology), say in U , variety X is a product of a nonsingular variety Z with a (possibly) singular one, say Y , then $N(X)$ over U is isomorphic to the product $Z \times N(Y)$ of Z with $N(Y)$. Nash blow up either separates any pair of smooth local étale irreducible components, or reduces the *contact* between them. (With I_j , $j = 1, 2$, being the ideals of local étale irreducible components in the completion $\hat{\mathcal{O}}$ of local ring $\mathcal{O}$ of the ambient manifold contact is the largest integer l such that $I_1 + \hat{\mathfrak{m}}^l = I_2 + \hat{\mathfrak{m}}^l$, where $\hat{\mathfrak{m}}$ is the maximal ideal of $\hat{\mathcal{O}}$.) Thus the sequence of Nash blowings up of a variety with smooth local étale irreducible components terminates separating 'Nash liftings' of these components.

1.4. **Singularities vis-a-vis structure of binomial varieties. Locations guidance.** In Sections 2 and 5 we state and prove Theorem 2.7 on the structure of *affine binomial* (shortly $\mathcal{AB}$-) varieties, which are the Zariski closures $\hat{V}$ in $\mathbb{A}^N$ of the vanishing on the standard torus $\mathbb{T}^N := \cap_j \{w_j \neq 0\} \subset \mathbb{A}^N$ collections of binomials, where $\mathbb{A}^N := \operatorname{Spec} \mathbb{K}[w]$ and $\mathbb{K}[w]$ is the ring of polynomials in N variables w_j with coefficients in a field $\mathbb{K}$ with not vanishing in $\mathbb{K}$ number $\underline{d}$ specified in part **C** of the theorem. (For an algebraically closed field $\mathbb{K}$ in Sections 2 and 5 one may replace $\mathbb{A}^N$ by $\mathbb{K}^N$.) Identifying coordinates z_1 , ... , $z_{\underline{N}}$ not vanishing at any point of $\hat{V}$ and morphism $\pi : \hat{V} \to \pi(\hat{V}) \hookrightarrow \mathbb{A}^{\underline{N}}$ with $'z$-coordinates' as components we refer to varieties $(\pi)^{-1}(W)$ for a nonsingular $W \hookrightarrow \pi(\hat{V})$ as $\hat{V}$*-admissible.* As a byproduct for *generalized affine binomial* (shortly $\mathcal{GAB}$-) varieties, i. e. $\hat{V}$-admissible with $\hat{V}$ in $\mathcal{AB}$ class, the singularities of the irreducible components of the local étale germs are essentially 'the same' as those of the respective $\mathcal{AB}$-variety, see Claim 2.14 . The $\mathcal{GAB}$ class includes all *quasi-binomial varieties*, i. e. allowing as binomials any linear combinations of pairs of monomials.

Other consequences of Theorem 2.7 include a reduction of Nash (respectively normalized Nash) desingularizations of $\mathcal{GAB}$-varieties to the respective desingularizations of irreducible binomial varieties passing through the origins of the (appropriate) ambient affine coordinate charts as well as simple criteria of nonsingularity for all toric varieties in terms of the exponents of monomial parametrizations of their dense tori and, as a consequence, for blowings up of smooth affine spaces at the ideals generated by monomials, see Criterion 2.18 and Remark 2.19.

Affine toric varieties are the closures in $\mathbb{A}^N$ of the images - tori $\phi_{\mathcal{E}}(\mathbb{T}^m)$ of the standard tori $\mathbb{T}^m \simeq \underline{\mathbb{T}}^m := \cap_j\{x_j \cdot \tilde{x}_j = 1\} \hookrightarrow \mathbb{A}^{2m}$ under monomial bijections $\phi_{\mathcal{E}} : \mathbb{T}^m \to \phi_{\mathcal{E}}(\mathbb{T}^m) \hookrightarrow \mathbb{T}^N$ (with $\mathcal{E} \subset \mathbb{Z}^m$ being the set of the exponents of the monomial components of $\phi_{\mathcal{E}}$). Toric varieties are binomial, but not necessarily normal, e. g. Whitney Umbrella $\{x^2 - z \cdot y^2 = 0\} \subset \mathbb{C}^3$. Moreover, Nash blowings up of normal varieties with open dense tori may fail to be normal, e. g. Nash blow up of surface $S := \overline{\phi_{\mathcal{E}}(\mathbb{T}^2)} \subset \mathbb{C}^3$, where $\phi_{\mathcal{E}} : (x_1, x_2) \mapsto (x_1 \cdot x_2 \ , \ x_1 \cdot x_2^2 \ , \ x_1^3 \cdot x_2^2)$, fails to be normal in spite that S is a normal surface. Indeed, normality of the latter is a consequence (due to a criterion in Section 2.1 of [5]) of the property of the exponents $\mathcal{E} = \{(1,1) \ , \ (1,2) \ , \ (3,2)\} \subset (\mathbb{Z}_+)^2$ of monomial map $\phi_{\mathcal{E}}$ to generate over $\mathbb{Z}_+$ all points of its integral lattice within the (positive) cone that the respective exponents span in $\mathbb{R}^2$, see Example 6.3 for the failure of normality for $N(S)$. Consequently we refer to the varieties with a dense torus as *toric* (as in [14] or [1]), while in [5] they are refered to as toric only when normal.

It turns out that Nash blow up of essential variety is a finite union of affine charts which are essential, see Claim 4.6 . The latter allows to establish (Section 4) a 'combinatorial bookkeeping' of the progress in Nash (respectively normalized Nash) sequence of blowings up for essential varieties leading to a multidimensional Euclidean division. When $\dim Y_V = 2$ we state (in Section 3) and prove (in Section 7) an apriori bound $2 \cdot \log_2 D$ (with D from Section 1.1) on the lengths of the sequences of normalized Nash blowings ups resulting in a desingularization. Moreover, it turns out that if *essential dimension* (i. e. dimension of essential subvariety) equals 2 the normalized Nash desingularization, as well as separately the normalization of binomial varieties, are of a polynomial complexity in terms of the binary size of the initial input, see Theorem 3.1 , Corollaries 7.6 , 7.7 and Corollary 7.5. In Section 8 we establish (local) invariance of $D = D_o$ at a point $o \in Y$ with respect to local isomorphisms that preserve hypersurfaces invariant under the action of the torus of Y and contain o .

Part 1. Arbitrary dimension.

2. Reduction to essential toric case.

We consider algebraic varieties (so called binomial) that admit (Zariski) open coverings by 'affine binomial' varieties, i. e. closures $\hat{V}$ in $\mathbb{A}^N$ of sets $V^*(\hat{f}) := \{w \in \mathbb{T}^N : \hat{f}_j(w) = 0, 1 \le j \le M \}$, where $(\hat{f})$ are the ideals in the ring $\mathbb{K}[w]$ of polynomials in $w := (w_1, \dots, w_N)$ with coefficients in a field $\mathbb{K}$ generated by binomials

$$\hat{f}_j := w_1^{\hat{\alpha}_{j1}} \cdots w_N^{\hat{\alpha}_{jN}} - w_1^{\hat{\beta}_{j1}} \cdots w_N^{\hat{\beta}_{jN}} . \tag{2.1}$$

With an *exponents matrix* $\hat{E}$ of $\hat{V}$ having entries $\hat{\alpha}_{ji} - \hat{\beta}_{ji}$ we adopt notations $\hat{V}^* := \hat{V} \cap \mathbb{T}^N = V^*(\hat{f}) = \{w \in \mathbb{T}^N : w^{\hat{E}} = \mathbb{I}_M\}$, $\mathbf{0} \in \mathbb{A}^M$ is the origin, $\mathbb{I}_M := (1, \dots, 1) \in \mathbb{A}^M$ and vector (j) has the only nonzero j-th coordinate equal to one. Finally, we split all w-coordinates into y- , whenever $\{w_j = 0\} \cap \hat{V} \neq \emptyset$, and z-coordinates, $w = (y , z)$, and let $\pi : \mathbb{A}^N \to \mathbb{A}^{N-L}$ be the linear map defined by z-coordinates.

Denote Id_N the unit matrix of size $N \times N$ and $\mathbb{R}_+ \subset \mathbb{R}$, $\mathbb{Q}_+ \subset \mathbb{Q}$, $\mathbb{Z}_+ \subset \mathbb{Z} \setminus \{\mathbf{0}\}$ the subsets of non negative real, rational and integral numbers respectively. We refer to the closure in $\mathbb{A}^N$ of the image of a bijective monomial map $\phi_{\mathcal{E}} : \mathbb{T}^m \to X^*_{\mathcal{E}} := \phi_{\mathcal{E}}(\mathbb{T}^m) \subset \mathbb{A}^N$ (with the exponents in $\mathcal{E} \subset \mathbb{Z}^m$) as an affine toric variety and denote the latter $X_{\mathcal{E}}$. For the sake of convenience we denote $(A||B)$ the matrix with columns of A followed by the columns of B and the matrix with rows from the exponents set $\mathcal{E}$ by the same letter, i. e. $\phi_{\mathcal{E}}(x) = x^{\mathcal{E}}$, while both the set of columns and transpose matrix of a matrix T by T^{tr} (in particular $\pi \circ \phi_{\mathcal{E}} = \phi_{(\pi(\mathcal{E}^{tr}))^{tr}}$). We refer to $\Delta \subset \mathbb{Z}^N$ with $\#(\Delta) = \mathrm{rank}\,(\Delta)$ and $\mathrm{Span}_{\mathbb{Z}}(\Delta) = \mathrm{Span}_{\mathbb{Q}}(\Delta) \cap \mathbb{Z}^N$ as a $\mathbb{Z}$-basis (of $\mathrm{Span}_{\mathbb{Q}}(\Delta)$) and denote by $\mathrm{Conv}\,(\mathcal{E}) \subset \mathbb{R}^m$ the convex hull of $\mathcal{E}$.

Remark 2.1. Applying 'Gauss elimination' let Λ , λ be square matrices with entries in $\mathbb{Z}$ and $\det(\Lambda) = 1 = \det(\lambda)$ such that matrix $\tau := \Lambda \cdot \hat{E} \cdot \lambda$ has vanishing entries except in the upper-left corner on a 'diagonal' of length $r = \mathrm{rank}\,\hat{E}$ (while for the successive integral entries $d_q \in \mathbb{Z}_+$, $q = 1, \dots, r$, the ideals generated in $\mathbb{Z}$ by the $q \times q$ minors of matrix $\hat{E}$ and, respectively, by $d_1 \cdot ... \cdot d_q$ coincide, the so called Smith normal form). Denote $d(\hat{E}) := |d_1 \cdot ... \cdot d_r|$. Of course $\dim \hat{V} = N - r$, solutions of $w^{\hat{E}} = \mathbb{I}_M$ and of $\tilde{w}^{\tau} = \mathbb{I}_M$ in $\mathbb{T}^N$ are related by an automorphism ϕ_λ of $\mathbb{T}^N$, $d(\hat{E}) = \#((\mathrm{Span}_{\mathbb{Q}}(\hat{E}^{tr}) \cap \mathbb{Z}^M)/\mathrm{Span}_{\mathbb{Z}}(\hat{E}^{tr}))$ and $\hat{V}^*$ has $[d(\hat{E})]$ irreducible components (with $[d(\hat{E})] := d(\hat{E})$ or $:= d(\hat{E}) \cdot p^{-s} \in \mathbb{Z} \setminus (p \cdot \mathbb{Z})$ depending on $\mathbb{K}$ being of characteristic $p = 0$ or $p > 0$). Finally, if $N = r$ then morphism

$\phi_{\hat{E}} : \mathbb{T}^N \ni w \mapsto w^{\hat{E}} \in \mathbb{T}^M$ is a parametrization when $d(\hat{E}) = 1$, is étale when $[d(\hat{E})] = d(\hat{E})$ and is finite of degree $d(\hat{E})$ if $X^*_{\hat{E}} = X_{\hat{E}}$ (by exploiting that ϕ_Λ is an automorphism of $\mathbb{T}^M$).

Consequently the irreducible component $V^* \ni \mathbb{I}_N$ of $\hat{V}^*$ is a torus $V^* = X^*_{\mathcal{E}}$ with the choices for parametrizing V^* exponents $\mathcal{E} \subset \mathbb{Z}^n$, $n := N - r$, such that the columns of $\mathcal{E}$ as a matrix should form a $\mathbb{Z}$-basis of $\operatorname{Ker} \hat{E} \cap \mathbb{Z}^N$. (Indeed, for $\{\tilde{w} \in \mathbb{T}^N : \tilde{w}^\tau = \mathbb{I}_M\}$ parametrizations $x \mapsto \tilde{w} = x^{\tilde{\mathcal{E}}}$ are determined by the $\mathbb{Z}$-bases of $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^N$, implying the claim by means of the automorphism $w = \tilde{w}^\lambda$ of $\mathbb{T}^N$ and correspondence $\mathcal{E} := \lambda \cdot \tilde{\mathcal{E}}$.) Finally,

Property A. Cosets $[g] \in \Gamma := \hat{V}^*/V^*$ of $g \in \hat{V}^*$ list the irreducible components $g \cdot V$ of $\hat{V}$, where $V := \overline{V^*}$, and $\hat{V}^* \subset \operatorname{Reg} \hat{V}$.

Remark 2.2. Affine toric variety $X_{\mathcal{E}} \ni \mathbf{0}$ iff $\operatorname{Conv}(\mathcal{E}) \not\ni \mathbf{0}$. Indeed, the 'only if' follows since if $\operatorname{Conv}(\mathcal{E}) \ni \mathbf{0}$ then there are $\mathcal{E}' \subset \mathcal{E}$ and $\{p_e \in \mathbb{Z}_+\}_{e \in \mathcal{E}'}$ such that $\sum_{e \in \mathcal{E}'} p_e \cdot e = \mathbf{0}$, which implies that $X_{\mathcal{E}} \subset \{w : \prod_{e \in \mathcal{E}'} w_e^{p_e} = 1\}$. 'If' follows by choosing $\eta \in \mathbb{Z}^m \subset (\mathbb{R}^m)^{dual}$ with $\eta(e) > 0$ for $e \in \mathcal{E}$ since then $\mathbf{0} \in X_{\eta(\mathcal{E})} \subset X_{\mathcal{E}}$.

The proofs of claims of this section are in Section 5 unless included.

Claim 2.3. *Torus $X \cap \mathbb{T}^N$ of an affine m-dimensional toric variety X admits parametrization $\phi_{\mathcal{E}}$ with exponents $\mathcal{E} \subset (\mathbb{Z}_+)^m$ iff $\mathbf{0} \in X$.*

Lemma 2.4. *Variable w_j , is not a z-variable (equivalently is a y-variable) for $\hat{V}$ iff there is $\vec{\xi} \in \operatorname{Ker} \hat{E} \cap (\mathbb{Z}_+ \cup \{\mathbf{0}\})^N$ with $(\vec{\xi})_j > 0$.*

Corollary 2.5. *Exists $\vec{\xi^+} \in \operatorname{Ker} \hat{E} \cap (\mathbb{Z}_+ \cup \{\mathbf{0}\})^N$ with $(\vec{\xi^+})_j > 0$ iff w_j is a y-variable. Hence $(\mathbf{0}, \mathbb{I}_{N-L}) \in X_{\mathcal{E}^+} \subset \hat{V}$ for $\mathcal{E}^+ := \{(\vec{\xi^+})_j\}_j \subset \mathbb{Z}$.*

For the sake of completeness we include the following

Claim 2.6. *Polynomial $P \in \mathbb{K}[w]$ vanishes on $\hat{V}$ if and only if $(y_1 \cdot \ \ldots \ \cdot y_L)^l \cdot P \in (\hat{f})$ for some $l \in \mathbb{Z}_+$.*

Theorem 2.7. *For any affine binomial variety $\hat{V} \hookrightarrow \mathbb{A}^N$*

B. *Variety $\pi(\hat{V}) = \overline{\pi(\hat{V}^*)}$ is binomial and closed in $\mathbb{A}^{N-L}$, while $\hat{V} \cap (\mathbb{A}^L \times \mathbb{I}_{N-L}) = V^*(\hat{f}) \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})$ and has a common irreducible toric component $Y =: X_{\mathcal{E}_Y}$, $\mathcal{E}_Y \subset \mathbb{Z}^{\dim Y}$, with $\hat{Y} := V \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})$.*

C. *Tori $Z := X^*_{\mathcal{E}_Z} \hookrightarrow V^* = X^*_{(\mathcal{E}_Y || \mathcal{E}_Z)}$ exist and must be closed in $\mathbb{A}^N$. Morphisms $\pi|_Z : Z \to \pi(V)$ and multiplication $\mu : Z \times \hat{Y} \to V$ are surjective, finite of degree $\underline{d} := d(\pi(\mathcal{E}_Z^{tr}))$ with all fibres of size equal $[\underline{d}] = \#(\hat{Y}^*/Y^*)$. Both morphisms are étale if $\underline{d} \neq 0$ in $\mathbb{K}$.*

Also, $\mu|_{Z \times (g \cdot Y)}$ for $g \in \hat{Y}^*$ are surjective and finite of degree $\underline{d}$.

Remark 2.8. Degree of μ in **C.** is $\dim_{\mu^*(\mathbb{K}(V))} \mathbb{K}[Z\times\hat{Y}]\cdot S^{-1}$, where $\mathbb{K}(V)$ is the field of rational functions on V and $S := \mu^*(\mathbb{K}[V]\setminus\{\mathbf{0}\})$.

Example 2.9. Note that $\mu|_{Z\times Y} : Z\times Y \to V$ need not be étale, e. g. if $V := \{y_1^2 = z_1\cdot y_2^2\ ,\ z_1\cdot z_2 = 1\}$ then $Y = \{z_1 = z_2 = 1\ ,\ y_1 = y_2\}$ ($Z := \{z_1 = y_1 = y_2^2\ ,\ z_1\cdot z_2 = 1\}$ satisfies the assumptions of Theorem 2.7 **C.**) and there are two étale irreducible components of V at the points of $V\cap\{y_1 = y_2 = 0\}$, while $Z\times Y$ is nonsingular, and hence étale irreducible at every point. Nevertheless the local étale irreducible components of an affine binomial variety $\hat{V}$ are isomorphic to the respective étale germs of $Z\times Y$ due to Theorem 2.7.

Note that μ and $\mu|_{Z\times Y}$ are finite since $\mathbb{K}[Z\times Y]\simeq\mathbb{K}[t\ ,\ s\ ,\ s^{-1}]$ and $\mathbb{K}[Z\times\hat{Y}]\simeq\mathbb{K}[y_1\ ,\ y_2\ ,\ s\ ,\ s^{-1}]/(y_1^2-y_2^2)$ are integral over $\mathbb{K}[t\cdot s^2\ ,\ t\cdot s\ ,\ s^2\ ,\ s^{-2}]\simeq\mu|^*_{Z\times Y}(\mathbb{K}[V])\hookrightarrow\mathbb{K}[Z\times Y]$ and, respectively, $\mathbb{K}[y_1\cdot s^2\ ,\ y_2\cdot s\ ,\ s^2\ ,\ s^{-2}]/(y_1^2-y_2^2)\simeq\mu|^*_{Z\times\hat{Y}}(\mathbb{K}[V])\hookrightarrow\mathbb{K}[Z\times\hat{Y}]$.

Remark 2.10. Identifying V^* with $\mathbb{T}^n$ via bijection $\phi_{(\mathcal{E}_Y||\mathcal{E}_Z)}$ (the latter valid due to assumption on $\mathcal{E}_Z\subset\mathbb{Z}^{n-m}$ in Theorem 2.7 **C.**) both properties $\#(\hat{Y}^*/Y^*) = d(\pi(\mathcal{E}_Z^{tr}))$ and $\#(\pi|_Z^{-1}(\mathbb{I}_{N-L})) = d(\pi(\mathcal{E}_Z^{tr}))$ follow from Remark 2.1 by replacing matrix $\hat{E}$ by $\pi(\mathcal{E}_Z^{tr})^{tr}$. Equivalently, all irreducible components of $\hat{Y}^*$ are of the form $g\cdot Y^*$ for some $g\in\hat{Y}^*$ (Property **A.** of Remark 2.1) and multiplication $\mu|_{Z\times Y^*} : Z\times Y^*\to V^*$ is a bijection since $\phi_{(\mathcal{E}_Y||\mathcal{E}_Z)} : \mathbb{T}^n\to V^*$ is a bijection, implying first $Y^*\cap(\pi|_Z)^{-1}(\mathbb{I}_{N-L}) = \{\mathbb{I}_N\}$ and $Z\cap g\cdot Y^*\neq\emptyset$ for any $g\in V^*$, and as a consequence that points of subgroup $\tilde{\Gamma} := (\pi|_Z)^{-1}(\mathbb{I}_{N-L})\hookrightarrow Z$ belong to distinct irreducible components of $\hat{Y}$ and, respectively, that every irreducible component intersects $\tilde{\Gamma}$. Summarizing, the distinct irreducible components of $\hat{Y}$ are $g\cdot Y$ for $g\in\tilde{\Gamma}$ and $\#(\tilde{\Gamma}) = \#(\hat{Y}^*/Y^*)$. (One may also determine $\#(\hat{Y}^*/Y^*)$ by means of any exponents matrix $E_{\hat{Y}}$ of equations of $\hat{Y}$ as $[d(E_{\hat{Y}})]$ by making use of Remark 2.1 with matrix $E_{\hat{Y}}$ replacing matrix $\hat{E}$. Note that $\hat{V} = V$ when $d(\hat{E}) = 1$ and then one may choose $E_{\hat{Y}}$ having rows of $\hat{E}$ followed by the rows of matrix $(0||\mathrm{Id}_{N-L})$.)

Also, since multiplication by $g\in\tilde{\Gamma}\hookrightarrow Z$ is an isomorphism of $Z\to Z$, of $V\to V$ and of $Y\to g\cdot Y$ surjectivity and finiteness of $\mu|_{Z\times Y}$ implies analogous properties for each $\mu(Z\times(g\cdot Y))\ ,\ g\in\tilde{\Gamma}$.

Remark 2.11. Obviously $\pi((\mathcal{E}_Y)^{tr}) = \{\mathbf{0}\}$. Then for any $\mathbb{Z}$-basis $\tilde{\mathcal{E}}^{tr}$ its 'z-coordinates', i. e. $\pi(\tilde{\mathcal{E}}^{tr})$, with $X^*_{\tilde{\mathcal{E}}} = V^*$ generate over $\mathbb{Z}$ a sublattice $\mathrm{Span}_{\mathbb{Z}}(\pi(\tilde{\mathcal{E}}^{tr}))\subset\mathbb{Z}^{N-L}\cap\mathrm{Span}_{\mathbb{Q}}(\pi(\tilde{\mathcal{E}}^{tr})) = \mathbb{Z}^{N-L}\cap\pi(\mathrm{Ker}\,\hat{E})$ that depends only on V^* implying that $d(\pi(\tilde{\mathcal{E}}^{tr}))$ is well defined as an invariant of $V^* \hookrightarrow \mathbb{T}^N$ and coincides with number $d(\pi(\mathcal{E}_Z^{tr}))$. Of course $\hat{Y}_V := \hat{Y}$ is binomial (due to **A.** and **B.**), while $Y_V := Y$ is an irreducible component of $\hat{Y}$ containing $\mathbb{I}_N$ (toric due to **A.**).

Remark 2.12. The sets of exponents parametrizing the tori of Y and V are the rows of the matrices whose columns must be $\mathbb{Z}$-bases $\mathcal{E}_Y$ of $\operatorname{Ker}\hat{E}\cap(\mathbb{Z}^L\times\mathbf{0})$ and $\mathcal{E}_V := (\mathcal{E}_Y||\mathcal{E}_Z)$ of $\operatorname{Ker}\hat{E}$ (Remark 2.1). Hence morphism $\phi_{(\pi(\mathcal{E}_Z^{tr}))^{tr}} = \pi|_Z\circ\phi_{\mathcal{E}_Z}$ implying (when $\pi(V^*) = \pi(V)$ and Z are closed) that the properties of $\pi|_Z : Z \to \pi(V^*)$ listed in part **C.** are equivalent to the analogous properties of $\phi_{(\pi(\mathcal{E}_Z^{tr}))^{tr}}$. Of course $\pi(\mathcal{E}_Z^{tr})$ is a $\mathbb{Q}$-basis of $\pi(\operatorname{Ker}\hat{E}) \cap \mathbb{Z}^{N-L}$, but (as in the Example 2.9) it need not be a $\mathbb{Z}$-basis. Respectively $\pi|_Z$ need not be an isomorphism, but is only a finite map of degree $\underline{d}$ as in part **C.**. Finally, the properties of μ listed in **C** follow from the respective properties of $\pi|_Z$ by making use of the coordinatewise multiplication action by Z on V (the missing details are in Section 5).

We refer to affine toric subvariety $Y \hookrightarrow \hat{V}$ as *essential* and, if $Y = \hat{V}$ to $\hat{V}$ as *essential* variety (e. g. due to Corollary 2.5 Y is).

