Height of Heegner Cycles and Derivatives of L-series

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1 Introduction

In [13], Gross and Zagier proved an identity on modular curves between the heights pairings of certain Heegner points and coefficients of certain cusp form of weight 2. As a consequence, they obtained a remarkable formula which relates the first derivative of the L-function of a modular form f of weight 2 on $\Gamma_0(N)$(over suitable imaginary quadratic field $Q(\sqrt{D})$) and the Néron-Tate height of a Heegner divisor $y_{\sqrt{D}}$(over a suitable imaginary quadratic field) on the f-part of the Jacobian of the modular curve $X_0(N)$. If w_N is the Fricke involution, $X_0(N)^* = X_0(N)/w_N$, y_D^* the image of y_D on $X_0(N)^*$, f comes from a modular form with weight $\frac{3}{2}$ under the Shimura lifting, by combining with Waldspurger [23]'s result, then Gross-Zagier formula say that the height of the f-component of Heegner point y_D^* is equal, up to a simple factor, to $L'(f,1)c(D)^2$, where $c(D)$ is the D-th coefficient of the half-integer weight modular form associate to f. In [12], Gross Kohnen and Zagier generalized this relation to the height pairings of the f-component of Heegner divisors with different discrimint.

At the end of [12] [13], Gross and Zagier discussed the possible extension of their work to higher weight modular forms. They mentioned that Deligne has suggested some definitions for CM-cycles and their local pairings. The details of these definitions have been worked out by Brylinski[5]. Since the lack of the global theory of pairings, as well as the theory of self-pairing(ie, adjunction formula), the results of [12] and [13] are difficult to extend to higher weight modular forms by using the theory of Deligne and Brylinski. Recently Zhang[24] used Arakelov theory to extend Gross and Zagier[13]'s calculation to higher weight modular forms. More precisely,

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Zhang defined the (global) height pairings between CM-cycles in certain Kuga-Sato varieties, and established an identity between the height pairings of Heegner cycles and the coefficients of certain cusp forms of higher weights. Zhang [24] [25] also proved an analogue of Gross-Zagier's formula for higher weight modular forms and conjecture of Belinson[2] and Bloch[4] by combining with the work of Nekovár[18] or Besser[3]. Nekovár and Besser generalized the Kolyvagin's work to higher weight modular forms. With the assumption that the height pairing on CM-cycles is positively definite(which is a consequence of the index conjectures of Beilinson conjecture[2] [4] and Gillet-Soule[10]), Zhang also proved the positivity of derivatives of L-series for the modular forms with higher weight.

In this paper, we will try to generalize Gross, Kohnen and Zagier's computation to higher weights modular forms by using Zhang's method. More precisely, we will compute the height pairings between CM cycles in certain Kuga-Sato variety and prove an identity (theorem(1.1)) between the height pairings and coefficients of certain cusp forms of higher weight which comes from Jacobian forms; our theorem(1.2) says that the height pairings of f-component of CM-cycles with different discriminants is equal, up to a simple factor, to the product of derivatives of L-series with certain coefficients of Jacobian form associate to modular forms f, and the theorem(1.4) says that the Gross, Kohnen and Zagier's algebraicity conjecture is right. With the assumption that the height pairing on on CM-cycles is positively definite, the theorem(1.3) says that the f-eigencomponents of Heegner cycle $s_k(P_{D,r})$ all lie a single line in $(S_k(X^*)_f \otimes R)_f$ and the subspace they generate is non-trivial precisely when $L'(f,k) \neq 0$.

In [20], Perrin-Riou proved a p-adic version of Gross-Zagier's formula; and Nékovar[19] has extended the Perrin-Riou's work to higher weight modular forms.

In the following, we will describe our identity and its application.

Let $N \geq 1, k \geq 1, D_0 \leq 0, D_1 \leq 0$ be integers and $r_0 r_1 = \rho (mod 2N)$ with $r_i^2 = D_i (mod 4N)$.Let $\Delta = \sqrt{D_0 D_1}$. Let $S_{2k}(N)^-$ be the space of cusp forms of weight $2k$ on $\Gamma_0(N)$ with eigenvalues -1 under the action of Fricke involution w_N. Let $f = \sum_{n=1}^{\infty} a(n) e^{2n\pi z} \in S_{2k}(N)^-$ be normalized newform. According to Skorruppa and Zagier[22], there is a non-zero Jacobian form $\phi = \sum c(n,r) q^n \zeta^r \in J_{2k,N}^{cusp}$ with the same eigenvalues as f under all the Hecke operators T_m, $(m, N) = 1$ and $c(n, r_0)$ are real. Define the L-series associate to f by

$$L(f,s) = \sum_{n=1}^{\infty} a(n) n^{-s}$$

Gross, Kohnen and Zagier[13] proved that $L(f,k) = 0$ and the linear functional

$$f \longrightarrow \frac{(2k-3)!! N^{k-1} \|f\|^2}{2^{k+1} \pi^k \Delta^{\frac{k-1}{2}} \|\phi\|^2} L'(f,k) c(n_0, r_0) c(n_1, r_1)$$

on the hermitian space $S_{2k}(N)^-$ with Peterson product can be represented by an element $F_{k,N,\Delta,\rho,D_0} = \sum_{m\geq 1} a_m q^m \in S_{2k}(\Gamma_0(N)^-$. When $k=1$, in [12], Gross, Kohnen, Zagier constructed heegner divisors P_{D_0,r_o} and P_{D_1,r_1} on $X(N)$ and proved that

$$< P^*_{D_0,r_0} - H(D_0)\infty^*, T_m(P^*_{D_1,r_1} - H(D_1)\infty^*) >= a_m$$

for $(m,N)=1$, where $<,>$ denotes Neron-Tate height pairing on Jacobian of $X_0(N)/\omega_N$, $H(D_i)$ is the class number of the field $Q(\sqrt{D_i})$.

The generalization of Gross, Kohnen and Zagier's identity in this paper is as follows

For each CM- point x^* on $X^*_Q := X_0(N)_{Q[\zeta_N]}/\omega_N$, we will construct an CM-cycle $s_k(x^*)$ on certain Kuga-Sato variety Y^* with codimension k, and these CM-cycle generated a subgroup of $CH^k(Y^*)_0$, we denote the subgroup by $S_k(X^*)$. We can also define the height pairing $<,>$ on the group $S_k(X^*)$ of CM-cycles. We will prove that

Theorem 1.1 *Let $s_k(P^*_{D,r}) = \sum_{x^*\in P_{D,r}} s_k(x^*)$, a_m the Fourier coefficient of F_{k,N,Δ,ρ,D_0}. Then we have*

$$(D_0D_1)^{\frac{k-1}{2}} < s_k(P^*_{D_0,r_0}), T_m s_k(P^*_{D_1,r_1}) >= a_m \tag{1.1}$$

for $(m,N)=1$.

Let V be the subspace of Heegner cycles generated by $\{T_m s_k(P_{D_0,r_0}), T_m s_k(P_{D_1,r_1})\}$ for all $(m,N)=1$, V' be the quotient of V modulo the null subspace with respect to the height pairing, $s'_k(P_{D_0,r_0})$ be the image of $s_k(P_{D_0,r_0})$ in V'. The identity (1.1) will leads to a formula

$$(D_0D_1)^{\frac{k-1}{2}} < s'_k(P^*_{D_0,r_0})_f, s'_k(P^*_{D_1,r_1})_f >= \frac{(f, F_{k,N,\Delta,\rho,D_0})}{(f,f)} \tag{1.2}$$

where $s'_k(P_{D,r})_f$ is the f-eigencomponents of $s'_k(P^*_{D,r})$, $f\in S_{2k}(N)^- = S_{2k}(\Gamma^*_0(N))$ a normalized newforms and $<,>$ has been extended to $S_k(X^*)\times\mathbf{R}$ by linearity. From (1.2) and [12], we obtain

Theorem 1.2 *Let $D_0, D_1 \leq 0$ be the coprime fundamental discriminants, $D_i = {r_i}^2 - 4n_iN$ for $i=0,1$, $f\in S_{2k}(\Gamma^*_0(N))$ a normalized newform and ϕ_f the Jacobi form associated to f. Then*

$$< s'_k(P^*_{D_0,r_0})_f, s'_k(P^*_{D_1,r_1})_f >= \frac{(2k-3)!!N^{k-1}L'(f,k)}{2^{k+1}\pi^k\|\phi\|^2\Delta^{\frac{k-1}{2}}} c(n_0,r_0)c(n_1,r_1). \tag{1.3}$$

Where $c(n_i,r_i)$ are the coefficients of ϕ_f.

