# The theory of logarithmic differential forms and its applications. Torsion differentials and residues

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**Abstract:** This paper deals with the notion of logarithmic differential forms founded by Kyoji Saito. Relations between the properties of logarithmic differential forms and torsion holomorphic differentials on a singular hypersurface are established. Making use of the theory of logarithmic and regular meromorphic differential forms, a natural extension of the classical Poincaré residue to the case of singular hypersurfaces is obtained. A variant of Grothendieck duality for non-normal reduced hypersurfaces is described. The logarithmic de Rham complex is studied and its cohomology groups for hypersurfaces with isolated singularities are computed.

## Introduction

A general notion of logarithmic differential forms appeared in 1975. It was introduced by Kyoji Saito in his famous preprinted notes (see [27]). In these notes he established the basic properties of such forms and considered applications to computing the Gauss-Manin connection associated with the versal deformation of a simple hypersurface singularities of types $A_2$ and $A_3$. For unclear and unknown reasons, this preprint was reproduced only two years later in 1977 in a volume of the RIMS-publication series [28], which is not accessible to many of those interested in the subject. In 1980, K.Saito published, at last, a paper [29] containing some part of the material of the mentioned preprint. Among other things, in this paper the general notion and a clear description of the basic properties of logarithmic differential forms and vector fields are given. Somewhat later, after the RIMS-publication of Saito in 1978, the same notion of logarithmic differential forms was introduced in the book by Ph.Griffiths and J.Harris [18], with no references to the previous work. It should be remarked that in their book, they used this notion only in the case of hypersurfaces which are normal crossings of smooth ones. In

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fact, this case had been also treated in earlier works by P.Deligne [11], J.Steenbrink [32], F.Pham [25], R.Gérard and A.Levelt [15], and others.

However, at present time it is the *general notion* of logarithmic differential forms which is exploited fruitfully in many fields of modern mathematics. Among them, one can mention the following:

algebraic geometry (the cohomology theory of algebraic varieties and Hodge theory [12], [10], [32], etc.),

topology and geometry (the theory of arrangements of real and complex hyperplanes [24], [8], the fundamental group of the complement of a singular hypersurface [21], etc.),

the theory of singularities, the deformation theory and the theory of Gauss-Manin connexion [28], [30], [7], etc.,

the theory of $\mathcal{D}$-modules, the microlocal analysis, the theory of differential equations [11], [25], the theory of flat coordinate systems [31], etc.,

complex analysis (the theory of Abel's integrals [17], Torelli theorems, the theory of primitive forms and their periods [18], etc.),

the theory of special functions (generalized hypergeometric functions [14], etc.),

mathematical and theoretical physics (the theory of Frobenius varieties and the topological field theory [13], [23], etc.)

Of course, this list is quite incomplete and can be easily extended by the specialists in related fields of mathematics.

These notes deal with the theory of logarithmic differential forms and with some of the less known applications of this theory. In section 1 we recall the basic notations, definitions and properties of logarithmic differential forms along a reduced hypersurface in a complex analytic manifold. In section 2 we consider some properties of logarithmic differential forms in relation to properties of the torsion holomorphic differentials on a singular hypersurface. In the next two sections we recall the definition of a Poincaré residue, consider its properties, and discuss its relation to a logarithmic residue introduced by K.Saito in [27] and [29]. Following D.Barlet [9], we then apply the theory of regular meromorphic differential forms to the special case of hypersurfaces. Among other things, we obtain a highly elegant description of these modules on an arbitrary singular hypersurface $D$ in terms of the residue of logarithmic differential forms. In the section 5, a variant of the Grothendieck duality for non-normal reduced hypersurfaces is described. In the final section we study the logarithmic de Rham complex and compute its cohomology for a hypersurface with isolated singularities.

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# 1. Logarithmic differential forms

Let $U$ be an open subset of $\mathbf{C}^m$, and let $D$ be a hypersurface defined by an equation $h(z)=0$, where $h(z)=h(z_1,\ldots,z_m)$ is a holomorphic function in $U$, and $z_1,\ldots,z_m$ is a system of coordinates. Suppose that $D$ is *reduced,* that is, $h(z)$ has no multiple factors. The following basic definition is due to K.Saito (see [27], [29]).

**Definition 1.1.** A meromorphic differential $q$-form $\omega$, $q\geq 0$, on $U$ is called *logarithmic* (along a divisor $D$) if $\omega$ and its differential $d\omega$ have poles along $D$ at worst of the first order. It means that $h\omega$ and $hd\omega$ are *holomorphic* differential forms on $U$.

In practical computations, the second condition is usually replaced by the condition "$dh\wedge\omega$ is a *holomorphic* differential form on $U$"; both conditions are equivalent, in view of the identity $d(h\omega)=dh\wedge\omega+h\cdot d\omega$.

Let $S$ be an $m$-dimensional complex manifold, and let $\Omega^\bullet_S=(\Omega^q_S,d)_{q=0,1,\ldots}$ be the de Rham complex of germs of holomorphic differential forms on $S$, whose terms, locally at the point $0\in S$, are defined as follows:

$$\Omega^q_{S,0}=\mathcal{O}_{S,0}\langle\ldots,dz_{i_1}\wedge\ldots\wedge dz_{i_q},\ldots\rangle,\quad (i_1,\ldots,i_q)\in[1,m].$$

Let $D$ be a reduced hypersurface of $S$, and let $h_x=0$ be an equation of $D$, locally at the point $x\in D$. A meromorphic $q$-form $\omega$ is logarithmic along $D$ at $x$, if $h_x\omega$ and $h_xd\omega$ are holomorphic. We denote the $\mathcal{O}_{S,x}$-module of germs of logarithmic $q$-form at $x$ and the corresponding *sheaf* of logarithmic differential $q$-form on $S$ by $\Omega^q_{S,x}(\log D)$ and $\Omega^q_S(\log D)$, respectively. Thus, the $\mathcal{O}_S$-module $\Omega^1_S(\log D)$ is a submodule of $\Omega^1_S(\star D)$, consisting of all the "differential forms with polar singularities along $D$." Obviously, the sheaves $\Omega^q_S(\log D)$ and $\Omega^q_S$ coincide off the divisor $D$, for all $q\geq 0$. By definition,

$$\Omega^0_{S,x}(\log D)\cong\Omega^0_{S,x}\cong\mathcal{O}_{S,x},\quad \Omega^m_{S,x}(\log D)\cong\frac{1}{h_x}\Omega^m_{S,x}.$$

In what follows, when we consider the local situation the point $x$ will be taken to be 0 for simplicity. We shall also assume that $U$ is an open subset of $\mathbf{C}^m$ containing the origin.

**Example 1.2.** Suppose $D\subset U$ be a hyperplane or, more generally, a smooth hypersurface defined by the equation $z_1=0$. Then

$$\Omega^1_{S,0}(\log D)\cong\mathcal{O}_{S,0}\left\langle\frac{dz_1}{z_1},dz_2,\ldots,dz_m\right\rangle$$

is a *free* $\mathcal{O}_{S,0}$-module of rank $m$, generated by the forms $dz_1/z_1,dz_2,\ldots,dz_m$. Moreover,

$$\Omega^q_{S,0}(\log D)\cong\bigwedge^q\Omega^1_{S,0}(\log D),\quad 1\leq q\leq m.$$

**Example 1.3.** More generally, let us consider the case when $D$ is the union of $k \le m$ coordinates hyperplanes in $S = \mathbf{C}^m$. In other words, $D$ is a *normal crossing.* It is the case considered in many works published before Saito's preprint [27]. Then the defining equation of $D$ is: $h = z_1 \cdots z_k = 0$, and an easy calculation shows that

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\Big\langle \frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}, dz_{k+1}, \dots, dz_m \Big\rangle,$$

and for all $1 \le q \le m$ there are the following isomorphisms

$$\Omega^q_{S,0}(\log D) \cong \bigwedge^q \Omega^1_{S,0}(\log D).$$

Thus, $\Omega^q_{S,0}(\log D)$ is a *free* $\mathcal{O}_{S,0}$-module of rank $\binom{m}{q}$.

