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SERIES OF RATIONAL MODULI COMPONENTS OF STABLE RANK 2 VECTOR BUNDLES ON $\mathbb{P}^3$

ALEXEY KYTMANOV, ALEXANDER TIKHOMIROV, AND SERGEY TIKHOMIROV

Abstract. We study the problem of rationality of an infinite series of components, the so-called Ein components, of the Gieseker-Maruyama moduli space $M(e,n)$ of rank 2 stable vector bundles with the first Chern class $e=0$ or -1 and all possible values of the second Chern class n on projective 3-space. The generalized null correlation bundles constituting open dense subsets of these components are defined as cohomology bundles of monads whose members are direct sums of line bundles of degrees depending of nonnegative integers $a,b,c,$ where $b\ge a$ and $c>a+b$. We show that, in the wide range when $c>2a+b-e,\ b>a,\ (e,a)\ne(0,0)$, the Ein components are rational, and in the rest cases they are at least stably rational. As a consequence, the union of the spaces $M(e,n)$ over all $n\ge1$ contains an infinite series of rational components. An explicit construction of rationality of Ein components under the above conditions on e,a,b,c and, respectively, of their stable rationality in the rest cases, is given. In case of rationality, we also construct universal families of generalized null correlation bundles over certain open subsets of Ein components showing that these components are fine moduli spaces. Thus, the union of the spaces the $M(e,n)$ over all $n\ge1$ contains an infinite series of rational components. As a by-product of this construction, for $c_1=0$ and n even, these open subsets provide, perhaps the first known, examples of fine moduli spaces not satisfying a usual sufficient condition "n is odd" for fineness.

1. Introduction

For $e\in\{-1,0\}$ and $n\in\mathbb{Z}_+$ let $M(e,n)$ be the Gieseker-Maruyama moduli space of stable rank 2 algebraic vector bundles with Chern classes $c_1=e,\ c_2=n$ on the projective space $\mathbb{P}^3$. R. Hartshorne [12] showed that $M(e,n)$ is a quasi-projective scheme, nonempty for arbitrary $n\ge1$ in case $e=0$ and, respectively, for even $n\ge2$ in case $e=-1$, and the deformation theory predicts that each irreducible component of $M(e,n)$ has dimension at least $8n-3+2e$.

In this article we study the problem of rationality of irreducible components of $M(e,n)$. Since 70ies not so much has been known about this problem. In particular, in case $e=0$ it is known by now (see [12], [10], [5], [7], [24], [25]) that the scheme $M(0,n)$ contains an irreducible component I_n of expected dimension $8n-3$, and this component is the closure of the open subset of $M(0,n)$ constituted by the so-called mathematical instanton vector bundles. Moreover, $M(0,n)$ is irreducible (hence coincides with I_n) and rational for $n=1$ and 2 [12]. The rationality of I_3 and of I_5 was proved in [10] and [18], respectively, and for $n=4$ and $n\ge6$ the rationality of I_n is still a challenging open question. Note that $M(0,n)$ is reducible for $n\ge3$, and the exact number of irreducible components of $M(0,n)$ is nowadays known only up to $n=5$. We enumerate these components in Section 8.

In case $e=-1$, for each $n\ge1$, the space $M(-1,2n)$ contains at least one irreducible component Y_{2n} of expected dimension $16n-5$ [12]. In particular,

$M(-1,2) = Y_2$ is a rational variety of the expected dimension 11 by [15]. The space $M(-1,4)$ is also known – it contains, besides the rational component Y_4 of expected dimension 27, one more rational component of dimension 28. For $n \geq 6$ the exact number of irreducible components of $M(0,n)$ is unknown by now. See details in Section 8.

In 1978 W. Barth and K. Hulek [6] found, for each integer $k \geq 1$, a rational family $\tilde{Q}_k$ of dimension $3k^2+10k+8$ of vector bundles from $M(0,2k+1)$, and G. Ellingsrud and S. A. Strømme in [10, (4.6)-(4.7)] showed that the image of $\tilde{Q}_k$ under the modular morphism $\tilde{Q}_k \to M(0,2k+1)$ is an open subset of an irreducible component Q_k distint from the instanton component I_{2k+1}. Besides, from the definition of Q_k it follows that it is (at least) unirational. Later in 1984 V. K. Vedernikov [27] constructed, for each pair of integers k,l with $1 \leq l \leq k$, two families $V_i(k,l)$ of bundles from $M(0,n_i)$, $i=1,2$, and one family $V_3(k,l)$ of bundles from $M(-1,n_3)$, where n_1,n_2,n_3 are certain polynomials on k,l. In his subsequent paper [28] one more family $V_4(k)$ of bundles from $M(0,(k+1)^2)$ was found for $k \geq 1$. In [27], [28] the construction of stable rationality of the family $V_1(k,l)$, respectively, of rationality of $V_2(k,l)$ and $V_4(k)$ was given – see Remark 15 below for details. Besides, the author asserted that these families are open parts of irreducible components of $M(e,n)$, though the proofs for these statements were not given. A more general series of rank 2 bundles depending on triples of integers a,b,c, appeared in 1984 in the paper of A. Prabhakar Rao [23] (cf. Remark 16). Soon after that, in 1988, L. Ein [9] independently studied these bundles (called in his paper the generalized null correlation bundles) and proved that they constitute open parts of irreducible components of $M(e,n)$ (called below the Ein components). Surprisingly, Ein components contain Vedernikov's families $V_1(k,l)$ and $V_4(k)$, respectively, $V_2(k,l)$ and $V_3(k,l)$ as their open subsets in special cases when $e=a=0$, respectively, $a=b$ (see details in Remark 15). Moreover, in case $e=a=0,\ b=k\geq 1,\ c=k+1$ the closure of Vedernikov's family $V_1(k,1)$ coincides with the component Q_k of Ellingsrud-Strømme, i. e. Q_k is also an Ein component.

The problem of rationality of Ein components is the main subject of this paper. We will prove their rationality in a wide range of parameters a,b,c when $(e,a) \neq (0,0),\ c > 2a+b-e$, and their (at least) stable rationality in the rest of the cases. In particular, we show that our results cover Vedernikov's results in case $e=0,\ a=b>0,\ c>3a$ and improve them in case $e=a=0,\ b>0$ (see Remark 15). Together with the rest of results of Vedernikov, this gives a complete answer to the problem of rationality or, otherwise, (at least) stable rationality of Ein components for all possible values of e,a,b,c. Before proceeding to precise formulations we recall briefly the definition of generalized null correlation bundles.

For any three integers a,b,c such that $b \geq a \geq 0,\ c > a+b$, consider the monad

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c+e) \to \mathcal{H} \to \mathcal{O}_{\mathbb{P}^3}(c) \to 0, \tag{1}$$

where

$$\mathcal{H} = \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(-a+e) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \oplus \mathcal{O}_{\mathbb{P}^3}(-b+e), \tag{2}$$

such that the cohomology sheaf E of this monad is locally free. According to [23, Prop. 3.1] (see also [9, Prop. 1.2(a)]) such monads exist and their cohomology rank 2 vector bundle E is stable. We call E the *generalized null correlation bundle* and denote by $N(e,a,b,c)^{\mathrm{nc}}$ the set of all generalized null correlation bundles for the above integers e, a, b, c. Ein shows in [9] that $N(e, a, b, c)^{\mathrm{nc}}$ is a dense Zariski open subset of an irreducible component $N(e, a, b, c)$ of the space $M(e, n)$, where $n = c^2 - a^2 - b^2 - e(c - a - b)$. We therefore call these moduli components $N(e, a, b, c)$ the *Ein components* of $M(e, n)$.

We give now a sketch of the contents of the paper. In Section 2 we construct a family $\mathbf{E}_\tau$ of generalized null correlation bundles on $\mathbb{P}^3$ with certain base $\mathbf{X}_\tau$ such that there is a principal PGL-bundle $\Phi : \mathbf{X}_\tau \to X_\tau$ with X_τ being a locally trivial projective bundle over a certain dense open subset $N(e, a, b, c)_\tau$ of $N(e, a, b, c)^{\mathrm{nc}}$ – see Theorem 1. In Section 3 we produce a family of reflexive sheaves on $\mathbb{P}^3$ obtained from generalized null correlation bundles of the above family by a reduction step in the sense of Hartshorne, cf. Remark 3. In Section 4 we study the properties of reflexive sheaves of this family. We then construct a certain locally trivial projective bundle $W_\tau \to X_\tau$ and a pullback $\tilde{\Phi} : \mathbf{W}_\tau \to W_\tau$ onto W_τ of the principal bundle Φ. In Section 5 we produce, using the Serre construction (see Remark 8), another family (with rational base) of reflexive sheaves of the above type on $\mathbb{P}^3$, and provide its local study in Section 6. We then use this second family of reflexive sheaves to construct via the dual reduction step a new family $\underline{\mathbf{E}}$ of generalized null correlation bundles, with certain rational base B. In Section 7 we relate the family $\underline{\mathbf{E}}$ to the pullback $\mathbf{E}_{\mathbf{W}_\tau}$ of the family $\mathbf{E}_\tau$ onto $\mathbb{P}^3 \times \mathbf{W}_\tau$. Namely, we show that the two families $\underline{\mathbf{E}}$ and $\mathbf{E}_{\mathbf{W}_\tau}$ are, roughly speaking, the same up to an appropriate lift (the exact relation between them is given in (108) below). More precisely, we construct a morphism $\mathbf{W}_\tau \to B$ which factors through the principal PGL-bundle $\tilde{\Phi}$ mentioned above, thus giving a morphism $W_\tau \to B$, and we show that this morphism is an isomorphism onto a certain open subset B_τ of B (see Theorem 12). This leads to the main result of the paper, Theorem 12, which states that each Ein component $N(e, a, b, c)$ of $M(e, n)$, $n = c^2 - a^2 - b^2 - e(c - a - b)$, is a rational variety and a fine moduli component if $c > 2a + b - e,\ b > a,\ (e, a) \neq (0, 0)$, and is at least stably rational otherwise. Thus, the union of the spaces $M(e, n)$ over all $n \geq 1$ contains an infinite series of rational components (see Corollary 13). As a by-product of Theorem 12 we show that, for $c_1 = 0$ and n even, the open subsets $N(e, a, b, c)_\tau$ of Ein components provide, perhaps the first known, examples of fine moduli components of rank 2 stable bundles not satisfying a usual sufficient condition "n is odd" for fineness – see Remark 14. As another application of Theorem 12, in Section 8 we give a list of known irreducible components of $M(e, n)$, including Ein components, for small values of n, up to $n = 20$, and specify those of Ein components which are rational, respectively, stably rational, for both $e = 0$ and $e = -1$, and give their dimensions.

Conventions and notation.

- Everywhere in this paper we work over the base field $\mathbf{k}$ of characteristic 0.
- $\mathbb{P}^3$ is a projective 3-space over $\mathbf{k}$.
- Given a morphism of schemes $f : X \to Y$ and a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^3 \times Y$, set

$$\mathcal{F}_X := (\mathrm{id}_{\mathbb{P}^3} \times f)^* \mathcal{F}.$$

  This notation will be systematically used throughout the paper.
- For any coherent sheaf $\mathcal{G}$ on a scheme X, we set $\mathbb{P}(\mathcal{G}) := \mathrm{Proj}(S^{\cdot}_{\mathcal{O}_X} \mathcal{G})$. Also, $\mathcal{O}_Y(1)$ denotes the Grothendieck invertible sheaf on $Y = \mathbb{P}(\mathcal{G})$.

- Given $m, n \in \mathbb{Z}$, $\mathbf{P}$ a projective space of arbitrary dimension, X a scheme, and $\mathcal{A}$ a coherent sheaf on $\mathbf{P} \times \mathbb{P}^3 \times X$, set
$$\mathcal{A}(m,n) := \mathcal{A} \otimes \mathcal{O}_{\mathbf{P}}(n) \boxtimes \mathcal{O}_{\mathbb{P}^3}(m) \boxtimes \mathcal{O}_X, \quad \mathcal{A}(m) := \mathcal{A}(m,0).$$
- For a stable rank 2 vector bundle E with $c_1(E) = n$ on $\mathbb{P}^3$, we denote by $[E]$ its isomorphism class in $M(e,n)$.

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2. Family $\mathbf{E}$ of generalized null correlation bundles on $\mathbb{P}^3$

In this Section, for an open subset $N(e,a,b,c)^{\mathrm{nc}}$ of an arbitrary Ein moduli component $N(e,a,b,c)$, we construct a family $\mathbf{E}$ of generalized null correlation bundles on $\mathbb{P}^3$ with base $\mathbf{X}$ covering $N(e,a,b,c)^{\mathrm{nc}}$ under the modular morphism $\mathbf{X} \to N(e,a,b,c)$, $\mathbf{x} \mapsto \left[\mathbf{E}|_{\mathbb{P}^3 \times \{\mathbf{x}\}}\right]$ (see diagram (30)). We also introduce a certain irreducible open subset $\mathbf{X}_\tau$ of $\mathbf{X}$, together with a family $\mathbf{E}_\tau$ of generalized null correlation bundles with base $\mathbf{X}_\tau$ (see Theorem 1). These families will be used in subsequent sections.

