PURELY LOG TERMINAL BLOW-UPS OF INDEX 2

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Abstract. The purely log terminal blow-ups of index 2 of tree-dimensional terminal singularities were classified.

Classification of singularities, which appear in Log Minimal Models Programm is one of the most fundamental and hard to do problem in modern birational geometry. In present paper we continue our studying of three-dimensional terminal singularities by describing purely log terminal blow-ups which was started in [12].

In this paper purely log terminal blow-ups of index 2 of three-dimensional terminal singularities are classified.

1. Basic definitions and results

All varieties are defined over $\mathbb{C}$. Basic definitions are contained in [4], [7] and [6].

Definition 1.1. Let X be a normal log canonical algebraic variety, $f\colon Y \to X$ be some extraction. Let an exceptional locus of f is consists just from one irreducible divisor E ($\mathrm{Exc}(f) = E$). Then $f\colon (Y, E) \to X$ is called *purely log terminal blow-up*, if divisor $K_Y + E$ is purely log terminal and $-E$ is f-ample.

Remark 1.2. In the definition 1.1 we need the divisor E to be a $\mathbb{Q}$-Cartier. So, variety Y has $\mathbb{Q}$-Gorenstein singularities.

Remark 1.3. We have the following properties of purely log terminal blow-ups:

(1) If X has log terminal singularities, then divisor $-(K_Y + E)$ is f-ample. It follows from numerical equivalence
$K_Y + E \equiv (a(E, 0) + 1)E$ over X.

(2) If variety X has strictly log canonical singularities then $a(E, 0) = -1$.

(3) [7] If variety X is $\mathbb{Q}$-factorial, then Y is also $\mathbb{Q}$-factorial and $\rho(Y/X) = 1$. Here we see that in the definition 1.1 in $\mathbb{Q}$-factorial case we not need to ask about relative ampleness of divisor

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$-E$, because it always holds. Note that every component of exceptional locus of every birational contraction to a $\mathbb{Q}$-factorial point has codimension one.

(4) [7] Let $f_i \colon (Y_i, E_i) \to (X \ni P)$ be two purely log terminal blow-up. If E_1 and E_2 defines one discrete valuation of function field $\mathcal{K}(X)$, then blow-ups f_1 and f_2 are isomorphic.

Definition 1.4. Consider a cyclic quotient singularity $X := \mathbb{C}^n/\mathbb{Z}_m(a_1, \dots, a_n)$, where $a_i \in \mathbb{N}$ and $\gcd(a_1, \dots, a_n) = 1$ (the case $m = 1$, i. e. $X \simeq \mathbb{C}^n$, is also possible). Let $x_1, \dots, x_n$ be eigen-coordinates in $\mathbb{C}^n$, for $\mathbb{Z}_m$. *The weighted blow-up* of X with weights $a_1, \dots, a_n$ is a projective birational morphism $f \colon Y \to X$ such that Y is covered by affine charts $U_1, \dots, U_n$, where

$$U_i = \mathbb{C}^n_{y_1,\dots,y_n}/\mathbb{Z}_{a_i}(-a_1, \dots, \underset{\substack{\uparrow \\ i}}{m}, \dots, -a_n).$$

The coordinates in X and in U_i are related by

$$x_i = y_i^{a_i/m}, \qquad x_j = y_j y_i^{a_j/m}, \quad j \neq i.$$

The exceptional set E of f is an irreducible divisor and $E \cap U_i = \{y_i = 0\}/\mathbb{Z}_{a_i}$. The morphism $f \colon Y \to X$ is toric, i.e. there is an equivariant natural action of $(\mathbb{C}^*)^n$. It is easy to show that E is the weighted projective space $\mathbb{P}(a_1, \dots, a_n)$ and $\mathcal{O}_E(bE) = \mathcal{O}_{\mathbb{P}}(-mb)$, if b is divisible by $\operatorname{lcm}(a_1, \cdots, a_n)$ (and then bE is a Cartier divisor).

Note that the blow-up constructed above depends on a choice of numbers $a_1, \dots, a_n$, and not just on their values $\mod m$.

Let X be a hypersurface in $\mathbb{C}^n$. By *weighted blow-up* of X with weights $(a_1, \dots, a_n)$ we will mean the restriction of the weighted blow-up of $\mathbb{C}^n$ with weights $(a_1, \dots, a_n)$ on X.

2. Birational contractions of index 2

Here we will study some properties of 2-Cartier divisors which can be contracted to terminal singularities

Proposition 2.1. *Let $f : X \to Z$ be some divisorial contraction, such that its exceptional locus is a irreducible and reduced 2-Cartier Weil divisor S, $\dim X = \dim Z = 3$, $K_X + S$ is purely log terminal, Z is a germ of terminal singularity $P \in Z$. Then the following is true:*

(1) *If Z is Gorenstein, then X is canonical;*
(2) *Discrepancy of S is minimal over Z.*

Proof. Let consider a hyperplane section $H \in |-K_Z|$. The pair K_Z+H is purely log terminal, since H has just canonical singularities. We have $K_X + H' + S = f^*(K_Z + H)$. Since H' is Cartier and the pair $K_Z + H$ is purely log terminal, then H does not pass through log terminal singularities of X. The property 1) holds since H is in a general position.