On the other hand the main result of [24] implies that

$$< s_k'(P^*_{D_0,r_0})_f, s_k'(P^*_{D_0,r_0})_f >= \frac{(2k-2)!N^{k-1}\|D_0\|^{1/2}}{2^{4k-1}\pi^{2k}\|f\|^2} L(f,D,k)L'(f,k)$$

, and by [12]'s Corollary 1 in Charp.2 ,

$$< s_k'(P^*_{D_0,r_0})_f, s_k'(P^*_{D_0,r_0})_f >= \frac{(2k-3)!!N^{k-1}L'(f,k)}{2^{k+1}\pi^k\|\phi\|^2\|D_0\|^{k-1}} C(n,r)^2 (D^2 = r^2 - 4Nn) \qquad (1.4)$$

If we assume further more that the height pairing on $S_k(X^*)$ is positively definite, then $V = V'$. eqs.(1.3) and (1.4) imply that $s_k(P^*_{D_1,r_1})_f$ and $s_k(P^*_{P_1,r_1})_f$ are collinear, and this gives the following theorem:

Theorem 1.3 *With the assumption that the height pairing on $S_k(X^*)$ is positively definite. Let $f \in S_{2k}(\Gamma_0^*(N))$ be a normalized newform. Then the Subspace of $S_k(X(N)^*_Q) \otimes R$ generated by the f-eigencomponents of all Heegner cycles $(s_k(P^*_{D,r}))$ with $(D, 2N) = 1$ has dimension 1 if $L'(f,k) \neq 0$ and 0 if $L'(f,k) = 0$. More precisely, $s_k(D^*_{D,r})_f = \frac{c(r^2-D/4N,r)}{|D|^{k-1}} s_k(P)_f$, where $c(n,r)$ is the coefficient of $q^n\zeta^r$ in Jacobi form $\phi_f \in J_{2k,N}$ and $s_k(P)_f \in S_{2k}(\Gamma_0^*(N)_Q) \otimes R$ is independent of D and r with $L'(f,k) = \frac{2^{k+1}\pi^k\|\phi_f\|^2}{(2k-3)!!N^{k-1}} < s_k(P)_f, s_k(P)_f >$*

It would be interesting to discuss the relation between the formula $L'(f,k) = \frac{2^{k+1}\pi^k\|\phi_f\|^2}{(2k-3)!!N^{k-1}} < s_k(P)_f, s_k(P)_f >$ and Bloch-Kato's Tamagawa number conjecture by combining Jannsen[15] with Scholl[21] and Besser[3].

Algebraic conjecture. In [13] Gross-Zagier defined certain Green's function $G_{N,s,m}(x,y)$ on the complex points of $X_0(N)$ for $Res \geq 1$. Let

$$G_{N,s,m}(P_{D_0,r_0}, P_{D_1,r_1}) = \sum_{\tau_0 \in P_{D_0,r_0}, \tau_1 \in P_{D_1,r_1}} G_{N,s,m}(\tau_0, \tau_1)$$

$$G^{\pm}_{N,s,m}(P_{D_0,r_0}, P_{D_1,r_1}) = G_{N,s,m}(P_{D_0,r_0}, P_{D_1,r_1}) + G_{N,s,m}(P_{D_0,r_0}, P_{D_1,-r_1}).$$

If $\{\lambda_m\}_{m\geq 1}$ is a finite collection of integers such that the Hecke correspondence $T = \sum_{m\geq 1} \lambda_m T_m$ vanishes on the space $S^{new}_{2k}(\Gamma(N))^-$. For Heegner divisors P_{D_0,r_0}, P_{D_1,r_1}, in [12], Gross-Kohnen-Zagier conjectured that there is a rational number a and an algebraic number b_{τ_0,τ_1} for $\tau_0 \in P_{D_0,r_0}$, $\tau_1 \in P_{D_1,r_1}$ such that

$$\ln a = (D_0 D_1)^{\frac{k-1}{2}} \sum_m \lambda_m G^{\pm}_{N,k,m}(P_{D_0,r_0}, P_{D_1,r_1}) \qquad (1.5)$$

$$\ln b_{\tau_0,\tau_1} = (D_0 D_1)^{\frac{k-1}{2}} \sum_m \lambda_m G^{\pm}_{N,k,m}(\tau_0, \tau_1) \qquad (1.6)$$

and they proved that the conjecture is true for $k = 1$. We will show

Theorem 1.4 *Assume that $k \geq 2$ and T vanishes on $S_k(X_Q^*)$. Then there is a rational number a such that the (1.5) is true and there exists an algebraic number b_{τ_0,τ_1} for $\tau_0 \in P_{D_0,r_0}$, $\tau_1 \in P_{D_1,r_1}$ such that (1.6) is right.*

The analogue of the question for $D_0 = D_1, r_0 = r_1$ was discussed by Zhang [24].

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2 Kuga-Sato Varieties and CM-Cycles

In this section, we will review some basic facts about Kuga-Sato varieties and CM-cycles.

2.1 Universal semi-stable elliptic curves

Let $N \geq 3$ be an integer and ζ_N a primitive N-th root of the unity. Let $X(N)_{Q(\zeta_N)}$ be the compactification of the moduli of elliptic curves E over $Q[\zeta_N]$-scheme S, with a canonical full level N-structure, i.e. , an isomorphism of groups

$$\phi : (Z/N)^2 \to E(N)$$

over S, such that the Weil pairing of $\phi(1,0)$ and $\phi(0,1)$ is ζ_N. Let $\mathcal{E}^0_{Q[\zeta_N]}$ be the universal elliptic curve over the noncuspidal part $X(N)^0_{Q[\zeta_N]}$ of $X(N)_{Q[\zeta_N]}$, and $\mathcal{E}(N)_{Q[\zeta_N]}$ be the Kodaira-Neron minimal compactification of $\mathcal{E}^0(N)_{Q[\zeta_N]}$, which is a semistable elliptic curve over $X(N)_{Q[\zeta_N]}$ with N-polygons at cusps. Let $\pi_{Q[\zeta_N]}$ denote the structure morphism $\mathcal{E}(N)_{Q[\zeta_N]} \to X(N)_{Q[\zeta_N]}$. Following [7] and [16], S. Zhang[24] showed that if N is a product of two relatively prime integers $N_1 \geq 3$ and $N_2 \geq 3$, then the morphism $\pi_{Q[\zeta_N]}$ can be uniquely extended to a morphism $\pi = \pi_{Z[\zeta_N]} : \mathcal{E}(N) \to X(N)$ of regular, flat and projective $Z[\zeta_N]$-schemes, such that π makes $\mathcal{E}(N)$ a semistable elliptic curves over $X(N)$. In [24], Zhang also described the fibers of $X(N)$ and $\mathcal{E}(N)$ over the closed point $spec(k) \to Spec(Z[\zeta_N])$. Let p be the character of k. If $p \not| N$, then X_k is smooth and the cuspidal divisor is reduced, so $\mathcal{E}_k$ is smooth. If p/N, then $X(N)$ is the disjoint union of some smooth curves over k, with crossing at the supersingular points.

Let w_N be the Fricke involution of $X(N)_{Q[\zeta_N]}$. By [16], w_N can be extended to $X(N)$. Let $X(N)^{**}$ be the minimal regulization of $X(N)/w_N$, $\mathcal{E}(N)'$ the pullback of $X(N)^*$ and $\mathcal{E}(N)/w_N$

over $X(N)/w_N$, at a cusp of $X(N)^*$, the fiber of $\mathcal{E}(N)'$ has several components. By blowing up $\mathcal{E}(N)'$ along intersections of the components several times, we will obtain a scheme $\mathcal{E}(N)^{**}$ which is regular, flat and projective $Z[1/N]$-scheme, and the morphism $\pi^{**} : \mathcal{E}(N)^{**} \to X(N)^{**}$ makes $\mathcal{E}(N)^{**}$ a semistable elliptic curves over $X(N)^{**}$. If N is a product of two relatively prime integers N_1 andN_2 which are great than three, by the same argument in the proof of theorem (2.1.1) in [24], we can glue π^{**} to obtain a morphism $\pi^* : \mathcal{E}(N)^* \to X(N)^*$ of regular, flat and projective Z-schemes, the morphism makes $\mathcal{E}(N)^*$ a semistable elliptic curves over $X(N)^*$ and extends the structure morphism $\pi_Q^* : \mathcal{E}(N)/w_N \to X_Q^*$. The fibers of $X(N)^*$ and $\mathcal{E}(N)^*$ over the closed point $Spec(k) \to Spec(Z)$ can be also described as follows, let p be the characteristic of k . If $p \not| N$, then X_k^* is smooth, and the cuspidal divisor is reduced, so $\mathcal{E}(N)_k^*$ is smooth. If p/N, then $X(N)_k^*$ is the disjoint union of some smooth curves over k, with crossing at the supersingular points and the fibers of $\mathcal{E}(N)^*$ over these smooth curves are semistable elliptic curves.