The following statement is a direct consequence of the basic definition (see [4], or [5], §1).

**Claim 1.4.** *Let $D \subset U$ be a reduced hypersurface defined by the equation $h = 0$. Then for any $q \ge 1$ there exists a natural isomorphism of $\mathcal{O}_{S,0}$-modules*

$$\Omega^q_{S,0} \bigcap ((dh/h) \wedge \Omega^{q-1}_{S,0}) \cong dh \wedge \Omega^{q-1}_{S,0}(\log D).$$

**Proof.** Let us remark at first that there is a natural inclusion

$$\Omega^q_{S,0} \bigcap ((dh/h) \wedge \Omega^{q-1}_{S,0}) \hookrightarrow dh \wedge \Omega^{q-1}_{S,0}(\log D).$$

For, if an element $\omega \in \Omega^q_{S,0}$ belongs to the $\mathcal{O}_{S,0}$-module on the left side, then it is of the form $\omega = (dh/h) \wedge \eta$ for some $\eta \in \Omega^{q-1}_{S,0}$. Hence, by definition,

$$(\eta/h) \in \Omega^{q-1}_{S,0}(\log D) \Rightarrow \omega \in dh \wedge \Omega^{q-1}_{S,0}(\log D),$$

and we obtain the desirable left arrow. On the other hand, $h \cdot \Omega^{q-1}_{S,0}(\log D) \subseteq \Omega^{q-1}_{S,0}$. Multiplication by $\wedge dh$ induces the map

$$dh \wedge \Omega^{q-1}_{S,0}(\log D) \longrightarrow \frac{dh}{h} \wedge \Omega^{q-1}_{S,0}.$$

Obviously this gives us the inverse map to the first inclusion. This completes the proof of Claim.

The following reformulation of the basic definition is also due to K.Saito (see [29], $(1.1), iii)$).

**Lemma 1.5.** *Let $\omega$ be a meromorphic $q$-form on $U$, $q \geq 0$, and let $D \subset U$ be a hypersurface as above. Then $\omega$ is logarithmic along $D$ if and only if there exist a holomorphic function $g$ defining a hypersurface $V \subset U$, a holomorphic $(q-1)$-form $\xi$ and a holomorphic $q$-form $\eta$ on $U$ such that*

*a)* $\dim_{\mathbf{C}} D \cap V \leq m-2$,

*b)* $g\omega = \dfrac{dh}{h} \wedge \xi + \eta$.

**Proof.** For simplicity let us consider the case $m = 2$. Suppose that $\omega$ is a logarithmic $q$-form. Then we have

$$\omega = \frac{a_1 dz_1 + a_2 dz_2}{h}, \quad dh \wedge \omega = \frac{h_1' a_2 - h_2' a_1}{h} dz_1 \wedge dz_2 \stackrel{\text{def}}{=} b(z) dz_1 \wedge dz_2,$$

where $a_1, a_2$ and $b(z)$ are holomorphic, and $h_i' = \partial h / \partial z_i,\ i = 1, 2$. Further,

$$h_1' \omega = \frac{h_1' a_1 dz_1 + h_1' a_2 dz_2}{h} =$$

$$\frac{h_1' a_1 dz_1 + h_2' a_1 dz_2}{h} + \frac{h_1' a_2 - h_2' a_1}{h} dz_2 = \frac{dh}{h} \wedge a_1 + b(z) dz_2.$$

It means that

$$h_1' \omega = \frac{dh}{h} \wedge a_1 + b(z) dz_2.$$

There are analogous presentations for $h_2' \omega$, and hence for any $g \in J(h)$, where $J(h) = (h_1', h_2')$ is the Jacobian ideal of $h$. Since $D$ is reduced, there is a function $g \in J(h)$ defining a non zero divisor in $\mathcal{O}_D/(h)$ as required in the condition *a*).

The converse is simpler. Indeed, the equation *b*) implies that

$$h\omega = dh \wedge \frac{\xi}{g} + \frac{\eta}{g},$$

that is, $h\omega$ and $dh \wedge \omega$ are holomorphic at least in codimension $\geq 2$, hence, in virtue of the Riemann extension theorem, they are holomorphic everywhere. This completes the proof when $m = 2$. The general case can be considered analogously.

**Corollary 1.6.**

*i)* $\Omega^q_{S,0}(\log D)$, $q = 0, 1, \ldots, m$, *are* $\mathcal{O}_{S,0}$*-modules of finite type,*

*ii)* $\oplus_{q=0}^m \Omega^q_{S,0}(\log D)$ *is an* $\mathcal{O}_{S,0}$*-exterior algebra closed under the exterior differentiation.*

As a consequence, we see that $\Omega^q_S(\log D)$, $q = 0, 1, \ldots, m$, are *coherent* sheaves of $\mathcal{O}_S$-modules.

# 2. Torsion differentials

In this section we consider simple relations between logarithmic differential forms and torsion holomorphic differentials on a hypersurface with singularities.

Given a commutative ring $A$ with the total ring of fractions $F$, and an $A$-module $N$ of finite type, we call the kernel of the canonical map

$$\iota\colon N \to N \otimes_A F$$

the *torsion* submodule $\operatorname{Tors} N$ of $N$. Thus the torsion submodule consists of all the elements of $N$ which are killed by the nonzero elements of $A$.

By definition, set $\mathcal{O}_{D,0} = \mathcal{O}_{S,0}/(h)\mathcal{O}_{S,0}$, and

$$\Omega^q_{D,0} = \Omega^q_{S,0}/(h \cdot \Omega^q_{S,0} + dh \wedge \Omega^{q-1}_{S,0}), \quad q \geq 1.$$

Then $\Omega^q_{D,0}$ is the $\mathcal{O}_{D,0}$-module of germs of holomorphic differential forms on the *hypersurface* $D$ at the point $0 \in D$. This module is usually called the module of Kähler regular differentials of the $q$-th order. The standard differentiation $d$ induces the action on $\Omega^q_{D,0}$ denoted by the same symbol. Thus, the de Rham complex of sheaves of germs of holomorphic differential forms on the whole $D$ is well defined:

$$\Omega^\bullet_D = (\Omega^q_D, d)_{q=0,1,\dots}.$$

**Example 2.1.** Suppose $S = \mathbf{C}^2$ and consider the hypersurface $D$ defined by the equation $h = x^2 - y^3 = 0$. It is a curve with a cusp singularity at the origin. Easy calculations show that

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\Big\langle \frac{dh}{h}, \ \frac{2ydx - 3xdy}{h} \Big\rangle, \ \ \Omega^2_{S,0}(\log D) \cong \mathcal{O}_{S,0}\Big\langle \frac{dx \wedge dy}{h} \Big\rangle$$

are *free* $\mathcal{O}_{S,0}$-modules of rank 2 and 1, respectively.

