Cauchy Type Integrals of Algebraic Functions

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Abstract We consider Cauchy type integrals $I(t) = \frac{1}{2\pi i}\int_\gamma \frac{g(z)dz}{z-t}$ with $g(z)$ an algebraic function. The main goal is to give constructive (at least, in principle) conditions for $I(t)$ to be an algebraic function, a rational function, and ultimately an identical zero near infinity. This is done by relating the monodromy group of the algebraic function g, the geometry of the integration curve γ, and the analytic properties of the Cauchy type integrals. The motivation for the study of these conditions is provided by the fact that certain Cauchy type integrals of algebraic functions appear in the infinitesimal versions of two classical open questions in Analytic Theory of Differential Equations: the Poincaré Center-Focus problem and the second part of the Hilbert 16-th problem.

1. Introduction

In this paper we study integrals

$$I(t) = I(\gamma, g, t) = \frac{1}{2\pi i}\int_\gamma \frac{g(z)dz}{z-t}\ , \tag{1.1}$$

where γ is a curve in the complex plane $\mathbb{C}$ and $g(z)$ is an algebraic function. More accurately, we assume that after removing from γ a finite set of points Σ (which includes all the double points of γ and its end points) on each segment of $\gamma \setminus \Sigma$ the function $g(z)$ is given by an analytic continuation of a germ of an algebraic function. Accordingly, it is always assumed below that the branches of $g(z)$ on each segment of $\gamma \setminus \Sigma$ are chosen in advance and in this sense $g(z)$ is univalued on γ. Of course, analytic continuation of $g(z)$ outside $\gamma \setminus \Sigma$ may ramify. Furthermore, we assume that at the points of Σ the function $g(z)$ can ramify but does not have poles.

The main problem considered in this paper is to give conditions for the identical vanishing of $I(t)$ near infinity. Since for $|t| \gg 1$,

$$I(t) = -\frac{1}{2\pi i}\sum_{k=0}^{\infty} m_k t^{-k-1}$$

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with $m_k = \int_\gamma z^k g(z)dz$ this is equivalent to finding conditions for all the moments m_k to vanish. By the reasons explained below, we call this problem a "Moment problem" for integral (1.1). Notice that our Moment problem overlaps with the classical Moment problems (described for example in [1]) only to a rather limited extent.

The motivation for the vanishing problem for the Cauchy type integrals of algebraic functions is provided by two classical open questions in Analytic Theory of Differential Equations: the Poincaré Center-Focus problem and the second part of the Hilbert 16-th problem (see Sections 1.1.1 and 1.1.3 below, where the appearance of the Moment problem in Differential Equations is explained).

As we show below, it is natural to consider the vanishing problem for $I(t)$ together with the conditions for $I(t)$ to be an algebraic or a rational function.

Recall that for a closed curve γ without self-intersection the vanishing of all the moments m_k is a necessary and sufficient condition for a function g to be the boundary value of a certain holomorphic function G in the compact domain bounded by γ. On the other hand, if γ is a non-closed curve without self-intersection then the identical vanishing of $I(t)$ near infinity can happen only for $g(z) \equiv 0$ on γ (see, for example, [38]). Nevertheless, the condition of *algebraicity* of $I(t)$ even in these simplest situations turns out to be a rather delicate requirement on the global monodromy group of g and its local ramifications at the points of Σ. We give this condition and some examples in Section 4 below.

For γ a curve with self-intersections the situation becomes much more complicated already for the vanishing problem. Of course, a classical "homological" condition for the identical vanishing of $I(t)$ near infinity (i.e. that g on γ bounds a holomorphic chain) remains valid. However, this condition does not provide in general a "constructive" answer as we would like to have in the Moment problem. Indeed, the classical homological condition is not easy to translate into an explicit condition on our finite-dimensional input data (the algebraic function g, the set Σ, and the homotopy class of γ). Even in the simplest case of the "Polynomial Moment problem" (see Section 1.1.1 and Section 6 below) the explicit answer is not known in spite of a very classical setting of the question. Moreover, the results of [39-41] as well as some of the results of the present paper imply that this answer can not be too simple. As we shall see below, the problem of algebraicity of $I(t)$ brings certain additional difficulties related to the "global" behavior of the monodromy of g with respect to γ.

The main goal of the present paper is to give a constructive (at least, in principle) answer to the Moment and the Algebraicity problems. This is done by relating the Monodromy group of the algebraic function g, the geometry of the curve γ, and the analytic properties of the Cauchy type integrals. Let us describe shortly the main results.

Let $\mathbb{C} \setminus \gamma$ be the union of the domains D_i (with D_0 being the infinite domain). The expression (1.1) defines $I(t)$ as a collection of regular analytic functions I_i on the domains D_i. A simple classical description exists for the behavior of $I(t)$ in the process of crossing the curve γ: for the adjacent domains D_i and D_j the function I_j is obtained from I_i by the analytic continuation into D_j combined with the addition of the local branch of g at the crossing point (also analytically continued into D_j). This last operation (as extended to several crossings of γ) is "combinatorial" in its nature. It is captured by the notion of the "combinatorial monodromy of I" introduced in Section 4. The combinatorial monodromy depends only on the monodromy of g and on the geometry of γ and in principle it can be explicitly computed.

We analyze the analytic continuation of $I_i(t)$ from each of the domains D_i and show that it is essentially described by the combinatorial monodromy. On this base we get a necessary and sufficient condition for $I(t)$ to be algebraic: its combinatorial monodromy must be finite. A necessary and sufficient condition for rationality is that the combinatorial monodromy is trivial. Finally, the vanishing of $I(t)$ is provided by the additional condition of the absence of the poles in certain sums of the branches of g.

We translate the above conditions into certain local (and local-global) branching conditions of g with respect to γ. Most of these conditions have a form of the vanishing of a certain sum of the branches of g. Besides, we give an accurate analytic description of $I(t)$ at its singular points.

We give a number of examples illustrating the above results. Some of them we consider as rather unexpected. This includes a non-closed γ with nonzero g and $I(t) \equiv 0$ near ∞, a non-algebraic $I(t)$ with a finite ramification of all its branches at each singular point, and g with non-trivial "jumps" on a closed γ and with $I(t) \equiv 0$ near ∞.

As the first specific application we give an essentially complete solution of the "Rational Double Moment problem" on the non-closed curve. Remind that in the case of the closed integration curve the answer is given by the classical result of Wermer and Harwey-Lawson: double moments vanish if and only if the path bounds a complex 1-chain (see [2,23,30,59,60] and Section 1.1.3 below). We show that on a non-closed curve the vanishing of the double moments (and in fact just an algebraicity of the appropriate generating functions) is equivalent to a certain composition factorization of the integrand functions which "closes up" the integration path, combined together with the Wermer and Harwey-Lawson condition for their "left factors".

Another application provided is a significant extension of the class of "definite" polynomials (those for which the answer to the Polynomial Moment problem is given by the Composition condition–see Section 1.1.2 and Section 5 below). As it is shown in [9,17,18,61], definite polynomials play an important role in the explicit analysis of the Center-Focus problem for the Abel equation. We characterize some classes of definite polynomials P through the geometry of the images $P(\Gamma)$ of the curves Γ joining a and b. This leads also to an interesting geometric invariant of complex polynomials.

In some aspects the present paper provides just the approach to (or the first examples of) the phenomena which we expect to be of a major importance in the circle of the problems considered. This concerns first of all the fact that the Cauchy type integrals of algebraic functions satisfy Fuchsian linear differential equations. We do not prove this fact in the present paper (providing just the idea of the proof in Remark 2 after Theorem 4.4

of Section 4) but all the necessary tools are prepared here. Another case is the following: in the present paper apparently new examples appear of a specific type of functions arising as Cauchy Integrals of algebraic functions: those with an infinite global ramification but with a finite branching of each of its leaves at each of the finite number of singular point (see Example 5 of Section 4). It turns out that such functions are closely related to certain Kleinian groups and automorphic functions. Once more, in the present paper we restrict ourselves to a short discussion of this phenomenon in a remark after Example 5 of Section 4, not providing the proofs. We plan to present separately the rigorous results in these directions.

We hope also that the tools introduced in this paper around the notion of the combinatorial monodromy can be further developed to provide a really strong approach for the investigation of the analytic and algebraic properties of the Cauchy type integrals of algebraic functions. In the remarks at the end of Section 4 we outline some of the natural directions of such development. We believe that ultimately it may provide a much deeper understanding of the structure of these integrals and of their role in the open questions of the Analytic Theory of Differential Equations. The present paper is the first step in this direction.

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1.1. Motivations

In this section we discuss some questions in Analytic Theory of Differential Equation where the Moment problem naturally appears.

1.1.1. Classical Center-Focus problem and Moments

Our study of the Moment Problem for the Cauchy type integrals is motivated by the classical Poincaré Center-Focus Problem for plane polynomial vector fields.

Let $F(x,y)$, $G(x,y)$ be analytic functions of x, y in a neighborhood of the origin in $\mathbb{R}^2$ vanishing at 0 together with their first derivatives. Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + F(x,y) \\ \dot{y} = x + G(x,y) \end{cases} \tag{A}$$

The system (A) has a center at the origin if all its solutions around zero are closed. The (part of the) classical Center-Focus problem is to find conditions on F and G which are necessary and sufficient for the system (A) to have a center at the origin. See [6,7,32,42,46,50,51,53,55,62] for a detailed discussion of this problem and of a closely related second part of Hilbert's 16-th problem (which asks for the maximal possible number of isolated closed trajectories (limit cycles) of (A)).

It was shown in [20] that one can reduce the system (A) with homogeneous polynomials F, G of degree d to the *trigonometric Abel equation*

$$r' = p(t)r^2 + q(t)r^3, \quad t \in [0, 2\pi], \tag{B}$$

where $p(t)$, $q(t)$ are polynomials in $\sin t$, $\cos t$ of the degrees $d+1$, $2d+2$ respectively. Then (A) has a center if and only if (B) has all the solutions $r = r(t)$ periodic on $[0, 2\pi]$, i.e. satisfying $r(0) = r(2\pi)$. So the classical Center-Focus problem is to find for the trigonometric Abel equations (B) (obtained via the Cherkas transformation [20]) the necessary and sufficient condition on p and q for all its solutions $r = r(t)$ to be periodic on $[0, 2\pi]$. A natural generalization of the classical Center-Focus problem is to find for any trigonometric Abel equation (not only for those obtained via the Cherkas transformation [20] from the plane vector fields) the necessary and sufficient conditions on p and q for all the solutions of (B) to be periodic.

In turn, the trigonometric Abel equation (B) can be transformed by an exponential substitution into the equation

$$y' = p(x)y^2 + q(x)y^3$$

with p and q Laurent polynomials on the unit circle S^1. The Center problem becomes in this setting a problem of non-ramifying of all the solutions on S^1.

Finally, deviating slightly from the original setting we can consider the Abel equation

$$y' = p(x)y^2 + q(x)y^3 \tag{C}$$

with meromorphic p, q on any curve, not necessarily closed.

Let Γ be a curve in $\mathbb{C}$ avoiding poles of p and q and joining two points $a, b \in \mathbb{C}$. The points a and b are called *conjugated with respect to (C) along a curve* Γ if $y(a) = y(b)$ for any solution $y(x)$ of (C) analytically continued from a to b along Γ, with the initial value $y(a)$ sufficiently small. Equivalently, we shall say that *(C) has a center at* (a, b) *along* Γ. The condition on p and q under which (C) has a center is called the Center condition. For $a = b$ this means that the solutions of (C) do not ramify on the closed curve Γ. In this case we say that *(C) has a center along* Γ.

The Hilbert 16-th problem can be reformulated in this setting as follows: we say that a solution $y(x)$ of the equation (C) (analytically continued from a to b along Γ) is *periodic* if $y(a) = y(b)$. The problem is to bound the possible number of isolated periodic solutions of (C).

In both the Center and the Hilbert problems the key tool is the Poincaré first return mapping $G(y_a) = y_b$ which associates to each y_a the value y_b of the solution $y(x)$ of the equation (C) satisfying $y(a) = y_a$ and analytically continued from a to b along Γ. The periodic solutions of (C) along Γ correspond exactly to the fixed points y_a of G (so that $G(y_a) = y_a$) and the Center condition is that $G(y_a) \equiv y_a$.

Although the Center and the Hilbert problems for equation (C) on an interval (or in general on a non-closed curve) do not correspond directly to the classical setting, they present an interest by their own and they have been intensively investigated in [3-5,19-22,25,36,37,54,58] and in many other publications. It is a general belief that the Center and Hilbert problems for a polynomial Abel equation (C) on the interval present all the main difficulties of the classical ones while possibly simplifying essential technical details.

In [3,4] the Center problem for the trigonometric Abel equation has been related to the composition factorization of the coefficients. Recently in [9-18,21,47,61] the Center problem for both the trigonometric and the polynomial Abel equations has been related to the problem of the vanishing of certain generalized moments on one hand and to the Composition algebra of univariate analytic functions on the other. This approach has been further developed in [19,31,58].

For p and q as above let $P = \int p$, $Q = \int q$. Consider the "one-sided" moments $m_k(P,Q)$ defined by

$$m_k(P,Q) = \int_\Gamma P^k(x)Q(x)p(x)dx \ , \tag{1.2}$$

and the Moments generating function

$$H(y) = \int_\Gamma \frac{QdP}{y-P} = \sum_{k=0}^{\infty} m_k(P,Q)y^{-k-1}. \tag{1.3}$$

It is shown in [14-16] (see also [9,17,18,61]) that if we consider a parametric version of the equation (C)

$$y' = p(x)y^2 + \epsilon q(x)y^3 \tag{D}$$

then the infinitesimal center conditions with respect to ϵ for (D) at $\epsilon = 0$ are given by the vanishing of the one-sided moments. Essentially, the Moment generating function $H(y)$ of (1.3) is the derivative with respect to ϵ (at $\epsilon = 0$) of the Poincaré first return mapping $G(y,\epsilon)$ of the equation (D) (see [14-16,17,18]). Hence, $H(y) \equiv 0$ is the infinitesimal or the tangential Center condition (corresponding to the fact that the center of the equation (D) for $\epsilon = 0$ survives in first approximation also for nonzero ϵ).

The Moment generating function $H(y)$ of (1.3) defines also the behavior of the periodic solutions of the equation (D) for small ϵ. One can show (see [18]) that this periodic solutions correspond to the zeroes of $H(y)$. This agrees with the standard fact in the analysis of perturbed plane Hamiltonian systems, where the derivative with respect to ϵ (at $\epsilon = 0$) of the Poincaré first return mapping is given by the Abelian integrals along the level curves of the Hamiltonian. The periodic trajectories of the perturbed system for small ϵ correspond to the zeroes of these Abelian integrals (see [6,7,27,32,46]).

Consequently, the Moment generating function $H(y)$ plays in the analysis of the Abel equation the same role as the Abelian integrals play in the investigation of the perturbed plane Hamiltonian systems, i.e. the most central role. The study of the conditions for the identical vanishing of $H(y)$ (which is one of the main problems of the present paper) corresponds to the study of the identical vanishing of the Abelian integrals in [26]. Essentially, this is the study of the infinitesimal (or "tangential") Center problem for (D). In turn, the study of the distribution of zeroes of $H(y)$ (which has been recently started in [18]) corresponds to the study of zeroes of the Abelian integrals. This last problem is one of the most active research areas in the Analytic Theory of Differential Equations in the last two decades (see [6,7,26-29,32-35,43,46,56,57]).

Assume now that p, q are polynomials. Performing a change of variables $P(x) = z$, $p(x)dx = dz$ we obtain

$$m_k(P,Q) = \int_\gamma z^k g(z)dz, \qquad H(y) = \int_\gamma \frac{g(z)dz}{y-z} = -2\pi i I(y)\ ,$$

with $\gamma = P(\Gamma)$ and $g(z) = Q(P^{-1}(z))$. Thus the Moment generating function $H(y)$ is a special case of the Cauchy integral (1.1). As it will be clear below this special case is not much simpler than the general one. To our opinion this justifies a detailed investigation of the general Cauchy-type integrals of algebraic functions. The relation of $H(y)$ with the Poincaré first return mapping motivates also the study of general analytic properties of $I(y)$ (singularities, analytic continuation, etc.) that we start in the present paper.

1.1.2. Center-Focus problem and Compositions

Let us explain now the role of the Composition Algebra of univalent functions in the study of the Center-Focus Problem for the Abel equation (C) (and in particular in the study of the Moment generating function $H(y)$). It turns out that a basic sufficient condition for the equation (C) to have a center (as well as for the vanishing of the one-sided moments (1.2) and thus for the identical vanishing of $H(y)$) is provided by what we call a "Composition condition". Let as above $p = P', q = Q'$.

Composition condition. *$P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$, where W maps $\mathbb{C}$ into a Riemann Surface X in such a way that $W(\Gamma)$ is a closed curve δ in X (in particular, if $a \neq b$ then $W(a) = W(b)$), δ is contained in a simply-connected domain D in X, and $\tilde{P}$ and $\tilde{Q}$ are regular in D.*

A special form of this condition with $X = \mathbb{C}$ is the Polynomial Composition condition (PCC) below. In [10,11] (and in Section 1.1.3 below) the Composition condition appears with X a rational curve in $\mathbb{C}^2$. The case of X an elliptic curve is considered in [12].

The sufficiency of the Composition condition for the Center problem follows from the fact that after performing a change of variables in (C) and taking W an independent variable, we get a regular equation in a simply-connected domain D in X whose solutions cannot ramify along a closed path δ. The same consideration provides sufficiency of the Composition condition also for the Moment problem i.e. for the vanishing of the moments (1.2).

In the case where $p = P', q = Q'$ are polynomials the Composition condition takes the following simple form which we call the Polynomial Composition condition (PCC):

$$P(x) = \tilde{P}(W(x))\ ,\ Q(x) = \tilde{Q}(W(x))\ , \qquad (PCC)$$

with $\tilde{P}, \tilde{Q}, W$ - polynomials and with an additional requirement that $W(a) = W(b)$.

A specific case of the Composition condition has been introduced to the study of the Center-Focus problem for the Abel equation in [3,4]. The condition (PCC) has been introduced and intensively studied in [9-18,21,47,61] (see also [19,22,31,58]). There is a growing evidence supporting the major role played by the Polynomial Composition condition (and in general, by the polynomial Composition Algebra) in the structure of the Center conditions for the polynomial Abel equation (C). In particular, we have no counterexamples to the following "Composition conjecture":

Composition conjecture. *The Abel equation (C) on the interval with p, q polynomials has a center if and only if (PCC) holds.*

This conjecture has been verified for small degrees of p and q and in many special cases in [9-19,21,58,61].

For some time it was conjectured that (PCC) is a necessary and sufficient condition also for the Polynomial Moment problem i.e. for the vanishing of the polynomial moments (1.2) (the "Moment Composition conjecture"). However, it was recently shown in [39] that the Polynomial Composition condition (PCC) is only sufficient but *not necessary* for the vanishing of one-sided moments (1.2). The counterexample given in [39] exploits rather subtle composition properties of univariate polynomials, in particular, some classical results of Ritt ([44,45,52]). This fact stresses the role of the Composition Algebra in the structure of the moments and of the Center equations and illuminates some important features of the Cauchy-type integrals of algebraic functions (see Sections 4,5,6 below).

The appearance of counterexamples to the "Moment Composition conjecture" (together with the recent results of [9,17,18]) underlines also the role of those polynomials P for which the vanishing the one-sided moments (1.2) with any given q *does imply* the Composition condition. Following [9,61] we call such P *definite*. We consider characterization of definite polynomials as an important problem. Indeed, the role of definite polynomials in the local Center-Focus problem has been demonstrated in [9,18,61]. Shortly it can be explained as follows: consider the Abel equation

$$y' = p(x)y^2 + q(x)y^3, \tag{C}$$

where p is fixed, and q is a variable polynomial in the space V_d of the univariate polynomials of a given degree d. Then for $P = \int p$ definite the local geometry of the Center set $CS \subset V_d$ of the equation (C) near the origin is completely described by the Moment vanishing equations. In particular, this implies that CS near $0 \in V_d$ coincides with the Composition set ([9]). Moreover, it turns out that in this case the so-called Bautin ideal I (generated by all the Taylor coefficients of the Poincaré mapping in the local ring of the polynomials on V_d near the origin) is in fact generated by the moments. This in turn provides an important information on the fixed points of the Poincaré mapping (i.e. on the periodic solutions of the Abel equation). See [9,18].

Even more important role definite polynomials play in the global study of the Center equations near infinity (as presented in [17]). One of important conclusions of [17] is the following: consider a projectivization PV_d of V_d. Then the structure of the Center set $CS \subset PV_d$ at and near the infinite hyperplane in PV_d is very accurately described by the zero set of the "dual" moments $\tilde{m}_k = \int Q^k p$. In many cases this description can be extended from the neighborhood of the infinite hyperplane in PV_d to the affine part V_d and thus it includes the original Center set CS.