We have $K_X = f^*K_Z + aS$, where $a > 0$ because Z is terminal. Let there is a divisor E with a discrepancy $b < a$ over Z. Then there is a morphism $\pi : Y \to X \to Z$ such that E is realised on Y. We have $b = b_X + ak > a$, where $k \in \frac{1}{2}\mathbb{Z}$ is a multiplicity of S in a center of E on X, $b_X > 0$. Let consider two cases:

(1) Z is Gorenstein;
(2) Z is non-Gorenstein.

In the first case $a \in \mathbb{Z}$. If $a = 1$, then we are done. Otherwise, $a \geq 2$. We get $b = b_X + ak > 1$ and contradiction with a fact, that there is a divisor with a discrepancy 1.

In the second case $a = \frac{i}{m}$ for some $i \in \mathbb{Z}$, m is an index of Z. In the same way if $a = \frac{1}{m}$, then we are done. Otherwise, $a = \frac{i}{m}$ for $i \geq 2$. Then we also have $b = b_X + ak > \frac{1}{m}$. Contradiction with a fact that there is a divisor with discrepancy $\frac{1}{m}$. □

3. Z is a germ of Gorenstein singularity

3.1. **cA case.** If Z is a germ of a cA point (it can be smooth also), then the following variants (S, Z, f) are possible. We will see that f always will be weighted blow up and we will write just it's weights.

(1) $(\mathbb{P}^2, \mathbb{C}^3, (2,1,1))$ - smooth case;
(2) $((ty + g_3(x,z) = 0) \subset \mathbb{P}(2,1,1,1), (ty + g_{\geq 3}(x,z) = 0) \subset \mathbb{C}^4, (2,1,1,1))$ - cA$_2$ case;
(3) $((ty + g_4(x,z) = 0) \subset \mathbb{P}(2,2,1,1), (ty + g_{\geq 4}(x,z) = 0) \subset \mathbb{C}^4, (2,2,1,1))$ - cA$_3$ case.

It follows from [12] that there are no other variants. In fact, for cA$_n$ singularities, with $n \geq 4$ all divisors with minimal discrepancy have index ≥ 3. Over cA$_1$ singularities all divisors with minimal discrepancy are Cartier.

3.2. **cD$_4$ case.** To deal with this case we need the classification (up to an analytic isomorphism) of normal forms of cubi curves in $\mathbb{P}^2$ (see. [1])

Theorem. Here are normal forms (relative to GL_3) of cubic curves in $\mathbb{P}^2$:

(1) $g_3 = y^3$;

(2) $g_3 = z^2y$;
(3) $g_3 = zy(z+y)$;
(4) $g_3 = xyz$;
(5) $g_3 = (x^2+yz)y$;
(6) $g_3 = (x^2+yz)x$;
(7) $g_3 = (y^2z+x^3)$;
(8) $g_3 = (y^2z+x^3+x^2z)$;
(9) $g_3 = (x^3+y^3+z^3)+axyz,\ a^3+27 \neq 0$.

Remark. Note that Z is defined by the equation $(t^2 + g_3(x,y,z) + g_{\geq 4}(x,y,z) = 0) \subset \mathbb{C}^4$

3.2.1. *Forms 1 and 2.* This cases are taken to singularities of cD_n type, for $n \geq 5$, which will be considered later (see 3.3).

3.2.2. *Form 3.* Let consider at first the case when the equation of singularity can be written in the following way $t^2 + z^2y + y^3 + z^3 + g_{wt\geq 6}(x,y,z) = 0$, where wt corresponds to the following weight $(wtt, wty, wtz, wtx) = (3,2,2,1)$.

Remark. Note, that the form $zy(z+y)$ always can be brought to the form $z^3+y^3+yz^2$ (and vice versa), because the homogeneous polynomial $z^3+y^3+yz^2$ can be factorized into linear factors (by the main algebra theorem) and third multiplier is the sum of the first two.

Weighted blow-up f with weights $(3,2,2,1)$ of this singularities will be purely log terminal blow-up of index 2. Indeed, let $h : \overline{X} \to Z$ be this blow-up, E be h-exceptional divisor. Then $\overline{X}$ is covered by four affine cards U_1, U_2, U_3, U_4:

$$U_1 = \{1 + yz^2 + y^3 + z^3 + \cdots = 0\}/\mathbb{Z}_3(1,1,1,-1),$$
$$U_2 = \{t^2 + z^2 + 1 + z^3 + \cdots = 0\}/\mathbb{Z}_2(1,1,1,1),$$
$$U_3 = \{t^2 + y + y^3 + 1 + \cdots = 0\}/\mathbb{Z}_2(1,1,1,1),$$
$$U_4 = \{t^2 + yz^2 + z^3 + y^3 + \cdots = 0\} \subseteq \mathbb{C}^4.$$

Variety $\overline{X}$ has three singularity $\frac{1}{2}(1,1,1)$, and the exceptional divisor E, which is 2-Cartier passes through them.

There are no other purely log terminal blow-ups of index 2, because there are no divisors, which passes through just one singularity of the variety $\overline{X}$ and does nor pass through the others singularities.

