On Bănică sheaves and Fano manifolds

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Dedicated to the memory of Constantin Bănică

Edoardo Ballico and Jarosław A. Wiśniewski

Introduction.

Reflexive sheaves are nowadays a common tool to study projective varieties. In the present paper we apply reflexive sheaves to study projective morphisms. Given a projective map $\varphi : X \to Y$ and an ample line bundle $\mathcal{L}$ on X one may consider an associated coherent sheaf $\mathcal{F} := \varphi_*\mathcal{L}$ on Y. The knowledge of the sheaf $\mathcal{F}$ allows sometimes to understand some properties of the variety X and of the map φ. This is a typical way to study cyclic coverings (or, more generally, finite maps) and projective bundles. In the latter case one may choose the bundle $\mathcal{L}$ to be a relative $\mathcal{O}(1)$-sheaf so that $X = \mathbf{P}(\mathcal{F})$. A similar approach can be applied to study equidimensional quadric bundles: again, choosing $\mathcal{L}$ as the relative $\mathcal{O}(1)$, one produces a projective bundle $\mathbf{P}(\mathcal{F})$ in which X embedds as a divisor of a relative degree two. Note that, in all the above examples, if X and Y are smooth then the map φ is flat and the resulting sheaf $\mathcal{F}$ is locally free. In the present paper we want to extend the method also to non-flat maps. In particular, we will consider varieties which arise as projectivizations of coherent sheaves.

Our motivation for this study was originally two-fold: firstly we wanted to understand the class of varieties called by Sommese (smooth) scrolls — they occur naturally in his adjunction theory — and secondly we wanted to complete a classification of Fano manifolds of index r, dimension $2r$ and $b_2 \geq 2$ — the task which was undertaken by the second named author of the present paper. As our understanding of the subject develloped we have realised that many other points and applications of the theory of projective fibrations are also very interesting and deserve proper attention. However, for the sake of clarity of the paper we refrained from dealing with most of the possible extensions of the theory. Therefore, in the present paper we will deal mostly with coherent sheaves whose projectivizations are smooth varieties. This class of sheaves is related to the class of *smooth sheaves* which were studied by Constantin Bănică in one of his late papers. Thus we decided to name the class of the sheaves studied in the present paper after Bănică to commemorate his name.

The paper is organised as follows: in the first two sections we introduce some pertinent definitions and constructions and subsequently we examine their basic properties. In particular we prove that *Bănică sheaves* of rank $\geq n$ (where n is the dimension of the base) are locally free, and subsequently we discuss a version of a cojecture of Beltrametti and Sommese on smooth scrolls. In Section 3 we gathered a number of examples which illustrate some aspects of the theory. From section 4 on we deal with *Bănică sheaves* of rank $n-1$: first we discuss when they can be extended to locally free sheaves and examine numerical properties of extensions. In the remaining two sections we apply this to study ampleness of the divisor adjoint to a *Bănică sheaf* and then to classify Fano manifolds of large index which are projectivizations of non-locally free sheaves.

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Notation and assumptions. We adopt standard notation, see Hartshorne's texbook [H1]. We will frequently identify divisors and line bundles on smooth varieties. We assume that all varieties are defined over complex numbers, though the definitions and some results are also valid for varieties over an algebraically closed field.

1. Projectivization.

First, let us recall the definition of a projectivization of a coherent sheaf $\mathcal{E}$ over a scheme V, see [G] and [H1] for details.

(1.0). We start with a local description. Let A be a noetherian ring and M a finitely generated A-module. We will also usually assume that the ring A is an integrally closed domain, though it is not needed for the definitions. Let B denote the symmetric algebra of M

$$B := Sym(M) = \bigoplus_{m \geq 0} S^m(M)$$

where $S^m M$ is the m-th symmetric product of the module M. The A-algebra B has a natural gradation $B_m = S^m(M)$ and we define $\mathbf{P}_A(M) := Proj(B)$. Such defined projective scheme is a generalisation of the projective space over A. The scheme $\mathbf{P}(M)$ has a natural affine covering defined by elements of M: for a non-zero $f \in M$ consider

$$\mathcal{D}_+(f) = \{q \in Proj(B) : f \notin q\},$$

the scheme $\mathcal{D}_+(f)$ is then isomorphic to an affine scheme $Spec(B_{(f)})$, where $B_{(f)}$ denotes the zero-graded part of the localisation B_f of B with respect to the element f. The embedding $A = S^0 M \subset Sym(M)$ yields a projection map

$$p : \mathbf{P}(M) \to SpecA.$$

Graded modules over B give rise to coherent sheaves over $\mathbf{P}(M)$. In particular, on $\mathbf{P}(M)$ there are invertible sheaves $\mathcal{O}(k)$ associated to graded B-modules $B(k)$, with $B(k)_m = B_{k+m} = S^{m+k}(M)$, where the sub-index denotes the gradation shifted by k with respect to the gradation of B. Note that sections of the sheaf $\mathcal{O}(1)$ are isomorphic to the module M.

The above local definition of $\mathbf{P}$ allows us to define projectivization for any coherent sheaf $\mathcal{E}$: If $Sym\mathcal{E} := \bigoplus_{m \geq 0} S^m \mathcal{E}$ is the symmetric algebra of sections of coherent sheaf $\mathcal{E}$ over a normal variety V then we define

$$\mathbf{P}(\mathcal{E}) := Proj_V(Sym\mathcal{E}).$$

The inclusion $\mathcal{O}_V \cong S^0\mathcal{E} \to Sym\mathcal{E}$ yields the projection morphism $p : \mathbf{P}(\mathcal{E}) \to V$. We will always assume that the morphism p is surjective, or equivalently, that the support of $\mathcal{E}$ coincides with V. The local definition of $\mathcal{O}(1)$ gives rise to a globally defined invertible sheaf and thus over $\mathbf{P}(\mathcal{E})$ there exisits an invertible sheaf $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ such that $p_*\mathcal{O}(1) = \mathcal{E}$.

In the present section we want to understand some basic properties of this construction. The first one is about irreducibility.

Lemma 1.1. *If $\mathbf{P}(\mathcal{E})$ is an integral scheme then $\mathcal{E}$ is torsion-free.*

Proof. The assertion is local. Note that $\mathcal{O}(1)$ is locally free of rank 1 on an integral scheme and therefore it has no torsions. Consequently $\mathcal{E}$, being locally the space of sections of $\mathcal{O}(1)$, is torsion-free.

The converse of the above lemma is not true, see the example (3.2).

Therefore, from now on we will assume that all the sheaves whose projectivizations we will consider are torsion free.

Lemma 1.2. (cf. [H2, 1.7]) *Let $\mathcal{E}$ be a torsion-free sheaf over a normal variety Y and let $p : \mathbf{P}(\mathcal{E}) \to Y$ be the projectivization of $\mathcal{E}$. Assume that $\mathbf{P}(\mathcal{E})$ is a normal variety and no Weil divisor in $\mathbf{P}(\mathcal{E})$ is contracted to a subvariety of Y of codimension ≥ 2. Then the sheaf $\mathcal{E}$ is reflexive.*

Proof. We claim that the sheaf $\mathcal{E}$ is normal (in the sense of [OSS, II,1] or [H2]). This is because any section of $\mathcal{E}$ over open subset $U \setminus D$ of Y, where D is of codimension ≥ 2, is associated to a section of $\mathcal{O}(1)$ over $p^{-1}(U \setminus D)$. This, however, extends uniquely over $p^{-1}(U)$ because $\mathbf{P}(\mathcal{E})$ is normal and $p^{-1}(D)$ is of codimension ≥ 2 (c.f. [H2, 1.6]).

The above argument works for any projective surjection $\varphi : X \to Y$ of normal varieties. If φ contracts no Weil divisor on X to a codimension ≥ 2 subset of Y then a push-forward $\varphi_*\mathcal{F}$ of any reflexive sheaf $\mathcal{F}$ on X is reflexive on Y, see [H2, 1.7]. This is used in the following

Lemma 1.3. *Let $\mathcal{E}$ be a reflexive sheaf over a normal variety Y satisfying the assumptions of the previous lemma. Then*

$$p_*(\mathcal{H}om(\Omega_{\mathbf{P}(\mathcal{E})/Y}, \mathcal{O}(-1))) \cong \mathcal{H}om(\mathcal{E}, \mathcal{O}_Y).$$

Proof. Note that the isomorphism is true if $\mathcal{E}$ is locally free (one can use relative Euler sequence to prove it). The sheaf $\mathcal{H}om(\Omega_{\mathbf{P}(\mathcal{E})/Y}, \mathcal{O}(-1))$ is reflexive as a dual on a normal variety. Then, similarly as above we prove that its push-forward is reflexive as well. Thus we have isomorphism of the two reflexive sheaves defined outside of a codimension 2 subset of Y. Therefore the sheaves are isomorphic.