Remark 2.13. With π as above and a convention of identifying $\mathbb{A}^L\times\mathbb{I}_{N-L} \simeq \mathbb{A}^L$ and $\mathbf{0}\times\mathbb{I}_{N-L} \simeq \mathbf{0}$ variety Y is essential iff $\mathbf{0} \in Y$. Essential variety is distinguished by the property of having *the origin* as its most singular point (in the sense that the singularities of any of its germs occur in any neighbourhood of its origin). Indeed, consider the automorphisms of Y induced by the coordinatewise multiplication by $g \in X_{\mathcal{E}+}$ with $X_{\mathcal{E}+}$ from Corollary 2.5. Then for any point $\mathcal{P} \in Y \setminus Y^*$ the germs of Y at $g\cdot\mathcal{P}$, $g \in X^*_{\mathcal{E}+}$, are isomorphic and the origin of Y coincides with $(\mathcal{P}\cdot X_{\mathcal{E}+}) \setminus (\mathcal{P}\cdot X^*_{\mathcal{E}+})$, as claimed.

Applications of Theorem 2.7 include

Claim 2.14. *The irreducible components of the local étale germs of a $\mathcal{GAB}$-variety $\tilde{V}$ that occurs as the $\hat{V}$-admissible subvariety of an $\mathcal{AB}$-variety $\hat{V}$ are isomorphic to the products of nonsingular germs with respective germs of subvariety $\hat{Y}_V$ of $\hat{V}$ (from Remark 2.11 , Theorem 2.7* **B***). Hence these components are nonsingular iff $\hat{Y}_V$ is not singular and the conclusions of Remark 1.2 and of Corollary 2.15 apply to all $\mathcal{GAB}$-varieties. Any quasi-binomial variety is in $\mathcal{GAB}$ class.*

For Nash/normalized Nash blowings up Theorem 2.7 implies

Corollary 2.15. *It follows that the 'towers' of Nash (as well as normalized Nash) blowings up starting with varieties $g\cdot V$ for $g \in \Gamma$*

are mutually isomorphic and therefore it suffices to study the effect of this process on a single irreducible component V *to make them all smooth. Moreover, Remark 1.2 implies that the stabilization of the sequence of Nash blowings up (respectively normalized Nash blowings up) of an affine binomial variety is equivalent to the stabilization of the respective sequence for its essential toric subvariety.*

Theorem 2.7 C. also implies a criterion of nonsingularity for arbitrary affine toric variety $X_{\tilde{\mathcal{E}}}$ in terms of the exponents $\tilde{\mathcal{E}} \subset \mathbb{Z}^n$ of an arbitrary monomial parametrization of the torus $X^*_{\tilde{\mathcal{E}}}$ of $X_{\tilde{\mathcal{E}}}$. In the simpler case of $X_{\tilde{\mathcal{E}}}$ being an essential variety, which in terms of $\tilde{\mathcal{E}}$ means $\mathbf{0} \notin \operatorname{Conv}(\tilde{\mathcal{E}})$ (Remark 2.2), the criterion is

Remark 2.16. Essential toric variety $Y := X_{\tilde{\mathcal{E}}}$ is not singular iff the exponents $\tilde{\mathcal{E}} \subset \mathbb{Z}^n$ of an arbitrary monomial parametrization of the torus of Y are generated over $\mathbb{Z}_+$ by $\dim Y$ among them. Of course the 'if' implication is obvious. For the 'only if' implication note that under the nonsingularity assumption Y near $\mathbf{0}$ coincides with a graph of an étale map-germ, say $\mathcal{G}$, at $\mathbf{0}$ and, also, that Y is the closure of the image under a monomial parametrization, say $\phi_{\mathcal{E}^+}$, of the torus of Y with exponents $\mathcal{E}^+ \subset (\mathbb{Z}_+)^n$ (Claim 2.3). It follows, by making use of the uniqueness of the Taylor series expansion of the composite $\mathcal{G} \circ \phi_{\mathcal{E}^+}$, that map $\mathcal{G}$ is monomial and thus obviously implies the conclusion of the 'only if' implication.

The latter criterion of nonsingularity of Y depends on the assumption $\mathbf{0} \notin \operatorname{Conv}(\tilde{\mathcal{E}})$, i. e. on Y being essential, as demonstrates

Example 2.17. The closure $X_{\tilde{\mathcal{E}}}$ of $\phi_{\tilde{\mathcal{E}}}((\mathbb{C}^*)^4) \subset \mathbb{C}^6$ for a monomial map $(\mathbb{C}^*)^4 \ni x \mapsto \phi_{\tilde{\mathcal{E}}}(x) := (x_1 , x_2 , x_3 , x_4 , x_3^{-1} , x_3 \cdot x_4^{-1}) \in (\mathbb{C}^*)^6$ is nonsingular, 4-dimensional and its essential subvariety $Y = \mathbb{C}^2 \times \{\mathbb{I}_4\}$. But $\tilde{\mathcal{E}}$ is not generated over $\mathbb{Z}_+$ by any subset of 4 vectors.

Curiously Theorem 2.7 C allows to derive a criterion of nonsingularity for a toric variety $X_{\tilde{\mathcal{E}}} \hookrightarrow \mathbb{A}^N$ from the one for its essential subvariety Y . To that end note that the subset of 'y-coordinates' for $X_{\tilde{\mathcal{E}}}$ among coordinates w_e , $e \in \tilde{\mathcal{E}} \subset \mathbb{Z}^n$, on $\mathbb{A}^N$ can be identified as

$$\mathcal{E}' = \{e : \exists \eta \in (\mathbb{Q}^n)^{dual} \ , \ \eta(e) > 0 \ , \ \eta|_{\tilde{\mathcal{E}}} \geq 0\} \ , \tag{2.2}$$

due to Corollary 2.5 . As a straightforward consequence of the definitions the subset of 'z-coordinates' $\tilde{\mathcal{E}} \setminus \mathcal{E}' \supset \mathcal{E}'' := \cup_{l \geq 1} \mathcal{E}_l$, where subsets $\mathcal{E}_l \setminus \mathcal{E}_{l-1} \subset \tilde{\mathcal{E}} \setminus \mathcal{E}_{l-1}$, $l \geq 1$, are minimal with respect to $\operatorname{Conv}(\mathcal{E}_l \setminus \mathcal{E}_{l-1}) \cap \operatorname{Span}_{\mathbb{Q}}(\mathcal{E}_{l-1}) \neq \emptyset$, $l \geq 2$, and, respectively, $\operatorname{Conv}(\mathcal{E}_1) \supset \mathcal{E}_0 := \{\mathbf{0}\}$. Of course then $\operatorname{Conv}(\tilde{\mathcal{E}} \setminus \mathcal{E}'') \cap \operatorname{Span}_{\mathbb{Q}}(\mathcal{E}'') = \emptyset$

implying that exists $\eta \in (\mathbb{Q}^n)^{dual}$ vanishing on set $\mathcal{E}''$ and positive on $\tilde{\mathcal{E}} \setminus \mathcal{E}''$ and then the values of η on the rows of $\tilde{\mathcal{E}}$ provide the $\vec{\xi}^+$ of Corollary 2.5 . Consequently, Lemma 2.4 implies

$$\tilde{\mathcal{E}} = \mathcal{E}' \cup \mathcal{E}'' \; . \tag{2.3}$$

(The latter algorithm is single exponential, while that of identifying $\mathcal{E}'$ in $\tilde{\mathcal{E}}$ via formula (2.2) is polynomial, cf. Section 7.2.) Finally

Criterion 2.18. *Variety $V := X_{\tilde{\mathcal{E}}}$ is nonsingular iff its local étale irreducible components are nonsingular and it is étale irreducible. Due to Theorem 2.7* **C.** *and Remark 2.11 criterion for the étale irreducibility of $\hat{Y}$ is ($d(\pi(\tilde{\mathcal{E}}^{tr})) = d(\pi(\mathcal{E}_Z^{tr})) = 1$ or) that $\rho \times \rho$ minors of matrix $\mathcal{E}''$ generate the unit ideal, where $\rho := \operatorname{rank}(\mathcal{E}'')$. Let $m := n - \rho$.*

Then étale irreducible components of $X_{\tilde{\mathcal{E}}}$ are nonsingular iff over $\mathbb{Z}_+$ set $\mathcal{E}' \subset \mathbb{Z}^n$ is generated (mod $\operatorname{Span}_{\mathbb{Q}}(\mathcal{E}'')$) by its m elements.

Proof. The case of $\mathcal{E}' = \tilde{\mathcal{E}}$ is fully explained in Remark 2.16. Reduction to the $\mathcal{E}' = \tilde{\mathcal{E}}$ case is by identifying the torus Y^* of the toric component Y of $V \cap (\cap_{e \in \mathcal{E}''} \{w_e = 1\})$ and by means of a parametrization of $Y^* \overset{\chi}{\hookrightarrow} \mathbb{T}^{\#(\mathcal{E}')} \hookrightarrow \mathbb{T}^N$. (Note that $\phi_{\mathcal{E}'} = \chi \circ \phi_{\tilde{\mathcal{E}}}$.) Let matrix $\mathcal{M}$ be of size $n \times m$ with entries in $\mathbb{Z}$ and as columns a $\mathbb{Z}$-basis of the orthogonal complement to $\operatorname{Span}_{\mathbb{Q}}(\mathcal{E}'') \subset \mathbb{R}^n$. Then (due to Remark 2.1) map $\phi_{\mathcal{M}}$ is a parametrization of $\phi_{\tilde{\mathcal{E}}}^{-1}(Y^*) \hookrightarrow \mathbb{T}^n \tilde{\to} V^*$ implying that $\phi_{\tilde{\mathcal{E}}} \circ \phi_{\mathcal{M}}$ is a parametrization of $Y^* \hookrightarrow \mathbb{T}^N$. It follows that set $\mathcal{E} \subset \mathbb{Z}^m$ of the rows of the product matrix $\mathcal{E}' \cdot \mathcal{M}$ provides the exponents of a monomial parametrization $\phi_{\mathcal{E}}$ of $\chi(Y \cap \mathbb{T}^N)$ (since $\chi \circ \phi_{\tilde{\mathcal{E}}} \circ \phi_{\mathcal{M}} = \phi_{\mathcal{E}'} \circ \phi_{\mathcal{M}} = \phi_{\mathcal{E}' \cdot \mathcal{M}}$). Of course m rows of matrix $\mathcal{E}'$ generating over $\mathbb{Z}_+$ all rows of $\mathcal{E}'$ modulo $\operatorname{Ker} \mathcal{M}^{tr} = \operatorname{Span}_{\mathbb{Q}}(\mathcal{E}'')$ exist iff exist m rows of matrix $\mathcal{E}' \cdot \mathcal{M}$ generating over $\mathbb{Z}_+$ all rows of $\mathcal{E}' \cdot \mathcal{M}$. But the former is the criterion of nonsingularity (as stated above) for the local étale irreducible components of V and the latter is the criterion of nonsingularity for Y , which by the special case considered first is equivalent to Y being nonsingular and combined with the criterion of étale irreducibility of V via Theorem 2.7 **C.** (as explained above) is equivalent to variety V being nonsingular. $\square$

Remark 2.19. For the blowing up $\sigma_{\mathcal{I}} : X \to \mathbb{A}^n$ with center at an ideal $\mathcal{I}$ generated by monomials x^e (with $e \in \overline{\mathcal{E}} \subset (\mathbb{Z}^n \cap (\mathbb{Q}_+)^n) \setminus \{\mathbf{0}\}$ and a minimal $\{x^e\}_{e \in \overline{\mathcal{E}}}$, i. e. its proper subsets do not generate $\mathcal{I}$) a criterion for nonsingularity of X in terms of set $\overline{\mathcal{E}}$ is a consequence. Indeed, by definition X is the closure of the graph of the monomial map $\Psi_{\overline{\mathcal{E}}} := \mathbb{T}^n \ni x \mapsto [\dots : x^e : \dots]_{e \in \overline{\mathcal{E}}} \in \mathbb{P}^N$, where $N := \#\overline{\mathcal{E}} - 1$, and $\sigma_{\mathcal{I}}$ is the restriction to X of projection $\mathbb{A}^n \times \mathbb{P}^N \to \mathbb{A}^n$. Of course

$\mathbb{P}^N$ is the union of affine charts $U_e := \mathbb{P}^N \setminus \{w_e = \mathbf{0}\} \simeq \mathbb{A}^N$, where w_e's, $e \in \overline{\mathcal{E}}$, are the homogeneous coordinates on $\mathbb{P}^N$. Consequently $X = \cup_{e\in\overline{\mathcal{E}}} X_e$ with each $X_e := X \cap (\mathbb{A}^n \times U_e)$ being the closure of $\phi_{\mathcal{E}_e}(\mathbb{T}^n)$ in $\mathbb{A}^{n+N}$, where $\mathcal{E}_e := \{e' - e : e' \in \overline{\mathcal{E}}\} \cup \{(j) : 1 \le j \le n\}$.

Moreover, $X = \cup_{e\in\Gamma_{\overline{\mathcal{E}}}} X_e$, where $\Gamma_{\overline{\mathcal{E}}} \subset \overline{\mathcal{E}}$ is the set of vertexes of the Conv $(\cup_{e\in\overline{\mathcal{E}}}(e + \mathbb{R}^n_+))$ since whenever $e_0 \in \overline{\mathcal{E}} \cap (\mathrm{Conv}\,(\Gamma_{\overline{\mathcal{E}}}) + \mathbb{R}^n_+)$ it follows that there is a nonempty $I_0 \subset \Gamma_{\overline{\mathcal{E}}}$ with $\{q_e\}_{e\in I_0} \subset \mathbb{Z}_+$ and $\omega \in \mathbb{Z}^n \cap \mathbb{Q}^n_+$ satisfying $\left(\prod_{e\in I_0}(w_e/w_{e_0})^{q_e} \cdot x^\omega\right)\big|_{U_{e_0}} = 1$. Consequently chart $U_{e_0} \subset U_e$ for an $e \in I_0 \subset \Gamma_{\overline{\mathcal{E}}}$. Then X is nonsingular iff all X_e , $e \in \Gamma_{\overline{\mathcal{E}}}$, are nonsingular, and the nonsingularity Criterion 2.18 in terms of sets $\mathcal{E}_e$, $e \in \Gamma_{\overline{\mathcal{E}}}$, applies. But the special case at hand provides substantial simplifications since among exponents $\mathcal{E}_e$ for $e \in \Gamma_{\overline{\mathcal{E}}}$ exponents corresponding to the 'z-coordinates' (as in the definition of $\mathcal{E}''$ following (2.2)) do not occur and therefore a simpler criterion of Remark 2.16 applies, i. e. that over $\mathbb{Z}_+$ set $\mathcal{E}_e$ is generated by its n elements. Indeed, otherwise set $(\mathcal{E}_e)'' \neq \emptyset$ implying that there is a nonempty $I_e \subset (\mathcal{E}_e)''$ with $\sum_{\vec{v}\in I_e} q_{\vec{v}} \cdot \vec{v} = \mathbf{0}$ and $\{q_{\vec{v}}\}_{\vec{v}\in I_e} \subset \mathbb{Z}_+$. Then $I_e \cap \{e' - e : e' \in \overline{\mathcal{E}}\} \neq \emptyset$ and so $e \in (\mathrm{Conv}\,(\Gamma_{\overline{\mathcal{E}}}) + \mathbb{R}^n_+) \setminus \Gamma_{\overline{\mathcal{E}}}$, contrary to our assumption.

3. A sharp apriori bound in essential dimension 2.

Let $\hat{V}$ be an affine binomial variety, $\hat{E}$ an associated matrix and $\{\vec{\delta}_i \times \mathbf{0}\}_i \in \mathbb{Z}^N$ be a $\mathbb{Z}$-basis of $\mathrm{Ker}\,\hat{E} \cap (\mathbb{Q}^L \times \mathbf{0}) \subset \mathbb{Q}^N$, where $\{\vec{\delta}_i\}_{1\le i\le m} \subset \mathbb{Z}^L$ (with splittings $w = (y, z)$ and $\mathbb{K}^N = \mathbb{K}^L \oplus \mathbb{K}^{N-L}$ as in the previous section, while $\mathbb{K}$ being an algebraically closed field of zero characteristic, cf. Section 1.2). Our main estimate is

Theorem 3.1. Complexity bound on desingularization when $m = 2$.
(i) The convex hull of $\{((\vec{\delta}_1)_l\,,\ (\vec{\delta}_2)_l)\}_{1\le l\le L}$ *does not contain* $\mathbf{0} \in \mathbb{R}^2$.
(ii) Let D *be the size of the coordinate of* $\vec{\delta}_1 \wedge \vec{\delta}_2$ *at* $(l) \wedge (k)$, $1 \le l\ ,\ k \le L$, *for which the cone in* $\mathbb{R}^2$ *spanned over* $\mathbb{R}_+$ *by* $((\vec{\delta}_1)_l\,,\ (\vec{\delta}_2)_l)$ *and* $((\vec{\delta}_1)_k\,,\ (\vec{\delta}_2)_k)$ *contains all vectors* $((\vec{\delta}_1)_j\,,\ (\vec{\delta}_2)_j)$, $1 \le j \le L$. *Then after at most* $2 \cdot \log_2 D$ *of normalized Nash blowings up starting with variety* $\hat{V}$ *the process stabilizes.*

Remark 3.2. The first claim of the preceding theorem (for any m) is a consequence of Remark 2.2 (cf. Remark 4.1 below). Note that for any integral basis $\{\ \vec{\delta}_i\}_{1\le i\le m}$, as considered preceding Theorem 3.1, the coordinates of $\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m$ in the standard basis are unique up to a sign and can simply be found by choosing *any* $\mathbb{Q}$-basis $\{\vec{v}_i\}_{1\le i\le m}$ with the same $\mathbb{Q}$-span as that of the $\{\vec{\delta}_i\}_{1\le i\le m}$, then multiplying the respective coordinates of $\vec{v}_1 \wedge \cdots \wedge \vec{v}_m$ by their least common denominator and subsequently dividing obtained integers by their g.c.d. . For $m = 2$ we may, moreover, determine the bound D of Theorem 3.1 up to a sign by detecting which $(l) \wedge (k)$ coordinate of the resulting sequence of integers to take. To that end the criterion of detecting pair (l, k) of Theorem 3.1 does not depend on the choice of a basis and can be applied as well with a basis $\{\vec{v}_i\}_{1\le i\le m}$. In particular, it follows by making use of Lemma 8.3 and Corollary 8.5 that integer D introduced in Theorem 3.1 is a local invariant of $\hat{V}$ at $\mathbf{0}$.

We placed the proof of Theorem 3.1 (ii) in Section 7 and of claims of invariance of Remark 3.2 in Section 8.

4. Reduction of Nash algorithm to a combinatorial one.

Field $\mathbb{K}$ here is algebraically closed and of zero characteristic.

4.1. Gauss map and Nash blow up of an essential subvariety.

Let $\{\vec{\delta}_i \times \mathbf{0}\}_{1\le i\le m} \subset \mathbb{Z}^N$, where $\vec{\delta}_i := (\delta_{1i}, \dots, \delta_{Li})$, generate the integral lattice of $\mathrm{Ker}\,\hat{E} \cap (\mathbb{Q}^L \times \mathbf{0}) \subset \mathbb{Q}^N$ over $\mathbb{Z}$ and denote $\mathcal{E} := \{\vec{\Delta}_j\}_{1\le j\le L} \subset \mathbb{Z}^m$, where each $\vec{\Delta}_j := (\delta_{j1}, \dots, \delta_{jm})$. Then

$$(4.1)\quad (\phi_{\mathcal{E}})_j(x) := \prod_{1\le i\le m} x_i^{\delta_{ji}},\ 1 \le j \le L; \quad (\phi_{\mathcal{E}})_s \equiv 1,\ L < s \le N ,$$

are components of isomorphism $\phi_{\mathcal{E}} : (\mathbb{K}^*)^m \to Y^* := Y \cap (\mathbb{K}^*)^N \hookrightarrow \mathbb{K}^L$ of tori ($\phi := \phi_{\mathcal{E}}$ in this section). The closure $Y \hookrightarrow \mathbb{K}^L$ of Y^* contains $\mathbf{0} \in \mathbb{K}^L$ (Corollary 2.5) and one may choose $\delta_{ji} \in \mathbb{Z}_+$ (Claim 2.3).

Remark 4.1. Map $(\phi|_{(\mathbb{R}_+ \setminus \{\mathbf{0}\})^m} : (\mathbb{R}_+ \setminus \{\mathbf{0}\})^m \to Y \cap (\mathbb{R}_+ \setminus \{\mathbf{0}\})^N$ and, therefore, also its tangent (at $\mathbb{I}_m \in \mathbb{R}^m$) map

$$(\mathbb{R}^m)^{dual} \ni h \mapsto (h(\vec{\Delta}_1), ..., h(\vec{\Delta}_L)) \times \mathbf{0} \in \mathrm{Ker}\,\hat{E} \cap (\mathbb{R}^L \times \{\mathbf{0}\})$$

are isomorphisms. Hence due to the choice of vector $\vec{\xi}^+$ from Corollary 2.5 there is a functional $h^+ \in (\mathbb{Q}^m)^{dual}$ such that each $h^+(\vec{\Delta}_j) = (\vec{\xi}^+)_j > 0$. Hence $\mathbb{R}^m \supset \mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$. We refer to $\mathcal{E} \subset \mathbb{Z}^m$ with the latter property as *essential. It enables recording of the process of Nash (and/or normalized Nash) blow ups as a 'combinatorial' algorithm.*

To 'control' the closure of torus Y^* we prove in Section 5

Lemma 4.2. *One can 'reach' all points* $\mathcal{P} \in Y \setminus (\mathbb{K}^*)^N$ *by means of* $g \cdot X^*_{\mathcal{E}^+} \hookrightarrow Y^*$ *with* $g \in Y^*$ *, i. e.* $\{\mathcal{P}\} = g \cdot (X_{\mathcal{E}^+} \setminus X^*_{\mathcal{E}^+})$ *, where the exponents set* $\mathcal{E}^+ = \{(\vec{\xi})_j\}_{1\le j\le N} \subset \mathbb{Z}$ *for* $\vec{\xi} \in \mathrm{Ker}\,\hat{E} \cap ((\mathbb{Z}_+)^L \times \{\mathbf{0}\})$

and coordinates $(\vec{\xi})_j$ *of* $\vec{\xi}$ *are in* $\mathbb{Z}_+$ *or equal to zero depending on the respective coordinate of* $\mathcal{P}$ *being equal to zero or not.*

Remark 4.3. *Limits and criteria of being an essential variety.* Whenever there are exponents $\delta_{ji} < 0$ map ϕ would not extend to all of $\mathbb{K}^m$ and even if all $\delta_{ji} > 0$, as in Claim 2.3 , map $\phi : \mathbb{K}^m \to Y \hookrightarrow \mathbb{K}^L$ may not be surjective. Nevertheless one may reach all points $\mathcal{P} \in Y \setminus (\mathbb{K}^*)^N$ by means of $g \cdot X^*_{\mathcal{E}+} \hookrightarrow Y^*$ with $g \in Y^*$ (and $\vec{\xi}$), as in the preceding Lemma 4.2. As in Remark 4.1 we may pick a functional $h \in (\mathbb{Q}^m)^{dual} \cap \mathbb{Z}^m$ and $q \in \mathbb{Z}_+$ such that each $h(\vec{\Delta}_j) = q \cdot (\vec{\xi})_j \in \mathbb{Z}_+ \cup \{\mathbf{0}\}$ with $\mathcal{E}^+ = \{(\vec{\xi})_j\}_j \subset \mathbb{Z}$. Of course replacing $\vec{\xi}$ by $q \cdot \vec{\xi}$ (and set $\mathcal{E}^+ = h(\mathcal{E})$ by $q \cdot \mathcal{E}^+$) preserves the outcome $\{\mathcal{P}\} = g \cdot (X_{h(\mathcal{E})} \setminus X^*_{h(\mathcal{E})})$ of Lemma 4.2 . Note that given an $h \in (\mathbb{Q}^m)^{dual} \cap \mathbb{Z}^m$ set $X_{h(\mathcal{E})} \setminus X^*_{h(\mathcal{E})} \neq \emptyset$ iff $h(\vec{\Delta}_j) \geq 0$ for all j .

In particular, by identifying $\mathbb{K}^L$ with $\mathbb{K}^L \times \mathbb{I}_{N-L} \hookrightarrow \mathbb{K}^N$ and by making use of Corollary 2.5 and Remark 2.2 , it follows that the origin of $\mathbb{K}^L$ is in Y . Equivalently, there is an $h^+ \in (\mathbb{Q}^m)^{dual}$ such that for $1 \leq j \leq L$ values $h^+(\vec{\Delta}_j) = (\vec{\xi^+})_j > 0$, which is also equivalent to $\mathcal{E} := \{\vec{\Delta}_j\}_{1 \leq j \leq L} \subset \mathbb{Z}^m$ being essential, i. e. $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$. *This property is proved in Claim 4.6 to be hereditary for an appropriate choice of affine charts covering Nash blow up of* Y .