Note that the numerator of the second generator of $\Omega^1_{S,0}(\log D)$, the differential 1-form $\theta = 2ydx - 3xdy$, represents an element of the *torsion* submodule $\operatorname{Tors} \Omega^1_{D,0} \subset \Omega^1_{D,0}$. Indeed, in our case

$$A = \mathcal{O}_{D,0} \cong \mathbf{C}\langle t^2, t^3 \rangle, \quad N = \Omega^1_{D,0}, \quad F = \mathbf{C}(t),$$

and the mapping $\iota$ is defined by the normalization map $x = t^3$, $y = t^2$ of the curve $D$. Thus,

$$\iota(\theta) = \iota(2ydx - 3xdy) = 2t^2dt^3 - 3t^3dt^2 = 6t^4dt - 6t^4dt = 0,$$

that is, $\theta \in \operatorname{Ker}(\iota) \cong \operatorname{Tors} \Omega^1_{D,0}$.

Equivalently, take a non-zero element $x \in \mathcal{O}_{D,0}$. One then obtains

$$x \cdot \theta = 2xydx - 3x^2dy = ydh - 3hdy \equiv 0$$

in $\Omega^1_{D,0} = \Omega^1_{S,0}/(h\cdot\Omega^1_{S,0} + dh\wedge\mathcal{O}_{S,0})$. Further calculations show (cf. [34]) that

$$\operatorname{Tors}\Omega^1_{D,0} \cong \mathcal{O}_{D,0}\langle\theta\rangle \cong \mathbf{C}\langle\theta,\ y\cdot\theta\rangle\ .$$

In the terminology of local cohomology theory $\operatorname{Tors}\Omega^1_{D,0} \cong H^0_{\{0\}}(\Omega^1_{D,0})$, so the torsion submodule is an initial term of the standard long exact sequence:

$$0 \longrightarrow H^0_{\{0\}}(\Omega^1_{D,0}) \longrightarrow \Omega^1_{D,0} \longrightarrow j_*j^*\Omega^1_{D,0} \longrightarrow H^1_{\{0\}}(\Omega^1_{D,0}) \longrightarrow 0,$$

where $j\colon D\setminus\{0\} \longrightarrow D$ is the canonical embedding of the non-singular part of $D$.

Analogously, one can consider a plane curve with a node which is a particular case of Example 1.3.

**Example 2.2.** Suppose $m = 2$, that is $S = \mathbf{C}^2$, and the hypersurface $D$ is defined by the equation $h = xy = 0$. It is a curve with a *node* singularity at the origin. Then

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dx}{x},\ \frac{dy}{y}\right\rangle,\ \ \Omega^2_{S,0}(\log D) \cong \bigwedge^2 \Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dx\wedge dy}{xy}\right\rangle$$

are again *free* $\mathcal{O}_{S,0}$-modules of rank 2 and 1, respectively. In this case there is a similar presentation of $\Omega^1_{S,0}(\log D)$ with $\theta = ydx - xdy$ :

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dx}{x},\ \frac{dy}{y}\right\rangle \cong \mathcal{O}_{S,0}\left\langle \frac{dh}{h},\ \frac{\theta}{h}\right\rangle,$$

where $h = xy$ and $\theta \in \operatorname{Tors}\Omega^1_{D,0}$. Taking a non-zero element $x \in \mathcal{O}_{D,0} = \mathcal{O}_{S,0}/(xy)$ we obtain in $\Omega^1_{D,0}$ two such identities:

$$x\cdot\theta = xydx - x^2dy = -xdh + 2hdx \equiv 0.$$

In this case, $\operatorname{Tors}\Omega^1_{D,0} \cong \mathcal{O}_{D,0}\langle\theta\rangle \cong \mathbf{C}\langle\theta\rangle$ .

These observations lead us to the following statement (cf. [4]).

**Proposition 2.3.** *For $q = 1,\dots,m$, there are exact sequences of $\mathcal{O}_{S,0}$-modules*

$$0 \longrightarrow \Omega^{q-1}_{S,0}/h\cdot\Omega^{q-1}_{S,0}(\log D) \xrightarrow{\wedge dh} \Omega^q_{S,0}/h\cdot\Omega^q_{S,0} \longrightarrow \Omega^q_{D,0} \longrightarrow 0,$$

$$0 \longrightarrow \Omega^q_{S,0}/dh\wedge\Omega^{q-1}_{S,0}(\log D) \xrightarrow{\cdot h} \Omega^q_{S,0}/dh\wedge\Omega^{q-1}_{S,0} \longrightarrow \Omega^q_{D,0} \longrightarrow 0,$$

$$0 \longrightarrow \Omega^q_{S,0} + \frac{dh}{h}\wedge\Omega^{q-1}_{S,0} \longrightarrow \Omega^q_{S,0}(\log D) \xrightarrow{\cdot h} \operatorname{Tors}\Omega^q_{D,0},$$

*where we denote by $\wedge dh$ and by $\cdot h$ the homomorphisms of exterior and usual multiplication, respectively. Moreover, when $D$ is irreducible at $0\in D$ then the third exact sequence will be exact from the right:*

$$0 \longrightarrow \Omega^q_{S,0} + \frac{dh}{h}\wedge\Omega^{q-1}_{S,0} \longrightarrow \Omega^q_{S,0}(\log D) \xrightarrow{\cdot h} \operatorname{Tors}\Omega^q_{D,0} \longrightarrow 0.$$

**Proof.** Consider a differential $q$-form $\omega \in \Omega^{q-1}_{S,0}$ which represents a certain element of the quotient $\Omega^{q-1}_{S,0}/h \cdot \Omega^{q-1}_{S,0}(\log D)$ and suppose that $dh \wedge \omega = h \cdot \eta,\ \eta \in \Omega^q_{S,0}$. We set $\tilde{\omega} = \omega/h$. It is obvious that $h \cdot \tilde{\omega}$ and $dh \wedge \tilde{\omega}$ are holomorphic, hence $\tilde{\omega} \in \Omega^{q-1}_{S,0}(\log D)$ by definition. Thus the first sequence is exact from the left. Evidently it is exact from the right too. In the same way, one can easily prove the exactness of the second sequence.

The exactness from the left of the third one follows by definition. Besides, in virtue of the basic Lemma 1.5, it is clear that $\mathrm{Im}\,(\cdot h) \subseteq \mathrm{Tors}\,\Omega^q_{D,0}$ :

$$g\omega = \frac{dh}{h} \wedge \xi + \eta \Rightarrow gh\omega = dh \wedge \xi + h\eta \equiv 0.$$

Now suppose that $D$ is irreducible so that $\mathcal{O}_D$ has no zero divisors. Take an element $\omega \in \mathrm{Tors}\,\Omega^q_{D,0}$. This means that there is a non-zero element $g \in \mathcal{O}_{D,0}$ such that $g \cdot \omega = 0$. We will denote by $g$ and $\omega$ their representatives in $\mathcal{O}_{S,0}$ and $\Omega^q_{S,0}$, respectively. Then we have

$$g \cdot \omega = dh \wedge \xi + h \cdot \eta, \quad \xi \in \Omega^{q-1}_{S,0}, \quad \eta \in \Omega^q_{S,0}.$$

Since $g$ is not a zero divisor, the condition $b)$ of the Lemma 1.5 is satisfied. This implies that $\omega/h = \tilde{\omega} \in \Omega^q_{S,0}(\log D)$, that is $\omega \in \mathrm{Im}\,(\cdot h)$.