One of the applications of the methods developed in the present paper is a description in Section 5 below of some apparently new (with respect to the results of [9-16,21,40-41,47,61]) and natural classes of definite polynomials.

1.1.3. Double moments and centers on a closed curve

Here we describe shortly an additional question arising naturally on the way to the Center-Focus problem. This question concerns the center conditions for the Abel equation on a *closed* curve Γ and their relation to the vanishing of the one-sided and double moments. As it was explained above, the Center problem for the Abel equation on a closed curve provides a rather accurate approximation of the classical Center-Focus problem. Also in the case of a closed Γ the Composition algebra plays a central role connecting the vanishing of the moments with the center conditions.

As we pass from an open interval to a closed curve Γ the vanishing of the one-sided moments (1.2) becomes a much less restrictive condition. Although the Composition condition is still sufficient for this vanishing it is now far from being necessary. A natural stronger analytic condition on P and Q to be considered (and compared with the Center one for a closed Γ) is the vanishing of the double moments.

Generalizing situation a little bit let us consider the double moments

$$m_{i,j} = \int_\Gamma P^i(x)Q^j(x)p(x)dx, i, j = 0, 1, \ldots, \tag{1.4}$$

where Γ is a path - closed or non-closed - and P and Q are only assumed to be holomorphic in a neighborhood of Γ.

The double moments (1.4) appear in several fields of Analysis, Several Complex variables and Banach Algebras. The classical result of Wermer and Harwey-Lawson (See [2,23,30,59,60]) implies that *if the image of Γ under the map $z \to (P(z), Q(z)) \in \mathbb{C}^2$ is a closed curve σ then the vanishing of all the moments $m_{i,j}$ is equivalent to the fact that the curve σ bounds a compact analytic one-chain in $\mathbb{C}^2$.*

As it was mentioned above the Composition algebra still plays an important role in relating the moments vanishing, the Wermer and Harwey-Lawson condition, and the topology of the rational curve $Y \in \mathbb{C}^2$ parametrized by $P(z), Q(z)$. In [10] we give a simple and constructive necessary and sufficient condition for vanishing of double moments (1.4) for P and Q rational functions. It is obtained as a combination of the composition approach with the general Wermer-Harwey-Lawson theorem. Assume that the curve Γ is closed. Let W be the right composition greatest common factor (CGCF) of P and Q, i.e. $P = \tilde{P}(W)$,

$Q = \tilde{Q}(W)$ with $\tilde{P}$, $\tilde{Q}$ having no composition right factor of a positive degree. (We shall call such rational functions $\tilde{P}$, $\tilde{Q}$ relatively prime in composition sense).

Theorem 1.1.3. *For P, Q rational functions $m_{ij} = 0$ for all $i, j \geq 0$ if and only if all the poles of $\tilde{P}$ and $\tilde{Q}$ lie on one side of $W(\Gamma)$. In particular, if P, Q are relatively prime in composition sense rational functions then $m_{ij} = 0$ for all $i, j \geq 0$ if and only if all the poles of P and Q lie on one side of Γ.*

For Γ with self-intersections the "sides" of Γ are accurately defined in the next section.

The examples of the centers in homogeneous planar systems (A) of degrees 2 and 3 (which lead via the Cherkas transformation to the Abel equations (C) with P and Q - Laurent polynomials) show that *in general on a closed curve Γ the Center condition for the Abel equation (C) with rational coefficients p and q does not imply the vanishing of the double moments* (see [12]).

On the other hand, the vanishing of the double moments implies Center in many special cases. In particular, in [10,11] it is shown that this is true for P and Q – Laurent polynomials and $\Gamma = S^1$. For P and Q - general rational functions the validity of the above implication depends on the geometry of the curve Γ and its image $W(\Gamma)$ under the Composition Greatest Common Factor W of P and Q. In particular, if Γ and $W(\Gamma)$ are simple closed curves then the vanishing of the double moments implies Center.

The proof of these facts in [10-11] relays on the general Composition condition given in Section 1.1.2 above. The Riemann surface X in our situation is the rational algebraic curve $Y = (P, Q)(\mathbb{C}) \subset \mathbb{C}^2$. P and Q provide a rational parametrization of Y and map Γ into the curve δ in Y. The vanishing of the double moments implies (via the general Wermer-Harwey-Lawson theorem) that δ is homological to zero in Y. But in many cases a loop δ on an affine *rational* complex curve Y which is homological to zero in Y must be contained in a certain simply-connected domain in Y. By the general Composition condition this implies Center.

In contrast, consider the example of the Abel equation (C) with P and Q the Weierstrass function and its derivative, respectively, and Γ a small circle around zero in $\mathbb{C}$. It is shown in [12] that in this case all the double moments vanish while the Abel equation (C) does not have Center. The difference with the rational case is that for P and Q the Weierstrass function and its derivative the curve $Y = (P, Q)(\mathbb{C}) \subset \mathbb{C}^2$ is an affine *elliptic* curve. The loop $\delta = (P, Q)(\Gamma)$ is still homological to zero in Y. But it bounds in Y a *not simply-connected domain.*

All these results stress the role of the vanishing of double moments and of the Composition factorizations in the case of the closed curve Γ. Having this in mind we can now step back and ask for the vanishing condition for the *double* moments on the *non-closed* curve Γ. This question is considered in Section 6 below.

1.2. Organization of the paper

In Section 2 a convenient classical description of the partition of $\mathbb{C} \setminus \gamma$ is given. Then as the initial example we give (by a direct computation) a description of $I(t)$ for g a rational function.

In Section 3 we first remind some classical results on the behavior of the Cauchy type integrals near the integration curve. Then we obtain an analytic description of $I(t)$ at the ramification points of g (up to addition of a regular analytic germ) in terms of the Puiseux expansion of g. As a result we obtain our first necessary condition for algebraicity of $I(t)$ (which naturally overlaps with the results of Section 4).

Section 4 is devoted to the global analytic properties of $I(t)$. In particular, here we obtain our main conditions for algebraicity, rationality, and vanishing of $I(t)$.

We start with the introduction of the notion of the "sum of branches" of g across γ and along an auxiliary curve S. Then we define the notion of the "combinatorial monodromy" and prove that it essentially defines the usual monodromy of the analytic continuation of $I(t)$.

Next the necessary and sufficient conditions are given for algebraicity, rationality, and vanishing of $I(t)$ in terms of the combinatorial monodromy. Later in this section we obtain more explicit local necessary (sometimes also sufficient) conditions for each of the properties in question.

Finally, a number of examples is considered in reasonable details.

In Section 5 the results of Sections 3 and 4 are applied to the study of the Polynomial Moment problem. In particular, we produce some new classes of definite polynomials. We also introduce and study a certain geometric invariant of complex polynomials.

In Section 6 we study the double moments of rational functions on a non-closed γ. We show that in this case the vanishing of the double moments (in fact, only algebraicity of the appropriate Moment generating functions) is equivalent to a certain composition factorization of the integrand functions which "closes up" the integration path, together with the Wermer and Harwey-Lawson condition for the left factors. The last condition can be interpreted via Theorem 1.1.3 above.

2. Partition of $\mathbb{C}$ by γ. An example: $I(g, \gamma, t)$ for g-polynomials and rational functions.

Below we always assume the curve γ to be oriented, piecewise-smooth, and to have only transversal self-intersections. In this section we also assume that γ is closed. A classical description of the geometry of γ given below closely follows [2,59,60].

The curve γ subdivides $\mathbb{C}$ into a finite number of open domains D_i. One of these domains which we denote by D_0 is unbounded and the rest are bounded and simply connected. For a point $z \in \mathbb{C}\backslash\gamma$ define $\mu(\gamma, z)$ as the rotation number of γ around z. Clearly, $\mu(\gamma, z)$ is constant on each D_i and we will denote this constant by μ_i, $\mu_0 = 0$. Alternatively, $\mu(\gamma, z)$ can be defined as the (signed) number of the intersection points of γ with any path joining ∞ to z or as the linking number of the curve γ and the point z. According to this last definition for any complex one-dimensional chain Z in $\mathbb{C}$ with

$\gamma = \partial Z$ the number $\mu(\gamma, z)$ is the (signed) "intersection number of Z with z" or, in other words, simply the number of times the chain Z covers the point z. Figure 1 illustrates this construction.

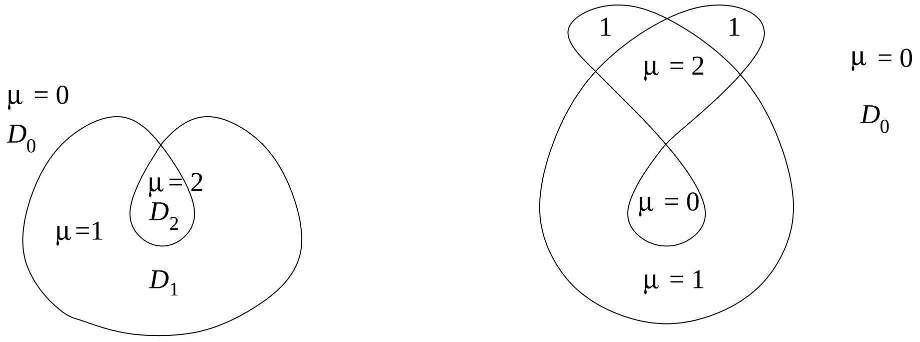

Figure 1

It is natural to call the union of D_j with $\mu_j = 0$ "an outside part" of the curve γ and the union of D_j with $\mu_j \neq 0$ "an inside part".

Let $\Delta = (z_1, \ldots, z_m)$ be a finite collection of points in $\mathbb{C}$.

Lemma 2.1. *The curve γ is homologous to zero in $\mathbb{C}\backslash\Delta$ if and only if $\mu(\gamma, z_i) = 0$ for each $z_i \in \Delta$, i.e. all z_i belong to the outside part of γ.*

PROOF. If $\gamma = \partial Z$ with $Z \subseteq \mathbb{C}\backslash\Delta$ then by the last definition of $\mu(\gamma, z)$ this function is zero for any $z_i \in \Delta$. Conversely, we always have $\gamma = \partial\left(\sum_{i=0} \mu_i \overline{D}_i\right)$ and if $\mu_i = 0$ on any domain D_j containing the points of Δ, then the 1-chain $\sum_{i=0} \mu_i \overline{D}_i$ is contained in $\mathbb{C}\backslash\Delta$.

Now we are ready to describe Cauchy integrals of polynomials and rational functions on closed curves.

Proposition 2.2. *For $g(z)$ - a polynomial in z and for γ - closed, $I(\gamma, g, t) = \mu(\gamma, t) \cdot g(t)$.*

PROOF. A function $\frac{g(z)}{z-t}$ in z has the only pole at $z = t$ with a residue $g(t)$ while the curve γ makes exactly $\mu(\gamma, t)$ turns around t.

Now let $g(z)$ be a general rational function with the poles of the orders $k_1, \ldots, k_l, k_i \geq 1$ at the points $z_1, \ldots, z_l$, respectively. Let for $i = 1, \ldots, l$ the "essential part" $R_i(z)$ of the function $g(z)$ at z_i (i.e the negative part of the Laurent polynomial of $g(z)$ at the pole z_i) be given by $R_i(z) = \sum_{j=1}^{k_i} \frac{\alpha_{i,j}}{(z-z_i)^j}$.

Proposition 2.3. *The Cauchy integral $I(\gamma, g, t)$ is given by*

$$I(\gamma, g, t) = \mu(\gamma, t)g(t) - \sum_{i=1}^{\ell} \mu(\gamma, z_i)R_i(t). \tag{2.1}$$

PROOF. Represent $g(z)$ as $g(z) = g_0(z) + \sum_{i=1}^{\ell} \sum_{j=1}^{k_i} \frac{\alpha_{i,j}}{(z-z_i)^j}$ with $g_0(z)$ a polynomial. We have

$$I(\gamma, g, t) = I(\gamma, g_0, t) + \sum_{i=1}^{\ell} \sum_{j=1}^{k_i} \alpha_{i,j} I(\gamma, \frac{1}{(z-z_i)^j}, t)\ . \tag{2.2}$$

By Proposition 2.2, $I(\gamma, g_0, t) = \mu(\gamma, t)g_0(t)$. Now, representing the integrand $\frac{1}{(z-t)(z-z_i)^j}$ in the Cauchy integral $I(\gamma, \frac{1}{(z-z_i)^j}, t)$ as the sum of the elementary fractions one obtains

$$\frac{1}{(z-t)(z-z_i)^j} = (\frac{1}{t-z_i})^j (\frac{1}{z-t} - \frac{1}{z-z_i}) + \sum_{s=2}^{j} \beta_{i,s} \frac{1}{(z-z_i)^s} \ . \tag{2.3}$$

Integrating along γ with respect to z and noticing that the last sum in (2.3) does not contribute to the integral we get:

$$I(\gamma, \frac{1}{(z-z_i)^j}, t) = (\frac{1}{t-z_i})^j (\mu(\gamma, t) - \mu(\gamma, z_i)). \tag{2.4}$$

Finally,

$$I(\gamma, g, t) = \mu(\gamma, t)g_0(t) + \mu(\gamma, t) \sum_{i=1}^{\ell} \sum_{j=1}^{k_i} \frac{\alpha_{i,j}}{(t-z_i)^j} - \sum_{i=1}^{\ell} \mu(\gamma, z_i) \sum_{j=1}^{k_i} \frac{\alpha_{i,j}}{(t-z_i)^j} =$$

$$= \mu(\gamma, t)g(t) - \sum_{i=1}^{\ell} \mu(\gamma, z_i) R_i(t).$$

This completes the proof of Proposition 2.3.

Corollary 2.4. *Let $g(z)$ be a rational function with only the first order poles at $z_1, \ldots, z_\ell$, each with the residue α_i, $i = 1, \ldots, \ell$, respectively. Then*

$$I(\gamma, g, t) = \mu(\gamma, t)g(t) - \sum_{i=1}^{\ell} \frac{\mu(\gamma, z_i)\alpha_i}{t - z_i} \ . \tag{2.5}$$

Corollary 2.5. *If all the poles of g belong to the outside part of γ then $I(\gamma, g, t) = \mu(\gamma, t)g(t)$. In particular, this is a necessary and sufficient condition for $I(\gamma, g, t)$ to vanish identically for t near infinity.*

Remark. Computations of Propositions 2.2 and 2.3 did not use in any essential form the rationality of the function g. Exactly in the same way we obtain the corresponding results for regular and meromorphic functions, respectively. Let $U \subset \mathbb{C}$ be a simply-connected domain containing the closed integration curve γ.

Proposition 2.6. *For the integrand $g(z)$ being extendable to a holomorphic function in U $\quad I(\gamma, g, t) = \mu(\gamma, t) \cdot g(t)$.*

Of course, for such $g(z)$ the Cauchy integral $I(\gamma, g, t)$ always vanishes identically on the exterior domain D_0.

Now let U be as above and let $g(z)$ be a meromorphic function in U with the finite number of poles at the points $z_1, \ldots, z_l$ with the orders $k_1, \ldots, k_l$, $\quad k_i \geq 1$, respectively.

Proposition 2.7. *The Cauchy integral $I(\gamma, g, t)$ is given by*

$$I(\gamma, g, t) = \mu(\gamma, t)g(t) - \sum_{i=1}^{\ell} \mu(\gamma, z_i)R_i(t),$$

where for $i = 1, \ldots, l$ $R_i(z)$ denotes the essential part of $g(z)$ at the pole z_i. In particular, for such $g(z)$ the Cauchy integral $I(\gamma, g, t)$ is a rational function on the exterior domain D_0.

3. Local structure of $I(t)$

Let γ be a curve (closed or non-closed) and let $z \in \gamma$. We say that the integrand function *g has a "jump" at z if the branches g_0 and g_1 of g on the two sides of z on γ cannot be obtained from one another by a local analytic continuation (i.e. a continuation along a curve inside any given neighborhood of z). Equivalently, the full local germs of g_0 and g_1 at a jump point z do not coincide.*

Let us remind that we have denoted by Σ the set containing the end-points of γ, all its multiple points, and all the points z on γ where the integrand function g has either a jump or a ramification point. (In this paper we exclude the possibility for g to have poles on γ).

Lemmas 3.1-3.3 below provide an elementary description of the behavior of the Cauchy type integral near the integration curve γ. For a convenience of the reader we give some proofs and explanations.

Consider first $z_0 \in \gamma$ and $z_0 \notin \Sigma$. In particular, g is regular at z_0.

Lemma 3.1. *$I(\gamma, g, t)$ near z_0 is represented by two regular analytic functions: $I_-(t)$ on the left side of γ and $I_+(t)$ on the right side. Both I_- and I_+ are analytically extendible into an entire neighborhood U of z_0 and $I_+ = I_- + g$ in U.*

PROOF. This is the usual property of Cauchy type integrals (see [38]) taking into account that $g|\gamma$ is a restriction on γ of a regular analytic function g in U. Indeed, using analytic continuation of g into the neighborhood U of z_0 and deforming the integration path we obtain the required extension of I_- and I_+ just by the original expression (1.1).

Let now z_0 be a double point of γ and let γ_0 and γ_1 denote the two local segments of γ crossing at z_0. We assume that the restrictions of g to γ_0 and γ_1 are both regular at z_0.

Lemma 3.2. *$I(\gamma, g, t)$ in a neighborhood U of z_0 is represented as $I(t) = I^0_\pm + I^1_\pm$, where the combination of the signs is chosen according to the part of $U \backslash \gamma$ considered, and $I^0_\pm$ and $I^1_\pm$ have with respect to γ_0 and γ_1 all the properties stated in Lemma 3.1.*

PROOF. Up to a regular addition we have $I(\gamma, g, t) = I(\gamma_0, g, t) + I(\gamma_1, g, t) = I^0 + I^1$. Application of Lemma 3.1 to each of these integrals proves the lemma. Figure 2 illustrates this construction.

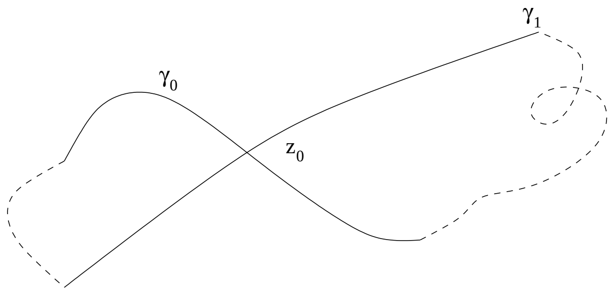

Figure 2

Assume now that z_0 is the end-point of γ with the positive (integration) direction from z_0 along γ. Let g be regular near z_0.

Lemma 3.3. *In a neighborhood of z_0,*

$$I(t) = \tilde{I}(t) - \frac{1}{2\pi i} g(t) \log(t - z_0)$$

with $\tilde{I}(t)$ regular near z_0.

PROOF. Once more, this is the standard property of the Cauchy type integrals (see [38]). We give a proof in order to illustrate the approach used below in a singular situation.

Up to a regular addition, for t near z_0 we can write

$$I(t) = \frac{1}{2\pi i} \int_{z_0}^{z_1} \frac{g(z)dz}{z-t} = \frac{1}{2\pi i} \int_{z_0}^{z_1} \frac{(g(z) - g(t))dz}{z-t} + \frac{1}{2\pi i} \int_{z_0}^{z_1} \frac{g(t)dz}{z-t}.$$

But $\frac{g(z)-g(t)}{z-t} = R(z,t)$ is a regular function of z and t for z, t near z_0. Hence,

$$I(t) = \frac{1}{2\pi i} \int_{z_0}^{z_1} R(z,t)dz + \frac{g(t)}{2\pi i} \log \frac{t - z_1}{t - z_0} = \tilde{I}(t) - \frac{1}{2\pi i} g(t) \log(t - z_0).$$

The next result describes the local structure of $I(t)$ near the end point z_0 of γ which is also a ramification point of g. We believe this result is new. It gives a very accurate description of $I(t)$ near z_0 starting with a very accurate description of g: its Puiseux series at z_0. Let

$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^{k/n} \tag{3.1}$$

be a Puiseux series of $g(z)$ at z_0. We denote by $g_r(z)$ a “regular part” of $g(z)$ at z_0:

$$g_r(z) = \sum_{\ell=0}^{\infty} a_{n\ell} (z - z_0)^{\ell} \ . \tag{3.2}$$

Denote by $\tilde{g}(u)$ a regular function

$$\tilde{g}(u) = \sum_{k=0}^{\infty} a_k u^k \ . \tag{3.3}$$

The expression (3.1) does not specify any individual branch of the algebraic function g at z_0, but rather represents the full local germ of g at z_0 (and in this way all its local branches). However, in the definition (1.1) of the Cauchy type integral a certain specific branch g_0 of g on γ near z_0 has been fixed. So let U be a sufficiently small simply-connected neighborhood of a part of $\gamma \setminus z_0$ near z_0. Denote by $h(z)$ those branch in U of the inverse function to $(z-z_0)^n$ for which g_0 on γ near z_0 is given by

$$g_0(z) = \tilde{g}(h(z)). \tag{3.4}$$

Let U_0 be a (simply-connected) part of U lying on one side of $\gamma \setminus 0$ (say, the part in the counter-clockwise direction from γ). Now we fix the branch t_0 of $(t-z_0)^{1/n}$ in U_0 given by $t_0 = h(t)$ and denote by t_j, $j = 0, \ldots, n-1$, all the other n-th roots of $t - z_0$, $t_j = \epsilon^j t_0$, $\epsilon = e^{2\pi i/n}$. The functions $t_j = t_j(t)$ are univalued functions of $t \in U_0$. Remind that in the expression (1.1) of the Cauchy type integral $I(t)$ the argument t is not allowed to be in γ.