If we can not to reduce the equation of our singularity to the equation $t^2 + z^2y + y^3 + z^3 + g_{wt\geq 6}(x,y,z) = 0$, then there are no purely log terminal blow-ups of index 2. Indeed, in this case $wt(g) \leq 5$:

(1) if $wt(g) = 4$, then x^4 appears in $g_{wt=4}$. Only the following variant is possible:
   (a) $((yz^2 + z^3 + y^3 = 0 \subset \mathbb{P}(2,1,1,1)), (t^2 + yz^2 + z^3 + y^3 + x^4 + g_{\geq 4}(x,y,z) = 0 \subset \mathbb{C}^4, (2,1,1,1))$;
(2) if $wt(g) = 5$, then one of the monomials can appear x^5, x^3y, x^3z in $g_{wt=5}$. The only variant is possible:
   (a) $((yz^2 + z^3 + y^3 = 0 \subset \mathbb{P}(2,1,1,1)), (t^2 + yz^2 + z^3 + y^3 + g_{\geq 5}(x,y,z) = 0 \subset \mathbb{C}^4, (2,1,1,1))$.

In all of the variants there are no plt blow-ups of index 2, because singularities of the exceptional divisors are worth than log terminal, and since there are no other divisors with discrepancy 1, then relative anti ampleness gives us the fact (see [18]).

3.2.3. *Form 4.* In the case there are plt blow-ups of index 2. In fact we can blow-up Z with weights $(2,2,1,1)$ $h : \overline{X} \to Z$. Only one divisor with discrepancy $a = 2+2+1+1-4-1 = 1$ will appear. Let check singularities of $\overline{X}$. Variety $\overline{X}$ is covered by four affine cards U_1, U_2, U_3, U_4.

$$U_1 = \{1 + y^2z + x^3 + \cdots = 0\}/\mathbb{Z}_2(1,1,1,1),$$
$$U_2 = \{t^2 + xz + \cdots = 0\}/\mathbb{Z}_2(0,1,1,1),$$
$$U_3 = \{t^2 + xy + \cdots = 0\} \subseteq \mathbb{C}^4,$$
$$U_4 = \{t^2 + yz + \cdots = 0\} \subseteq \mathbb{C}^4.$$

There are two non-Gorenstein singularities on $\overline{X}$, both are placed in the card U_2. One is cA/2 point, and another is $\frac{1}{2}(1,1,1)$ point. The exceptional divisor E of morphism h is defined by $E \simeq (t^2 + xyz + g_{wt=4} = 0) \subset \mathbb{P}(2,2,1,1)$.

Hayakawa showed that there just one divisor with minimal discrepancy over cA/2 point ([6]). This divisor has discrepancy 1 over Z. It was showed by Kawamata, that over another singularity there is just one divisor with minimal discrepancy. It also has discrepancy 1 over Z. All this divisors can be obtained by weighted blow-ups of Z with weights $(2,1,2,1)$ and $(2,1,1,2)$.

3.2.4. *Form 5.* In the case there are plt blow-ups of index 2. In fact we can blow-up Z with weights $(2,2,1,1)$ $h : \overline{X} \to Z$. Only one divisor with discrepancy $a = 2+2+1+1-4-1 = 1$ will appear. Let check singularities of $\overline{X}$. Variety $\overline{X}$ is covered by four affine cards

U_1, U_2, U_3, U_4.

$$U_1 = \{1 + x^2y + \cdots = 0\}/\mathbb{Z}_2(1,1,1,1),$$
$$U_2 = \{t^2 + x^2 + yz + \cdots = 0\}/\mathbb{Z}_2(0,1,1,1),$$
$$U_3 = \{t^2 + x^2y + \cdots = 0\} \subseteq \mathbb{C}^4,$$
$$U_4 = \{t^2 + y + \cdots = 0\} \subseteq \mathbb{C}^4.$$

There is just one non-Gorenstein point on $\overline{X}$ and it is placed in the card U_2 (it is cA/2 point). The exceptional divisor E of morphism h is defined by $E \simeq (t^2 + x^2y + g_{wt=4} = 0) \subset \mathbb{P}(2,2,1,1)$. Hayakawa showed that there just one divisor with minimal discrepancy over cA/2 point ([6]). This divisor has discrepancy 1 over Z. It can not be realized as plt blow-up over Z. Indeed, it is easy to see that if for some pair $(\overline{X}, D_X)$ it realizes with discrepancy -1, then if we will consider them on Z it will also have discrepancy -1. But in this case the divisor E will also have discrepancy -1. So it will be not a plt blow-up.