**Remark 2.4.** When $D$ is smooth, then $\mathrm{Tors}\,\Omega^q_{D,0} = 0,\ q \geq 0$, and from the above Proposition we get isomorphisms of the Example 2.1 :

$$\Omega^q_{S,0}(\log D) \cong \Omega^q_{S,0} + \frac{dh}{h} \wedge \Omega^{q-1}_{S,0}, \quad q \geq 0.$$

In fact, it is well-known from the singularity theory (see, for example, [16]) that

$$\mathrm{Tors}\,\Omega^q_{D,0} = 0,\ 0 < q < \mathfrak{c},$$

where $\mathfrak{c} = \mathrm{codim}\,(\mathrm{Sing}\,D, D)$. In particular, when $D$ is a *normal* hypersurface the third exact sequence of Proposition 2.3 yields the following isomorphism:

$$\Omega^1_{S,0}(\log D) \cong \Omega^1_{S,0} + \frac{dh}{h}\mathcal{O}_{S,0}$$

since $\mathfrak{c} \geq 2$ (cf. [27], [28]). This is clearly true when the hypersurface $D$ has isolated singularities only and $\dim D \geq 2$.

**Corollary 2.5.** *Suppose that $D$ is irreducible at $0 \in D$. Then there is a long exact sequence of $\mathcal{O}_{S,0}$-modules*

$$0 \longrightarrow dh \wedge \Omega^{q-1}_{S,0}(\log D) \longrightarrow \Omega^q_{S,0} \oplus \frac{dh}{h} \wedge \Omega^{q-1}_{S,0} \longrightarrow$$

$$\longrightarrow \Omega^q_{S,0}(\log D) \xrightarrow{\cdot h} \mathrm{Tors}\,\Omega^q_{D,0} \longrightarrow 0.$$

**Proof.** This is an immediate consequence of Proposition 2.3 and Claim 1.4.

**Remark 2.6.** It is well-known that torsion differentials play important role in the description of the geometry and topology of singularities. For example, in the case of an isolated singularity, the torsion modules are finite dimensional vector spaces. For a quasihomogeneous singularity of a curve we have:

$$\dim_{\mathbf{C}} \mathrm{Tors}\, \Omega^1_{D,0} = \mu(D),$$

where $\mu(D)$ is the Milnor number of the singularity. The latter is the basic topological invariant of singularities. A similar remark concerns hypersurfaces or, more generally, complete intersections with an isolated singularity of arbitrary dimension.

The above Corollary is very useful in computing the torsion modules in the case when $\Omega^q_{S,0}(\log D)$ is a *free* $\mathcal{O}_{S,0}$-module. Indeed, Corollary 2.5 gives us an $\mathcal{O}_{S,0}$-free *resolution* of the torsion module.

Following P.Cartier a divisor $D \subset S$ is called a *Saito-divisor,* or sometimes, a *free* divisor if the $\mathcal{O}_S$-module $\Omega^q_S(\log D)$ for some $q \geq 1$ is locally free. For example, the discriminants of the minimal versal deformations of isolated hypersurface or complete intersections singularities are free Saito divisors. In fact, the theory of such type divisors is very interesting and rich. Here we only note that for a quasihomogeneous free Saito divisor it is possible to compute all its Milnor numbers $\mu_i,\ i = 1, \ldots, \dim D$, in terms of the homology groups of a Lie algebra consisting of vector fields logarithmic along $D$ and annihilating the defining ideal of $D$ (see details in [2]).

# 3. The Poincaré residue map

We start with the definition of the classical Poincaré residue following [17]. In fact, this notion appears in the works of Georges de Rham (1954) and Jean Leray (1959) and was inspired by the ideas of H. Poincaré who in 1887 introduced the notion of 1-form-residue of any rational 2-form in $\mathbf{C}^2$.

Let $\omega$ be a meromorphic differential $m$-form on an $m$-dimensional complex analytic *manifold* $S$ with a polar divisor $D \subset S$. Thus, locally we have a representation:

$$\omega = \frac{f(z) dz_1 \wedge \ldots \wedge dz_m}{h(z)}.$$

By definition, the Poincaré residue $\mathrm{rés}_D(\omega)$ is a meromorphic $(m-1)$-form on $D$ whose singularities are contained in the singular locus $\mathrm{Sing}\, D \subset D$. To define this form explicitly, let us note that at each point $x \in D \setminus \mathrm{Sing}\, D$ one of the derivatives does not vanish:

$$\frac{\partial h}{\partial z_i}(x) \neq 0.$$

Hence there is a locally defined expression

$$\mathrm{rés}_D(\omega) = (-1)^{m-i} \frac{f(x) dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_m}{\frac{\partial h}{\partial z_i}(x)} \Bigg|_D .$$

It is not difficult to verify that this restriction depends neither on the index $i$ nor on the local coordinates and on a defining equation of $D$. Moreover, the Poincaré residue $\mathrm{rés}_D(\omega)$ is holomorphic on the complement $S \setminus D$. When $D$ is *smooth,* one can take $h(z) = z_m$, and then

$$\mathrm{rés}_D \left( \frac{f(x) dz_1 \wedge \ldots \wedge dz_m}{z_m} \right) = f(x) dz_1 \wedge \ldots \wedge dz_{m-1}$$

coincides with the usual residue.

For brevity, the Poincaré residue $\mathrm{rés}_D(\omega)$ will be denoted by $\mathrm{rés}(\omega)$. When $D \subset S$ is a *nonsingular* hypersurface, the Poincaré residue induces an exact sequence

$$0 \longrightarrow \Omega_S^m \longrightarrow \Omega_S^m(D) \xrightarrow{\mathrm{rés}} \Omega_D^{m-1} \longrightarrow 0 \ ,$$

where $\Omega_S^m(D)$ denotes the set of meromorphic forms on $S$ having a pole of the first order along the divisor $D$. In particular we obtain that the germ of every holomorphic $(m-1)$-form on $D$ is a Poincaré residue. It is obvious that this is true globally when the first cohomology group vanishes: $H^1(S, \Omega_S^m) = 0$.

More generally, one can apply the above construction to meromorphic differential $q$-forms on $S$ with a *smooth* polar divisor $D \subset S$ for any $1 \leq q \leq m$, and write down locally for any differential $q$-form $\omega$ with poles of the first order the following presentation:

$$\omega = \frac{dh}{h} \wedge \xi + \eta,$$

where $\xi$ and $\eta$ are holomorphic. Then the restriction of $\xi$ to $D$ is a holomorphic form which is independent of the choice of a local equation $h = 0$ of $D$, and is denoted by $\mathrm{rés}(\omega)$ as before. As a result one gets (see [26]) an exact sequence

$$0 \longrightarrow \Omega_S^q \longrightarrow \Omega_S^q(D) \xrightarrow{\mathrm{rés}} \Omega_D^{q-1} \longrightarrow 0, \quad 1 \leq q \leq m,$$

which is equivalent, since the divisor $D$ is a *smooth* hypersurface, to the sequence

$$0 \longrightarrow \Omega_S^q \longrightarrow \Omega_S^q(\log D) \xrightarrow{\mathrm{rés}} \Omega_D^{q-1} \longrightarrow 0 \ .$$

However, an extension of the notion of Poincaré residue to the case when the divisor $D$ has singularities requires more delicate considerations. Here we only note that already de Rham and Leray remarked that the above definition could be applied to the case when the singular locus $\mathrm{Sing}\, D$ of the divisor $D$ consists of isolated double quadratic points only, or when the divisor $D$ is the union of smooth hypersurfaces in general position. It should be also remarked that influenced by Leray's works M.Herrera, P.Liberman and their followers, have developed the theory of the residue-currents.