Remark 4.4. *Gauss map in local coordinates.* Consider the composite of the Gauss map G_Y of Y on Y^* with a monomial parametrization (4.1) of Y^* and identify $G_Y(\phi(x)) \in \mathcal{G}_m(\mathbb{K}^L) \hookrightarrow \mathbb{KP}^{\binom{L}{m}-1}$, where the latter is the embedding of the Grassmanian $\mathcal{G}_m(\mathbb{K}^L)$ of the m-dimensional subspaces of $\mathbb{K}^L$ by means of Plücker coordinates, with the image $T_{\phi(x)}Y$ of $T_x\mathbb{K}^m \simeq \mathbb{K}^m$ by the tangent map to ϕ at $x \in (\mathbb{K}^*)^m$. The homogeneous (Plücker) coordinates $\tilde{w} = [\ \dots : \tilde{w}_J : \dots\ ]$ of $G_Y(\phi(x)) = \mathrm{Im}\,\frac{\partial \phi}{\partial x}(x)$ are the subdeterminants $\det_J(\mathcal{J}_\phi)(x)$ of the $m \times m$ size submatrices of the jacobian matrix $\mathcal{J}_\phi(x)$ of map $y = \phi(x)$ and are listed by the choices of $J = \{j_1, \dots, j_m\} \subset \{1, \dots, L\}$ of m distinct rows of the $L \times m$ matrix $\mathcal{J}_\phi$, i. e. $\tilde{w}_J = \det_J(\mathcal{J}_\phi(x)) = \det_J(\delta) \cdot x^{\sum_{j \in J} \vec{\Delta}_j} / (x_1 \cdot \ldots \cdot x_m)$, where $\det_J(\delta)$ are the respective subdeterminants of the exponents matrix δ in (4.1) . Denote $\mathcal{S} := \mathcal{S}(\mathcal{E}) := \{J : \det_J(\delta) \neq 0\}$ and $L^* := \#\mathcal{S} - 1$ (notation $\mathcal{S}(\mathcal{E})$ is justified since $\dim_{\mathbb{Q}} \mathrm{Span}_{\,\mathbb{Q}}\{\vec{\Delta}_j\}_{j \in J} = m$ iff $\det_J(\delta) \neq 0$). Let $\mathbb{KP}^{L^*} := \bigcap_{\{J : \det_J(\delta) = 0\}} \{\tilde{w}_J = 0\} \hookrightarrow \mathbb{KP}^{\binom{L}{m}-1}$. Then $G_Y \circ \phi(x) \in \mathbb{KP}^{L^*}$ for all $x \in (\mathbb{K}^*)^m$. Moreover, then $G_Y \circ \phi : (\mathbb{K}^*)^m \to \cap_{J \in \mathcal{S}} \{\tilde{w}_J \neq 0\} =: T$.

Of course each $\mathcal{W}_J := \{\tilde{w}_J \neq 0\} \simeq \mathbb{K}^{L^*}$ and via this isomorphism T identifies with $(\mathbb{K}^*)^{L^*} \subset \mathbb{K}^{L^*}$. In abuse of notation let then $\mathcal{W}_J^*$ denote $T \hookrightarrow \mathcal{W}_J$. Similarly, denote $\mathcal{U}_J := \mathbb{K}^L \times \mathcal{W}_J$, $\mathcal{U}_J^* := (\mathbb{K}^*)^L \times \mathcal{W}_J^*$ and also $N(Y)_J := N(Y) \cap \mathcal{U}_J$, $N(Y)_J^* := N(Y) \cap \mathcal{U}_J^*$. Of course $N(Y)_{J_0}^* = \cap_{J \in \mathcal{S}} N(Y)_J$ for any $J_0 \in \mathcal{S}$. For the sake of convenience we replace coordinates $\tilde{w}_J$ by $w_J := (\det_J(\delta))^{-1} \cdot \tilde{w}_J$.

Remark 4.5. *Essential affine charts of* $N(Y)$. Then $\mathcal{U}_J^* \hookrightarrow \mathcal{U}_J$ is isomorphic to $(\mathbb{K}^*)^{L+L^*} \hookrightarrow \mathbb{K}^{L+L^*}$ and affine toric variety $N(Y)_J$ is the closure of the image $N(Y)_J^*$ of torus $(\mathbb{K}^*)^m \subset \mathbb{K}^m$ under an algebraic group monomorphism $x \mapsto \psi(x) := (\phi(x) , G_Y \circ \phi(x))$. For $J \in \mathcal{S}$ let $\vec{\Delta}_J := \sum_{j \in J} \vec{\Delta}_j$. Explicit formula of Remark 4.4 for the Gauss map monomorphism ψ (in the affine coordinates of chart $\mathcal{U}_{J_0}$, for $J_0 \in \mathcal{S}$) is the monomial map $\phi_{\mathcal{E}_{J_0}}$ whose exponents set is $\mathcal{E}_{J_0} := \{\vec{\Delta}_j\}_{1 \leq j \leq L} \cup \{\vec{\Delta}_J - \vec{\Delta}_{J_0}\}_{J \in \mathcal{S} \setminus \{J_0\}}$.

Remark 4.3 implies that for any $J_0 \in \mathcal{S}$ one may reach all points $\tilde{\mathcal{P}} \in N(Y)_{J_0} \setminus N(Y)_{J_0}^*$ by means of $g \cdot X_{h(\mathcal{E}_{J_0})}^* \hookrightarrow Y^*$ with $g \in Y^*$, i. e. $\{\mathcal{P}\} = g \cdot (X_{h(\mathcal{E}_{J_0})} \setminus X_{h(\mathcal{E}_{J_0})}^*)$, where $h \in (\mathbb{Q}^m)^{dual} \cap \mathbb{Z}^m$, and that $h(\mathcal{E}_{J_0}) \subset \mathbb{Z}_+ \cup \{\mathbf{0}\}$ since $X_{h(\mathcal{E}_{J_0})} \setminus X_{h(\mathcal{E}_{J_0})}^* \neq \emptyset$. Moreover, affine chart $N(Y)_{J_0}$ contains the origin of $\mathcal{U}_{J_0} \simeq \mathbb{K}^{L+L^*}$, i. e. is essential, iff there is $\tilde{h} \in (\mathbb{Q}^m)^{dual}$ such that $\tilde{h}(\mathcal{E}_{J_0}) \subset \mathbb{Z}_+$ and is equivalent (Lemma 2.4) to all coordinates on $\mathcal{U}_{J_0}$ being 'y-variables' for $N(Y)_{J_0}$. Equivalently (Remark 4.3) $\mathrm{Conv}\,(\mathcal{E}_{J_0}) \not\ni \mathbf{0}$.

Claim 4.6. *Assuming* $\mathbf{0} \in Y = X_{\mathcal{E}} \hookrightarrow \mathbb{K}^L \simeq \mathbb{K}^L \times \mathbb{I}_{N-L}$ *it follows that* $N(Y) = \cup_{J \in \mathcal{S}'} N(Y)_J$ *, where* $\mathcal{S}'$ *is the subset of all* $J \in \mathcal{S}$ *such that affine charts* $N(Y)_J$ *are essential.*

Proof. Due to Remarks 2.2 , 4.3 our assumption is $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$. Let cone $\tilde{\mathcal{C}} := \{h \in (\mathbb{R}^m)^{dual} : h_{|\mathcal{E}} \geq 0\}$ and, likewise, for every $J \in \mathcal{S}$ let $\tilde{\mathcal{C}}_J := \{h \in \tilde{\mathcal{C}} : h_{|\mathcal{E}_J} \geq 0\}$. Then h^+ from Remark 4.3 is in the interior of cone $\tilde{\mathcal{C}}$ (in particular $\dim_{\mathbb{R}} \tilde{\mathcal{C}} = m$). We refer to $h = (h_1, \ldots, h_m) \in (\mathbb{R}^m)^{dual}$ with $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\{h_1, \ldots, h_m\} = m$ as an *irrational* point of $(\mathbb{R}^m)^{dual}$. For any irrational $h \in \tilde{\mathcal{C}}$ there is (and unique) $J \in \mathcal{S}$ such that h is in the interior of $\tilde{\mathcal{C}}_J$. Therefore $\dim_{\mathbb{R}} \tilde{\mathcal{C}}_J = m$ iff $\mathrm{Conv}\,(\tilde{\mathcal{C}}_J) \not\ni \mathbf{0}$, while the latter is equivalent to $J \in \mathcal{S}'$ implying $\tilde{\mathcal{C}} = \cup_{J \in \mathcal{S}'} \tilde{\mathcal{C}}_J$.

Consider any $J_0 \in \mathcal{S}$. Torus $N(Y)_{J_0}^*$ coincides with the image $\psi((\mathbb{K}^*)^m) \subset \cap_{J \in \mathcal{S}'} N(Y)_J$. Let $\mathcal{P} \in N(Y)_{J_0} \setminus N(Y)_{J_0}^*$. Then, as in Remark 4.3 , there are $g \in N(Y)_{J_0}^*$ and $h \in (\mathbb{Q}^m)^{dual} \cap \mathbb{Z}^m$ such that $\{\mathcal{P}\} = g \cdot (X_{h(\mathcal{E}_{J_0})} \setminus X_{h(\mathcal{E}_{J_0})}^*)$ Moreover, values $h(\vec{\Delta}_j)$, $1 \leq j \leq L$, and of all $h(\vec{\Delta}_J - \vec{\Delta}_{J_0})$, $J \in \mathcal{S} \setminus \{J_0\}$, are positive or vanish depending on the respective coordinate of $\mathcal{P}$ being equal to zero or not (Lemma 4.2). Thus $h \in \tilde{\mathcal{C}} = \cup_{J \in \mathcal{S}'} \tilde{\mathcal{C}}_J$ and, therefore, there exists $J_1 \in \mathcal{S}'$ such that $h \in \tilde{\mathcal{C}}_{J_1}$. As a consequence $h(\vec{\Delta}_{J_0}) = h(\vec{\Delta}_{J_1})$. It follows that the ratio w_{J_0}/w_{J_1} of the homogeneous coordinates is identically one on $X^*_{h(\mathcal{E}_{J_0})}$ and is constant and coincides with the ratio $w_{J_0}(g)/w_{J_1}(g)$ on $g \cdot X^*_{h(\mathcal{E}_{J_0})}$. Hence $\mathcal{P} \in N(Y)_{J_1} \setminus N(Y)^*_{J_1}$, as required. □

In the next two sections we summarize our 'translation' of Nash and of normalized Nash blowings up into respective combinatorial versions in terms of the smallest (in every reasonable sense) subsets of generators for additive semigroups $\mathbb{Z}_+(\mathcal{E})$ generated by finite sets $\mathcal{E} \subset \mathbb{Z}^m$ with $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$ and for $\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}} := \mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \setminus \{\mathbf{0}\}$.

For an additive semigroup without zero, say G_+ , we introduce a notion of the set $\mathcal{E}xt(G_+)$ of all *Z_+-extremal points* of G_+ , i. e. of all $g \in G_+$ such that $g \neq g_1 + g_2$ for any g_1 , $g_2 \in G_+$.

Let $\nabla(J) := \mathrm{Conv}\,(J \cup \{\mathbf{0}\})$ and $\mathrm{int}(\nabla(J)) :=$ the interior of $\nabla(J)$.

Remark 4.7. Assume that set $\mathcal{E} \subset \mathbb{Z}^m$ is finite and essential.

(i) Obviously set $\mathcal{E}xt(\mathbb{Z}_+(\mathcal{E}))$ is finite and generates $\mathbb{Z}_+(\mathcal{E})$, while for $\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}$ a similar claim is a consequence of Gordon's lemma (Prop.1 in 1.2 [5]) since $\mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E})$ coincides with the dual cone $(\tilde{\mathcal{C}})^{dual}$ of its own dual cone $\tilde{\mathcal{C}}$ and $\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}$ is the set of its integral points (meaning points in $\mathrm{Span}_{\mathbb{Z}}(\mathcal{E})$).

(ii) Note that $\mathcal{E}' = \mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ implies, by making use of (i), that $\mathbb{Z}_+(\mathcal{E}') = \mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}} \subset \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) = \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}')$. Hence $\mathbb{Q}_+(\mathcal{E}')_{\mathbb{Z}} = \mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}$ and therefore $\mathcal{E}' = \mathcal{E}xt(\mathbb{Q}_+(\mathcal{E}')_{\mathbb{Z}})$.

(iii) Assume $\mathcal{E} = \mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ and $J \in \mathcal{S}'$ (with notations from Claim 4.6). Then $\mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}} = \emptyset$. Indeed, if otherwise and $\vec{a} \in \mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}$ let us choose an irrational $h \in \tilde{\mathcal{C}}$, as in Claim 4.6, such that $h(\vec{\Delta}_J) = \min_{J' \in \mathcal{S}} h(\vec{\Delta}_{J'})$ and let $j_0 \in J$ be such that $h(\vec{\Delta}_{j_0}) = \max_{j \in J} h(\vec{\Delta}_j)$. Then, to begin with, $\vec{a} \notin \mathcal{E}$, since otherwise collection $J_0 := (J \cup \{\vec{a}\}) \setminus \{\vec{\Delta}_{j_0}\}$ is in $\mathcal{S}$, but $h(\vec{\Delta}_{J_0}) < h(\vec{\Delta}_J)$. Then $\vec{a} \in \mathbb{Z}_+(\mathcal{E})$, due to assumption on $\mathcal{E}$, and therefore there is a vector $\vec{b} \in \mathcal{E}$ such that $J_1 := (J \cup \{\vec{b}\}) \setminus \{\vec{\Delta}_{j_0}\}$ is in $\mathcal{S}$, but $h(\vec{\Delta}_{J_1}) < h(\vec{\Delta}_J)$ (since if $\vec{a} \in \vec{b} + \mathbb{Z}_+(\mathcal{E})$ then inequalities $h(\vec{\Delta}_{j_0}) > h(\vec{a}) > h(\vec{b})$ hold), contrary to the choice of h .

4.2. **Multidimensional Euclidean division as a bookkeeping.** In this section we complete translation of the process of Nash blowings up into a *combinatorial* tree-like branching algorithm on finite essential subsets of $\mathbb{Z}^m$. To that end we choose $\{(\delta_{1i}, \ldots, \delta_{Li})\}_{1 \le i \le m} \subset \mathbb{Z}^L$

as in (4.1). The input of this algorithm is collection $\mathcal{E}xt(\mathbb{Z}_+(\mathcal{E}))$, where $\mathcal{E} = \{\vec{\Delta}_j = (\delta_{j1}, \ldots, \delta_{jm})\}_{1 \le j \le L}$ is the essential collection (see Remark 4.3) of exponents of a monomial parametrization of torus Y^* of an essential variety Y , we may assume that $\mathcal{E} = \mathcal{E}xt(\mathbb{Z}_+(\mathcal{E}))$.

In notations of Claim 4.6 the record of changes (derived in section 4.1) in the collections of exponents parametrizing the tori of the essential charts of Nash blowings up starting with variety Y is the

Multidimensional Euclidean algorithm on essential collections: *with $\mathcal{S} = \mathcal{S}(\mathcal{E})$ being the set of all m-tuples of linearly independent vectors in a finite essential (input) collection $\mathcal{E} = \{\vec{\Delta}_j\}_j \subset \mathbb{Z}^m$ we augment set $\mathcal{E}$ to a collection $\mathcal{E}_J$ by adjoining set $\{\vec{\Delta}_{J'} - \vec{\Delta}_J\}_{J \neq J' \in \mathcal{S}}$ provided that $J \in \mathcal{S}' := \{J \in \mathcal{S} : \mathcal{E}_J$ is essential $\}$. Finite essential set $N_J(\mathcal{E}) := \mathcal{E}xt(\mathbb{Z}_+(\mathcal{E}_J))$ generates semigroup $\mathbb{Z}_+(\mathcal{E}_J)$ and is the output of an algorithm branching according to the choices of $J \in \mathcal{S}'$.*

A branch of this algorithm terminates at a node with an associated to the node collection $\mathcal{E} = \{\vec{a}_j\}_j \subset \mathbb{Z}^m$ whenever $\#(\mathcal{E}) = m$.

Remark 4.8. Note that differences $\vec{\Delta}_{J'} - \vec{\Delta}_J$ with $\#(J' \setminus J) = 1$ generate over $\mathbb{Z}_+$ all other differences in collections $\mathcal{E}_J$, i. e. it suffices to include in $\mathcal{E}_J$ only them. Indeed, matrix $(a_{ji})_{j \in J' ,\, i \in J}$ transforming basis J of $\mathbb{Q}^m$ into basis J' is not degenerate implying existence of a bijection $J' \ni j \mapsto i = i(j) \in J$ with all $a_{j\ i(j)} \neq 0$ and $\vec{\Delta}_{J'} - \vec{\Delta}_J = \sum_{j \in J'} (\vec{\Delta}_j - \vec{\Delta}_{i(j)}) = \sum_{j \in J'} (\vec{\Delta}_{J \cup j \setminus i(j)} - \vec{\Delta}_J)$, as required.

Nash desingularization of essential affine toric subvariety Y of an affine binomial variety $\hat{V}$ leads to a Nash desingularization of $\hat{V}$ by making use of Property **A.**, Theorem 2.7 **C.** and of Remark 1.2 . Variety Y' resulting from a sequence of Nash blowings up of Y is a union of its essential affine charts $Y' \cap \mathcal{U}' \hookrightarrow \mathcal{U}' \simeq \mathbb{K}^{L'}$ due to Claim 4.6 . Every affine chart $Y' \cap \mathcal{U}'$ corresponds to a *node* of a branch of our combinatorial 'bookkeeping' algorithm. With $\{\vec{a}_j\}_{1 \le j \le L'} \subset \mathbb{Z}^m$ being the essential collection associated with the latter node it follows that the essential affine toric variety $Y' \cap \mathcal{U}'$ corresponding to the node admits a monomial parametrization of its torus by $(\mathbb{K}^*)^m$ in coordinates y'_j , $1 \le j \le L'$, on $\mathcal{U}'$ as follows: $y'_j = (\Phi)_j(x) := x^{\vec{a}_j}$, $1 \le j \le L'$. We finally show the equivalence of stabilization of the sequence of Nash blowings up of Y to the termination of our combinatorial algorithm

Claim 4.9. *A branch $\mathcal{B}$ of the multidimensional analogue of Euclidean division algorithm terminates iff the essential affine chart $Y' \cap \mathcal{U}'$ corresponding to the terminal node of $\mathcal{B}$ is nonsingular.*

Proof. Say $\mathcal{E}' = \{\vec{a}_j\}_{1\le j\le k}$ is the collection corresponding to a node of branch $\mathcal{B}$ and $Y' \cap \mathcal{U}' \hookrightarrow \mathcal{U}' \simeq \mathbb{K}^{L'}$ is the corresponding essential affine chart. Then exponents of monomial parametrization $y'_j = x^{\vec{a}_j}$, $1 \le j \le L'$, of torus $(Y' \cap \mathcal{U}')^* = (Y' \cap \mathcal{U}') \cap (\mathbb{K}^*)^{L'}$ include collection $\mathcal{E}'$ and, moreover, are in $\mathbb{Z}_+(\mathcal{E}')$, i. e. can be expressed as nonnegative integral linear combinations $\vec{a}_j = \sum_{1\le l\le k} n_{jl} \cdot \vec{a}_l$, $k+1 \le j \le L'$.

Therefore, if branch terminates, i. e. collection $\mathcal{E}'$ associated with its *terminal node* is of size m , then $Y' \cap \mathcal{U}'$ is nonsingular being the graph of map $y'_j = (y'_1)^{n_{j1}} \cdot \ldots \cdot (y'_m)^{n_{jm}}$, $m+1 \le j \le L'$.

Conversely, as in Remark 2.16, if $Y' \cap \mathcal{U}'$ is nonsingular at the origin of $\mathcal{U}'$, it follows that it is a graph of an étale map-germ $\mathcal{G}$ at the origin over a coordinate subspace $\mathbb{K}^m \subset \mathbb{K}^{L'}$. Since the closure $\overline{Y' \cap \mathcal{U}'}$ of torus $(Y' \cap \mathcal{U}')^*$ contains the origin $\mathbf{0}$ of $\mathcal{U}' \simeq \mathbb{K}^{L'}$ Claim 2.3 implies that there is a monomial parametrization $y'_j = x^{\vec{\omega}_j}$, $1 \le j \le L'$, of $(Y' \cap \mathcal{U}')^*$ with $\{\vec{\omega}_j\}_{1\le j\le L'} \subset \mathbb{Z}^m_+$. Then (uniqueness of Taylor series expansion of the composite of $\mathcal{G}$ with the components of parametrization $y'_{j_l} = x^{\vec{\omega}_{j_l}}$, $1 \le l \le m$, associated with the aforementioned coordinate subspace $\mathbb{K}^m$ implies that) map-germ $\mathcal{G}$ is monomial. We may conclude now that vectors $\vec{a}_j$, $1 \le j \le L'$, are generated over $\mathbb{Z}_+$ by their subset (of size m) corresponding to the coordinate subspace $\mathbb{K}^m$ of the previous sentence. □

Remark 4.10. The proof of Claim 4.9 shows that essential toric variety is nonsingular iff it is nonsingular at the origin.

4.3. **Effect of normalization.** Normalization $\mathcal{N}(Y)$ of essential affine variety Y adjoins as regular functions on $\mathcal{N}(Y)$ all monomials $\mathcal{M}$ in coordinates y_j , $1 \le j \le L$, on $\mathbb{K}^L$ whenever $\mathcal{M}^d$ for some $d \in \mathbb{Z}_+$ coincides on Y with another monomial $\mathcal{M}'$ in y_j's with non negative integral exponents (see Section 2.1 in [5]). Since torus Y^* is parametrized by monomials $y_j = x^{\vec{\Delta}_j}$, $1 \le j \le L$,

normalization translates into a combinatorial algorithm: *augment an essential input set* $\mathcal{E} = \{\vec{\Delta}_j\}_j \subset \mathbb{Z}^m$ *to a semigroup* $\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}$ *generated by its finite essential subset* $\mathcal{N}(\mathcal{E}) := \mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ *(Remark 4.7 (i)) - the output of combinatorial normalization.*

Of course a sequence of composites of normalized Nash blowings up followed by normalization coincides with normalization followed by the sequence of Nash blowings up composed with normalizations. For the convenience of exposition (and reflecting the latter) essential collection $\mathcal{N}(\mathcal{E})$, with $\mathcal{E} = \{\vec{\Delta}_j\}_{1\le j\le L}$ from (4.1), is the initial input for

normalized multidimensional Euclidean division algorithm: *whose input for each step is an essential collection* $\mathcal{E} = \mathcal{N}(\mathcal{E})$ *and essential collections* $\mathcal{N}(N_J(\mathcal{E}))$ *for* $J \in \mathcal{S}(\mathcal{E}')$ *are the output.*

The latter records a sequence of normalized Nash blow ups (followed by normalization) of an essential toric variety Y . By definition a branch of this tree-like algorithm terminates at a node with an essential collection $\mathcal{E}$ provided that the size of $\mathcal{E}$ is m .

The proof of Claim 4.9 applies to show that a branch $\tilde{\mathcal{B}}$ of normalized multidimensional Euclidean division terminates iff the essential chart corresponding to the terminal node of $\tilde{\mathcal{B}}$ is nonsingular. Since normalization separates all local étale irreducible components (and due to Property **A.**, Theorem 2.7 **C.** and Remark 1.2) the lengths of the normalized Nash desingularization of the essential subvariety Y of an affine binomial variety $\hat{V}$ and that of $\hat{V}$ coincide.

5. Structure of binomial varieties, proofs.

We consider affine binomial varieties $\hat{V} := \overline{V^*(\hat{f})}$ in $\mathbb{A}^N$ determined by a set $\hat{f} := \{\hat{f}_j\}_{1\le j\le M}$ of binomials from (2.1).