We will need the following

Lemma 1.4. *Let (A, m) be a regular local ring which is an algebra over its residue field $k = A/m$. Assume that M is an A-module which is not free and which comes from an exact sequence*

$$0 \longrightarrow A \xrightarrow{s} A^{r+1} \longrightarrow M \longrightarrow 0.$$

Let us write $s(1) = (s_0, \ldots, s_r)$ where $s_i \in m \subset A$. Then $\mathbf{P}_A(M)$ is regular if and only if the classes of elements $s_0, \ldots, s_r$ are k-linearly independent in m/m^2.

Proof. The ideal of $\mathbf{P}(M)$ in $\mathbf{P}^r_A = Proj(A[t_0, \ldots, t_r])$ is generated by an element $\sum s_i t_i$. Therefore, in an affine subset $U_0 = SpecA[t'_1, \ldots, t'_r]$ (where $t'_i = t_i/t_0$) its equation is $s_0 + \sum s_i t'_i = 0$. Thus, $\mathbf{P}(M)$ is smooth at $t'_1 = \ldots = t'_r = 0$ if and only if s_0 is non-zero in m/m^2. The above argument can be repeated for any k-linear transformation of coordinates in m/m^2 which proves that $s_0, \ldots, s_r$ are linearly independent in m/m^2.

2. Bănică sheaves, first properties.

In one of his last papers [B], Constantin Bănică considered a special class of reflexive sheaves.

Definition 2.0. *A reflexive sheaf $\mathcal{E}$ of rank r over a smooth variety V is called smooth if $\mathcal{E}xt^q(\mathcal{E}, \mathcal{O}) = 0$ for $q \geq 2$ and $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}) = \mathcal{O}_v/(t_1, \ldots, t_{r+1})$ for some choice $(t_1, \ldots, t_n)$ of regular parameter system at a point v of singularity of $\mathcal{E}$.*

Smooth sheaves are convenient for studying subvarieties of smooth varieties, see also [H2], [BC] and [HH].

In the present paper we will deal with another special class of coherent sheaves over normal varieties. As it will be seen this class is a generalisation of the one studied by Bănică and therefore we name these sheaves after him.

Definition 2.1. *A coherent sheaf $\mathcal{E}$ of rank $r \geq 2$ over a normal variety Y is called Bănică sheaf if its projectivization is a smooth variety.*

The assumption on smoothness of the projectivization is very strong as the following lemma shows.

Lemma 2.2. *If $\mathcal{E}$ is a Bănică sheaf then it is reflexive and moreover the map $p : \mathbf{P}(\mathcal{E}) \to Y$ is an elementary, or extremal ray contraction. Furthermore $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is p-ample and generates $PicX$ over $PicY$ so that we have a sequence*

$$0 \longrightarrow PicY \longrightarrow Pic\mathbf{P}(\mathcal{E}) \longrightarrow \mathbf{Z}[\mathcal{O}(1)] \longrightarrow 0.$$

Moreover every Weil divisor on Y is Cartier and, in particular, Y is Gorenstein.

Proof. First, note that since $\mathbf{P}(\mathcal{E})$ is irreducible, $\mathcal{E}$ is torsion-free (1.1). To prove that p is an elementary contraction note that every fiber of p over a point $y \in Y$ is a projective space $\mathbf{P}(\mathcal{E}_y \otimes k(y))$ (where $k(Y)$ denotes the residue field). Taking a line in a generic fiber and deforming it, we obtain a non-trivial curve in a special fiber, too (actually a line), therefore all curves contracted are numerically proportional, hence p is an extremal ray contraction in the sense of Mori theory and consequently: $\mathcal{O}(1)$ is p-ample. Moreover there is an exact sequence

$$0 \longrightarrow PicV \longrightarrow Pic\mathbf{P}(\mathcal{E}) \longrightarrow \mathbf{Z}[\mathcal{O}(1)] \longrightarrow 0.$$

The sequence is exact even at the last place because $\mathcal{O}(1)$ has intersection 1 with a line in the fiber. If a prime Weil divisor in $\mathbf{P}(\mathcal{E})$ is contracted to a proper subset of Y then it has trivial intersection with curves contracted by p and thus it is a pull-back of a Cartier divisor from Y. Now the reflexivity of $\mathcal{E}$ follows because of lemma (1.2). The last assertion of the lemma follows similarly: the inverse image of a Weil divisor from Y is Cartier on X and has intersection 0 with curves contracted by p and thus it is a pull-back of a Cartier divisor from Y.

(2.3). One motivation to study Bănică sheaves comes from *smooth scrolls* which are defined by Sommese as follows: A pair $(X, \mathcal{L})$ consisting of a smooth variety X and an ample line bundle $\mathcal{L}$ is called a *scroll* if there exists a morphism $p : X \to Y$ onto a normal variety Y of smaller dimension such that $K_X \otimes \mathcal{L}^{\otimes(dimX - dimY+1)}$ is a pull-back of an ample line bundle from Y.

A smooth scroll is over a general point a projective bundle, this follows from Kodaira vanishing and Kobayashi—Ochiai characterisation of the projective space. Obviously, projective bundles and, more generally, projectizations of coherent sheaves are examples of smooth scrolls. Conversly, if all fibers of the map p are of the same dimension then the scroll is a projective bundle, [F1, 2.12] and [I]. We have also examples of scrolls which do not belong to any of these two classes; their fibers may be Grassman varieties of large dimension with respect to the dimension of a general fiber, see the example (3.2). If we assume that the smooth scroll is a projectivization of a sheaf, the dimension of special fibers can not jump so much:

Lemma 2.4. *Let $\mathbf{P}(\mathcal{E}) \to Y$ be a projectivization of a rank-r Bănică sheaf. Let F be a fiber of dimension $> r-1$. Then $dimF \leq dimY$.*

Proof. Let $\Pi \cong \mathbf{P}^{r-1} \subset F \cong \mathbf{P}^k$ be a specialization of a general fiber. We have then a sequence of normal bundles

$$0 \longrightarrow N_{\Pi/F} \cong \mathcal{O}(1)^{k-r+1} \longrightarrow N_{\Pi/\mathbf{P}(\mathcal{E})} \longrightarrow N_{(F/\mathbf{P}(\mathcal{E}))|\Pi} \longrightarrow 0.$$

Since $N_{\Pi/\mathbf{P}(\mathcal{E})}$ is a specialization of a trivial bundle it has a trivial total Chern class, therefore

$$c_t(N_{(F/\mathbf{P}(\mathcal{E}))|\Pi}) = \big(c_t(\mathcal{O}(1))\big)^{r-k-1}.$$

Consequently,

$$n + r - 1 - k = rank(N_{F/\mathbf{P}(\mathcal{E})}) \geq dim\Pi = r - 1$$

and the inequality follows.

On the other hand, if we assume that the jump of the dimension of fibers in a scroll is small then we can apply Theorem 4.1 from [AW] to get the following

Proposition 2.5. *Let $(X, \mathcal{L})$ be a smooth scroll. Assume that for any fiber F of the map $p : X \to Y$ it holds $dimF \leq dimX - dimY$. Then Y is smooth and $X = \mathbf{P}(p_*\mathcal{L})$ (so that $p_*\mathcal{L}$ is a Bănică sheaf. Moreover, if $dimX \geq 2dimY$ then p is a projective bundle.*

For smooth scrolls which are projectivization of sheaves there holds a conjecture of Beltrametti and Sommese; namely we have

Theorem 2.6. *Let $\mathcal{E}$ be a Bănică sheaf of rank r over a normal variety Y. If $r \geq dimY$ then Y is smooth and $\mathcal{E}$ is locally free. If $r = dimY - 1$ then Y is smooth.*

Proof. The first part follows immediately from Lemma 2.4 and Fujita's result [F, 2.12]. Then, the second part follows then from 2.5.

