# Lusternik-Schnirelmann category of classifying spaces.

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**Abstract**

Let $X$ be a finite simply connected CW-complex. In this paper, we show that the Lusternik-Schnirelmann category of the classifying space $B\,aut\,X$ is infinite if $X = S^n \vee Y$.

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## 1 Introduction

In this paper $X$ will denote a simply connected CW-complex of finite type, that is, $H^n(X,\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-vector space, for each $n$. Recall that the Lusternik-Schnirelmann category of a topological space, $cat(X)$, is the least integer $n$ such that $X$ can be covered by $(n+1)$ open subsets contractible in $X$, and is infinite if no such $n$ exists.

Denote by $X_0$ the localization of $X$ at zero, the rational Lusternik-Schnirelmann category, $cat_0(X)$, is defined by $cat_0(X) = cat(X_0)$. It verifies $cat_0(X) \leq cat(X)$ [5].

In this paper we will use the theory of Quillen models. The Quillen minimal model of $X$ is a free chain Lie algebra $(\mathbb{L}(V), \delta)$ satisfying $\delta V \subset \mathbb{L}^{\geq 2}V$ and the graded vector space $V$ is related to the cohomology of $X$ by $V_n \cong H^{n+1}(X,\mathbb{Q})$ [9, 1].

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Fibrations of fiber in the homotopy type of $X$ are obtained, up to fiber homotopy equivalence, as pull back of the universal fibration $X \longrightarrow B\,aut^{\bullet}\,X \longrightarrow B\,aut\,X$ [3, 4]; here $aut\,X$ denotes the topological monoid of all self-homotopy equivalences of $X$, $aut^{\bullet}X$ is the submonoid of $aut\,X$ consisting of pointed self-homotopy equivalences of $X$, and $B$ is the Dold-Lashof functor [2]. Let $\tilde{B}\,aut\,X \xrightarrow{f} B\,aut\,X$ be the universal covering, the induced fibration $X \longrightarrow \tilde{B}\,aut^{\bullet}X \longrightarrow \tilde{B}\,aut\,X$ is universal for fibrations with simply connected base spaces [4, Proposition 4.2].

This work deals with the calculation of the Lusternik-Schnirelmann category of $B\,aut\,X$ under restrictions on $X$. The computation of $cat(B\,aut\,X)$ is of great interest as shown by the next result.

**Proposition 1** *Let $X$ be a 1-connected CW-complex and $G$ a connected compact Lie group acting on $X$. If the Lusternik-Schnirelmann category of $\tilde{B}\,aut\,X$ is finite, then the Borel fibration $X \longrightarrow EG \times_G X \longrightarrow BG$ is rationally trivial.*

*Proof.* Let $f : BG \longrightarrow \tilde{B}\,aut\,X$ be the classifying map of the Borel fibration $X \longrightarrow EG \times_G X \longrightarrow BG$. Consider the map $H^*(f, \mathbb{Q}) : H^*(\tilde{B}\,aut\,X, \mathbb{Q}) \longrightarrow H^*(BG, \mathbb{Q}) = \Lambda V$, where $V$ is concentrated in even degrees. Suppose now that $cat(\tilde{B}\,aut\,X)$ is finite. Then $H^*(f, \mathbb{Q})$ is trivial, otherwise the nilpotency index of $H^*(\tilde{B}\,aut\,X, \mathbb{Q})$ is infinite. It is then easily seen that the map $f : BG \longrightarrow \tilde{B}\,aut\,X$ is rationally trivial. ■

We briefly recall the construction of a model of $\tilde{B}\,aut\,X$ that we will use to compute $cat(\tilde{B}\,aut\,X)$.

Define the Lie algebra of derivations $(Der\mathbb{L}(V), D)$ as follows: $Der\mathbb{L}(V) = \bigoplus_{k\geq 1} Der_k(\mathbb{L}(V))$, where $Der_k(\mathbb{L}(V))$ is the vector space of derivations which increase the degree by $k$ with the restriction that $Der_1(\mathbb{L}(V))$ is the vector space of derivations of degree one which commute with the differential $\delta$.

Given two derivations $\theta$ and $\theta'$, the Lie bracket is defined by $[\theta, \theta'] = \theta\theta' - (-1)^{|\theta||\theta'|}\theta'\theta$ and the differential $D$ is defined by $D\theta = [\delta,\, \theta]$.

Define the differential Lie algebra $(s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V),\, \tilde{D})$ as follows:

- $s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V)$ is isomorphic to $s\mathbb{L}(V) \oplus Der\mathbb{L}(V)$ as a graded vector space,

- If $\theta, \theta' \in Der\, \mathbb{L}(V)$ and $sx, sy \in s\mathbb{L}(V)$, then $[\theta, \theta'] = \theta\theta' - (-1)^{|\theta||\theta'|}\theta'\theta$, $[\theta, sx] = (-1)^{|\theta|}\, s\theta(x),\ [sx, sy] = 0$,
- $\tilde{D}(\theta) = [\delta, \theta], \tilde{D}(sx) = -s\delta x + ad\, x$, where $ad\, x$ is the derivation of $\mathbb{L}(V)$ defined by $(ad\ x)(y) = [x, y]$.