Given integers e, a, b, c with $e \in \{-1, 0\}$ and $b \geq a \geq 0$, $c > a + b$, consider the Ein component $N(e,a,b,c)$. As it is known from [9, (2.2.B) and Section 3] (see also [4, Section 5]),
$$\dim N(e,a,b,c) = h^0(\mathcal{H}(c-e)) - h^0(S^2\mathcal{H}(-e)) - 1.$$
Substituting here $\mathcal{H}$ from (2) we obtain:
$$\begin{aligned}\dim N(e,a,b,c) &= \binom{c+a-e+3}{3} + \binom{c+b-e+3}{3} + \binom{c-a+3}{3} \\ &+ \binom{c-b+3}{3} - \binom{a+b-e+3}{3} - \binom{b-a+3}{3} - \binom{2a-e+3}{3} \\ &- \binom{2b-e+3}{3} - 3 - t(e,a,b),\end{aligned} \tag{3}$$
where
$$t(0,a,b) = \begin{cases} 4, & \text{if } a = b = 0, \\ 1, & \text{if } 0 = a < b \quad \text{or } a = b > 0, \\ 0, & \text{otherwise.} \end{cases} \qquad t(-1,a,b) = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$
Besides, for any generalized null correlation bundle $[E] \in N(e,a,b,c)^{\mathrm{nc}}$, formulas (1) and (2) yield:
$$h^1(E(m)) = h^0(\mathcal{O}_{\mathbb{P}^3}(c+m)) - h^0(\mathcal{O}_{\mathbb{P}^3}(a+m)) - h^0(\mathcal{O}_{\mathbb{P}^3}(b+m)), \quad m \leq -1.$$

In particular,

$$h^1(E(-c)) = 1, \tag{5}$$

$$h^1(E(-c-1)) = 0, \tag{6}$$

$$h^1(E(-b)) = \binom{c-b+3}{3} - 1, \quad b > 0. \tag{7}$$

Consider an arbitrary Ein component $N(e,a,b,c)$ and its open dense subset $N(e,a,b,c)^{\mathrm{nc}}$ of generalized null correlation bundles. There exists a big enough positive integer m such that all bundles from $N(e,a,b,c)^{\mathrm{nc}}$ are m-regular in the sense of Mumford-Castelnuovo [17, Ch. 4.3]. Let $P \in \mathbb{Q}[x]$ be the Hilbert polynomial $P(k) = \chi(E(k))$, $[E] \in N(e,a,b,c)^{\mathrm{nc}}$, and let $\mathcal{H} := \mathbf{k}^{N_m} \otimes \mathcal{O}_{\mathbb{P}^3}(-m)$, where $N_m := P(m)$. Consider the Quot-scheme $Q := \mathrm{Quot}_{\mathbb{P}^3}(\mathcal{H}, P)$, together with the universal quotient morphism $\mathcal{H} \boxtimes \mathcal{O}_Q \twoheadrightarrow \mathbb{E}$. Then the scheme

$$Y = \left\{ y \in Q \mid \left[\mathbb{E}|_{\mathbb{P}^3 \times \{y\}}\right] \in N(e,a,b,c)^{\mathrm{nc}} \right\}$$

is an open subscheme of Q, together with a family

$$\mathbb{E}_Y$$

of generalized null correlation bundles over Y. Then, according to the GIT-construction [17, Ch. 4] of $N(e,a,b,c)^{\mathrm{nc}}$, the modular morphism

$$\varphi : \ Y \to N(e,a,b,c)^{\mathrm{nc}} = Y//PGL(N_m), \quad y \mapsto \left[\mathbb{E}_Y|_{\mathbb{P}^3 \times \{y\}}\right] \tag{8}$$

is a geometric $PGL(N_m)$-quotient and a principal $PGL(N_m)$-bundle.

Since by Serre duality for any $[E] \in N(e,a,b,c)^{\mathrm{nc}}$ one has $h^2(E(c-e-4)) = h^1(E(-c))$, $h^2(E(b-e-4)) = h^1(E(-b))$, using (5) and (7) and the base change we obtain that the sheaves

$$L = R^2 p_{2*}\mathbb{E}_Y(c-e-4), \quad L' = R^2 p_{2*}\mathbb{E}_Y(b-e-4), \tag{9}$$

where $p_2 : \mathbb{P}^3 \times Y \to Y$ is the projection, are locally free $\mathcal{O}_Y$-sheaves of ranks

$$\mathrm{rk}\, L = 1, \quad \mathbf{r} := \mathrm{rk}\, L' = \binom{c-b+3}{3} - 1. \tag{10}$$

Set

$$\mathbf{P} := |\mathcal{O}_{\mathbb{P}^3}(c-b)| \tag{11}$$

and let

$$\Gamma = \left\{ (S,x) \in \mathbf{P} \times \mathbb{P}^3 \mid x \in S \right\} \tag{12}$$

be the universal family of surfaces of degree $c-b$ in $\mathbb{P}^3$. There is an exact triple on $\mathbf{P} \times \mathbb{P}^3 \times Y$:

$$0 \to \mathcal{O}_{\mathbf{P}}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(b-c) \boxtimes \mathcal{O}_Y \to \mathcal{O}_{\mathbf{P}} \boxtimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_Y \to \mathcal{O}_{\Gamma \times Y} \to 0. \tag{13}$$

Tensoring it with the sheaf $\mathbb{E}_Y(c-e-4) \boxtimes \mathcal{O}_{\mathbf{P}}$ and applying to the resulting exact triple the functor $R^i pr_{13*}$, where $pr_{13} : \mathbf{P} \times \mathbb{P}^3 \times Y \to \mathbf{P} \times Y$, in view of the base change and the equalities $h^3\big(E(b-e-4)\big) = 0$ we obtain an exact triple

$$\mathcal{O}_{\mathbf{P}}(-1) \boxtimes L' \xrightarrow{\psi} \mathcal{O}_{\mathbf{P}} \boxtimes L \to R^2 pr_{13*}\left(\mathcal{O}_{\mathbf{P}} \boxtimes \mathbb{E}_Y(c-e-4)|_{\Gamma \times Y}\right) \to 0. \tag{14}$$

Now take an arbitrary point $y \in Y$ and let

$$[E_y] := \varphi(y).$$

Resricting the triple (14) onto $\mathbf{P} \times \{y\}$ and using (10) and the base change we obtain an exact triple

$$\mathbf{r}\mathcal{O}_{\mathbf{P}}(-1) \xrightarrow{\psi\otimes\mathbf{k}(y)} \mathcal{O}_{\mathbf{P}} \to \mathrm{coker}\big(\psi \otimes \mathbf{k}(y)\big) \to 0. \tag{15}$$

where by the base change we have for any surface $S \in \mathbf{P}$:

$$\mathrm{coker}\big(\psi\otimes\mathbf{k}(y)|_{\{(S,y)\}}\big) = R^2pr_{23*}\left(\mathbb{E}(c-e-4)\boxtimes\mathcal{O}_{\mathbf{P}}|_{\Gamma\times Y}\right)|_{\{(S,y)\}} = H^2\big(E_y(c-e-4)|_S\big). \tag{16}$$

From the triple (15) it follows that

$$h^2\big(E_y(c-e-4)|_S\big) \leq 1. \tag{17}$$

On the other hand, the Grothendieck-Serre duality for a locally free $\mathcal{O}_S$-sheaf $E_y|_S$ shows that

$$h^2\big(E_y(c-e-4)|_S\big) = h^0\big(E_y(-b)|_S\big). \tag{18}$$

Next, the triple (15) shows that

$$\mathbf{P}(y) := \mathrm{Supp}(\mathrm{coker}(\psi\otimes\mathbf{k}(y))) \tag{19}$$

is a linear subspace of codimension at most $\mathbf{r} = \dim\mathbf{P}$ in $\mathbf{P}$. Hence this subspace $\mathbf{P}(y)$ is always nonempty, and (16)–(18) give the following explicit description of $\mathbf{P}(y)$:

$$\mathbf{P}(y) = \{S \in \mathbf{P}|\ h^0(E_y(-b)|_S) = 1\} \tag{20}$$

Set $\tau(y) = \dim\mathbf{P}(y)$ and let

$$\tau := \min_{y\in Y}\tau(y), \quad Y_\tau := \{y \in Y|\ \tau(y) = \tau\}. \tag{21}$$

Since Y is irreducible, the semicontinuity yields that Y_τ is a dense open subset of Y. Moreover, from (8) it follows that there exists a dense open subset $N(e,a,b,c)_\tau$ of $N(e,a,b,c)^{\mathrm{nc}}$ such that

$$Y_\tau = \varphi^{-1}\big(N(e,a,b,c)_\tau\big)$$

and $\varphi : Y_\tau \to N(e,a,b,c)_\tau$ is a principal $PGL(N_m)$-bundle. Namely, the set $N(e,a,b,c)_\tau$ is explicitly described as follows. For any point $[E] \in N(e,a,b,c)^{\mathrm{nc}}$ consider the exact triple

$$0 \to E(b-c)\boxtimes\mathcal{O}_{\mathbf{P}}(-1) \to E\boxtimes\mathcal{O}_{\mathbf{P}} \to E\boxtimes\mathcal{O}_{\mathbf{P}}|_\Gamma \to 0$$

and apply to it the functor R^ipr_{2*}, where $pr_2 : \mathbb{P}^3\times\mathbf{P} \to \mathbf{P}$ is the projection. Then similar to (15) we obtain an exact triple

$$\mathbf{r}\mathcal{O}_{\mathbf{P}}(-1) \overset{\psi_E}{\to} \mathcal{O}_{\mathbf{P}} \to \mathrm{coker}\,\psi_E \to 0.$$

Similar to the above, set $\mathbf{P}\big([E]\big) := \mathrm{Supp}(\mathrm{coker}\,\psi_E)$, $\tau_E = \dim\mathbf{P}_E$. Then, as in (20)–(21), we have

$$\mathbf{P}\big([E]\big) = \{S\in\mathbf{P}|\ h^0(E(-b)|_S) = 1\}, \qquad \min_{[E]\in N(e,a,b,c)}\tau_E = \tau, \tag{22}$$

and

$$N(e,a,b,c)_\tau = \{[E] \in N(e,a,b,c)|\ \tau_E = \tau\}.$$

Now consider the subscheme $\mathbf{X}$ of $\mathbf{P}\times Y$, together with the projection $\boldsymbol{\theta} : \mathbf{X} \to Y$, defined as

$$\mathbf{X} := \{\mathbf{x} = (S,y) \in \mathbf{P}\times Y\ |\ S \in \mathbf{P}(y)\}, \quad \boldsymbol{\theta} : \mathbf{X} \to Y,\ (S,y) \mapsto y. \tag{23}$$

Remark that, as $\operatorname{rk} L = 1$ by (10), the triple (14) twisted by $\mathcal{O}_{\mathbf{P}}(1) \boxtimes L^\vee$ can be rewritten as

$$\mathcal{O}_{\mathbf{P}} \boxtimes (L' \otimes L^\vee) \xrightarrow{\psi} \mathcal{O}_{\mathbf{P}}(1) \boxtimes \mathcal{O}_Y \to \mathbb{G} \to 0, \tag{24}$$

where

$$\mathbb{G} := \mathcal{O}_{\mathbf{P}}(1) \boxtimes \mathcal{O}_Y|_{\mathbf{X}}$$

is a line bundle on $\mathbf{X}$. The fibre of $\mathbb{G}$ over an arbitrary point $\mathbf{x} = (S, y) \in \mathbf{X}$ has in view of (20) and (23) the description

$$\mathbb{G} \otimes \mathbf{k}(\mathbf{x}) = H^0(E_y(-b)|_S). \tag{25}$$

Applying to (24) the functor p_{2*}, where $p_2 : \mathbf{P} \times Y \to Y$ is the projection, we obtain an exact triple

$$L' \otimes L^\vee \xrightarrow{f} S^{c-b}V \otimes \mathcal{O}_Y \to \mathbb{U} \to 0, \quad \mathbb{U} = p_{2*}\mathbb{G}, \tag{26}$$

where $V = H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee$, $f = p_{2*}\psi$ and $\mathbf{X} = \mathbb{P}(\mathbb{U})$. In addition, $\mathbb{G} = \mathcal{O}_{\mathbb{P}(\mathbb{U})}(1)$ and there is the canonical epimorphism

$$p_2^*\mathbb{U} \twoheadrightarrow \mathbb{G}. \tag{27}$$

Remark that, since $\mathbb{E}$ has a natural $GL(N_m)$-linearization as a sheaf over Q, the sheaf $L' \otimes L^\vee$ has an induced $GL(N_m)$-linearization, and the sheaf $S^{c-b}V \otimes \mathcal{O}_Y$ also has a (trivial) $GL(N_m)$-linearization. Hence by (26) the sheaf $\mathbb{U}$ also inherits $GL(N_m)$-linearization. It follows that $\mathbf{X}$ inherits $PGL(N_m)$-action such that $\boldsymbol{\theta} : \mathbf{X} \to Y$ is a $PGL(N_m)$-equivariant morphism. Hence the geometric quotient

$$X := \mathbf{X}//PGL(N_m)$$

is well-defined, and the canonical projection

$$\Phi : \ \mathbf{X} \to X$$

is a principal $PGL(N_m)$-bundle.

Furthermore, comparing (20) with (22) we see that, for any $[E] \in N(e,a,b,c)^{\mathrm{nc}}$ and any $y \in \varphi^{-1}\big([E]\big)$ the fibre $\boldsymbol{\theta}^{-1}(y) = \mathbf{P}(y)$ as a subspace of $\mathbf{P}$ coincides with a subspace $\mathbf{P}\big([E]\big)$ of $\mathbf{P}$, and hence depends only on $[E]$. This implies that: (i) $\boldsymbol{\theta}$ is a $PGL(N_m)$-equivariant morphism and therefore induces a morphism of categorical quotients $\theta : X \to N(e,a,b,c)^{\mathrm{nc}}$; (ii) a fibre $\theta^{-1}\big([E]\big)$ is a subspace $\mathbf{P}\big([E]\big)$ of $\mathbf{P}$.

Next, since $f = p_{2*}\psi$, we can rewrite (19) as

$$\boldsymbol{\theta}^{-1}(y) = \mathbf{P}(y) = P(\operatorname{coker}(\psi \otimes \mathbf{k}(y))), \quad y \in Y.$$

Set

$$X_\tau := \theta^{-1}(N(e,a,b,c)_\tau). \tag{28}$$

By definition, $\theta : X_\tau \to N(e,a,b,c)_\tau$ is a morphism with the fibre $\theta^{-1}([E])$ over an arbitrary point $[E] \in N(e,a,b,c)_\tau$ being a subspace $\mathbf{P}([E]) \simeq \mathbb{P}^\tau$ of $\mathbf{P}$. Hence $\theta : X_\tau \to N(e,a,b,c)_\tau$ is a $\mathbb{P}^\tau$-subfibration of the trivial fibration $\mathbf{P} \times N(e,a,b,c)_\tau \to N(e,a,b,c)_\tau$. Hence it is locally trivial. Furthermore, as $N(e,a,b,c)_\tau$ is irreducible, it follows that X_τ is also irreducible.

We are now led to the following result.

Theorem 1. *(i) Let X_τ be defined in (28). There is a locally trivial $\mathbb{P}^\tau$-fibration $\theta: X_\tau \to N(e,a,b,c)_\tau$ with a fibre $\mathbf{P}([E]) = \theta^{-1}([E])$ over an arbitrary point $[E] \in N(e,a,b,c)_\tau$ given by (22). In other words, the set of closed points of the scheme X_τ is described as*

$$X_\tau = \{(S,[E]) \in \mathbf{P} \times N(e,a,b,c)_\tau \mid h^0(E(-b)|_S) = 1\}.$$

In particular,

$$\dim X_\tau = \dim N(e,a,b,c) + \tau, \tag{29}$$

where $\dim N(e,a,b,c)$ is given by formula (3).
(ii) There are cartesian diagrams

$$\begin{tikzcd} \mathbf{X} \arrow[r, "\Phi"] \arrow[d, "\boldsymbol{\theta}"'] & X \arrow[d, "\theta"] \\ Y \arrow[r, "\varphi"] & N(e,a,b,c)^{\mathrm{nc}}, \end{tikzcd} \quad \begin{tikzcd} \mathbf{X}_\tau \arrow[r, "\Phi"] \arrow[d, "\boldsymbol{\theta}"'] & X_\tau \arrow[d, "\theta"] \\ Y_\tau \arrow[r, "\varphi"] & N(e,a,b,c)_\tau, \end{tikzcd} \tag{30}$$

in which horizontal maps are principal $PGL(N_m)$-bundles. Here the second diagram is obtained from the first via the base change $N(e,a,b,c)_\tau \hookrightarrow N(e,a,b,c)^{\mathrm{nc}}$. Furthermore, vertical maps in the second diagram are locally trivial $\mathbb{P}^\tau$-fibrations.
(iii) The compositions $\mathbf{X} \xrightarrow{\boldsymbol{\theta}} Y \hookrightarrow Q$ and $\mathbf{X}_\tau \xrightarrow{\boldsymbol{\theta}} Y_\tau \hookrightarrow Y \hookrightarrow Q$ induce families

$$\mathbf{E} := \mathbb{E}_{\mathbf{X}}, \quad \mathbf{E}_\tau := \mathbb{E}_{\mathbf{X}_\tau}$$

of generalized null correlation bundles, respectively, where $\mathbb{E}$ is the universal quotient sheaf on $\mathbb{P}^3 \times Q$.