To formulate our result it remains to fix the branches of the logarithm appearing in the expression (3.5) below. We fix in an arbitrary way a certain branch of the $\log(t-z_0)$ for t in the simply-connected domain U_0 and denote it by *Log*. Let $c = h(z_1)$ with $z_1 \in \gamma \setminus z_0$. Fix the branch of the log near c satisfying $\log c = \frac{1}{n} Log(z_1 - z_0)$ for the branch *Log* of $\log(t-z_0)$ chosen above, and extend it into a small simply-connected neighborhood V of c. We denote this extended branch of the logarithm in V by Log_1. Finally, for t small also $t_j = t_j(t)$ are small. Therefore $c - t_j \in V$ and we use the chosen branch Log_1 of $\log(c - t_j)$ in V for each $j = 0, \ldots, n-1$.

Theorem 3.4. *For any t in the domain U_0,*

$$I(t) = R(t) - \sum_{j=0}^{n-1} (\frac{j}{n} - \frac{1}{2}) \tilde{g}(t_j) + \frac{1}{2\pi i} \sum_{j=0}^{n-1} \tilde{g}(t_j) Log_1(c - t_j) - \frac{1}{2\pi i} g_r(t) Log(t - z_0). \tag{3.5}$$

Here $R(t)$ is a regular function and $c = h(z_1)$ with $z_1 \in \gamma \setminus 0$ is a nonzero complex constant. The branches $t_j = t_j(t)$ and the branches of the logarithms are chosen as described above.

PROOF. To simplify notations, we assume that $z_0 = 0$. As in the proof of Lemma 3.3, up to addition of a regular function we have

$$I(t) = \frac{1}{2\pi i} \int_0^{z_1} \frac{g_0(z) dz}{z - t} .$$

Here $z_1 \in \gamma \setminus 0$. Make a change of variables in this integral: $z = u^n$. Since γ is a smooth curve near z_0 the inverse transformation $u = z^{1/n}$ splits γ into n smooth curves γ_i at 0. Order these n curves in such a way that γ_0 be the one given by the branch $h(z)$ of $u = z^{1/n}$ defined above and $\gamma_j = \epsilon^j \gamma_0$ for $j = 0, \ldots, n-1$ (where as above $\epsilon = e^{2\pi i/n}$). For any t in U_0 the values of $t_j = t_j(t)$, $j = 0, \ldots, n-1$, belong to the simply-connected domains Ω_j lying on the "left" side of the curves $\gamma_j \setminus 0$. See Figure 3.

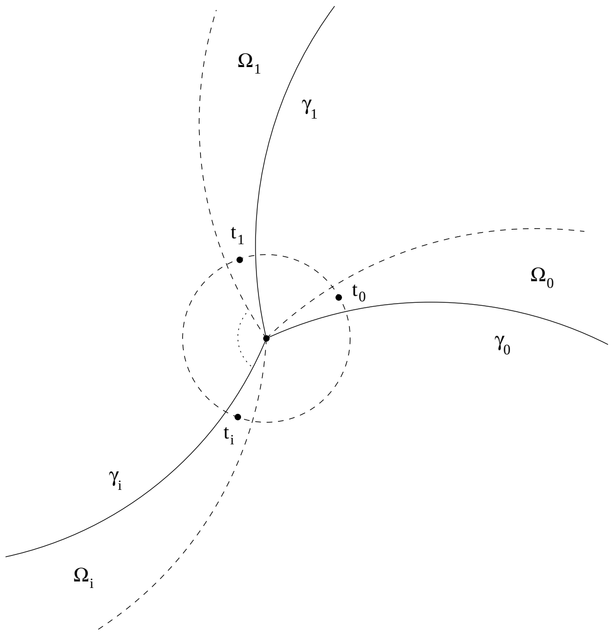

Figure 3

So the integration after the change of variables goes along γ_0 from 0 to $u_1 = c = h(z_1) = z_1^{1/n}$. By (3.4) we get

$$I(t) = \frac{n}{2\pi i} \int_0^{u_1} \frac{\tilde{g}(u) u^{n-1} du}{u^n - t}. \tag{3.6}$$

We have:

$$\frac{1}{u^n - t} = \sum_{j=0}^{n-1} \frac{A_j}{u - t_j(t)}$$

with $A_j = \frac{1}{n t_j^{n-1}(t)}$. Hence (writing shortly $t_j(t)$ as t_j) we obtain

$$2\pi i I(t) = \sum_{j=0}^{n-1} \int_0^{u_1} \frac{\tilde{g}(u) u^{n-1} du}{t_j^{n-1}(u - t_j)} = \sum_{j=0}^{n-1} \int_0^{u_1} \left[\frac{\tilde{g}(u) u^{n-1} - \tilde{g}(t_j) t_j^{n-1}}{t_j^{n-1}(u - t_j)} + \frac{\tilde{g}(t_j)}{u - t_j} \right] du. \tag{3.7}$$

Integration and summation of the terms in (3.7) with only $u - t_j$ in the denominator gives (up to the factor $\frac{1}{2\pi i}$)

$$\sum_{j=0}^{n-1} \tilde{g}(t_j)[\log(u_1 - t_j) - \log(-t_j)] = \sum_{j=0}^{n-1} \tilde{g}(t_j) \log(u_1 - t_j) - \sum_{j=0}^{n-1} \tilde{g}(t_j) \log(-t_j). \tag{3.8}$$

Now we transform the expression (3.8) taking into account the choice of the branches of the logarithm fixed in the formula (3.5) of Theorem 3.4. The differences $\log(u_1 - t_j) - \log(-t_j)$ on the left hand side of (3.8) do not depend on the choice of the branches of the logarithm but do depend on the integration path (which becomes the continuation path of the logarithm). So let us take in each of these differences $\log(u_1 - t_j)$ to be given by the branch of the logarithm $Log_1(u_1 - t_j)$ (where Log_1 is the branch in the neighborhood V of u_1 fixed in Theorem 3.4). With this choice the first sum on the right hand side of (3.8) enters as the third term into the expression (3.5) of Theorem 3.4.

To compute the second sum on the right hand side of (3.8) (and thus to get the second and the fourth terms in (3.5)) let us specify for each summand in (3.8) the continuation path of the chosen branch of the logarithm. As u goes from zero to u_1 along γ_0, $u - t_j$ goes along the curve γ_0^j obtained from γ_0 by the shift to $-t_j$. See Figure 4 which depicts also an auxiliary path σ obtained from γ_0 by the shift to t_0.

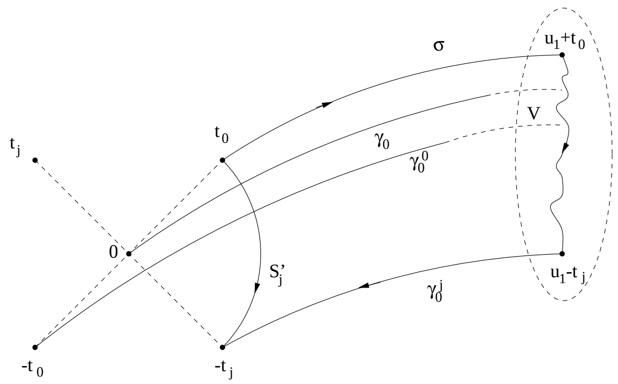

Figure 4

Notice that the curves γ_0^j, $j = 0, \dots, n-1$, do not pass through 0. (Indeed, since t is not allowed to be in γ, t_j cannot be in any of the branches γ_i, in particular, in γ_0). Some of these curves pass (near zero) on the same side of γ_0 as σ and some pass on the other side. An easy computation (see Figure 4) shows that on the same side pass the curves with $j = [\frac{n-1}{2}] + 1, \dots, n-1$, and on the other side those with $j = 0 \dots, [\frac{n-1}{2}]$.

Now, according to the integration path in (3.7) the branch of the logarithm in each of the summands $\log(-t_j)$ in (3.8) is obtained as follows: the branch Log_1 of the logarithm in a neighborhood V of u_1 is taken. Then this branch is continued from $u_1 - t_j$ (which belongs to V) to $-t_j$ along the path γ_0^j. Notice that by the choice of the branch Log_1 above its continuation to the domain Ω_0 (which we denote also Log_1) satisfies for each $t_0(t) \in \Omega_0$ the equality $Log_1(t_0) = \frac{1}{n} Log(t)$.

Therefore we can obtain the branches of each of the summands $\log(-t_j(t))$ in (3.8) by continuing Log_1 from t_0 to $-t_j$ along the path S_j obtained as follows: we go from t_0 back to $u_1 + t_0 \in V$ along the path σ, then join $u_1 + t_0$ to $u_1 - t_j$ inside V, and then continue from $u_1 - t_j$ to $-t_j$ along the path γ_0^j. (See Figure 4). Now for $j = 0 \dots, [\frac{n-1}{2}]$ the path S_j can be deformed into the part S'_j of the small circle going from t_0 to $-t_j$ in a clockwise direction, while for $j = [\frac{n-1}{2}] + 1, \dots, n-1$ the path S'_j goes from t_0 to $-t_j$ in a counter-clockwise direction. In each case the path S'_j presents the rotation from t_0 to the angle $-\pi + (\frac{2\pi}{n})j$ (see Figure 4). Starting with the equality $Log_1(t_0) = \frac{1}{n} Log(t)$ we finally obtain $\log(-t_j) = \frac{1}{n} Log(t) + (\frac{2\pi i}{n})j - \pi i$.

Remark. The same result gives a "naive" computation

$$\log(-t_j) = \log(-\epsilon^j t_0) = \log(-\epsilon^j t^{1/n}) = \frac{1}{n}\log t + (\frac{2\pi i}{n})j - \pi i.$$

Hence, the second sum on the right hand side of the expression (3.8) gives

$$\frac{1}{2\pi i n} Log(t) \sum_{j=0}^{n-1} \tilde{g}(t_j) + \sum_{j=0}^{n-1} (\frac{j}{n} - \frac{1}{2})\tilde{g}(t_j). \tag{3.9}$$

To bring the last term to the form it takes in (3.5) it remains to notice that the sum $\sum_{j=0}^{n-1} \tilde{g}(t_j)$ which is the sum of all the local branches of g at 0 is equal to n times the regular part $g_r(t)$ of g. Indeed,

$$\sum_{j=0}^{n-1} \tilde{g}(t_j) = \sum_{j=0}^{n-1}\sum_{k=0}^{\infty} a_k(\epsilon^j t_0)^k = \sum_{k=0}^{\infty} a_k t_0^k \sum_{j=0}^{n-1} \epsilon^{jk} = n \sum_{k=n\ell} a_k t_0^k = n g_r(t). \tag{3.10}$$

This gives the second and the fourth terms in the expression (3.5) of Theorem 3.4 and completes the computation of the sum of the terms in (3.7) with only $u - t_j$ in the denominator.

To complete the proof of Theorem 3.4 it remains to show that the sum of the terms in (3.7) with the denominator $t_j^{n-1}(u - t_j)$ is a regular function in t near 0. Since $\tilde{g}(u)$ is given by a convergent power series in nonnegative integer powers of u, it is enough to prove the statement for each term of this series separately. We have

$$\sum_{j=0}^{n-1} \frac{u^m - t_j^m}{t_j^{n-1}(u - t_j)} = \frac{1}{t}\sum_{j=0}^{n-1} t_j(u^{m-1} + u^{m-2}t_j + \cdots + t_j^{m-1}) = \frac{1}{t}\sum_{j=0}^{n-1}\sum_{s=1}^{m} u^{m-s}t_j^s =$$

$$= \frac{1}{t}\sum_{s=1}^{m} u^{m-s}\sum_{j=0}^{n-1} t_j^s = \frac{1}{t}\sum_{s=\ell n, 0<s\le m} u^{m-s}\cdot n t^{\ell} = n \sum_{0<\ell\le[m/n]} u^{m-\ell n}t^{\ell-1}.$$

This completes the proof of Theorem 3.4.

Remark 1. The expression (3.5) of Theorem 3.4 with the branches t_j and the branches of the logarithms specified as explained above, represent the actual value of the Cauchy integral $I(t)$ given by (1.1) in a one-sided neighborhood U_0 of $\gamma \setminus z_0$. However, the same formula gives also the local analytic continuation of $I(t)$ as t is allowed to vary on the full punctured neighborhood of z_0. In this case the branches of t_j and of the logarithms in (3.5) should be interpreted as the appropriate analytic continuations of the original ones.

Remark 2. If we fix each of the branches of the logarithms appearing in the expression (3.5) of Theorem 3.4 in an arbitrary way, this expression takes the following form:

$$I(t) = R(t) - \sum_{j=0}^{n-1}(\frac{j}{n} - \frac{1}{2} + m_j)\tilde{g}(t_j) + \frac{1}{2\pi i}\sum_{j=0}^{n-1}\tilde{g}(t_j)Log_1(c - t_j) - \frac{1}{2\pi i}g_r(t)Log(t - z_0),$$

with m_j – integers, $j = 0, \ldots, n-1$. Indeed, the addition of $(2\pi i)m$ to $Log(t - z_0)$ brings a regular correction $(2m\pi i)g_r(t)$ which enters the term $R(t)$. The additions of $(2\pi i)m_j$ to the branches of $Log_1(c - t_j)$ are reflected in the formula above. The computations in the proof of Theorem 3.4 show how to fix the branches of the logarithms in a coherent way in order to eliminate the unspecified constants m_j.

Remark 3. Theorem 3.4 can be considered as a generalization and sharpening of computations of C. Christopher [21] and M. Briskin [13].

Corollary 3.5. *In a neighborhood of z_0, $I(t) = \hat{I}(t) - \frac{1}{2\pi i} g_r(t) \log(t - z_0)$, where the analytic continuation of the function $\hat{I}(t)$ has a finite ramification of the order at most n at z_0.*

PROOF. The term $R(t)$ in the expression of Theorem 3.4 is regular in t near z_0. The second and the third terms return to the original branch as t makes n turns around z_0. Indeed, as t is close to z_0, t_j are near 0 but $\log(c - t_j)$ is regular near 0 since $c \neq 0$. $\tilde{g}$ is a regular function by definition. Hence as t_j return to their original values after t turns n times around z_0, the same do the second and the third terms in the expression of Theorem 3.4.

Corollary 3.6. *If g_r at z_0 is not identically zero then the analytic continuation of $I(t)$ has an infinite ramification around z_0. In particular, $I(t)$ cannot be an algebraic function.*

PROOF. For g_r not identically zero the last term in Corollary 3.5 implies an infinite ramification of $I(t)$.

In particular, this happens if $g(z_0) \neq 0$ since in this case $g_r(z_0) = g(z_0) \neq 0$.

Example 1. The following simple example illustrates the statement and the proof of Theorem 3.4 (as well as some of the results of the next section):

$$I(t) = \int_0^1 \frac{\sqrt{z}dz}{z - t} \ .$$

The positive branch of $\sqrt{z}$ for z on $[0, 1]$ is chosen. Then performing the change of variables $z = u^2, \quad dz = 2udu$ we get

$$I(t) = 2 \int_0^1 \frac{u^2 du}{u^2 - t} = 2 \int_0^1 du + 2 \int_0^1 \frac{tdu}{u^2 - t}.$$

Denote by $\sqrt{t}$ the branch which is positive for positive t (so t_0 and t_1 in the proof of Theorem 3.4 are $\sqrt{t}$ and $-\sqrt{t}$). We have

$$\frac{2t}{u^2 - t} = t \left(\frac{1/\sqrt{t}}{u - \sqrt{t}} - \frac{1/\sqrt{t}}{u + \sqrt{t}} \right) = \frac{\sqrt{t}}{u - \sqrt{t}} - \frac{\sqrt{t}}{u + \sqrt{t}} \ .$$

Thus

$$I(t) = 2 - [-\sqrt{t} \log(u - \sqrt{t}) + \sqrt{t} \log(u + \sqrt{t})]|_0^1 =$$

$$= 2 + \sqrt{t} \log\left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right) - \sqrt{t}(\log(-\sqrt{t}) - \log(\sqrt{t})) =$$

$$= 2 + \sqrt{t} \log\left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right) - \sqrt{t}(\log(-1)) = 2 - \pi i \sqrt{t} + \sqrt{t} \log\left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right).$$

The term with a logarithmic ramification around zero is absent since $g(z) = \sqrt{z}$ has a regular part $g_r(z)$ equal to zero at $z = 0$. Notice that I(t) still has a logarithmic branching around $t = 1$.

We need also a description of $I(t)$ near a simple interior point z_0 of γ at which g may have a jump. Denote by g_0 and g_1 the branches of g on γ before and after z_0, respectively.

Theorem 3.7. *In a neighborhood of z_0 functions $I_\pm(t)$ can be represented as*

$$I_\pm(t) = \hat{I}_\pm(t) + \frac{1}{2\pi i}(g_{r_1}(t) - g_{r_0}(t)) \log(t - z_0) \ ,$$

with $\hat{I}_\pm(t)$ having a finite ramification at z_0. Here g_{r_0} and g_{r_1} denote the regular parts of g_0 and g_1 at z_0 respectively.

PROOF. Up to a regular addition the functions $I_\pm(t)$ in a neighborhood of z_0 are given by the sum of the Cauchy integrals on the "semi-curves" γ_0 and γ_1 having z_0 as the end points. Now the required representation follows directly from Corollary 3.5.

Remark 3. One can write in the situation of Theorem 3.7 a full expression completely analogues to the formula (3.5) of Theorem 3.4. However, this expression becomes rather complicated since the ramification orders of g on the two sides of z_0 may be different and most of the sums in (3.5) must appear twice. A simplified version of this formula given by Theorem 3.7 is sufficient for our applications.

In the same way as Corollary 3.6 we now obtain the following result:

Corollary 3.8. *If $g_{r_0} \not\equiv g_{r_1}$ then the analytic continuations of both $I_\pm(t)$ have an infinite ramification around z_0. In particular, this happens if $g_0(z_0) \neq g_1(z_0)$.*

The regular part of g at its ramification point has been defined above in terms of the Puiseax series of g as the sum of all the terms in this series with the integer exponent. In the proof of Theorem 3.4 it was shown that in fact

$$\sum_{j=0}^{n-1} \tilde{g}(t_j) = n g_r(t) \ .$$

In other words, the regular part $g_r(t)$ of $g(t)$ is an average (or a "normalized sum") of all the local branches of g.

Let us summarize the results of Corollaries 3.6 and 3.8 as follows:

Corollary 3.9 *Let $I(t)$ be algebraic. Then at each interior point $z_0 \in \Sigma$ the regular parts of the branches of g on the two sides of z_0 coincide. In other words, the normalized sums* *of the local branches of g on the two sides of z_0 are equal to one another. If z_0 is the endpoint of γ then the sum of the local branches of g at z_0 must be zero.*

Notice that in the statement of the results of Section 3 it is not essential that the number of the local branches of g is exactly the denominator n in the exponents of the Puiseux series. Some of these branches may coincide between themselves - the normalized sum of the branches remains the same.

4. Global structure of $I(t)$: continuation, algebraicity, rationality, and vanishing

Integral representation (1.1) defines $I(t)$ as a collection of univalent regular functions $I_i(t)$ in each domain D_i of the complement of γ in $\mathbb{C}$. In this section we study the relation between $I_i(t)$ in the neighboring domains D_i and on this base analyze their global analytic continuation.

Denote by γ_s the segments of $\gamma \setminus \Sigma$. So g is regular in a neighborhood of each interior point of γ_s. According to Lemma 3.1 for two adjoint domains D_i and D_j separated by their common segment γ_s of the curve γ, I_j is obtained from I_i as follows:

a. I_i is analytically continued through γ_s into a certain neighborhood Ω of γ_s in D_j.

b. An algebraic function g_s in Ω (obtained by the analytic continuation to Ω of the branch g_s of g on γ_s) is added to I_i (multiplied by -1 if the crossing orientation of γ is negative).

c. $I_i + g_s$ is analytically extended from Ω to the entire domain D_j. I_j is equal to this continued function $I_i + g_s$.