3.2.5. *Form 6.* In the case there are plt blow-ups of index 2. In fact we can blow-up Z with weights $(2,1,1,2)$ $h: \overline{X} \to Z$. Only one divisor with discrepancy $a = 2 + 2 + 1 + 1 - 4 - 1 = 1$ will appear. Let check singularities of $\overline{X}$. Variety $\overline{X}$ is covered by four affine cards U_1, U_2, U_3, U_4.

$$U_1 = \{1 + x^3t^2 + \cdots = 0\}/\mathbb{Z}_2(1,1,1,1),$$
$$U_2 = \{t^2 + x^3y^2 + zx + \cdots = 0\} \subseteq \mathbb{C}^4,$$
$$U_3 = \{t^2 + x^3z^2 + \cdots = 0\} \subseteq \mathbb{C}^4,$$
$$U_4 = \{t^2 + x^2 + yz + \cdots = 0\}/\mathbb{Z}_2(0,1,1,1).$$

There is just one non-Gorenstein point on $\overline{X}$ and it is placed in the card U_2 (it is cA/2 point). The exceptional divisor E of morphism h is defined by $E \simeq (t^2 + xyz + g_{wt=4} = 0) \subset \mathbb{P}(2,1,1,2)$. Hayakawa showed that there just one divisor with minimal discrepancy over cA/2 point ([6]). This divisor has discrepancy 1 over Z. It can not be realized as plt blow-up over Z. Indeed, it is easy to see that if for some pair $(\overline{X}, D_X)$ it realizes with discrepancy -1, then if we will consider them on Z it will also have discrepancy -1. But in this case the divisor E will also have discrepancy -1. So it will be not a plt blow-up.

3.2.6. *Form 7.* In this case there are no plt blow-ups of index 2. Indeed, let blow-up Z with weights $(2,1,1,1)$ $h: \overline{X} \to Z$. One divisor with discrepancy $a = 2 + 1 + 1 + 1 - 3 - 1 = 1$ will appear. Let check singularities on $\overline{X}$. Variety $\overline{X}$ is covered by four affine cards U_1, U_2, U_3, U_4.

$$U_1 = \{t + y^2z + x^3 + \cdots = 0\}/\mathbb{Z}_2(1, 1, 1, 1),$$
$$U_2 = \{t^2y + z + x^3 + \cdots = 0\} \subseteq \mathbb{C}^4,$$
$$U_3 = \{t^2z + y^2 + x^3 + \cdots = 0\} \subseteq \mathbb{C}^4,$$
$$U_4 = \{t^2x + y^2z + 1 + \cdots = 0\} \subseteq \mathbb{C}^4.$$

There are no non-Gorensten singularities on $\overline{X}$, so the exceptional divisor E is unique one with discrepansy 1. On the other hand E can be defined by the equation $E \simeq (y^2z + x^3 = 0) \subset \mathbb{C}^4$. This surface has worse then log terminal singularities and by 1.3(4) we are done.

3.2.7. *Form 8.* In this case there are no plt blow-ups of index 2. Indeed, let blow-up Z with weights $(2, 1, 1, 1)$ $h : \overline{X} \to Z$. One divisor with discrepancy $a = 2 + 1 + 1 + 1 - 3 - 1 = 1$ will appear. Let check singularities on $\overline{X}$. Variety $\overline{X}$ is covered by four affine cards U_1, U_2, U_3, U_4.

$$U_1 = \{t + y^2z + x^3 + x^2z + \cdots = 0\}/\mathbb{Z}_2(1, 1, 1, 1),$$
$$U_2 = \{t^2y + z + x^3 + x^2z + \cdots = 0\} \subseteq \mathbb{C}^4,$$
$$U_3 = \{t^2z + y^2 + x^3 + x^2 + \cdots = 0\} \subseteq \mathbb{C}^4,$$
$$U_4 = \{t^2x + y^2z + 1 + z + \cdots = 0\} \subseteq \mathbb{C}^4.$$

There are no non-Gorensten singularities on $\overline{X}$, so the exceptional divisor E is unique one with discrepansy 1. On the other hand E can be defined by the equation $E \simeq (y^2z + x^3 = 0) \subset \mathbb{C}^4$. This surface has worse then log terminal singularities and by 1.3(4) we are done.

3.2.8. *9.* In this case there are no plt blow-ups of index 2. Indeed, let blow-up Z with weights $(2, 1, 1, 1)$ $h : \overline{X} \to Z$. One divisor with discrepancy $a = 2+1+1+1-3-1 = 1$ will appear. Let check singularities on $\overline{X}$. Variety $\overline{X}$ is covered by four affine cards U_1, U_2, U_3, U_4.

$$U_1 = \{t + y^3 + x^3 + z^3 + axyz + \cdots = 0\}/\mathbb{Z}_2(1, 1, 1, 1),$$
$$U_2 = \{t^2y + z^3 + x^3 + 1 + axz + \cdots = 0\} \subseteq \mathbb{C}^4,$$
$$U_3 = \{t^2z + y^3 + x^3 + 1 + axy + \cdots = 0\} \subseteq \mathbb{C}^4,$$
$$U_4 = \{t^2x + y^3 + 1 + z^3 + ayz + \cdots = 0\} \subseteq \mathbb{C}^4.$$

There are no non-Gorensten singularities on $\overline{X}$, so the exceptional divisor E is unique one with discrepansy 1. On the other hand E can be defined by the equation $E \simeq (x^3 + y^3 + z^3 + axyz = 0) \subset \mathbb{C}^4$. This surface has worse then log terminal singularities and by 1.3(4) we are done.