Remark 5.1. Let $r := \operatorname{rank} \hat{E}$, $n := N - r$. Denote by $E = \{E_{ji}\}$ a matrix of size $r\times N$ with rows being a basis over $\mathbb{Z}$ of $(\hat{E})^{tr}(\mathbb{Q}^M)\cap\mathbb{Z}^N$. Then ideal generated in $\mathbb{Z}$ by $r \times r$ minors of matrix E is the unit ideal, i. e. $d(\operatorname{Ker} E) = 1$ (Remark 2.1), which is equivalent to

$$(\mathbb{Z}) \qquad \{\xi \in \mathbb{R}^N : E\xi \in \mathbb{Z}^r\} = \operatorname{Ker} E \cap \mathbb{R}^N + \mathbb{Z}^N \subset \mathbb{R}^N .$$

Let $\alpha_{ji} := \max\{E_{ji} , 0\}$, $\beta_{ji} := -\min\{E_{ji} , 0\}$ and denote $V^*(f) := \{w \in \mathbb{T}^N : f_j(w) = 0 , 1 \le j \le r\}$, where binomials

$$f_j := w_1^{\alpha_{j1}} \cdots w_N^{\alpha_{jN}} - w_1^{\beta_{j1}} \cdots w_N^{\beta_{jN}} . \tag{5.1}$$

Both $V^*(f) \subset V^*(\hat{f})$ are subgroups of $\mathbb{T}^N$ (and $V^*(\hat{f}) \subset \operatorname{Reg} \hat{V}$). Since $\operatorname{Ker} E = \operatorname{Ker} \hat{E}$ the sets of exponents parametrizing V^* and $V^*(f)$ coincide (Remark 2.1) and $V = \overline{V^*(f)}$.

Remark 5.2. In the special case that $\mathbb{K} = \mathbb{C}$ let us introduce subgroups $G := \{w \in \hat{V}^* : |w| = 1\}$, where $|w| \in \mathbb{R}^N$ is a point with coordinates being the absolute values $|w_j|$ of coordinates of $w \in \mathbb{C}^N$, and $G_0 := \{w = \exp(2\pi\sqrt{-1} \cdot h) : h \in \mathbb{R}^N , Eh = \mathbf{0}\}$, where $\exp((h_1, \dots , h_N)) := (e^{h_1}, \dots , e^{h_N})$, of $\hat{V}^*$. Then property ($\mathbb{Z}$) of matrix E implies that $G_0 = G \cap V^*(f)$ and $\Gamma \simeq G/G_0$ (since $g := w \cdot |w|^{-1} \in G$ and $|w| \in V^*(f)$ whenever $w \in V^*(\hat{f})$). Map $\xi \mapsto \exp(2\pi\sqrt{-1}\cdot\xi)$ provides a bijection onto Γ of an additive group

$\Gamma_* := \{\xi \in \mathbb{R}^N : \hat{E}(\xi) \in \mathbb{Z}^M\}/(\mathbb{Z}^N + \operatorname{Ker} E)$ and Γ_* is finite (since for any choice of a basis $\{\vec{h_j}\}_{1\le j\le r}$ of $\hat{E}(\mathbb{R}^N)\cap\mathbb{Z}^M$ over $\mathbb{Z}$ there is a choice of $\{\vec{\xi_j}\}_j \subset \mathbb{Q}^N$ with each $\vec{h_j} = \hat{E}(\vec{\xi_j})$).

We will make use of the following

Claim 5.3. *Assume* $\mathcal{P} \in \hat{V} \setminus V^*(\hat{f})$ *and that upon splitting all variables* w_j , $1 \le j \le N$, *into* $w = (u, v)$ *the* u*-coordinates of* $\mathcal{P}$ *vanish while* $b := v(\mathcal{P}) \in \mathbb{T}^{N''}$. *Then there are point* $a \in \mathbb{T}^{N'}$, *where* $N' := N - N''$, *and* $\vec{\xi} \in (\mathbb{Z}_+)^{N'} \times \{\mathbf{0}\}$ *such that* $g \cdot X^*_{\mathcal{E}^+} \hookrightarrow V^*(\hat{f})$, *where* $g := (a, b) \in \mathbb{T}^N$ *and* $\mathcal{E}^+ := \{(\vec{\xi})_j\}_{1\le j\le N} \subset \mathbb{Z}$.

In particular, point $\{\mathcal{P}\} = (g \cdot X_{\mathcal{E}^+}) \setminus \left(g \cdot X^*_{\mathcal{E}^+}\right)$.

Proof. Let $X \hookrightarrow \hat{V}$ be an irreducible curve with $\mathcal{P} \in X$ Then normalization $\mathcal{N}(X)$ of X is a nonsingular curve and morphism $\mathcal{N}_X : \mathcal{N}(X) \to X$ is finite and surjective. Let point $\mathcal{Q} \in (\mathcal{N}_X)^{-1}(\mathcal{P})$. Since $\mathcal{N}(X)$ at $\mathcal{Q}$ is nonsingular it follows that the completion $\hat{\mathcal{O}}$ (in Krull topology) of the local ring $\mathcal{O} \hookrightarrow \hat{\mathcal{O}}$ of $\mathcal{N}(X)$ at $\mathcal{Q}$ is the ring $\mathbb{F}[[t]]$ of the formal power series expansions in one variable, say t , with coefficients in the residue field $\mathbb{F}$ of $\mathcal{O}$ (hence $[\mathbb{F} : \mathbb{K}] < \infty$). Denote by $\gamma_j(t) \in \mathbb{F}[[t]]$ the pull back $(\mathcal{N}_X)^*(w_j|_X) \in \mathcal{O} \hookrightarrow \hat{\mathcal{O}}$ of the restriction $w_j|_X$ of the w_j-coordinate to X . It follows that $\gamma(t)^{\hat{E}} = \mathbb{I}_M$ in $\mathbb{F}[[t]]^M$ and that $w(\mathcal{P}) = (\mathbf{0}, b) = \gamma(0)$. Let for each j , $1 \le j \le N'$, the initial form of $\gamma_j(t)$ to be $in(\gamma_j) = a_j \cdot t^{\xi_j}$, $a_j \in \mathbb{F}^*$ and $\xi_j \in \mathbb{Z}_+$. Then $a := (a_1, \ldots, a_{N'}) \in (\mathbb{F}^*)^{N'}$ and $\vec{\xi} := (\xi_1, \ldots, \xi_{N'}, 0, \ldots, 0) \in \mathbb{Z}^N$ satisfy $X^*_{\mathcal{E}^+} \hookrightarrow V^*(\hat{f})$ and $(a, b)^{\hat{E}} = \mathbb{I}_M$, i. e. are as required. □

Corollary 5.4. *Claim 5.3 implies (a) equality* $\hat{Y} = \overline{V^*(\hat{f}) \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})}$ *of Theorem 2.7 B., (b) Lemma 4.2 and (c) Lemma 2.4 :*

Proof. Indeed, starting with a proof of (a) and applying Claim 5.3 to a point $\mathcal{P} \in \hat{Y} := \hat{V} \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})$ it follows that there are $g \in V^*(\hat{f})$ and $\vec{\xi} \in \mathbb{Z}^N$ with coordinates in $\mathbb{Z}_+$ or vanishing depending on the respective coordinate of $\mathcal{P}$ vanishing or not such that g and $\vec{\xi}$ satisfy the conclusions of Claim 5.3 and thus imply that $\mathcal{P} \in \overline{V^*(\hat{f}) \cap (\mathbb{A}^L \times \mathbb{I}_{N-L})}$, as required.

Moreover, items (b) and (c) follow by applying the proof of (a) with an appropriate choice of point $\mathcal{P}$. □

Remark 5.5. Equality $\operatorname{Ker} \hat{E} = \operatorname{Ker} E$ and Lemma 2.4 imply that the splitting of variables w into y and z-variables for variety $\hat{V} \subset \mathbb{A}^N$ and for the irreducible component $V \ni \mathbb{I}_N$ of $\hat{V}$ coincide.

Let matrices $(\hat{\Omega} \quad \hat{\Xi}) := \hat{E}$ and $(\Omega \quad \Xi) := E$ with the columns of $\hat{\Omega}$ and Ω corresponding to y and, respectively, the columns of $\hat{\Xi}$ and Ξ to z -variables. Claim following implies that $\pi(V)$ is a closed binomial variety and **completes the proof of Theorem 2.7 B.** (using Property **A.** of Remark 2.1)

Claim 5.6. *$\pi(V^*(\hat{f})) = \pi(\hat{V})$, is closed in $\mathbb{A}^{N-L}$ and is binomial.*

Proof. Let matrix T of size $M' \times M$, $M' := M - \operatorname{rank}(\hat{\Omega})$, have as rows a basis over $\mathbb{Z}$ of $\operatorname{Ker}(\hat{\Omega})^{tr} \cap \mathbb{Z}^M$. Then $\operatorname{Ker} H = \pi(\operatorname{Ker} \hat{E})$ for $H := T \cdot \hat{\Xi}$. Moreover

Lemma 5.7. $\pi(V^*(\hat{f})) = \{z \in \mathbb{T}^{N-L} : z^H = \mathbb{I}_{M'}\}$.

Proof. Matrix T admits (cf. Remark 5.1) a right inverse matrix $\mathcal{L}$ with entries in $\mathbb{Z}$, i. e. $T \cdot \mathcal{L} = \mathrm{Id}_{M'}$. Therefore $T \cdot (\mathrm{Id}_M - \mathcal{L} \cdot T) = 0$, $(\mathrm{Ker}\ T) \cap (\mathrm{Im}\ \mathcal{L} \cdot T) = \{0\}$, $\mathrm{Im}\ \mathcal{L} = \mathrm{Im}\ \mathcal{L} \cdot T$. Hence $\mathbb{Q}^M = \mathrm{Im}\ (\mathrm{Id}_M - \mathcal{L} \cdot T) \oplus \mathrm{Im}\ \mathcal{L} \cdot T$ implies $\mathrm{Im}\ (\mathrm{Id}_M - \mathcal{L} \cdot T) = \mathrm{Ker}\ T = \mathrm{Im}\ \hat{\Omega}$. Of course there are square matrices Λ and λ with entries in $\mathbb{Z}$, $\det(\Lambda) = 1 = \det(\lambda)$ and such that matrix $\tau := \Lambda \cdot \hat{\Omega} \cdot \lambda$ has a diagonal upper left corner of size $M' \times M'$ and zero entries otherwise. Then $\mathrm{Im}\ \tau = \mathrm{Im}\ \Lambda \cdot \hat{\Omega} = \mathrm{Im}\ \theta$, where $\theta := \Lambda \cdot (\mathrm{Id}_M - \mathcal{L} \cdot T)$, implying for any $v \in \mathbb{T}^M$ existence of $y_* \in \mathbb{T}^L$ with $y_*^\tau = v^\theta$, which for $v := z^{\hat{\Xi}}$ with $z^H = \mathbb{I}_{M'}$ and $y := y_*^{-\lambda}$ implies $y^{-\hat{\Omega}} = z^{\hat{\Xi}}$, i. e. if $z^H = \mathbb{I}_{M'}$ then $z \in \pi(V^*(\hat{f}))$, while the converse is obvious. □

In other words $\pi(V^*(\hat{f}))$ is the vanishing set of binomials and H is a matrix associated with variety $\hat{W} = \overline{\pi(V^*(\hat{f}))}$ for which all variables are the 'z-variables' (follows using $\vec{\xi}^+$ of Corollary 2.5). Therefore $\pi(V^*(\hat{f}))$ is a closed binomial variety and coincides with $\pi(\hat{V})$. □

Corollary 5.8. *It follows that $\pi(V) = \pi(V^*(f)) = \pi(V^*) \hookrightarrow \mathbb{T}^{N-L}$ is a torus (Remark 2.1) closed in $\mathbb{A}^{N-L}$ and, being nonsingular, is a connected component of $\pi(\hat{V})$.*

Next we prove Theorem 2.7 C.

Proof. We start by showing the claim of existence in part **C.**. Namely, following the arguments of Criterion 2.18 let $V^* = X^*_{\tilde{\mathcal{E}}}$ and split the exponents of set $\tilde{\mathcal{E}} \subset \mathbb{Z}^n$ into subsets $\mathcal{E}'$ and $\mathcal{E}''$ according to the splitting of all coordinates w on $\mathbb{A}^N$ into y and z-coordinates. Let matrix $\widetilde{\mathcal{M}}$ complete matrix $\mathcal{M}$ of Criterion 2.18 to a square size matrix with entries in $\mathbb{Z}$ and $\det(\widetilde{\mathcal{M}}) = 1$ by attaching matrix $\underline{\mathcal{M}}$

of size $n\times(n-m)$ as the last $n-m$ columns. Then, respectively, the columns of matrices $\mathcal{E}_Y := \tilde{\mathcal{E}}\cdot\mathcal{M}$ and $\mathcal{E}_V := \tilde{\mathcal{E}}\cdot\widetilde{\mathcal{M}}$ form $\mathbb{Z}$-bases of $\operatorname{Ker} E\cap(\mathbb{Z}^L\times\{\mathbf{0}\})$ and $\operatorname{Ker} E\cap\mathbb{Z}^N$ implying that $Y^* = X^*_{\mathcal{E}_Y}\hookrightarrow\mathbb{T}^N$ and $X^*_{\tilde{\mathcal{E}}} = X^*_{\mathcal{E}_V}$. Moreover, letting $\mathcal{E}_Z := \tilde{\mathcal{E}}\cdot\underline{\mathcal{M}}$ it follows that $\mathcal{E}_Z$ is a $\mathbb{Z}$-basis and that as the set of exponents $\mathcal{E}_V = (\mathcal{E}_Y||\mathcal{E}_Z)$, as required.

We next prove that torus $Z^* := X^*_{\mathcal{E}_Z}$ is closed in $\mathbb{A}^N$. Applying projection π to columns of matrices $\mathcal{E}_V$ and $\mathcal{E}_Z$ it follows that $\operatorname{Span}_{\mathbb{Z}}(\pi(\mathcal{E}_V^{tr})) = \operatorname{Span}_{\mathbb{Z}}(\pi(\mathcal{E}_Z^{tr}))$ implying $\dim_{\mathbb{Q}}(\operatorname{Span}_{\mathbb{Q}}(\pi(\mathcal{E}_Z^{tr}))) = \dim_{\mathbb{Q}}(\operatorname{Span}_{\mathbb{Q}}(\mathcal{E}_V^{tr}))-\dim_{\mathbb{Q}}(\operatorname{Span}_{\mathbb{Q}}(\mathcal{E}_Y^{tr})) = \dim_{\mathbb{Q}}(\operatorname{Span}_{\mathbb{Q}}(\mathcal{E}_Z^{tr}))$, although $\pi(\mathcal{E}_Z^{tr})$ is not necessarily a $\mathbb{Z}$-basis of $\operatorname{Span}_{\mathbb{Q}}(\mathcal{E}_Z^{tr})$ as Example 2.17 demonstrates. Inclusion $\pi(\overline{Z})\subset\pi(V)\subset\mathbb{T}^{N-L}$ (Corollary 5.8) implies that all 'z-variables' for $\hat{V}$ are the 'z-variables' for Z and then the criterion of the iterative construction preceding (2.3) implies that all w_j , $1\le j\le N$, variables are the 'z-variables' for $\overline{Z}$, i. e. $\mathbb{T}^N\supset\overline{Z}=Z$, as required.

Properties of morphism $\pi|_Z : Z\to\pi(V)$ follow (Remark 2.12) from the analogous properties of $\phi_{(\pi(\mathcal{E}_Z^{tr}))^{tr}} : \mathbb{T}^{n-m}\to\pi(V)$. Surjectivity of the latter is a consequence of Corollary 5.8, while the properties of morphism $\pi|_Z$ being finite of degree $\underline{d} = d((\pi(\mathcal{E}_Z^{tr}))^{tr})$ with the size of all fibres equal $[\underline{d}] = \#(\hat{Y}^*/Y^*)$ (cf. Remark 2.10) and the property of being étale if $\underline{d}\neq 0$ in $\mathbb{K}$ follow from Remark 2.1 by replacing matrix $\hat{E}$ by $(\pi(\mathcal{E}_Z^{tr}))^{tr}$.

Next we establish the properties of $\mu : Z\times\hat{Y}\to V$ and of $\mu|_{Z\times Y}$ listed in Theorem 2.7 **C.**. Surjectivity and the quasifiniteness of both with all fibres of μ of the same size $[\underline{d}]$ as those of morphism $\pi|_Z$ are straightforward consequences of the surjectivity of $\pi_{|Z} : Z\to\pi(V)$ as a group homomorphism and of the definition of $\hat{Y} := V\cap(\mathbb{A}^L\times\mathbb{I}_{N-L})$.

Besides morphism μ being étale (which we prove at the very end) it remains to show that both μ and $\mu|_{Z\times Y}$ are finite morphisms of the same degree $\underline{d}$ as $\pi|_Z$. The proof is similar to the calculation in the special case of Example 2.9 and so we carry it only in the case of morphism μ . Indeed, since $Z\hookrightarrow\mathbb{A}^N$ is isomorphic to a closed torus $\underline{\mathbb{T}}^{n-m}\hookrightarrow\mathbb{A}^{2\cdot(n-m)}$ the ring of regular functions on Z is $\mathbb{K}[Z]\simeq\mathbb{K}[s_1 , \ldots , s_{n-m} , s_1^{-1} , \ldots , s_{n-m}^{-1}]$, while $\mathbb{K}[Z\times\hat{Y}]\simeq(\mathbb{K}[Z])[y]/\mathcal{I}$, where $\mathcal{I}$ is the ideal in $(\mathbb{K}[Z])[y]$ generated by equations defining $\hat{Y}$ in $\mathbb{A}^L$. Splitting the exponents $e\in\mathcal{E}_Z$ of parametrization $\mathbb{T}^{n-m}\ni s\to\phi_{\mathcal{E}_Z}(s)\in Z$ of Z according to the y and z-coordinates, say $\phi^*_{\mathcal{E}_Z}(y_j) = s^{e'_j}$, $1\le j\le L$, and $\phi^*_{\mathcal{E}_Z}(z_i) = s^{e''_i}$, $1\le i\le N-L$, where $s=(s_1 , \ldots , s_{n-m})$, it follows that

$$\mathbb{K}[Z]\simeq\mathbb{K}[s^{e'_1} , \ldots , s^{e'_L} , s^{e''_1} , \ldots , s^{e''_{N-L}}] ,$$

$$\mathbb{K}[\pi(Z)] \simeq \mathbb{K}[s^{e''_1}, \ldots, s^{e''_{N-L}}] \text{ and}$$
$$\mu^*(\mathbb{K}[V]) \simeq \mathbb{K}[\pi(Z)][y_1 \cdot s^{e'_1}, \ldots, y_L \cdot s^{e'_L}]/\mathcal{I} \hookrightarrow (\mathbb{K}[Z])[y]/\mathcal{I} .$$

(Recall that $\pi(Z) = \pi(V)$ and $(\pi|_V)^* : \mathbb{K}[\pi(Z)] \hookrightarrow \mathbb{K}[V]$.) Then $\mathbb{K}[Z\times\hat{Y}]$ is integral over $\mu^*(\mathbb{K}[V])$ since $\mathbb{K}[Z]$ is integral over $\mathbb{K}[\pi(Z)]$ (the finiteness of $\pi|_Z$) and each element $s^{-e'_j} \in \mathbb{K}[Z]$, $1 \le j \le L$.

Next, degree of $\pi|_Z$ is $\underline{d}$ means that $\dim_{\mathbb{F}} \mathbb{K}[Z]\cdot\tilde{S}^{-1} = \underline{d}$, where $\tilde{S} := \mathbb{K}(\pi(Z)) \setminus \{\mathbf{0}\}$ and $\mathbb{F} := \mathbb{K}(\pi(Z))$. Note that $\mathbb{K}[Z]\cdot\tilde{S}^{-1} \simeq \mathbb{F}[s^{e'_1}, \ldots, s^{e'_L}]$ and $\mu^*(\mathbb{K}[V])\cdot\tilde{S}^{-1} \simeq \mathbb{F}[y_1\cdot s^{e'_1}, \ldots, y_L\cdot s^{e'_L}]/\mathcal{I}$. Also $(y_j\cdot s^{e'_j}) \in \mu^*(\mathbb{K}[V])$, $y_j \in \mathbb{K}[Z\times\hat{Y}]$, $1 \le j \le L$ and element $y_j \in (y_j\cdot s^{e'_j})\cdot\mathbb{K}[Z] \subset (y_j\cdot s^{e'_j})\cdot\mathbb{K}[Z]\cdot\tilde{S}^{-1} \subset \mathbb{K}[Z\times\hat{Y}]\cdot\tilde{S}^{-1}$. Then $\mathbb{K}[Z]\cdot\tilde{S}^{-1} \otimes_{\mathbb{K}(\pi(Z))} \mathbb{K}(V) \simeq \mathbb{K}[Z\times\hat{Y}]\cdot S^{-1}$, implying that $\dim_{\mu^*(\mathbb{K}(V))}\mathbb{K}[Z\times\hat{Y}]\cdot S^{-1} = \dim_{\mathbb{K}(\pi(Z))}\mathbb{K}[Z]\cdot\tilde{S}^{-1} = \underline{d}$, cf. Remark 2.8 .

Finally, the property of morphism μ to be étale is a consequence of the analogous property for $\pi|_Z : Z \to \pi(V)$. In a special case of $\mathbb{K} = \mathbb{C}$ the étale inverse $(\pi_{Z,a})^{-1}$ is an analytic map of $\pi(Z)$ to Z (of a neighbourhood in the classical topology of $\pi(a)$ to that of a). Then the étale inverse $(\hat{\mu}_{(a,b)})^{-1}$ of μ as an analytic map germ (at (a,b)) is

$$V_{\mu(a,b)} \ni v \mapsto ((\pi_{Z,a})^{-1}(\pi(v)) \times [(\pi_{Z,a})^{-1}(\pi(v))]^{-1}\cdot v) \in (\mathbb{Z}\times\hat{Y})_{(a,b)} ,$$

where $[g]^{-1} : v \to [g]^{-1}\cdot v$ is the action of $g := (\pi_{Z,a})^{-1}(\pi(v)) \in Z$ on V and $V_{\mu(a,b)}$, $(\mathbb{Z}\times\hat{Y})_{(a,b)}$ are the germs as analytic sets at the respective points $\mu(a,b) \in V$, $a \in Z$ and $b \in \hat{Y}$. In the general case we exploit the calculations of the previous two paragraphs.

Indeed, we must show that for any prime ideal $\mathfrak{p} \in \mathrm{Spec}\,(\mathbb{K}[Z\times\hat{Y}])$ and $\mathfrak{q} := \mathfrak{p} \cap \mu^*(\mathbb{K}[V]) \in \mathrm{Spec}\,(\mu^*(\mathbb{K}[V]))$ the respective localizations at $\mathfrak{p}$ and $\mathfrak{q}$ followed by the completions in the Krull topologies leads to isomorphic rings. (Since $\pi|_Z$ is étale the analogous procedure starting with prime ideals $\tilde{\mathfrak{p}} := \mathfrak{p} \cap \mathbb{K}[Z]$ and $\tilde{\mathfrak{q}} := \mathfrak{q} \cap \mathbb{K}[\pi(Z)]$ leads to the same ring, say $\hat{\mathcal{O}}$.) It suffices to show that adjoining $(\mathbb{K}[Z]\setminus\tilde{\mathfrak{p}})^{-1}$ to $\mathbb{K}[Z\times\hat{Y}]$ and $(\mathbb{K}[\pi(Z)]\setminus\tilde{\mathfrak{q}})^{-1}$ to $\mu^*(\mathbb{K}[V])$ followed by the completions in the Krull topologies induced by the powers of the ideals generated by $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{q}}$ in the respective rings leads to isomorphic rings (even prior to localizing at the full $\mathfrak{p}$ and $\mathfrak{q}$ followed the respective completions). But the partial localizations followed by the respective completions of the previous sentence transform rings $\mu^*(\mathbb{K}[V]) \hookrightarrow \mathbb{K}[Z\times\hat{Y}]$ into the pair of rings $\hat{\mathcal{O}}[y_1\cdot s^{e'_1}, \ldots, y_L\cdot s^{e'_L}]/\mathcal{I} \hookrightarrow \hat{\mathcal{O}}[y]/\mathcal{I}$, which are of course isomorphic since each element $s^{-e'_j} \in \mathbb{K}[Z] \hookrightarrow \hat{\mathcal{O}}$, $1 \le j \le L$. This completes the proof of Theorem 2.7 **C.** . □

We now prove (in the respective order) Claims 2.14 , 2.3 and 2.6 .

Proof. **Claim 2.14 .** Binomial variety $\pi(\hat{V}) = \pi(\hat{V}^*) \subset \mathbb{T}^{N-L}$, hence is nonsingular and, consequently, its irreducible components are disjoint and smooth. To prove the first statement of Claim 2.14 it suffices (due to property **A.** of Remark 2.1) to consider a nonsingular subvariety W of component $\pi(V)$ and a respective subvariety $\tilde{V}$ of V , obtained by restricting the original z-variables (due to Remark 5.5) to a nonsingular subvariety W . Similarly, let $\tilde{Z} \hookrightarrow Z$ be obtained by restricting z-variables to W . Then $\tilde{Z}$ is nonsingular (due to the étale property of μ , Theorem 2.7 **C.**) and, moreover, morphisms $\pi|_{\tilde{Z}} : \tilde{Z} \to W$ and of coordinatewise multiplication $\mu : \tilde{Z} \times \hat{Y}_V \to \tilde{V}$ are surjective étale morphisms and $\pi|_{\tilde{Z}}$ is finite (again due to Theorem 2.7 **C.**), which imply the first half of Claim 2.14 .