The operation of an addition of an algebraic function and its continuation (as extended to several crossings of γ) is "combinatorial" in its nature. It depends only on the monodromy of g and on the geometry of γ and in principle it can be explicitly computed. To define this operation accurately let us consider curves S (or $S_{c,d}$) starting at $c \in \mathbb{C} \setminus \gamma$ and ending at $d \in \mathbb{C} \setminus \gamma$. Say that S is admissible if it avoids singularities of g and crosses γ transversally and only at the interior points of the segments γ_s. For any admissible curve $S_{c,d}$ denote by $S^*_{c,d}$ *the operator of the analytic continuation along $S_{c,d}$ of the analytic germs at c to the analytic germs at d.*

Now let $S_{c,d}$ be an admissible curve with $c \in D_i$ and $d \in D_j$. Suppose that $S \cap \gamma = \{a_1, a_2, ..., a_r\}$ and let $\{g_1, g_2, ..., g_r\}$ be the germs of g at a_i, $1 \leq i \leq r$.

Define a *sum of branches $g(S_{c,d}, \gamma)$ of g along $S_{c,d}$ across* γ as follows: it is a regular algebraic germ at d defined by

$$g(S_{c,d}, \gamma) = \sum_{i=1}^{r} sgn(a_i) S^*_{a_i,d}(g_i), \tag{4.1}$$

where $S_{a_i,d}$ denotes the part of S connecting a_i and d and $sgn(a_i)$ is equal to plus or minus one according to the orientation of the crossing of S and γ at a_i.

The following property of the sum of branches along S is immediate:

Proposition 4.1. *Let the admissible curve $S_{c,d}$ be the union of the admissible curves $S_{c,e}$ and $S_{e,d}$. Then*

$$g(S_{c,d}, \gamma) = S^*_{e,d}(g(S_{c,e}, \gamma)) + g(S_{e,d}, \gamma). \tag{4.2}$$

Denote by Σ_1 the set of those singular points of (all the branches) of g that do not lie on γ (and hence do not belong to Σ). Denote by U the complex plane $\mathbb{C}$ with Σ and Σ_1 deleted. By definition, admissible curves S lie entirely in U.

Proposition 4.2. *The sum of branches along $S_{c,d}$ depends only on the homotopy class (with the fixed end-points) of this curve in U.*

PROOF. As we deform S in U preserving the transversality of the intersection of S and γ, each term in (4.1) remains the same, being the analytic continuation of the same function along a continuously deformed path. Now for a generic deformation of S at each moment of a non-transversal intersection of S and γ a couple of transversal intersections appears (or disappears). The corresponding terms in (4.1) cancel one another.

Remark. The result follows also from Lemma 4.3 below since the analytic continuation of I_i depends only on the homotopy class of S in U.

The following lemma shows that the sum of branches along S measures the difference between the analytic continuation $S^*(I_i)$ of I_i into the domain D_j and the function I_j. Let $S_{c,d}$ be any admissible curve with $c \in D_i$ and $d \in D_j$.

Lemma 4.3. *The germ of I_i at c can be analytically continued along S. The resulting germ $S^*(I_i)$ at d satisfies*

$$S^*(I_i)(t) = I_j(t) - g(S, \gamma)(t). \tag{4.3}$$

PROOF. It follows by induction on the number r of the intersection points of S and γ. If we write (4.3) in the form

$$I_j(t) = S^*(I_i)(t) + g(S, \gamma)(t) \tag{4.4}$$

then for the first crossing of S and γ the equality (4.4) follows directly from the description of the behavior of I on γ given in the steps a, b, c above. Assuming that (4.4) holds after l crossings of S and γ and combining the above description with the definition of the sum of branches along S, we get that (4.4) is valid also after $l + 1$ crossings.

Let $S_{c,c}$ be a closed admissible curve with $c \in D_i$. According to Proposition 4.2, the sum of branches along S across γ depends only on the element $\hat{S}$ of the fundamental group $\pi_1(U, c)$ defined by S. We define a *combinatorial monodromy of I_i at c* as the mapping A_i of $\pi_1(U, c)$ to the analytic germs at c which associates to each $\hat{S} \in \pi_1(U, c)$ the germ $A_i(\hat{S})$ at c equal to the sum of branches along S across γ.

We say that *the combinatorial monodromy of I_i at c is finite* if the image of $\pi_1(U, c)$ under A_i is finite, and we say that *the combinatorial monodromy of I_i at c is trivial* if $A_i(\hat{S}) = 0$ for any $\hat{S} \in \pi_1(U, c)$.

The combinatorial monodromy depends only on the monodromy of g and on the geometry of γ and in principle it can be explicitly computed. In the present paper we mostly restrict ourselves to the local behavior of the combinatorial monodromy. See Remarks 1-3 at the end of this section where we outline a certain global algebraic approach that captures naturally the combinatorial monodromy and simplifies its computation.

Now we are ready to prove the main results of this section. The following Theorem 4.4 provides a description of the complete analytic continuation of the Cauchy integral $I_i(t)$ from the domain D_i where it was initially defined by the expression (1.1). So fix a point $c \in D_i$ and let $I_i(t)$ be the function in D_i defined by (1.1). Remind that the usual *monodromy mapping* MA_i of the fundamental group $\pi_1(U, c)$ to the analytic germs at c is given by

$$MA_i(\hat{S}) = S^*(I_i(t)),$$

for any $\hat{S} \in \pi_1(U, c)$.

Theorem 4.4. *The function $I_i(t)$ allows for a complete analytic continuation as a regular multivalued function $\hat{I}_i(t)$ in U. For any admissible curve $S_{c,d}$ with $c \in D_i$ and $d \in D_j$ the analytic continuation $S^*_{c,d}(I_i)$ of I_i along $S_{c,d}$ is given by $S^*_{c,d}(I_i) = I_j - g(S, \gamma)$. In particular, the monodromy mapping MA_i of $I_i(t)$ is given by $MA_i(\hat{S}) = I_i(t) - A_i(\hat{S})$ for any $\hat{S} \in \pi_1(U, c)$, where A_i is the combinatorial monodromy of I_i.*

At the singular points of g in Σ_1 any leave of the function $\hat{I}_i(t)$ may have only the finite order ramifications. In fact, singularities of $\hat{I}_i(t)$ at the points of Σ_1 are those of certain sums of the branches of g, up to a regular addition.

At each point of Σ all the leaves of the function $\hat{I}_i(t)$ have simultaneously either a finite or an infinite order of the local branching. The analytic representation of each of the leaves of the function $\hat{I}_i(t)$ at these points (up to addition of an algebraic germ) is given by Theorem 3.4.

PROOF. The analytic continuation of $I_i(t)$ along any admissible curve S and its expression via the sum of branches of g across γ is provided by Lemma 4.3. Applying this expression to the closed curve $S = S_{c,c}$ we get the required description of the monodromy action MA_i.

Now let us take S with the end-point d near a singular point $w_0 \in \Sigma_1$ of g. By the representation above we get the leave of the function $\hat{I}_i(t)$ obtained by the analytic continuation along S as the difference of the regular function $I_j(t)$ and a certain sum of the branches of g. This implies the required description of the singularities of the leaves of $\hat{I}_i(t)$ at the points of Σ_1.

Taking S with the end-point d near a point $w_1 \in \Sigma$ we get the corresponding leave of $\hat{I}_i(t)$ as the difference of the possibly singular at w_1 function $I_j(t)$ and a certain finite sum of the branches of g. Therefore the property of this leave to have a finite or an infinite local ramification at w_1 depends only on the local branching of $I_j(t)$. Hence all the leaves simultaneously have at w_1 either a finite or an infinite order of the local branching. Since the analytic representation of $I_j(t)$ at w_1 is given by Theorem 3.4, this provides the required analytic description of the singularities of of all the leaves of $\hat{I}_i(t)$ at the points of Σ, up to addition of an algebraic germ. This completes the proof of Theorem 4.4.

Remark 1. Simple examples show that the combinatorial monodromy of I_i may be infinite. This happens for instance in Example 1 of Section 3 and in Examples 2 and 5 below. In this case we still get certain finite sums of branches of g on each step of forming the sum of branches across γ. However, it is exactly this step of forming sums that may lead ultimately to infinite branching.

Remark 2. Let us return for a moment to the comparison between the functions $I(t)$ and the Abelian integrals. One of the most important analytic properties of the the Abelian integrals is the fact that they satisfy certain Fuchsian linear differential equations with rational coefficients (see [6,7,26-29,32-35,43,46,56,57]). Theorem 4.4 allows us to show that the same is true for the Cauchy type integrals of algebraic functions. Indeed, a necessary and sufficient condition for a multivalued function $y(x)$ to satisfy a linear differential equation with univalued coefficients is that the linear space spanned by the germs of all the branches of $y(x)$ at each point x is finite dimensional. Now by Theorem 4.4 all the branches of the analytic continuation of $I_i(t)$ over the domain D_j are obtained by adding to $I_j(t)$ certain sums of the branches of the algebraic function g. Hence the linear space spanned by the germs of all the branches of the analytic continuation of $I_i(t)$ over the domain D_j always has the basis consisting of $I_j(t)$ and of all the branches of g. More detailed analysis allows one to show that $I(t)$ satisfies in fact a Fuchsian linear differential equations with rational coefficients. It would be important to construct such an equation explicitly and to investigate its relation to the differential equation which is satisfied by the algebraic function g itself.

Remark 3. There is another natural way to compute the monodromy of the Cauchy type integrals of algebraic functions. As the argument t of the function $I_i(t)$ follows the loop S and approaches γ we start to deform γ in order to avoid the crossing of γ by t. Under these circumstances the integral expression (1.1) on the deformed curve γ defines the analytic continuation of $I_i(t)$. After t completes the full loop S, the integration contour γ must be modified to γ' by adding certain loops related to S. The analytic continuation $S^*(I_i(t))$ is given by the Cauchy integral (1.1) over γ'. Of course, the explicit computation of the resulting additions leads to the same formula with the sum of the branches of g as in Theorem 4.4.

We would like to thank L. Gavrilov for pointing out the facts and questions mentioned in Remarks 2 and 3.

Remark 4. The results above provide also a comparison of the analytic continuations of the functions I_i and I_j from two different domains D_i and D_j: these continuations differ by a certain sum of the branches of g.

Theorem 4.4 allows us to give a necessary and sufficient condition for $I_i(t)$ to be an algebraic function. Let us remind shortly some basic definitions. A (multivalued) function $y(z)$ is called *algebraic* if it satisfies an equation $y^d + a_{d-1}(z)y^{d-1} + \ldots + a_1(z)y + a_0(z) = 0$ with $a_i(z)$ rational functions. A singular point z_0 of a multivalued analytic function $f(z)$ is called *algebroid* if $f(z)$ has a finite ramification at z_0 and an absolute value of $f(z)$ near z_0 is bounded by a certain negative power of $|z-z_0|$. A multivalued analytic function $f(z)$ is called *locally algebroid* if all its singularities are algebroid. As we show in a moment, under some additional finiteness assumptions locally algebroid functions are in fact algebraic.

Let $V = \mathbb{C}\backslash\{z_0, \ldots, z_N\}$. Let $f(z)$ be a multivalued regular analytic function in V. We say that $f(z)$ *has a globally finite branching in* V if the number of the different univalued branches of $f(z)$ over each simply-connected domain Ω in V is finite. (This number does not depend on Ω). A basic fact here is the following: *a locally algebroid function $f(z)$ which has a globally finite branching in V is algebraic.* Let us give for completeness a very short proof of this fact. Let $y_1(z), \ldots, y_d(z)$ be the values of all the branches of $f(z)$ at $z \in V$ (taken in any order). Consider the product $(y - y_1(z)) \ldots (y - y_d(z)) = y^d + a_{d-1}(z)y^{d-1} + \ldots + a_1(z)y + a_0(z)$. The coefficients $a_{d-1}(z), \ldots, a_0(z)$ are symmetric functions of $y_1(z), \ldots, y_d(z)$ so they are univalued on V. Since the singularities of all the branches $y_1(z), \ldots, y_d(z)$ of $f(z)$ at the points $z_0, \ldots, z_N$ are algebroid the same is true for $a_{d-1}(z), \ldots, a_0(z)$. Now, *a univalued function on V with all the singularities at $z_0, \ldots, z_N$ algebroid is rational.* Indeed, if we multiply such a function by a polynomial with zeroes of a sufficiently high order at all the finite singularities, we get a regular function on $\mathbb{C}$ with a polynomial growth at infinity, i.e. another polynomial.

Now we are ready to give a criterion for algebraicity of $I_i(t)$.

Theorem 4.5. *The function $I_i(t)$ (and hence all the $I_j(t)$) is algebraic if and only if the combinatorial monodromy of I_i is finite.*

PROOF. Assume that $I_i(t)$ is algebraic (which is the same as to assume that its full analytic continuation $\hat{I}_i(t)$ is algebraic). In this case the image of the monodromy mapping MA_i is finite. By Theorem 4.4 the same is true for the combinatorial monodromy of I_i. Conversely, assume that the combinatorial monodromy of I_i is finite. Then by Theorem 4.4 the monodromy mapping MA_i has a finite image. Hence $\hat{I}_i(t)$ has globally only a finite branching. The singularities of all the branches of $\hat{I}_i(t)$ at the points of Σ_1 are algebraic (up to a regular addition) by Theorem 4.4. In our case of the finite branching the singularities of all the branches of $\hat{I}_i(t)$ at the points of Σ are algebroid. Indeed, by Theorem 4.4 each of these singularities at a certain point $z_0 \in \Sigma$ is given by the sum of the algebraic germ and the germ of the Cauchy integral I_j at z_0, where D_j is the domain adjacent to γ near z_0. Since $\hat{I}_i(t)$ has a finite branching at z_0 the same is true also for $I_j(t)$ at z_0. Consider for simplicity the case of the endpoint z_0 of γ. The case of the interior jump point z_0 is treated exactly in the same way.

According to the representation of $I_j(t)$ given by Theorem 3.4, it has the form

$$I(t) = R(t) - \sum_{j=0}^{n-1} (\frac{j}{n} - \frac{1}{2})\tilde{g}(t_j) + \frac{1}{2\pi i} \sum_{j=0}^{n-1} \tilde{g}(t_j) Log_1(c - t_j) - \frac{1}{2\pi i} g_r(t) Log(t - z_0). \quad (4.5)$$

In the case of the finite branching of $I_j(t)$ the last term in (4.5) containing $\log(t - z_0)$ must disappear. Now, the first three terms of (4.5) are bounded near z_0 and have at this point a finite ramification (remind that the constant c in the third term is different from zero). Hence the singularity of $I_j(t)$ at z_0 is algebroid. Therefore the same is true for the singularity of $\hat{I}_i(t)$ at z_0.

Also at infinity the singularity of any branch of $\hat{I}_i(t)$ is algebroid. Indeed, $\hat{I}_i(t)$ has there a finite branching. Moreover, up to addition of an algebraic germ at infinity any branch of $\hat{I}_i(t)$ coincides there with the regular germ $I_0(t)$ given by (1.1). Thus $\hat{I}_i(t)$ has globally a finite branching and all its singularities algebroid. By the classical description of algebraic functions given above this implies that $\hat{I}_i(t)$ is algebraic. This completes the proof of Theorem 4.5.

Using the same approach we obtain some *local* conditions on g that are necessary for algebraicity of $I(t)$. Below γ may be open or closed.

Corollary 4.6. *Let $I(t)$ be algebraic. Then at each "jump" point $z_0 \in \Sigma$ (including each of the endpoints of γ) a certain nontrivial integer linear combination of the local branches of g must be zero.*

PROOF. Consider the case of the endpoint. Let z_0 be one of the endpoints of γ and let the adjacent domain to γ near z_0 be D_i. Denote by σ a small closed loop going around z_0 in the counter-clockwise direction from a certain point c near γ, and let σ^* denote as above the operator of the analytic continuation along σ. Let g_1 be the branch of g on γ near z_0. By Proposition 4.1 the sum of the branches along σ repeated n times is given by

$$g_1 + \sigma^*(g_1) + {\sigma^*}^2(g_1) + \ldots + {\sigma^*}^n(g_1). \tag{4.5}$$

Now $I_i(t)$ is algebraic and its ramification at z_0 is finite. Fix the smallest n for which ${\sigma^*}^n(I_i(t)) = I_i(t)$. Then the formula (4.3) of Lemma 4.3 shows that for this n the sum (4.5) is zero. This provides the required relation. The proof for the interior point z_0 is essentially the same. We get an equality of certain sum of the branches of g on the two sides of z_0 on γ. Since z_0 was assumed to be a "jump" point of g, the branches of g on the two sides of z_0 on γ cannot be transformed one into another by the local monodromy of g and hence the resulting sum of the branches of g is non-trivial. (We would like to thank the referee for suggesting the above calculation).

Remark 1. Corollary 3.9 is formally stronger than Corollary 4.6 since it provides *the specific vanishing sum* of the local branches of g. A modification of the arguments above allows one to get the same specific vanishing sum via the approach of this section.

Remark 2. Example 5 below shows that in general the vanishing of the sums of the local branches of g given by Corollary 3.9 *does not imply the global finiteness* of the combinatorial monodromy. One can hope that the algebraic approach to the representation and computing the combinatorial monodromy described in the remark at the end of this section can provide a unified way to producing all the necessary and sufficient finiteness conditions in terms of the vanishing of certain sums of the branches of g (local and global).

Let us continue by providing conditions for $I_i(t)$ to be a rational function. Notice that in contrast with the algebraicity case these conditions depend on the specific function $I_i(t)$

(and the domain D_i) we start with. Indeed, if $I_i(t)$ is rational then the functions $I_j(t)$ for $i \neq j$ will be usually only algebraic and not rational (unless g itself is rational).

Theorem 4.7. *$I_i(t)$ is a rational function if and only if the combinatorial monodromy of I_i is trivial.*

PROOF. Assume that $I_i(t)$ is rational. This is the same as to say that its full analytic continuation $\hat{I}_i(t)$ is rational. In this case the image of the monodromy mapping MA_i is the germ $I_i(t)$. Hence by Theorem 4.4 the image of the combinatorial monodromy of I_i is zero. Conversely, assume that the combinatorial monodromy of I_i is trivial. Then by Theorem 4.4 the image of the monodromy mapping MA_i is the germ $I_i(t)$. In particular, this implies that $\hat{I}_i(t)$ is univalued over U. It was shown in the proof of Theorem 4.5 that all the singular points of $\hat{I}_i(t)$ (including infinity) are algebroid. By the basic result on rational functions presented above in this case $\hat{I}_i(t)$ must be rational.

Of course, the local conditions of Corollary 4.6 are satisfied if $I_i(t)$ is rational. Let us present some stronger local conditions for rationality (which in contrast to the algebraicity local conditions are both necessary and sufficient). These conditions just express the non-branching of $\hat{I}_i(t)$ at each of its singularities. We consider separately three cases: the endpoint of γ, the interior "jump" point, and the singular point of g not in γ.

Let z_0 be the endpoint of γ, belonging to the domain D_j, and let $S_{c,d}$ be an admissible curve with $c \in D_i$ and with $d \in D_j$ close to a certain point w on the curve γ. (See Figure 5). The sum of branches $g(S_{c,d}, \gamma)$ of g (along S across γ) is an algebraic germ at d which we extend to the algebraic function F (multivalued in general) defined in D_j. Denote by σ a small closed loop going around z_0 in the counter-clockwise direction from the point w to itself. Finally, let g_1 be the branch of g on γ near z_0.

Proposition 4.8. *If $I_i(t)$ is a rational function then in a neighborhood of w we have $g_1 = F - \sigma^*(F)$. In particular, if $D_j = D_i$ then $F \equiv 0$ and therefore $g_1 \equiv 0$ on γ.*

PROOF. Denote by $\tilde{S}$ the curve following S and then going from d to w and let $T = T_{c,w}$ be the curve following $\tilde{S}$ and then following σ. If $I_i(t)$ is rational then by Lemma 4.3 the sum of branches along any two admissible curves leading to the same point w must be the same. Hence $g(T, \gamma) = g(\tilde{S}, \gamma) = F$. Applying Proposition 4.1 we obtain: $g(T, \gamma) = g_1 + \sigma^*(F)$ or $g_1 = F - \sigma^*(F)$.

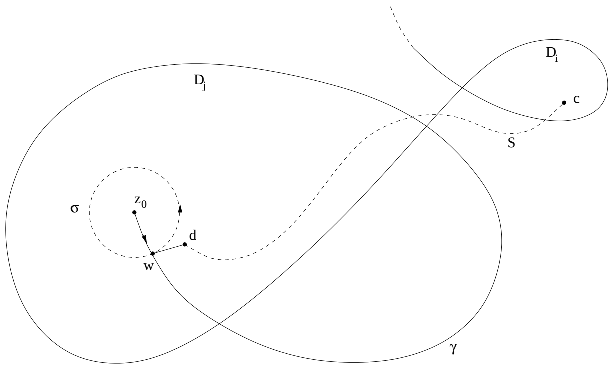

Figure 5

Remark. If the endpoint z_0 of γ belongs to the initial domain D_i and I_i is rational then Proposition 4.8 implies that $g_1 \equiv 0$ on γ. This fact follows immediately also from the possibility to reach both sides of γ near z_0 from the same point c in D_i. Indeed, g_1 on γ is the difference of the continuations of $I_i(t)$ on the two sides of γ. See Figure 6.