Remark 2. *(i) Let $\overline{X}_\tau$ be the closure of X_τ in X. Since $N(e,a,b,c) \setminus N(e,a,b,c)_\tau$ is a proper closed subset of $N(e,a,b,c)$, $\overline{X}_\tau$ is an irreducible component of X. This component is uniquely distinguished among all other possible components of X by the property that it dominates $N(e,a,b,c)$ via the morphism θ. Respectively, for $\mathbf{X}_\tau = \Phi^{-1}(X_\tau)$ its closure $\overline{\mathbf{X}}_\tau$ in $\mathbf{X}$ is the unique irreducible component of $\mathbf{X}$ dominating $N(e,a,b,c)$ under the morphism $\theta \circ \Phi$.*
(ii) Since $N(e,a,b,c)_\tau$ is an open subset of $N(e,a,b,c)^{\mathrm{nc}}$, it follows from Thm 1() that X_τ (respectively, $\mathbf{X}_\tau$) is an open subset of X (respectively, of $\mathbf{X}$).

3. Family of reflexive sheaves $\mathbf{F}$ on $\mathbb{P}^3$ related to the familiy $\mathbf{E}$

In this section we construct a familiy of reflexive sheaves $\mathbf{F}$ related to the family of generalized null correlation bundles $\mathbf{E}$ and study properties of sheaves from this family. Consider the incidence variety

$$\boldsymbol{\Gamma} = (\Gamma \times Y) \times_{\mathbf{P} \times Y} \mathbf{X}$$

with the natural projection $\rho : \boldsymbol{\Gamma} \to \mathbf{X}$ and let

$$\mathbf{L} = L_{\mathbf{X}},$$

where the invertible $\mathcal{O}_Y$-sheaf L was defined in (9). Consider the family of generalized null correlation bundles $\mathbf{E}$ defined in Theorem 1(iii). The first equality in (9) and the base change imply

$$\mathbf{L} = R^2\rho_*(\mathbf{E}|_{\boldsymbol{\Gamma}}(c-e-4)),$$

so that the relative Serre duality for the projection ρ yields

$$\mathbf{L}^\vee \simeq \rho_*(\mathbf{E}|_{\mathbf{\Gamma}}(-b)). \tag{31}$$

Respectively, for an arbitrary point $\mathbf{x} \in \mathbf{X}$ and a surface $S_{\mathbf{x}} := \mathbf{\Gamma} \times_{\mathbf{X}} \{\mathbf{x}\} \in \mathbf{P}$, we have

$$\mathbf{L}^\vee \otimes_{\mathcal{O}_{\mathbf{X}}} \mathbf{k}(\mathbf{x}) = H^0(\mathbf{E}(-b)|_{S_{\mathbf{x}}}). \tag{32}$$

Since any $[E] \in N(e,a,b,c)^{\mathrm{nc}}$ is a stable bundle and $b \geq 0$, one has $h^0(E(-b-1)) = 0$. Hence, given a surface $S \in \mathbf{P}$, in view of (6) it follows from the exact triple $0 \to E(-c-1) \to E(-b-1) \to E(-b-1)|_S \to 0$ that $h^0(E(-b-1)|_S) = 0$. In particular,

$$H^0(\mathbf{E}(-b-1)|_{S_{\mathbf{x}}}) = 0, \quad \mathbf{x} \in \mathbf{X}. \tag{33}$$

Denote $\mathbf{L}_\rho := \rho^*\mathbf{L}$. The isomorphism (31) induces a section $s_{\mathbf{\Gamma}} \in H^0(\mathbf{E}|_{\mathbf{\Gamma}}(-b) \otimes \mathbf{L}_\rho)$ defined as

$$s_{\mathbf{\Gamma}} : \ \mathcal{O}_{\mathbf{\Gamma}} = \rho^*\rho_*(\mathbf{E}|_{\mathbf{\Gamma}}(-b)) \otimes \mathbf{L}_\rho \xrightarrow{ev} \mathbf{E}|_{\mathbf{\Gamma}}(-b) \otimes \mathbf{L}_\rho.$$

Let

$$\mathcal{Z} = (s_{\mathbf{\Gamma}})_0$$

be the zero scheme of this section. By the base change for any $\mathbf{x} \in \mathbf{X}$ the scheme $Z_{\mathbf{x}} = \mathcal{Z} \cap S_{\mathbf{x}}$ is the zero set of the section $s_{\mathbf{\Gamma}}|_{S_{\mathbf{x}}} \in H^0(\mathbf{E}(-b)|_{S_{\mathbf{x}}})$, hence in view of (33) we have $2 = \operatorname{codim}_{S_{\mathbf{x}}} Z_{\mathbf{x}} = \operatorname{codim}_{\mathbf{\Gamma}} \mathcal{Z}$, so that

$$\operatorname{codim}_{\mathbb{P}^3 \times \mathbf{X}} \mathcal{Z} = \operatorname{codim}_{\mathbb{P}^3 \times \{\mathbf{x}\}} Z_{\mathbf{x}} = 3. \tag{34}$$

Use (33) and the relation

$$\mathbf{E}^\vee \simeq \mathbf{E}(-e) \tag{35}$$

and consider the composition $\boldsymbol{\varepsilon} : \ \mathbf{E} \overset{\otimes \mathcal{O}_{\mathbf{\Gamma}}}{\twoheadrightarrow} \mathbf{E}|_{\mathbf{\Gamma}} \overset{s_{\mathbf{\Gamma}}^\vee}{\twoheadrightarrow} \mathcal{I}_{\mathcal{Z},\mathbf{\Gamma}}(e-b) \otimes \mathbf{L}_\rho$. Setting

$$\mathbf{F} := \ker \boldsymbol{\varepsilon},$$

we obtain an exact triple

$$0 \to \mathbf{F} \to \mathbf{E} \xrightarrow{\boldsymbol{\varepsilon}} \mathcal{I}_{\mathcal{Z},\mathbf{\Gamma}}(e-b) \otimes \mathbf{L}_\rho \to 0. \tag{36}$$

Remark 3. *(i) Take any point $\mathbf{x} \in \mathbf{X}$ and restrict the last triple onto $\mathbb{P}^3 \times \{\mathbf{x}\}$. We will obtain the triple*

$$0 \to F \to E \xrightarrow{\varepsilon} \mathcal{I}_{Z,S}(e-b) \to 0, \tag{37}$$

where $E = \mathbf{E}|\mathbb{P}^3 \times \{\mathbf{x}\}$ is a generalized null correlation bundle, $S = S_{\mathbf{x}}$ is a surface from the linear series $\mathbf{P}$, $Z = Z_{\mathbf{x}}$, and $F = \ker \varepsilon$. This triple is an example of the so-called reduction step in the sense of Hartshorne [13, Prop. 9.1]*, hence F, and therefore also $\mathbf{F}$, is a reflexive sheaf.*
(ii) From (36) it follows that the sheaf $\mathbf{F}$ is determined by the sheaf $\mathbf{E}$ uniquely up to an isomorphism. Hence, since $\mathbf{E}$ inherits a $GL(N_m)$-linearization as a quotient sheaf over (an open part of) the Quot-scheme, the sheaf $\mathbf{F}$ also inherits a $GL(N_m)$-linearization.

Note also that, as Z is the zero-set of the section $s = \varepsilon^\vee$ of the bundle $E(-b)|_S$, a standard computation using the relations $c_1(E) = e$, $c_2(E) = c^2 - a^2 - b^2 - e(a-b-c)$ and $\deg S = c - b$ shows that

$$\begin{aligned} c_1(F) &= e + b - c, \\ c_2(F) &= c^2 - a^2 - bc - e(c-a-b), \\ c_3(F) &= l(Z) = c_2(E(-b)|_S) = (c-a)(c-b)(c+a-e). \end{aligned} \tag{38}$$

Since by construction

$$F = \mathbf{F}|_{\mathbb{P}^3 \times \{\mathbf{x}\}}, \quad \mathbf{x} \in \mathbf{X}, \tag{39}$$

and $\det \mathbf{E} \simeq \mathcal{O}_{\mathbb{P}^3 \times \mathbf{X}}(e)$, it follows from (38) that

$$\det \mathbf{F} \simeq \mathcal{O}_{\mathbb{P}^3 \times \mathbf{X}}(e + b - c). \tag{40}$$

As $\mathbf{F}$ is a rank 2 reflexive sheaf on $\mathbb{P}^3 \times \mathbf{X}$ by Remark 3(i), (40) implies

$$\mathbf{F}^\vee = \mathbf{F}(c - e - b). \tag{41}$$

Next, from (13) follows the relation $N_{\mathbf{\Gamma}/\mathbb{P}^3 \times \mathbf{X}} \simeq \mathcal{O}_{\mathbf{\Gamma}}(c - b, 1)$, and (34) implies

$$\mathcal{E}xt^1(\mathcal{I}_{\mathcal{Z},\mathbf{\Gamma}}(e-b) \otimes \mathbf{L}_\rho, \mathcal{O}_{\mathbb{P}^3 \times \mathbf{X}}) = \mathcal{E}xt^1(\mathbf{L}_\rho(e-b), \mathcal{O}_{\mathbb{P}^3 \times \mathbf{X}}) = \mathbf{L}_\rho^\vee(c - e, 1).$$

Thus, dualizing the triple (36) and using (35) and (41) we obtain an exact triple

$$0 \to \mathbf{E}(b - c) \to \mathbf{F} \xrightarrow{\boldsymbol{\psi}} \mathbf{L}_\rho^\vee(b, 1) \to 0. \tag{42}$$

Note that the restriction of (42) onto $\mathbb{P}^3 \times \{\mathbf{x}\}$ for any $\mathbf{x} \in \mathbf{X}$ yields an exact triple

$$0 \to E(b-c) \to F \xrightarrow{\psi} \mathcal{O}_S(b) \to 0, \quad E = \mathbf{E}|_{\mathbb{P}^3 \times \{\mathbf{x}\}}, \quad F = \mathbf{F}|_{\mathbb{P}^3 \times \{\mathbf{x}\}}. \tag{43}$$

4. Properties of reflexive sheaves of the family $\mathbf{F}$

In this section we study more closely reflexive sheaves F of the family $\mathbf{F}$ – see (39). Note that an arbitrary sheaf F is obtained from a generalized null correlation bundle $[E] \in N(e, a, b, c)_\tau$ by the triple (37).

Here by definition (see (1)-(2)) the sheaf E is the cohomology sheaf of the monad $0 \to \mathcal{O}_{\mathbb{P}^3}(-c + e) \xrightarrow{\lambda} \mathcal{H} \xrightarrow{\mu} \mathcal{O}_{\mathbb{P}^3}(c) \to 0$, where

$$\lambda = (f_2, -f_1, f_4, -f_3)^t, \quad \mu = (f_1, f_2, f_3, f_4), \quad \mu \circ \lambda = 0, \tag{44}$$

$$\begin{aligned} &f_1 \in H^0(\mathcal{O}_{\mathbb{P}^3}(c-a)), \quad f_2 \in H^0(\mathcal{O}_{\mathbb{P}^3}(c+a-e)), \\ &f_3 \in H^0(\mathcal{O}_{\mathbb{P}^3}(c-b)), \quad f_4 \in H^0(\mathcal{O}_{\mathbb{P}^3}(c+b-e)). \end{aligned} \tag{45}$$

Moreover, since μ is surjective, it follows that the subset $\cap_{i=1}^4 \{f_i(x) = 0\}$ of $\mathbb{P}^3$ is empty. In particular, polynomials f_1 and f_3 do not have common factors of positive degree. In other words, the surfaces

$$S := \{f_3(x) = 0\} \quad \text{and} \quad S' := \{f_1(x) = 0\} \tag{46}$$

intersect in a curve

$$C := S \cap S' \tag{47}$$

which is a complete intersection curve with a conormal sheaf $N^\vee_{C/\mathbb{P}^3} \simeq \mathcal{O}_C(a - c) \oplus \mathcal{O}_C(b - c)$. Besides, (44)–(47) yield:

$$\mathcal{O}_S(C) \simeq \mathcal{O}_{\mathbb{P}^3}(S')|_S \simeq \mathcal{O}_S(c - a). \tag{48}$$

Furthermore, by [23, Example 3.3], there is a well defined quotient sheaf $\mathcal{O}_C(a+b-e)$ of $N^\vee_{C/\mathbb{P}^3}$,

$$N^\vee_{C/\mathbb{P}^3} = \mathcal{O}_C(a-c) \oplus \mathcal{O}_C(b-c) \twoheadrightarrow \mathcal{O}_C(a+b-e), \tag{49}$$

which determines a double scheme structure $\tilde{C}$ on C with the following properties:

(i) the curve $\tilde{C}$ is a locally complete intersection curve satisfying the exact triple

$$0 \to \mathcal{O}_C(a+b-e) \to \mathcal{O}_{\tilde{C}} \to \mathcal{O}_C \to 0 \tag{50}$$

(ii) $\tilde{C}$ is the zero-scheme of some section of the sheaf $E(c-a-b)$:

$$\tilde{C} = (s)_0, \quad 0 \neq s \in H^0(E(c-a-b)). \tag{51}$$

Remark that (51) implies an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a+b-c) \xrightarrow{s} E \xrightarrow{\alpha} \mathcal{I}_{\tilde{C}}(c-a-b+e) \to 0. \tag{52}$$

This consideration leads to the following theorem.