3.3. **$\mathbf{cD}_n$ case for $n \geq 5$.** In this case plt blow-ups of index 2 always exist. We can blow up $f : \overline{X} \to Z$ with weights $(wt(t), wt(y), wt(z), wt(x)) = (2, 2, 1, 1)$. Variety $\overline{X}$ is covered by four affine cards $U_1,\ U_2,\ U_3,\ U_4$, where U_i is a hypersurface in $\mathbb{C}^4$ defined by

$$U_1 = \{1 + yz^2 + \cdots = 0\}/\mathbb{Z}_2(1, 0, 1, 1),$$
$$U_2 = \{t^2 + z^2 + \cdots = 0\}/\mathbb{Z}_2(0, 1, 1, 1),$$
$$U_3 = \{t^2 + y + \cdots = 0\}/ \subseteq \mathbb{C}^4,$$
$$U_4 = \{t^2 + yz^2 + \cdots = 0\} \subseteq \mathbb{C}^4.$$

The exceptional divisor E of morphism f has discrepancy 1 and is defined by the equation:

$$E = \{t^2 + yz^2 + g_{wt=4}(x, y, z) = 0\} \subseteq \mathbb{P}(2, 2, 1, 1).$$

$\overline{X}$ has terminal singularities in cards $U_1,\ U_2,\ U_3$, and at worse canonical in the card U_4.

We will need the following definition to define the number of divisors with minimal discrepancy.

Definition 3.4. Let define the sequence of weights σ_i in the following way:

(1) $\sigma_0(t, y, z, x) = (2, 2, 1, 1)$;
(2) $\sigma_1(t, y, z, x) = (3, 2, 2, 1)$;
(3) $\sigma_j(t, y, z, x) = (j + 2, 2, j + 1, 1)$.

Remark 3.5. Note, that every such a blow-up is plt and if $wt(g) = 2j+4$, then an irreducible divisor F with discrepancy 1 will appear. Indeed, $F \simeq (t^2 + yz^2 + g_{min} = 0 \subset \mathbb{P}(j + 2, 2, j + 1, 1))$, so its irreducibility is obvious. Discrepancy $a(F) = j + 2 + 2 + j + 1 + 1 - 2j - 4 - 1 = 1$. For σ_0 it was already checked.

Proposition 3.6. *Let consider a set of weighted blow-ups $w_i : Z_i \to Z$ with weights σ_i. Then Z_i has terminal singularities iff i has the following properties:*

(1) $\sigma_i wt(g) + 1 \geq \sigma_{i+1} wt(g)$.
(2) $\sigma_i wt(g) \geq 2i + 4;\ \sigma_{i+1} wt(g) < 2i + 6$.

Remark 3.7. The first property means that the monomial which has a minimal weight g_{min} (with weight σ_i) can not be written as $z^2 h$, where h is a polynomial. The second property means in some sense maximality of this blow-up.

Proof. Variety Z_i is covered by four affine cards $U_1,\ U_2,\ U_3,\ U_4$, where U_i is a hypersurface in $\mathbb{C}^4$ defined by

$$U_1 = \{1 + yz^2 + \cdots = 0\}/\mathbb{Z}_{i+2}(1, -2, -i-1, -1),$$
$$U_2 = \{t^2 + z^2 + \cdots = 0\}/\mathbb{Z}_2(i+2, 1, -i-1, -1),$$
$$U_3 = \{t^2 + y + \cdots = 0\}/\mathbb{Z}_{i+1}(-i-2, -2, 1, -1)$$
$$U_4 = \{t^2 + yz^2 + \cdots = 0\} \subseteq \mathbb{C}^4.$$

In cards $U_1,\ U_2,\ U_3$ singularities are at worst terminal (in the card U_1 variety is smooth, in the card U_2 there is a singularity of a type cA/2 or cAx/2, in the card U_3 there is a singularity of a type $\frac{1}{i+1}(1, -1, 1)$). If in the card U_4 singularities are isolated, then they are at worst terminal cDv points.

At first let check that if i is satisfied to the conditions (1),(2), then singularities on Z_i are isolated. Properties (1) and (2) means that there is a monomial $\alpha = x^a y^b z^c$, where $c \leq 1$, such that $\sigma_k wt(\alpha) = \sigma_k wt(g)$ and $\sigma_k wt(\alpha) < 2k + 4$ for every $k > i$. Then, if $\sigma_i wt(\alpha) = 2i + 4$, then U_4 will be defined by $U_4 = \{t^2 + yz^2 + y^b z^c + \cdots = 0\} \subseteq \mathbb{C}^4$. In this case singularities are isolated. If $\sigma_i wt(\alpha) = 2i + 5$ and $c \neq 0$, then this i does not satisfied to condition (2), and changing i to $j = i + 1$, we will get the desired blow-up. If $c = 0$, then U_4 will be defined by $U_4 = \{t^2 + yz^2 + xy^b + \cdots = 0\} \subseteq \mathbb{C}^4$. In this case singularities are isolated.

For $j < i$, Z_j will have strictly canonical singularities. It is sufficient to show that in the card U_4 singularities will be non-isolated. Indeed, U_4 will be defined by $U_4 = \{t^2 + yz^2 + x^d(h(x, y, z)) = 0\} \subseteq \mathbb{C}^4$, where $d \geq 2$. We get non-isolated singularity which is defined in general point by $A_n \times \mathbb{C}^1$ for some n, y is a parameter. □

Proposition 3.8. *Let $\sigma_i : Z_i \to Z$ be the weighted blow-up (from definition 3.4), with maximal i. Then over Z there are at most $i+3$ divisors with minimal discrepancy. We can realize just $i+1$ divisors as a purely log terminal blow-up in canonical category. These plt blow-ups have index 2 only for $i = 0, 1$.*

Remark 3.9. Z_i has terminal singularities only for maximal i.