We shall describe another approach, based on the notions of *logarithmic* and *regular meromorphic* differential forms, which leads to a natural generalization of the Poincaré residue to the general case of a singular divisor $D$.

# 4. A logarithmic residue

Now recall another important definition due to K.Saito [28], [29]. Let $D \subset U$ be a hypersurface, and let the sheaf $\mathcal{M}_D$ be the $\mathcal{O}_D$-Module of germs of meromorphic functions on $D$.

**Definition 4.1** (see [29], (2.2)). The (logarithmic) residue morphism is a homomorphism of $\mathcal{O}_S$-modules

$$\text{res.}: \ \Omega^q_S(\log D) \longrightarrow \mathcal{M}_D \otimes_{\mathcal{O}_D} \Omega^{q-1}_D,$$

defined locally as follows. By the basic Lemma 1.5, for any $\omega \in \Omega^q_{S,0}(\log D)$ there exist a holomorphic function $g$ defining a hypersurface $V \subset U \subset S$, a holomorphic $(q-1)$-form $\xi$ and a holomorphic $q$-form $\eta$ in $U$ such that

*a*) $\dim_{\mathbf{C}} D \cap V \leq m-2$,

*b*) $g\omega = \dfrac{dh}{h} \wedge \xi + \eta$.

Then $\text{res.}\,\omega$ is given by the germ of the meromorphic $(q-1)$-form $\dfrac{1}{g}\xi$ in the module $\mathcal{M}_{D,0} \otimes_{\mathcal{O}_{D,0}} \Omega^{q-1}_{D,0}$.

**Claim 4.2** (see [29], (2.5)). *Let $D \subset S$ be a hypersurface. Then for any $q \geq 1$ there exists an exact sequence of $\mathcal{O}_S$-modules*

$$0 \longrightarrow \Omega^q_S \longrightarrow \Omega^q_S(\log D) \xrightarrow{\text{res.}} \mathcal{M}_D \otimes_{\mathcal{O}_D} \Omega^{q-1}_D.$$

**Proof.** Making use of the presentation of logarithmic forms as in the definition of the symbol res. above, we have

$$\text{res.}\,\omega = 0 \Leftrightarrow g\omega \in \Omega^q_{S,0} \Leftrightarrow \omega \in \Omega^q_{S,0}.$$

This completes the proof. In particular, for $q = 1$ we have

$$0 \longrightarrow \Omega^1_S \longrightarrow \Omega^1_S(\log D) \xrightarrow{\text{res.}} \mathcal{M}_D \cong \mathcal{M}_{\widetilde{D}},$$

where $\widetilde{D}$ is the normalization of $D$, that is, $\pi\colon \widetilde{D} \longrightarrow D$. K.Saito remarked, that in the case when $D$ is the normal crossing outside of a subvariety of codimension at least 2, then the image Im (res. ) contains $\pi_*(\mathcal{O}_{\widetilde{D}})$ consisting of so-called *weakly holomorphic* function on $D$ (see [29], (2.8)), that is, meromorphic functions, whose preimage becomes holomorphic on the normalization. Some interesting properties of the class of weakly holomorphic functions are investigated in [33].

**Remark 4.3.** By this way we can consider the image of the logarithmic residue $\mathrm{res.}\,\Omega^q_S(\log D)$ as an $\mathcal{O}_D$-Module. Indeed, the definition of logarithmic forms implies that

$$h\cdot(\Omega^q_{S,0}(\log D)/\Omega^q_{S,0}) = 0.$$

Hence the multiplication by $h\cdot$ annihilates Im (res. ).

Following D.Barlet [9], let us consider the $\mathcal{O}_D$-modules of *regular meromorphic* differential forms $\omega^q_D,\ q\geq 0$, on the hypersurface $D$, making use of the following local representation:

$$0\longrightarrow \omega^q_{D,0}\xrightarrow{\mathcal{C}}\mathrm{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega^{q+1}_{S,0})\xrightarrow{\wedge dh}\mathrm{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega^{q+2}_{S,0}),\quad q\geq 0,$$

where $\omega^q_{D,0}\subset j_*j^*\Omega^q_{D,0}$ and $\mathcal{C}$ is induced by the multiplication on the fundamental class of $D$ in $S$. Thus $\mathcal{C}(v)$ corresponds to a Čech cocycle $w/h$ such that $w = v\wedge dh$. It is easy to see, that there is an isomorphism

$$\omega^{m-1}_{D,0}\cong \mathrm{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega^m_{S,0}).$$

Hence $\omega^{m-1}_{D,0}$ is the Grothendieck *dualizing module* of the $(m-1)$-dimensional germ of the hypersurface $(D,0)$.

The following theorem shows that the logarithmic residue introduced by K.Saito is highly suitable for a good extension or a generalization of the classical notion of the Poincaré residue to the case of *singular* hypersurfaces.

**Theorem 4.4** (see [5], §4)**.** *Let $D\subset S$ be a reduced hypersurface. Then for any $q\geq 0$ there are an isomorphism of $\mathcal{O}_D$-modules*

$$\omega^q_D\cong \mathrm{res.}\,\Omega^{q+1}_S(\log D),$$

*and an exact sequence*

$$0\longrightarrow\Omega^{q+1}_S\longrightarrow\Omega^{q+1}_S(\log D)\xrightarrow{\mathrm{res.}}\omega^q_D\longrightarrow 0\ .$$

**Proof.** It is sufficient to verify both statements locally. Let $\mathcal{K}.(h)$ be the usual Koszul complex associated with $h$

$$0\longrightarrow\mathcal{O}_{S,0}e_0\xrightarrow{d_0}\mathcal{O}_{S,0}\xrightarrow{d_{-1}}\mathcal{O}_{D,0}\longrightarrow 0\ ,$$

where $\mathcal{K}_1(h) = \mathcal{O}_{S,0}e_0,\ \mathcal{K}_0(h) = \mathcal{O}_{S,0}$ and $d_0(e_0) = h,\ \ d_{-1}(1) = 1$. Then we have the following piece of the dual exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_0(h),\ \Omega^{q+1}_{S,0}) \xrightarrow{d^0} \mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_1(h),\ \Omega^{q+1}_{S,0}) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega^{q+1}_{S,0}) \longrightarrow 0.$$

This means that any element of $\mathrm{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega^{q+1}_{S,0})$ can be presented as a Čech 0-cochain (more explicitly, a 0-cocycle)

$$\nu/h \in \mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_1(h),\ \Omega^{q+1}_{S,0}) \cong C^0_{(1)}(\Omega^{q+1}_{S,0}),$$

where $\nu \in \Omega^{q+1}_{S,0}$.