Next we show that a quasi-binomial variety, say $\tilde{X}$, is a special case of the preceding construction. Without loss of generality we may assume that quasi-binomial equations defining $\tilde{X}$ are the linear combinations of two monomials with the first coefficients being equal 1 . We start by replacing the 'second' coefficients of quasi-binomial equations (one per each) by minus a variable, say $-c_j$, introducing simultaneously another variable $\tilde{c}_j$ and a binomial equation $c_j \cdot \tilde{c}_j = 1$. We thus construct a binomial variety, say X , with all of the just introduced new variables being among the 'z-variables' for X . Obviously it suffices to show that the intersection W of the projection $\pi(X)$ (of binomial variety X to the affine subspace of its z-variables) with the specialization of variables c_j (according to their values in the quasi-binomial equations defining variety $\tilde{X}$) is nonsingular, thus reducing to a special case of the construction of the previous paragraph. Due to Theorem 2.7 **B.** $\pi(X) = \pi(X^*)$ and is a closed binomial variety (implying that W is a quasi-binomial variety). Obviously $\pi(X) = \pi(X^*) \subset \mathbb{T}^{N-L}$ implies $W = W^* := W \cap \mathbb{T}^{N-L} \subset \operatorname{Reg} W$ (the latter follows by making use of the algebraic group structure of $\mathbb{T}^{N-L}$ similarly to the 'Gauss elimination' argument of Remark 2.1 and from the analogous claim $\hat{V}^* \subset \operatorname{Reg} \hat{V}$ of Remark 2.1), as required. □

Proof. **Claim 2.3 .** The 'only if' implication is obvious. Assume that $\mathbf{0} \in X$. It follows that there are no z-coordinates and Corollary 2.5 implies existence of $\vec{\xi}^+ \in \operatorname{Ker} E \cap (\mathbb{Z}_+^N)$. Say $m := \dim X = N - \operatorname{rank} E$. To construct a monomial parametrization of the torus of X with positive integral exponents $\mathcal{E} = \{\vec{\Delta}_j\}_{1 \le j \le N} \subset \mathbb{Z}^m$ it suffices to find a $\mathbb{Z}$-basis $\{\vec{\delta}_i\}_{1 \le i \le m}$ of $\operatorname{Ker} E \cap \mathbb{Z}^N$ with positive coordinates, as in Remark 2.1. Construction of the latter provides lemma below. □

Lemma 5.9. *For any matrix E of size $M \times N$ with entries in $\mathbb{Q}$ and $m := N - \operatorname{rank} E$ the following properties are equivalent:*

(i) there is $\vec{v} \in \operatorname{Ker} E \cap (\mathbb{Z}_+^N)$;

(ii) there is a $\mathbb{Q}$-basis $\{\vec{\delta}_i\}_{1\le i\le m} \subset \mathbb{Z}_+^N$ of $\operatorname{Ker} E \cap \mathbb{Q}^N$;

(iii) there is a $\mathbb{Z}$-basis $\{\vec{\delta}_i\}_{1\le i\le m}$ of $\operatorname{Ker} E \cap \mathbb{Z}^N$ with all positive coordinates (equivalently, there exists a $\mathbb{Q}$-basis $\{\vec{\delta}_i\}_i \subset \mathbb{Z}_+^N$ of $\operatorname{Ker} E \cap \mathbb{Q}^N$ such that $I = \mathbb{Z}$, where $I = I(\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m)$ is the ideal generated in $\mathbb{Z}$ by all coordinates of $\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m$ in the standard basis $\{(j_1) \wedge \cdots \wedge (j_m)\}_{1\le j_1<\cdots<j_m\le N}$).

Remark 5.10. Due to a theorem of Gordan [2] (communicated to us by Dima Pasechnik) property (i) is equivalent to $(\text{Im } E^{tr}) \cap \mathbb{Q}_+^N = \{\mathbf{0}\}$.

Proof. Our proof is based on a simple linear algebra. To prove (i) implies (ii) it suffices to choose any basis $\{\vec{v}_i\}_i \subset \mathbb{Z}^N$ of $\operatorname{Ker} E \cap \mathbb{Q}^N$ with $\vec{v}_1 := \vec{v}$ and then letting $\vec{\delta}_1 := \vec{v}$ and $\vec{\delta}_i := t \cdot \vec{v} + \vec{v}_i$, $i > 1$, (ii) follows for a sufficiently large $t \in \mathbb{Z}_+$.

The remaining implication "(iii) follows from (ii)" is slightly harder. Starting with a $\mathbb{Q}$-basis $\{\vec{\delta}_i\}_{1\le i\le m} \subset \mathbb{Z}_+^N$ of $\operatorname{Ker} E \cap \mathbb{Q}^N$ let $s \in \mathbb{Z}_+$ be the generator of ideal I , i. e. $(s \cdot \mathbb{Z}) = I$. If $s = 1$ we are done. Otherwise, we modify basis $\{\vec{\delta}_i\}_{1\le i\le m}$ reducing the size of s , which would suffice. Pick a prime factor p of s . Denote field $\mathbb{Z}/(p \cdot \mathbb{Z})$ by $\mathbb{F}_p$. Now our collection of vectors $\{\vec{\delta}_i\}_{1\le i\le m}$ considered modulo ideal $(p \cdot \mathbb{Z})$ in $(\mathbb{F}_p)^N$ is linearly dependent, i. e. $\sum_{1\le i\le m} \lambda_i \cdot \vec{\delta}_i = \mathbf{0}$ in $(\mathbb{F}_p)^N$ for a collection of coefficients $\{\lambda_i\}_{1\le i\le m} \subset (\mathbb{F}_p)^m \setminus \{\mathbf{0}\}$. Choose $\tilde{\lambda}_i \in \mathbb{Z}$ so that $\lambda_i = \tilde{\lambda}_i \ (mod\ p)$ and $0 \le \tilde{\lambda}_i < p$, $1 \le i \le m$. Then $\tilde{\lambda}_{i_0} \ne 0$ for some i_0 , $1 \le i_0 \le m$, and $\vec{\delta}_0 := (1/p) \cdot \sum_{1\le i\le m} \tilde{\lambda}_i \cdot \vec{\delta}_i \ \in \ \mathbb{Z}_+^N$. It follows that all coordinates of the modified $\mathbb{Q}$-basis of $\operatorname{Ker} E \cap \mathbb{Q}^N$ obtained by replacing vector $\vec{\delta}_{i_0}$ of $\{\vec{\delta}_i\}_{1\le i\le m}$ by vector $\vec{\delta}_0$ are positive integers and that $I(\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_{i_0-1} \wedge \vec{\delta}_0 \wedge \vec{\delta}_{i_0+1} \wedge \cdots \wedge \vec{\delta}_m) = \tilde{\lambda}_{i_0} \cdot (s/p) \cdot \mathbb{Z}$. Due to the choice of $\{\tilde{\lambda}_i\}_{1\le i\le m}$ in $\mathbb{Z}^m$ the size of $\tilde{\lambda}_{i_0} \cdot (s/p)$ is smaller than the size of s , which suffices. □

Remark 5.11. Complexity of construction of a basis satisfying property (iii) of the algorithm '(ii) implies (iii)' is polynomial in the maxima of the absolute values of the coordinates of $\vec{\delta}_1 \wedge \cdots \wedge \vec{\delta}_m$ in the standard basis for the initial $\mathbb{Q}$-basis $\{\vec{\delta}_i\}_{1\le i\le m}$, i. e. is exponential in the binary size of the input (unlike construction of a basis $\{\vec{\delta}_j\}_{1\le j\le m}$ of (ii) which is a typical problem of linear programming and carries a polynomial cost in the binary size of the input). Of course we do not need the output with property (iii) for the algorithms of this article.

Proof. **Claim 2.6 .** The 'if' implication is obvious. We first prove the 'only if' implication in the case that there are no y-coordinates, i. e. we must show that in this case $(\hat{f})$ is a radical ideal when $\hat{V} = \hat{V} \cap \mathbb{T}^N = V^*(\hat{f})$. Of course $V^*(\hat{f}) \subset \mathrm{Reg}\, \hat{V}$ (as we have explained in Remark 2.1). Therefore, assuming that polynomial $P \in \mathbb{K}[w]$ vanishes on $\hat{V}$ it follows that polynomial P belongs to the ideals $I_{\mathfrak{m}}$ generated by ideal $(\hat{f})$ in the local rings $\mathcal{O}_{\mathfrak{m}}$ of the localizations of the polynomial ring $\mathbb{K}[w]$ at its maximal ideals $\mathfrak{m}$. The result follows by the standard 'partition of unity' argument of commutative algebra. (Indeed, for every $\mathfrak{m}$ there is a polynomial $Q_{\mathfrak{m}} \in \mathbb{K}[w]$ with $Q_{\mathfrak{m}} \notin \mathfrak{m}$ such that $Q_{\mathfrak{m}} \cdot P \in (\hat{f})$. Since the ideal generated by all $Q_{\mathfrak{m}}$ in $\mathbb{K}[w]$ is not in any maximal ideal $\mathfrak{m}$ of $\mathbb{K}[w]$ it follows that it coincides with $\mathbb{K}[w]$ and therefore there is a finite linear combination $\sum_k h_k \cdot Q_{\mathfrak{m}_k} = 1$, for an appropriate choice of polynomials $h_k \in \mathbb{K}[w]$, commonly refered to as a partition of unity. Expressing inclusions $Q_{\mathfrak{m}_k} \cdot P \in (\hat{f})$ as equalities $Q_{\mathfrak{m}_k} \cdot P = \sum_j G_{\mathfrak{m}_k,j} \cdot \hat{f}_j$ it follows that $P = \sum_k h_k \cdot Q_{\mathfrak{m}_k} \cdot P = \sum_j (\sum_k h_k \cdot G_{\mathfrak{m}_k,j}) \cdot \hat{f}_j$.)

Finally, we reduce to the previously considered special case. Let $v := (v_1, \ldots, v_L)$ and $g_i := y_i \cdot v_i - 1$ denote auxiliary variables and polynomials. Of course $\hat{V} \cap \{(y,z) \in \mathbb{A}^N : y_1 \cdot \ldots \cdot y_L \neq 0\} = V^*(\hat{f})$ (by definition of the y-variables). Therefore assumption that $P \in \mathbb{K}[w]$ vanishes on $\hat{V}$ (and equivalently on $V^*(\hat{f})$) implies that polynomial $P \in \mathbb{K}[w] \subset \mathbb{K}[w,v]$ vanishes on $V^*(\hat{f}, g) \subset \mathbb{A}^{N+L}$. Obviously all (w,v) variables for the collection $\mathcal{F}$ of binomials $\{\hat{f}_j\}_j \cup \{g_i\}_i$ are, as we refer to them, the 'z-variables'. Therefore the case we considered first implies that polynomial $P(w)$ is in the ideal generated by polynomials from $\mathcal{F}$ in the ring $\mathbb{K}[w,v]$. Substitution of $v_j = 1/y_j$, $1 \leq j \leq L$, in the equality expressing the inclusion of the previous sentence, followed by 'clearing' the denominators, i. e. (in our setting) by multiplying by a sufficiently high power of $y_1 \cdot \ldots \cdot y_L$, completes the proof. □

Part 2. Essential dimension $= 2$.

6. Termination of normalized Euclidean division: dim$=$ **2**.

Conjecture 6.1. *Tree* $\overline{\mathcal{T}}$ *associated with the multidimensional Euclidean algorithm is finite for any initial data.*

By König's lemma the latter is equivalent to the property that the algorithm terminates along every branch of tree $\overline{\mathcal{T}}$. In dimension > 2 'normalized' version of 6.1 is the following

Conjecture 6.2. *Tree $\mathcal{T}$ associated with the normalized multidimensional Euclidean algorithm is finite for any initial data.*

We start with an example from Introduction of a normal toric surface in $\mathbb{C}^3$ whose Nash blow up is not normal

Example 6.3. With $\phi : (x_1, x_2) \mapsto (x_1 \cdot x_2 \ , \ x_1 \cdot x_2^2 \ , \ x_1^3 \cdot x_2^2)$ let $S := \overline{\phi(\mathbb{T}^2)} \subset \mathbb{C}^3$. Exponents $\mathcal{E} := \{(1,1) \ , \ (1,2) \ , \ (3,2)\} \subset \mathbb{Z}^2$ generate over $\mathbb{Z}_+$ integral points $\mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E})$ of cone $\mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \subset \mathbb{Q}^2$ spanned by $\mathcal{E}$, because $\det((3,2) \ , \ (1,1)) = 1 = \det((1,1) \ , \ (1,2))$ implies that cones $\mathrm{Span}_{\mathbb{Q}_+}(\{(3,2) \ , \ (1,1)\})$ and $\mathrm{Span}_{\mathbb{Q}_+}(\{(1,1) \ , \ (1,2)\})$ are, respectively, generated by pairs of vectors $(3,2)$, $(1,1)$ and $(1,1)$, $(1,2)$ and because the union of these two cones is exactly the cone generated by $\mathcal{E}$. Then due to a criterion of Section 2.1 in [5] it follows that surface S is normal. Next, with reference to Section 4.2 there are exactly two elements in the set $\mathcal{S}(\mathcal{E})'$, namely: $J_1 = \{(1,1) \ ; \ (1,2)\}$ and $J_2 = \{(1,1) \ ; \ (3,2)\}$, - and the Nash blow up $N(S)$ of S is covered by two respective affine charts $N(S)_{J_j}$, $j = 1, 2$, as explained in Claim 4.6 . (In the remainder we make use of notations of Remark 4.5 .) It turns out $N(S)_{J_1} \subset \mathbb{C}^5$ is not normal, i. e. collection of exponents $\mathcal{E}_{J_1}$ of monomial parametrization

$$\psi : (x_1, x_2) \mapsto (x_1 \cdot x_2 \ , \ x_1 \cdot x_2^2 \ , \ x_1^3 \cdot x_2^2 \ , \ x_1^2 \cdot x_2 \ , \ x_1^2)$$

of torus $N(S)^*_{J_1}$ does not generate $\mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}_{J_1})$ over $\mathbb{Z}_+$, because obviously point $(1,0) \in \mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}_{J_1}) \setminus \mathbb{Z}_+(\mathcal{E}_{J_1})$, but $(1,0) \notin \mathbb{Z}_+(\mathcal{E} \cup \{(2,1) \ , \ (2,0)\})$, implying $N(S)$ is not normal. (Note, that $\psi_3(x) = \psi_1(x) \cdot \psi_4(x)$, i. e. exponent $(3,2)$ is generated over $\mathbb{Z}_+$ by 'others', illustrating passage from $\mathcal{E}_J$ to $\mathcal{E}xt(\mathbb{Z}_+(\mathcal{E}_J))$ in the combinatorial algorithm recording Nash blowing up.)

Consider a node τ of a tree $\mathcal{T}$ associated with normalized multidimensional Euclidean division for initial essential collection $\mathcal{N}(\mathcal{E})$ with $\mathcal{E}$ from Remark 4.1 . Let $C_\tau \subset \mathbb{Z}^2$ denote the associated with node τ essential collection. In abuse of notation we will not indicate the dependence of $\mathcal{S}_\tau := \mathcal{S}(C_\tau)$ and $\mathcal{S}'_\tau := \mathcal{S}(C_\tau)'$ on τ (for $\mathcal{S}(\mathcal{E})$ and $\mathcal{S}'$ see Remark 4.4 and Claim 4.6). Note that $\mathrm{int}(\nabla(J)) \cap \mathrm{Span}_{\mathbb{Z}}(C_\tau) = \mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(C_\tau)_{\mathbb{Z}}$ for $J \in \mathcal{S}_\tau$ and that $J \in \mathcal{S}'_\tau$ implies that $\mathrm{int}(\nabla(J)) \cap \mathbb{Q}_+(C_\tau)_{\mathbb{Z}} = \emptyset$, see Remark 4.7 (ii) , (iii) . Of course $\mathrm{Span}_{\mathbb{Z}}(C_\tau) = \mathrm{Span}_{\mathbb{Z}}(\mathcal{E})$ for any node τ . We may assume that $\mathbb{Z}^m = \mathrm{Span}_{\mathbb{Z}}(\mathcal{E})$, otherwise we 'rescale' replacing the latter span by

$\mathbb{Z}^m$. Finally, we refer to the initial node τ_0 of $\mathcal{T}$ as its root and to the collection of 'immediate descendants' of τ in $\mathcal{T}$ as *child nodes* of τ - terms commonly used in the 'theory of trees'.

6.1. **An apriori bound in (essential) dimension $m = 2$ on the length of desingularization by normalized Nash blow ups.** Below we assume that $m = 2$, nodes τ_0 and τ are not terminal and with node τ associate an integer $\mathcal{V}(\tau) := 2 \times$ the area of $\mathrm{Conv}\,(C_\tau)$. We refer to vectors $\{\vec{\Delta}_{j_i}\}_{i=1,2} \subset \mathcal{E} := \{\vec{\Delta}_j\}_{1\le j\le L} \subset \mathbb{Z}^2$ minimal on the intersection of $\mathcal{E}$ with two extremal rays of the cone generated by $\mathcal{E}$ over $\mathbb{R}_+$ as the *extremal vectors* of $\mathcal{E}$. Of course extremal vectors of the input $\mathcal{N}(\mathcal{E})$ for the normalized 2-dimensional Euclidean division are the same vectors. Integer D of Theorem 3.1 (ii) equals $|\det(\vec{\Delta}_{j_1}\ ,\ \vec{\Delta}_{j_2})|$. In abuse of notation we will not distinguish in this section between the subsets $J \in \mathcal{S}_\tau$ of indices of vectors in collections C_τ and the sets of the respective vectors themselves. Let $b_1,\ ,\ b_2 \in C_\tau$ be the extremal vectors of C_τ . Denote $D(\tau) := |\det(b_1\ ,\ b_2)|$ and pick a 2-tuple $J := \{u_j\}_{j=1,2} \in \mathcal{S}'$. In other words J corresponds to a child node $\overline{\tau}$ of τ and determines the branching of $\mathcal{T}$ at node τ . Of course $C_\tau = \mathcal{E}xt(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$.

Every $J \in \mathcal{S}'$ is a *frame*, i. e. is a collection of linearly independent vectors, and moreover is a *minimal frame of* C_τ . By minimal we mean that for an irrational functional h positive on the convex hull of collection $C_{\overline{\tau}} \subset \mathbb{Z}^2$ the value of $h(\vec{\Delta}_J)$, where $\vec{\Delta}_J := u_1 + u_2$, is smaller than the value of $h(\vec{\Delta}_{J'})$ for any other choice of $J' \in \mathcal{S}$. This property of frames $J \in \mathcal{S}'$ does not depend on the choice of irrational h being positive on the convex hulls of collections $C_{\overline{\tau}} \subset \mathbb{Z}^2$, corresponding to $\overline{\tau}$ and provides a bijective correspondence between the minimal frames of C_τ and the child nodes $\overline{\tau}$ of τ , cf. Claim 4.6 . We identify in explicit geometric terms sets involved in the proof below of an apriori bound Theorem 3.1 (ii) in the following

Claim 6.4. *Generators* $\mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_\mathbb{Z})$ *of any subset* $\mathcal{E} \subset \mathbb{Z}^2$ *with* $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$ *and* $\mathrm{Span}_\mathbb{Z}(\mathcal{E}) = \mathbb{Z}^2$ *are the integral points of bounded edges* Γ *of* $K := \mathrm{Conv}\,(\mathbb{Q}_+(\mathcal{E})_\mathbb{Z})$ *. For any node* τ *of tree* $\mathcal{T}$

$$D(\tau) - \mathcal{V}(\tau) = \#(C_\tau) - 1 \tag{6.1}$$

Proof. Inclusion of the integral points of bounded edges Γ of K in $\mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_\mathbb{Z})$ is obvious. To show the opposite inclusion we pick any pair J of adjacent integral points $\{u_1\ ,\ u_2\}$ on any bounded edge Γ of K . Then the only integral points of triangle $\nabla(u_1\ ,\ u_2)$ are its vertices. Therefore the only integral points in the parallelogramm $P(J)$

spanned by vectors u_1 , u_2 are its extremal points, which implies (by tiling of $\mathbb{R}^2$ by translations of $P(J)$) that $\mathrm{Span}_Z(J) = \mathbb{Z}^2$. Consequently $\mathbb{Z}^2 \cap \mathrm{Span}_{\mathbb{Q}_+}(J) \setminus \{\mathbf{0}\} = \mathbb{Z}_+(J)$ and $\mathrm{Span}_{\mathbb{Q}_+}(J) \cap \mathcal{E} = J$, which is equivalent to $1 = |\det(u_1 , u_2)| = 2 \cdot \mathrm{area}(\nabla(u_1 , u_2))$ for any pair of adjacent integral points u_1 , u_2 of any bounded edge Γ of $\mathrm{Conv}\,(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ implying (6.1) for any node τ . Also the remainder of the claim ("the opposite inclusion") follows by making use of $\mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) = \cup_J \mathrm{Span}_{\mathbb{Q}_+}(J)$, where the union is over pairs J of the adjacent integral points of the bounded edges of K . □

Remark 6.5. Any $J = \{u_1 , u_2\} \in \mathcal{S}(\mathcal{E})'$ must lie on a bounded edge Γ of $\mathrm{Conv}\,(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$. Moreover, frame J is a minimal frame iff u_1 , $u_2 \in \Gamma$ are adjacent integral points of edge Γ and at least one of them is a vertex of Γ , since $J \in \mathcal{S}(\mathcal{E})'$ iff $\dim \tilde{\mathcal{C}}_J = 2$ (see proof of Claim 4.6). Of course $|\det(u_1 , u_2)| = 1$ for any pair $\{u_1 , u_2\}$ of adjacent integral points on a bounded edge of $\mathrm{Conv}\,(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ is a byproduct of the proof of Claim 6.4 above. Moreover, the converse also holds. Namely, let $u_1, \ldots , u_k \in \mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}$ be such that u_1 , u_k are extremal vectors of $\mathcal{E}$. Assume that $|\det(u_i , u_{i+1})| = 1$, $1 \leq i < k$, and that $|\det(u_i , u_j)| \geq 2$ whenever $i \geq j + 2$. Then $\mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}) = \{u_1, \ldots , u_k\}$ and points u_i , u_{i+1} , $1 \leq i < k$, are the adjacent integral points on a bounded edge of $\mathrm{Conv}\,(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$.

Of course $\mathcal{V}(\tau) = 0$ for a terminal node τ and if node τ is not terminal but $\mathcal{V}(\tau) = 0$, then there are exactly two child nodes of node τ and both are terminal due to a simple argument of the case 2a of the proof in Section 6.2 of the following weak version of Theorem 3.1 (ii)

Theorem 6.6. *Assume $\overline{\tau}$ is not terminal. With every step of normalized 2-dimensional Euclidean algorithm integer $\mathcal{V}(\tau)$ decreases, i. e. $\mathcal{V}(\tau) > \mathcal{V}(\overline{\tau})$.*

Corollary 6.7. *Normalized 2-dimensional Euclidean algorithm terminates after at most $\mathcal{V}(\tau_0) + 1 \leq D(\tau_0) - 1$ steps.*

We derive Theorem 3.1 (ii) as a consequence of the following

Theorem 6.8. *Assume $\overline{\tau}$ is not terminal. It follows that either $\mathcal{V}(\overline{\tau}) < \mathcal{V}(\tau)/2$ or $\mathcal{V}(\overline{\overline{\tau}}) \leq \mathcal{V}(\overline{\tau})/2 < \mathcal{V}(\tau)/2$.*

Of course Theorem 3.1 (ii) follows, namely

Corollary 6.9. *Normalized 2-dimensional Euclidean algorithm terminates after at most $2 \cdot \log_2(\mathcal{V}(\tau_0) + 2) \leq 2 \cdot \log_2 D(\tau_0)$ steps.*

Claim 6.10. *For any node $\tau \neq \tau_0$ collection C_τ contains at most 6 vectors. Moreover, $Conv(\mathbb{Q}_+(C_\tau)_{\mathbb{Z}})$ contains at most 3 bounded* *edges. If there are at least 2 bounded edges then no edge can have more than 4 integral points. If there are just 3 bounded edges then the middle edge among them has exactly two integral points and no edge can have more than 3 integral points. Finally, at most 3 child nodes of τ can be nonterminal.*

We begin with a proof of a weaker bound of Theorem 6.6. The proofs of Theorem 6.8 and Claim 6.10 we placed in Section 7.