2.2 Kuga-Sato varieties

Let k be a positive integer ≥ 2, N the product of two relatively prime integers $N_1 \geq 3$ and $N_2 \geq 3$. Let $\mathcal{E}_{Q[\zeta_N]}^{2k-2}$ $(\mathcal{E}^*{}_Q^{2k-2})$ be $2k-2$ tuple fiber product over $X(N)(X(N)^*)$ of $\mathcal{E}(N)(\mathcal{E}(N)_Q^*)$ with itself, and let $Y(N)_{Q[\zeta_N]}(Y(N)_Q^*)$ be the canonical desingularization described in [6]. Let $\mathcal{E}_{Z[\zeta_N]}^{2k-2}$ $(\mathcal{E}^*{}_Z^{2k-2})$ be $2k-2$ tuple fiber product over $X(N)(X(N)^*)$ of $\mathcal{E}(N)(\mathcal{E}(N)_Z^*)$ with itself, and let $Y(N)_{Z[\zeta_N]}(Y(N)_Z^*)$ be the regular resolution described in [24] 2.2. It is obvious that $Y(Y^*)$ is the regular model of $Y_{Q[\zeta_N]}(Y_Q^*)$ over $Z[\zeta_N](Z)$.

We recall the geometric realization of Hecke operators. For an integer $m \geq 1$ prime to N, let $X(N,m)$ be the (reduced) curve over Q parameterizing elliptic curves E with a full level N structure C and an isaigeny $\lambda : E \to E'$ of degree m. There are two finite maps $s, t : X(N,m) \to X(N)$ given by

$$s(E,C,\lambda) = (E,C), t(E,C,\lambda) = (E', \lambda_*(C))$$

Then we have the following commutative diagram

$$\begin{array}{ccccccc}
\mathcal{E}^{2k-2} & \xleftarrow{\phi_1} & \mathcal{E}^{2k-2} \otimes_{X(N)^s} X(N,m) & \xrightarrow{\psi} & \mathcal{E}^{2r-2} \otimes_{X(N)^t} X(N,m) & \xrightarrow{\phi_2} & \mathcal{E}^{2k-2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X(N) & \longleftarrow & X(N,m) & = & X(N,m) & \to & X(N)
\end{array}$$

in which ψ is induced by λ.

On Y,the m-th Hecke correspondence T_m is defined as the Zariski closure of the graph of

$$(\phi_1, \phi_2\psi) : \mathcal{E}^{2k-2} \otimes_{X(N)^s} X(N,m) \to \mathcal{E}^{2k-2} \times \mathcal{E}^{2k-2}$$

which commutes with the Scholl projector ϵ in [21] and the Fricke involution w_N. So the T_m also defines a correspondence on Y^*, we also denote this correspondence by T_m, and also call it by Hecke correspondence.

It is obvious that the construction of Scholl is also suited to $Y^* \to X^*$, and the results are also right for this case. We still use symbol ϵ to denote the associated Scholl projector.

2.3 CM-cycles

2.3.1 Cycle $s_k(E)$

Let E be a CM elliptic curve over an integral ring R whose generic point has characteristic zero. The ring $End_R(E) \otimes Q$ which depends only on the isogeny class of E, is isomorphic to the imaginary quadratic extension. Fix an embedding $\tau_E : End_S(E) \times Q \to C$, If $\sqrt{D}(D < 0) \in End_S(E)$, let $\Gamma_{\sqrt{D}}$ be the graph of the multiplication by $\sqrt{D}$. One defines the divisor

$$Z(E, \sqrt{D}) = \Gamma_E \times \{0\} - |D| \{0\} \times E - E \times \{0\} \in CH^1(E \times E)$$

Now $Z(E)^{k-1}$ defines a cycle of codimension $k-1$ in E^{2k-2}. Let S_{2k-2} be the symmetric group which acts on E^{2k-2} by permuting the factors. We define

$$w_k(E, \sqrt{D}) = \textstyle\sum_{g \in S_{2k-2}} sgn(g) g^*(Z(E)^{k-1})$$

and

$$s_k(E, \sqrt{D}) = c w_k(E, \sqrt{D})$$

where c is a positive constant such that the self intersection of $s_k(E, \sqrt{D})_\eta$ in E_η^{2k-2} is $(-1)^{k-1}$. Zhang[24] also proved that the class $[s_k(E, \sqrt{D})]$ of $s_k(E, \sqrt{D})$ in $CH^{k-1}(E^{2k-2})$ does not depend on the choice of $\sqrt{D}$ when τ is fixed, if τ is changed to its complex conjugate, then the class $[s_k(E)]$ is changed to $(-1)^{k-1}[s_k(E)]$ in the Chow group. Thus we can simply denote $s_k(E, \sqrt{D})$ by $s_k(E)$.

2.3.2 CM-cycles

Let N be a positive integer. Let R be an algebraic integer ring which is ramified over all the primes dividing N. Let $x(x^*)$ be an irreducible CM divisor in $X(X^*)$, by which we mean that $x(x^*)$ is flat over R and the corresponding elliptic curves pair has complex multiplication. We will recall the construction of $s_k(x)(s_k(x^*))$ as follows.

We first assume that N is the product of two relatively prime integers ≥ 3. Write $\mathcal{E} := \mathcal{E}(N) \otimes R$, $\mathcal{E}^* := \mathcal{E}(N)^* \otimes R$ and $Y := Y(N) \otimes R$, $Y^* := Y(N)^* \otimes R$. So one has the following morphisms of schemes

$$X \leftarrow \mathcal{E}^{2k-2} \leftarrow Y$$
$$X^* \leftarrow \mathcal{E}^{*2k-2} \leftarrow Y^*$$

For irreducible CM divisor $x(x^*)$ of $X(X^*)$, the corresponding elliptic curve $E = \mathcal{E}_x(E^* = \mathcal{E}^*{}_{x^*})$ has a complex multiplication. Write $s_k(x)(s_k(x^*))$ for the class of $s_k(E)(s_k(E^*))$ in $CH^k(Y)$ $(CH^k(Y^*))$, then we call $s_k(x)(s_k(x^*)$ the CM-Chow-cycle over $x(X^*)$.

We define the space of CM-cycles $S_k(X)(S_k(X^*))$ to be the subspace of $CH^k(Y)(CH^k(Y^*))$ generated by the classes of $w_k(x)(w_k(x^*)$ over all CM-divisors $x(x^*)$ Of $X(X^*)$. So $s_k(x) \in s_k(X) \otimes \mathcal{R}$, $s_k(x^*) \in s_k(X^*) \otimes \mathcal{R}$, and $\pm s_k(x)(\pm s_k(x^*))$ depends only on $x(x^*)$. S.Zhang also proved the functoriality of CM-cycles. By the functoriality, Zhang also gave the definitions of $s_k(x)$ and $S_k(X)$ for N is not the product of two relative prime positive integers great than 3. Similarly we can define $s_k(x^*)$ and $S_k(X^*)$ for the general case. For the details, see Zhang[24]'s 2.4.4. In the following sections, we will use the same symbol $s_k(x)$ to denote the cycle and its class in Chow group.

2.3.3 Action of Hecke operator.

Let N, m be a positive integers, $(N, m) = 1$ and x^* an irreducible CM-divisor of $X(N)^*$. Then one has

$$T_m(s_k(x^*)) = \sum_i m^{k-1} s_k(x_i^*) \tag{2.1}$$

if $T_m x^* = \sum x_i^*$.

Following Zhang[24], it is obvious that the action of Δ_{2k-2} on $S_k(X^*)$ has character ϵ.

2.3.4 An Intersection Formula

In the above construction of $s_k(E, \sqrt{D})$, if $R = k$ is of characteristic $p \neq 0$ and k is algebraic closed field. E is a supsingularial elliptic curve over k, then $End(E) \otimes Q$ is a quaternion algebra over Q, not an imaginary quadratic field. For different α, $\beta \in End(E)$, we can construct two Chow-cycles $s_k(E, \alpha) \in CH^{k-1}(E^{2k-2})$, $s_k(E, \beta) \in CH^{k-1}(E^{2k-2})$ and they are usually different. By Nekovár[19], or Brylinsky[5], or Zhang[24], we can calculate their intersection number by the following formula

$$s_k(E, \alpha) s_k(E, \beta) = P_{k-1}\left(\frac{tr(\alpha\beta)}{\sqrt{N(\alpha)N(\beta)}}\right) \tag{2.2}$$

where $P_{k-1}(x)$ is the first kind Legrande Polynomial, $N(\alpha)$ $(tr(\alpha))$ is the norm number (trace) of α in the quaternion algebra $End_k(E) \otimes Q$. This can be obtained directly from Nekovár[19]'s proposition (3.4) also.