Proof. Let $\sigma_i : Z_i \to Z$ be a weighted blow-up in a terminal category (i.e. Z_i has terminal singularities) with an exceptional divisor E_i. Z_i is covered by four affine cards $U_1,\ U_2,\ U_3,\ U_4$, where U_i is hypersurface in $\mathbb{C}^4$ defined by the equation

$$U_1 = \{1 + yz^2 + \cdots = 0\}/\mathbb{Z}_{i+2}(1, -2, -i-1, -1),$$
$$U_2 = \{t^2 + z^2 + \cdots = 0\}/\mathbb{Z}_2(i+2, 1, -i-1, -1),$$
$$U_3 = \{t^2 + y + \cdots = 0\}/\mathbb{Z}_{i+1}(-i-2, -2, 1, -1)$$
$$U_4 = \{t^2 + yz^2 + \cdots = 0\} \subseteq \mathbb{C}^4.$$

In the card U_1 variety is smooth, in the card U_2 there is a singularity of a type either cAx/2 (generally) or cA/2, in the card U_3 there is singularity of a type $\frac{1}{i+1}(1,-1,1))$, in the card U_4 there are isolated cDv points.

Let count the number of divisors with discrepancy 1 over Z. Such a divisors may lye just over non-Gorenstein singularities. Let define over O_2 (correspondingly O_3) the singular point in a card U_2 (correspondingly in U_3).

Let consider the point O_2. It has type either cAx/2 or cA/2. Hayakawa has showed that over cAx/2 there are at most two divisors with discrepancy $\frac{1}{2}$, and over cA/2 just one ([6]).

There are i divisors with discrepancy 1 (over Z) over point O_3. We can "economically" resolve it (see [2]). That is there is a projective morphism $\nu : \bar{Y} \to Z_i$ such that

$$K_{\bar{Y}} = \nu^* K_{Z_i} + \sum_{i=1}^{a-1} \frac{i}{a} F_i,$$

where $\sum F_i$ is the exceptional divisor over O_3. E_i is defined by equation $z = 0$ (near O_3) so we have:

$$\nu^*(E_i) = \nu^{-1}(E_i) + \sum_{i=1}^{a-1} \frac{a-i}{a} F_i.$$

We also have:

$$a(F_i, X) = \frac{i}{a} + \frac{a-i}{a} = 1,$$

So, totally, there are at most $i+3$ divisors with discrepancy 1 over Z (and at least $i+2$).

We can get all the divisors with minimal discrepancy over O_3 as plt blow-up. It is easy to see that all exceptional surfaces will have just log terminal singularities, so all the morphisms will be plt blow-ups. In the card U_4 varieties Z_i will have non-isolated singularities for almost all i (except the maximal i). But exceptional divisor on variety Z_i will have index $i+1$. So such a morphisms will be plt blow-ups with index 2 only for $i = 0, 1$.

It is easy to see that there are no divisors over point O_2 which can be realized as plt blow-up of index 2. If there is such a plt blow-up with an exceptional divisor S, then there is a $\mathbb{Q}$-divisor D, such that the pair (Z, D) realize S with discrepancy -1. If check the image of D on Z_0, we get that it a sum of Cartier divisors with rational coefficients. It is easy to see that if there are such a divisors which realize a divisor with discrepancy -1, then divisor E_0 will also appear with discrepancy -1. $\square$

3.10. **cEn case.** It was showed by Markushevich [3], that Z has type cEn iff $Z \simeq (t^2 + y^3 + az^4 + byz^3 + cz^5 + g_{\geq 6}(x, y, z) = 0) \subset \mathbb{C}^4$, where at least one of a, b, c is non zero. Let consider three cases:

(1) $a \neq 0$. In this case there are plt blow-ups of index 2. Indeed, let blow up $f : \overline{X} \to Z$ with weights $(wt(t), wt(y), wt(z), wt(x)) = (3, 2, 2, 1)$. Variety $\overline{X}$ is covered by four affine cards $U_1,\ U_2,\ U_3,\ U_4$, where U_i is a hypersurface in $\mathbb{C}^4$ defined by the equation

$$U_1 = \{1 + \cdots = 0\}/\mathbb{Z}_3(1, -2, -2, -1),$$
$$U_2 = \{t^2 + 1 + \cdots = 0\}/\mathbb{Z}_2(1, 1, 0, 1),$$
$$U_3 = \{t^2 + y^3 + z^2 + \cdots = 0\}/\mathbb{Z}_2(1, 0, 1, 1),$$
$$U_4 = \{t^2 + y^3 + \cdots = 0\} \subseteq \mathbb{C}^4.$$

The exceptional divisor E of morphism f has discrepancy 1 and defined by the equation:

$$E = \{t^2 + y^3 + g_{wt=6}(x, y, z) = 0\} \subseteq \mathbb{P}(3, 2, 2, 1).$$

To find other possible plt blow-ups we need to consider some cases. Let we have some weighted blow up $h : Y \to Z$ with weight σ, which realizes a log terminal surface with discrepancy 1. Then one of the following possibilities holds