6.2. **Proof of Theorem 6.6 .**

Proof. Fix an irrational h and by reindexing arrange that $h(b_1) < h(b_2)$. Let b'_1 , $b'_2 \in C_{\overline{\tau}}$ be the extremal vectors of $C_{\overline{\tau}}$ and $\tilde{b}'_1$, $\tilde{b}'_2 \in N_J(C_\tau)$ be the minimal vectors in the intersection of $N_J(C_\tau)$ with two extremal rays of the cone generated by $N_J(C_\tau)$ over $\mathbb{R}_+$. Of course the latter cone does not change under 'normalization', i. e. coincides with the cone generated by $C_{\overline{\tau}}$ over $\mathbb{R}_+$, see Section 4.3 . In particular, it follows that (after an appropriate choice of indices) extremal vectors $\tilde{b}'_1$, $\tilde{b}'_2$ preceding normalization are proportional to the extremal vectors b'_1 , b'_2 with coefficients from $\mathbb{Z}_+$.

Remark 6.11. Node τ is terminal iff $|\det(b_1 , b_2)| = 1$ iff $\#(C_\tau) = 2$ iff $\{b_1 , b_2\}$ is a minimal frame in C_τ . To establish the only nonobvious implication (i. e. that the last property implies the first) it suffices to apply Claim 6.4 . The latter reference and node τ not being terminal also imply that if $J \not\subset \mathrm{int}\nabla(b_1 , b_2)$ then $\#(\{b_1 , b_2\} \cap J) = 1$ and $b_2 \notin J$ (otherwise $h(b_1) < \min h|_J < h(b_2)$ contrary to the choice of the irrational functional $h \in \tilde{\mathcal{C}}_J$).

Plan : Our proof of decrease of $\mathcal{V}(\tau)$ splits into several cases identified below. First we consider the case that $J \subset \mathrm{int}\nabla(b_1 , b_2)$ and otherwise $b_1 \in J$, $b_2 \notin J$ (due to Remark 6.11) and, also, $b_1 \in \{b'_1 , b'_2\}$ follows by making use of $\mathrm{Span}_{\mathbb{Q}_+}(J) \cap C_\tau = J$ established in Claim 6.4 , cf Figures 1 , 2 and 3 . Say $b'_1 = b_1$ and $u_1 = b_1$. The remaining cases are split according to either $u_2 \notin \mathrm{int}\nabla(b_1 , b_2)$ (and then $\overline{\tau}$ is terminal contrary to our assumption) or otherwise and then according to $\#(C_\tau) = 3$ (when $\#(C_\tau) = 2$ node is terminal) or $\#(C_\tau) \geq 4$. We show that in the latter case $\#(\mathbb{Z}^2 \cap \Gamma) > 2$ for the bounded edge $\Gamma \supset J$ of $\mathrm{Conv}\,(\mathbb{Q}_+(C_\tau)_{\mathbb{Z}})$ implies that node $\overline{\tau}$ must be terminal, which is contrary to our assumption. In the previous case of $u_2 \in \mathrm{int}\nabla(b_1 , b_2)$ and $\#(C_\tau) = 3$ the arguments of our proof differ depending on $D(\tau)$ being even or odd : if $D(\tau) = 2k - 1$ is odd then it turns out that $C_{\overline{\tau}} = \{b_1 , u_2 , b_2 - (k-1) \cdot u_2 , b_2 - b_1\}$ and $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = 1$, on the other hand if $D(\tau) = 2k$ is even then

$C_{\overline{\tau}} = \{b_1\ ,\ u_2\ ,\ (b_2 - b_1)/2\}$ and $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = \mathcal{V}(\tau)/2 + 1$. In each of the cases (with nodes τ and $\overline{\tau}$ not being terminal) we establish that (after 'normalization') integer $\mathcal{V}(\tau)$ decreases. We now start with

1. Points $u_1\ ,\ u_2$ in the interior of $\nabla(b_1\ ,\ b_2)$.

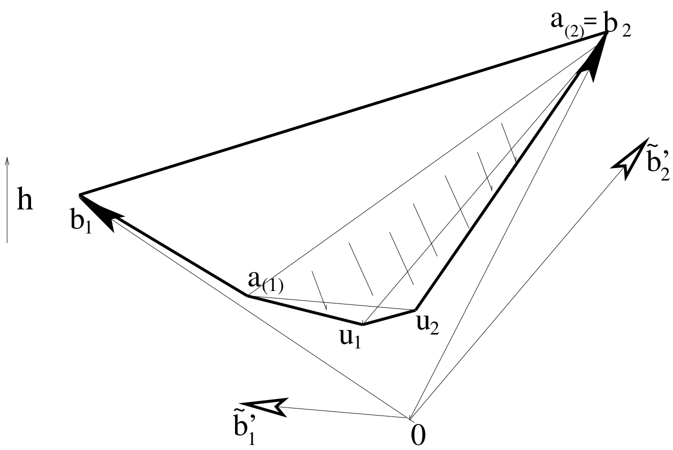

FIGURE 1. $C_\tau = \{\ b_1\ ,\ a_{(1)}\ ,\ u_1\ ,\ u_2\ ,\ b_2\ \}$.

Then after one step of 2-dimensional Euclidean division (and prior to normalization) each extremal vector $\tilde{b}'_l = a_{(l)} - u_{j_l}$ for appropriate points $a_{(l)} \in C_\tau \cap\ (\mathrm{int}(\nabla(b_1\ ,\ b_2)) \cup \{b_1\ ,\ b_2\})$, $l = 1\ ,\ 2$, $j_l \in \{1\ ,\ 2\}$, and after one step of normalized 2-dimensional Euclidean algorithm extremal vectors $b'_1\ ,\ b'_2$ are proportional to their respective counterparts $\tilde{b}'_1\ ,\ \tilde{b}'_2$ with positive coefficients majorated by 1 , so that $D(\overline{\tau}) \le |\det(\tilde{b}'_1\ ,\ \tilde{b}'_2)|$. Denote by H and $\mathcal{A}_H$ the convex hull of $\{a_{(1)}\ ,\ a_{(2)}\ ,\ u_{j_1}\ ,\ u_{j_2}\}$ and its area. Of course the areas of triangles $\nabla(b_1\ ,\ b_2)$ and $\nabla(b'_1\ ,\ b'_2)$ are $D(\tau)/2$ and, respectively, $D(\overline{\tau})/2$. Then claimed inequality follows from

$$\mathcal{V}(\overline{\tau}) < D(\overline{\tau}) \le |\det(\tilde{b}'_1\ ,\ \tilde{b}'_2)| = 2 \cdot \mathcal{A}_H \le \mathcal{V}(\tau)\ .$$

Remark 6.12. In the proofs of Theorem 6.8 and Claim 6.10 we will distinguish between the following subcases of case **1**.

1a Minimal frame $\{u_1\ ,\ u_2\} \subset \Gamma$ is not the set of all integral points of a bounded edge Γ of $Conv(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$.

Then, due to Remark 6.5 , we may assume that u_2 is an endpoint of Γ and that points $u_1\ ,\ u_2$ are adjacent integral points of Γ . Then there is also an integral point $a_{(1)}$ in Γ adjacent to u_1 and of course $a_{(1)} - u_1 = u_1 - u_2$. Also, there is a bounded edge $\Gamma' \ni u_2$ of $Conv(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$ and an integral point, say $a_{(2)} \in \Gamma'$, adjacent to u_2 . Then $u_1 + a_{(2)} = l \cdot u_2$ for an integer $l \geq 3$ since due to Remark 6.5 $\det(u_1 + a_{(2)} , u_2) = 0$ and $\det(u_1 , u_1 + a_{(2)}) \geq 3$. We will refer to subcases of **1a** with integer l being even or odd as **1a+** and, respectively, **1a–** .

1b $\{u_1 , u_2\} = \mathbb{Z}^2 \cap \Gamma$ **for a bounded edge** Γ **of** $Conv(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$ **.**

Then there are bounded edges $\Gamma_i \ni u_i$, $i = 1$, 2 , of $Conv(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$ distinct from edge Γ . Say $a_{(i)} \in \Gamma_i$ are the integral points adjacent to u_i , $i = 1$, 2 . Once again due to Remark 6.5 there are integers l_1 , l_2 ≥ 3 such that $u_2 + a_{(1)} = l_1 \cdot u_1$, $u_1 + a_{(2)} = l_2 \cdot u_2$. We refer to subcases of **1b** with l_1 , l_2 being even or both odd as **1b++** and, respectively, **1b– –** . Otherwise it is subcase **1b+ –** .

If case **1** does not hold then

2. Extremal vector $b_1 \in \{u_1 , u_2\}$ **.**

Since τ is not terminal $b_2 \notin J = \{u_1 , u_2\}$ and $b_1 \in \{b'_1 , b'_2\}$ (see 'Plan'). Set both $b'_1 = b_1$, $u_1 = b_1$, i. e. $b'_1 = \tilde{b}'_1 = b_1 = u_1$ for the remainder of the proof. **Case 2.** we split into several starting with

2a. Assume $u_2 \notin$ **int**$\nabla(b_1 , b_2)$ **.**

Then, with reference to Claim 6.4 , u_2 is in the open edge (b_1 , b_2) (i. .e. excluding endpoints b_1 , b_2) of triangle $\nabla(b_1 , b_2)$ and therefore $C_\tau \subset [b_1 , b_2] := (b_1 , b_2) \cup \{b_1 , b_2\}$. Then $\tilde{b}'_2 = a - u_2 \neq 0$ for the adjacent to u_2 point $a \in C_\tau \cap [u_2 , b_2]$ implying $b'_2 = \tilde{b}'_2 = u_2 - u_1$. Hence, with reference to Claim 6.4 , $|\det(b'_1 , b'_2)| = |\det(u_1 , u_2)| = 1$ and $\overline{\tau}$ is terminal (Remark 6.11).

In the remaining subcases of case **2** $u_2 \in \mathrm{int}\nabla(b_1 , b_2)$ and the assumptions of the next one imply that $\overline{\tau}$ is terminal.

2b. Assume $u_2 \in$ **int**$\nabla(b_1 , b_2)$, $\#(C_\tau) \geq 4$ **and** $\#(\mathbb{Z}^2 \cap \Gamma) > 2$ **for the bounded edge** $\Gamma \supset J$ **of** Conv $(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$ **.**

Then, with reference to Claim 6.4 , $\tilde{b}'_2 = a - u_2 \neq 0$ for the adjacent to u_2 point $a \in C_\tau \cap \Gamma \setminus \{u_1\}$ implying (as in the previous case) that $b'_2 = \tilde{b}'_2 = u_2 - u_1$, that $|\det(b'_1 , b'_2)| = |\det(u_1 , u_2)| = 1$ and, finally, that $\overline{\tau}$ is a terminal node, contrary to initial assumption.

2c. Assume $u_2 \in \mathbf{int}\nabla(b_1\ ,\ b_2)$, $\#(C_\tau) \geq 4$ **and** $\#(\mathbb{Z}^2 \cap \Gamma) = 2$ **for the bounded edge** $\Gamma \supset J$ **of** $\mathrm{Conv}\,(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$ **.**

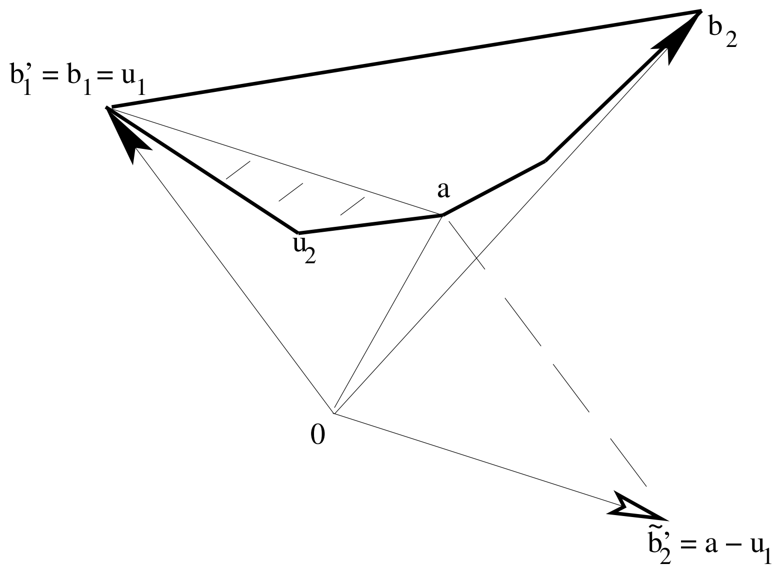

FIGURE 2. The area of $\mathrm{Conv}\,(C_\tau \setminus \{u_2\}) \geq 1$.

Then $\mathbb{Z}^2 \cap \Gamma = J$, $\#(C_\tau \setminus J) \geq 2$ and, with reference to Remark 6.5 , there is a bounded edge $\Gamma' \ni u_2$ of $\mathrm{Conv}\,(\mathbb{Q}_+(C_\tau)_\mathbb{Z})$ distinct from Γ and an integral point $a \in \Gamma'$ adjacent to u_2 with $\tilde{b}'_2 = a - u_1$. Therefore integer $\mathcal{V}(\tau) - 2 \cdot \mathrm{area}(u_1 + \nabla(u_2 - u_1\ ,\ a - u_1)) > 0$ implying $|\det(\tilde{b}'_1\ ,\ \tilde{b}'_2)| = 2 + 2 \cdot \mathrm{area}(u_1 + \nabla(u_2 - u_1\ ,\ a - u_1)) \leq 2 + (\mathcal{V}(\tau) - 1)$. Combining with (6.1) and Remark 6.11 proves inequality $\mathcal{V}(\overline{\tau}) < \mathcal{V}(\tau)$, as required:

$$2 + \mathcal{V}(\overline{\tau}) \leq D(\overline{\tau}) \leq |\det(\tilde{b}'_1\ ,\ \tilde{b}'_2)| \leq 1 + \mathcal{V}(\tau) \quad .$$

Remark 6.13. With a from case **2c** above and again due to Remark 6.5 (similarly to the argument in Remark 6.12 **1a**) there is an integer $l \geq 3$ with $u_1 + a = l \cdot u_2$. In the proofs of Theorem 6.8 and Claim 6.10 we will refer to subcases of case **2c** with integer l being even or odd as **2c+** and, respectively, as **2c−** .

2d. Assume $u_2 \in \mathbf{int}\nabla(b_1\ ,\ b_2)$ **and** $\#(C_\tau) = 3$ **.**

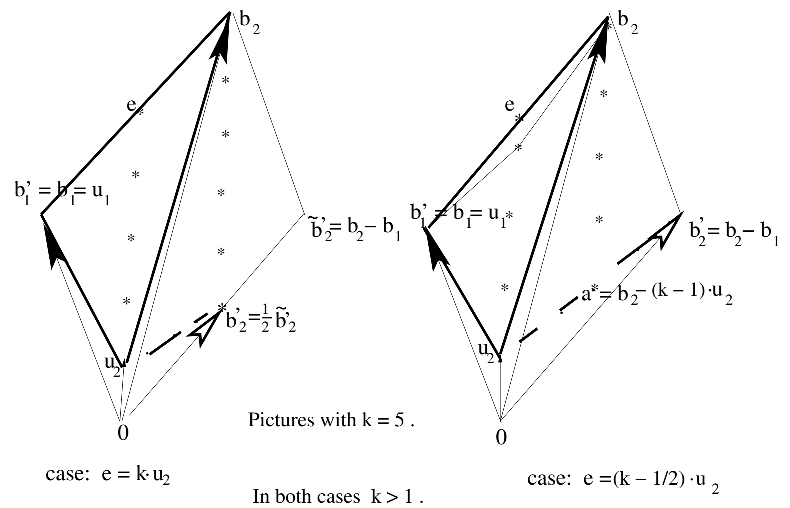

FIGURE 3. $D(\tau) = 2k$ or $2k - 1$ $\Rightarrow \#C_{\overline{\tau}} = 3$ or 4 respectively.

Let e be the point of intersection of edge $(b_1\ ,\ b_2)$ with ray $\mathbb{R}_+ \cdot u_2$, say $\lambda{\cdot}u_2 = e$, $\lambda > 0$. Due to Claim 6.4 $\nabla(b_1\ ,\ b_2) \cap \mathbb{Z}^2 \setminus \{\mathbf{0}\ ,\ b_1\ ,\ b_2\} \subset \mathbb{Z}_+ \cdot u_2$ and $|\det(b_2\ ,\ u_2)| = 1 = |\det(u_2\ ,\ b_1)|$ implying $\tilde{b}_2' = b_2 - b_1$ and that the areas of triangles $\nabla(b_2\ ,\ e)$ and $\nabla(b_1\ ,\ e)$ coincide. Hence $e = (b_1 + b_2)/2$ and, also, $\lambda = |\det(e\ ,\ b_1)| = D(\tau)/2$. The arguments in the remainder depend on $D(\tau)$ being even or odd and accordingly we split case **2d** into the following two subcases.

2d+ Assume $D(\tau)$ **is even and let** $k := D(\tau)/2$ **.**

Then $b_2' = \tilde{b}_2'/2$ since $\{(b_2 - b_1)/2\} = \mathbb{Z}^2 \cap (\mathbf{0}\ ,\ \tilde{b}_2')$. Therefore $|\det((b_2 - b_1)/2\ ,\ u_2)| = |(\det(b_2\ ,\ u_2) + \det(u_2\ ,\ b_1))/2| = 1$ implies that $C_{\overline{\tau}} = \{b_1\ ,\ u_2\ ,\ (b_2 - b_1)/2\}$ (Remark 6.5).

Remark 6.14. Claim 6.10 in case 2d+ is a consequence.

Finally, with reference to (6.1), it follows that

$$\mathcal{V}(\overline{\tau}) + 2 = D(\overline{\tau}) = |\det(b_1\ ,\ (b_2 - b_1)/2)| = D(\tau)/2 = (\mathcal{V}(\tau) + 2)/2$$

implying that $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = \mathcal{V}(\tau)/2 + 1$, as required.

Remark 6.15. Of course Theorem 6.8 in case 2d+ follows.

2d– Assume $D(\tau)$ **is odd and let** $k := (D(\tau) + 1)/2$ **.**

Then there are no integral points on edge $(b_1\ ,\ b_2)$ (as well as on ‘interval’ $(\mathbf{0}\ ,\ \tilde{b}_2')$) implying that $b_2' = \tilde{b}_2' = b_2 - b_1$. Denote point

$a := b_2 - (k-1) \cdot u_2 = (u_2 + b_2')/2$. Then, since $|\det(b_2' , u_2)| = 2$, it follows that $|\det(b_2' , a)| = |\det(a , u_2)| = 1$. Now, with reference to Remark 6.5 it follows that $C_{\overline{\tau}} = \{b_1 , u_2 , b_2 - (k-1) \cdot u_2 , b_2 - b_1\}$.

Remark 6.16. Therefore Claim 6.10 in case 2d– follows.

The latter formula for $C_{\overline{\tau}}$ and (6.1) imply that

$$\mathcal{V}(\overline{\tau}) + 3 = D(\overline{\tau}) = |\det(b_1' , b_2')| = D(\tau) = \mathcal{V}(\tau) + 2 .$$

Therefore $\mathcal{V}(\tau) - \mathcal{V}(\overline{\tau}) = 1$, which completes the proof of Theorem 6.6.

□

7. Sharp apriori bound and polynomial complexity. Proofs.

7.1. **Proofs of Theorem 6.8 and Claim 6.10.** Following the notation for the splitting into cases introduced in the course of the proof of Theorem 6.6 and starting with **1a+** we establish both results separately in each (exluding the cases already covered by Remarks 6.14 , 6.15 , 6.16 and cases **2a** and **2b**, when $\overline{\tau}$ is terminal).

Under the assumptions of case **1a+** and due to Remark 6.5

$$C_{\overline{\tau}} = \{ \ u_1 - u_2 \ , \ u_2 \ , \ a_{(2)} - k u_2 = (a_{(2)} - u_1)/2 \ \}$$

(unless $k - 1 = |\det(u_1 - u_2 , (a_{(2)} - u_1)/2)| = 1$ which implies that

$$C_{\overline{\tau}} = \{ \ u_1 - u_2 \ , \ (a_{(2)} - u_1)/2 \ \}$$

and, due to Remark 6.11 , that $\overline{\tau}$ is terminal). The latter proves Claim 6.10 in case **1a+** . Moreover, then also

$$\mathcal{V}(\overline{\tau}) = |\det(u_1 - u_2 \ , \ (a_{(2)} - u_1)/2)| - 2 =$$

$$k - 3 < k - 1 = |\det(u_1 - u_2 \ , \ a_{(2)} - u_2)|/2 < \mathcal{V}(\tau)/2$$

(unless $k = 2$ and $\overline{\tau}$ is terminal, as we showed above), which establishes Theorem 6.8 in case **1a+** .

Under the assumptions of case **1a–** and due to Remark 6.5 it follows that

$$C_{\overline{\tau}} = \{ \ u_1 - u_2 \ , \ u_2 \ , \ a_{(2)} - (k-1)u_2 \ , \ a_{(2)} - u_1 \ \}$$

with points u_2 , $a_{(2)} - (k-1)u_2$, $a_{(2)} - u_1$ lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $a_{(2)} - (k-1)u_2 = (u_2 + (a_{(2)} - u_1))/2$ (unless $2k - 3 = |\det(u_1 - u_2 , a_{(2)} - u_1)| = 1$ which implies that

$$C_{\overline{\tau}} = \{ \ u_1 - u_2 \ , \ a_{(2)} - u_1 \ \}$$

and, due to Remark 6.11 , that $\overline{\tau}$ is terminal). This proves Claim 6.10 in case **1a–** . Then

$$\mathcal{V}(\overline{\tau}) = |\det(u_1 - u_2 \ , \ a_{(2)} - u_1)| - 3 = 2k - 6 < 2k - 3 =$$

$$|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_2)|/2 \le \mathcal{V}(\tau)/2$$

(unless $k = 2$ and $\overline{\tau}$ is terminal, as proved above), which establishes Theorem 6.8 in case **1a–** .

Under the assumptions of case **1b++** and due to Remark 6.5

$$C_{\overline{\tau}} = \{ \ a_{(1)} - k_1 \cdot u_1 = (a_{(1)} - u_2)/2 \ , \ u_1 \ , \ u_2 \ , \ a_{(2)} - k_2 \cdot u_2 = (a_{(2)} - u_1)/2 \ \}$$

(unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 4$ in which case

$$C_{\overline{\tau}} = \{ \ (a_{(1)} - u_2)/2 \ , \ (a_{(2)} - u_1)/2 \ \}$$

and, due to Remark 6.11 , that $\overline{\tau}$ is terminal). This proves Claim 6.10 in case **1b++** . Then

$$\mathcal{V}(\overline{\tau}) = |\det((a_{(1)} - u_2)/2 \ , \ (a_{(2)} - u_1)/2)| - 3 <$$

$$|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)|/4 \le \mathcal{V}(\tau)$$

(unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 4$ and $\overline{\tau}$ is terminal, as we proved), which establishes Theorem 6.8 in case **1b++** .

Under the assumptions of case **1b+–** and due to Remark 6.5

$$C_{\overline{\tau}} = \{ \ a_{(1)} - u_2 \ , \ a_{(1)} - (k_1 - 1) \cdot u_1 \ , \ u_1 \ , \ u_2 \ , \ a_{(2)} - (k_2 - 1) \cdot u_2 = \frac{a_{(2)} - u_1}{2} \ \}$$

with the first three points $a_{(1)} - u_2$, $a_{(1)} - (k_1 - 1) \cdot u_1$, u_1 lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $a_{(1)} - (k_1 - 1) \cdot u_1 = (a_{(1)} - u_2 + u_1)/2$ (unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 2$ in which case

$$C_{\overline{\tau}} = \{ \ a_{(1)} - u_2 \ , \ (a_{(2)} - u_1)/2 \}$$

and, due to Remark 6.11 , that $\overline{\tau}$ is terminal). This proves Claim 6.10 in case **1b+–** . Then

$$\mathcal{V}(\overline{\tau}) = |\det(a_{(1)} - u_2 \ , \ (a_{(2)} - u_1)/2)| - 4 <$$

$$|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)|/2 \le \mathcal{V}(\tau)/2$$

(once again unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 2$ and, consequently, $\overline{\tau}$ is terminal), which establishes Theorem 6.8 in case **1b+–** .

Under the assumptions of case **1b– –** and due to Remark 6.5

$$C_{\overline{\tau}} = \{ \ a_{(1)} - u_2 \ , \ a_{(1)} - (k_1 - 1) \cdot u_1 \ , \ u_1 \ , \ u_2 \ , \ a_{(2)} - (k_2 - 1) \cdot u_2 \ , \ a_{(2)} - u_1 \ \}$$

with the first three points $a_{(1)} - u_2$, $a := a_{(1)} - (k_1 - 1) \cdot u_1$, u_1 lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $a = (a_{(1)} - u_2 + u_1)/2$ as well as all of the last three points u_2 , $b := a_{(2)} - (k_2 - 1) \cdot u_2$, $a_{(2)} - u_1$ lying on one bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $b = (u_2 + a_{(2)} - u_1)/2$ (unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 1$ in which case

$$C_{\overline{\tau}} = \{ \ a_{(1)} - u_2 \ , \ a_{(2)} - u_1 \}$$

and, due to Remark 6.11, that $\overline{\tau}$ is terminal). Therefore

$$\mathcal{V}(\overline{\tau}) = |\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| - 5 \leq \mathcal{V}(\tau) - 5$$

(unless $|\det(a_{(1)} - u_2 \ , \ a_{(2)} - u_1)| = 1$ and $\overline{\tau}$ is terminal, as we proved) and Claim 6.10 is proved in case **1b**$- -$. It remains to prove Theorem 6.8 (passing from node $\overline{\tau}$ to $\overline{\overline{\tau}}$), but we will need to examine several options in choosing minimal frames $J' = \{u'_1 \ , \ u'_2\}$ of $C_{\overline{\tau}}$ associated with the child node $\overline{\overline{\tau}}$ of $\overline{\tau}$ (unlike in the previously considered cases).