**Theorem 2** [10, 12] *A model of the universal fibration $X \longrightarrow \tilde{B}\, aut^{\bullet} X \longrightarrow \tilde{B}\, aut\, X$ is given by*

$$(\mathbb{L}(V), \delta) \longrightarrow (Der\mathbb{L}(V), D) \longrightarrow (s\mathbb{L}(V) \underset{\sim}{\oplus} Der\mathbb{L}(V), \tilde{D}).$$

A model of $\tilde{B}\, aut\, X$ from derivations of the Sullivan minimal model of $X$ is described in [11].

We will suppose henceforth that $X$ is a finite simply connected CW-complex. We know that the LS category of $B\, aut\, X$ is not finite in various cases, among them when $X$ is an elliptic space (i.e. $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional), a wedge of spheres or a product space $X = Y \times Z$ [6, 7].

One may expect, by dualty, the LS-category of $B\, aut\, X$ be finite when $X = Y \vee Z$. We show that it is the case if $Z$ is a wedge of spheres.

# 2 The theorem

**Theorem 3** *The Lusternik-Schnirelmann categoy of $B\, aut\, X$ is infinite if $X = Y \vee Z$, where $Z$ is a wedge of spheres.*

## Proof of the theorem

## Case 1: $X = Y \vee S^{2n}$.

Let $F \longrightarrow E \longrightarrow B$ be a fibration, then $cat(E) \leq (cat(B)+1).(cat(F)+1)-1$. Applying this to the universal fibration $X \longrightarrow B\, aut^{\bullet}\, X \longrightarrow B\, aut\, X$, we get $cat(B\ aut^{\bullet} X) \leq (cat(B\ aut\ X) + 1).(cat(X) + 1) - 1$.

As $cat(X)$ is finite, we deduce that $cat(B\ aut\ X)$ is infinite whenever $cat(B\ aut^{\bullet} X)$ is infinite.

The Quillen minimal model of $X$ is $(\mathbb{L}(V), \delta) = (\mathbb{L}(W \oplus \mathbb{Q}.x_{2n-1}), \delta)$ where $\delta(x_{2n-1}) = 0$ and $\delta(W) \subset \mathbb{L}(W)$.

Let $\theta$ be the derivation defined by $\theta(x_{2n-1}) = [x_{2n-1}, x_{2n-1}]$, $\theta(W) = 0$. Let us show that $\theta$ is a cycle in $(Der\mathbb{L}(V), D)$. Obviously $[\delta, \theta](x_{2n-1}) = 0$ and if $w \in W$, then $[\delta, \theta](w) = \delta\theta(w) + \theta(\delta w) = \theta(\delta w)$. But $\delta(w) \in \mathbb{L}(W)$, therefore $\theta(\delta w) = 0$. Moreover, $\theta$ cannot be a boundary. If it is, then there exists a derivation $\theta'$ such that $[\delta, \theta'](x_{2n-1}) = \delta\theta'(x_{2n-1}) = \theta(x_{2n-1}) = [x_{2n-1}, x_{2n-1}]$; what should imply that $[x_{2n-1}, x_{2n-1}]$ is a boundary in $(\mathbb{L}(V), \delta)$.

As $[\theta, \theta] = 0$, the injection of the Lie subalgebra generated by $\theta$ provides a morphism $K(\mathbb{Q}, 2n) \longrightarrow (\tilde{B}\, aut^{\bullet}\, X)_0$ that induces an injective map in homotopy. Therefore, applying the mapping theorem [5], $cat(\tilde{B}\, aut^{\bullet}\, X)$ is infinite.

### Case 2: $X = Y \vee S^{2n+1}$.

Consider the univesal principal $G$-bundle $S^1 \longrightarrow PK(\mathbb{Z}, 2) \longrightarrow K(\mathbb{Z}, 2)$. The circle $S^1 = \{z \in \mathbb{C},\ |z| = 1\}$ acts on $S^{2n+1} = \{(z_1, \cdots_{n+1}) \in \mathbb{C}^{n+1},\ |z_1|^2 + \cdots + |z_{n+1}|^2 = 1\}$ by $z.(z_1, \cdots z_{n+1}) = (zz_1, \cdots zz_{n+1})$.

The resulting Borel fibration is not trivial and rationally equivalent to the homotopy fibration $S_0^{2n+1} \longrightarrow (\mathbb{C}P(n))_0 \longrightarrow K(\mathbb{Q}, 2)$. Extend the action of $S^1$ on $Y \vee S^{2n+1}$ by setting it being trivial on $Y$. Therefore the resulting Borel fibration is not rationally trivial. Hence, applying Proposition 1, we deduce that $cat(\tilde{B}\, aut\, X)$ is infinite.

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