3 Heights for Heegner Cycles

In this chapter, we first review the idea of height pairing for CM-cycles defined by Zhang[24], then define the Heegner cycles $s_k(P^*_{D,r})$ over $X(N)^*$ over Q, these are indexed by negative discriminants D of conductor prime to N and classes $r \pmod{2N}$ with $r^2 \equiv D \pmod{4N}$. Their precise definition will be given in sect 2. Our main aim in this chapter is to compute the global height pairing $< s_k(P^*_{D_0,r_0}), s_k(P^*_{D_1,r_1}) >$ on $S_k(X^*)$ over Q. In the Height computation, we shall assume that D_0 and D_1 are relatively prime. With arithmetic intersection theory, we will express the pairing as a sum of local symbols $< s_k(P^*_{D_0,r_0}), s_k(P^*_{D_1,r_1}) >_v$ indexed by the places v of Q. We will compute the non-archimedean symbols using intersection theory on a regular model Y^* over Z and the archimedean symbols using Green's functions for the Riemann surface $X(N)^*$.

3.1 Height of CM-cycles

In this section, we will recall some back facts on heights of CM-cycles from [24]. Zhang[24] defined the Néron-Tate type height pairing for $S_k(X)$ by using arithmetic intersection theory[9]. This theory was developed by Arakelov[1] and Faltings[8].

Let N be a positive integer, F a number field in which all prime factors of N are unramified. Write $X = X(N) \otimes \mathcal{O}_F$, Zhang defined a height pairings on $S_k(X_F)$. Let X^* be the minimal desingularization of the quotient of X by the Fricke involution w_N. Similar to Zhang[]'s method, we can define a height pairings on $S_k(X^*_F)$. More preciously, If N is a product of two relatively prime integers ≥ 3, let x^*, y^* be two CM-divisor on X^*_F . Let $\bar{x}^*$, $\bar{y}^*$ be their closure on X^*. We obtain integer CM-cycles $s_k(\bar{x}^*)$, if we let $g_k(x^*)$ be the green current for cycle $s_k(x^*)$ such that

(a) $\frac{\partial\bar{\partial}}{\pi i} g_k(x^*) = \delta_{s_k(x^*)}$ and that

(b) The integration $\int g_k(x^*)\eta$ is 0 and any $\frac{\partial\bar{\partial}}{\pi i}$ closed form η on $Y(\mathbf{C})^*$, such $g_k(x^*)$ is determined up to images of ∂ and $\bar{\partial}$. We define the arithmetic CM-cycles over x as

$$\hat{s}_k(x^*) = (s_k(\bar{x}^*), g_k(x^*))$$

Now the height pairing of CM-cycles of $s_k(x^*)$ and $s_k(y^*)$ is

$$< s_k(x^*), s_k(y^*) >= (-1)^k \hat{s}_k(\bar{x}^*) \cdot \hat{s}_k(\bar{y}^*) >$$

If N is not the product of two relatively prime integers ≥ 3. We can also define the height pairing by using projection formula for arithmetic intersection. For the details, see [24].

Positivity. Similar to [24]'s 3.1.2, we can prove the positive definiteness of height pairing on the groups $S_k(X^*)$ of CM-cycles under the Arithmetic Hodge Index Conjecture of Gillet-Soule[10] or under the Height Pairing Conjecture of Beilinson[2] and Bloch[4].

Local decomposition. Let the notations and assumption be given as above, we can decompose the global height into local heights

$$< s_k(x^*), s(y^*) >= \sum_p < s_k(x^*), s_k(y^*) >_p \mathcal{E}_p$$

where

$$\mathcal{E}_p = \left\{ \begin{array}{ll} \log p & \text{if } p \text{ is finite} \\ 1 & \text{if } p \text{ is real} \\ 2 & \text{if } p \text{ is complex} \end{array} \right\}$$

where we can define

$$< s_k(x^*), s_k(y^*) >_p= (-1)^k (\hat{s}_k(x^*), \hat{s}_k(y^*))_p$$

If $x^* \neq y^*$. If $x^* = y^*$, the cycle $\hat{s}_k(\bar{x}^*)$ is supported in the fiber E^{2k-2} of Y^* over $\bar{x}^*$,where E is an elliptic curve corresponding to $\bar{x}^*$, since E^{2k-2} is an abelian variety and $w_k(\bar{x}^*)$ is a sum of abelian varieties, By the []'s Chapter 1, Theorem 1.3.2, we can define local intersection number $(\hat{s}_k(x^*), \hat{s}_k(x^*))_p$ by choosing the same local coordinate as Gross and Zagier[13], for the details see [24].

Since the local decomposition here is compatible with the morphism of modular curves, by the projective formula, one can even define local decomposition of heights even when N is not the product of two relatively prime integers ≥ 3.

3.2 Heegner Cycle

Let D be a negative discriminant which is a square $\pmod{4N}$ and has a conductor prime to N and let r be a class $\pmod{2N}$ with $r^2 \equiv D \pmod{4N}$, Gross-Kohnen-Zagier[12] defined a rational divisor $P_{D,r}$ on $X_0(N)$ over $K = Q(\sqrt{D})$. Recall that the affine points of $X_0(N)$ are given by $x = (\pi : E \to E')$ where E and E' are elliptic curves and π a cyclic N-isogeny. Let $\mathcal{O}_D = Z + Z\frac{r+\sqrt{D}}{2}$ be the ring of discriminant D in K and $\mathcal{N}$ the primitive ideal $ZN + Z\frac{r+\sqrt{D}}{2}$ of index N. The point x in $P_{D,r}$ are those which admit endmorphisms by $\mathcal{O}_D$[i.e., $\mathcal{O}_D$ maps into

$End(E)$ and $End(E')$ with the obvious diagram commuting] and such that the kernel of π is annihilated by $\mathcal{N}$. The $P_{D,r} = \sum_{x \in P_{D,r}} \frac{1}{e} x$, where e is the order of $Aut(x)/\{\pm 1\}$. Then we can define

$$s_k(P_{D,r}) = \sum_{x \in P_{D,r}} \frac{1}{e} s_k(x)$$

which is a rational K-cycle of Y_K.

If we denote by c the non-trivial automorphism of K over Q, then $(P_{D,r})^c = P_{D,-r} = w_N(P_{D,r})$, and c induces a map on Y_K, we also denote it by c, then $s_k(P_{D,r})^c = s_k(P_{D,-r}) = w_N(s_k(P_{D,r}))$. If we let $P^*_{D,r}$ be the image of $P_{D,r}$ on the quotient curve $X^*_0(N) = X_0(N)/w_N$, then $P^*_{D,r}$ is rational over Q and depends only on $\pm r \pmod{2N}$, It is obvious that $s_k(P^*_{P,r})$ is the image of $s_k(P_{D,r})$ on the quotient variety $Y^* = Y_k/w_N$. and it is also rational Q-cycle. Finally, by [24]'s Proposition 2.4.2, and Gross-Kohnen-Zagier[]'s IV 1, we see that

$$T_m(s_k(P_{D,r})) = (m)^{k-1} \sum_{m=dd'} (\frac{D}{d'}) s_k(P_{Dd^2,rd})$$

for D fundamental and $(m, N) = 1$. Since T_m commutes with w_N, the same formula holds for $s_k(P^*_{D,r})$. In the following, we will compute the globe height pairing of Heegner cycles on Y^*_Q as a sum over places of Q, i.e.,

$$< s_k(P^*_{D_0,r_0}), s_k(P^*_{D_1,r_1}) = \sum_p < s_k(P^*_{D_0,r_0}, s_k(P^*_{D_1,r_1}) >_p$$

for $(D_0, D_1) = 1$, $r_0 \neq r_1$.