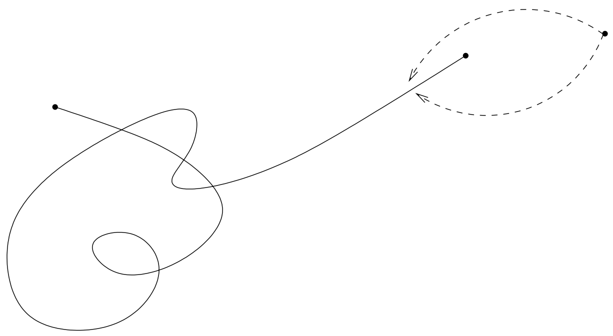

Figure 6

Now let $z_0 \in \Sigma$ be a simple interior point of γ. According to the definition of Σ the function g at z_0 may have a jump and/or a branching point. Denote by g_0 and g_1 the branches of g on γ before and after z_0, respectively. See Figure 7.

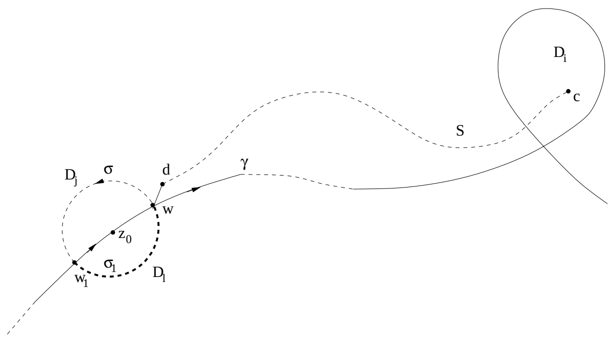

Figure 7

Let the partition domains on the left and on the right of γ near z_0 be D_j and D_l, respectively. Let $S_{c,d}$ be an admissible curve with $c \in D_i$ and with $d \in D_j$ close to a certain point w on the curve γ after z_0 (see Figure 7). The sum of branches $g(S_{c,d}, \gamma)$ at d we extend to the algebraic function F defined in D_j. Denote by σ a small closed loop going around z_0 in the counter-clockwise direction from the point w to itself and let σ_1 be the part of σ in D_l (so σ_1 goes in D_l from a certain point $w_1 \in \gamma$ before z_0 to the point $w \in \gamma$ near d (see Figure 7).

Proposition 4.9. *If $I_i(t)$ is a rational function then in a neighborhood of w we have $g_1 - {\sigma_1}^*(g_0) = F - \sigma^*(F)$. In particular, if $D_j = D_i$ then $F \equiv 0$ and therefore $g_1 \equiv {\sigma_1}^*(g_0)$ on γ.*

PROOF. We use the same auxiliary curves $\tilde{S}$ and $T = T_{c,w}$ as in the proof of Proposition 4.8. In our case application of Proposition 4.1 gives $g(T, \gamma) = \sigma^*(F) - {\sigma_1}^*(g_0) + g_1$ and from the equality $g(T, \gamma) = g(\tilde{S}, \gamma) = F$ we get $g_1 - {\sigma_1}^*(g_0) = F - \sigma^*(F)$.

Remark. An important special case of the situation described in Proposition 4.9 occurs when $i = 0$, $I_0(t) \equiv 0$ and z_0 is on the boundary of the infinite domain D_0. Since $D_j = D_i = D_0$ the proposition gives $g_1 \equiv \sigma_1{}^*(g_0)$. In the specific case considered this follows directly from the fact that both g_0 and g_1 are boundary values of the function I_l in D_l. See Figure 8.

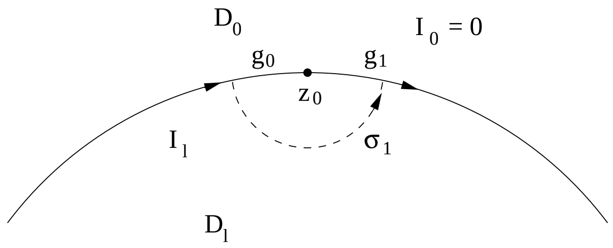

Figure 8

Finally, let $z_0 \in \Sigma_1$ be a singular point of g inside the domain D_j. Let σ, $S_{c,d}$ and F be as above, with d near z_0. See Figure 9.

Proposition 4.10. *If $I_i(t)$ is a rational function then $F - \sigma^*(F) = 0$. In other words, F does not ramify at z_0.*

PROOF. We use the original curve S and the auxiliary curve $T = T_{c,d}$ obtained as S followed by σ. As above, we denote $g(S, \gamma)$ by F. In our case σ does not cross γ and by Proposition 4.1 $g(T, \gamma) = \sigma^*(F)$. Hence the equality $g(T, \gamma) = g(S, \gamma) = F$ proves the required result.

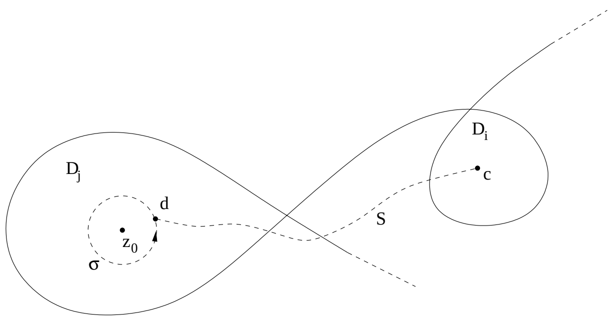

Figure 9

Remark 1. Of course, this result follows directly from Lemma 4.3: $F = S^*(I_i(t)) - I_j(t)$ with $I_j(t)$ regular at z_0 and $S^*(I_i(t))$ rational (being the analytic continuation of the rational function I_i). In particular, the algebraic function F in D_j may have only poles as singularities.

Remark 2. The condition of a non-ramification of F at z_0 can be translated into the vanishing of certain sums of the branches of g (as well as most of the conditions in this section). Indeed, applying σ^* to each of the summands F_r in F and equating the result to F we get the vanishing of a certain sum of the branches of g.

Remark 3. A natural question is under what conditions we get here a *nontrivial* sum of the branches of g. In general, it would be important to find the mutual dependencies between the local conditions of Propositions 4.8-4.10.

Now the necessary conditions for rationality of $I_i(t)$ given by the three propositions above turn out to be also sufficient. Let $z_k,\ k = 1, \dots, N$, be all the points of Σ (including the endpoints of γ) and of Σ_1, taken in a certain order. Let us fix a point $c \in D_i$ and for each z_k let us fix an admissible curve S^k leading to a neighborhood of z_k and a small loop σ^k around z_k (in each case as described in Propositions 4.8-4.10, respectively).

Theorem 4.11 *Assume that at each $z_k,\ k = 1, \dots, N$, (and for the chosen S^k and σ^k) the conclusion of Proposition 4.8 (respectively, 4.9 or 4.10) is satisfied. Then the function $I_i(t)$ is rational.*

PROOF. Denote by $\tilde{S}^k = \tilde{S}^k_{c,c}$ the loop following S^k then σ^k and then returning via S^k in the opposite direction. If the conclusion of the appropriate proposition above is satisfied then the sum of branches along the loop $\tilde{S}^k$ is zero. Indeed, this conclusion expresses the fact that the sum of branches along S^k and along S^k followed by σ^k is the same. But then the sum of branches along the loop $\tilde{S}^k$ is the same as for S^k passed forward and then back, and the last path is homotopic to the constant one. Now the loops $\tilde{S}^k,\ k = 1, \dots, N$, generate the fundamental group $\pi_1(U)$. By Proposition 4.1 if the sum of branches along the loops $\tilde{S}^k,\ k = 1, \dots, N$, is zero the same is true for the products of these loops. Therefore, the combinatorial monodromy of I_i is trivial and by Theorem 4.7 the function $I_i(t)$ is rational.

Finally we come to the conditions for the identical vanishing of $I_i(t)$. It is more convenient to characterize first the property of $I_i(t)$ being identically constant. We shall consider not the complex plane $\mathbb{C}$ but the Riemann sphere $\mathbb{C}P^1$.

Proposition 4.12. $I_i(t) \equiv$ Const *in D_i if and only if the combinatorial monodromy of I_i is trivial and the following additional condition is satisfied: for any $j \neq i$ and for any admissible curve $S_{c,d}$ with $c \in D_i$ and $d \in D_j$ the sum of branches $F_j = g(S, \gamma)$ is regular in D_j. For $i = 0$ (i.e. for the exterior domain D_0 and the Cauchy integral $I_0(t)$ on it) the same conditions are necessary and sufficient for $I_0(t) \equiv 0$.*

PROOF. In one direction the result follows from Theorem 4.7 and Lemma 4.3. Indeed, $I_i(t)$ being identically constant is rational. Hence the combinatorial monodromy of I_i is trivial. Since the analytic continuation $\hat{I}_i(t)$ is the same constant, the relation $F_j = \hat{I}_i(t) - I_j(t)$ of Lemma 4.3 with $I_j(t)$ regular implies regularity of F_j in D_j. In the opposite direction, if the combinatorial monodromy of I_i is trivial then $I_i(t)$ is rational by Theorem 4.7. Since F_j are regular in D_j for any $j \neq i$ the same relation of Lemma 4.3 implies regularity of $\hat{I}_i(t)$ in each D_j. But in D_i itself $\hat{I}_i(t) = I_i(t)$ is regular by definition. Hence the global rational function $\hat{I}_i(t)$ is regular everywhere on $\mathbb{C}P^1$ so it is constant. Since by (1.1) I_0 is equal to zero at infinity I_0 is constant if and only if it is identically zero.

Of course, the condition of Proposition 4.12 essentially coincides with the classical vanishing condition for the Cauchy-type integrals (i.e. that $g|\gamma$ bounds a holomorphic one-chain). This chain is provided by the sums of branches F_j: the definition of the sum of branches shows immediately that $g|\gamma$ is a boundary of $\sum F_j$.

Remark. Regularity of the sum of branches $F_j = g(S,\gamma)$ in D_j is equivalent to the cancellation of the negative Laurent terms of F_j at each singular point of g in D_j. This provides a set of local conditions at the singularities of g in D_j expressed by certain linear equations on the branches of g. By Proposition 4.12 these conditions (together with the requirement that the combinatorial monodromy of I_i be trivial) are equivalent to $I_i(t)$ being identically constant. However, these conditions (in contrast to the "sum of branches" vanishing conditions) are not *a priori* invariant under the monodromy action on g. It would be important to understand the role of these "regularity conditions" and their relation to the rest of the properties investigated above.

One way to explicitly verify conditions of Proposition 4.12 is to check the position of the poles of the algebraic function g.

Corollary 4.13. *If the combinatorial monodromy of I_i is trivial and all the poles of g are in D_i then $I_i(t) \equiv$ Const.*

PROOF. If the combinatorial monodromy of I_i is trivial then for any $j \neq i$ the sum of branches $F_j = g(S,\gamma)$ is a univalued algebraic function in D_j. Since g has no poles in D_j the same is true also for F_j (which is the sum of certain branches of g in D_j). Hence F_j is regular in D_j.

Remark. It is interesting to compare this result with the direct computations for rational functions given in Section 2 above. For g rational and for any γ the sum of branches along any S starting in the exterior domain D_0 and ending in some D_j is equal to $\mu_j g$ (in notations of Section 2). Consequently, the condition of Proposition 4.12 is satisfied if and only if all the poles of g belong to the "outside" of γ i.e. to the domains D_j with $\mu_j = 0$. (In Corollary 2.4 above this result was obtained by a direct computation). A natural question is whether it is possible to relax accordingly the conditions of Corollary 4.13.

Corollary 4.14. *Let all the poles of g belong to the exterior domain D_0. Then the complete analytic continuation $\hat{I}_0(t)$ cannot be univalued on U unless it is identical zero. In particular, $I_0(t)$ at infinity cannot be a nonzero germ of a polynomial, a rational or a meromorphic in $\mathbb{C}$ function.*

PROOF. Theorem 4.4 implies that if $\hat{I}_0(t)$ is univalued on U then the combinatorial monodromy of $I_0(t)$ is trivial. But since all the poles of g belong to the exterior domain D_0, by Corollary 4.13 this implies that $I_0(t) \equiv 0$ at infinity.

Let us consider now some examples. Returning to the Example 1 given in Section 3, we see that at the endpoint 0 of $\gamma = [0,1]$ the sum of local branches of $g(z) = \sqrt{z}$ is zero while at the endpoint 1 the sum of local branches is not zero. Consequently, $I(t)$ has a logarithmic ramification at $z = 1$ while each branch of $I(t)$ has a finite ramification (of order 2) at $z = 0$. On can see this also from the explicit expression for $I(t)$ given in Section 3.

Example 2. Let γ be the unit circle S^1 and let $g(z) = \sqrt{z}$, with $g(1) = 1$, analytically continued along S^1 in a counter-clockwise direction. So g has a jump at 1. The sum of the branches of g across S^1 and along the curve S shown by a dotted line on Figure 10-a, is twice the positive branch of the $\sqrt{z}$ at $z = 2$. In the same way we can see that the sum of branches along S repeated n times is $+2n\sqrt{z}$. Hence the combinatorial monodromy of $I(t)$ is infinite in this example. The analytic continuation of $I(t)$ both from inside and from outside S^1 has a logarithmic branching at $z = 1$.

Example 3. For the same $g(z) = \sqrt{z}$ and a curve γ going twice around 0, $I_0(t) \equiv 0$ at ∞. Indeed, after the substitution $z = w^2$ we get $I(t) = \frac{1}{\pi i} \int_{S^1} \frac{w^2 dw}{w^2 - t}$ and for t near infinity the integrand is regular inside S^1. Hence in this example the combinatorial monodromy is trivial. The sum of branches starting in the exterior domain D_0 gives $+\sqrt{z}$ in the domain D_1 and the identical zero in the domain D_2 containing the origin (see Figure 10-b). Accordingly, the Cauchy integral which coincides in our case with the sum of branches gives $I_0(t) \equiv 0$, $I_1(t) = +\sqrt{t}$, and $I_2(t) \equiv 0$.

Example 4. Let $g(z) = \sqrt{z(z-1)}$ and let the curve γ go around 0 and 1 in an "∞" shape (see Figure 10-c). We continue along γ the positive branch of g at $z = 2$. Here the continuation "closes up" and g does not have jumps on γ. Nevertheless, the sum of branches across γ and along a curve S shown by a dotted line on Figure 10-c gives twice the positive branch of the $\sqrt{z(z-1)}$ at $z = 3$. To simplify the notations we denote the germ of this branch at $z = 3$ by a. Since $a \neq 0$ the combinatorial monodromy of I_0 is not trivial and $I_0(t)$ is not rational. Here the obstruction is not in the jump point of g on γ but rather in the branching of g at $z = 0$ and $z = 1$.

Let us show that in this example $I(t)$ is in fact an algebraic function. To do this consider a second loop S' going from $z = 3$ around the origin as shown on Figure 10-c. An easy computation gives $g(S', \gamma) = 2a$. Applying Proposition 4.1 we obtain that the sum of branches along SS, $S'S$, SS', and $S'S'$ is zero as well as along other products of any two loops S, S' or their inverses. (We use the notation for the product of the loops in the fundamental group of $\mathbb{C} \setminus \{0, 1\}$ where the loops in the product are passed in the order they are written from the left to the right). Remind that the monodromy of g along both S and S' is given by a multiplication by -1. Now application of Proposition 4.1 to any product of S and S' in the fundamental group of $\mathbb{C} \setminus \{0, 1\}$ shows that the sum of branches along this product is either $2a$ or zero. Hence the combinatorial monodromy of I_0 is of order 2 and by Theorem 4.5 I_0 is algebraic (as well as the other I_j) but not rational.

Let us stress once more that in this example the curve γ is closed and the function g is regular at each point of γ. In particular, g has no "jump point" on γ. So the set Σ contains exactly one point – the double point of γ. But since g is regular near this points on both the crossing segments of the curve γ, any branch of $I_0(t)$ at this point is regular (by Lemma 3.2 and Theorem 4.4). Therefore, in this example all the conditions of the Propositions 4.8 and 4.9 are automatically satisfied. This is the condition of Proposition 4.10 that is violated at the singular points 0 and 1 of g and that prevents $I_0(t)$ from being rational.

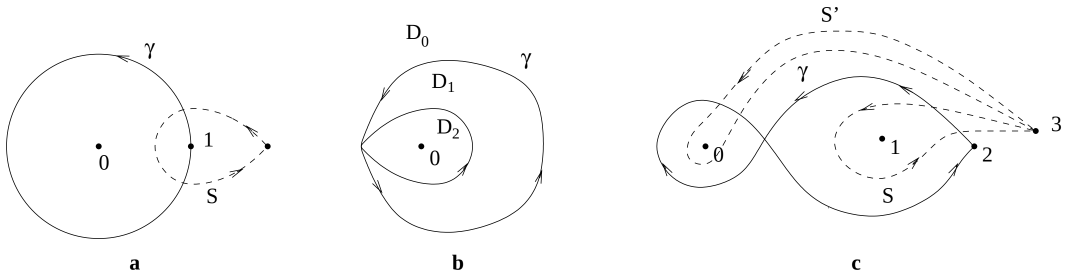

Figure 10

Example 5. Let γ be the interval $[-1,1]$ and let g on $[-1,1]$ be given by the positive branch of $(1-z^2)^{\frac{1}{r}}$. We shall show that for $r=2$ the Cauchy integral $I(t)$ is a non-rational algebraic function, while for any integer $r \geq 3$ the function $I(t)$ is a non-algebraic locally algebroid function. First of all let us notice that $g(z)$ satisfies the equation $g^r-(1-z^2)=0$. Hence, the sum of all the branches of $g(z)$ is identically zero (being equal to the $(r-1)$-th coefficient of the equation defining $g(z)$). Since the local germs of $g(z)$ at -1 and 1 contain *all its branches* the sum of the local branches of $g(z)$ at each of its ramification points -1 and 1 is zero. By Theorem 3.4 and Theorem 4.4 this implies that for any leave of the full analytic continuation $\hat{I}(t)$ of the Cauchy integral $I(t)$ its branching at -1 and 1 is finite and the growth is bounded. Therefore all the singularities of all the leaves of $\hat{I}(t)$ are algebroid and hence $\hat{I}(t)$ is locally algebroid.

Now consider two loops S and S' going in a counter-clockwise direction around -1 and 1, respectively, from a fixed point c on the negative part of the imaginary axis near the interval $[-1,1]$ (see Figure 11).

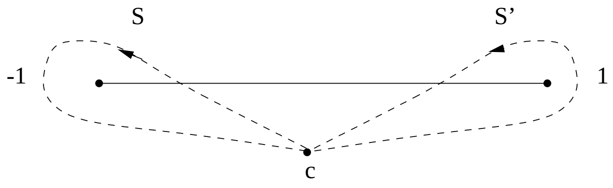

Figure 11

The monodromy of g along both S and S' is given by a multiplication by $\epsilon = \exp(\frac{2\pi i}{r})$. Denote by a the germ at c of the branch of $(1-z^2)^{\frac{1}{r}}$ that takes positive values on the interval $[-1,1]$. On can easily see that the sum of branches along S and S' is equal to ϵa and $-a$, respectively.

Now let us fix $r=2$. In this case $\epsilon=-1$ and the sum of branches gives $-a$ along both S and S'. Proposition 4.1 then shows that for the sums of branches along the products SS' and $S^{-1}S'$ we have $g(SS',\gamma)=0$ and $g(S^{-1}S',\gamma)=0$. Also the sums of branches along the products of any other two loops S, S' or their inverses gives zero. Using this fact we show exactly as in Example 4 above that the sum of branches along any product of S and S' in the fundamental group of $\mathbb{C} \setminus \{-1,1\}$ is either $-a$ or zero. Hence in this case the combinatorial monodromy of I_0 is of order 2 and by Theorem 4.5 the function $I_0(t)$ is a non-rational algebraic function.

Finally, to show that $I_0(t)$ is not algebraic for $r \geq 3$ consider the loop $S^{-1}S'$ (see Figure 11). The monodromy of g along this loop is trivial. An easy computation shows that the sum of branches $g(S^{-1}S', \gamma)$ of g along $S^{-1}S'$ is equal to $b = -a(1+\epsilon) \neq 0$. Let us formulate the last step of our computation as a lemma.

Lemma 4.15. *Let $S = S_{c,c}$ be an admissible loop. Assume that the monodromy of g along S is identical. If the sum of branches $g(S, \gamma) = a$ then the sum of branches $g(nS, \gamma) = na$ for any natural n.*

PROOF. This is a direct consequence of Proposition 4.1.

Applying Lemma 4.15, we see that the sum of branches along the loop $S^{-1}S'$ passed n times is nb. Since $b \neq 0$ we conclude that the combinatorial monodromy of I is infinite and therefore by Theorem 4.5 $I(t)$ is not algebraic for $r \geq 3$.