Theorem 4. *For any $[E] \in N(e,a,b,c)_\tau$ the following statements hold.*
(i) There exists a surface $S \in \theta^{-1}\big([E]\big)$ such that the reflexive sheaf F defined by the pair $\big(S,[E]\big)$ as in Remark 3 satisfies the conditions

$$h^0(F(c-a-b)) = \begin{cases} 1, & \text{if } (e,a) \neq (0,0), \\ 2, & \text{if } e=a=0,\ b>0, \\ 3, & \text{if } e=a=b=0, \end{cases} \tag{53}$$

$$h^1(F(c-a-b)) = 0. \tag{54}$$

(ii) For any $0 \neq s \in H^0(F(c-a-b))$ there an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} F(c-a-b) \to \mathcal{I}_C(c-2a-b+e) \to 0, \tag{55}$$

where $C = (ev)_0$ is a complete intersection curve $C = S \cap S'$, where S' is certain surface of degree $c-a$ in $\mathbb{P}^3$.
(iii) The space $\mathrm{P}(H^0(F(c-a-b)))$ is naturally identified with a linear subspace of the linear series $|\mathcal{O}_S(C)| = |\mathcal{O}_S(c-a)|$.

Proof. Note first that, since $c-a-e>0$, it follows that, in (49), the quotient sheaf $\mathcal{O}_C(a+b-e)$ does not coincide with the direct summand $\mathcal{O}_C(b-c)$ of the conormal sheaf $N^\vee_{C/\mathbb{P}^3}$, so that the curve $\tilde{C}$ defined in (49)–(51) is not a subscheme of the surface S. Therefore, the sheaf $\kappa = \ker(\mathcal{O}_{\bar{C}} \twoheadrightarrow \mathcal{O}_C)$, where $\bar{C} := \tilde{C} \cap S$, has dimension at most zero:

$$0 \to \kappa \to \mathcal{O}_{\bar{C}} \to \mathcal{O}_C \to 0, \qquad \dim \kappa \leq 0.$$

This together with (48) implies an exact triple

$$0 \to \mathcal{I}_{Z,S}(e-b) \to \mathcal{O}_S(c-a-b+e) \to \mathcal{O}_{\bar{C}}(c-a-b+e) \to 0 \tag{56}$$

and a relation $\kappa \simeq \mathcal{O}_Z$ for some subscheme of S of dimension at most zero:

$$\dim Z \leq 0. \tag{57}$$

The exact triples

$$0 \to \mathcal{O}_{\mathbb{P}^3}(e-a) \xrightarrow{\cdot S} \mathcal{O}_{\mathbb{P}^3}(c-a-b+e) \to \mathcal{O}_S(c-a-b+e) \to 0,$$

$$0 \to \mathcal{I}_{\tilde{C}}(c-a-b+e) \to \mathcal{O}_{\mathbb{P}^3}(c-a-b+e) \to \mathcal{O}_{\tilde{C}}(c-a-b+e) \to 0$$

together with (56) extend to a commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{I}_C(e-a) \arrow[r] \arrow[d] & \mathcal{I}_{\tilde{C}}(c-a-b+e) \arrow[r, "\beta"] \arrow[d] & \mathcal{I}_{Z,S}(e-b) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(e-a) \arrow[r, "\cdot S"] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(c-a-b+e) \arrow[r] \arrow[d] & \mathcal{O}_S(c-a-b+e) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_C(e-a) \arrow[r] \arrow[d] & \mathcal{O}_{\tilde{C}}(c-a-b+e) \arrow[r] \arrow[d] & \mathcal{O}_{\bar{C}}(c-a-b+e) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}
$$

Now the composition of morphisms $\beta \circ \alpha$, where α is taken from (52) and β is defined in the above diagram, decomposes as

$$\beta \circ \alpha : \ E \overset{\otimes \mathcal{O}_S}{\twoheadrightarrow} E|_S \overset{\gamma}{\twoheadrightarrow} \mathcal{I}_{Z,S}(e-b). \tag{58}$$

for some epimorphism $\gamma : \ E|_S \overset{\gamma}{\twoheadrightarrow} \mathcal{I}_{Z,S}(e-b)$. Show that it coincides with the morphism ε in (37). Indeed, (57) implies the equalities $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z, \mathcal{O}_{\mathbb{P}^3}) = 0, \ i = 1, 2,$ which together with the exact sequence $\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z, \mathcal{O}_{\mathbb{P}^3}) \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z, \mathcal{O}_S) \to \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z, \mathcal{O}_{\mathbb{P}^3}(b-c))$ obtained from the exact triple $0 \to \mathcal{O}_{\mathbb{P}^3}(b-c) \xrightarrow{\cdot S} \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0$ yield

$$\mathcal{E}xt^1_{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_S) = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z, \mathcal{O}_S) = 0. \tag{59}$$

Applying the functor $\mathcal{E}xt^{\cdot}_{\mathcal{O}_S}(-, \mathcal{O}_S)$ to the exact triple $0 \to \mathcal{I}_{Z,S} \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$ and using (59) we obtain

$$\mathcal{I}^{\vee}_{Z,S} \simeq \mathcal{O}_S. \tag{60}$$

Dualizing the morphism γ in (58) and using (60) and the isomorphism $(E|_S)^{\vee} \simeq (E|_S)(-e)$, after twisting it by $\mathcal{O}_S(e-b)$ we obtain a morphism $\mathbf{s} = (\gamma)^{\vee}(e-b) : \mathcal{O}_S \to E(-b)|_S$ i.e. a section $0 \neq \mathbf{s} \in H^0(E(-b)|_S)$. This section is a subbundle morphism on $S \setminus Z$, hence in view of (57) it extends to the Koszul exact triple

$$0 \to \mathcal{O}_S \xrightarrow{\mathbf{s}} E(-b)|_S \xrightarrow{\mathbf{s}^{\vee} \otimes \wedge^2 \mathbf{s}} \mathcal{I}_{Z,S}(e-2b) \to 0.$$

This triple shows that $\gamma = \mathbf{s}^{\vee}$ and $Z = (\mathbf{s})_0$. Besides, the equality $h^0(E(-b)|_S) = 1$ follows from (17) and (18). Hence, $H^0(E(-b)|_S)$ is spanned by $\mathbf{s}$. We thus have proved that (58) coincides with the morphism ε in (37).

Thus, to finish the proof of Theorem, we have to show that the equalities (53)-(54) are true. For this, remark that, by (52), the composition $\beta \circ \alpha \circ s$ is zero. Hence the triple (52) and the upper horizontal triple of the above diagram extend to a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_{\mathbb{P}^3}(a+b-c) \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(a+b-c) \arrow[d, "s"] & & \\
0 \arrow[r] & F \arrow[r] \arrow[d] & E \arrow[r, "\beta\circ\alpha"] \arrow[d, "\alpha"] & \mathcal{I}_{Z,S}(e-b) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{I}_C(e-a) \arrow[r] \arrow[d] & \mathcal{I}_{\tilde{C}}(c-a-b+e) \arrow[r, "\beta"] \arrow[d] & \mathcal{I}_{Z,S}(e-b) \arrow[r] & 0 \\
 & 0 & 0, & &
\end{tikzcd}$$

in which the leftmost vertical triple twisted by $\mathcal{O}_{\mathbb{P}^3}(c-b-a)$ coincides with (55).

Next, since C is a complete intersection (47), it follows that the sheaf $\mathcal{I}_C(c-2a-b+e)$ has the following locally free $\mathcal{O}_{\mathbb{P}^3}$-resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(e-c-a) \to \mathcal{O}_{\mathbb{P}^3}(e-2a) \oplus \mathcal{O}_{\mathbb{P}^3}(e-a-b) \to \mathcal{I}_C(c-2a-b+e) \to 0. \tag{61}$$

Passing to sections in the triples (61) and (55) we obtain (53) and (54).

Now remark that, since $\mathcal{I}_{C,S}(c-2a-b+e) \simeq \mathcal{O}_S(e-a-b)$, the triple (55) extends to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r, "s"] \arrow[d] & F(c-a-b) \arrow[r] \arrow[d] & \mathcal{I}_C(c-2a-b+e) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{Z,S}(c-a) \arrow[r] \arrow[d] & F_S(c-a-b) \arrow[r] \arrow[d] & \mathcal{O}_S(e-a-b) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

where $\mathrm{Supp}Z = \mathrm{Sing}F$. Now the last statement (iii) of Theorem follows from the identification of the space $P(H^0(F(c-a-b)))$ with a subspace of a linear series $|\mathcal{O}_S(C)| = |\mathcal{O}_S(c-a)|$. Indeed, any section $s \in H^0(F(c-a-b)$ defines the left vertical morphism $\mathcal{O}_{\mathbb{P}^3} \to \mathcal{I}_{Z,S}(c-a) \hookrightarrow \mathcal{O}_S(c-a)$ in this diagram, this morphism being the composition $\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\otimes\mathcal{O}_S} \mathcal{O}_S \xrightarrow{\cdot C} \mathcal{O}_S(c-a)$. $\square$

Let $p : \mathbb{P}^3 \times \mathbf{X} \to \mathbf{X}$ be the projection, and set

$$\mathbf{W} := \mathbb{P}((p_*\mathbf{F}(c-a-b))^\vee) \xrightarrow{\boldsymbol{\pi}} \mathbf{X}. \tag{62}$$

Note that, by (53), (54) and the base change, $p_*\mathbf{F}(c-a-b)$ is a locally free sheaf of rank

$$\mathrm{rk}(p_*\mathbf{F}(c-a-b)) = h^0(F(c-a-b)), \tag{63}$$

where $h^0(F(c-a-b))$ is given in (53). Hence $\boldsymbol{\pi} : \mathbf{W} \to \mathbf{X}$ is a locally trivial projective bundle, and there is a canonical epimorphism of vector bundles on $\mathbf{W}$

$$\epsilon : \ \boldsymbol{\pi}^*((p_*\mathbf{F}(c-a-b))^\vee) \twoheadrightarrow \mathcal{O}_{\mathbf{W}}(1). \tag{64}$$

Consider the $PGL(N_m)$-action on $\mathbf{X}$ making the projection $\Phi : \mathbf{X} \to X$ a principal $PGL(N_m)$-bundle (see Theorem 1(ii)). It follows from the definition of $\mathbf{W}$ and Remark 3(ii) that this action lifts to a $PGL(N_m)$-action on $\mathbf{W}$ such that $\boldsymbol{\pi}$ is a $PGL(N_m)$-invariant morphism. We thus obtain a cartesian diagram of principal $PGL(N_m)$-bundles

$$\begin{tikzcd} \mathbf{W} \arrow[r, "\tilde{\Phi}"] \arrow[d, "\boldsymbol{\pi}"'] & W \arrow[d, "\pi"] \\ \mathbf{X} \arrow[r, "\Phi"] & X, \end{tikzcd} \tag{65}$$

where $W = \mathbf{W}//PGL(N_m)$ is a geometric factor, $\tilde{\Phi} : \mathbf{W} \to W$ is a canonical projection, and $\pi : W \to X$ is the induced morphism. This diagram and the base change $X_\tau \hookrightarrow X$ yield a cartesian diagram, in which $W_\tau = W \times_X X_\tau$, $\mathbf{W}_\tau = \mathbf{W} \times_X X_\tau$:

$$\begin{tikzcd} \mathbf{W}_\tau \arrow[r, "\tilde{\Phi}"] \arrow[d, "\boldsymbol{\pi}"'] & W_\tau \arrow[d, "\pi"] \\ \mathbf{X}_\tau \arrow[r, "\Phi"] & X_\tau. \end{tikzcd} \tag{66}$$

Let $\mathbf{W} \xleftarrow{\tilde{p}} \mathbb{P}^3 \times \mathbf{W} \xrightarrow{\tilde{\boldsymbol{\pi}}} \mathbb{P}^3 \times \mathbf{W}$ be the induced projections. The canonical epimorphism ϵ from (64) induces a morphism

$$\mathbf{s} :\ \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathbf{W}}(-1) \xrightarrow{\tilde{p}^*(\epsilon^\vee)} \tilde{p}^* \boldsymbol{\pi}^* p_* \mathbf{F}(c-a-b) = \tilde{\boldsymbol{\pi}}^* p^* p_* \mathbf{F}(c-a-b) \xrightarrow{\tilde{\boldsymbol{\pi}}^* ev} \mathbf{F}_{\mathbf{W}}(c-a-b). \tag{67}$$

(Note that here, $\mathbf{F}_{\mathbf{W}} = \tilde{\boldsymbol{\pi}}^* \mathbf{F}$, according to our agreement on notation.)

Theorem 5. *(i) The variety W is described as $W = \{(x, C) \mid x = (S, [E]) \in X$, and $C = (s)_0$ for some $0 \neq s \in H^0(F(c-a-b))$, where F is determined by the pair $x = (S, [E])$ via the reduction step (37)$\}$. In addition, the morphism $\pi : W \to X$ is given by $(x, C) \mapsto x$, and $\pi^{-1}(x) = P(H^0(F(c-a-b)))$.*

(ii) The vertical maps $\boldsymbol{\pi} : \mathbf{W}_\tau \to \mathbf{X}_\tau$ and $\pi : W_\tau \to X_\tau$ in (66) are locally trivial $\mathbb{P}^m$-fibrations, where

$$m = m(e, a, b, c) := h^0(F(c-a-b)) - 1, \tag{68}$$

and $h^0(F(c-a-b))$ is given by (53). Therefore,

$$\dim W_\tau = \dim X_\tau + m(e, a, b, c) \tag{69}$$

In particular, if $(e, a) \neq (0, 0)$, then there is an isomorphism

$$\pi : W_\tau \xrightarrow{\simeq} X_\tau. \tag{70}$$

(iii) There is an exact $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{W}_\tau}$-triple

$$0 \to \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathbf{W}_\tau}(-1) \xrightarrow{\mathbf{s}} \mathbf{F}_{\mathbf{W}_\tau}(c-a-b) \to \mathcal{I}_{\boldsymbol{\mathcal{C}}, \mathbb{P}^3 \times \mathbf{W}_\tau}(c-2a-b+e) \to 0, \tag{71}$$

where $\mathbf{s}$ is defined in (67) and $\boldsymbol{\mathcal{C}} = (\mathbf{s})_0$ is a codimension 2 subscheme of $\mathbb{P}^3 \times \mathbf{W}_\tau$. This triple being restricted onto $\mathbb{P}^3 \times \{\mathbf{w}\}$, for an arbitrary point $\mathbf{w} \in \mathbf{W}_\tau$, coincides with the triple (55) where $s = \mathbf{s} \otimes \mathbf{k}(\mathbf{w})$, $F = \mathbf{F}_{\mathbf{W}_\tau}|_{\mathbb{P}^3 \times \{\mathbf{w}\}}$, and $C = \boldsymbol{\mathcal{C}} \cap \mathbb{P}^3 \times \{\mathbf{w}\}$.