Using the above Proposition 2.5 and Remark 4.12 from [AW] we get the following

Lemma 2.7. *Let $\mathcal{E}$ be a Bănică sheaf of rank r over a normal variety Y. If $r \geq dimY - 1$ or, if for any point $y \in Y$, $dim_k \mathcal{E}_y \otimes k(y) \leq r + 1$, then Y is smooth and locally $\mathcal{E}$ is a quotient of a trivial sheaf by a rank-1 subsheaf, that is, we have a sequence*

$$0 \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{Y,y}^{r+1} \longrightarrow \mathcal{E}_y \longrightarrow 0.$$

If we now combine lemmata (1.4) and (2.7) we get the following

Corollary 2.8. *Any smooth sheaf (in the sense of Bănică) is a Bănică sheaf. If a Bănică sheaf $\mathcal{E}$ over a normal variety Y satifies the condition*

$$\text{for any } y \in Y : dim_k \mathcal{E}_y \otimes k(y) \leq rank\mathcal{E} + 1$$

then it is smooth.

3. Examples.

The simplest examples of scrolls are projective bundles. In particular, if the base Y is smooth then any locally free sheaf is a *Bănică sheaf*. Also, if a locally free sheaf $\mathcal{F}$ over a smooth Y is spanned by global sections then a general section s of $\mathcal{F}$ will yield a *Bănică sheaf* as a quotient:

$$0 \longrightarrow \mathcal{O} \xrightarrow{s} \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0.$$

The singular set of $\mathcal{E}$ coincides with the zero locus of the section s. The local condition on the sheaf $\mathcal{E}$ to be *Bănică sheaf* is described in Lemma (1.4).

More generally, we can consider arbitrary morphisms of vector bundles over smooth base

Lemma 3.1. (cf. [B, Thm. 2]) *Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves over a smooth variety Y of rank f and g, respectively. Assume that $f \geq g + 2$ and the sheaf $\mathcal{H}om(\mathcal{G}, \mathcal{F})$ is spanned by global sections. Then, for a generic $\sigma \in Hom(\mathcal{G}, \mathcal{F})$ we have an exact sequece*

$$0 \longrightarrow \mathcal{G} \xrightarrow{\sigma} \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

with the quotient $\mathcal{E}$ being Bănică sheaf of rank $f - g$.

Proof. We have a natural isomorphism (see [H1, II.5])

$$Hom_{\mathbf{P}(\mathcal{F})}(p^*\mathcal{G}, \mathcal{O}(1)) \cong Hom_Y(\mathcal{G}, \mathcal{F}).$$

The zero locus of a section

$$\sigma \in Hom(p^*\mathcal{G}, \mathcal{O}(1)) = H^0(\mathbf{P}(\mathcal{F}), p^*\mathcal{G}^*(1))$$

coincides with the projectivization of the cokernel $\mathcal{E}$ of the map $\sigma : \mathcal{G} \to \mathcal{F}$ embedded into $\mathbf{P}(\mathcal{F})$ by the map associated to the epimorphism $\mathcal{F} \to \mathcal{E}$. Therefore the lemma follows from Bertini theorem.

Not all scrolls arise as the projectivizations of sheaves.

Example 3.2. Consider the Grassmann variety $G(2,n)$ of linear planes of a given linear space W of dimension n. Over $G(2,n)$ we have the universal quotient bundle $\mathcal{Q}$ whose projectivization is a flag variety

$$F(1,2,n) = \{(x,l) \in \mathbf{P}^{n-1} \times G(2,n) : x \in l\}$$

with a projection onto $\mathbf{P}^{n-1}$. The projection has a natural structure of $\mathbf{P}^{n-2}$-bundle. Now take a bundle $\mathcal{Q} \oplus \mathcal{O}$, its projectivization $q : \mathbf{P}(\mathcal{Q} \oplus \mathcal{O}) \to G(2,n)$ maps onto $\mathbf{P}^n$, $p : \mathbf{P}(\mathcal{Q} \oplus \mathcal{O}) \to \mathbf{P}^n$ so that all fibers but one are isomorphic to $\mathbf{P}^{n-2}$. The exceptional fiber (call it F_0) is associated to the $\mathcal{O}$-factor of the bundle q and it is isomorphic to $G(2,n)$, so that it is of dimension $2(n-2)$. One checks easily that the variety has a structure of a smooth scroll, however it is not a projectivization of a sheaf as the special fiber is not a projective space (for $n \geq 4$). Let us consider the sheaf $\mathcal{F} := p_* q^* \mathcal{O}_{G(2,n)}(1)$ where $\mathcal{O}_{G(2,n)}(1)$ is the positive generator of $PicG(2,n)$. The sheaf $\mathcal{F}$ is locally free outside one point where it has a fiber isomorphic to $\Lambda^2 W$. It is not hard to check that it is reflexive though its projectivization is a reducible variety consisting of two components: the dominant one which is the original scroll and the special fiber $\mathbf{P}(\Lambda^2 W)$, the fiber F_0 embedded in $\mathbf{P}(\Lambda^2 W)$ via Plücker embedding.

If we allow the projectivization have some singularities, even mild ones, some of the statements from the previous section are not true (e.g. 2.4).

Example 3.3. (Sommese [S, 3.3.3]) Take a smooth surface S and blow it up $\beta : S' \to S$ at a point $s \in S$. Let E denote the exceptional divisor. Let L be a pull-back to S' of an ample line bundle from S. We may assume (possibly replacing L by its power) that $L \otimes \mathcal{O}(-E)$ is ample and spanned on S'. Over S' we consider a projective bundle $p' : \mathbf{P}(L \oplus (L \otimes \mathcal{O}(-E))) \to S'$. The $\mathcal{O}(1)$-sheaf on the projectivization is clearly nef and ample outside the inverse image of E. The unique curve with which the $\mathcal{O}(1)$-sheaf has trivial intersection is the section of the projective bundle over E (a smooth rational curve) associated to the splitting $\mathcal{O} \oplus \mathcal{O}(1) \to \mathcal{O}$. The smooth rational curve is easily seen to have normal bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ and it can be contracted to an isolated singular point by the morphism coming from the evaluation of $\mathcal{O}(1)$ (since the bundle $L \oplus (L \otimes \mathcal{O}(-E))$ is spanned). The singularity is Gorenstein since the canonical bundle on the projectivization has intersection 0 with the contracted curve. By V let us call the resulting 3-fold obtained by contracting the section to a point. The 3-fold V maps onto S and the map makes V a scroll. There exists a unique exceptional fiber of the scroll which is isomorphic to $\mathbf{P}^2$ and which contains the singular point. On the other hand, the threefold V can be described as a projectivization of a sheaf $\mathcal{E} := (\beta \circ p')_* \mathcal{O}(1)$, and it is not hard to see that the singularity of $\mathcal{E}$ at the point s is of the type $\mathcal{O} \oplus J_s$, where J_s is the ideal of the point s.

In the present paper we will also deal with Fano manifolds arising as projectivization of sheaves. We have:

Lemma 3.4. *Let $\mathcal{E}$ be a Bănică sheaf over a normal variety Y. Assume that a singular set of $\mathcal{E}$ is of dimension ≤ 1 or $\rho(Y) = 1$. If $\mathbf{P}(\mathcal{E})$ is a Fano manifold then $-K_Y$ is ample, that is, Y is a Gorenstein Fano variety.*

Proof. The argument is similar as in the proof of [W, 4.3], compare also with [SW, 1.6] and [KMM]. We are only to prove that

$$p^*(-K_Y) = \left(-K_{\mathbf{P}(\mathcal{E})}\right) + \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-rank\mathcal{E}) + p^*(-det\mathcal{E})$$

has positive intersection with any extremal rational curve C in $\mathbf{P}(\mathcal{E})$ not contracted by p. We claim that the curve C may be chosen so that $p(C)$ is not contained in the singular locus of $\mathcal{E}$. Indeed, if it were, then the whole locus of the ray $\mathbf{R}^+[C]$ would be contracted by p to a set of dimension 1, thus all fibers of the contraction of the ray would be of dimension 1, hence the locus would be a divisor, [W1, 1.1]. This, however, contradicts the fact that p contracts no Weil divisors to set of codimension ≥ 2. Once the curve C is assumed not to be in the singular locus of the map p we conclude as in [SW, 1.6], or as in [KMM].