To begin with we assume that u_1 , u_2 are the endpoints of a bounded edge Γ of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$.

A choice of $J' = \{u_1 \ , \ u_2\}$ is an option (see Remark 6.5). Then

$$\mathcal{V}(\overline{\overline{\tau}}) < |\det((a_{(1)} - u_2 + u_1)/2 - u_2 \ , \ (a_{(2)} - u_1 + u_2)/2 - u_1)| <$$
$$|\det(a_{(1)} - 2 \cdot u_2 \ , \ a_{(2)} - 2 \cdot u_1)|/2 \leq \mathcal{V}(\overline{\tau})/2 \ ,$$

which establishes Theorem 6.8 in this subcase of case **1b**$- -$.

With the same assumption on $\{u_1 \ , \ u_2\}$ another possibility for the choice of a minimal frame J' of $C_{\overline{\tau}}$ is $u'_1 = (a_{(1)} - u_2 + u_1)/2$, $u'_2 = u_1$. Then, with reference to **1a** (passing from node $\overline{\tau}$ to node $\overline{\overline{\tau}}$),

$$\mathcal{V}(\overline{\overline{\tau}}) < |\det((a_{(1)} - u_2 + u_1)/2 - u_1 \ , \ (a_{(1)} - u_2 + u_1)/2 - u_2)| =$$
$$|\det(a_{(1)} - 2 \cdot u_2 \ , \ u_2 - u_1)|/2 < \mathcal{V}(\overline{\tau})/2 \ ,$$

implying Theorem 6.8 in this subcase.

Once again with the same assumption on $\{u_1 \ , \ u_2\}$ and for the choice of $J' := \{u'_1 = a_{(1)} - u_2 \ , \ u'_2 = (a_{(1)} - u_2 + u_1)/2\}$ it follows with reference to case **2b** (passing from node $\overline{\tau}$ to $\overline{\overline{\tau}}$) that node $\overline{\overline{\tau}}$ is terminal. With the same assumption on $\{u_1 \ , \ u_2\}$ the remaining options for a choice of a minimal frame J' and, consequently, of a child node $\overline{\overline{\tau}}$ are either $J' := \{(u_2 + a_{(2)} - u_1)/2 \ , \ a_{(2)} - u_1\}$, which is similar to the case just considered, or $J' := \{u_2 \ , \ (u_2 + a_{(2)} - u_1)/2\}$, which is similar to the case considered in the previous paragraph. Consequently, in these cases Theorem 6.8 follows by means of analogous arguments.

To complete the proof of Theorem 6.8 in the case **1b**$- -$ it remains to consider the case when u_1 , u_2 are not the end points of one bounded edge of $K := Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and then, following the constrains established in the first paragraph of case **1b**$- -$, there are at most 2 bounded edges of K . In the case that there is exactly one bounded edge Γ of K there are, due to Remark 6.5 , exactly two possible choices of minimal frames $\{u'_1 \ , \ u'_2\}$ of $C_{\overline{\tau}}$ both leading to case **2a** (passing from node $\overline{\tau}$ to node $\overline{\overline{\tau}}$) and therefore $\mathcal{V}(\overline{\tau}) = 0$ establishing Theorem 6.8 in this case. In the case that there are exactly two bounded edges of K it follows by making use of Remark 6.5

that there are exactly 4 possible choices of minimal frames $J' := \{u'_1\ ,\ u'_2\}$ of $C_{\overline{\tau}}$. We distiguish these choices only by the property of the intersection of the two edges being in J' or not. The latter case is the case **2b** (passing from node $\overline{\tau}$ to node $\overline{\overline{\tau}}$) and, consequently, implies that node $\overline{\overline{\tau}}$ is terminal establishing Theorem 6.8 in this case. In the former case we are in the setting of case **1a** (but passing from node $\overline{\tau}$ to node $\overline{\overline{\tau}}$). Inequalities on the values of $\mathcal{V}(\cdot)$ proved in both subcases of **1a** applied in our setting imply the second alternative of Theorem 6.8 in this last subcase of **1b**$-$ $-$, as required

The remaining cases to consider are **2c+** , **2c–** and **2d–**.

Under the assumptions of case **2c+** and due to Remark 6.5

$$C_{\overline{\tau}} = \{\ u_1\ ,\ u_2\ ,\ a - k \cdot u_2 = (a - u_1)/2\ \}\ ,$$

which proves Claim 6.10 in case **2c+**. Then

$$\mathcal{V}(\overline{\tau}) = |\det(u_1\ ,\ (a - u_1)/2)| - 2 = k - 2 < (l - 1)/2 =$$

$$(|\det(u_1 - u_2, a - u_2)| + 1)/2 \leq \mathcal{V}(\tau)/2\ ,$$

which establishes Theorem 6.8 in case **2c+**.

Under the assumptions of case **2c–** and due to Remark 6.5

$$C_{\overline{\tau}} = \{\ u_1\ ,\ u_2\ ,\ a - (k - 1) \cdot u_2\ ,\ a - u_1\ \}$$

with points u_2 , $a - (k - 1) \cdot u_2$, $a - u_1$ lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $a - (k - 1) \cdot u_2 = (u_2 + a - u_1)/2$. This proves Claim 6.10 in case **2c–**.

To establish Theorem 6.8 in the latter case we, once again, will examine options for choosing of minimal frames $J' := \{\ u'_1\ ,\ u'_2\ \}$ in $C_{\overline{\tau}}$ and, consequently, corresponding child nodes $\overline{\overline{\tau}}$ of node $\overline{\tau}$ (with the exception of the case that $k = 2$ when $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ has a single bounded edge with four integral points from $C_{\overline{\tau}}$ implying that we are in the case **2a** for node $\overline{\tau}$ and therefore node $\overline{\overline{\tau}}$ is terminal).

There are three options for the choice of a minimal frame J' .

The first choice is $u'_1 = u_1$, $u'_2 = u_2$. For an integer $l_1 \geq 3$ vector $u_1 + (a - (k - 1) \cdot u_2) = l_1 \cdot u_2$. If l_1 is even then with the reference to case **2c+** $\mathcal{V}(\overline{\overline{\tau}}) < \mathcal{V}(\overline{\tau})/2$. If $l_1 = 2 \cdot k_1 - 1$ is odd then the passage from $\overline{\tau}$ to $\overline{\overline{\tau}}$ is similar to the considered above in case **2c–** (of the passage from τ to $\overline{\tau}$), hence with reference to Remark 6.5 (and assuming $k \neq 2$)

$$C_{\overline{\overline{\tau}}} = \{\ u_1\ ,\ u_2\ ,\ a - (k - 1) \cdot u_2 - (k_1 - 1) \cdot u_2\ ,\ a - (k - 1) \cdot u_2 - u_1\ \}$$

with points u_2 , $a - (k - 1) \cdot u_2 - (k_1 - 1) \cdot u_2$, $a - (k - 1) \cdot u_2 - u_1$ lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $a - (k - 1) \cdot u_2 - (k_1 - 1) \cdot u_2 =$

$(u_2 + a - (k-1)\cdot u_2 - u_1)/2$. Consequently

$$\mathcal{V}(\overline{\overline{\tau}}) + 1 = |\det(u_1\ ,\ a - (k-1)\cdot u_2 - u_1)| - 2 = l_1 - 2 =$$

$$|\det(u_1 - u_2\ ,\ a - (k-1)\cdot u_2 - u_1)| = \frac{|\det(u_1 - u_2\ , a - 2\cdot u_1)|}{2} = \frac{\mathcal{V}(\overline{\tau})}{2}$$

and Theorem 6.8 follows in this subcase of **2c–** .

Another option for the choice of J' is $u'_1 = a - (k-1)\cdot u_2$, $u'_2 = a - u_1$ which leads to case **2b** (of node $\overline{\tau}$) and therefore it follows that $\mathcal{V}(\overline{\overline{\tau}}) = 0$, which once again suffices. The final option for the choice of J' is $u'_1 = u_2$, $u'_2 = a - (k-1)\cdot u_2$ and leads to case **1a** (of node $\overline{\tau}$) for which in both of its subcases we derived the required to establish Theorem 6.8 inequality $\mathcal{V}(\overline{\overline{\tau}}) < \mathcal{V}(\overline{\tau})/2$.

This completes the proof of Claim 6.10 , but to complete the proof of Theorem 6.8 it remains to consider case **2d–** (Remark 6.16 takes care of Claim 6.10 in this case). Under the assumptions of case **2d–**

$$C_{\overline{\tau}} = \{b_1\ ,\ u_2\ ,\ b_2 - (k-1)\cdot u_2\ ,\ b_2 - b_1\}$$

with points u_2 , $b_2 - (k-1)\cdot u_2$, $b_2 - b_1$ lying on a bounded edge of $Conv(\mathbb{Q}_+(C_{\overline{\tau}})_{\mathbb{Z}})$ and $b_2 - (k-1)\cdot u_2 = (u_2 + b_2 - b_1)/2$, which is the setting of case **2c–** implying its conclusion $\mathcal{V}(\overline{\overline{\tau}}) \le \mathcal{V}(\overline{\tau})/2$, which fully completes the proofs of both Claim 6.10 and Theorem 6.8 . □

Example 7.1. Example below demonstrates that the bound of Theorem 3.1 (ii) (and of Corollary 6.9) is sharp. In notations of case **2d+** consider u_1 , u_2 , $b_2 \in \mathbb{Z}^2$ with $|\det(u_1\ ,\ u_2)| = |\det(u_2\ ,\ b_2)| = 1$ and $u_1 + b_2 = 2^l \cdot u_2$ for an integer $l > 0$, e. g. say $u_1 = (-1, 1)$, $u_2 = (0, 1)$, $b_2^{(l)} := (1\ ,\ 2^l - 1)$. Then $\mathcal{V}(\tau_0) = 2^l - 2$. With a choice of $\{u_1\ ,\ u_2\}$ as a minimal frame and following the arguments of case **2d+** $C_{\overline{\tau_0}} = \{u_1\ ,\ u_2\ ,\ b_2^{(l-1)}\}$ with $b_2^{(l-1)} = b_2^{(l)} - 2^{l-1}\cdot u_2 = (b_2^{(l)} - u_1)/2$. Then $\mathcal{V}(\overline{\tau_0}) = 2^{l-1} - 2$. Therefore in this example normalized 2-dimensional Euclidean algorithm terminates after $l = \log_2 D(\tau_0)$ steps.

7.2. **Complexity issues.** We have constructed an algorithm by means of Lemma 2.4 (via linear programming) and subsequently in section 4.1, whose input is the exponents matrix $\hat{E}$ (from (2.1)) and the output is an essential collection $\mathcal{E} = \{\vec{\Delta}_j\}_{1\le j\le L}$ of the exponent vectors of a monomial parametrization of (4.1). Complexity of the designed algorithm is polynomial in the binary size of the input relying on the following two subroutines, namely:

(i) The first one by means of linear programming [12] separates variables w_j on $\mathbb{K}^N$ into two groups of z-variables and y-variables.

(ii) The second ([4]) yields a $\mathbb{Z}$-basis $\{(\vec{\delta}_{1i}\ ,\ \dots\ ,\ \vec{\delta}_{Li}) \times \mathbf{0}\}_{1\le i\le m}$ of the integral lattice in $\operatorname{Ker} \hat{E} \cap (\mathbb{Q}^L \times \{\mathbf{0}\}) \subset \mathbb{Q}^N$ and vectors from collection $\mathcal{E}$ by formulae $\vec{\Delta}_j = (\delta_{j1}\ ,\ \dots\ ,\ \delta_{jm})$ for each j .

Combination of the latter two subroutines results in an algorithm whose input being an exponents matrix of an affine binomial variety $\hat{V} \subset \mathbb{K}^N$ provides exponents $\vec{\Delta}_j \in \mathbb{Z}^m$, $1 \le j \le L$, of a monomial parametrization $\mathbb{T}^m \to Y \cap \mathbb{T}^N \hookrightarrow \hat{V} \cap (\mathbb{T}^L \times \mathbb{I}_{N-L})$ of torus of the essential toric subvariety $Y \hookrightarrow \hat{V}$, defined by formulae $y_j = x^{\vec{\Delta}_j}$, $1 \le j \le L$. As explained in Corollary 2.15 normalized Nash desingularization of variety Y implies normalized Nash desingularization of the same length of variety $\hat{V}$. We also observe that Criterion 2.18 invokes just subroutins (i),(ii) and thereby one can verify nonsingularity of an affine binomial variety within polynomial complexity.

When $m = 2$ the sequence of normalizations followed by Nash blowings up stabilizes, as is proved in this section, and provides normalized Nash desingularization of Y . This process is recorded by means of a combinatorial algorithm on the exponents of monomial parametrizations of the dense tori of the successive composites of the normalized Nash blowings up starting with the normalization of the essential toric variety Y and followed by the normalized 2-dimensional Euclidean algorithm (described in section 4.3 and in great detail here).

Complexity of both procedures is estimated below in terms of the number D that appears in our Abstract (see for the normalized Euclidean algorithm Remark 7.2 and for the normalizing algorithm Corollary 7.5). Consequently, the complexity of the normalized Nash desingularizatiom of Y is polynomial in the binary size of the input (i. e. of the exponents of binomial equations defining an affine binomial variety whose essential toric subvariety is of dimension $m \le 2$).

Remark 7.2. After each step of the normalized 2-dimensional Euclidean algorithm the maximal binary size of points of the input (set $\mathcal{E} = \mathcal{N}(\mathcal{E}) \subset \mathbb{Z}^2$ of the algorithm in Section 4.3) increases at most by an additive constant. Since the length of any branch of the algorithm is bounded by $2 \cdot \log_2 D$ (Theorem 3.1 (ii)) and $\log_2 D$ is polynomial in the binary size of the initial input (combining the bounds for the subroutines considered above), it follows that the complexity of a single step of the algorithm as well as the complexity along a single branch are polynomial in the binary size of the initial input.

7.3. **Polynomial complexity of normalization.** Finally we establish a polynomial complexity bound for constructing normalization

$\mathcal{N}(\mathcal{E})$ starting with an initial essential collection $\mathcal{E} \subset \mathbb{Z}^2$. Let $K := Conv(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ have k bounded edges with $l_1 , \dots , l_k$ integral points, respectively. We denote these points by

$$v_{1,1} , v_{1,2} , \dots , v_{1,l_1} := v_{2,1} , v_{2,2} , \dots , v_{2,l_2} := v_{3,1} , v_{3,2} , \dots , v_{k,l_k} ,$$

where each pair of the consecutive points consists of the adjacent integral points, say A , B , on the boundary of K with $det(A , B) = -1$ (cf. Remark 6.5) and points $v_{i,1} , v_{i,2} , \dots , v_{i,l_i}$ lie on the i-th bounded edge with $v_{i,1} , v_{i,l_i}$ being its endpoints.

Denote $v_i := v_{i,2} - v_{i,1} = \dots = v_{i,l_i} - v_{i,l_i-1}$. Then

Remark 7.3. Point v_{i,l_i} is a common vertex of two bounded edges of K whenever $v_{i,l_i} + v_i \notin K$. Moreover, then $v_{i+1,2} = B(s) := v_{i+1,1} + v_i + s \cdot v_{i+1,1}$ for $s = \lambda := det(v_i , v_{i+1}) \in \mathbb{Z}_+$, implying that λ is the smallest integer with $B(\lambda) \in \mathbb{Q}_+(\mathcal{E})$. (Because if $B(s) = v_{i+1,1} + t \cdot v_{i+1}$ for some $t , s \in \mathbb{R}$ then $t = det(B , v_{i+1,1}) = det(v_i , v_{i+1,1}) = 1$ and $1 + s = det(v_i , B) = 1 + det(v_i , v_{i+1})$.) Finally $D = -\det(v_{1,1} , v_{k,l_k})$.

Proposition 7.4. $l_1 \cdots l_i \leq |\det(v_{1,1} , v_{i,l_i})| , 1 \leq i \leq k$.

Proof. By induction on i . The base of induction $l_1 = |\det(v_{1,1} , v_{1,l_1})|$ is a consequence of Remark 6.5. For $v \in \mathbb{R}^2$ let $h(v)$ be the distance from v to the line $Span_{\mathbb{R}}(v_{1,1})$. Then the inductive hypothesis is

$$2 \cdot ||v_{1,1}|| \cdot h(v_{i,l_i}) = |\det(v_{1,1} , v_{i,l_i})| \geq l_1 \dots l_i$$

(Note that $h(v_{i+1,2} - v_{i,l_i}) = h(v_{i+1,2}) - h(v_{i,l_i})$.) With $\lambda \geq 1$ it follows

$$h((v_{i,l_i} + v_i) + \lambda \cdot v_{i,l_i}) > h((\lambda + 1) \cdot v_{i,l_i}) \geq 2 \cdot h(v_{i,l_i}) ,$$

implying $|\det(v_{1,1} , v_{i+1,2})| \geq 2 \cdot |\det(v_{1,1} , v_{i,l_i})|$. Similarly, for $j \geq 2$, $h(v_{i+1,j}) = h(v_{i,l_i} + (j-1) \cdot v_{i+1}) = h(v_{i,l_i}) + (j-1) \cdot h(v_{i+1,2} - v_{i,l_i}) > j \cdot h(v_{i,l_i})$, implying $|\det(v_{1,1} , v_{i+1,j})| > j \cdot |\det(v_{1,1} , v_{i,l_i}| > j \cdot l_1 \cdots l_i$ and for $j = l_{i+1}$ the inductive step of the proof. □

Corollary 7.5. *The number* k *of edges of* K *does not exceed* $\log_2 D$.

Next we describe (in dimension $m = 2$) in a greater detail the normalization algorithm of Section 4.3 , whose input is $\mathcal{E} \subset \mathbb{Z}^m$ with $\mathrm{Conv}(\mathcal{E}) \not\ni \mathbf{0}$ and the output $\mathcal{N}(\mathcal{E}) := \mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}) \subset \mathbb{Z}^m$. To carry out subsequently the normalized Euclidean algorithm with the initial input $\mathcal{N}(\mathcal{E})$ with $\mathcal{E}$ of the previous sentence it suffices to indicate on the i -th bounded edge of K the endpoints $v_{i,1} , v_{i,l_i}$ and, also, point $v_{i,2}$, which then yields $v_i = v_{i,2} - v_{i,1}$ and $v_{i,l_i-1} = v_{i,l_i} - v_i$. The normalized Euclidean algorithm then starts by choosing a minimal frame $J \in \mathcal{S}(\mathcal{N}(\mathcal{E}))'$, which are (Remark 6.5) of the form

$J = \{v_{i,1}\ ,\ v_{i,2}\}$ or $J = \{v_{i,l_i-1}\ ,\ v_{i,l_i}\}$ with $1 \le i \le k$, and the output of its first step for the choice of J is $\mathcal{N}(N_J(\mathcal{N}(\mathcal{E})))$.

The normalization algorithm first finds by means of linear programming $v'_{1,1}\ ,\ v' \in \mathcal{E}$ such that $\mathbb{Q}_+(\mathcal{E}) = \mathbb{Q}_+(v'_{1,1}\ ,\ v')$ and then (by dividing the coordinates of the points by their greatest common divisors) the minimal integral non-zero points $v_{1,1}\ ,\ v$ on the corresponding rays $\mathbb{Q}_+(v'_{1,1})\ ,\ \mathbb{Q}_+(v')$, i. e. the outcome is $v = v_{k,l_k}$ of the first paragraph of this subsection.

We execute the normalizing algorithm by recursion on i starting with points $v_{1,1}\ ,\ v$. For the base of recursion of the algorithm we first find (by means of the integer programming on the plane) an integral point $v'_{1,2} \in \mathbb{Q}_+(\mathcal{E})$ such that $|\det(v_{1,1}\ ,\ v'_{1,2})| = 1$ and then set $v_{1,2} := v'_{1,2} + \lambda \cdot v_{1,1}$ for the minimal integer λ such that $v'_{1,2}+\lambda\cdot v_{1,1} \in \mathbb{Q}_+(\mathcal{E})$, cf. Remark 7.3. Next, once again by means of the integer programming, we construct $v_{1,l_1} := v_{1,1} + (l_1 - 1)\cdot(v_{1,2} - v_{1,1})$ for the largest integer l_1 such that $v_{1,l_1} \in \mathbb{Q}_+(\mathcal{E})$. Of course the integral points of the edge of K passing through $v_{1,1}\ ,\ v_{1,2}$ are points $v_{1,j} = v_{1,1} + (j-1)\cdot(v_{1,2} - v_{1,1}) \in \mathbb{Q}_+(\mathcal{E})\ ,\ 1 \le j \le l_1$.

Assuming point v_{i,l_i} , and vector v_i for an $i \ge 1$ being constructed we set (applying the integer programming) $v_{i+1,2} := (\lambda+1)\cdot v_{i,l_i} + v_i$ for the smallest integer λ such that $(\lambda+1)\cdot v_{i,l_i} + v_i \in \mathbb{Q}_+(\mathcal{E})$ (and resulting with $\lambda \ge 1$), cf Remark 7.3 . Therefore $v_{i+1} = v_{i+1,2} - v_{i,l_i}$ and then (again applying the integer programming and) following our algorithm we set $v_{i+1,l_{i+1}} := v_{i,l_i} + (l_{i+1} - 1)\cdot v_{i+1}$ for the largest l_{i+1} such that $v_{i+1,l_{i+1}} \in \mathbb{Q}_+(\mathcal{E})$. Once again the integral points of the edge of K passing through $v_{i+1,1}\ ,\ v_{i+1,2}$ are points $v_{i+1,j} = v_{i+1,1} + (j-1)\cdot v_{i+1} \in \mathbb{Q}_+(\mathcal{E})\ ,\ 1 \le j \le l_{i+1}$, which completes the recursive step and the description of the normalizing algorithm.

Points $v_{i,1}\ ,\ v_{i,2}\ ,\ v_{i,l_i-1}\ ,\ v_{i,l_i}$ provided by the algorithm lie in triangle $\nabla(v_{1,1}\ ,\ v_{k,l_k})$ implying that the binary sizes of these points are polynomial in the binary sizes of the input data. Now Corollary 7.5 combined with Remark 7.2 imply that the complexity of the algorithm of normalization is polynomial, as well as that of the normalized 2-dimensional Euclidean division algorithm.

Corollary 7.6. *Complexity of the normalized* 2*-dimensional Euclidean division algorithm along a single branch (or equivalently of the normalized Nash desingularization of affine binomial varieties of essential dimension* $m \le 2$ *) is polynomial in the binary size of the input.*

Finally, Corollary 7.5 combined with Claim 6.10 imply

Corollary 7.7. *The tree* $\mathcal{T}$ *(of Section 6) associated with the normalized* 2*-dimensional Euclidean algorithm applied to normalization* $\mathcal{N}(\mathcal{E}) \subset \mathbb{Z}^2$ *of* $\mathcal{E} \subset \mathbb{Z}^2$ *with* $\mathrm{Conv}\,(\mathcal{E}) \not\ni \mathbf{0}$ *contains less than* $O(D^{2\cdot\log_2 3} \cdot \log D) < O(D^{3.2})$ *nodes.*

We conclude this Section with two examples. The first one shows that the bounded edges of K can contain more than $D/2$ integral points, while the normalization algorithm of this section should not (and does not as we have described it) produce too many integral points on the edges, in order to proceed within the polynomial complexity (in fact it would construct at most four points on each edge).

Example 7.8. Let $v_{1,1} := (1\ ,\ 2)$, $v_{2,l_2} := (l_2\ ,\ 1)$. Obviously $D = 2 \cdot l_2 - 1$. Then K has just two bounded edges, the first of which contains two integral points $(1\ ,\ 2)$, $(1\ ,\ 1)$, while the second of which contains l_2 integral points $(i\ ,\ 1)$, $1 \le i \le l_2$.

The second example demonstrates the sharpness of the bound in Corollary 7.5.

Example 7.9. Denote $\Phi_1 := \Phi_2 := 1$ and by Φ_i the i-th Fibonacci number. Set $v_{1,1} := (\Phi_2\ ,\ \Phi_1)$, $v_{k,2} := (\Phi_{2k+2}\ ,\ \Phi_{2k+1})$. Then K has k bounded edges, i-th among them contains just two integral points (being its endpoints) $(\Phi_{2i}\ ,\ \Phi_{2i-1})$, $(\Phi_{2i+2}\ ,\ \Phi_{2i+1})$.