Proof. Statement (i) follows from the base change and the definition of $\mathbf{W}$ and W. In (ii), the local triviality of the fibration $\boldsymbol{\pi}$ is clear, and Theorem 4(iii) yields the local triviality of the fibration π. The isomorphism (70) is a corollary of (53). Statement (iii) follows from the definition of the morphism $\mathbf{s}$ in (67). □

Remark 6. *According to Theorem 1(ii) and Theorem 5(ii),* $\mathbf{W}_\tau \xrightarrow{\boldsymbol{\pi}} \mathbf{X}_\tau \xrightarrow{\boldsymbol{\theta}} Y_\tau \xrightarrow{\varphi} N(e,a,b,c)^{\rm nc}$ *is a composition of two projective bundles and of a principal bundle. Hence, since* $N(e,a,b,c)_\tau$ *is a reduced scheme by [9], it follows that* $\mathbf{W}_\tau$ *is a reduced scheme.*

(ii) Applying the functor $\tilde{\boldsymbol{\pi}}^*$ *to the epimorphism* $\boldsymbol{\psi}$ *in (42) we obtain an epimorphism* $\boldsymbol{\psi}_{\mathbf{W}}:\ \mathbf{F}_{\mathbf{W}} \twoheadrightarrow (\mathbf{L}_\rho^\vee(b,1))_{\mathbf{W}}$*, hence also an epimorphism*

$$\boldsymbol{\psi}_{\mathbf{W}_\tau}:\ \mathbf{F}_{\mathbf{W}_\tau} \twoheadrightarrow (\mathbf{L}_\rho^\vee(b,1))_{\mathbf{W}_\tau}. \tag{72}$$

5. Another family $\underline{\mathbf{F}}$ of reflexive sheaves

In this section we construct a new family $\underline{\mathbf{F}}$ of reflexive sheaves with Chern classes (38) and with the same properties as that of the sheaves of the family $\mathbf{F}$ – see Theorem 9. As above, we fix the numbers e,a,b,c which determine an Ein component $N(e,a,b,c)$ of $M(e,n)$, where $n=c^2-a^2-b^2-e(c-a-b)$. Consider the projective space $\mathbf{P}=|\mathcal{O}_{\mathbb{P}^3}(d)|$ (see (11)) and the set

$$\mathbf{R} := \{(S,C)\in \mathbf{P}\times \mathrm{Hilb}_{\mathbb{P}^3} \mid C\in|\mathcal{O}_S(c-a)|\}, \tag{73}$$

together with a natural projection $r:\mathbf{R}\to\mathbf{P},\ (S,C)\mapsto S$. The projection $r:\mathbf{R}\to\mathbf{P}$ is a locally trivial projective fibration with fibre

$$r^{-1}(S)=|\mathcal{O}_S(c-a)|,\quad \dim r^{-1}(S)=\binom{c-a+3}{3}-\binom{b-a+3}{3}-1.$$

Whence,

$$\dim\mathbf{R}=\dim\mathbf{P}+\dim r^{-1}(S)=\binom{c-b+3}{3}+\binom{c-a+3}{3}-\binom{b-a+3}{3}-2. \tag{74}$$

Take an arbitrary point $(S,C)\in\mathbf{R}$ and compute the number $h^0(\mathcal{O}_C(c+a-e))$. Since C is a complete intersection curve $C=S\cap S'$ (see (73)), we obtain the equality

$$\det N_{C/\mathbb{P}^3}=\mathcal{O}_C(2c-a-b) \tag{75}$$

and the exact triples

$$0\to\mathcal{I}_C(c+a-e)\to\mathcal{O}_{\mathbb{P}^3}(c+a-e)\to\mathcal{O}_C(c+a-e)\to 0,$$

$$0\to\mathcal{O}_{\mathbb{P}^3}(2a+b-c-e)\to\mathcal{O}_{\mathbb{P}^3}(a+b-e)\oplus\mathcal{O}_{\mathbb{P}^3}(2a-e)\to\mathcal{I}_C(c+a-e)\to 0.$$

These triples yield

$$h^0(\mathcal{O}_C(c+a-e))=\binom{c+a-e+3}{3}-\binom{a+b-e+3}{3}-\binom{2a-e+3}{3}+\delta(e,a,b,c), \tag{76}$$

where

$$\delta(e,a,b,c)=\begin{cases}\binom{2a+b-c-e+3}{3}, & \text{if } c\le 2a+b-e,\\ 0, & \text{if } c>2a+b-e.\end{cases} \tag{77}$$

For an arbitrary point $x = (S, C) \in \mathbf{R}$ consider the groups

$$\mathrm{Ext}^i(x) := \mathrm{Ext}^i(\mathcal{I}_C(c - 2a - b + e), \mathcal{O}_{\mathbb{P}^3}), \quad i = 0, 1.$$

From (75) it follows that

$$\mathcal{E}xt^1(\mathcal{I}_C(c - 2a - b + e), \mathcal{O}_{\mathbb{P}^3}) = \mathcal{E}xt^2(\mathcal{O}_C(c - 2a - b + e), \mathcal{O}_{\mathbb{P}^3}) = \det N_{C/\mathbb{P}^3}(2a + b - c - e) \simeq \mathcal{O}_C(c + a - e). \tag{78}$$

Since $h^i(\mathcal{H}om(\mathcal{I}_C(c - 2a - b + e), \mathcal{O}_{\mathbb{P}^3})) = h^i(\mathcal{O}_{\mathbb{P}^3}(2a + b - c - e)) = 0,\ i = 0, 1, 2,$ from (76), (78) and the spectral sequence of local-to-global Ext's we obtain

$$\dim \mathrm{Ext}^0(x) = h^0(\mathcal{O}_{\mathbb{P}^3}(2a + b - c - e)), \tag{79}$$

$$\dim \mathrm{Ext}^1(x) = h^0(\mathcal{O}_C(c + a - e)) = \binom{c + a - e + 3}{3} - \binom{a + b - e + 3}{3} - \binom{2a - e + 3}{3} + \delta(e, a, b, c). \tag{80}$$

Remark 7. *Consider the incidence subscheme $\Sigma \subset \mathbb{P}^3 \times \mathbf{R}$ defined as*

$$\Sigma := \{(x, S, C) \in \mathbb{P}^3 \times \mathbf{R} \mid x \in C\}.$$

In view of (79)–(80) the dimensions of the groups $\mathrm{Ext}^1(x)$ do not depend on the point $x = (S, C) \in \mathbf{R}$, so that the sheaves

$$\mathcal{E}_i := \mathcal{E}xt^i_{p_2}(\mathcal{I}_{\Sigma, \mathbb{P}^3 \times \mathbf{R}}(c - 2a - b + e), \mathcal{O}_{\mathbb{P}^3 \times \mathbf{R}}), \quad i = 0, 1,$$

by [3] *commute with the base change in the sense of* [20, Remark 1.5]*. In particular, the sheaf $\mathcal{E}_1$ is a locally free $\mathcal{O}_{\mathbf{R}}$-sheaf of rank*

$$\mathrm{rk}\, \mathcal{E}_1 = h^0(\mathcal{O}_C(c + a - e))$$

and for any $x = (S, C) \in \mathbf{R}$ one has the base change isomorphism $\mathcal{E}_1 \otimes \mathbf{k}(x) \overset{\simeq}{\to} \mathrm{Ext}^1(x)$.

Consider the rational variety

$$\mathbf{T} := \mathbb{P}(\mathcal{E}_1^\vee) \tag{81}$$

with its structure morphism $\mu : \mathbf{T} \to \mathbf{R}$ which is a locally trivial projective fibration with fibre of dimension $h^0(\mathcal{O}_C(c + a - e)) - 1$. We thus obtain from (74) and (80) the formula for the dimension of $\mathbf{T}$:

$$\dim \mathbf{T} = \binom{c - b + 3}{3} + \binom{c - a + 3}{3} - \binom{b - a + 3}{3} + \binom{c + a - e + 3}{3} - \binom{a + b - e + 3}{3} - \binom{2a - e + 3}{3} + \delta(e, a, b, c) - 3. \tag{82}$$

By construction, $\mathbf{T}$ is set-theoretically described as

$$\mathbf{T} = \{t = (x, \xi) \mid x = (S, C) \in \mathbf{R},\ \xi \in P(\mathrm{Ext}^1(x))\},$$

and each point $t = (S, C, \xi) \in \mathbf{T}$ defines a non-trivial (class of proportionality of an) extension of $\mathcal{O}_{\mathbb{P}^3}$-sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a + b - c) \to F_t \to \mathcal{I}_C(e - a) \to 0. \tag{83}$$

Remark 8. *This is the well-known Serre construction – cf.* [12], [13], [22]*. In particular, F_t is a reflexive sheaf with Chern classes given by (38).*

Globalizing over $\mathbf{T}$ the triple (83) we obtain the following result.

Theorem 9. *On $\mathbb{P}^3 \times \mathbf{T}$ there is a sheaf $\underline{\mathbf{F}}$ defined as the universal extension sheaf*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a+b-c) \boxtimes \mathcal{O}_{\mathbf{T}}(1) \to \underline{\mathbf{F}} \to \mathcal{I}_{\mathbf{\Sigma},\mathbb{P}^3\times\mathbf{T}}(e-a), \to 0, \quad \mathbf{\Sigma} = \Sigma \times_{\mathbf{R}} \mathbf{T}. \tag{84}$$

The sheaf $\underline{\mathbf{F}}$ is a family of reflexive sheaves (83) on $\mathbb{P}^3$ with the base $\mathbf{T}$.

6. Properties of sheaves of the family $\underline{\mathbf{F}}$. A family of generalized null correlation bundles $\underline{\mathbf{E}}$ associated to $\underline{\mathbf{F}}$.

In this section we study closely reflexive sheaves of the family $\underline{\mathbf{F}}$. Given a point $t = (S, C, \xi) \in \mathbf{T}$, the sheaf $F = F_t$ of the family $\underline{\mathbf{F}}$ over the point t is defined as an extension (83). We will show that to the sheaf F_t there corresponds a family of generalized null correlation bundles E such that the reflexive sheaf F_t is recovered from each bundle E of this family uniquely by the reduction step (37) (see Remark 3). Relativizing these families over $\mathbf{T}$, we then obtain the family $\underline{\mathbf{E}}$ of generalized null correlation bundles with base $\mathbf{T}$ (see (102)).

A hint for producing a family of generalized null correlation bundles E from the reflexive sheaf F is given by the triple (43). In this triple a generalized null correlation bundle E is obtained from F by the "inverse" reduction step (cf. Remark 3(i)) as a kernel of an epimorphism $F \twoheadrightarrow \mathcal{O}_S(b)$. In fact, this construction leads to the following theorem.

Theorem 10. *Consider a subset T of $\mathbf{T}$ consisting of those points $t = (S, C, \xi) \in \mathbf{T}$ for which there exists an epimorphism $\psi : F_t \twoheadrightarrow \mathcal{O}_S(b)$, with F_t given by an extension (83), such that $E = \ker \psi$ is locally free. Then T is nonempty and E is a generalized null correlation bundle, $[E] \in N(e, a, b, c)^{\mathrm{nc}}$.*

Proof. Clearly, T is nonempty: it is enough to take a point $\mathbf{x} = (S, y) \in \mathbf{X}$ and set $[E] = \varphi(y)$, so that the data (F, C, ξ) are determined by the pair $\big(S, [E]\big)$ as in Theorem 4; in particular, ξ is defined as the extension class of the triple (55). Then for the point $t = (S, C, \xi)$ by (55) the sheaf $F_t = F$ coincides with the sheaf $F_{\mathbf{x}}$ in the triple (43), and this triple showes that $t \in T$.

Now take $t = (S, C, \xi) \in T$ and consider the triple (43) twisted by $\mathcal{O}_{\mathbb{P}^3}(c-b+m)$:

$$0 \to E(m) \to F(c-b+m) \xrightarrow{\psi} \mathcal{O}_S(c+m) \to 0, \quad m \in \mathbb{Z}. \tag{85}$$

Respectively, the triple (83) twisted by $\mathcal{O}_{\mathbb{P}^3}(c-b+m)$ yields

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a+m) \xrightarrow{i} F(c-b+m) \xrightarrow{\theta} \mathcal{I}_C(m+c-a-b+e) \to 0. \tag{86}$$

Besides we have a standard exact triple

$$\begin{aligned} &0 \to \mathcal{O}_{\mathbb{P}^3}(-c+e+m) \to \mathcal{O}_{\mathbb{P}^3}(-b+e+m) \oplus \mathcal{O}_{\mathbb{P}^3}(-a+e+m) \to \\ &\mathcal{I}_C(m+c-a-b+e) \to 0. \end{aligned} \tag{87}$$

Substituting $m \le b-c$ into (86) and (87) and using the inequalities $c > a+b,\ e \le 0$ we obtain $h^0(F(m)) \le 0,\ m \le 0$. Besides, since $Z \ne \emptyset$ and $e-b \le 0$, $h^0(\mathcal{I}_{Z,S}(e-b+m)) = 0,\ m \le 0$. Hence the triple (37) twisted by $\mathcal{O}_{\mathbb{P}^3}(m)$ implies

$$h^0(E(m)) = 0, \quad m \le 0. \tag{88}$$

In particular, $h^0(E) = 0$, i. e. E is stable.

Now consider the triples (85) and (86) and the morphisms ψ and i therein. If the composition $\psi \circ i$ is zero then i becomes a section of E which contradicts to (88). Hence, the composition $\psi \circ i$ factors as

$$\psi \circ i : \ \mathcal{O}_{\mathbb{P}^3}(a+m) \overset{\psi'}{\twoheadrightarrow} \mathcal{O}_S(a+m) \xrightarrow{i'} \mathcal{O}_S(c+m).$$

Denote $U = \mathbb{P}^3 \setminus \mathrm{Sing} F$. Since by (86) the morphism $i|_U : \mathcal{O}_U(a+m) \to F(c-b+m)|_U$ is a section of the locally free sheaf $F(c-b+m)|_U$ vanishing at the curve $C \cap U$, it follows that $i' : \mathcal{O}_S(a+m) \to \mathcal{O}_S(c+m)$ is a multiplication by the equation of the divisor C in S. Hence $\operatorname{coker} i' = \mathcal{O}_C(c+m)$ and we obtain a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(a+b-c+m) \arrow[r, "\cdot S"] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(a+m) \arrow[r, "\psi'"] \arrow[d, "i"] & \mathcal{O}_S(a+m) \arrow[r] \arrow[d, "i'"] & 0 \\
0 \arrow[r] & E(m) \arrow[r] \arrow[d] & F(c-b+m) \arrow[r, "\psi"] \arrow[d, "\theta"] & \mathcal{O}_S(c+m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{\tilde{C}}(c-a-b+e+m) \arrow[r] \arrow[d] & \mathcal{I}_C(c-a-b+e+m) \arrow[r, "\psi''"] \arrow[d] & \mathcal{O}_C(c+m) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

in which ψ'' is induced by the morphisms ψ and ψ', and $\tilde{C}$ is a certain double scheme structure on the curve C. Consider the bottom horizontal and left vertical triples in this diagram:

$$0 \to \mathcal{I}_{\tilde{C}}(c-a-b+e+m) \to \mathcal{I}_C(c-a-b+e+m) \xrightarrow{\psi''} \mathcal{O}_C(c+m) \to 0, \tag{89}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a+b-c+m) \to E(m) \to \mathcal{I}_{\tilde{C}}(m+c-a-b+e) \to 0, \tag{90}$$

The triple (89) by [23, Example 3.3] shows that the cohomology $H^0_*(\mathcal{O}_{\mathbb{P}^3})$-module $H^1_*(\mathcal{I}_{\tilde{C}})$ as a graded module over the graded ring $H^0_*(\mathcal{O}_{\mathbb{P}^3}) \simeq \mathbf{k}[x_0, x_1, x_2, x_3]$ has one generator. Hence the triple (90) implies that the cohomology $H^0_*(\mathcal{O}_{\mathbb{P}^3})$-module $H^1_*(E)$ also has one generator. This together with [9, Prop. 1.3] shows that E is a generalized null correlation bundle. □

Theorem 11. *In conditions and notation of Theorem 10, let* $t = (S, C, \xi) \in T$ *and* $F = F_t$. *Then the following statements hold.*

(i) $\dim \mathrm{Hom}(F, \mathcal{O}_S(b)) = \binom{b+c-e+3}{3} - \binom{2b-e+3}{3} + 1$.