Choose now an element $\nu \in \Omega^{q+1}_{S,0}$ such that

$$\frac{\nu}{h} \wedge dh \in \mathrm{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega^{q+2}_{S,0}),$$

corresponds to the trivial element (see the definition of $\omega^q_{D,0}$). That is, $\nu \wedge dh/h$ is defined by an element of

$$d^0(\mathrm{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_0(h),\ \Omega^{q+2}_{S,0})).$$

This means that $\nu\wedge dh = h\cdot\eta$ for some form $\eta \in \Omega^{q+2}_{S,0}$. From the first exact sequence in Proposition 2.3, it follows that $\nu \in h \cdot \Omega^{q+1}_{S,0}(\log D)$. Hence, $\tilde{\nu} = \mathcal{C}^{-1}(\nu/h)$. Set in the definition $v = \tilde{\nu},\ w = \nu$; it follows that $\mathcal{C}(\tilde{\nu})$ corresponds to a Čech cocycle $\nu/h$ such that $\nu = \tilde{\nu} \wedge dh$. This implies $\nu/h = \tilde{\nu} \wedge dh/h$, and $\mathrm{res.}\,(\nu/h) = \tilde{\nu}$. Thus, for any element $\tilde{\nu} \in \omega^q_D$ there is a preimage under the logarithmic residue map represented by $\nu/h$.

**Remark 4.5.** In the case when $D$ is a *normal* hypersurface, the sheaves $\omega^q_D$ have the following well-known interpretation:

$$\omega^q_D \cong j_*j^*\Omega^q_D,$$

where $j\colon D \setminus \mathrm{Sing}\, D \longrightarrow D$ is the canonical embedding. More precisely, in such a case for any $q \geq 0$ there is an exact sequence of local cohomology groups with supports in $Z = \mathrm{Sing}\, D$ :

$$0 \longrightarrow H^0_Z(\Omega^q_D) \longrightarrow \Omega^q_D \longrightarrow \omega^q_D \longrightarrow H^1_Z(\Omega^q_D) \longrightarrow 0.$$

It should be remarked also that in general $\omega^0_{D,0}$, as well as $\mathrm{res.}\,\Omega^1_{S,0}(\log D)$, contains all the germs of *locally bounded meromorphic* functions on $D$ (see [9], §2 ex.i)). Moreover, these two sets coincide if the local fundamental group of the complement $S \setminus D$ is abelian (see [27]).

**Example 4.6** (see [9], end of §1)**.** In the above Example 2.1 of a cusp singularity, it is easy to compute that $\mathrm{res.}\,\Omega^1_{S,0}(\log D)$ and $\omega^0_{D,0}$ are generated by 1 and $y/x$ which one can think of as elements belonging to $\mathcal{M}_{S,0}$ or $j_*j^*\Omega^0_{D,0} \cong j_*j^*\mathcal{O}_{D,0}$ respectively, while $\omega^1_{D,0}$ and $\mathrm{res.}\,\Omega^2_{S,0}(\log D)$ are generated by $dy/2x = dx/3y^2$ which also belongs to $\mathcal{M}_{S,0} \otimes \Omega^1_{S,0}$ or to $j_*j^*\Omega^1_{D,0}$.

# 5. Duality

Following [2], §5 we now explain that there is a natural extension of Grothendieck duality for isolated singularities to the case of non-normal hypersurfaces. The main idea is to use localization of the germ of such a hypersurface, relative to the ideal which defines its singular locus.

More precisely, let $X$ be a reduced irreducible Cohen-Macaulay germ of an $n$-dimensional complex variety, $\mathcal{O} = \mathcal{O}_X$ its (dual) analytic algebra and let $Z \subset X$ be any subgerm defined by the ideal $\mathcal{J} \subset \mathcal{O}$. Suppose that $Z$ is *pure-dimensional* and $\dim Z = d$. In other words, the ideal $\mathcal{J}$ is *unmixed* (i.e. without embedded prime ideals) and its height is equal to $n - d \geq 1$. We denote by $\{\wp_1, \dots, \wp_h\}$ the set of all the minimal prime divisors of the ideal $\mathcal{J}$. Then every ideal $\wp_i$, $i = 1, \dots, h$, corresponds to some $d$-dimensional irreducible subgerm $Z_i \subset X$, hence $Z_{\text{red}} = \bigcup\limits_{i=1}^{h} Z_i$ is the irreducible decomposition of the support $Z$.

Further let $\mathcal{L} = \mathcal{O} \setminus \bigcup\limits_{i=1}^{h} \wp_i \subset \mathcal{O}$ be the multiplicative subset of the local ring $\mathcal{O}$. Then the localization $\mathcal{L}^{-1}\mathcal{O}$ of the ring $\mathcal{O}$ by this multiplicative system is an $(n-d)$-dimensional *reduced* semilocal rings (and a $\mathbf{C}$-algebra at the same time) with maximal ideals $\mathfrak{M}_i$ equal to $\mathcal{L}^{-1}\wp_i$, $i = 1, \dots, h$, and besides $\mathcal{L}^{-1}\mathcal{O}$ is a Cohen-Macaulay ring as well as $\mathcal{O}$. We shall call $\mathcal{L}^{-1}\mathcal{O}$ the *semilocal* ring associated with the reducible germ $Z$ and denote it by $\mathcal{O}_{\mathcal{Z}}$.

Under our assumptions, all the fields $\mathbf{C}(Z_i) \cong \mathcal{O}_{\wp_i}/\mathfrak{M}_i$, $i = 1, \dots, h$, have the same transcendence degree over $\mathbf{C}$, namely $n - d$. Let $\tilde{x}_i = \{\tilde{x}_{i_1}, \dots, \tilde{x}_{i_{n-d}}\}$ be any transcendental basis of $\mathbf{C}(Z_i)/\mathbf{C}$ and let $x_i = \{x_{i_1}, \dots, x_{i_{n-d}}\}$ be any system of representatives of $\tilde{x}_i$ in $\mathcal{O}_{\wp_i}$ (cf. [20], §1). It is known that there exists a field of representatives $\mathcal{T}_i$ for $\mathbf{C}(Z_i)$ in the completion $\hat{\mathcal{O}}_{\wp_i}$. Thus, $\mathbf{C}(x_{i_1}, \dots, x_{i_{n-d}})$ belongs to $\mathcal{T}_i$. Throughout this section we shall assume that $\mathcal{T}_i \subset \mathcal{O}_{\wp_i}$. For instance, this is true for a quasi-homogeneous germ $Z$. In addition, we will suppose (but this is not essential) that $\mathcal{O}$ is a *domain*. Hence, we have the equality $\mathcal{L}^{-1}\mathcal{O} = \bigcap\limits_{i=1}^{h} \mathcal{O}_{\wp_i}$ in the field of fractions of $\mathcal{O}$, and one can define the field of representatives $\mathcal{T} = \cap \mathcal{T}_i$ for the semilocal ring $\mathcal{O}$.

**Theorem 5.1** (cf. [20], (2.3)). *Let $Z \subset X$ be a Cohen-Macaulay germ of pure dimension d. Under the above assumptions, there exist $\mathcal{F}$-functorial isomorphisms in the category of coherent $\hat{\mathcal{O}}_{\mathcal{Z}}$-modules:*

$$\mathcal{E}xt^i_{\hat{\mathcal{O}}_{\mathcal{Z}}}(\hat{\mathcal{F}}_{\mathcal{Z}},\ \hat{\omega}_{\mathcal{Z}}) \cong \mathcal{H}om_{\mathcal{T}}(H^{d-i}_Z(\mathcal{F}),\ \Omega^{n-d}_{\mathcal{T}/\mathbf{C}}), \quad i = 0, 1, \dots, d.$$

*Here we denote the Grothendieck dualizing module of the germ $X$ by $\omega = \omega^n_X$, the completion of $\mathcal{O}_{\mathcal{Z}}$ with respect to the radical $\mathfrak{R} = \cap\ \mathfrak{M}_i$ by $\hat{\mathcal{O}}_{\mathcal{Z}}$, $\hat{\mathcal{F}}_{\mathcal{Z}} = \mathcal{F} \otimes_{\mathcal{O}} \hat{\mathcal{O}}_{\mathcal{Z}}$ (just the same $\hat{\omega}_{\mathcal{Z}}$), the local cohomology functor with supports in $Z_{red}$ by $H^*_Z(-)$, and the module of meromorphic differential $(n-d)$-forms on $\mathcal{T}/\mathbf{C}$ by $\Omega^{n-d}_{\mathcal{T}/\mathbf{C}}$.*

**Proof.** Under our assumptions this Theorem can be proved as in [20], (2.2) taking into consideration the following. Instead of the algebraically geometric objects of [20] we can consider in a similar way, analytic objects, and we can take the system of regular parameters (in the semilocal ring $\mathcal{O}_{\mathcal{Z}}$) which belongs to the radical $\mathfrak{R} \subset \mathcal{O}_{\mathcal{Z}}$, as well as the corresponding system in the case of an irreducible germ $Z$.