8. Invariance of termination bounds.

This section is entirely devoted to the issue of the invariance of the integer D introduced in Sections 3 and 6 in terms of which the termination and complexity bounds are expressed (though has no evident bearing on the problem of termination of neither normalized multidimensional Euclidean division nor of its geometric counterpart for $m > 2$). Considered in both sections in the case of dimension $m = 2$ and associated with a monomial parametrization $\mathbb{T}^m \ni x \mapsto y = \phi_{\mathcal{E}}(x) \in Y^*$ (with components $y_j = (\phi_{\mathcal{E}})_j(x) := x^{\vec{\Delta}_j}$) of the torus Y^* of an essential toric subvariety Y of a binomial variety $\hat{V} \subset \mathbb{A}^N$ number D is expressed in terms of the exponents $\mathcal{E} = \{\vec{\Delta}_j\}_{1 \le j \le L} \subset \mathbb{Z}^m$ of map $\phi_{\mathcal{E}}$ as the area of a parallelogram generated by the extremal vectors, i. e. the least points of $\mathrm{Span}_{\mathbb{Z}}(\mathcal{E})$ on the (two) extremal rays of the cone spanned over $\mathbb{R}_+$ by the exponents in $\mathcal{E}$, see Section 6 .

Due to Theorem 2.7 , Corollary 2.5 and Claim 2.3 we may, as well, assume all exponents to be strictly positive, i. e. that $\mathcal{E} \subset \mathbb{Z}^m_+$. Also, we may assume without loss of generality that $\mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) = \mathbb{Z}^m$. Recall that Y is ‘essential’ means that $Y \ni \mathbf{0}$ and is equivalent to

$\mathrm{Conv}\,(\mathbb{Z}_+(\mathcal{E})) \not\ni \mathbf{0}$, Sections 2 and 4 . By *extremal vectors* for any m we (similarly) mean the subset $\mathcal{E}xt(\mathcal{E}) \subset \mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$, where $\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}} = \mathrm{Span}_{\,\mathbb{Z}}(\mathcal{E}) \cap \mathrm{Span}_{\,\mathbb{Q}_+}(\mathcal{E}) \setminus \{\mathbf{0}\}$, of all minimal in size points of $\mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ on the extremal rays of cone $\mathrm{Span}_{\,\mathbb{Q}_+}(\mathcal{E})$ and 'normality' property of Y is equivalent in terms of exponents $\mathcal{E}$ to $\mathbb{Z}_+(\mathcal{E}) = \mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}$ and (by construction) is valid for normalized algorithms (Nash and/or 2-dimensional Euclidean) of Section 6 for which termination is proved. We may also (without loss of generality) assume that $\mathcal{E} = \mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$ since the 'left out' exponents (and corresponding affine coordinates) are in $\mathbb{Z}_+(\mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}}))$ (and, respectively, coincide on Y with monomials in the coordinates corresponding to elements in $\mathcal{E}xt(\mathbb{Q}_+(\mathcal{E})_{\mathbb{Z}})$). Number D admits a natural extension for an arbitrary m in terms of set $\mathcal{E}$ as the smallest $D = D(\mathcal{E}) \in \mathbb{Z}_+$ such that $D \cdot \vec{\Delta}_j \in \mathbb{Z}_+(\mathcal{E}xt(\mathcal{E}))$ for all $\vec{\Delta}_j \in \mathcal{E}$.

Next we restate the definition of *denominator* $D(\mathcal{E})$ as a local invariant of Y (as well as of any of the isomorphic irreducible components of $\hat{V}$, say of V) at any point $o \in Y$. Invariance we consider is with respect to the germs at o of local étale isomorphisms preserving coordinate hyperplanes that contain o . We restrict variety $X := Y$ (or respectively $X := V$) to affine charts $\mathcal{U}_o$ obtained by exclusion of all coordinate hyperplanes off o , which we refer to as the *origin* (recall, Section 2 and Remark 5.5 , that 'y-variables' of varieties Y , V and even of $\hat{V}$ coincide). To be precise charts $\mathcal{U}_o$ are constructed by introducing a 'double' $\tilde{z}_j$ of every affine coordinate $z_j := w_j$ with $w_j(o) \neq 0$, say $j = 1, ..., \tilde{L}$, and

$$\mathcal{U}_o := \{(z, \tilde{z}) \in \mathbb{A}^{2\tilde{L}} : z_j \cdot \tilde{z}_j = 1\ ,\ 1 \leq j \leq \tilde{L}\} \times \mathbb{A}^{L_o} \hookrightarrow \mathbb{A}^{L_o + 2\tilde{L}}\ ,$$

with y-variables of variety X being the remaining L_o variables induced by the original y-coordinates with $y_j(o) = 0$.

Then, according to Theorem 2.7 and Remark 2.12 , the germ X_o of variety X at o is isomorphic to a product of a germ Z_a of a nonsingular subvariety Z at $a \in Z$ with a germ at $b \in (\pi|_X)^{-1}(\mathbb{I}_{2\tilde{L}}) =: \hat{Y}$ of a union $\hat{Y}$ of, possibly several, mutually isomorphic subvarieties (including the germ Y_b at b of the essential toric subvariety of X) and $o = \mu(a, b)$. Moreover, germ Z_a is 'étale identified' with $\pi(Z_a) = \pi(X_o) \hookrightarrow \mathbb{A}^{2\tilde{L}}$ for projections $\pi : \mathbb{A}^{L_o + 2\tilde{L}} \to \mathbb{A}^{2\tilde{L}}$, whose components are the z-coordinates (Theorem 2.7 **C**).

Therefore (using Krull completion) morphisms $\mathcal{O}_{\pi(X_o)} \hookrightarrow \hat{\mathcal{O}}_{\pi(X_o)} \tilde{\to} \hat{\mathcal{O}}_{Z_a}$ and $(\pi|_{X_o})^* : \mathcal{O}_{\pi(X_o)} \hookrightarrow \mathcal{O}_{X_o}$ allow to consider the following base change $\mathcal{R}_o := (\mathcal{O}_{X_o} \otimes_{\mathcal{O}_{\pi(X_o)}} \hat{\mathcal{O}}_{Z_a}) \otimes_{\hat{\mathcal{O}}_{Z_a}} \mathbb{F}$, where $\mathbb{F}$ is the field of fractions of $\hat{\mathcal{O}}_{Z_a}$. Morphism $\mu : Z_a \times \hat{Y}_b \to X_o$ is étale and $\pi|_{X_o} \circ \mu$ coincides with $\pi|_{Z_a} : Z_a \times \hat{Y}_b \ni (u \times v) \mapsto \pi|_{Z_a}(u) \in \pi|_{Z_a}(Z_a)$, while $Z_a \times Y_b$ is an irreducible component of $Z_a \times \hat{Y}_b$ and is a product of germs at a , b of torus Z and, respectively, of the essential toric subvariety of X . Consequently, the base change above corresponds (via étale morphism μ) to a base change of $Z_a \times \hat{Y}_b$ and is isomorphic to a very simple base change $\tilde{X}_b$ of $\hat{Y}_b$ via $- \otimes_{\mathbb{K}} \mathbb{F}$. Thus $\mathcal{R}_o$ is the local ring of a germ at b $(= \mathbf{0} \in \mathbb{A}^{L_o})$ of variety $\tilde{X}$ obtained from $\hat{Y}$ by means of the base change via $- \otimes_{\mathbb{K}} \mathbb{F}$ and the germ at b of the base change $\tilde{Y}$ of Y via $- \otimes_{\mathbb{K}} \mathbb{F}$ is an essential toric variety and a component of $\tilde{X}$, cf. Ch.1 [7]. We use these constructions below.

By attaching subscript o indicating the dependence on the new origin $o \in X$ we will assume below that all notations (and assumptions) of the second paragraph of this section (including of the sets of exponents $\mathcal{E}_o$ associated with essential subvariety Y of $X \hookrightarrow \mathcal{U}_o$ and of the extremal vectors $\mathcal{E}xt(\mathcal{E}_o) \subset \mathcal{E}_o$, as well as of the numbers $m_o := \dim Y_o$ and $D_o := D(\mathcal{E}_o)$ are associated with toric variety $X \hookrightarrow \mathcal{U}_o$. By reindexing y_j's we may assume that $\mathcal{E}xt(\mathcal{E}_o) = \{y_j\}_{1 \le j \le L'_o}$. In abuse of notation we will write below $j \in \mathcal{E}xt(\mathcal{E}_o)$ instead of $y_j \in \mathcal{E}xt(\mathcal{E}_o)$.

For the sake of invariance we must consider notions allowing to define denominator $D(\mathcal{E}_o)$ in the respective local ring $\mathcal{O}_{X,o}$ (i.e. with X being the 'original' variety Y and/or V from the first paragraph of this section), while in $\mathcal{O}_{X,o}$ its 'defining equations' are no longer binomial, i. e. binomials do not generate the ideal of relations between local parameters (even though we include among the latter all affine coordinates y_j with $y_j(o) = 0$, which we do since we examine the invariance with respect to the germs of local isomorphisms preserving all germs of sets $\{y_j = 0\}$). To overcome this problem we consider a base change (as above) passing to a germ $\tilde{X}_b$ of binomial variety $\tilde{X}$ (defined over field $\mathbb{F}$) and its local ring $\mathcal{R}_o$, whose maximal ideal $\mathfrak{m}_o$ is generated by the classes $\overline{y}_j$ in $\mathcal{R}_o$ of all affine coordinates y_j with $y_j(o) = 0$. Of course, collection (of 'parameters') $\mathcal{P}ar(\mathcal{R}_o) := \{\overline{y}_j\}_{1 \le j \le L_o} \subset \mathfrak{m}_o$ induces a set that spans $\mathfrak{m}_o/\mathfrak{m}_o^2$ over field $\mathbb{F}$.

Remark 8.1. Sets $\mathcal{E}xt(\mathcal{P}ar(\mathcal{R}_o)) \subset \mathcal{P}ar(\mathcal{R}_o)$ can be defined in terms of collection $\mathcal{P}ar(\mathcal{R}_o) \subset \mathcal{R}_o$ as follows: $j \in \mathcal{E}xt(\mathcal{P}ar(\mathcal{R}_o))$ **iff**
(i) $\overline{y}_i^p = \overline{y}_j^q$, $(p, q) \in \mathbb{Z}_+^2$, $i \neq j$, implies $p < q$, and
(ii) $\overline{y}_j$ is not in the integral closure in $\mathcal{R}_o$ of the subring of $\mathcal{R}_o$ generated by $\overline{y}_i$'s such that $\overline{y}_i^p \neq \overline{y}_j^q$ for any $(p, q) \in \mathbb{Z}_+^2$.

Note that ring $\mathcal{R}_o$ is the integral closure of its subring $\mathcal{R} \hookrightarrow \mathcal{R}_o$ generated by $\overline{y}_j$'s with $j \in \mathcal{E}xt(\mathcal{P}ar(\mathcal{R}_o))$ (using Section 2.1 of [5]). We may therefore introduce in terms of collection $\mathcal{P}ar(\mathcal{R}_o)$ the smallest positive integer $D = D(\mathcal{P}ar(\mathcal{R}_o))$ such that for all j , $\overline{y}_j^D \in \mathcal{R}$. Obviously, the value of *denominator* D of $\mathcal{P}ar(\mathcal{R}_o)$ coincides with $D_o = D(\mathcal{E}_o)$, where $\mathcal{E}_o$ is the collection of exponents $\{\vec{\Delta}_j\}_j$ of any monomial map $\phi_{\mathcal{E}_o}$ (including with nonpositive exponents) parametrizing torus Y^* of the essential subvariety Y of X , i. e. $D(\mathcal{E}_o)$ is a local invariant due to the definition of $D = D(\mathcal{E}_o)$ being stated entirely in terms of collection $\mathcal{P}ar(\mathcal{R}_o)$.

Remark 8.2. With reference to Section 4.3 normalization $\mathcal{N}(Y)$ of $Y \subset \mathbb{A}^L$ is a toric variety in $\mathbb{A}^{L'}$ whose torus $\mathcal{N}(Y)^* := \mathcal{N}(Y) \cap \mathbb{T}^{L'}$ is parametrized by a map $\phi_{\mathcal{E}'} : \mathbb{T}^m \ni x \mapsto y = \phi_{\mathcal{E}'}(x) \in \mathcal{N}(Y)^*$ with components $y_j = (\phi_{\mathcal{E}'})_j(x) := x^{\vec{\Delta}_j}$, and the collection of exponents, say $\mathcal{E}' := \{\vec{\Delta}_j\}_{1 \le j \le L'} \subset \mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \subset \mathbb{Z}_+^m$, augmenting set $\mathcal{E} = \{\vec{\Delta}_j\}_{1 \le j \le L}$ so that $\mathbb{Z}_+(\mathcal{E}') = \mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \setminus \{\mathbf{0}\}$. It follows that $\mathbb{Z}_+(\mathcal{E}') = \mathrm{Span}_{\mathbb{Z}}(\mathcal{E}') \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}') \setminus \{\mathbf{0}\}$. In short, all assumptions of the lemma following (except on the size of $\mathcal{E}xt(\mathcal{E})$ when $m > 2$) are satisfied for Y replaced by its normalization $\mathcal{N}(Y)$. Of course elements of $\mathcal{E}xt(\mathcal{E}')$ and of $\mathcal{E}xt(\mathcal{E})$ span the same extremal rays with the extremal vectors of $\mathcal{E}xt(\mathcal{E}')$ being (equal or) shorter than their respective counterparts in $\mathcal{E}xt(\mathcal{E})$.

For a matrix $\mathcal{M}$ of size $m \times m$ with entries in $\mathbb{Z}$ let $\mathrm{den}\,(\mathcal{M}) \in \mathbb{Z}_+$ denote the least $d \in \mathbb{Z}_+$ with the entries of $d \cdot \mathcal{M}^{-1}$ being integers. Obviously, entries of matrix $d \cdot \mathcal{M}^{-1}$ generate a unit ideal in $\mathbb{Z}$ and if also $m = 2$ and the entries of $\mathcal{M}$ have no common divisor then $\mathrm{den}\,(\mathcal{M}) = |\det(\mathcal{M})|$. Recall that a matrix whose columns are elements of collection $\mathcal{E} \subset \mathbb{Z}^m$ we denote by the same letter $\mathcal{E}$.

Lemma 8.3. *If* $\mathrm{Span}_{\mathbb{Z}}(\mathcal{E}) = \mathbb{Z}^m$, $\mathbb{Z}^m \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \setminus \{\mathbf{0}\} = \mathbb{Z}_+(\mathcal{E})$ *and* $\#(\mathcal{E}xt(\mathcal{E})) = m$ *it follows that* $D(\mathcal{E}) = \mathrm{den}\,(\mathcal{E}xt(\mathcal{E}))$.

Remark 8.4. Of course, if $\#(\mathcal{E}xt(\mathcal{E})) = m$ and $D(\mathcal{E}) = 1$ affine variety Y being of dimension m must be nonsingular. Also, if $m = 2$, then obviously $\#(\mathcal{E}xt(\mathcal{E})) = m$ and $D(\mathcal{E}) = |\det(\mathcal{E}xt(\mathcal{E}))|$.

Proof. Inclusion $\mathrm{den}\,(\mathcal{E}xt(\mathcal{E})) \in D(\mathcal{E}) \cdot \mathbb{Z}$ is a simple consequence of the definitions. It therefore suffices to show that for any prime number p and $s \in \mathbb{Z}_+$ it follows from $\mathrm{den}\,(\mathcal{E}xt(\mathcal{E})) \in p^s \cdot \mathbb{Z}$ that $D(\mathcal{E}) \in p^s \cdot \mathbb{Z}$. Let $\mathcal{M} := \mathrm{den}\,(\mathcal{E}xt(\mathcal{E})) \cdot \mathcal{E}xt(\mathcal{E})^{-1}$. Then there is a column $\vec{\lambda}$ of matrix $\mathcal{M}$ with a nonvanishing $\mod p$ entry and modifying the latter column to $\vec{\lambda}' := \vec{\lambda} + p^s \cdot t \cdot \mathbb{I}_m$ with a sufficiently large positive

$t \in \mathbb{Z}_+$ so as to make all entries of $\vec{\lambda}'$ positive it follows that $\vec{\lambda}' \neq \mathbf{0}$ $(mod\ p)$. Therefore vector $\mathcal{E}xt(\mathcal{E}) \cdot \vec{\lambda}' \in (p^s \cdot \mathbb{Z}^m) \cap \mathrm{Span}_{\mathbb{Q}_+}(\mathcal{E}) \setminus \{\mathbf{0}\}$. It follows that $D(\mathcal{E}) \in p^s \cdot \mathbb{Z}$, as required. □

Corollary 8.5. *Denominator* $D(\mathcal{E})$ *of essential subvariety of a binomial variety* $\hat{V}$ *is the bound* D *appearing in our abstract for* $m = 2$ *(and is a local integral invariant of* $\hat{V}$ *).*

9. **Appendix:** Length bound $1 + \log_2(\#\Gamma)$ on normalized Nash resolution with Γ dual graph of the minimal one of a minimal surface singularity - **by M. Spivakovsky.**

Let (S, ξ) be a normal surface singularity and $\pi : X \to S$ its minimal desingularization.

Definition 9.1. The set $\pi^{-1}(\xi) \subset X$ is called **the exceptional divisor** of the resolution of singularities π.

The exceptional divisor is a curve on X, which may, in general, be reducible. Let $\pi^{-1}(\xi) = \bigcup\limits_{i=1}^{n} E_i$ be its decomposition into irreducible components. Two basic combinatorial invariants are usually associated to the singularity (S, ξ): the dual graph and the intersection matrix. The **dual graph** has vertices $\{x_i\}_{1 \leq i \leq n}$, one for each irreducible exceptional curve E_i; two vertices x_i and x_j are connected by an arc if and only if $E_i \cap E_j \neq \emptyset$. The **intersection matrix** is the $n \times n$ matrix $(E_i.E_j)$. Since (S, ξ) is normal, Zariski's main theorem implies that the exceptional divisor, and hence also the dual graph, are connected. By a well-known theorem of Mumford and Grauert, the intersection matrix $(E_i.E_j)$ is negative definite.

Remark 9.1. We note the following consequences of the Mumford–Grauert theorem:

(1) We have $E_i^2 < 0$ for all $i \in \{1, \dots, n\}$.
(2) Take an index $i \in \{1, \dots, n\}$ and assume that $E_i \cong \mathbb{P}^1$. Then $E_i^2 \leq -2$. Indeed, if we had $E_i^2 = -1$ then such an exceptional curve could be contracted to a non-singular point by Castelnuovo's criterion, which would contradict the minimality of the desingularization π.
(3) There exists a cycle of the form

$$Z = \sum_{i=1}^{n} m_i E_i, \tag{9.1}$$

such that all the m_i are strictly positive integers and $Z.E_i \leq 0$ for all $i \in \{1, \dots, n\}$.

Among all the cycles Z satisfying (9.1), we can choose one which is *componentwise* minimal. Such a cycle is uniquely determined by the intersection matrix; it is called **the fundamental cycle** of the singularity (S, ξ).

Definition 9.2. The singularity (S, ξ) is called **minimal** if $E_i \cong \mathbb{P}^1$ for all $i \in \{1, \dots, n\}$, the intersections $E_i \cap E_j$ are transverse (whenever $E_i \cap E_j \neq \emptyset$), the dual graph of (S, ξ) is simply connected and the fundamental cycle Z is reduced (that is, $m_i = 1$ for all $i \in \{1, \dots, n\}$).

For more information on minimal singularities, we refer the reader to the article [9] by Janos Kollar where they were originally defined.

Definition 9.3. The singularity (S, ξ) is **a cyclic quotient** if each exceptional curve E_i intersects at most two other exceptional curves.

It follows easily from the definitions and Remark 9.1 (2) that every cyclic quotient singularity is minimal. The cyclic quotient singularities are precisely the toric ones among normal surface singularities (that is, they are precisely those normal surface singularities which can be defined by a binomial ideal in the ambient space). As the name suggests, they are also characterized by the fact that they can be obtained as quotients of a non-singular point by the action of a finite cyclic group.

Let (S, ξ) be a minimal singularity. For a graph Γ, the notation $\#\Gamma$ will stand for the number of vertices of Γ. For example, if Γ is the dual graph of ξ, we have $\#\Gamma = n$.

Theorem 9.2. ([13], Lemma 2.5, p. 442) Let $\sigma : S' \to S$ denote the normalized Nash blowing up of S, let ξ' be a singular point of S' and Γ' its dual graph. Then (S', ξ') is also a minimal singularity and

$$\#\Gamma' \leq \frac{n}{2}. \tag{9.2}$$

This bound is sharp in the sense that there are many examples for which equality holds in (9.2).

The simplest example of equality in (9.2) is the following. Let (S, ξ) be the A_n singularity with n even. This is the singularity defined in the three dimensional space by the equation $xy - z^{n+1}$. It can be obtained as the quotient of the two-dimensional space with coordinates (u, v) by the cyclic group action $(u, v) \to (\zeta u, \zeta^{-1} v)$, where ζ is the n-th root of unity. The dual graph of this singularity consists of n vertices, arranged in a straight line. The intersection matrix is given by

$$E_i^2 = -2, \quad i \in \{1, \dots, n\}; \tag{9.3}$$

$$E_i.E_{i+1} = 1 \quad \text{for } i \in \{1, \dots, n-1\} \tag{9.4}$$

$$E_i.E_j = 0 \quad \text{for all the other choices of } i, j \in \{1, \dots, n\}. \tag{9.5}$$

As is shown in [6], the normalized Nash blowing S' up of (S, ξ) has two singular points ξ_1, ξ_2 of multipliciy three, and the dual graph of each of the singularities (S', ξ_1), (S', ξ_2) has $\frac{n}{2}$ vertices.

Corollary 9.3. The singularity (S, ξ) is resolved after at most $[\log_2 n]+$ 1 normalized Nash blowings up.

Proof of the Corollary: Let $l = [\log_2 n]+1$. Consider the sequence

$$S_l \xrightarrow{\sigma_l} S_{l-1} \xrightarrow{\sigma_{l-1}} \ldots \xrightarrow{\sigma_2} S_1 \xrightarrow{\sigma_1} S$$

of normalized Nash blowing up. We claim that S_l is non-singular. To see this, we will assume that S_l contains a singular point ξ_l and deduce a contradiction. Let ξ_i denote the image of ξ_l in S_i, $0 \leq i \leq l$ (we adopt the convention that $S_0 = S$ and $\xi_0 = \xi$). Let n_i denote the number of vertices in the dual graph of ξ_i. Since ξ_l is assumed to be singular, we have $n_l \geq 1$. By Theorem 9.2 and descending induction on i, we obtain $n_i \geq 2^{l-i}$ so, in particular, $n \geq 2^l$, that is, $l \leq \log_2 n$. This contradicts the definition of l. □

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References

1. E. Bierstone and P .D. Milman, *Desingularization of toric and binomial varieties*, J. Algebraic Geom., **15** (2006), no. 3, 448–486.
2. G. B. Dantzig and M. N. Thapa, *Linear Programming: 2: Theory and Extensions*, Springer, 1997.
3. D. Eisenbud, *Commutative algebra*, Springer, New York, 1995
4. M. Frumkin, *An application of modular arithmetic to the construction of algorithms for the solution of systems of linear equations*, Soviet Math. Dokl., **229** (1976), no. 5, 1067–1070.
5. W. Fulton, *Introduction to toric varieties*, Princeton University Press, 1993.
6. G. Gonzalez-Sprinberg, *Résolution de Nash des points doubles rationnels.* Ann. Inst. Fourier, **32**, (1982), no. 2, 111–178
7. A. Grothendieck, *Eléments de géometrie algébrique (rédigés avec la collaboration de Jean Dieudonné) I. Le langage des schémas.* Publ. Math. I.H.E.S., **4**, (1960), 5–228.
8. H. Hironaka, *On Nash blowing-up*, in Arithmetic and Geometry II, Birkhäuser, 1983, 103–111.
9. J. Kollár, *Toward moduli of singular varieties.* Compositio Mathematica, **56**, (1985), no. 3, 369–398.
10. J. Lipman, *On the Jacobian ideal of the module of differentials*, Proc. Amer. Math. Soc., **21** (1969), 422–426.
11. A. Nobile, *Some properties of Nash blowing-up*, Pacific J. Math., **60** (1975), 297–305.

12. A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, 1986.
13. M. Spivakovsky, *Sandwiched singularities and desingularization of surfaces by normalized Nash transformations*, Ann. of Math. (2) **131** (1990), no. 3, 411–491.
14. B. Sturmfels, *Groebner bases and convex polytopes*, University Lecture Series, 8. American Mathematical Society, Providence, RI, 1996.

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