(ii) the set $P(\mathrm{Hom}(F, \mathcal{O}_S(b)))^* = \{\mathbf{k}\psi \in P(\mathrm{Hom}(F, \mathcal{O}_S(b))) \mid \psi : \ F \to \mathcal{O}_S(b)$ is surjective and $\ker \psi$ is locally free$\}$ *is nonempty, hence dense open in* $P(\mathrm{Hom}(F, \mathcal{O}_S(b)))$.

(iii) For any point $\mathbf{k}\psi \in P(\mathrm{Hom}(F, \mathcal{O}_S(b)))^*$, *the sheaf*

$$E_\psi := (\ker(F \twoheadrightarrow \mathcal{O}_S(b)))(c-b)$$

is a generalized null correlation bundle, $[E_\psi] \in N(e, a, b, c)^{\mathrm{nc}}$.

Proof. (i) Note that the natural epimorphism $\rho : \mathcal{I}_C(e-a) \twoheadrightarrow \mathcal{I}_{C,S}(e-a) \simeq \mathcal{O}_S(-C)(e-a) \simeq \mathcal{O}_S(e-c)$ composed with the epimorphism $\theta : F \twoheadrightarrow \mathcal{I}_C(e-a)$ from the triple (86) for $m = b-c$ gives an epimorphism

$$\rho \circ \theta : F \twoheadrightarrow \mathcal{O}_S(e-c).$$

Restricting it onto S yields an exact triple: $0 \to \mathcal{I}_{Z,S}(b) \to F|_S \to \mathcal{O}_S(e-c) \to 0$. This triple together with the triple $0 \to \mathcal{I}_{Z,S}(b) \to \mathcal{O}_S(b) \to \mathcal{O}_Z \to 0$ by push-out yield two exact triples:

$$0 \to \mathcal{O}_S(b) \xrightarrow{u} (F|_S)^{\vee\vee} \to \mathcal{O}_S(e-c) \to 0, \tag{91}$$

$$0 \to F|_S \to (F|_S)^{\vee\vee} \to \mathcal{O}_Z \to 0, \tag{92}$$

where $(F|_S)^{\vee\vee} = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{H}om_{\mathcal{O}_S}(F|_S, \mathcal{O}_S), \mathcal{O}_S)$. On the other hand, restricting onto S the epimorphism $\psi : F \twoheadrightarrow \mathcal{O}_S(b)$ from the triple (85) with $m = b-c$ we obtain an exact triple $0 \to \mathcal{I}_{Z,S}(e-c) \to F|_S \to \mathcal{O}_S(b) \to 0$. As above, by push-out this triple yields an exact triple

$$0 \to \mathcal{O}_S(e-c) \to (F|_S)^{\vee\vee} \xrightarrow{v} \mathcal{O}_S(b) \to 0. \tag{93}$$

Now consider the morphisms u and v in the triples (91) and (93). If their composition $v \circ u : \mathcal{O}_S(b) \to \mathcal{O}_S(b)$ iz zero, this implies that there exists a nonzero morphism $\mathcal{O}_S(b) \to \mathcal{O}_S(e-c)$, contrary to the condition that $e-c-b<0$. Hence $v \circ u : \mathcal{O}_S(b) \to \mathcal{O}_S(b)$ is an isomorphism. This means that both triples (91) and (93) split. Thus

$$(F|_S)^{\vee\vee} \simeq \mathcal{O}_S(b) \oplus \mathcal{O}_S(e-c). \tag{94}$$

Remark that, since $\dim Z = 0$, it follows that $\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_S(b)) = \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_S(b)) = 0$, the triple (92) yields the isomorphisms $\mathrm{Hom}(F, \mathcal{O}_S(b)) \simeq \mathrm{Hom}(F|_S, \mathcal{O}_S(b)) \simeq \mathrm{Hom}(\mathcal{O}_{(F|_S)^{\vee\vee}}, \mathcal{O}_S(b))$. This together with (94) shows that

$$\mathrm{Hom}(F, \mathcal{O}_S(b)) = H^0(\mathcal{O}_S) \oplus H^0(\mathcal{O}_S(b+c-e)).$$

Whence, (i) follows.

Statements (ii) and (iii) are immediate consequences of Theorem 10. □

Now return to the family $\underline{\mathbf{F}}$ of reflexive sheaves on $\mathbb{P}^3 \times \mathbf{T}$, and recall that $\mathbf{T}$ is a rational variety (see (81)) with the projection $r \circ \mu : \mathbf{T} \to \mathbf{P}$. Let $\underline{\Gamma} := \Gamma \times_{\mathbf{P}} \mathbf{T} \subset \mathbb{P}^3 \times \mathbf{T}$ be the family of surfaces in $\mathbb{P}^3$ with base $\mathbf{T}$, together with the natural projection $\underline{\Gamma} \to \mathbf{T}$, the fibre of which over an arbitrary point $t = (S, C, \xi) \in \mathbf{T}$ is a surface S. Consider an $\mathcal{O}_{\mathbf{T}}$-sheaf

$$\mathcal{A} = \mathcal{E}xt^0_{\mathrm{pr}_2}(\underline{\mathbf{F}}, \mathcal{O}_{\underline{\Gamma}}(b)),$$

where $\mathrm{pr}_2 : \mathbb{P}^3 \times \mathbf{T} \to \mathbf{T}$ is the projection. The base change and Theorem 11(i) show that

$$\mathcal{A} \otimes \mathbf{k}(t) = \mathrm{Hom}(F_t, \mathcal{O}_S(b)), \quad F_t = \underline{\mathbf{F}}|_{\mathbb{P}^3 \times \{t\}}, \tag{95}$$

and $\mathcal{A}$ is a locally free $\mathcal{O}_{\mathbf{T}}$-sheaf of rank

$$\mathrm{rk}\, \mathcal{A} = \binom{b+c-e+3}{3} - \binom{2b-e+3}{3} + 1. \tag{96}$$

Since $\mathbf{T}$ is a rational variety, the scheme

$$\mathbf{B} := \mathbb{P}(\mathcal{A}^\vee)$$

is a rational variety and its structure morphism $\lambda : \mathbf{B} \to \mathbf{T}$ is a locally trivial projective fibration with fibre of dimension $\mathrm{rk}\,\mathcal{A} - 1$. Thus by (82) and (96):

(97)

$$\dim \mathbf{B} = \binom{c-b+3}{3} + \binom{c-a+3}{3} - \binom{b-a+3}{3} + \binom{c+a-e+3}{3} - \binom{a+b-e+3}{3} - \binom{2a-e+3}{3} + \binom{b+c-e+3}{3} - \binom{2b-e+3}{3} + \delta(e,a,b,c) - 3.$$

In view of (95) we have the set-theoretic description of $\mathbf{B}$ as:

$$\mathbf{B} = \{(t, \mathbf{k}\psi) \mid t = (S, C, \xi) \in \mathbf{T},\ \mathbf{k}\psi \in P(\mathrm{Hom}(F_t, \mathcal{O}_S(b)))\}.$$

Let $\mathcal{O}_{\mathbf{B}}(1)$ be the Grothendieck sheaf on $\mathbf{B}$, $\mathbf{A} := \lambda^*\mathcal{A}$, and let

$$j :\ \mathcal{O}_{\mathbf{B}} \to \mathbf{A} \otimes \mathcal{O}_{\mathbf{B}}(1)$$

be the tautological subbundle morphism. Next, set

$$\Gamma_{\mathbf{B}} := \underline{\Gamma} \times_{\mathbf{T}} \mathbf{B}$$

and let

$$\mathrm{can} : \underline{\mathbf{F}}_{\mathbf{B}} \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{A} \to \mathcal{O}_{\Gamma_{\mathbf{B}}}(b)$$

be the canonical (evaluation) morphism. Consider the universal morphism

$$\mathbf{\Psi} :\ \underline{\mathbf{F}}_{\mathbf{B}} \to \mathcal{O}_{\Gamma_{\mathbf{B}}}(b) \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathbf{B}}(1) \tag{98}$$

defined as the composition

$$\mathbf{\Psi} :\ \underline{\mathbf{F}}_{\mathbf{B}} \xrightarrow{\mathrm{id}\otimes j} \underline{\mathbf{F}}_{\mathbf{B}} \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes (\mathbf{A} \otimes \mathcal{O}_{\mathbf{B}}(1)) \xrightarrow{\mathrm{can}\otimes\mathrm{id}} \mathcal{O}_{\Gamma_{\mathbf{B}}}(b) \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathbf{B}}(1).$$

Now Theorem 11(ii) says that the set

(99)

$$\begin{aligned} B &:= \{\mathbf{b} = (t, \mathbf{k}\psi) \in \mathbf{B} \mid t = (S, C, \xi) \in \mathbf{T},\ \mathbf{k}\psi \in P(\mathrm{Hom}(\underline{\mathbf{F}} \otimes \mathbf{k}(t), \mathcal{O}_S(b)))^*\} \\ &= \{\mathbf{b} = (t, \mathbf{k}\psi) \in \mathbf{B} \mid t = (S, C, \xi) \in \mathbf{T},\ \mathbf{\Psi} \otimes \mathbf{k}(\mathbf{b}) :\ \underline{\mathbf{F}} \otimes \mathbf{k}(t) \to \mathcal{O}_S(b) \text{ is} \\ &\quad \text{surjective and } \ker(\mathbf{\Psi} \otimes \mathbf{k}(\mathbf{b})) \text{ is locally free}\}. \end{aligned}$$

is a dense open subset of $\mathbf{B}$, i. e. is a rational variety of dimension given by formula (97). In addition, comparing (97) with (3) we obtain

$$\dim B = \dim N(e,a,b,c) + \delta(e,a,b,c) + t(e,a,b). \tag{100}$$

From (29) and (100) it follows that

$$\dim X_\tau - \dim B = \tau - \delta(e,a,b,c) - t(e,a,b). \tag{101}$$

Set

$$\mathcal{O}_B(1) := \mathcal{O}_{\mathbf{B}}(1)|_B, \quad \Gamma_B := \Gamma_{\mathbf{B}} \times_{\mathbf{B}} B, \quad \Psi := \mathbf{\Psi}|_{\mathbb{P}^3 \times B}.$$

By the definition of B the morphism Ψ is surjective, and we set

$$\underline{\mathbf{E}} := (\ker \Psi)(c-b). \tag{102}$$

Thus the following triple is exact:

$$0 \to \underline{\mathbf{E}}(b-c) \to \underline{\mathbf{F}} \xrightarrow{\Psi} \mathcal{O}_{\Gamma_B}(b) \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_B(1) \to 0. \tag{103}$$

Note that, for an arbitrary point $y = (S, C, \xi, \mathbf{k}\psi) \in B$, we have by (103) and Theorem 11(iii)

$$\underline{\mathbf{E}}|_{\mathbb{P}^3 \times \{\mathbf{b}\}} \simeq E_\psi, \quad [E_\psi] \in N(e,a,b,c)^{\mathrm{nc}}. \tag{104}$$

Therefore there is a well-defined morphism

$$q:\ B \to N(e,a,b,c)^{\mathrm{nc}}, \quad \mathbf{b} \mapsto [\ker \Psi|_{\mathbb{P}^3 \times \{\mathbf{b}\}}]. \tag{105}$$

7. Relation between $\underline{\mathbf{E}}$ and $\mathbf{E}$. Proof of the main result

We are now ready to prove the main result of the paper, Theorem 12, which follows from the relation between the families $\underline{\mathbf{E}}$ and $\mathbf{E}$. (The exact form of this relation is the isomorphism (108).)

Theorem 12. *(i) There is a dense open subset B_τ of B and an isomorphism*

$$f : W_\tau \xrightarrow{\simeq} B_\tau. \tag{106}$$

Hence $N(e,a,b,c)$ is at least stably rational. Furthermore, on $\mathbb{P}^3 \times W_\tau$ there exists a family of generalized null correlation bundles $\underline{\mathbf{E}}_{W_\tau}$ for which the corresponding modular morphism $W_\tau \to N(e,a,b,c)_\tau$ is surjective.

(ii) Assume $(e,a) \neq (0,0)$, $c > 2a+b-e$, and $b > a$. Then $\tau = 0$, $N(e,a,b,c)$ is a rational variety, and its open dense subset $N(e,a,b,c)_0 := N(e,a,b,c)_{\tau=0} \simeq X_0$ is a fine moduli space, i.e. the family $\underline{\mathbf{E}}_{W_0}$ is a universal family of generalized null correlation bundles on $\mathbb{P}^3 \times N(e,a,b,c)_0$.