3.3 The Contribution from the Finite Places

We fix a finite prime p of Q. To calculate the local height $< s_k(P^*_{D_0,r_0}), s_k(P^*_{D_1,r_1}) >_p$ in terms of intersection theory, we need regular model X^* for $X_0(N)^*$, and regular model Y^* for $Y(N)^*$ over the valuation ring Z_p of Q_p. It is obvious that $\mathcal{E}^*_{Z_p} = \mathcal{E}^* \otimes Spec(Z_p)$ is the regular model of $\mathcal{E}_Q$ over the valuation ring Z_p of Q_p, $X^*_{Z_p} = X^* \otimes Spec(Z_p)$ be the regular model of $X^*_0(N)_Q$ over the valuation ring Z_p of Q_p, $Y^*_{Z_p} = Y^* \otimes Spec(Z_p)$ is the regular model of Y^*_Q over the valuation ring Z_p of Q_p. When $p \not| N$, $X^*_{Z_p}$ is smooth over Z_p , $\mathcal{E}^*_{Z_p}$ is smooth over $X^*_{Z_p}$, and $Y^*_{Z_p}$ is smooth over Z_p and $X^*_{Z_p}$. When p/N, the curve $X^* \otimes Z/p$ may have several components, and over every component C of $X^* \otimes Z/p$, the fiber of $\mathcal{E}^*$ over C is regular and semistable curve over C. Let $\bar{P}^*_{D,r}$ be the closure of $P^*_{D,r}$ in $X^*_{Z_p}$. By the definition of local height, one has

$$< s_k(P^*_{D_0,r_0}), s_k(P^*_{D_1,r_1}) >_p = (-1)^k (s_k(\bar{P}^*_{D_0,r_0}) \cdot s_k(\bar{P}^*_{D_1,r_1}))_p \log p. \tag{3.1}$$

Where $(s_k(\bar{P}^*_{D_0,r_0})\cdot s_k(\bar{P}^*_{D_1,r_1}))_p$ is the intersection number of two cycles $(s_k(\bar{P}^*_{D_0,r_0})$ and $(s_k(\bar{P}^*_{D_1,r_1})$ over $Y^*_{Z_p}$. Now we turn to the calculation of the relevant intersection multiplicities. First we have

Proposition 3.1 *If p/N, then* $(s_k(\bar{P}^*_{D_0,r_0})\cdot s_k(\bar{P}^*_{D_1,r_1}))_p = 0$

Proof. We may assume that $p \not| D_0$, the points in P_{D_0,r_0} have ordinary reduction on X. If p is split in $Q(\sqrt{D_1})$, the same is true for points in P_{D_1,r_1}. By the Deuring's theory[3], singular points with ordinary reduction and distinct quadratic fields of multiplication reduction reduce to distinct points (mod p) and they define different $s_k(\bar{P}^*_{D_0,r_0})_p$ and $s_k(\bar{P}_{D_1,r_1})_p$ which have no intersection . If p is not split in $Q(\sqrt{D_1})$, the points in P_{D_1,r_1} have supersingular reduction on X, they are disjoint from the points in the reduction of P_{D_0,r_0} which is ordinary, so $s_k(\bar{P}^*_{D_0,r_0})$ and $s_k(\bar{P}^*_{D_1,r_1})$ have no intersection. Hence the intersection number is zero in all cases.

When $p \not| N$, X_{Z_p} is smooth over Z_p, $\mathcal{E}_{Z_p}$ is smooth over Z_p and smooth over X_{Z_p}, and $X^*_{Z_p} \cong X_{Z_p}/w_N$, $\mathcal{E}^*_{Z_p} \cong \mathcal{E}_{Z_p}/w_N$, and one has the following commutative diagram

$$\begin{array}{ccc} (\mathcal{E}_{Z_p})^k & \xrightarrow{F_{\mathcal{E}}} & (\mathcal{E}^*_{Z_p})^k \\ \downarrow & & \downarrow \\ X_{Z_p} & \xrightarrow{F_X} & X^*_{Z_p}. \end{array}$$

and

$$F_{\mathcal{E}*}(s_k(\bar{P}_{D,r})) = s_k(\bar{P}^*_{D,r})$$
$$F_{\mathcal{E}}{}^*(s_k(\bar{P}^*_{D,r})) = s_k(\bar{P}_{D,r}) + s_k(\bar{P}_{D,-r}).$$

Since D_0 and D_1 are relatively prime, we may re-order them such that $p \not| D_0$. Let W be the completion of the maximal unramified extension of Z_p, fix a square root $\sqrt{D_0}$ of D_0 in W. A Heegner point $x = (\phi : E \to E')$ lies in P_{D_0,r_0} if $\mathcal{O}_0 = Z + Z\frac{D_0+\sqrt{D_0}}{2}$ embeds into $End(x)$ and $\alpha_0 = \frac{r_0+\sqrt{D_0}}{2} \in End(x)$ annihilates $ker(\phi)$. Here and the follows, we will write $\sqrt{D_0}$ for the endmorphism which gives multiplication by our fixed square root of D_0 on the tangent space. The points in the P_{D_0,r_0} are all rational over W, for $x \in P_{D_0,r_0}$, the cycle $s_k(x)$ is defined over W. A point y lies in the divisor $P_{D_1,r_1} + P_{D_1,-r_1}$ if the order $\mathcal{O}_1$ of discriminant D_1 act on y and the cyclic subgroup of order N in its diagram is annihilated by an element $\alpha_1 \in \mathcal{O}_1$ of trace r_1(hence also by $-\alpha_1$ of trace $-r_1$. The divisor $P_{D_0,r_0} + P_{D_1,r_1}$ is always rational over W, so $s_k(P_{D_0,r_0}) + s_k(P_{D_1,r_1})$ is a rational W-cycle; $s_k(y)$ are all rational W-cycle iff $p \not| D_1$. So one has

$$(s_k(\bar{P}^*_{D_0,r_0})\cdot s_k(\bar{P}^*_{D_1,r_1}))_p = (s_k(\bar{P}_{D_0,r_0})(s(P_{D_1,r_1}) + s_k(\bar{P}_{D_1,-r_1}))_W \qquad (3.2)$$

and we now turn to the intersection product on $(\mathcal{E})^k$ over W.

First we assume that that $p \not| D_1$. Then we can fix a square root $\sqrt{D_1}$ in W. A point $y = (\phi : F \to F')$ lies in the divisor P_{D_1,r_1} over W if the endmorphism $\alpha_1 = \frac{r_1+\sqrt{D_1}}{2}$ annihilates $ker(\phi)$. If $(xy)_W > 0$ for $x \in P_{D_0,r_0}, y \in P_{D_1,r_1}$, then our diagrams reduce to the same isogeny $z \equiv (\phi : E \to E') \equiv (\psi : F \to F')$ on $X \otimes W/pW$. Write $R = End_{W/pW}(z)$ the reduction of endmorphisms gives injections $End(x) \to R$, $End(y) \to R$. It follows from Deuring's theory that z is a supersingular point on $X \otimes W/pW$, and that $R \otimes Q$ is a quaternion algebra over Q ramified only at p and at infinity. By Gross-Kohnen-Zagier[12], we know that the embedings of $End_W(x)$ and $End_W(y)$ give elements $\sqrt{D_i}$ and $\alpha_i = (r_i + \sqrt{D_i})/2 (i = 0,1)$ in R satisfy

$$\sqrt{D_0}\sqrt{D_1} + \sqrt{D_1}\sqrt{D_0} = 2n, \text{for some } n \in Z,\ n \equiv r_0 r_1 \pmod{2N},\ n \leq r_0 r_1.$$

Lemma 3.1 *Let x,y, n, N, D_0, D_1, r_0, r_1, R be as above and $R^* = \{\pm 1\}$, $\Delta = D_0 D_1$. Then*

$$(s_k(\bar{x}) \cdot s_k(\bar{y}))_W = \frac{1}{2} ord_p(\frac{D_0 D_1 - 4n^2}{4N}) P_{k-1}(\frac{n}{\sqrt{\Delta}})$$

where $P_k(t)$ is the first kind Legendre function.