Example 3.5. The assumption on the singular set of $\mathcal{E}$ is indespencible. Let

$$Y := \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(1)^{\oplus 3} \oplus \mathcal{O}_{\mathbf{P}^2}).$$

Then $-K_Y = 4\eta$ where η denotes the relative $\mathcal{O}(1)$ of the projectivization over $\mathbf{P}^2$. The line bundle η is spanned but not ample, so Y is not Fano. The morphism associated to $|\eta|$ contracts to a point the unique section of $Y \to \mathbf{P}^2$ associated to the $\mathcal{O}$-factor, call this set Z. Let H be the pullback of the hyperplane from $\mathbf{P}^2$ to Y. The line bundle asociated to $\eta - H$ is spanned off Z by three sections. Thus we have a morphism $\mathcal{O}_Y^{\oplus 3} \to \mathcal{O}_Y(\eta - H)$ which yields a sequence

$$0 \longrightarrow \mathcal{O}_Y(-\eta + H) \longrightarrow \mathcal{O}_Y^{\oplus 3} \longrightarrow \mathcal{E} \longrightarrow 0$$

with a rank-2 sheaf $\mathcal{E}$ which is free outside Z. The variety $\mathbf{P}(\mathcal{E})$ is the coincidence variety of divisors from the linear system $|\eta - H|$ and each one of the divisors is isomorphic to $\mathbf{P}(\mathcal{O}_{\mathbf{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^2}(-1))$. Therefore $\mathbf{P}(\mathcal{E})$ is smooth. Moreover

$$-K_{\mathbf{P}(\mathcal{E})} = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(2) \otimes p^*(\mathcal{O}_Y(3\eta + H)).$$

Therefore $\mathbf{P}(\mathcal{E})$ is a smooth Fano variety.

4. Extensions to locally free sheaves, nefness

We want to find conditions to realise globally the projectivization of a *Bănică sheaf* as a divisor in a projective bundle. For simplicity we introduce the following definition.

Definition 4.0. *We say that a coherent sheaf $\mathcal{E}$ over a normal variety Y extends to a locally free sheaf $\mathcal{F}$ if there exists a sequence of $\mathcal{O}_Y$-modules*

$$0 \longrightarrow \mathcal{O} \xrightarrow{s} \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0.$$

In other words, $\mathcal{E}$ is obtained by dividing $\mathcal{F}$ by a non-zero section s. The singular locus of $\mathcal{E}$ coincides with the zero locus of s. Alternatively, $\mathbf{P}(\mathcal{E})$ is a divisor in $\mathbf{P}(\mathcal{F})$ from the linear system $|\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)|$.

In the present section we will also discuss numerical properties of coherent sheaves. Let us recall that a sheaf $\mathcal{E}$ is ample (resp. nef) if $\mathcal{O}(1)$ is ample (resp. nef) on $\mathbf{P}(\mathcal{E})$; this makes sense also if we multiply $\mathcal{E}$ by a $\mathbf{Q}$-divisor.

For a coherent sheaf $\mathcal{E}$ by $\mathcal{E}^*$ we will denote its dual sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{O})$.

Lemma 4.1. *Let $\mathcal{E}$ be a Bănică sheaf of rank $n-1$ over a smooth projective variety Y of dimension n. If $H^2(Y, \mathcal{E}^*) = 0$ then $\mathcal{E}$ extends to a localy free sheaf; in particular $\mathcal{E} \otimes \mathcal{L}^{-m}$ extends for $\mathcal{L}$ an ample line bundle and $m \gg 0$.*

Proof. Because of (2.7) we know that the extension exists locally. To prove the existence of a global extension consider the spectral sequence relating local $\mathcal{E}xt$ and global Ext. Then we have the following exact sequence

(4.1.1) $\quad H^1(Y, \mathcal{H}om(\mathcal{E}, \mathcal{O})) \to Ext^1_Y(\mathcal{E}, \mathcal{O}) \to H^0(Y, \mathcal{E}xt^1(\mathcal{E}, \mathcal{O})) \to H^2(Y, \mathcal{H}om(\mathcal{E}, \mathcal{O})).$

The support of $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O})$ consists of isolated points of singularity of $\mathcal{E}$. For any such point y, $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O})_y \cong \mathcal{O}_y$ and the unit represents the extension to a free module. Thus the vanishing of $H^2(Y, \mathcal{E}^*)$ yields the existence of an extension in $Ext^1(\mathcal{E}, \mathcal{O})$ to a locally free sheaf.

Therefore, frequently we will be interested in the vanishing of the latter term in the sequence (4.1.1). To this end we have.

Lemma 4.2. *Let $\mathcal{E}$ be a Bănică sheaf of rank $n-1$ over a smooth projective variety Y of dimension n. Assume that $\mathcal{L}$ is an ample line bundle on Y and let $H \in |\mathcal{L}|$ be a smooth divisor which does not meet the singular set of $\mathcal{E}$. If, for $k \geq 1$ and $i = 1, 2$ $H^i(H, (\mathcal{E}^* \otimes \mathcal{L}^k)_{|H}) = 0$ then $\mathcal{E}$ extends.*

Proof. The vanishing of $H^2(Y, \mathcal{E}^*)$ follows from the vanishing of cohomology on H which, beause of the divisorial sequence for H, implies that

$$H^2(Y, \mathcal{E}^* \otimes \mathcal{L}^k) = H^2(Y, \mathcal{E}^* \otimes \mathcal{L}^{k+1})$$

for $k \geq 0$.

On the other hand, the non-vanishing of $H^2(Y, \mathcal{E}^* \otimes \mathcal{L}^k)$ for $k \ll 0$ can be used to estimate $c_n(\mathcal{E})$, that is, the number of singular points of $\mathcal{E}$. The following lemma was suggested to us by Adrian Langer whom we owe our thanks for finding a mistake in a previous version of this paper.

Lemma 4.2.1. *Let $\mathcal{E}$ be a Bănică sheaf of rank $n-1$ over a smooth projective variety Y of dimension n and let $\mathcal{L}$ be an ample line bundle over Y. Then, for $k \gg 0$, we have $H^2(Y, \mathcal{E}^* \otimes \mathcal{L}^{-k}) = c_n(\mathcal{E})$.*

Proof. We have a global duality [H1, III.7.6]:

$$Ext^i_Y(\mathcal{E} \otimes \mathcal{L}^k, \mathcal{O}_Y) \cong H^{n-i}(Y, \mathcal{E} \otimes \mathcal{L}^k \otimes K_Y)^*$$

and the latter term vanishes for $k \gg 0$ and $i < n$. Therefore the spectral sequence relating Ext and $\mathcal{E}xt$ converges to a trivial one. This yields that $H^0(Y, \mathcal{E}xt^1(\mathcal{E} \otimes \mathcal{L}^k, \mathcal{O}_Y)) = H^2(Y, \mathcal{E}^* \otimes \mathcal{L}^{-k})$ (c.f. 4.1.1) and we are done.

Making similar argument as is the proof of 4.2 we get the following

Corollary 4.2.2. *Let $\mathcal{E}$ be a Bănică sheaf of rank $n-1$ over a smooth projective variety Y of dimension n. Assume that $\mathcal{L}$ is an ample line bundle on Y and let $H \in |\mathcal{L}|$ be a smooth divisor which does not meet the singular set of $\mathcal{E}$. If, for any $k \in \mathbf{Z}$ and $i = 1, 2$ the groups $H^i(H, (\mathcal{E}^* \otimes \mathcal{L}^k)_{|H})$ vanish then $\mathcal{E}$ is locally free.*

We will need also the following version of the lemmata 4.1 and 4.2 for arbitrary sheaves with isolated singularities.

Lemma 4.3. *Let $\mathcal{E}$ be coherent sheaf with isolated singularities over a smooth variety Y, $dim Y \geq 3$. Let $\mathcal{L}$ be an ample line bundle over Y and let $H \in |\mathcal{L}|$ be a smooth divisor which does not meet the singular points of $\mathcal{E}$. Then:*
(a) if $\mathcal{E}$ extends to a locally free sheaf then $\mathcal{E} \otimes \mathcal{L}^{-1}$ extends as well,
(b) if $\mathcal{E} \otimes \mathcal{L}^{-1}$ extends to a locally free sheaf and $H^2(Y, \mathcal{E}^*) = 0$ then also $\mathcal{E}$ extends to a locally free sheaf.