Proof. (i) We first construct a $PGL(N_m)$-equivariant morphism

$$\mathbf{f}_B :\ \mathbf{W}_\tau \to B.$$

For this, consider the triple (71) and remark that the subscheme $\boldsymbol{\mathcal{C}}$ in this triple is a family with base $\mathbf{W}_\tau$ of complete intersection curves from $\mathbf{R}$ (see (73)). Thus by the universality of the Hilbert scheme there exists a morphism $\mathbf{f}_0 :\ \mathbf{W}_\tau \to \mathbf{R}$ such that $\boldsymbol{\mathcal{C}} = \Sigma \times_{\mathbf{R}} \mathbf{W}_\tau$. Hence,

$$\mathcal{I}_{\boldsymbol{\mathcal{C}},\mathbb{P}^3\times\mathbf{W}_\tau}(c-2a-b+e) \simeq (\mathrm{id}_{\mathbb{P}^3} \times \mathbf{f}_0)^* \mathcal{I}_{\Sigma,\mathbb{P}^3\times\mathbf{R}}(c-2a-b+e).$$

Now consider the triples (71) and (84) as families of extensions of $\mathcal{O}_{\mathbb{P}^3}$-sheaves with bases $\mathbf{W}_\tau$ and $\mathbf{T}$, respectively. Use Remark 7 and the fact that $\mathbf{W}_\tau$ is reduced (see Remark 6) to apply the universal property of the scheme $\mathbf{T}$ (see [20, Cor. 4.4]). By this universal property there is a uniquely defined morphism $\mathbf{f}_1 :\ \mathbf{W}_\tau \to \mathbf{T}$ such that $\mathbf{f}_0 = \mu \circ \mathbf{f}_1$ and such that the triple (71) is obtained by applying the functor $(\mathrm{id}_{\mathbb{P}^3} \times \mathbf{f}_1)^*$ to the triple (84). In particular,

$$\mathbf{F}_{\mathbf{W}_\tau} \simeq \underline{\mathbf{F}}_{\mathbf{W}_\tau}. \tag{107}$$

By (107) and the universal property of the scheme $\mathbf{B}$ over $\mathbf{T}$ there is a unique morphism $\mathbf{f}_B :\ \mathbf{W}_\tau \to \mathbf{B}$ such that $\mathbf{f}_1 = \mu \circ \mathbf{f}_B$ and such that the epimorhism $\boldsymbol{\psi}_{\mathbf{W}_\tau} :\ \mathbf{F}_{\mathbf{W}_\tau} \twoheadrightarrow (\mathbf{L}_\rho^\vee(b,1))_{\mathbf{W}_\tau}$ in (72) is obtained from the universal epimorphism $\boldsymbol{\Psi}$ in (98) by aplying the functor $(\mathrm{id}_{\mathbb{P}^3} \times \mathbf{f}_B)^*$. Since $\boldsymbol{\psi}_{\mathbf{W}_\tau}$ is surjective and $\mathbf{E}_{\mathbf{W}_\tau} = \ker \boldsymbol{\psi}_{\mathbf{W}_\tau}$ is a family of locally free sheaves on $\mathbb{P}^3$, it follows that

$$\mathbf{f}_B(\mathbf{W}_\tau) \subset B.$$

Moreover, (42), (103) and (107) yield

$$\mathbf{E}_{\mathbf{W}_\tau} \simeq \underline{\mathbf{E}}_{\mathbf{W}_\tau}. \tag{108}$$

Furthermore, as the $PGL(N_m)$-principal bundle $\tilde{\boldsymbol{\Phi}}:\ \mathbf{W}_\tau \to W_\tau$ is a categorical factor, and the morphism $\mathbf{f}_B : \mathbf{W}_\tau \to B$ by construction is $PGL(N_m)$-invariant, it follows that there exists a morphism

$$f:\ W_\tau \to B$$

such that $\mathbf{f}_B = f \circ \tilde{\boldsymbol{\Phi}}$.

We have to show that f is an isomorphism of W_τ onto a certain dense open subset B_τ of B. For this, remark that the sheaf $\mathbf{H} = pr_{2*}\underline{\mathbf{E}}(m)$, where $pr_2 : \mathbb{P}^3 \times B \to B$ is the projection, is a locally free $\mathcal{O}_B$-sheaf of rank N_m, and the evaluation morphism $ev : pr_2^*\mathbf{H} \to \underline{\mathbf{E}}(m)$ is surjective (see Section 2). Now consider a locally free $\mathcal{O}_B$-sheaf $\mathcal{K} = \mathcal{H}om(\mathbf{k}^{N_m} \otimes \mathcal{O}_B, \mathbf{H})$ and the corresponding scheme $\mathbf{V}(\mathcal{K}^\vee) = \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{K}^\vee)$. There is an open dense subset $\mathbf{V} = \mathrm{Isom}(\mathbf{k}^{N_m} \otimes \mathcal{O}_B, \mathbf{H})$ of $\mathbf{V}(\mathcal{K}^\vee)$ consisting of (fibrewise) invertible homomorphisms from $\mathbf{k}^{N_m} \otimes \mathcal{O}_B$ to $\mathbf{H}$, together with the projection $v:\ \mathbf{V} \to B$ and the canonical isomorphism can : $\mathbf{k}^{N_m} \otimes \mathcal{O}_{\mathbb{P}^3 \times \mathbf{V}} \xrightarrow{\sim} (\mathrm{id}_{\mathbb{P}^3} \times v)^*\mathbf{H}$. This isomorphism, being twisted by $\mathcal{O}_{\mathbb{P}^3}(-m) \boxtimes \mathcal{O}_{\mathbf{V}}$, together with the above epimorphism ev yields an epimorphism

$$\mathcal{H} \boxtimes \mathcal{O}_{\mathbf{V}} \xrightarrow{\mathrm{can}} (\mathrm{id}_{\mathbb{P}^3} \times v)^*\mathbf{H}(-m) \overset{ev}{\twoheadrightarrow} \underline{\mathbf{E}}_{\mathbf{V}},$$

where $\mathcal{H} = \mathbf{k}^{N_m} \otimes \mathcal{O}_{\mathbb{P}^3}(-m)$ (see Section 2). Thus, by the universal property of the Quot-scheme $Q = \mathrm{Quot}_{\mathbb{P}^3}(\mathcal{H}, P)$ introduced in Section 2, there exists a uniquely defined morphism $\tilde{\mathbf{q}}:\ \mathbf{V} \to Q$ such that

$$\underline{\mathbf{E}}_{\mathbf{V}} \simeq \mathbb{E}_{\mathbf{V}}, \tag{109}$$

where $\mathbb{E}$ is the universal quotient sheaf on $\mathbb{P}^3 \times Q$. Note that, by (104)-(105),

$$\varphi \circ \tilde{\mathbf{q}} = q \circ v, \tag{110}$$

where $\varphi : Y \to N(e,a,b,c)^{\mathrm{nc}}$ is a principal $PGL(N_m)$-bundle (8). In particular,

$$\tilde{\mathbf{q}}(\mathbf{V}) \subset Y.$$

Next, the group $\mathbf{k}^*$ naturally acts on $\mathbf{V}$ by homotheties, so that $\mathbf{Y} := \mathbf{V}//\mathbf{k}^*$ is a categorical quotient. Therefore, v as a principal $GL(N_m)$-bundle decomposes as

$$v = \boldsymbol{\Phi} \circ \nu,$$

where $\nu : \mathbf{V} \to \mathbf{Y}$ is a principal $\mathbf{k}^*$-bundle and $\boldsymbol{\Phi}:\ \mathbf{Y} \to B$ is a principal $PGL(N_m)$-bundle. Since the morphism $\tilde{\mathbf{q}}$ is $\mathbf{k}^*$-invariant it decomposes as

$$\tilde{\mathbf{q}} = \mathbf{q} \circ \nu,$$

where $\mathbf{q} : \mathbf{Y} \to Y$ is a $PGL(N_m)$-equivarint morphism. Thus, as the principal $PGL(N_m)$-bundles $\boldsymbol{\Phi} :\ \mathbf{Y} \to B$ and $\varphi :\ Y \to N(e,a,b,c)^{\mathrm{nc}}$ are categorical quotients, there exists a morphism $q:\ B \to N(e,a,b,c)^{\mathrm{nc}}$ making the diagram

$$\begin{tikzcd} Y \arrow[d, "\varphi"'] & \mathbf{Y} \arrow[l, "\mathbf{q}"'] \arrow[d, "\boldsymbol{\Phi}"] \\ N(e,a,b,c)^{\mathrm{nc}} & B \arrow[l, "q"'] \end{tikzcd} \tag{111}$$

cartesian. In addition, similar to (109) we see that the sheaf $\underline{\mathbf{E}}_{\mathbf{Y}}$ satisfies the relation

$$\underline{\mathbf{E}}_{\mathbf{Y}} \simeq \mathbb{E}_{\mathbf{Y}}. \tag{112}$$

Note that $\mathbf{Y}$ is irreducible, since B is irreducible.

We now construct the morphism

$$\mathbf{f}:\ \mathbf{W}_\tau \to \mathbf{Y}$$

making the square in the diagram

$$\begin{tikzcd} \mathbf{W}_\tau \arrow[r, "\mathbf{f}"] \arrow[d, "\tilde{\Phi}"'] \arrow[dr, "\mathbf{f}_B"] & \mathbf{Y} \arrow[d, "\mathbf{\Phi}"] \\ W_\tau \arrow[r, "f"] & B \end{tikzcd} \tag{113}$$

cartesian. For this, note that by the universal property of the Quot-scheme Q the family of generalized null correlation bundles $\mathbf{E}_{\mathbf{W}_\tau}$ on $\mathbb{P}^3 \times \mathbf{W}_\tau$ defines a morphism

$$\eta:\ \mathbf{W}_\tau \to Q \tag{114}$$

such that, by definition,

$$\eta(\mathbf{W}_\tau) \subset Y$$

and the diagram

$$\begin{tikzcd} \mathbf{W}_\tau \arrow[r, "\eta"] \arrow[d, "\tilde{\Phi}"'] & Y \arrow[d, "\varphi"] \\ W_\tau \arrow[r, "q\circ f"] & N(e,a,b,c)^{\mathrm{nc}} \end{tikzcd} \tag{115}$$

is cartesian. Now from the cartesian diagrams (111) and (115) and the transitivity of fibred products follows the existence of the desired morphism f satisfying (113).

Now consider the composition $B \xrightarrow{\lambda} \mathbf{T} \xrightarrow{\mu} \mathbf{R} \xrightarrow{r} \mathbf{P}$ of natural morphisms which were defined in Section 5, and the induced graph of incidence $\Gamma_B := \Gamma \times_{\mathbf{P}} B$, where Γ was defined in (12). Let $\gamma:\ \Gamma_B \to B$ be the projection and set

$$\mathbf{G} := (R^2\gamma_*(\underline{\mathbf{E}}(c-e-4)|_{\Gamma_B}))^\vee.$$

A standard base change and the Serre duality (cf. (18)) shows that $\mathbf{G}$ is a line bundle on B with a fibre over an arbitrary point $\mathbf{b} = (S, C, \xi, \mathbf{k}\psi) \in B$ (we use the notaion from (99)) given by

$$\mathbf{G} \otimes \mathbf{k}(\mathbf{b}) = H^0(E(-b)|_S),$$

where $E = \underline{\mathbf{E}}|_{\mathbb{P}^3\times\{\mathbf{b}\}}$. Comparing this with (25) and (27) and using (112) we obtain an epimorphism $\mathbf{q}^*\mathbb{U} \twoheadrightarrow \mathbf{G}$. Now by the universal property of $\mathbf{X} = \mathbb{P}(\mathbb{U}) \xrightarrow{\boldsymbol{\theta}} Y$ (see, e. g., [11, Ch. II, Prop. 7.12]) there is a morphism

$$\mathbf{g}:\ \mathbf{Y} \to \mathbf{X}$$

such that $\mathbf{q} = \boldsymbol{\theta} \circ \mathbf{g}$ and $\mathbf{G} \simeq \mathbf{g}^*\mathcal{O}_{\mathbb{P}(\mathbb{U})}(1)$. Therefore, in view of (112) we have

$$\underline{\mathbf{E}}_{\mathbf{Y}} \simeq (\mathrm{id}_{\mathbb{P}^3} \times \mathbf{g})^*\mathbf{E} = \mathbf{E}_{\mathbf{Y}}. \tag{116}$$

Futhermore, the morphism $\mathbf{g}:\ \mathbf{Y} \to \mathbf{X}$ is an equivariant morphism of $PGL(N_m)$-principal bundles $\mathbf{\Phi}:\ \mathbf{Y} \to B$ and $\Phi:\ \mathbf{X} \to X$. Hence there exists a morphism

$$g:\ B \to X$$

making the diagram

$$\begin{tikzcd} \mathbf{X} & \mathbf{Y} \arrow[l, "\mathbf{g}"'] \arrow[d, "\mathbf{\Phi}"] \\ X \arrow[from=u, "\Phi"'] & B \arrow[l, "g"] \end{tikzcd} \tag{117}$$

cartesian.

Next, we construct morphisms

$$\mathbf{h}:\ \mathbf{Y} \to \mathbf{W}, \quad h:\ B \to W$$

such that

$$\boldsymbol{\pi} \circ \mathbf{h} = \mathbf{g}, \qquad \pi \circ h = g \quad \text{and} \quad \tilde{\Phi} \circ \mathbf{h} = h \circ \Phi, \tag{118}$$

where $\tilde{\Phi} : \mathbf{W} \to W$ is a $PGL(N_m)$-principal bundle in the diagram (65). For this, remark that, since the sheaf $\mathbf{F_Y}$ (respectively, the sheaf $\underline{\mathbf{F}}_\mathbf{Y}$) is determined by the sheaf $\mathbf{E_Y}$ (respectively, by the sheaf $\underline{\mathbf{E}}_\mathbf{Y}$) uniquely up to an isomorphism (see Remark 3(ii)), the isomorphism (116) implies an isomorphism

$$\underline{\mathbf{F}}_\mathbf{Y} \simeq \mathbf{F_Y}.$$

Using this isomorphism, rewrite the left morphism in (84) twisted by $\mathcal{O}_{\mathbb{P}^3}(c-a-b) \boxtimes \mathcal{O}_\mathbf{T}$ and lifted onto $\mathbb{P}^3 \times \mathbf{Y}$ as

$$i:\ (\mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_\mathbf{T}(1))_\mathbf{Y} \to \underline{\mathbf{F}}_\mathbf{Y}(c-a-b) \simeq \mathbf{F_Y}(c-a-b).$$

Consider the diagram of natural projections

$$\begin{tikzcd} \mathbb{P}^3 \times \mathbf{X} \arrow[d, "p"'] & \mathbb{P}^3 \times \mathbf{Y} \arrow[l, "\mathrm{id}_{\mathbb{P}^3} \times \mathbf{g}"'] \arrow[d, "\mathbf{p}"] \\ \mathbf{X} & \mathbf{Y} \arrow[l, "\mathbf{g}"] \end{tikzcd}$$

and apply to the monomorphism i the functor $\mathbf{p}_*$. We obtain a subbundle morphism

$$\iota:\ \Theta^*\mathcal{O}_\mathbf{T}(1) \to \mathbf{p}_*\mathbf{F_Y}(c-a-b), \qquad \Theta := \lambda \circ \mathbf{\Phi}.$$

Note that $\mathbf{p}_*\mathbf{F_Y}(c-a-b)$ is a locally free sheaf (cf. (63)) for which the base change yields an isomorphism

$$\mathbf{p}_*\mathbf{F_Y}(c-a-b) \simeq \mathbf{g}^* p_* \mathbf{F}(c-a-b),$$

hence an epimorphism of locally free sheaves

$$\mathbf{g}^*(p_*\mathbf{F}(c-a-b))^\vee \twoheadrightarrow \Theta^*\mathcal{O}_\mathbf{T}(-1)$$

defined as the composition

$$\epsilon_\mathbf{Y}:\ \mathbf{g}^*(p_*\mathbf{F}(c-a-b))^\vee \simeq (\mathbf{g}^*p_*\mathbf{F}(c-a-b))^\vee \simeq (\mathbf{p}_*\mathbf{F_Y}(c-a-b))^\vee \xrightarrow{\iota^\vee} \Theta^*\mathcal{O}_\mathbf{T}(-1).$$

Comparing $\epsilon_\mathbf{Y}$ with the canonical epimorphism ϵ from (64), we obtain by the universal property of the projective bundle $\boldsymbol{\pi} : \mathbf{W} \to \mathbf{X}$ in (62) that there exists a morphism $\mathbf{h}:\ \mathbf{Y} \to \mathbf{W}$ satisfying the first relation (118) and such that $\mathbf{h}^*\epsilon = \epsilon_\mathbf{Y}$, $\mathbf{h}^*\mathcal{O}_\mathbf{W}(1) \simeq \Theta^*\mathcal{O}_\mathbf{T}(-1)$. By construction, the morphism $\mathbf{h}$ is $PGL(N_m)$-equivarint, so that it descends to the morphism $h:\ B \to W$ satisfying the last two relations in (118).