Proof By [24]'s Proposition 3.3.1(a), one have,

$$(s_k(\bar{x}) \cdot s_k(\bar{y}))_W = \frac{(-1)^k}{2} \sum_{\substack{n \geq 1 \\ \phi \in Isom_n(\bar{x},\bar{y})}} s_k(\bar{y})_0 \phi^* s_k(\bar{x})_0$$

where $s_k(\bar{x})_0$, $s_k(\bar{y})_0$ denote the reduction of $s_k(x)$ and $s_k(y)$ over $Y \otimes W/pW$. By the assumption and Gross-Kohnen-Zagier[12]'s calculation, we have

$$(s_k(\bar{x}) \cdot s_k(\bar{y}))_W = \frac{(-1)^k}{2} (ord_p(M) + 1) s_k(\bar{x})_0 s_k(\bar{y})_0$$

where $M = \frac{D_0 D_1 - n^2}{4N}$, $s_k(\bar{x})_0 = s_k(E_z, \sqrt{D_0})$, $s_k(\bar{y}) = s_k(E_z, \sqrt{D_1})$, by the (2.2), one has

$$(s_k(\bar{x}) \cdot s_k(\bar{y}))_W = \frac{(-1)^{k-1}}{2} (ord_p(M) + 1) P_{k-1}(\frac{n}{\sqrt{D_0 D_1}}).$$

Conversely, by [12] , for every supsingular point z on $X \otimes W/pW$ and let $R = End_{W/pW}(z)$, $I \subset R$ the ideal annihilating the kernel of cyclic N-isogeny defined by z. Fix a binary quadratic form $[D_0, 2n, D_1] = D_0 x^2 + 2nxy + D_1 y^2$ with $n^2 \leq D_0 D_1$ with $n^2 \leq D_0 D_1$ and $n \equiv r_0 r_1 \pmod{2N}$. Let $S = S_{[D_0,n,D_1]}$ be the associated Cliford order and $M = \frac{D_0 D_1 - n^2}{4N}$. For each embedding $\phi : S_{[D_0,n,D_1]} \to R$ normalized by insisting that $\phi(e_0) = \sqrt{D_0}$ on the tangent space, write $\phi(e_1) = \epsilon\sqrt{D_1}$ with $\epsilon = \pm 1$, where $e_0, e_1 \in S_{[D_0,n,D_1]}$ with $e_i^2 = D_i$ for $i = 0,1$ and $e_0 e_1 + e_1 e_0 = n$. The idea I contains the elements $\frac{s_0 + \sqrt{D_0}}{2}$ and $\frac{s_1 + \mathcal{E}\sqrt{D_1}}{2}$ for some integers s_0,s_1

(well defined modulo N), and $s_0 s_1 \equiv n \pmod{2N}$, and the induce embedding of $\mathcal{O}_0 = Q(\sqrt{D_0})$ and $\mathcal{O}_1 = Q(\sqrt{D_1})$ into R correspond to points $x \in P_{D_0,r_0}$, $y \in P_{D_1,r_1}$ which reduce to $z \pmod{p}$ and are congruent modulo p^l, where $l = (ord_p(M)+1)/2$. By the calculation of Gross-Kohnen-Zagier[12], one has,

$$
\begin{aligned}
& (s_k(\bar{P}_{D_0,r_0})(s_k(\bar{P}_{D_1,r_1}) + s_k(\bar{P}_{D_1,-r_1}))_W \\
= & \ \tfrac{1}{2^{t+1}} \sum_{\substack{n \equiv r_0 r_1 (2N) \\ n^2 \leq D_0 D_1}} \textstyle\sum_{R_i} \# \left\{ S_{[D_0,n,D_1]} \to R_i \pmod{R^*/\pm 1} \right\} \times \frac{ord_p(M)+1}{2} P_{k-1}(\frac{n}{\sqrt{\Delta}})
\end{aligned}
$$

where t is the number of prime divisor of N, $\# \left\{ S_{[D_0,n,D_1]} \to R_i \pmod{R)_i^*/\pm 1} \right\}$ is the order of set of equivalent class for the embedding $S_{[D_0,n,N_1]} \to R_i$ under the act of $R_i^*/\pm 1$, R_i is right order of left ideal classes for a fixed Eichler order of index N in the quaternion algebra $B(p)$ ramified at p and ∞. Similar to Gross-Kohnen-Zagier[12]'s discussion in page pp -pp , the formula is still true without the assumption that $R_i^* = \{\pm 1\}$ and $p \not| D_1$. We now combine (3.1) with formula (3.2) to obtain the following formula for the local height pairing:

$$
\begin{aligned}
< s_k(P_{D_0,r_0}), s_k(P_{D_1,r_1}) >_p \ = & \sum_{\substack{n \equiv r_0 r_1 \pmod{2N} \\ n^2 \geq D_0 D_1}} 2^{-t-1} \sum_{R_i} \# \{ \phi : S \to R_i \pmod{R_i^*/\pm 1} \} & (3.3) \\
& \times \frac{ord_p(M)+1}{2} P_{k-1}(\frac{n}{\sqrt{\Delta}})(-\log p) & (3.4)
\end{aligned}
$$

For $M = \frac{n^2 - D_0 D_1}{4N}$, any $p \not| NM$, we define $\epsilon(p)$ as follows, let D be any integer prime to p which can be represented by $q = D_0 x^2 + 2nxy + D_1 y^2$ and define $\epsilon(p) = (\frac{D}{p})$. This is independent of the choice of D and equals to 1 if p/N. For any positive divisor $d = \prod_i p_i^{a_i}$ of M, we let $\epsilon(d) = \prod_i \epsilon(p_i)^{a_i}$ and introduce the Dirichlet series

$$
l(s) = \sum_{d/M} \epsilon(d) d^s
$$

Since the algebra $B = S \otimes Q$ is definite and ramifies at at least finite place. Hence $l(0) = 0$ and the derivative $l'(0) = \sum_d \epsilon(d) \log d$ is nonzero precisely. In this case, Gross-Kohnen-Zagier showed in [12]'s Proposition 3 of Chap. 1, Sect.3, that

$$
l'(0) = 2^{-t-1} \sum_{R_i} \# \{ S \to R_i^* \pmod{R)_i/\pm} \} \frac{1}{2}(ord_p(M)+1)(-\log p).
$$

Comparing this with our formula (3.3) for the local height, we obtain our final result:

Theorem 3.1 *The total non-archimedean contribution to the height pairing of Heegner cycles $s_k(P^*_{D_0,r_0})$ and $s_k(P^*_{D_1,r_1})$ is given by*

$$
\sum_{p \ prime} < s_k(P^*_{D_0,r_0}), s_k(P^*_{D_1,r_1}) >_p = \sum_{\substack{n \equiv r_0 r_1 (2N) \\ n^2 \leq D_0 D_1}} (\sum_{d/M} \epsilon(d) \log d) P_{k-1}(\frac{n}{\sqrt{D_0 D_1}})
$$

3.4 Archimedean Contribution

In this section, we will, following Gross-Zagier[13] , Gross-Kohnen-Zagier[12] and Zhang[24], use Green function constructed in [13] to represent the archimedean height. We recall the definition as follows, for $s \in \mathcal{C}$ with $Re(s) > 1$, define the point-pair invariant

$$g_s(z_0, z_1) = -Q_{s-1}(1 + \frac{\|z_0 - z_1\|^2}{2Im(z_0)Im(z_1)})z_0, (z_1 \in \mathcal{H},\ z_0 \neq z_1.)$$

Where $Q_{s-1}(t)$ is the Legendre function of second kind. The Green function is defined by the convergent sum

$$G_{N,s}(z_0, z_1) = \sum_{\gamma \in \Gamma_0(N)} g_s(z_0, \gamma z_1)$$

Let $G^*_{N,s}(z_0, z_1)$ be the correspond function when $\Gamma_0(N)$ is replaced by $\Gamma_0^*(N)$, we have

$$G^*_{N,s} = G_{N,s}(z_0, z_1) + G_{N,s}(z_0, w_N z_1)$$

Proposition 3.2 *For any two CM-points x and y on X^*, one has*

$$< s_k(x), s_k(y) >= \frac{1}{2} G^*_k(x, y)$$

In particular,

$$\begin{aligned} < s_k(P_{D_0,r_0}), s_k(P_{D_1,r_1} >&= \frac{1}{2} G^*_k(P^*_{D_0,r_0}, P^*_{D_1,r_1}) \\ &= \frac{1}{2}[G_k(P_{D_0,r_0}, P_{D_1,r_1}) + G_k(P_{D_0,r_0}, P_{D_1,-r_1})] \end{aligned}$$

Proof Step 1. By the same argument as [24]'s 3.4.2, we can reduce to the case $N \geq 3$.

Step 2. By the same argument as in [24]'s (3.4.3), one can construct a function $H_k(z, z')$ on $X^* \times X^* -$ diagram such that $< s_k(x^*), s_k(y^*) >= H_k(x, y)$ for different CM-points $x, y \in X^*$. By using the same method in Zhang[24] we can prove that $H_k(x, y)$ satisfy

(1) $H_k(z, z')$ is killed by $\Delta - k(k-1)$ operate on the first variable where $\Delta = t^2(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2})$ where $z = s + ti$

(2)$H_k(z, z') = \log \|z - z'\| + O(1)$ as $z \to z'$

(3) In a neighborhood of a cusp $\gamma^{-1}\infty$, the function $im(\gamma z)^{k-1} H_k(z, z')$ extends to a continuous function.

step 3. By the same argument as in [13] or [22]'s 3.4.4, one can prove that $H_k(x, y) = \frac{1}{2} G_k(x, y)$.