**Example 5.2.** Suppose that $X = (D, 0)$ is a germ of *graded* Saito singularity of dimension 2 defined by a quasi-homogeneous polynomial $h$. Set $Z = \operatorname{Sing} D \subset D$. Then $\operatorname{codim}(Z, D) = 1$ and $Z$ is a *Cohen-Macaulay* germ (see [3], §2). Let $\mathcal{F} = \Omega^1_D$ be the $\mathcal{O}_D$-module of holomorphic differential 1-forms on $D$. We have

$$\omega = \omega^n_D \cong \mathcal{O}_D(dz/dh), \quad H^0_Z(\Omega^1_D) \cong \operatorname{Tors}\Omega^1_D, \quad \mathcal{E}xt^1_{\mathcal{O}_D}(\Omega^1_D,\ \mathcal{O}_D) \cong T^1(D),$$

where $T^1(D)$ is the space of the first cotangent cohomology of $D$. We remark that $T^1(D)$ and $Q_h = \mathcal{O}_D/J(h)$ are canonically isomorphic up to a shift of grading by $\deg h$, where $J(h)$ denotes the Jacobian ideal generated by the partial derivatives of quasihomogeneous polynomial $h$. Theorem 5.1 states that there exists a $\mathcal{T}$-linear perfect pairing of graded *infinite* dimensional $\mathbf{C}$-vector spaces:

$$\operatorname{Tors}\Omega^1_D \times T^1(D) \xrightarrow{\operatorname{Res}^{\mathcal{T}}} \Omega^1_{\mathcal{T}/\mathbf{C}}.$$

Thus, one can consider this pairing as an analog of Grothendieck's duality for $n$-dimensional hypersurface *isolated* singularities between *finite* dimensional $\mathbf{C}$-vector spaces (cf. also [6], Cor. 3):

$$\operatorname{Tors}\Omega^n_D \times T^1(D) \xrightarrow{\operatorname{Res}} \mathbf{C}\ .$$

For example (see [3], [5]), consider the discriminant $D$ of the versal deformation $\phi\ :\ V \longrightarrow S$ of the simple hypersurface singularity of type $A_3$. In this case $S \cong (\mathbf{C}^3,\ \mathfrak{o})$ and the subgerm $D \subset S$ is defined by the polynomial $h$ of degree 12, so that (see [27])

$$h = \frac{1}{48}(3\zeta^3 - \xi^2), \text{ where } \xi = \frac{8}{3}z_2^3 + 36z_3^2 - 96z_2z_4,\ \zeta = \frac{4}{3}z_2^2 + 16z_4,$$

and the free $\mathcal{O}_S$-module $\Omega^1_S(\log D)$ is generated by three logarithmic differential forms $dh/h$, $\theta_1/h$ and $\theta_2/h$, where

$$\vartheta_1 = (32z_2z_3z_4 - 9z_3^3)dz_2 + (6z_2z_3^2 + 64z_4^2 - 16z_2^2z_4)dz_3 - (4z_2^2z_3 + 48z_3z_4)dz_4\ ;$$

$$\vartheta_2 = (8z_2^2z_4 - 3z_2z_3^2 - 32z_4^2)dz_2 + (2z_2^2z_3 + 24z_3z_4)dz_3 + (16z_2z_4 - 4z_2^3 - 18z_3^2)dz_4.$$

Using the above notations, one obtains that

$$\wp_1 = (z_3,\ z_2^2 - 4z_4),\ \mathcal{T}_1 \cong \mathbf{C}(z_2), \quad \wp_2 = (\xi,\ \zeta),\ \mathcal{T}_2 \cong \mathbf{C}(\sqrt{z_2}),$$

$$\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2 \cong \mathbf{C}(z_2), \qquad \omega \cong \mathcal{O}_D(dz/dh) \cong \mathcal{O}_D(-3).$$

Making use of Proposition 2.3, we get that $\operatorname{Tors}\Omega^1_D$ can be generated as a $\mathbf{C}[z_2]$-module by the following set of 1-forms:

$$\{\ \theta_2,\ \theta_1,\ z_3\theta_2,\ z_4\theta_2,\ z_3z_4\theta_2\}$$

which have degrees 10, 11, 13, 14, 17, respectively. On the other hand, $Q_h = \mathcal{O}_D/J(h)$ is generated as a $\mathbf{C}[z_2]$-module by the elements

$$\{\ 1,\ z_3,\ z_4,\ z_3^2,\ z_3z_4\ \}$$

and their degrees (as elements of $T^1(D)$) are equal to $-12, -9, -8, -6, -5$, respectively. Let us denote by $\mathcal{C}_D$ the class of the 2-form $dz/dh$ in $\mathcal{L}^{-1}\omega$, where $\mathcal{L} = \mathcal{O} \setminus \{\wp_1 \cup \wp_2\} \subset \mathcal{O}$ and $\omega \cong \mathcal{O}_D(dz/dh)$.

Since $\mathcal{O}_{\mathcal{Z}} = \mathcal{L}^{-1}\mathcal{O}$ is an one-dimensional *reduced* semilocal Cohen-Macaulay ring, we can apply [20], (1.7), (2.1), and calculate

$$\operatorname{Res}^{\mathcal{T}}(\langle z_3z_4\theta_2, 1\rangle \wedge \mathcal{C}_D) = \operatorname{Res}^{\mathcal{T}}(z_3z_4\theta_2 \wedge \mathcal{C}_D) = \text{const} \cdot dz_2 \in \Omega^1_{\mathcal{T}/\mathbf{C}},$$

and so on. Moreover, we can also write out the *Poincaré series*

$$P(\operatorname{Tors}\Omega^1_D;\ t) = t^{10} \cdot (1 + t + t^3 + t^4 + t^7)/(1 - t^2)\ := \mathcal{P}(t)/(1 - t^2),$$

$$P(Q_h;\ t) = (1 + t^3 + t^4 + t^6 + t^7)/(1 - t^2),$$

$$P(T^1(D);\ t) = t^{-12} \cdot P(Q_h;\ t)\ := \mathcal{R}(t)/(1 - t^2),\ P(\mathbf{C}[z_2]) = 1/(1 - t^2).$$

The existence of the above-mentioned $\mathcal{T}$-linear perfect pairing in the case we have just considered means that $t^5 \cdot \mathcal{P}(t^{-1}) = \mathcal{R}(t),\ 5 = -\deg(\omega) + \deg(dz_2)$.