Proof. Consider a divisorial sequence associated to $H \in |\mathcal{L}|$:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_H \longrightarrow 0.$$

The morphism $\mathcal{O} \to \mathcal{L}$ from this sequence yields a commutative diagram with exact rows and columns coming from multiplying by a section defining H:

$$\begin{array}{ccccc} Ext^1_Y(\mathcal{E}, \mathcal{O}) & \to & H^0(Y, \mathcal{E}xt^1(\mathcal{E}, \mathcal{O})) & \to & H^2(Y, \mathcal{E}^*) \\ \downarrow & & \downarrow & & \downarrow \\ Ext^1_Y(\mathcal{E}, \mathcal{L}) & \to & H^0(Y, \mathcal{E}xt^1(\mathcal{E}, \mathcal{L})) & \to & H^2(Y, \mathcal{E}^* \otimes \mathcal{L}) \end{array}$$

On the other hand we know that

$$\mathcal{E}xt^i(\mathcal{E}, \mathcal{L}) \cong \mathcal{E}xt^i(\mathcal{E}, \mathcal{O}) \otimes \mathcal{L}$$

so that, because the singularities of $\mathcal{E}$ are isolated and H does not meet them, the vertical map in the center is an isomorphism. Therefore an extension in $Ext^1(\mathcal{E}, \mathcal{O})$ which gives a locally free sheaf will be mapped by the left-hand-side vertical map to an extension in $Ext^1(\mathcal{E},\mathcal{L})$ which produces a locally free sheaf, too. This proves (i). To get (ii) we make a similar agument, but this time applying vanishing of $H^2(Y,\mathcal{E}^*)$ to lift a local exension to a global one.

Now we want to compare ampleness and nefness of a rank r *Bănică sheaf* $\mathcal{E}$ with the same properties of a locally free sheaf $\mathcal{F}$ in whose projectivization $\mathcal{E}$ is embedded. Therefore, let us assume that $\mathcal{E}$ extends to $\mathcal{F}$, that is, we have the sequence 4.0. Obviously, if $\mathcal{E}$ is nef then also $\mathcal{F}$ is nef. As for the ampleness we have the following.

Lemma 4.4. *Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on a smooth variety Y satisfying the above assumptions. Assume moreover that $c_1 Y - c_1\mathcal{F}$ is nef and that $\mathcal{E}$ is ample. Then $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ is semiample, that is $|\mathcal{O}_{\mathbf{P}(\mathcal{F})}(m)|$ is base point free for $m \gg 0$. The exceptional set E of the morphism given by $|\mathcal{O}_{\mathbf{P}(\mathcal{F})}(m)|$, $m \gg 0$, if non-empty, contains all sections $Y \supset G \to \mathbf{P}(\mathcal{F}_G)$ associated to a splitting*

$$\mathcal{F}_{|G} \to \mathcal{O}_G \to 0$$

of the sequence (4.0) over any closed $G \subset Y$ of positive dimension. Moreover p maps E finite-to-one into $Y \setminus sing\mathcal{E}$.

Proof. The line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$ is nef and big. Since $K_{\mathbf{P}(\mathcal{F})} = \mathcal{O}(-r-1)\otimes p^*(K_Y + det\mathcal{F})$, it follows that $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(m)\otimes K^{-1}_{\mathbf{P}(\mathcal{F})}$ is nef and big for $m \gg 0$. Therefore, by the Kawamata-Shokurov contraction theorem $\mathcal{O}_{\mathbf{P}(\mathcal{F})}(m)$ is semiample. The morphism defined by $|\mathcal{O}_{\mathbf{P}(\mathcal{F})}(m)|$ is birational and its exceptional set does not meet $\mathbf{P}(\mathcal{E}) \subset \mathbf{P}(\mathcal{F})$ (because the divisor $\mathbf{P}(\mathcal{E}) \subset \mathbf{P}(\mathcal{F})$ has positive intersection with any curve meeting it). If the sequence (4.0) splits over a positive-dimensional set $G \subset Y$ then, clearly, the unique section of $\mathcal{F}$ over G is contained in E. And clearly $p(E)\cap sing(\mathcal{E}) = p(E\cap \mathbf{P}(\mathcal{E})) = \emptyset$.

Corollary 4.5. *In the above situation, if $G \subset Y$ is not contained in $p(E)$ then*

$$Ext^1_G(\mathcal{E}_G, \mathcal{O}_G) \neq 0.$$

We will also need the following.

Lemma 4.6. *Let $\mathcal{E}$ be an ample reflexive sheaf on a normal variety Y. Assume that some twist of $\mathcal{E}$ extends to a locally free sheaf so that we have a sequence*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

with $\mathcal{F}$ locally free and $\mathcal{L}$ a line bundle. Let $C \subset Y$ be a rational curve which is not contained in the singular locus of $\mathcal{E}$. Then

$$C.det\mathcal{E} \geq rank\mathcal{E} + \text{ number of singular points of } \mathcal{E} \text{ on } C.$$

Proof. First, let us note that, in the above situation,

$$C.det\mathcal{E} = (\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))^r.p^{-1}(C)$$

where r is the rank of $\mathcal{E}$ and $p : \mathbf{P}(\mathcal{E}) \to Y$ is the projection. Indeed, the formula is correct for projective bundles and is preserved for divisors in them which meet the cycle $p^{-1}(C)$ at the expected dimension. The cycle $p^{-1}(C)$ consists of "vertical" components over singular points (each being a projective space) and of the dominant component over C which is a projective bundle with a fibre $\mathbf{P}^{r-1}$. From the classification of bundles over $\mathbf{P}^1$ it follows that the latter component brings to the intersection at least r and therefore the inequality follows.

5. Adjunction.

In the present section we compare the determinant, or the first Chern class of a *Bănică sheaf* with the canonical sheaf of the variety over which the sheaf is defined. In case of locally free sheaves the question was considered in [YZ], [F2] and [ABW].

Theorem 5.1. *Let $\mathcal{E}$ be a Bănică sheaf of rank r over a smooth variety Y of dimension $n = r + 1 \geq 3$. Assume that $\mathcal{E}$ is ample and moreover that it is not locally free. Then*
(1) $K_Y + c_1\mathcal{E}$ is nef unless $Y \cong \mathbf{P}^n$ and $\mathcal{E}$ is a quotient of a decomposable sheaf:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus n} \longrightarrow \mathcal{E} \longrightarrow 0.$$

(2) if $K_Y + c_1\mathcal{E}$ is nef then it is also big unless
(2.1) Y is Fano and $K_Y + c_1\mathcal{E} = 0$, or
(2.2) Y has a structure of a projective bundle $\pi : Y \to B$ over a smooth curve B and $\mathcal{E}$ fits into a sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \pi^*\mathcal{G} \otimes \mathcal{O}_Y(1) \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{G}$ is a rank-n vector bundle over B and $\mathcal{O}_Y(1)$ a line bundle whose restriction to any fiber of π is $\mathcal{O}(1)$;
(3) if $K_Y + c_1\mathcal{E}$ is nef and big then it is also ample unless there exists a birational map $\pi : Y \to Y'$ supported by $K_Y + c_1\mathcal{E}$ onto a smooth variety Y' which blows-down disjoint exceptional divisors $E_i \cong \mathbf{P}^{n-1}$, such that $E_i \cap sing\mathcal{E} = \emptyset$. On Y' there exists an ample Bănică sheaf $\mathcal{E}'$ such that $\mathcal{E} \cong \pi^*\mathcal{E}' \otimes \mathcal{O}_Y(-\sum E_i)$ and $K_{Y'} + c_1\mathcal{E}'$ is ample.

Remark The case (2.1) of the theorem will be discussed thoroughly in the subsequent section. In particular, it will be shown that Y is either a projective space or a smooth quadric.