4 Heights and L-Series

In this section we will prove our main theorems and their consequence. In the first paragraph, we will give the main identity. After assuming the existence of f- isotropic components of heegner cycles, we give several consequences about the derivatives of L-Series and the heights of Heegner cycles in paragraph 4.2.

4.1 Identity.

Let $N \geq 1$ and $\Delta > 0$ be a discriminant which is a product of two negative discriminant D_0,D_1. $D_i = r_i^2 - 4Nn_i$, for $i = 0, 1$. ρ is an integer $\pmod{2N}$ with $\rho^2 \equiv \Delta \pmod{4N}$. Let P_{D_0,r_0}, P_{D_1,r_1} be the Heegner divisor defined in 3.2, $s_k(P^*_{D_0,r_0}), s_k(P^*_{D_1,r_1})$ the associated Heegner cycles. Let $S_{2k}(N)^-$ be the $(-1)^{k-1}$-eigenspace of w_N in the space of cusp forms of weight $2k$, $J^{\text{cusp}}_{k+1,N}$ the space of Jacobian cusp forms of weight $k+1$ and index N. By [22], for any normalized new forms $f = \sum_m a_m q^m \in S_{2k}(N)^-$, there is a non-zero Jacobian form

$$\Phi_f = \sum_{\substack{n,r \in Z \\ r^2 \leq 4Nn}} c(n, r) q^n \zeta^r \in J^{\text{cusp}}_{k+1,N}$$

which has the same eigenvalues as f under all Hecke operators T_m, $(m, N) = 1$ and $c(n, r)$ are all real. Define the L-series associate to f by $L(f, s) = \sum_{n \geq 1} a_m m^{-s}$. Gross-Kohnen-Zagier[12] proved the following theorem:

Theorem 4.1 *(1)* $L(f, k) = 0$

(2) There is a holomorphic cusp form $F_{k,N,\Delta,\rho,D_0} = \sum_{m \geq 1} a(m) q^m \in S_{2k}(\Gamma(N))^-$ *satisfies*

$$(f, F_{k,N,\Delta,\rho,D_0}) = \frac{(2k-3)!!N^{k-1}c(n_0, r_0)c(n_1, r_1)}{2^{k+1}\pi^k \|\phi\|} L'(f, k)$$

for all new form $f \in S^{new}_{2k}(\Gamma_0(N))^-$ *and with*

$$a(m) = (m\sqrt{\Delta})^{k-1}\Big(\sum_{\substack{|n| < m\sqrt{\Delta} \\ n \equiv m\rho(2N)}} \sigma'_\chi\big(\big(\tfrac{n+m\sqrt{\Delta}}{2}\big)\mathcal{N}^{-1}\big) P_{k-1}\big(\tfrac{n}{m\sqrt{\Delta}}\big) -$$

$$\sum_{\substack{n > m\sqrt{\Delta} \\ n \equiv m\rho(2N)}} \sigma_{0,\chi}\big(\big(\tfrac{n+m\sqrt{\Delta}}{2}\big)\mathcal{N}^{-1}\big) Q_{k-1}\big(\tfrac{n}{m\sqrt{\Delta}}\big) - \sum_{\substack{n > m\sqrt{\Delta} \\ n \equiv -m\rho(2N)}} \sigma_{0,\chi}\big(\big(\tfrac{n+m\sqrt{\Delta}}{2}\big)\mathcal{N}'^{-1}\big) Q_{k-1}\big(\tfrac{n}{m\sqrt{\Delta}}\big)\Big)$$

where $c(n_i, r_i)$ *are the coefficients of Jacobi forms* Φ_f, $\sigma'_{0,\chi}(a)$ *and* $\sigma_\chi(a)$ *for a an integral ideal of* $Q(\sqrt{\Delta})$ *denote the value and derivative respectively of*

$$\sigma_{s,\chi}(a) = \sum_{b|a} \chi(b) N(b)^s \tag{4.1}$$

at $s=0$, and $P_{k-1}(t)$ and $Q_{k-1}(t)$ are Legendre functions of first and second kinds, $\mathcal{N}=ZN+Z(\frac{\rho+\sqrt{\Delta}}{2})\subset\mathcal{O}_K$.

In the following, we will give an arithmetic interpretation of Fourier coefficients of F_{k,N,Δ,ρ,D_0} for $k>1$

Theorem 4.2 $<s_k(P^*_{D_0,r_0}),T_m s_k(P^*_{D_1,r_1})>=(\sqrt{\Delta})^{-k+1}a(m)$

proof. Step 1, In fact one has the following fact: For any decomposition $m=m_0m_1$ with m_0 and m_1 coprime, $(m_1,D_0)=1,(m_0,D_1)=1$ and any place p of Q, one has

$$<s_k(P^*_{D_0,r_0}),T_m s_k(P^*_{D_1,r_1})>_p=\sum_{\substack{m_0=d_0d_0'\\ m_1=d_1d_1'}}(\frac{D_0}{d_0'})(\frac{D_1}{d_1'})<s_k(P^*_{D_0d_0^2,r_0d_0}),s_k(P^*_{D_1d_1^2,r_1d_1})>_p$$

In fact, for any decomposition $m=m_0m_1$ as above, one have

$$<s_k(P^*_{D_0,r_0}),T_m s_k(P^*_{D_1,r_1})>_p=<T_{m_0}s_k(P^*_{D_0,r_0}),T_{m_1}s_k(P^*_{D_1,r_1})>_p$$

This follows that $T_m=T_{m_0}T_{m_1}$ and T_{m_0} is adjoint with respect to the local height pairing.

Step 2. Non-archimedean parts of the height pairing. First we recall a notation in [12]. For integers M, and all of whose prime factors p satisfy $(\frac{\Delta m^2}{p})\neq-1$, we define $l(M,s)=\sum_{d/M}\epsilon(d)d^s$, where ϵ is the multiplicative function on such integers given on primes p with $(\frac{\Delta m^2}{p})\neq-1$ by $\epsilon(p)=(\frac{D_i}{p})$ if $p\not|D_im$, By the theorem 3.1 and Step 1, one has

$$\begin{aligned}&\sum_p<s_k(P^*_{D_0,r_0}),T_m s_k(P^*_{D_1,r_1})>_p\\
=\;&m^{k-1}\sum_{\substack{m_0=d_0d_0',\\ m_1=d_1d_1'}}(\frac{D_0}{d_0'})(\frac{D_1}{d_1'})\sum_{\substack{x<m\sqrt{\Delta}\\ n\equiv\rho\Delta\pmod{2N}}}l'(\frac{\Delta d_0^2d_1^2-n^2}{4N},0)P_{k-1}(\frac{n}{m\sqrt{\Delta}})\\
=\;&m^{k-1}\sum_{\substack{n^2<m^2\Delta\\ n\equiv m\rho(2N)}}\sum_{\substack{d_0'/(n,m_0)\\ d_1'/(n,m_1)}}(\frac{D_0}{d_0'})(\frac{D_1}{d_1'})l'(\frac{\Delta d_0^2d_1^2-n^2}{4N},0)P_{k-1}(\frac{n}{m\sqrt{\Delta}})\\
=\;&m^{k-1}\sum_{\substack{n^2<m^2\Delta\\ n\equiv m\rho(2N)}}\sigma_\chi'((\frac{n+m\sqrt{\Delta}}{2})\mathcal{N}^{-1})P_{k-1}(\frac{n}{m\sqrt{\Delta}})\end{aligned}$$

The last step is followed from [12], Chap V's Lemma.

Step 3. The archimedean part of the height pairing.By the Step 1 and theorem(3.2) one has

$$\begin{aligned}<s_k(P^*_{D_0,r_0}),\quad&T_m s_k(P^*_{D_1,r_1})>_\infty=\sum_{\substack{m=d_0d_0',\\ m_1=d_1d_1'}}(\frac{D_0}{d_0'})(\frac{D_1}{d_1'})<s_k(P^*_{D_0d_0^2,r_0d_0}),s_k(P^*_{D_1d_1^2,r_1d_1})>_\infty\\
&=m^{k-1}\sum_{\substack{m=d_0d_0',\\ m_1=d_1d_1'}}(\frac{D_0}{d_0'})(\frac{D_1}{d_1'})(G_{N,k}(P_{D_0d_0^2,r_0d_0},P_{D_1d_1^2,r_1d_1})+G_{N,k}(P_{D_0d_0^2,r_0d_0},P_{D_1d_1^2,-r_1d_1}))\end{aligned}$$

By Gross-Kohnen-Zagier[12] chapter IV's proposition 2 and chap V's formula (6), one has

$$< s_k(P^*_{D_0,r_0}), T_m s_k(P^*_{D_1,r_1}) >_\infty =$$
$$- \sum_{\substack{n > m\sqrt{\Delta} \\ n \equiv m\rho(2N)}} \sigma_{0,\chi}\left(\left(\tfrac{n+m\sqrt{\Delta}}{2}\right)\mathcal{N}^{-1}\right) Q_{k-1}\left(\tfrac{n}{m\sqrt{\Delta}}\right) - \sum_{\substack{n > m\sqrt{\Delta} \\ n \equiv -m\rho(2N)}} \sigma_{0,\chi}\left(\left(\tfrac{n+m\sqrt{\Delta}}{2}\right)\mathcal{N}^{-1}\right) P_{k-1}\left(\tfrac{n}{m\sqrt{\Delta}}\right)$$

Combining step 2 and step 3, we complete the proof of the theorem.