# 6. Logarithmic de Rham complex

One can easily verify that the usual exterior differential $d$ defined on the de Rham complex $\Omega^\bullet_S$ on a complex manifold $S$ induces an action on $\Omega^q_S(\log D)$. So the logarithmic de Rham complex

$$\Omega^\bullet_S(\log D) = (\Omega^q_S(\log D), d)_{q=0,1,\ldots}$$

is well defined. In this connection, there arises the problem of calculating the cohomology groups of logarithmic de Rham complex $H^p(S, \Omega^\bullet_S(\log D))$, as well as the corresponding cohomology group sheaves $\mathcal{H}^p(\Omega^\bullet_S(\log D))$ whose stalk at a point $x \in S$ is defined as follows:

$$\mathcal{H}^p(\Omega^\bullet_S(\log D))_x := H^p_{\mathrm{DR}}(\Omega^\bullet_{S,x}(\log D)).$$

For example, we get $\dim_{\mathbf{C}} H^0_{\mathrm{DR}}(\Omega^\bullet_{S,0}(\log D)) = 1$. Furthermore, as a direct consequence of the holomorphic Poincaré lemma one obtains (see [26]) that in the case of a *smooth* hypersurface $D$ the cohomology sheaves of $\Omega^\bullet_S(\log D)$ satisfy

$$\mathcal{H}^0(\Omega^\bullet_S(\log D)) = \mathbf{C}_S, \quad \mathcal{H}^1(\Omega^\bullet_S(\log D)) = \mathbf{C}_D, \quad \mathcal{H}^q(\Omega^\bullet_S(\log D)) = 0, \quad q \geq 2.$$

This implies that the subcomplex $\Omega^\bullet_S(\log D)$ of the complex $\Omega^\bullet_S(\star D)$ of meromorphic forms on $S$ with poles along $D$ of any order has the same cohomology sheaves as the full complex. Hence the hypercohomology groups of $\Omega^\bullet_S(\log D)$ give us the complex cohomology of the complement $S \setminus D$.

**Proposition 6.1.** *Let $D \subset S$ be a reduced hypersurface. Then there exists an exact sequence of complexes of $\mathcal{O}_S$-modules*

$$0 \longrightarrow \Omega^\bullet_S \longrightarrow \Omega^\bullet_S(\log D) \xrightarrow{\text{res.}} \omega^{\bullet-1}_D \longrightarrow 0$$

*which induces a natural isomorphism*

$$H^p_{\mathrm{DR}}(\Omega^\bullet_S(\log D)) \cong H^{p-1}_{\mathrm{DR}}(\omega^\bullet_D)$$

*for any $p \geq 1$, that is, there is a quasi-isomorphism of complexes with shifting grading by $-1$.*

**Proof.** It is easy to see that the map res. commutes with the exterior differential $d$ (see [29], (2.6)), and we get the first assertion from the Theorem 4.4 above. Let us apply the cohomology functor to this exact sequence. The acyclicity of the de Rham complex in positive dimensions on $S$ gives us the isomorphism

$$H^p_{\mathrm{DR}}(\Omega^\bullet_S(\log D)) \cong H^{p-1}_{\mathrm{DR}}(\omega^\bullet_D), \quad p \geq 2,$$

and the long exact sequence

$$0 \longrightarrow H^1_{\mathrm{DR}}(\Omega^\bullet_S(\log D)) \longrightarrow H^1_{\mathrm{DR}}(\omega^{\bullet-1}_D) \longrightarrow$$
$$\longrightarrow H^0_{\mathrm{DR}}(\Omega^\bullet_S) \longrightarrow H^0_{\mathrm{DR}}(\Omega^\bullet_S(\log D)) \longrightarrow 0.$$

But the last two terms of that sequence have the same dimension one. This completes the proof.

**Remark 6.2.** When $D$ is any reduced hypersurface germ and $h$ is the number of irreducible components of $D$, then $\dim_{\mathbf{C}} H^0_{\mathrm{DR}}(\omega^\bullet_{D,0}) = h$ (see [19], (4.1)), and the long exact sequence from the proof of Proposition 6.1 yields $\dim_{\mathbf{C}} H^1_{\mathrm{DR}}(\Omega^\bullet_{S,0}(\log D)) = h$ (cf. [28], §1).

**Corollary 6.3.** *Assume that the singular locus of $D$ consists of quotient singularities by finite groups only. Then*

$$H^p_{\mathrm{DR}}(\Omega^\bullet_S(\log D)) = 0, \quad p \geq 2.$$

**Proof.** Under our assumptions the complex $\omega_D^\bullet$ is exact (see [19], (4.3)).

**Corollary 6.4.** *Suppose that $D \subset S$ is a germ of a hypersurface with an isolated singularity, $\dim D = n \geq 2$, $\mu$ and $\tau$ are the Milnor number and the Tjurina number of $D$, respectively. Then*

$$\dim_{\mathbf{C}} H^1_{\mathrm{DR}}(\Omega_S^\bullet(\log D)) = 1,\ H^p_{\mathrm{DR}}(\Omega_S^\bullet(\log D)) = 0 \ \text{ for any } 1 < p < n,$$

$$\dim_{\mathbf{C}} H^{n+1}_{\mathrm{DR}}(\Omega_S^\bullet(\log D)) - \dim_{\mathbf{C}} H^n_{\mathrm{DR}}(\Omega_S^\bullet(\log D)) = \mu - \tau.$$

**Proof.** This corollary is a direct consequence of the above Proposition and [19], (4.6).

**Corollary 6.5.** *Suppose in addition that $D$ is a quasi-homogeneous germ, defined by a polynomial, of degree $d \in \mathbf{Z}$ relative to the variables $z_i$ of weights $w_i$, $i = 1, \ldots, m$, respectively. Then*

$$\dim_{\mathbf{C}} H^{n+1}_{\mathrm{DR}}(\Omega_S^\bullet(\log D)) = \dim_{\mathbf{C}} H^n_{\mathrm{DR}}(\Omega_S^\bullet(\log D)) =$$

$$= \mathrm{res}_{t=0} t^{-d-1} \cdot \Pi(t^d - t^{w_i})/(t^{w_i} - 1).$$

**Proof.** From [19], (4.8) it follows that $H^n_{\mathrm{DR}}(\omega^\bullet_{D,0}) \cong (\omega^n_{D,0})_0$, where the subscript 0 denotes the zero-homogeneous piece of the graded $\mathbf{C}$-algebra $\omega^n_{D,0}$. Analogously, since $\mathrm{Tors}\,(\Omega^n_{D,0})_0 = (\Omega^n_{D,0})_0 = 0$, from the exact sequence

$$0 \longrightarrow H^0_{\{0\}}(\Omega^n_{D,0}) \longrightarrow \Omega^n_{D,0} \longrightarrow \omega^n_{D,0} \longrightarrow H^1_{\{0\}}(\Omega^n_{D,0}) \longrightarrow 0$$

we get $(\omega^n_{D,0})_0 \cong (H^1_{\{0\}}(\Omega^n_{D,0}))_0$. The Poincaré series of $H^1_{\{0\}}(\Omega^n_{D,0})$ has been written down (see [1]). The second equality follows from the well-known isomorphism

$$H^1_{\{0\}}(\Omega^n_{D,0}) \cong H^1_{\{0\}}(\Omega^{n-1}_{D,0}).$$

In fact, there are other interesting relations between logarithmic and regular meromorphic differential forms, especially in the case of the free Saito divisors.

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