Remark. Example 5 presents a sequence of functions

$$I^r(t) = \int_{-1}^{1} \frac{(1-z^2)^{\frac{1}{r}} dz}{z-t} \tag{4.6}$$

with I^2 algebraic and I^r transcendental but "locally algebroid" (according to the definition above) for any natural $r \geq 3$. In fact, the functions I^r also for $r \geq 3$ possess a number of remarkable properties which put them very close to the algebraic ones. First of all, the full analytic continuation $\hat{I}^r$ of I_0^r is regular in the domain $U = \mathbb{C} \setminus \{-1, 1, \infty\}$. Secondly, each of the infinite number of the leaves of $\hat{I}^r$ has at the points $-1, 1, \infty$ algebroid singularities with the branching of order r. Finally, the monodromy action of the fundamental group $\pi_1(U)$ on the branches of $\hat{I}^r$ can be represented in a relatively simple way via the combinatorial monodromy of I^r (see Theorem 4.4 and the computations in Example 5 above).

The functions with the same properties as I^r appear as Cauchy Integrals of algebraic functions in many important cases. In general, by Theorem 4.4 this happens if and only if the local sums of the branches of g (given by Corollary 3.9) vanish at each of the jump points of g on γ (including the end points of γ), while the combinatorial monodromy is infinite. In particular, the first condition is satisfied if γ is closed and g is regular on γ. Especially important example of this sort is provided by the rational Moment generating function (1.3) on the closed path Γ – the case which corresponds to the classical Center-Focus problem. However, in this case one can easily show that this function is in fact algebraic.

It turns out that the functions as above are closely related to certain Kleinian groups and automorphic functions. Indeed, their ramification properties are rather similar to that of the inverse to the factorization mapping by certain Kleinian groups. Composing the functions as above with this factorization mapping we get a single-valued meromorphic function whose behavior on the shifts of the fundamental domain is described via the combinatorial monodromy of the original Cauchy integral. We plan to present separately the rigorous results in this direction.

We would like to thank S. Natanzon for pointing out the relations mentioned in the above remark.

Example 6. Let us return once more to the case of g itself being a rational function. The direct computations given in Section 2 show that in this case $I_0(t)$ as well as each $I_i(t)$ are rational functions. As it was mentioned in the remark above, for g rational and for any γ the sum of branches along any S starting in the exterior domain D_0 and ending in some D_j is equal to $\mu_j g$ (in notations of Section 2). As we can expect, the combinatorial monodromy is trivial (although the nontrivial sums of the branches of g do appear. These sums reflect just the geometry of γ). So $I_0(t)$ is always rational. Now for each domain D_j the sum of branches F_j (obtained along any curve going fromD_0 to D_j) is equal in D_j to $\mu_j g$. By Proposition 4.12 $I_0(t) \equiv 0$ if and only if the functions F_j are regular in D_j for any $j \neq 0$. Now for $\mu_j = 0$ this is automatic and for $\mu_j \neq 0$ the only way for F_j not to have poles in D_j is that g itself does not have poles. Consequently, $I_0(t) \equiv 0$ if and only if all the poles of g belong to the "outside" of γ i.e. to the domains D_j with $\mu_j = 0$.

Example 7. In this example γ is the union of the interval $[-1, 1]$ and of the circle S_2 of radius 2 centered at the origin (Figure 12-a). The function g on $[-1, 1]$ is the same as in Example 5 (for $r = 2$) i.e. the positive branch of $\sqrt{(1 - z^2)}$. The function g on S_2 is the branch of $\frac{1}{2}\sqrt{(1 - z^2)}$ taking the values with the positive real part above the interval $[-1, 1]$. Now the direct computation shows that the conclusion of Proposition 4.8 is satisfied at each endpoint -1 and 1 of $[-1, 1]$ (which are also the only singular points of g). There are no jump points of g on γ. Hence the conclusions of Proposition 4.9 and 4.10 are automatically satisfied. By Theorem 4.11 we obtain that $I_0(t) \equiv 0$. We see also directly that g on γ bounds the chain F equal to the branch of $\frac{1}{2}\sqrt{(1 - z^2)}$ taking the values with the positive real part above the interval $[-1, 1]$. (This statement can be interpreted also by replacing the interval $[-1, 1]$ with its two copies passed in the opposite directions and with g equal to the positive branch of $\frac{1}{2}\sqrt{(1 - z^2)}$ on the "upper" interval and equal to the negative branch of $\frac{1}{2}\sqrt{(1 - z^2)}$ on the "bottom" one).

One can modify slightly the construction above and get a *connected* non-closed curve γ' for which g is not zero on γ' but $I_0(t) \equiv 0$. The curve γ' starts at -1, goes along $[-1, 1]$ till 0, then goes up till S_2, encircles S_2 in the clock-wise direction, returns to the interval $[-1, 1]$ at 0, and finally goes along $[-1, 1]$ till 1 (see Figure 12-b). The piecewise algebraic function g on γ' is defined as above on the parts of γ' belonging to $[-1, 1]$ and S_2 and it is defined as a linear interpolation of the end values on the inserted parts. As the integration on the two inserted intervals goes in the opposite directions, the Cauchy integral of the extended g on γ' is the same as the original integral on γ. An important question here is whether such an example (with $\gamma = P(\Gamma)$ non-closed, $g = Q(P^{-1})$, and $I_0(t) \equiv 0$) can appear in the polynomial Moment problem.

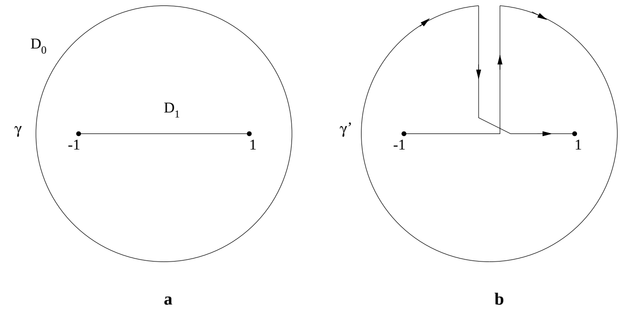

Figure 12

Example 8. This rather unexpected example presents a curve γ and an algebraic function g which does have a "jump" on γ and still $I(\gamma, g, t) \equiv 0$ for t near ∞. As it was shown above under this condition the regular parts (or the normalized sums of the branches) of g on both sides of the jump point must coincide. However, in the situation below the branches themselves on the two sides cannot be transformed one into another by any local analytic continuation. Equivalently one can say that the full local germs of g on both sides of the jump point are different.

From the point of view of Proposition 4.9 above, we see that in this example the difference between the branch g_0 of g on one side of 0 and the local analytic continuation $\sigma_1^*(g_1)$ of the branch g_1 on the other side of 0 is nonzero. However, it is compensated by the "monodromy shift" $F_1 - \sigma^*(F_1)$ at 0 of the Cauchy integral $I(t) = F_1$ (as defined in the domain D_1 next to the exterior domain D_0).

This example seems to us to be quite unexpected. It is based on a recent counterexample ([39]) to the "Moment Composition conjecture" (which asserted that the vanishing of the moments (1.2) is equivalent to the Composition condition (PCC). See Section 1.1.2 above).

Theorem 4.16. *Let γ be the curve in $\mathbb{C}$ shown in Figure 13 below, and let $g(z) = Q(P^{-1}(z))$ with $Q(x) = T_2(x) + T_3(x)$ and $P(x) = T_6(x)$, where $T_n(x) = \cos(n \arccos(x))$ is the n-th Chebyshev polynomial. The function g is analytically continued from 0 along γ in the positive direction, starting with $Q(-\sqrt{3}/2)$. Then $I(\gamma, g, t) \equiv 0$ for t near ∞, while the branches of g on the two sides of $0 \in \gamma$ cannot be obtained from one another by any local analytic continuation.*

PROOF. First of all, one can easily check that the curve γ shown in Figure 13 is equal to $P(\Gamma)$, with Γ obtained from the real interval $[-\sqrt{3}/2, \sqrt{3}/2]$ by a small shift (preserving the ends) into the positive imaginary direction. Hence $I(t)$ is given by

$$2\pi i I(t) = \int_a^b \frac{Q(x)p(x)dx}{P(x) - t} \qquad (*),$$

with $a = -\sqrt{3}/2$, $b = \sqrt{3}/2$ and P and Q as above. Therefore $I(\gamma, g, t) \equiv 0$ for t near ∞. Indeed, for $Q = T_2$ and for $Q = T_3$ the corresponding integrals vanish since the composition condition (PCC) is satisfied: $T_6 = T_2(T_3) = T_3(T_2)$ and all these three polynomials take equal values at a and b. Then for the sum $Q = T_2 + T_3$ the integral $I(t)$ vanishes by linearity in Q. Now Claim 2 from [39] implies that P and Q do not have a common composition factor. Therefore, by a well-known characterization of the composition factors (see Lemma 5.1 and the "Gluing Condition" in Section 5 below) the branches of g on the two sides of $0 \in \gamma$ cannot be obtained from one another by any local analytic continuation. This completes the proof.

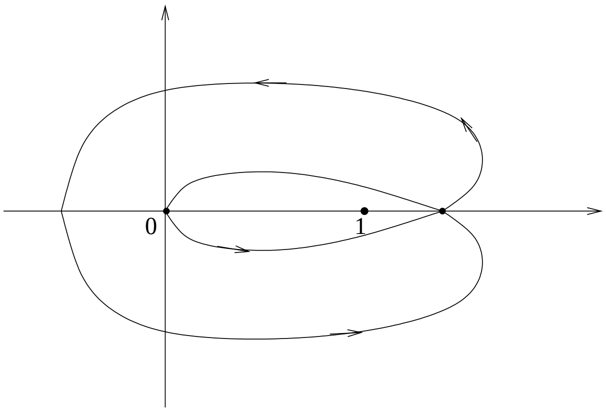

Figure 13

Notice that in the example above $g = g_1 + g_2$, with $g_1 = T_2(T_6^{-1})$ and $g_2 = T_3(T_6^{-1})$, respectively. For each of g_1 and g_2 separately, their branches on the two sides of $0 \in \gamma$ can be obtained from one another by a local analytic continuation, but along two different local paths. In fact, these two paths are the two parts of the small circle around 0. Each one joins the two pieces of γ on the two sides of $0 \in \gamma$, but in different domains D_j.

Let us conclude this section with three remarks outlining some further development directions for the tools introduced above.

Remark 1. In the present paper we consider the "sum of branches" of g across γ and along an auxiliary admissible curve S. This notion can be further extended to the "analytic continuation across γ and along S" which can be applied to any analytic germ u at the starting point c of S (not necessarily to the germ defined by the Cauchy integral). This operation consists in a continuation of u along S till the first crossing of S with γ. At this crossing we add the germ of g at the crossing point (with the sign corresponding to the orientation of the crossing). The resulting sum is analytically continued along S till the next crossing of S with γ. At this crossing we add the germ of g at the crossing point, and so on till the endpoint d of S. Lemma 4.3 shows that the functions I_i are obtained from one another via the analytic continuation across γ (and along any auxiliary admissible curve S). The notion of the continuation across γ allows one to extend the "combinatorial monodromy" (defined above as a *mapping* from the fundamental group $\pi_1(U)$ into the germs at a certain point) into the *action* of $\pi_1(U)$ on these germs.

Remark 2. In some cases it may be convenient to define the continuation across γ as a combinatorial process related to the functions in the domains D_j and not involving auxiliary curves S. For two adjacent domains D_i and D_j separated by their common segment γ_s of the curve γ and for a function u_i in D_i define its continuation across γ into the domain D_j as follows:

a. u_i is analytically continued through γ_s into a certain neighborhood Ω of γ_s in D_j.

b. An algebraic function g_s in Ω (obtained by the analytic continuation of the branch g_s of g on γ_s) is added to u_i (multiplied by -1 if the crossing orientation of γ is negative).

c. $u_i + g_s$ is analytically extended from Ω to the entire domain D_j. The function u_j is equal to this continued function $u_i + g_s$.

Let us call the process consisting of the steps a, b, c an "elementary combinatorial continuation across γ" and let us call any chain of subsequent elementary continuations a "combinatorial continuation across γ". Assuming that all the analytic continuations in the steps a, b, c are possible, the combinatorial continuation across γ can be applied to any analytic function defined in one of the domains D_i. In some cases this notion simplifies significantly the description of the combinatorial monodromy.

Remark 3. In the present paper we do not use a possibility to deform γ and to bring it to a certain simple standard form (without affecting the Cauchy integral $I(t)$ near infinity). We also do not try to push forward a unified algebraic framework where the combinatorial monodromy of I can be naturally represented. Both these tools are important for the study of the Cauchy-type integrals, and we plan to present them in detail separately, giving here only a very short and informal outline of our approach.

In the process of the deformation of γ it is natural in our setting to keep fixed the "jump" points Σ of g on γ. Besides this, γ can be deformed in an arbitrary way until it stays in U (defined as above as $\mathbb{C}$ with Σ and all the singularities of g deleted). In fact, in the process of the deformation certain crossings by γ of the points of Σ and of the singularities of g are also permitted (those which do not affect the local branch of g on γ at the crossing). This allows us to bring γ to a simple standard form which we call the Diagram of $I(t)$. For the Cauchy integrals $I(t)$ coming from the Polynomial moment problem, the Diagram of $I(t)$ can be computed quite effectively using the methods of the "Topological Theory of polynomials" involving a graphical representation of the monodromy group by means of the graphs obtained as the preimages of certain intervals, etc. Some applications of these methods to the Moment problem can be found in [40-41].

The sums of branches across γ (and more generally, the result of the continuation across γ as defined in the remarks 1 and 2 above) can be easily read off the Diagram. In these terms simple necessary and sufficient conditions for the identical vanishing of the Cauchy integrals $I(t)$ near infinity can be given.

Now let us describe a global algebraic framework for representing the combinatorial monodromy. It was shown above that the continuation across γ and along a given curve depends only on the homotopy class of S in U. Therefore, it can be described through a certain action of the fundamental group $\pi_1(U)$ on the branches of g. From the algebraic point of view, the object which naturally appears here is the $Z(G)$-module M consisting of all the "formal" finite sums of the branches of the algebraic function g at a given point $c \in U$. Here G is the monodromy group of g. There is also an "evaluation homomorphism" from M into the $Z(G)$-module of the germs of the analytic functions at c.

In terms of the continuation across γ a certain action A of the fundamental group $\pi_1(U, c)$ on the module M can be described. This action A provides an algebraic representation of the combinatorial monodromy of I. Consequently, the study of the algebraic structure of the module M and of the action A on it provides a natural and strong tool for a description of the global ramifications of the Cauchy integral $I(t)$ and of its singularities. In particular, various "vanishing sums of the branches" relations given in this section can be naturally expressed in this language. Indeed, all the vanishing sums of the branches of g form a $Z(G)$-submodule M_0 of M (the kernel of the evaluation homomorphism). The study of the algebraic structure of M_0 allows us to represent the "vanishing sums of the branches" relations in a uniform way and to determine their mutual dependence.

For the Cauchy integrals $I(t)$ coming from the Polynomial moment problem an important algebraic information on $M_0 \subset M$ and A can be obtained explicitly using the methods of the "Topological Theory of polynomials" mentioned above.

5. Polynomial moments on an interval

In this paper we investigate the polynomial moments (1.2)

$$m_k = m_k(P, Q) = \int_b^a P^k(x)Q(x)p(x)dx \tag{5.1}$$

and the Moment generating function (1.3)

$$H(y) = \int_a^b \frac{Q(x)p(x)dx}{t - P(x)} = \sum_{k=0}^{\infty} m_k y^{-k-1}. \tag{5.2}$$

Here $P(x)$ and $Q(x)$ are polynomials in $x \in \mathbb{C}$, $a, b \in \mathbb{C}$. As above, we denote by $p(x)$ and $q(x)$ the derivatives of $P(x)$ and $Q(x)$, respectively.

However, in most of the preceding papers [9-18,21,39-41,47,61] a slightly different definition for the moments was used:

$$\tilde{m}_k = \int_b^a P^k(x)q(x)dx, \quad k \geq 0. \tag{5.3}$$

The setting of (5.1) and (5.2) is more convenient for the purposes of the present paper since it leads to the Cauchy integral (1.1) with the function $g(z) = Q(P^{-1}(z))$ which does not have pole singularities at the finite points of $\mathbb{C}$ (see (5.5) below). In contrast, in the Cauchy integral for the generating function $\tilde{H}(y)$ of the moments (5.3) the function g is given by $g(z) = (\frac{q}{p})(P^{-1}(z))$ and it may have poles at the finite points of $\mathbb{C}$, in particular, on γ.

Let us show that the problems of the vanishing of the moments m_k and $\tilde{m}_k$ are essentially the same. We do not assume a priori that $P(a) = P(b)$ or $Q(a) = Q(b)$. Notice also that adding a constant in Q does not affect the moments $\tilde{m}_k$, while it may affect the moments m_k.

Claim. *The vanishing of the moments $\tilde{m}_k$ implies $Q(a) = Q(b)$. Assuming that the primitive function $Q = \int q$ is chosen to satisfy $Q(b) = Q(a) = 0$ all the moments m_k also vanish. In the opposite direction, if $Q(b) = Q(a) = 0$ then the vanishing of the moments m_k implies that of $\tilde{m}_k$.*

PROOF Set, as above,

$$H(t) = \sum_{k=0}^{\infty} m_k t^{-(k+1)}, \text{ and let } \tilde{\mathrm{H}}(\mathrm{t}) = \sum_{\mathrm{k}=0}^{\infty} \tilde{\mathrm{m}}_{\mathrm{k}} \mathrm{t}^{-(\mathrm{k}+1)}.$$

Then

$$H(t) = \int_a^b \frac{Q(z)P'(z)dz}{t - P(z)}, \qquad \tilde{H}(t) = \int_a^b \frac{q(z)dz}{t - P(z)}.$$

We have:

$$\frac{dH(t)}{dt} = -\int_a^b \frac{Q(z)P'(z)dz}{(t - P(z))^2} = -\int_a^b Q(z)d(\frac{1}{t - P(z)}) =$$

$$= -\frac{Q(z)}{t - P(z)}|_a^b + \int_a^b \frac{q(z)dz}{t - P(z)} = \frac{Q(a)}{t - P(a)} - \frac{Q(b)}{t - P(b)} + \tilde{H}(t). \tag{5.4}$$

Suppose now that $\tilde{m}_i = 0$ for all $i \geq 0$. Then in particular $\tilde{m}_0 = Q(b) - Q(a) = 0$ and hence for any choice of $Q(z) = \int q(z)dz$ the equality $Q(a) = Q(b)$ holds. Choose now $Q(z)$ such that $Q(a) = Q(b) = 0$. Then by (5.4) $\frac{dH(t)}{dt} = \tilde{H}(t)$. Therefore $\tilde{m}_i = 0$, $i \geq 0$, implies that $m_i = 0$, $i \geq 0$. In the opposite direction, under the assumption $Q(a) = Q(b) = 0$ the formula (5.4) shows that the vanishing of the moments m_k implies that of $\tilde{m}_k$.

Let us return now to our original expressions (5.1) and (5.2). A change of variables $P(x) = z$ brings (5.2) to the form

$$H(y) = -2\pi i I(y) = -\int_\gamma \frac{g(z)dz}{z - y}\ , \tag{5.5}$$

with $\gamma = P([a, b])$ and $g(z) = Q(P^{-1}(z))$. Notice that the requirement that for $z \in \gamma$ the point $P^{-1}(z)$ belongs to Γ defines the branch of P^{-1} on γ uniquely at any simple point of $z \in \gamma$. Therefore the above expression $g(z) = Q(P^{-1}(z))$ correctly defines a piecewise-algebraic function g on γ which satisfies all the requirements of (1.1).

Notice also that for $|t| \gg 1$ we can take in (5.2) any integration path joining a and b. We shall use this later.

To investigate the relation between the vanishing of the moments and the Composition condition we need the following property of the algebraic function $Q(P^{-1}(z))$ (see [21,39-41,44,45,47,52]):

Lemma 5.1. *Let P and Q be two rational functions. There exist rational $\tilde{P}$, $\tilde{Q}$, and W, $\deg W > 1$ such that*

$$P(x) = \tilde{P}(W(x)), \quad Q(x) = \tilde{Q}(W(x))$$

if and only if in a certain simply-connected domain Ω not containing critical values of P for two different branches $P_0^{-1}(z)$ and $P_1^{-1}(z)$ the following equality is satisfied: $Q(P_0^{-1}(z)) \equiv Q(P_1^{-1}(z))$.

Under the additional assumption that $P(a) = P(b) = z_0$ there exist rational $\tilde{P}$, $\tilde{Q}$, and W with

$$P(x) = \tilde{P}(W(x)), \quad Q(x) = \tilde{Q}(W(x)) \text{ and } \mathrm{W(a) = W(b)}$$

if and only if the full local germs of $g_0 = Q(P_0^{-1})$ and $g_1 = Q(P_1^{-1})$ at z_0 coincide. Here the branches P_0^{-1} and P_1^{-1} of P^{-1} take at z_0 the values a and b, respectively.