4.2 Consequence

Before we turn to the consequences, we first give a property about the actions of Hecke correspondences on cusp forms and CM-cycles which is analogous to theorem 1 in [25].

Proposition 4.1 *Let $v_1, v_2, \cdot, v_h \in S_k(X^*)$. Assume for each i, j there is an element $g_{i,j} \in S_{2k}(\Gamma(N))^-$ such that for any positive integer m prime to N, the number $, v_i, T_m v_j >$ is the m-th coefficient of $g_{i,j}$. Let V be the subspace of Heegner cycles generated by $\{T_m v_j, j = 1, \cdot, h\}$ for all $(m, N) = 1$. Let V' be the quotient of V modulo the null subspace with respect to the height pairing. Then the Hecke module V' is isomorphic to a sub-quotient module of $(S_{2k}(\Gamma_0(N))^-)^{+h}$.*

Let $D_i = r_i^2 - 4n_i N < 0 (i = 0, 1)$ be coprime fundamental discriminants and write $s(p_i^*)$ for $s_k(P_{D_i,r_i}) \in S_k(X^*)$. With the assumption that the Hecke algebra on $S_k(X^*)$ should be the the quotient of the Hecke algebra over $S_{2k}(N)^-$, the power series

$$G(z) = \sum_{m=1}^{\infty} < s_k(P_0^*), T_m s_k(P_1^*) > q^m (q^m = e^{2\pi i})$$

defines a modular form in $S_{2k}(\Gamma_0^*(N)) = S_{2k}(N)^-$. By the formal argument given in section 5 in [24] or on P306 of [13], the theorem (4.2) says that the coefficients of q^m in F and G agree for all m prime to N . In particular $F - G$ is an oldform , so $(G, f) = (F, f)$, for any normalized newform f in $S_{2k}(N)^-$. Applying theorem (4.2) and proposition (4.1) to cycles $\{s_k(P_0), s_k(P_1)\}$, we have that V' is a finite dimensional T_k algebra. Let $s'_k(P_0)$ be the image of $s_k(P_0)$ in V'. The arguments on P308 of [9] or [25] show that $(G, f) = (f, f)$ times the height pairing $< s'_k(P_0^*)_f, s'_k(P_1^*)_f >$. Where $s_k(P_i^*)_f$ is the f-eigencomponent of $s_k(P_i^*)$ on $S_k(X^*) \times \mathbf{R}$. Hence

$$< s'_k(P_0^*)_f, s'_k(P_1^*)_f > = \frac{1}{\|f\|^2}(F, f)$$

Let $\Phi_f = \sum c(n, r) q^n \zeta^r$ be the Jacobi form associated to f. By the theorems in the section 3 and section 4 of ChapterII in [12], one has

$$< s'_k(P_0^*)_f, s'_k(P_1^*)_f > = \frac{(2k-3)!! N^{k-1}}{2^{k+1} \pi^k \|\Phi_f\| \Delta^{\frac{k-1}{2}}} L'(f, k) c(n_0, r_0) c(n_1, r_1)$$

so we obtain the following theorem.

Theorem 4.3 *Let $D_0, D_1 < 0$ be coprime fundamental discriminants, $D_i \equiv r_i^2 \pmod{4N}$, and $f \in S_{2k}(\Gamma_0^*(N))$ a normalized newform. Then*

$$< s_k'(P_0^*)_f, s_k'(P_1^*)_f >= \frac{(2k-3)!!N^{k-1}}{2^{k+1}\pi^k \|\Phi_f\| \Delta^{\frac{k-1}{2}}} L'(f,k)c(n_0,r_0)c(n_1,r_1)$$

By S.Zhang[24], we have

$$< s_k'(P_0^*), s_k'(P_0^*) >= \frac{(2k-2)!|D_0|^{\frac{1}{2}}}{2^{4k-1}\pi^{2k}\|f\|^2} L(f,D_0,k)L'(f,k).$$

[where $L(f,D_0,s)$ denotes the twist of $L(f,s)$ by $(\frac{D_0}{\cdot})$] By ChapterII, section 3 Corollary 1 of [12]

$$\frac{c(n_0,r_0)^2}{\|\Phi\|^2} = \frac{(k-1)!|D_0|^{k-1/2}}{2^{2k-1}\pi^k N^{k-1}} \cdot \frac{L(f,D_0,k)}{\|f\|^2},$$

one gets

$$< s_k'(P_0^*)_f, s_k'(P_0^*)_f >= \frac{(2k-3)!!N^{k-1}}{2^{k+1}\pi^k \|\Phi_f\|^2} L'(f,k)c(n_0,r_0)^2$$

If we let $w_k(P^*) = (D)^{\frac{k-1}{2}} s_k(P^*)$, $L = \frac{(2k-3)!!N^{k-1}}{2^{k+1}\pi^k\|\Phi_f\|^2} L'(f,k)$, then,

$$< w_k'(P_0^*)_f, w_k'(P_0^*)_f >= c_0^2 L, < w_k'(P_1^*), w_k'(P_1^*) >= c_1^2 L$$
$$< w_k(P_0^*)_f, w_k'(P_1^*)_f >= c_0 c_1 L$$

With the assumption that the height pairing on $S_k(X^*) \otimes R$ is positive definite, then $V' = V$, the above formulas imply that $c_1 w_k(P_0^*) - c_0 w_k(P_1^*)$ has height 0, hence $s_k(P_0^*)_f$ and $s_k(P_1^*)_f$ is colinear. In order to obtain theorem (1.3), we need the same statement when D_0 and D_1 are not necessary coprime. But this follows, since by lemma 3.2 of [22] we can choose a fundamental discriminants $D_2 = r_2^2 - 4n_2N$ prime to D_0, D_1 and $2N$ with $c_2 = c(n_2,r_2) \neq 0$, and then $(w_k(P_0^*))_f = \frac{c_0}{c_2}(w_k(P_2^*))_f$, $(w_k(P_1^*))_f = \frac{c_1}{c_2}(w_k(P_2^*))_f$. By the special case of coprime discriminants, we deduce that $c_i^{-1}(w_k(P_i^*))_f$ is independent of i(for $c_i \neq 0$), i.e., there exists a vector $y_f \in S_k(X^*) \otimes R)_f$ with height $< y_f, y_f >= L$ such that

$$S_k(P_{D,r}^*) = \frac{c(n,r)}{\|D\|^{k-1}} y_f, (r^2 - 4nN = D) \tag{4.2}$$

for all fundamental discrimint D prime to $2N$, this is Theorem (1.3).

Similar to [13]'s argument, (4.1) is still true for D not fundamental, and D not prime to $2N$. If one knows (4.1) for all D and r, one could restate it in the form

$$\sum_{\substack{n,r \in Z \\ r^2 < 4Nn}} (4Nn - r^2)^{\frac{k-1}{2}} (s_k(P_{r^2-4Nn,r})_f q^n \zeta^r = \phi_f(\tau,z) y_f \tag{4.3}$$

Summing this over all new forms would say that the power series $\sum_{n,r} D^{\frac{k-1}{2}}(s_k(P_{D,r})_{new}q^n\zeta^r$ belongs to $S_k(X^*) \otimes J_{2k,N}^{cusp,new}$, where $D = r^2 - 4Nn$. Presumably the same relation would hold for the old parts by induction on N, this would give the following theorem which can be considered as an analogy of the theorem in [14].

Theorem 4.4 *With the assumption that the the height pairing on $S_k(X^*)\otimes R$ is positive definite, then the formal power series* $\sum_{\substack{r,n\in Z, \\ r^2-4Nn<0}} (4nN - r^2)^{k-1/2} s_k(P^*_{D,r})q^n\zeta^r \in S_k(X^*) \otimes J^{cusp}_{2k,N}$ *with non vanishing f-component if and only if $L'(f,k) \neq 0$*

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