Proof of the theorem. If $K_Y + c_1\mathcal{E}$ is not nef, then according to the cone theorem of Mori, there exists an extremal ray of Y which has negative intersection with this divisor. The length of the ray is at least n so that its locus coincides with Y, see [I, 0.4] or [W1, 1.1]. Therefore there exists a rational curve from the ray meeting the singular locus of $\mathcal{E}$. Because of (4.6), these curves have intersection at least $n+1$ with $-K_Y$. Consequently, by an argument on deformation of curves passing through a point (see e.g. [W1]), $PicY = \mathbf{Z}$ and we compute easily that $K_Y = (n+1)(K_Y + det\mathcal{E})$ and therefore by a theorem of Kobayashi-Ochiai $Y \cong \mathbf{P}^n$. The restriction of $\mathcal{E}$ to a generic hyperplane $H \subset \mathbf{P}^n$ is an ample vector bundle and $c_1(\mathcal{E}_H) = n + 1$, therefore we see that $\mathcal{E}_H \cong T\mathbf{P}^{n-1}$ or

$\mathcal{E}_H \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(n-1)}$, the latter possiblity ruled out because of 4.2.2. In the former case, we use 4.2 to produce an extension of $\mathcal{E}$ to a locally free sheaf $\mathcal{F}$; the only possible non-trivial extension on H leads to a decomposable bundle $\mathcal{O}_H(1)^{\oplus n}$ so the bundle $\mathcal{F}$ is decomposable as well (see e.g. [OSS]).

For the remaining cases the argument is similar. Assume first, that $K_Y + c_1\mathcal{E}$ is nef but not ample. Therefore there exists a ray of Y having intersection 0 with the divisor. The length of the ray is at least $n-1$, [W1, 1.1]. If the contraction of the ray is birational then it is actually divisorial and the ray has to have length $n-1$. In this case, however, the exceptional locus can not meet the singular locus of $\mathcal{E}$ because then we would find out (4.6) that the length is actually n which contradicts [I, 0.4]. Consequently, $\mathcal{E}$ is a vector bundle in a neighbourhood of E and the arguments from [ABW, 2.4] apply to conclude the description of the blow-down morphism and the sheaf $\mathcal{E}$ as in the case (3) of the theorem.

If the contraction of the ray in question is of fiber type then a fiber containing a singular point of $\mathcal{E}$ has to be a divisor (again, since $-K_Y.C \geq n$ for any rational curve passing through the singular point). Thus the contraction is either to a point (which is the case of (2.1)) or onto a smooth curve B. In the latter case we consider fibers which do not contain singularities of $\mathcal{E}$ and as in [ABW, 2.2] we prove that the fibers are projective spaces. Similarly, we conclude that Y has a structure of a scroll $\mathbf{P}(\mathcal{G}) \to B$ over the curve and $\mathcal{E}$ restricted to a general fiber F of the contraction is isomorphic to $T\mathbf{P}^{n-1}$. To complete the description of $\mathcal{E}$ we choose a smooth divisor $H \subset X$ which is a hyperplane in each fiber of the scroll and which does not meet the singular set of $\mathcal{E}$; the restriction of $\mathcal{E}$ to any fiber of $H \to B$ is then $T\mathbf{P}^{n-2} \oplus \mathcal{O}(1)$. After twisting $\mathcal{E}$ by a pull-back of a negative line bundle from B it will satisfy asssumptions of (4.2) on H, so that it will extend to a vector bundle $\mathcal{F}$. The description of $\mathcal{F}$ follows now easily, since its restriction to a general fiber has to be isomorphic to $\mathcal{O}(1)^n$.

To conclude the theorem note that the loci of extremal rays can not meet (because we would have a curve contracted by both contractions) and therefore the description of the adjoint morphism is as in (2) of the theorem.

6. Fano manifolds of middle index.

In the present section we want to complete the classification of Fano manifolds of index r and dimension $2r$ with second Betti number $b_2 \geq 2$. Let us recall that a smooth projective variety X is called Fano if its anti-canonical divisor $-K_X$ is ample. The index of the Fano variety is equal to the largest integer r for which $-K_X \equiv rH$, for some ample divisor H. Such varieties with projective and quadric bundle structure were studied in [PSW2] and [W2], respectively. To complete their classification one has to deal with these which are non-equidimensional scrolls [W2, Theorem I].

(6.0). Our set-up is as follows: X is a Fano manifold of index r and dimension $2r \geq 6$, and it is a projectivization of a non-locally free *Bănică sheaf* $\mathcal{E}$ over a smooth variety Y of dimension $r+1$. The projection $X \to Y$ we will denote by p; we may choose $\mathcal{E} := p_*(\mathcal{O}(H))$, so that the line bundle associated to H is $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. The variety X admits also another non-trivial map (a contraction) with connected fibers $\varphi : X \to Z$ onto a normal variety Z. In [W2, Thm. I] it was proved that all fibers of φ are of dimension $\leq r$ and thus, because of [AW, Thm. 4.1], Z is smooth and one of the possibilities occurs:

(i) $dimZ = r+1$ and $\varphi : X \to Z$ is a projectivization of a non-locally free sheaf;
(ii) $dimZ = 2r$ and $\varphi : X \to Z$ is a blow-down of a smooth divisor E in X to a smooth subvariety $T \subset Z$, $dimT = r-1$;
(iii) $dimZ = r$ and $\varphi : X \to Z$ is a quadric bundle;
(iv) $dimZ = r+1$ and $\varphi : X \to Z$ is a projective bundle.

(6.1). Fano manifolds with projective bundles were studied in [PSW2]; from the classification obtained in that paper it follows that the last possibility *(iv)* can not occur. Quadric bundles were studied in [W2] and it had turned out that two of the quadric bundles obtained there have also a structure of projectivization of non-locally free sheaf:

(a) a divisor of bidegree $(1,1)$ in the product $\mathbf{P}^r \times \mathbf{Q}^{r+1}$,
(b) a divisor of bidegree $(1,2)$ in the product $\mathbf{P}^r \times \mathbf{P}^{r+1}$.

¿From now on we assume that we are either in case *(i)* or *(ii)*, which we will call fibre and divisorial case, respectively.

Our arguments are similar to those from [PSW2]: we will use "big fibers" of the map φ, that is fibers of dimension r. We know that they are isomorphic to $\mathbf{P}^r$ and the restriction of H to each of them is $\mathcal{O}(1)$, see [AW, 4.1]. First we will deal with the case when φ is divisorial.

Lemma 6.2. (cf. [PSW2 (7.2)]) *Assume that φ is dvisorial. Then the restriction of $\mathcal{O}(E)$ to a fiber of p is isomorphic to $\mathcal{O}(1)$.*

Proof. Assume the contrary. Let us take a general fiber F of p such that E restricted to the fiber is a hypersurface of degree > 1. Let us take a line in F which is not contained in $F \cap E$; choose two points $x_1 \neq x_2$ such that $x_1,\ x_2 \in F \cap E$. Let $G_i := \varphi^{-1}(\varphi(x_i))$. We claim that there exists a curve $C \subset Y$, $p(F) \in C$, such that:

(*) for a general $c \in C$: $\#(p^{-1}(c) \cap (G_1 \cup G_2)) \geq 2$

Indeed, note first that $dim(G_i) = r$ and p maps G_i onto a divisor in Y. Therefore, if $\varphi(x_1) \neq \varphi(x_2)$ we take a curve in $p(G_1) \cap p(G_2) \ni p(F)$. If $G_1 = G_2$ we consider a curve in a set $\{y : \#(p^{-1}(y) \cap G_1) \geq 2\} \ni p(F)$ which again is of positive dimension.

Now over a generic $c \in C$ we choose a line L_c in $p^{-1}(c)$ such that L_c is not contained in E and L_c meets $G_1 \cup G_2$ at at least two points. This way we can construct a ruled surface over the normalisation of C which is mapped via φ to a two-dimensional variety and which contains a curve (or curves) contracted to point (or points) such that it contradicts the following:

Sublemma. *Let $\pi : S = \mathbf{P}(\mathcal{E}) \to C$ be a (geometrically) ruled surface (a $\mathbf{P}^1$-bundle) over a smooth curve C. Assume that there exists a map $\varphi : S \to \mathbf{P}^N$ such that the image of φ is of dimension 2 and φ contracts a curve $C_0 \subset S$ to a zero-dimensional set. Then C_0 is a unique section of π such that $C_0^2 < 0$.*

Proof. First, we claim that the curve C_0 is irreducible. Indeed, if C_1 and C_2 were two irreducible components of C_0 then $C_1^2 < 0$, $C_2^2 < 0$, $C_1C_2 \geq 0$ and $aC_1 - bC_2$ would be equivalent to a multiple of a fiber of π for a, $b > 0$ and thus $(aC_1 - bC_2)^2 = 0$, a contradiction. Let $i : B \to C_0 \subset S$ be the normalisation. Consider $\pi_B : S_B := \mathbf{P}((\pi \circ i)^*(\mathcal{E})) \to B$ a ruled surface over B obtained via base change; it has a section B_0 which comes from the epimorphism $(\pi \circ i)^*(\mathcal{E}) \to i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. The section B_0 is mapped birationally to C_0 under the induced map of projective bundles $j : S_B \to S$ and it is a component of $B_1 = j^{-1}(C_0)$ which is contracted by $\varphi \circ j$. Since B_1 is irreducible, it follows that $B_1 = B_0$ and

$$1 = deg(B_0 = B_1 \to C_0) = deg(S_B \to S) = deg(C_0 \to C)$$

and therefore we are done.