We shall call the property of the coincidence of the two local germs $g_0 = Q(P_0^{-1})$ and $g_1 = Q(P_1^{-1})$ at z_0 *the Gluing condition.* In the setting of the Cauchy integral (5.5) it is equivalent to the fact that *the branches $g_0(z)$ and $g_1(z)$ of $g(z) = Q(P^{-1}(z))$ on the two sides of z_0 on γ (corresponding to the branches P_0^{-1} and P_1^{-1} of P^{-1} on γ taking at z_0 the values a and b, respectively), can be obtained from one another by a local analytic continuation near z_0.*

Below in this section we always assume that P and Q are polynomials and $P(a) = P(b) = z_0$. In this case the Gluing condition is equivalent to the polynomial Composition condition (PCC) of Section 1.1.2:

$$P(x) = \tilde{P}(W(x)), \quad Q(x) = \tilde{Q}(W(x)), \quad W(a) = W(b),$$

with $\tilde{P}$, $\tilde{Q}$, and W polynomials.

As it was mentioned above, the "sum of the branches" condition provided by Corollary 3.9 (and necessary already for algebraicity of $I(t)$) plays a central role in the investigation of the Moment vanishing in [14-17,21,40-41,47,61]. The Gluing condition can be considered as a special case of the vanishing of the sum of the branches. Indeed, it corresponds to the case where there are exactly two branches of g in the sum (on each side of z_0 in γ) with the signs 1 and -1. For a and b – regular points of P this is the case already for the initial relation produced by Corollary 3.9 (see [21]). The main approach of [14-17] in the case of real polynomials and of [21,40-41,47,61] in general case is to produce the Gluing condition starting with a more complicated initial vanishing sum of the branches and using some additional algebraic (or analytic) considerations. A similar approach is used also in Section 6 below.

In the present section we concentrate on the consequences of the stronger (than the vanishing of the sum of the branches in Corollary 3.9) condition provided by Proposition 4.9. This condition is necessary for the rationality of $I(t)$ and consequently for its identical vanishing. It turns out that under some additional geometric assumptions on P this stronger condition leads directly to the Gluing condition for $Q(P^{-1})$.

In particular, we shall describe on this base some natural classes of "definite polynomials". Let us remind that *definite* polynomials P have been defined in Section 1.1.2 above as those for which the vanishing of the one-sided moments (1.2) implies (and hence is equivalent to) the composition condition (PCC) for any Q. The role of definite polynomials in the local Center-Focus problem is shown in [9,18,61]. Even more important role they play in the global study of the Center equations near infinity as presented in [17]. (At the end of Section 1.1.2 above we outline very shortly these applications). In the present section we describe some classes of definite polynomials P specified by the geometry of the curve $\gamma = P([a,b])$. In this connection a simple geometric invariant of complex univariate polynomials is introduced and some results and problems concerning this invariant are stated.

The starting result here is the following:

Theorem 5.2. *Let $P(x)$ be a complex polynomial, $P(a) = P(b) = z_0$. Assume that there exists a path $\Gamma \subset \mathbb{C}$ joining a and b such that z_0 is a simple point of $\gamma = P(\Gamma)$ and z_0 is on the boundary of the exterior domain D_0. Then for any polynomial Q the moments $m_k(P,Q)$ defined by (1.2) vanish if and only if the composition condition (PCC) is satisfied. In particular, $P(x)$ is definite on $[a,b]$.*

PROOF. We use Proposition 4.9. In a special case where z_0 is on the boundary of the exterior domain D_0, it implies that $g_0 = \sigma_1^*(g_1)$ for g_0 and g_1 the branches of g on the two sides of z_0 in γ and σ_1 a small path joining these two sides. Hence g_0 and g_1 can be obtained from one another by a local analytic continuation near z_0 and the Gluing condition is satisfied.

Figure 14 shows two types of γ with respect to $z_0 = P(a) = P(b)$. Another case of $P(a) = P(b) = z_0$ being "strongly inside" $\gamma = P(\Gamma)$ (and for any Γ joining a and b, as we shall see below) is given by Example 8 of Section 4 above (see Figure 13).

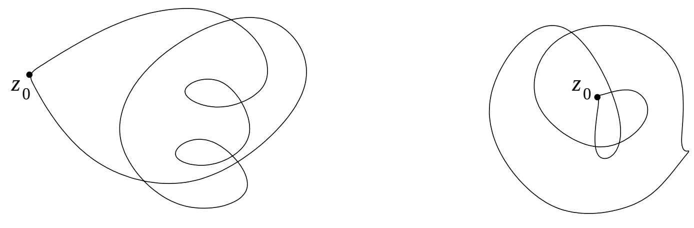

Figure 14

It would be important to characterize explicitly those polynomials P (and couples $a, b \in \mathbb{C}$, $z_0 = P(a) = P(b)$) for which *there is a path Γ joining a and b such that $z_0 = P(a) = P(b)$ is a simple point of $\gamma = P(\Gamma)$ and this point is on the boundary of the exterior domain D_0*. Let us call this *Property (E).*

In this context it is natural to generalize slightly this property and to introduce a certain invariant of complex polynomials measuring how far from the exterior domain D_0 lies the image of the endpoints. More accurately, *the invariant $D(P,a,b)$ of a polynomial $P(x)$ with respect to a, b, $P(a) = P(b) = z_0$, is the minimal "depth" of the point z_0 with respect to the curve $\gamma = P(\Gamma)$ for various $\Gamma \subset \mathbb{C}$ joining a and b. (The "depth" here is the* *minimal number of crossings γ necessary to join z_0 to infinity.)*

Let us summarize some properties of $D(P,a,b)$.

1. For some P, $D(P,a,b)$ may be strictly positive. For example, for $P=T_6$ - the 6-th Chebyshev polynomial, $a=-\sqrt{3}/2$, $b=\sqrt{3}/2$ the invariant $D(P,a,b)$ is equal to one. Indeed, as it was shown in [39] for $P=T_6$ and $Q=T_2+T_3$ $\quad I(t)\equiv 0$ at ∞ but (PCC) does not hold (see also Example 8, Section 4). If there exists a path $\Gamma\subset\mathbb{C}$ joining a and b for which $0=P(a)=P(b)$ lies on the boundary of the exterior domain D_0 with respect to $\gamma=P(\Gamma)$, then by Theorem 5.2 the Composition condition (PCC) must be satisfied for any polynomial Q for which the moments (1.2) vanish. In particular, this must be true for $Q=T_2+T_3$ – a contradiction. On the other hand, Figure 13 after Example 8 shows explicitly the curve $\gamma=P(\Gamma)$ with the depth of $P(a)=P(b)=z_0$ equal to one.

2. In many cases one can deform the path $\gamma=P(\Gamma)$ in order to reduce the depth of z_0 in such a way that this deformation is covered via P by the corresponding deformation of Γ. One of the possible constructions of such deformations is given below.

Let $S_{c,d}$ be a simple (without self-intersections) admissible curve with $c\in D_0$ and $d\in D_j$ where D_j is one of the domains containing z_0 in its boundary. We assume also that d is sufficiently close to z_0. Suppose that $S\cap\gamma=\{a_1,a_2,...,a_r\}$. Let $a_i=P(\alpha_i)$, $1\le i\le r$, with $\alpha_i\in\Gamma$. Notice that since S is an admissible curve then a_i, $1\le i\le r$, are simple points of γ. Therefore α_i are defined in a unique way. Let $\{u_1,u_2,...,u_r\}$ be the branches of P^{-1} at a_i, $1\le i\le r$, taking at these points the values α_i.

Denote by $S_{a_i,d}$ the part of S connecting a_i and d and consider the analytic continuations $h_i=S^*_{a_i,d}(u_i)$ of the germs u_i along S to d. So h_i represent certain branches of P^{-1} at d and they can be analytically extended to z_0. Assume that the germs $h_{i_1},\dots,h_{i_l}$ are regular at z_0 while the remaining $r-l$ germs $h_{i_{l+1}},\dots,h_{i_r}$ have singularities at z_0. We denote by $\nu(P,S,\gamma)$ the difference $r-l$.

Now assume that the polynomial P, the points $a,b\in\mathbb{C}$ with $z_0=P(a)=P(b)$, and the path Γ joining a and b are given, and $\gamma=P(\Gamma)$.

Proposition 5.3 *$D(P,a,b)$ does not exceed the minimum of $\nu(P,S,\gamma)$ over all the simple admissible curves $S_{c,d}$ with $c\in D_0$ and $d\in D_j$ where D_j is one of the domains containing z_0 in its boundary.*

PROOF. Fix an admissible curve $S_{c,d}$ as above. The deformation of γ (covered via P by the corresponding deformation of Γ) which reduces the depth of z_0 to $\nu(P,S,\gamma)=r-l$ is constructed as follows: consider one of the crossing points a_{i_s}, $s=1,\dots,l$, for which the germ h_{i_s} is regular at z_0. Now we deform γ along the curve $S_{a_{i_s},d}$ in such a way that the deformation is contained in a small neighborhood Ω of $S_{a_{i_s},d}$ and that in the final stage the deformed curve γ passes on another side of z_0 (see Figure 15). Since z_0 is a regular point of the branch h_{i_s} of P^{-1} and since S by assumptions does not contain other critical values of P then in fact the function h_{i_s} is regular and univalued in a whole simply-connected neighborhood Ω of $S_{a_{i_s},d}$ (and it coincides with u_{i_s} near the crossing point a_{i_s}). Moreover, since the curve $S_{a_{i_s},d}$ (and hence the domain Ω) do not have self-intersections, the function

h_{i_s} is one-to-one on Ω (as an inverse function to P). Let us define the domain Ω'_{i_s} as the image $h_{i_s}(\Omega)$ (or as the preimage $P^{-1}(\Omega)$ for the appropriate branch of P^{-1}). We obtain that P restricted to Ω'_{i_s} is a regular covering over Ω. Therefore the above deformation of γ can be lifted by $h_{i_s} = P^{-1}$ to the corresponding deformations of Γ. Repeating this operation for each crossing point $a_{i_1}, \ldots, a_{i_l}$ we get a new curve γ that crosses S only at the points $a_{i_{l+1}}, \ldots, a_{i_r}$. This completes the proof.

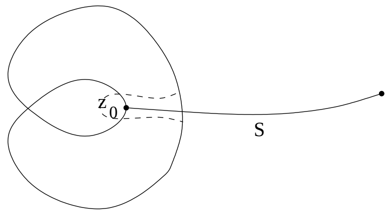

Figure 15

3. In general the deformations of γ reducing the depth of $z_0 = P(a) = P(b)$ to the minimum can be naturally analyzed in terms of the “diagrams” (shortly discussed in the concluding remarks in Section 4). As in other cases this analysis becomes quite explicit via the methods of the “Topological Theory of polynomials” mentioned above. We plan to present these results separately.

Let us return now to definite polynomials. Using the approach of Proposition 5.3 in many cases one can show that a given polynomial P possesses Property (E) and hence it is definite. As the first example let us give the following corollary:

Corollary 5.4. *If z_0 is a regular value of P and $P(a) = P(b) = z_0$ then $D(P, a, b) = 0$ and P possesses Property (E). In particular, P is a definite polynomial on $[a, b]$.*

PROOF. If z_0 is a regular value of P we get for any admissible curve $S_{c,d}$ as above $\nu(P, S, \gamma) = 0$. Proposition 5.3 now implies that $D(P, a, b) = 0$ and hence P possesses Property (E).

Corollary 5.4 follows also from the result of [21] that any P is definite with respect to any two its regular points a and b with $P(a) = P(b)$.

Consider now reals polynomials on the real line. An application of Proposition 5.3 provides the following result:

Corollary 5.5. *Let $a, b \in \mathbb{R}$ and let $P(x)$ be a real polynomial with $P(a) = P(b) = 0$. Assume that all the real zeroes x_i, $i = 1, \ldots, s$, belonging to the open interval (a, b) are simple. Then $P(x)$ possesses Property (E). In particular, it is definite on $[a, b]$.*

PROOF. Define Γ by shifting slightly the real interval $[a, b]$ into the upper half plane (and fixing a and b). See Figure 16-a. (The corresponding γ is shown in Figure 16-c. Notice that γ crosses the real axis near the the critical values d_j of P, on the side of each d_j which is determined by the sign of the second derivative of P at the corresponding critical point of P). We take as S the part of the imaginary axes going from $c = iD$, D real and sufficiently big, to zero. Each crossing a_{i_l} of S and γ, $l = 1, \ldots, r \leq s$, corresponds to the point α_{i_l} of Γ lying above and near one of the zeroes x_{i_l} of P where $P'(x_{i_l}) > 0$. (The parts of Γ lying above and near the zeroes x_j of P with $P'(x_j) < 0$ are mapped by P into the pieces of γ lying below zero and hence S does not cross these pieces of γ. Here we use the assumption that *all the real zeroes* x_i of $P(x)$ in the open interval (a, b) are simple, not only those where $P(x)$ changes sign from - to +).

Let u_{i_l} be the branch of P^{-1} taking at a_{i_l} the value α_{i_l} and let h_{i_l} be the analytic continuation of u_{i_l} along S from a_{i_l} to zero. The function h_{i_l} maps 0 into the root x_{i_l} of P and hence by the assumptions h_{i_l} is regular at 0 for $l = 1, \dots, r$. We conclude that $\nu(P, S, \gamma) = 0$. Proposition 5.3 now implies that $D(P, a, b) = 0$ and hence P possesses Property (E).

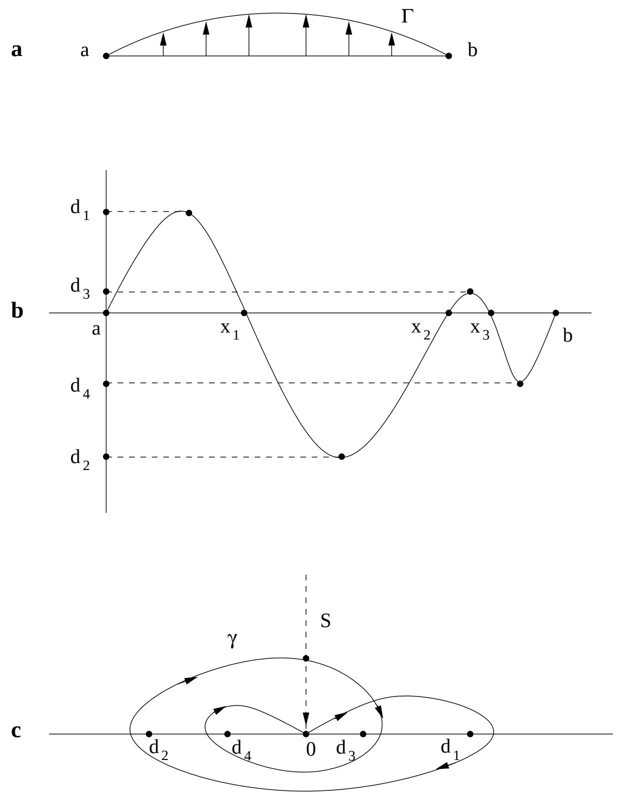

Figure 16

Remark 1. Apparently the result of Corollary 5.5 is not implied by the other known characterizations of definite polynomials. Indeed, it involves conditions only on the *real roots of* $P(x)$ between a and b, while most of the other results work for general complex polynomials and cannot take an advantage of $P(x)$ being real and of special properties of its real roots.

Remark 2. It seams plausible that the assumptions of Corollary 5.5 may be relaxed (if we want to show just that $P(x)$ is definite, without insisting on the "geometric" Property (E)). This is because in general the fact that if for a curve S as in Proposition 5.3 we have $\nu(P, S, \gamma) = 0$ then P is definite on $[a, b]$ follows also from Proposition 4.9. Indeed, the sum of branches F along S across γ for any integrand g of the form $g = Q(P^{-1})$ is given by

$$g(S_{c,d},\gamma) = \sum_{l=1}^{r} sgn(a_{i_l}) S^*_{a_{i_l},d}(Q(u_{i_l})) = \sum_{l=1}^{r} sgn(a_{i_l}) Q(h_{i_l}),$$

where the branches u_{i_l} and h_{i_l} of P^{-1} at a_{i_l} and at d, respectively, have been defined in the proof of Proposition 5.3. Since the branches h_{i_l} are regular at z_0 for $l = 1, \dots, r$ we conclude that F is regular for any polynomial Q. Remind now that Proposition 4.9 claims that if $I(t) \equiv 0$ at ∞ then the following equality between the branches g_0 and g_1 of $g = Q(P^{-1})$ on γ before and after z_0 is satisfied:

$$g_1 - \sigma_1^*(g_0) = F - \sigma^*(F),$$

where σ is a local loop around z_0 and σ_1 is the part of σ joining the segments of γ before and after z_0 (see Figure 7). An immediate consequence is that if F is regular at z_0 then the branches g_0 and g_1 can be obtained from one another by a local analytic continuation along σ_1. Hence, the Gluing condition is satisfied which implies the Composition condition on $P(x)$ and $Q(x)$.

However, the regularity of F at z_0 may follow from a certain cancellation effect and not just from the regularity of each of the branches h_{i_l} of P^{-1}. It would be interesting to find weaker conditions on P providing regularity of F for any polynomial Q, besides the conditions given in Corollary 5.5 (and generally in Proposition 5.3). As it was mentioned above, these last conditions imply the property (E) for P which is presumably stronger than just the regularity of F for any polynomial Q. A closely related conjecture is the following:

Conjecture. *Let P be a complex polynomial, $P(a) = P(b) = 0$. If all the roots of P besides a and b are simple then P is definite.*

Let us now describe some classes of polynomials P which possess the Property (E) by "geometric" reasons. The following simple observation works in many specific situations:

Proposition 5.6. *Let P be a complex polynomial with $P(a) = P(b) = 0$, $a, b \in \mathbb{C}$, let Γ be a piecewise-analytic curve in $\mathbb{C}$ joining a and b and let $\gamma = P(\Gamma)$. Assume that the open part $\gamma \setminus 0$ is contained in an open domain Ω with piecewise-analytic boundary and assume that 0 belongs to the exterior boundary of Ω. Then P possesses Property (E).*

PROOF. By the assumptions 0 already belongs to the exterior boundary of Ω and hence to the boundary of the domain exterior to γ. The only difficulty is that (as it happens in the examples below and in many other natural examples) for a specific given Γ the curve $\gamma = P(\Gamma)$ may be not in general position. So we need to perturb γ to provide all its self-intersections transversal. If we can do it in such a way that the point 0 remains fixed and that for the perturbed curve γ' the open part $\gamma' \setminus 0$ is still in Ω, then 0 belongs to the exterior boundary of γ' and the result follows. We can restrict the consideration to an arbitrarily small neighborhood of 0. Indeed, outside such a neighborhood γ is at a certain positive distance from the boundary of Ω and hence it can be brought there to a general position by any sufficiently small generic smooth perturbation.

Near zero we use the assumption that γ is a piecewise-analytic curve and that the boundary of Ω is also piecewise-analytic. We obtain that γ at 0 has two local analytic branches γ_0 and γ_1 "on the two sides of 0" (these branches may coincide with one another). Since the boundary of Ω is piecewise-analytic and $\gamma_0 \setminus 0$ and $\gamma_1 \setminus 0$ are contained in the open domain Ω, we can perform an analytic deformation of one of these branches (say, of γ_0) in such a way that the point 0 remains fixed, the deformed curve $\gamma_0' \setminus 0$ remains in Ω, and $\gamma_0' \setminus 0$ and $\gamma_1 \setminus 0$ do not intersect in a certain neighborhood of 0 (see Figure 17). Indeed, if γ_0 and γ_1 do not coincide with one another then no perturbation is necessary. If $\gamma_0 = \gamma_1$ then the required deformation of γ_0 can be achieved, for example, by adding to it (in an appropriate coordinate system) an analytic germ with a sufficiently high order of the vanishing at the origin.

Then we extend this deformation in a C^∞ way to the rest of the curve γ making all its self-intersections transversal. If this deformation is small enough then $\gamma' \setminus 0$ is still in Ω. This completes the proof.

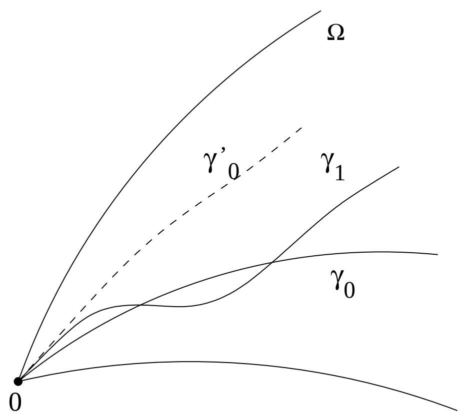

Figure 17

As the first application of Proposition 5.6 we prove the following corollary (which is also a special case of the Corollary 5.5 above):

Corollary 5.7. *Let $a, b \in \mathbb{R}$ and let $P(x)$ be a real polynomial with $P(a) = P(b) = 0$ and $P(x) > 0$ for $a < x < b$. Then property (E) holds for P.*

PROOF. Apply Proposition 5.6 with $\Gamma = [a, b]$ and Ω an open cone $-\delta < Arg(z) < \delta$ for some $\delta > 0$. Notice that in this situation $\gamma = P(\Gamma)$ is a real interval covered several times. So a perturbation is indeed necessary to bring γ into a general position.