(a) $\sigma wt(t^2) = \sigma wt(y^3) = \sigma wt(z^4) = \sigma wt(g)$, where g is the equation which defines Z;
(b) $\sigma wt(t^2) = \sigma wt(y^3) = \sigma wt(g) < \sigma wt(z^4)$;
(c) $\sigma wt(t^2) = \sigma wt(z^4) = \sigma wt(g) < \sigma wt(y^3)$;
(d) $\sigma wt(t^2) = \sigma wt(g) < \sigma wt(z^4)$ and $\sigma wt(g) < \sigma wt(y^3)$;

Remark 3.11. Note that t^2 must always have weight equal to the weight of g since we want to deal with plt blow-ups. Otherwise either the exceptional divisor will not have discrepancy 1 or the singularities of the exceptional surface will be worse than log terminal.

In the case a) $\sigma = (6, 4, 3, 1)$ if there are no monomials with σ-weight less than 12. Variety Y will be terminal, there are three non-Gorenstein singularities on Y: one of the type $\frac{1}{2}(1, 1, 1)$ and two of the type $\frac{1}{3}(1, -1, 1)$. So, this plt blow-up has index 3. In this case there are 6 divisors with discrepancy 1. If there are some monomials of σ-weight less than 12, then we are not in case a).

In the case b) we have just two possibilities for σ since we want to realize plt-blow-up of index 2: $\sigma = (3, 2, 2, 1)$, $\sigma = (6, 4, 3, 1)$. The second one has been considered in case a). The first one always exists and fits us properly.

In the case c) we have just one possibility $\sigma = (4, 3, 2, 1)$ (we will call the exceptional divisor E_2) if there are no monomials of σ-weight less than 8 and there is at least one monomial of degree at least 5 with σ-weight is equal to 8, i.e. one of the monomial $x^8, x^6z, x^5y, x^4z^2, x^2z^3$. If there is no such monomial then two surfaces will appear and it would not be a plt blow-up. Variety Y will be terminal, there will be two non-Gorenstein singularities on Y of a type $\frac{1}{2}(1, 1, 1)$, and totally 3 divisors with discrepancy 1. Now we will check whether they can be obtained as a plt-blow-up of index 2. Let blow up $f : X \to Z$ with weights $(3, 2, 2, 1)$, the exceptional divisor will be called E_1.

Remark 3.12. This weight does not belong to case c).

We see that there is a non-Gorenstein singularity of a type cA/2, and one canonical singularity $\mathrm{A}_1 \times \mathbb{C}$. So, there are totally 3 divisors with discrepancy 1. Divisor E_2 lyes over canonical singularity. The last divisor S with discrepancy 1 which lyes over cA/2 point can not be realized as a plt blow-up of index 2 because there is no divisor D such that the discrepancy $a(Z, D, S) = -1$ and $a(Z, D, E_1) > -1$.

In the case d) there are no variants for σ.

(2) $a = 0,\ b \neq 0$. Let consider some cases:
   (a) $wt(t^2) = wt(y^3) = wt(yz^3) = wt(g)$;
   (b) $wt(t^2) = wt(y^3) = wt(g) < wt(yz^3)$;
   (c) $wt(t^2) = wt(yz^3) = wt(g) < wt(y^3)$;
   (d) $wt(t^2) = wt(g) < wt(y^3)$ and $wt(t^2) = wt(g) < wt(yz^3)$.

In the case a) we have $\sigma = (9, 6, 4, 1)$ if the σ-weight of g will be 18. In this case the exceptional devisor will have index 4, it can not be plt blow-up of index 2.

In the case b) there are two possibilities $\sigma = (3,2,2,1)$, $\sigma = (6,4,3,1)$. The second variants does not fit us because the exceptional divisor has index 3. The first one fits us properly.

In the case c) there are two possibilities $\sigma = (5,4,2,1)$ and $\sigma = (7,5,3,1)$. In the first case the exceptional divisor has index 4 in the second case – index 5.

In the case d) there are no plt blow-ups of index 2.

(3) $a = b = 0, c \neq 0$. Let consider some cases:
   (a) $wt(t^2) = wt(y^3) = wt(z^5) = wt(g)$;
   (b) $wt(t^2) = wt(y^3) = wt(g) < wt(z^5)$;
   (c) $wt(t^2) = wt(z^5) = wt(g) < wt(y^3)$;
   (d) $wt(t^2) = wt(g) < wt(y^3)$ and $wt(t^2) = wt(g) < wt(z^5)$.

   In the case a) we have $\sigma = (15,10,6,1)$ if g has σ-weight 30. The exceptional divisor has index more than 2.

   In the case b) we have $\sigma = (3,2,2,1)$. This variant gives us plt blow-up of index 2.

   In the case c) we have $\sigma = (10,7,4,1)$ and $\sigma = (5,4,2,1)$. In both cases the exceptional divisor has index greater than 2.

   In the case d) no divisors with discrepancy 1 can be obtained as a plt blow-up of index 2.

Remark 3.13. There are no others plt blow-ups of index 2. It is easy to see that if there are such a divisors which realize a divisor with discrepancy -1, then some other divisor will also appear with discrepancy -1.