Remark. Note that this argument works also in case of Lemma 7.2 from [PSW2] to replace the original "lift-up" argument which is incomplete.

We continue with the divisorial case: As an immediate consequence of the preceeding lemma let us note that the good supporting divisors of φ and p (i.e. pullbacks of ample divisors from the targets of respective maps) may be chosen to be $H + E$ and $H - E$, respectively.

Let now M be the intersection of a r-dimensional fiber of p with E, it follows that $M \cong \mathbf{P}^{(r-1)}$ and $H_{|M} = E_{|M} = \mathcal{O}(1)$. Now since the map φ maps M onto T, by a result of Lazarsfeld [L] it follows that $T \cong \mathbf{P}^{(r-1)}$. Since $E + H$ is a pullback of a Cartier divisor $-K_Z/r$ from Z and $(E + H)_{|M} = \mathcal{O}(2)$ it follows that $-K_Z/r$ restricted to T is either $\mathcal{O}(2)$ or $\mathcal{O}(1)$. In the latter case, however, using the relation

$$-K_{Z|T} = -K_T - c_1 N^*_{T/Z}$$

we would find out that $c_1 N^*_{T/Z} = 0$. On the other hand, since $H = -E + (E + H)$ is ample on E it follows that $N^*_{T/Z} \otimes \mathcal{O}(-K_Z/r)$ is ample. Thus, if $c_1(N^*_{T/Z}) = 0$ and $\mathcal{O}(-K_Z/r)_{|T} = \mathcal{O}(1)$ the bundle $N^*_{T/Z}(1)$ would be isomorphic to $\bigoplus \mathcal{O}(1)^{r+1}$, a contradiction, since its projectivization would not have a dominant morphism on Y of dimension $r+1$. A similar argument done if $(-K_Z/r)_{|T} = \mathcal{O}(2)$ leads to the situation when $N^*_{T/Z}(2)$ is ample with first Chern class $\mathcal{O}(r+2)$ and therefore by splitting type (see e.g. [W2, 1.9]) $N^*_{T/Z}(2) \cong T\mathbf{P}^{r-1} \oplus \mathcal{O}(1)^2$. The projectivization of this latter bundle maps with connected fibers (because they are hyperplanes in fibers of p) onto Y. Therefore, we check that the morphism p restricted to E is given by the evaluation of the bundle $T\mathbf{P}^{r-1}(-1) \oplus \mathcal{O}^2$ and thus we get the following

Lemma 6.3. *If φ is divisorial then $Y \cong \mathbf{P}^{r+1}$ and any non-trivial fiber of φ is mapped by p isomorphically onto a hyperplane in $\mathbf{P}^{r+1}$. Moreover $T \cong \mathbf{P}^{r-1}$ and $N_{T/Z} \cong T\mathbf{P}^{r-1} \oplus \mathcal{O}(1)^2$.*

Now we deal with the case when both p and φ are of fiber type

Lemma 6.4. (Comparison Lemma [PSW2, 3.1]) *Assume that φ is of fiber type. Let*

$$r_Y := min\{-K_Y.C : \text{ where } C \text{ is rational on } X\}.$$

Then $r_Y \cdot H + p^*(K_Y)$ is a good supporting divisor for "the other" contraction φ.

The proof of the above lemma in case $r \geq 4$ is identical as in [PSW2]; for $r = 3$ and φ of fiber type the lemma will also work because φ has a fiber of dimension r, see Remark (3.4) in [PSW2].

Remark 6.5. Note that in the divisorial case we also have the comparison lemma since the pull-back of $\mathcal{O}(1)$ from $Y = \mathbf{P}^{r+1}$ to a fiber of φ is again $\mathcal{O}(1)$.

Corollary 6.6. *Assume that φ is either divisorial or of fiber type.*

(a) Let F be an r-dimensional fiber of φ. Then F is $\mathbf{P}^r$ and $\mathcal{E}_F(-1) := (p^*\mathcal{E}_{|F})(-1)$ is nef, and $c_1(\mathcal{E}_F(-1))$ is either 1 or 2.

(b) If $f \cong \mathbf{P}^{r-1}$ is a general hyperplane in F or — for φ of fiber type — a general fiber of φ then $\mathcal{E}_f(-1) := (p^*\mathcal{E}_{|f})(-1)$ is as described in [PSW1, Thm. 1] or [PSW2, 0.6].

Proof. We already noted that $F = \mathbf{P}^r$. The rest is proved exactly as (5.2) and (5.3) from [PSW2], the case $c_1 = 0$ ruled out because $\mathcal{E}$ is not locally free.

Lemma 6.7. *Assume that φ is of fiber type. Then both Y and Z are isomorphic to $\mathbf{P}^{r+1}$.*

Proof. We use notation from 6.6, i.e. F is a "big" fiber of φ while f is a general fiber of φ, or a general hyperplane in F. Let us consider a composition of maps

$$\mathbf{P}(\mathcal{E}_f) \longrightarrow \mathbf{P}(\mathcal{E}) \longrightarrow Z$$

where $\mathbf{P}(\mathcal{E}_f) \to \mathbf{P}(\mathcal{E})$ is induced by the change of the base $p : f \to Y$. We claim that the composition is surjective. Indeed, if this is not the case then $p^{-1}(p(f_1)) \cap f_2 = \emptyset$ for a sufficiently general choice of f_1 and f_2, so that the intersection of cycles $p^{-1}(p(f_1)) \cdot f_2$ is zero. But note that for a general choice of f_1 we have $dim(p^{-1}(p(f_1)) \cap F) = r - 2$ — because $p(F)$ is ample on Y — and thus $p^{-1}(p(f_1)) \cap f_2$ is non-empty of the expected dimension $r - 3 \geq 0$ for $f_2 \subset F$.

Therefore, for $r \geq 5$, looking up through the list from [PSW1], we find out that the only possibility when $\mathbf{P}(\mathcal{E}_f)$ admits a surjective map onto a $r + 1$ dimensional variety is

$$\mathcal{E}_f(-1) \cong \mathcal{O}^{r+2}/\mathcal{O}(-1)^2.$$

The map $\mathbf{P}(\mathcal{E}_f) \to Z$ factors through $\mathbf{P}^{r+1}$ and thus $Z = \mathbf{P}^{r+1}$. The reasoning is clearly symmetric with respect to the change of Z and Y so the lemma is proved in this case.

For r equal 3 and 4 we have to eliminate some other possibilities apart of $\mathcal{E}_f(-1) \cong \mathcal{O}^{r+2}/\mathcal{O}(-1)^2$, that is, possible sheaves $\mathcal{E}_f(-1)$ which occur in the classification [PSW1] such that $\mathbf{P}(\mathcal{E}_f)$ admits a morphism onto an $r+1$-dimensional variety. If $r=4$ the other possibility is a sheaf from the sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \Omega(2) \oplus \mathcal{O}^2 \longrightarrow \mathcal{E}_f(-1) \longrightarrow 0,$$

see [PSW1]. We claim that in this case $H^1(\mathbf{P}^3, \mathcal{E}_f^*(k)) = H^2(\mathbf{P}^3, \mathcal{E}_f^*(k)) = 0$ for $k \geq 2$ and therefore $\mathcal{E}_F(-1)$ extends to a locally free sheaf, see 4.2, 4.3. Indeed, the bundle $\mathcal{E}_f^*(k)$ is isomorphic to either $T\mathbf{P}^3(k-3) \oplus \mathcal{O}(k-1)$ or to $\mathcal{N}(k-2) \oplus \mathcal{O}(k-1)^2$, where $\mathcal{N}$ is a null-correlation bundle on $\mathbf{P}^3$; thus we check the vanishing easily. Now, to conclude this case, note that if $\mathcal{E}_F(-1)$ extends to a locally free sheaf $\mathcal{F}$ then $\mathcal{F}$ is nef on $\mathbf{P}^4$ and with Chern classes $(c_1, c_2, c_3) = (2,2,0)$, thus checking it with the list from [ibid] we arrive to a contradiction.