Remark. In this specific case we can get Γ with $\gamma = P(\Gamma)$ in general position and still inside Ω also by shifting slightly the real interval $[a, b]$ into the upper half plane while fixing a and b. See Figure 18.

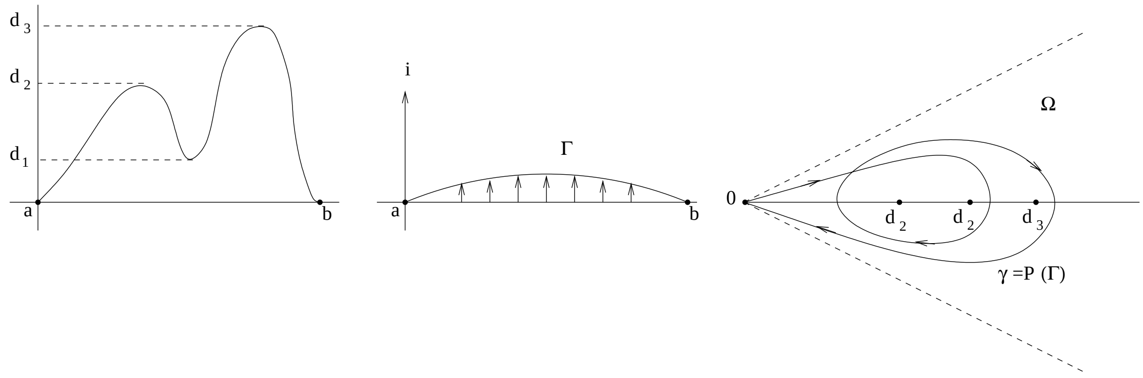

Figure 18

The following corollary provides a class of definite polynomials which are characterized directly by the geometry of their coefficients: these coefficients are assumed to belong to a certain convex set in $\mathbb{C}$ not containing zero. This assumption is not too restrictive from the algebraic point of view, so the polynomials P satisfying it may have zeroes of various multiplicities as well as various composition factorizations. In this sense the fact of these polynomials P being definite does not follow from the results of [14-16,21,40-41,47] and from the rest of the results of the present paper.

Corollary 5.8. *Let $a, b \in \mathbb{R}$, $0 \leq a < b$ and let $P(x) = (x-a)(b-x)P_1(x)$, with $P_1(x) = \sum_{k=0}^{n} a_k x^k$, $a_k \in \mathbb{C}$. If the convex hull CH of the coefficients a_k does not contain 0 then P has Property (E).*

PROOF. Apply Proposition 5.6 with $\Gamma = [a, b]$ and Ω some open cone $\alpha < Arg(z) < \beta$ containing the closed cone with the vertex at $0 \in \mathbb{C}$ generated by CH. For any $x \in (a, b)$, $P(x) \in \kappa(CH)$, with $\kappa > 0$. Hence for $\gamma = P(\Gamma)$ the open part $\gamma \setminus 0$ is in Ω. See Figure 19.

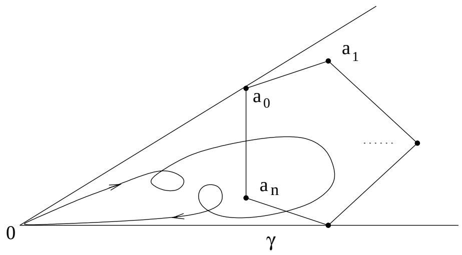

Figure 19

Remark. The results of the present section provide some explicit classes of definite polynomials. Other classes have been described in [14-16,21,40-41,47] in terms of the monodromy group, indecomposability, or through certain restrictions on their critical points. Yet another class of definite polynomials can be produced via the algebraic methods of [15,16] (see [17]). All these classes have almost no apparent intersections (besides the polynomials with regular zeroes at a and b, for the first two approaches). The only examples of the non-definite polynomials we know at present are provided by [39]. In particular, the polynomial $P = T_6$ is not definite on $[-\sqrt{3}/2, \sqrt{3}/2]$. Each counterexample to the "Moment Composition conjecture" given in [39] provides also an example of a non-definite polynomial P, and all these counterexamples are based on the composition relations $A(B) = C(D)$ classified in [44,45] (see also [52]). Once more, the appearance of these non-definite P is not easy to relate with the properties used in [14-17,21,47]. It would be important to understand the nature of definite polynomials and in particular to "unify" the approaches of the present paper, of [14-16,21], and of [39-41]. The first steps in this direction are given in [9,17].

We conclude this section with a discussion of the following problem: is it possible to relax the condition of the vanishing of all the one-sided moments (1.2) in the Polynomial Moment problem (requiring, for example, the vanishing of only a part of them) and still to get ultimately that all the m_k vanish? Some initial results in this direction can be derived from Corollary 4.14 above. This corollary states that if all the poles of the integrand g of the Cauchy integral $I(t)$ given by (1.1) belong to the exterior domain D_0 then the complete analytic continuation $\hat{I}_0(t)$ of the "exterior branch" I_0 of I cannot be univalued unless it is identical zero. In particular, $I_0(t)$ at infinity cannot be a nonzero germ of a polynomial, a rational or a meromorphic in $\mathbb{C}$ function.

The algebraic function $g = Q(P^{-1})$ on γ that appears in the Cauchy integral coming from the Polynomial Moment problem, does not have poles in the finite part of $\mathbb{C}$. Therefore by Corollary 4.14 the Moment generating function $H(y)$ defined by (1.3) (or by (5.1)) cannot be rational unless it vanishes identically. This proves the following:

Proposition 5.9. *If the moments $m_k = m_k(P, Q, \gamma)$ satisfy a linear recurrence relation*

$$m_j = \sum_{1}^{N} \alpha_i m_{k-i}$$

for each $j \geq N$ (in particular, if $m_j = 0$ for each $j \geq N$) then in fact $m_j = 0$ for each $j \geq 0$.

The next interesting question in this direction is whether $H(y)$ may be a non-rational algebraic function. In the Examples of Section 4 where $I(t)$ is a non-rational algebraic function the Cauchy integral does not come from the Polynomial Moment problem (i.e. not from the integral (5.2)). As it was mentioned in the remark after Example 5 in Section 4, for the closed integration path Γ the rational Moment generating function $H(y)$ is either algebraic or locally algebroid (i.e. has a finite ramification and a polynomially bounded growth at each of its singular point). On the other hand, the following proposition shows that "generically" the polynomial Moment generating function on a *non-closed* interval cannot be locally algebroid.

Proposition 5.10. *Let P and Q be polynomials and let a and b be regular points of P with $P(a) = P(b) = z_0$. If one of the branches of the analytic continuation $\hat{H}(y)$ of the Moment generating function $H(y)$ defined by (5.1) has a finite ramification at z_0 then in fact $H(y) \equiv 0$ near ∞. In particular, $H(y)$ near ∞ cannot be a nonzero algebraic function.*

PROOF. By Theorem 4.4 if one of the branches of $\hat{H}(y)$ has a finite ramification at z_0 then this is true for all the branches, and in particular, for the one represented by the expression in Theorem 3.7 of Section 3. Therefore, the logarithmic term in this expression must vanish, i.e the regular parts of $g = Q(P^{-1})$ on the two sides of z_0 in $\gamma = P([a,b])$ must coincide. Since a and b are regular points of P these regular parts are the corresponding regular branches of g themselves. We get a local coincidence of the branches of g i.e. the Gluing condition. This implies Composition condition (PCC) that in turn implies the vanishing of all the one-sided moments (1.2) and the identity $H(y) \equiv 0$ near ∞.

Remark 1. Proposition 5.10 can be formally restated as follows: it shows that $H(y)$ cannot be a nonzero algebraic function near ∞ for any P which is definite by the result of [21] (i.e. because a and b are regular points of P). Exactly in the same way we can show that $H(y)$ near ∞ cannot be a nonzero algebraic function for any P which is definite by the results of [40-41]. Indeed, in these papers the validity of the Composition condition (PCC) is shown under certain restrictions on P, starting with the vanishing of a certain sum of the branches of g which in turn is implied just by the algebraicity of $H(y)$ (or even by the property of $H(y)$ to be locally algebroid). It is not clear whether the same conclusion is true for the polynomials P which are definite by the results of [15,16,47] or of the present section.

Remark 2. The condition of algebraicity of $H(y)$ can be expressed in terms of its Taylor coefficients m_k, but not in a straightforward way (see [24,48,49]). It would be interesting to find an analogue of Proposition 5.10 with the assumptions given explicitly in terms of the moments m_k.

6. Rational Double moments

In this section we investigate the "Rational Double Moment problem" on the non-closed curve. This problem consists in providing necessary and sufficient conditions for the vanishing of the double moments (1.4)

$$m_{i,j} = \int_\gamma P^i(x) Q^j(x) p(x) dx, \quad i,j = 0,1,\ldots.$$

We assume that $P(x)$, $Q(x)$ are rational functions and the integration path Γ is *non-closed.* Remind that in the case of the closed integration curve the answer is given by the classical result of Wermer and Harwey-Lawson: double moments vanish if and only if the image curve $\delta = (P,Q)(\Gamma) \subset \mathbb{C}^2$ of the path Γ under (P,Q) bounds a compact complex 1-chain in $\mathbb{C}^2$ (see [2,23,30,59,60] and Section 1.1.3 above). We show that on a non-closed curve γ the vanishing of the double moments (and in fact just an algebraicity of the appropriate generating functions) is equivalent to a certain composition factorization of the integrand functions which "closes up" the integration path, combined together with the Wermer and Harwey-Lawson condition for their "left factors".

Next we show that under the additional assumption that the monodromy group of $P(x)$ is doubly transitive the vanishing of the one-sided moments only implies a composition factorization which closes up the integration path. Moreover, this composition factorization has a very special form: $Q(x) = \tilde{Q}(P(x))$.

The results of this section generalize the results of [40].

Theorem 6.1. *Let $P(x), Q(x)$ be rational functions and let Γ be a non-closed curve containing no poles of $P(x), Q(x)$ which starts at the point a and ends at the point b. Suppose that*

$$m_{i,j} = \int_{\Gamma} P^i(x)Q^j(x)p(x)\,dx = 0$$

for $0 \leq i \leq \infty$, $1 \leq j \leq d_a + d_b - 1$, where d_a (resp. d_b) is the multiplicity of the point a (resp. b) with respect to $P(x)$. Then there exist rational functions $\tilde{P}$, $\tilde{Q}$, W such that $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$, and $W(a) = W(b)$.

Note that if a, b are not critical points of $P(z)$ (that is if $d_a = d_b = 1$) then conditions of the theorem reduce to the vanishing of single moments and therefore theorem 6.1 can be considered as a wide generalization of the result of C. Christopher ([21]).

PROOF. Suppose first that $P(a) = P(b)$. Let U be a simply connected domain which contains no critical values of $P(x)$, and such that $P(a) = P(b) = z_0 \in \partial U$. Denote by $P^{-1}_{u_1}(z),\ P^{-1}_{u_2}(z), ..., P^{-1}_{u_{d_a}}(z)$ (resp. $P^{-1}_{v_1}(z),\ P^{-1}_{v_2}(z), ..., P^{-1}_{v_{d_b}}(z)$) the branches of $P^{-1}(z)$ defined in U which map points close to z_0 to points close to a (resp. b). Then for any j, $1 \leq j \leq d_a + d_b - 1$, Corollary 3.9 applied to the function $g(z) = Q^j(P^{-1}(z))$ implies that

$$d_b \sum_{s=1}^{d_a} Q^j(P^{-1}_{u_s}(z)) \equiv d_a \sum_{s=1}^{d_b} Q^j(P^{-1}_{v_s}(z)). \tag{6.1}$$

Clearly, this equality holds also for $j = 0$. Our assumption that the curve Γ is not closed (i.e. that $a \neq b$) implies that the branches of $P^{-1}(z)$ on the two sides of (6.1) are different. Hence the correspondent branches of $Q^j(P^{-1}(z))$ are different and therefore (6.1) provides a *non-trivial* relation between the branches of $Q^j(P^{-1}(z))$. (For $a = b$ the two sides of (6.1) are identically equal to one another).

Consider a Vandermonde determinant $D = \parallel Q^j(p_i^{-1}(t)) \parallel$, where $0 \leq j \leq d_a + d_b - 1$ and i ranges the set of indices $\{u_1, u_2, ..., u_{d_a}, v_1, ..., v_{d_b}\}$. Since system (6.1) implies that $D = 0$ we conclude that $Q(P_i^{-1}(t)) \equiv Q(P_j^{-1}(t))$ for some $i \neq j$, $1 \leq i, j \leq n$. (Here n is the degree of the rational function P). By Lemma 5.1 above the last condition is equivalent to the condition that there exist rational functions $\tilde{P}$, $\tilde{Q}$, W with $\deg W > 1$ such that $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$. Furthermore, without loss of generality we can suppose that $\tilde{P}$ and $\tilde{Q}$ have no nontrivial (of degree greater than 1) common right divisor in the composition algebra. Indeed, otherwise we compose W with this common right divisor and get a new W of a higher degree. The fact that W satisfies additionally the equality $W(a) = W(b)$ we prove below.

Let us suppose now that $P(a) \neq P(b)$. In this case instead of (6.1) Corollary 3.9 gives two systems

$$\sum_{s=1}^{d_a} Q^j(P^{-1}_{u_s}(z)) = 0, \quad \sum_{s=1}^{d_b} Q^j(P^{-1}_{v_s}(z)) = 0, \quad 1 \leq j \leq d_a + d_b - 1, \tag{6.2}$$

where $P_{u_1}^{-1}(z)$, $P_{u_2}^{-1}(z), ..., P_{u_{d_a}}^{-1}(z)$ (resp. $P_{v_1}^{-1}(z)$, $P_{v_2}^{-1}(z), ..., P_{v_{d_b}}^{-1}(z)$) denote the branches of $P^{-1}(z)$ defined in some neighborhood of $P(a)$ (resp. $P(b)$) which map points close to $P(a)$ (resp. $P(b)$) to points close to a (resp. b). Now the same reasoning as above applied to the system

$$\sum_{s=1}^{d_a} Q^j(P_{u_s}^{-1}(z)) = 0, \qquad 1 \le j \le d_a,$$

(taking into account that $d_a + d_b - 1 \ge d_a$) shows that $Q(P_i^{-1}(t)) = Q(P_j^{-1}(t))$ for two different branches $P_i^{-1}(z)$, $P_j^{-1}(z)$ of $P^{-1}(z)$. Once more, this implies the existence of rational functions $\tilde{P}$, $\tilde{Q}$, W with $\deg W > 1$ such that $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$ and such that $\tilde{P}$ and $\tilde{Q}$ have no nontrivial (of degree greater than 1) common right divisor in the composition algebra.

Let us show that such a W must satisfy $W(a) = W(b)$. Indeed, otherwise after the change of variable $x \to w = W(x)$ we get

$$m_{i,j} = \int_\delta \tilde{P}^i(w)\tilde{Q}^j(w)\tilde{P}'(w)\, dw = 0$$

for $0 \le i \le \infty$, $1 \le j \le d_a + d_b - 1$, where $\delta = W(\Gamma)$ is non-closed and d_a (resp. d_b) is (as above) the multiplicity of the point a (resp. b) with respect to $P(x)$. Taking into account that for any $c \in \mathbb{C}$ the multiplicity of c with respect to $P(x) = \tilde{P}(W(x))$ is greater or equal than the multiplicity of $W(c)$ with respect to $\tilde{P}(w)$, in the same way as above we would conclude that $\tilde{P}(w) = \bar{P}(U(w))$, $\tilde{Q}(w) = \bar{Q}(U(w))$ for some rational functions $\bar{P}, \bar{Q}, U$ with $\deg U > 1$. This contradicts the assumption that $\tilde{P}$, $\tilde{Q}$ have no common right divisor in the composition algebra.

Corollary 6.2. *Let $P(x), Q(x)$ be rational functions and let Γ be a non-closed curve containing no poles of $P(x), Q(x)$ which starts at the point a and ends at the point b. Then*

$$m_{i,j} = \int_\Gamma P^i(x)Q^j(x)p(x)\, dx = 0$$

for $0 \le i \le \infty$, $0 \le j \le \infty$ if and only if there exist rational functions $\tilde{P}$, $\tilde{Q}$, W such that $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$, $W(a) = W(b)$, and all the poles of $\tilde{P}$ and $\tilde{Q}$ lie on one side of the closed curve $\delta = W(\Gamma)$.

PROOF. Sufficiency of these conditions follows from Theorem 1.1.3 above, after we perform a change of variables $x \to w = W(x)$. Necessity is obtained as follows: assuming that the moments $m_{i,j}$ vanish, we apply Theorem 6.1 and get the factorization $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$ with $W(a) = W(b)$. Performing a change of variables $x \to w = W(x)$ we get the vanishing of the moments

$$\int_\delta \tilde{P}^i(w)\tilde{Q}^j(w)\tilde{P}'(w)\, dw$$

on the closed curve $\delta = W(\Gamma)$. Finally we apply Theorem 1.1.3.

Corollary 6.3. *If the moments $m_{i,j}$ vanish for $0 \le i \le \infty$, $1 \le j \le d_a + d_b - 1$ then $P(a) = P(b),\ Q(a) = Q(b)$.*

Notice that the results of Section 5 leave open the question whether the vanishing of the *one-sided* moments implies $P(a) = P(b)$ and $Q(a) = Q(b)$. It turns out that under the additional assumption that the monodromy group of $P(x)$ is doubly transitive the vanishing of the one-sided moments does imply the equality $P(a) = P(b)$ as well as a composition factorization of a very special form: $Q(x) = \tilde{Q}(P(x))$ (which of course closes up the integration path for $P(a) = P(b)$).

Theorem 6.4. *Let $P(x), Q(x)$ be non-zero rational functions and let Γ be a non-closed curve containing no poles of $P(x), Q(x)$ which starts at the point a and ends at the point b. Suppose that*

$$\int_\Gamma P^i(x)Q(x)p(x)\,dx = 0$$

for $i \ge 0$. If, additionally, the monodromy group of $P(x)$ is doubly transitive then the functions $P(x), Q(x)$ must satisfy $P(a) = P(b),\ Q(a) = Q(b)$ and there exists a rational function $\tilde{Q}$ such that $Q(x) = \tilde{Q}(P(x))$.

PROOF. Lemma 2 of [40] (see also [8]) states that if the monodromy group of $P(x)$ is doubly transitive and if the branches $Q(P_i^{-1}(z))$ satisfy

$$\sum_{i=1}^{n} a_i Q(P_i^{-1}(z)) \equiv 0$$

for some $a_i \in \mathbb{C}$ not all equal between themselves then there exists a rational function $\tilde{Q}$ such that $Q(x) = \tilde{Q}(P(x))$. (Here as above n is the degree of the rational function P).

Now as in the proof of theorem 6.1, the vanishing of the one-sided moments implies via Corollary 3.9 that the branches $Q(P_i^{-1}(z))$ are related either by relation (6.1) or by relation (6.2) (with $j = 1$). Notice that in each of these relations the coefficients are not all equal between themselves. This is immediate for (6.1). The only case where both the sums in (6.2) contain all the branches of $Q(P_i^{-1}(z))$ is when $P(x)$ can be reduced to x^n by the transformation $P(x) \to A(P(B(x)))$, where A, B are rational functions of the first degree. This possibility is excluded by the assumption that the monodromy group of P is doubly transitive.

It remains to show that P satisfies $P(a) = P(b)$. Let us perform a change of variable $x \to z = P(x)$. We get

$$\int_\gamma z^i \tilde{Q}(z) dz = 0 \tag{6.3}$$

for $i \ge 0$, where $\gamma = P([a, b])$. If $P(a) \ne P(b)$ then the curve γ is non-closed. In this case each of the relations (6.2) takes the form $\tilde{Q}(z) \equiv 0$ (at the points $P(a)$ and $P(b)$, respectively). Indeed, $\tilde{Q}(z)$, being a rational function, has only one branch at each point. In other words, the vanishing of the moments (6.3) for a non-closed curve γ is possible only for $\tilde{Q}(z) \equiv 0$. Since by the assumptions $Q(x) \neq 0$ also $\tilde{Q}$ cannot vanish identically. Hence $P(a) = P(b)$. This completes the proof of Theorem 6.4.

Remark. Corollary 3.9 requires only algebraicity of the Cauchy integral $I(t)$ (and not necessarily its identical vanishing) to get the vanishing of the local sum of the branches of g. Accordingly, we can replace the assumption of the vanishing of the double (the one-sided) moments in Theorem 6.2 (Theorem 6.4, respectively) by the assumption of the algebraicity of the corresponding moment generating functions.

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