4. Z is a germ of non-Gorenstein point

It is follows from [6] that there are no plt blow-ups of index 2.

5. Main Theorem

Let us formulate the main theorem:

Theorem 5.1. *These morphisms are only purely log terminal blow-ups of index 2 of three-dimensional terminal singularities (singularity, weight).*

1. *In cA case there are only variants*
   (a) $(\mathbb{C}^3, (2,1,1))$ - *smooth case;*
   (b) $((ty + g_{\geq 3}(x,z) = 0) \subset \mathbb{C}^4, (2,1,1,1))$ - cA_2 *case;*
   (c) $((ty + g_{\geq 4}(x,z) = 0) \subset \mathbb{C}^4, (2,2,1,1))$ - cA_3 *case.*
2. *In cD case there are only variants*
   (a) *In cD_4 case there are variants:*
      (i) $(t^2 + zy(z+y) + \cdots = 0 \subset \mathbb{C}^4, (3,2,2,1))$;

(ii) $(t^2 + xyz + \dots = 0 \subset \mathbb{C}^4, (2,2,1,1)$, $(t^2 + xyz + \dots = 0 \subset \mathbb{C}^4, (2,1,2,1)$, $(t^2 + xyz + \dots = 0 \subset \mathbb{C}^4, (2,1,1,2)$;

(iii) $(t^2 + (x^2 + yz)y + \dots = 0 \subset \mathbb{C}^4, (2,2,1,1)$;

(iv) $(t^2 + (x^2 + yz)x + \dots = 0 \subset \mathbb{C}^4, (2,1,1,2)$.

(b) *In cD_n for $n \geq 5$ there are variants:*

(i) *(every singularity of this type, (2,2,1,1);*

(ii) *(if there are no monomials x^4, x^5, x^3z, x^3y, (3,2,2,1).*

(3) *In cE case there are only variants:*

(a) *In cE_6 case there are variants:*

(i) *(every singularity of this type, (3,2,2,1));*

(ii) *(if there are no monomials $x^8, x^6z, x^5y, x^4z^2, x^2z^3$, (4,3,2,1)).*

(b) *In cE_7 case the only variant is (every singularity, (3,2,2,1);*

(c) *In cE_8 case the only variant is (every singularity, (3,2,2,1).*

References

[1] *Arnold V.I., Varchenko A.N, Gusein-Zade S.M.* Singularities of differential maps, Moscow, "Nauka", (1982)

[2] *Reid M.* Young person's guide to canonical singularities, Proc. Symp. in Pure Math. **46** (1987) 343–416

[3] *Markushevich D.G.* Canonical singularities of three-dimensional hypersurfaces, Izv. AN SSSR. Ser. Math. **49** (1985), 2, 334-368

[4] *Kawamata Y.,Matsuda K.,Matsuki K.* Introduction to the minimal model program // Algebraic Geometry,Sendai. Adv. Stud. Pure Math. 1987. **10.** 283–360

[5] *Kawamata Y.* Divisorial contractions to 3-dimentional terminal quotient singularities, Higher-dimentional complex varieties (Trento, 1994), de Gruyter, 1996, 241-246

[6] *Hayakawa T.* Blowing Ups of 3-dimentional Terminal Singularities, Rims, Kyoto Univ., **35** (1999), 515-570

[7] *Prokhorov Yu. G.* Blow-ups of canonical singularities, Algebra. Proc. Internat. Conf. on the Occasion of the 90th birthday of A. G. Kurosh, Moscow, Russia, May 25-30, 1998, Yu. Bahturin ed., Walter de Gruyter, Berlin (2000), 301-317

[8] *Hayakawa T.* Blowing Ups of 3-dimentional Terminal Singularities II, Rims, Kyoto Univ., **36** (2000), 423-456

[9] *Cutkosky S.* Elementary contractions of Gorenstein threefolds, Math. Ann., **280** (1988), 521-525

[10] *Kawamata Y.* The minimal discrepancy of 3-fold terminal singularity, Appendix to Shokurov V.V.,3-fold log flips, Russian Acad. Sci. Izv. Math. **40** (1993), 1, 93-202

[11] *Kawakita M.* General elements in anticanonical systems of threefolds with divisorial contractions and applications to classification, preprint,. math.AG/0110050

[12] *Fedorov I.Yu.* Blow-ups of three-dimensional terminal singularities: cA case, Math. Notes, 2002, v. 71, 3, 400-407

[13] *Shokurov V. V.* 3-folds log flips, Russian Acad. Sci. Izv. Math. 1993, v. 40, pp. 93-202
[14] *Shokurov V. V.* Complements on surfaces, J. of Math. Sci. 2000, v. 102, no. 2, pp. 3876-3932
[15] *Tziolas N.* Terminal 3-fold divisorial contractions of a surface to a curve I, preprint, arXiv:math.AG/0110074
[16] *Qiang Ye* On Gorenstein log del Pezzo surfaces, preprint, arXiv:math.AG/0109223
[17] *. A.* Purely log terminal blow-ups, Math. Notes 2001, v.69, No. 6, pp. 814-819
[18] *Kollar J. et al* Flips and abundance for algebraic threefolds, Astérisque 211,(1992).

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