The case $r=3$ (that is $f = \mathbf{P}^2$) is dealt with similarly: apart of $\mathcal{E}_f(-1) \cong \mathcal{O}^5/\mathcal{O}(-1)^2$ also decomposable bundles and a bundle with Chern classes $(c_1, c_2) = (2,2)$ admit morphism onto 4-dimensional variety, see the main theorem of [SW]. As above we check a vanishing to claim that $\mathcal{E}_F(-1)$ extends to a locally free sheaf $\mathcal{F}$ on $F = \mathbf{P}^3$, $\mathcal{F}$ is nef with Chern class $c_1 = 2$, thus globally generated, see [PSW1]. But $\mathbf{P}(\mathcal{F})$ contains $\mathbf{P}(\mathcal{E})$ which is mapped onto a 4-dimensional variety, so itself it has to be mapped onto a 5-dimensional variety. Again, by [ibid] the only possibilities for $\mathcal{F}$ are $\mathcal{O}^6/\mathcal{O}(-1)^2$ or $\Omega\mathbf{P}^3(2) \oplus \mathcal{O}$, or $\mathcal{N}(1) \oplus \mathcal{O}^2$, where $\mathcal{N}$ is a null-correlation; we are to exclude the latter two possibilities.

To this end note that $c_3(\Omega\mathbf{P}^3(2)) = 0$ and thus $\mathcal{E}_F$ is locally free. Now we can apply an argument from [PSW2, 5.5]: using the relative Euler sequence (because $\mathcal{E}$ is locally free at $p(F)$) we compute the total Chern class of $\Omega X_{|F}$:

$$c_t(\Omega X_{|F}) = c_t(p^*(\Omega Y)_{|F}) \cdot c_t(\mathcal{E}_F(-1)).$$

On the other hand, because of [AW, 4.9, 4.12] $N^*_{F/X} = T\mathbf{P}^r(-1)$ ($N_{F/X}$ denoting the normal bundle), and we compute

$$c_t(\Omega X_{|F}) = c_t(\Omega\mathbf{P}^r) \cdot c_t(N^*_{F/X}) = 1 - 3h + 3h^2 - h^3$$

and further

$$c_t(p^*(\Omega Y)_{|F}) = 1 - 5h + 11h^2 - 13h^3,$$

where h denotes the class of a plane in $\mathbf{P}^3$; in particular $p(F) \cdot c_3(\Omega Y)$ is not divisible by 5. In our case, however, the integer r_Y from 6.4 is equal to 5 so either $-K_Y$ is divisible by 5 in $PicY$ and then $Y = \mathbf{P}^4$ or $-K_Y$ generates $PicY$. In the latter case the intersection of any 1-cycle with any divisor would be divisible by 5 (this follows e.g. from deformation theory), a contradiction. On the other hand $c_t(\Omega\mathbf{P}^4) = 1 - 5h + 10h^2 - 10h^3 + 5h^4$ so comparing it with the above formula for $c_t(p^*(\Omega Y)_{|F})$ we arrive to a contradiction even if $Y = \mathbf{P}^4$. This completes the proof of 6.7.

To conclude the classification we will deal with the case $Y = \mathbf{P}^{r+1}$ in our set-up 6.0 (i)—(iii). Let $\mathcal{E}_H$ denote the restriction of $\mathcal{E}$ to a general hyperplane $H = \mathbf{P}^r$ in $\mathbf{P}^{r+1}$; $\mathcal{E}_H(-1)$ is then a nef vector bundle (see 6.4, 6.5) with $c_1 = 2$. Looking up through the list from [PSW1] we get the following possibilities depending on the dimension of Z:

(i) $\mathcal{E}_H(-1) = \mathcal{O}^{r+2}/\mathcal{O}(-1)^2$ and $\mathbf{P}(\mathcal{E}_H)$ has a contraction onto $\mathbf{P}^{r+1}$,

(ii) $\mathcal{E}_H(-1) = (T\mathbf{P}^r(-1) \oplus \mathcal{O}(1))/\mathcal{O}$ and $\mathbf{P}(\mathcal{E}_H)$ admits a birational morphism onto a quadric $\mathbf{Q}^{2r-1}$, the variety $\mathbf{P}(\mathcal{E}_H)$ is a blow-up of the quadric along a linear $\mathbf{P}^{r-1}$,

(iii) $\mathcal{E}_H(-1) = \mathcal{O}^{r+1}/\mathcal{O}(-2)$ and $\mathbf{P}(\mathcal{E}_H)$ is contracted onto $\mathbf{P}^r$.

The remaining cases appearing in [PSW1, Thm. 1] are excluded: decomposable bundles because of 4.2.2, the other ones because they do not admit maps onto varieties of dimension emerging in cases (i)—(iii) of 6.0.

Note that above three cases are in one-to-one correspondence with the cases (i)—(iii) from 6.0. The variety Z — the target of the contraction φ — is therefore $\mathbf{P}^{r+1}$, $\mathbf{Q}^{2r}$ and $\mathbf{P}^r$, respectively. If φ is divisorial or a quadric bundle, we obtain a description of X (and therefore of $\mathcal{E}$) immediately — see 6.3 and 6.1, respectively. If φ is of type (i) then note that $\mathcal{E}(-1)$ is spanned by $r+2$ sections, because $\mathcal{O}_{\mathbf{P}(\mathcal{E}(-1))}(1) = \varphi^*\mathcal{O}(1)$, and therefore we have an exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{O}^{r+2} \longrightarrow \mathcal{E}(-1) \longrightarrow 0$$

with $\mathcal{H}$ a reflexive sheaf of rank 2. Since $\mathcal{E}(-1)$ restricted to a hyperplane is $\mathcal{O}^{r+2}/\mathcal{O}(-1)^2$ it follows that $\mathcal{H} = \mathcal{O}(-1)^2$ and thus we have a description of $\mathcal{E}$ and of X.

We summarize the result in the following

Theorem 6.8. *Let X be a Fano manifold of index r and dimension $2r$. Assume that X is a projectivization of a sheaf $\mathcal{E}$, $p : X = \mathbf{P}(\mathcal{E}) \to Y$, and assume moreover that $\mathcal{E}$ is not locally free. Then one of the following holds (note that the top Chern class $c_{r+1}(\mathcal{E})$ is equal to the number of singular fibres of $\mathcal{E}$):*

(i) $Y \cong \mathbf{P}^{r+1}$, *X is an intersection of two divisors of bidegree $(1,1)$ in $\mathbf{P}^{r+1} \times \mathbf{P}^{r+1}$,*

$$\mathcal{E}(-1) = \mathcal{O}^{r+2}/\mathcal{O}(-1)^2, \quad c_{r+1}(\mathcal{E}) = r+2,$$

(ii) $Y \cong \mathbf{P}^{r+1}$ *and X is a blow-up of $\mathbf{Q}^{2r}$ along a linear $\mathbf{P}^{r-1} \subset \mathbf{Q}^{2r}$,*

$$\mathcal{E}(-1) = (T\mathbf{P}^{r+1}(-1) \oplus \mathcal{O}(1))/\mathcal{O}^2, \quad c_{r+1}(\mathcal{E}) = 2,$$

(iii) $Y \cong \mathbf{P}^{r+1}$, *X is a divisor of bidegree $(1,2)$ in $\mathbf{P}^{r+1} \times \mathbf{P}^r$ and*

$$\mathcal{E}(-1) = \mathcal{O}^{r+1}/\mathcal{O}(-2), \quad c_{r+1}(\mathcal{E}) = 2^{r+1},$$

(iii') $Y \cong \mathbf{Q}^{r+1}$, *X is a divisor of bidegree $(1,1)$ in $\mathbf{Q}^{r+1} \times \mathbf{P}^r$ and*

$$\mathcal{E}(-1) = \mathcal{O}^{r+1}/\mathcal{O}(-1), \quad c_{r+1}(\mathcal{E}) = 2.$$

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