Now remark that, by (113), $\mathbf{f}_B = \mathbf{\Phi}\circ\mathbf{f}$. Therefore, from (108) we obtain $\mathbf{E}_{\mathbf{W}_\tau} \simeq (\underline{\mathbf{E}}_{\mathbf{Y}})_{\mathbf{W}_\tau} = (\mathrm{id}_{\mathbb{P}^3}\times\mathbf{f})^*\underline{\mathbf{E}}_{\mathbf{Y}}$. This together with (116) yields:

$$\mathbf{E}_{\mathbf{W}_\tau} \simeq (\mathrm{id}_{\mathbb{P}^3}\times(\mathbf{h}\circ\mathbf{f}))^*\mathbf{E}_{\mathbf{W}_\tau}. \tag{119}$$

Now a standard argument shows that

$$\mathbf{h}\circ\mathbf{f} = \mathrm{id}_{\mathbf{W}_\tau}. \tag{120}$$

Indeed, consider the Quot-scheme

$$Q_{\mathbf{W}_\tau} := \mathrm{Quot}_{\mathbb{P}^3\times\mathbf{W}_\tau/\mathbf{W}_\tau}(\mathcal{H}\boxtimes\mathcal{O}_{\mathbf{W}_\tau}, P) \simeq \mathbf{W}_\tau\times Q \tag{121}$$

and the embedding

$$\Delta = (\mathrm{id},\eta):\ \mathbf{W}_\tau \to Q_{\mathbf{W}_\tau},\ \mathbf{w}\mapsto(\mathbf{w},\eta(\mathbf{w})),$$

where the morphism η is defined in (114). Then in view of the universal property of $Q_{\mathbf{W}_\tau}$ the relation (119) shows that the composition $\mathbf{W}_\tau \xrightarrow{\mathbf{h}\circ\mathbf{f}} \mathbf{W}_\tau \xrightarrow{\Delta} Q_{\mathbf{W}_\tau}$ coincides with Δ. Hence, since Δ is an embedding, (120) follows. From (120) it follows that

$$\mathbf{h}(\mathbf{Y}) \supset \mathbf{W}_\tau = \boldsymbol{\pi}^{-1}(\mathbf{X}_\tau). \tag{122}$$

Note that, by Remark 2, the closure $\overline{\mathbf{X}}_\tau$ of $\mathbf{X}_\tau$ in $\mathbf{X}$ is the only irrducible component of $\tilde{\mathbf{X}}$ dominating $N(e,a,b,c)$ under the morphism $\varphi\circ\boldsymbol{\theta}$. Therefore the irreducibility of $\mathbf{Y}$ and the inclusion (122) yield the inclusions

$$\mathbf{X}_\tau \subset \mathbf{g}(\mathbf{Y}) \subset \overline{\mathbf{X}}_\tau. \tag{123}$$

Now (122) and (123) show that

$$\mathbf{Y}_\tau := \mathbf{g}^{-1}(\mathbf{X}_\tau)$$

is a nonempty open subset of $\mathbf{Y}$ which is dense since $\mathbf{Y}$ is irreducible. By its construction, $\mathbf{Y}_\tau$ is $PGL(N_m)$-invariant, hence

$$B_\tau := \mathbf{\Phi}(\mathbf{Y}_\tau)$$

is a dense open subset of B and

$$\mathbf{\Phi}:\mathbf{Y}_\tau \to B_\tau$$

is a principal $PGL(N_m)$-bundle.

Consider the morphism

$$\mathbf{h}_\tau := \mathbf{h}|_{\mathbf{Y}_\tau}:\mathbf{Y}_\tau\to\mathbf{W}_\tau,$$

which is surjective because of the left inclusion in (123). Similar to (120) one shows that

$$\mathbf{f}\circ\mathbf{h}_\tau = \mathrm{id}_{\mathbf{Y}_\tau}. \tag{124}$$

(For this, use (108) to obtain, similar to (119), the isomorphism $\mathbf{E}_{\mathbf{Y}_\tau} \simeq (\mathrm{id}_{\mathbb{P}^3}\times(\mathbf{f}\circ\mathbf{h}_\tau))^*\mathbf{E}_{\mathbf{Y}_\tau}$, and then argue as in (121), with $Q_{\mathbf{Y}_\tau}$ instead of $Q_{\mathbf{W}_\tau}$, to achieve (124).) From (120) and (124) it follows that $\mathbf{h}_\tau = \mathbf{f}^{-1}$. In particular, $\mathbf{h}_\tau$ is an isomorphism, and we obtain a cartesian diagram of principal $PGL(N_m)$-bundles

$$\begin{tikzcd}
\mathbf{W}_\tau \arrow[d, "\Phi"'] & \mathbf{Y}_\tau \arrow[l, "\mathbf{h}_\tau"', "\simeq"] \arrow[d, "\mathbf{\Phi}"] \\
W_\tau & B_\tau, \arrow[l, "h_\tau"']
\end{tikzcd}$$

where $h_\tau := h|_{B_\tau}$ is an isomorphism which by construction is inverse to f. Whence $f : W_\tau \to B_\tau$ is the desired isomorphism.

The stable rationality of $N(e,a,b,c)$ now outcomes from the rationality of $W_\tau \simeq B_\tau$ and the local triviality of the $\mathbb{P}^m$-fibration $\pi : W_\tau \to X_\tau$ (Theorem 5(ii)) and of the $\mathbb{P}^\tau$-fibration $\theta : X_\tau \to N(e,a,b,c)_\tau$ (Theorem 1(i)). In addition, the isomorphism $f : W_\tau \xrightarrow{\simeq} B_\tau$ yields the family $\underline{\mathbf{E}}_{W_\tau}$ of generalized null correlation bundles on $\mathbb{P}^3 \times W_\tau$ for which by construction the corresponding modular morphism $W_\tau \to N(e,a,b,c)_\tau$ is just the composition of locally trivial projective bundles $\pi : W_\tau \to X_\tau$ and $\theta : X_\tau \to N(e,a,b,c)_\tau$.

(ii) From statement (i) and formulas (69) and (101) it follows that

$$\tau = \delta(e,a,b,c) + t(e,a,b) - m(e,a,b,c). \tag{125}$$

This together with (4), (68), (53) and (77) shows that, under the conditions $(e,a) \neq (0,0)$, $c > 2a+b-e$ and $b > a$, one has

$$\tau = 0.$$

Therefore, by Theorem 5(ii) (see 70) and Theorem 1(i) $W_0 \simeq X_0 \xrightarrow{\theta} N(e,a,b,c)_0$ is a $\mathbb{P}^0$-fibration, hence an isomorphism. Therefore, by the rationality of $B_0 \simeq W_0$, $N(e,a,b,c)_0$ is rational. In addition, $\underline{\mathbf{E}}_{W_0}$ is a universal family of generalized null correlation bundles over $N(e,a,b,c)_0$. □

From Theorem 12 now immediately follows

Corollary 13. *For both $e = 0$ and $e = -1$, the union of the spaces $M(e,n)$ over all $n \geq 1$ contains an infinite series of rational components.*

The following remarks are in order.

Remark 14. *Fine moduli for n even. There is a well-known sufficient condition for the (given component of the) moduli space of Gieseker-Maruyama moduli space to be fine – see* [17, Cor. 4.6.6]. *In case of $M(0,n)$ with n even this condition fails, and there were not known any examples of components of $M(0,n)$ when these moduli components were fine moduli spaces. (On the contrary, there are known certain components of $M(0,n)$ for n even, e. g., the instanton components which are not fine – see* [16].*) Theorem 12(ii) provides a series of fine (open dense subsets of) moduli components $N(0,a,b,c)_0$ for $c > 2a+b-e$, $b > a$, $(e,a) \neq (0,0)$, and $n = c^2 - a^2 - b^2$ even, this series clearly being infinite – see* [19].

Remark 15. *In 1984 V. K. Vedernikov* [27] *constructed, for each pair of integers k,l with $1 \leq l \leq k$, three families of vector bundles, namely, a family $V_1(k,l) \subset M(0, 2kl+2l-l^2)$, a family $V_2(k,l) \subset M(0, k^2+2k+1-l^2)$, and a family $V_3(k,l) \subset M(-1, k^2+3k+2+2l-2l^2)$. Later in 1987 (see* [28]*), he constructed one more family, $V_4(k) \subset M(0,(k+1)^2)$ for $k \geq 1$. From the results of L. Ein, 1988, see* [9]*, it follows that Ein components $N(e,a,b,c)$ with approriate a,b,c contain these Vedernikov's families $V_1(k,l)$ and $V_4(k)$, respectively, $V_2(k,l)$ and $V_3(k,l)$, as their open dense subsets in special cases when $e = a = 0$, respectively, $a = b$. More* *precisely,*

$$
\begin{aligned}
\overline{V_1(k,l)} &= N(0,a,b,c) && \textit{for} \quad a=0, \quad b=k+1-l, \quad c=k+1,\\
\overline{V_2(k,l)} &= N(0,a,b,c) && \textit{for} \quad a=b=l, \quad c=k+1,\\
\overline{V_3(k,l)} &= N(-1,a,b,c) && \textit{for} \quad a=b=l-1, \quad c=k+1,\\
\overline{V_4(k)} &= N(0,a,b,c) && \textit{for} \quad a=b=0, \quad c=k+1.
\end{aligned} \tag{126}
$$

In [27], *it is asserted that $V_1(k,l)$ is rational. However, the construction of rationality of $V_1(k,l)$ presented in* [27, Section 3] *coincides with ours and thus, by Theorem 12, yields only stable rationality of $V_1(k,l)$. Indeed, in this case, $\tau = 0$ by (125), but $m = m(0,0,k+1-l,k+1) = 1$ by (68) and (53), so that, $\pi : B_\tau \to V_1(k,l)$ is a locally trivial $\mathbb{P}^1$-bundle with B_τ rational. So the problem of rationality of $V_1(k,l)$ remains open.*

The construction of rationality of $V_2(k,l)$ provided in [27, Sections 5-6] *differs from ours. According to Theorem 12, the rationality of $V_2(k,l)$ is covered by our result in the range $k \geq 3l \geq 3$ and, respectively, not covered in the range $2 \leq 2l \leq k \leq 3l-1$.*

In [27, Section 7], *the rationality of $V_3(k,l)$ is asserted without proof. On the other hand, in this case the rationality (respectively, stable rationality) of $V_3(k,l)$ follows from Theorem 12 for $k \geq 3l-2$ (respectively, for $2l-2 \leq k \leq 3l-3$).*

Last, the rationality of $V_4(k)$ is proved in [28]. *It is not covered by Theorem 12. Indeed, in this case we obtain from (68) and (53) that $m = 2$, and Theorem 12 yields stable rationality of $V_4(k)$.*

Summarizing the above and using (126), we conclude that the result of Theorem 12 covers Vedernikov's (proven) results in case $e = 0$, $a = b > 0$, $c > 3a$ and improves them in case $e = a = 0$, $b > 0$.

Remark 16. *As it is known* [23, Prop. 3.1], [9], *the cohomology module $H^1_*(E)$ of a generalized null correlation bundle $[E] \in N(e,a,b,c)^{\mathrm{nc}}$ has one generator as a graded module over $\mathbf{k}[x_0,x_1,x_2,x_3]$. Using this, A. P. Rao in* [23, Prop. 3.1 and Remark 3.2] *constructed big enough rational families B of generalized null correlation bundles from $N(e,a,b,c)$ with a given cohomology module $H^1_*(E)$. It follows that $N(e,a,b,c)$ can be filled by unirational varieties $\Phi(B)$ of dimension big enough, where $\Phi : B \to N(e,a,b,c)$ is the modular morphism. This shows that $N(e,a,b,c)$ is at least rationally connected (which also follows from their stable rationality), and it possibly might give an alternative approach to the problem of rationality of Ein components.*

8. Components of the moduli space $M(e,n)$ for small n

In this section we enumerate the known components (including the Ein components) of the Gieseker-Maruyama moduli space $M(e,n)$ for small values of n, namely, for $n \leq 20$ in both cases (i) $e = 0$ and (ii) $e = -1$. We specify those of these components which are rational, respectively, stably rational. Their dimensions are also given.

(i) $e = 0$. The complete description of all the components of $M(0,n)$ up to now is known only for $n \leq 5$.

(i.1) $M(0,1)$ is irreducible: $M(0,1) \simeq \mathbb{P}^5 \setminus G$, where G is the Grassmannian $Gr(2,4)$ embedded in $\mathbb{P}^5$ by Plücker – see, e.g., [12] or [22]. Here $M(0,1)$ is an Ein component with $a = b = 0,\ c = 1$.

(i.2) $M(0,2)$ is an irreducible 13-dimensional rational variety, and any sheaf in $M(0,2)$ is an instanton bundle – see [12, Section 9]. Note that $M(0,2)$ is not an Ein component.

(i.3) $M(0,3)$ consists of two rational irreducible 21-dimensional components: the instanton component I_3 any sheaf of which is an instanton bundle, and the Ein component $N(0,0,1,2)$ any sheaf of which is a generalized null correlation bundle, i. e. $N(0,0,1,2) = N(0,0,1,2)^{\mathrm{nc}}$ – see [10].

(i.4) $M(0,4)$ consists of two irreducible 29-dimensional components: the instanton component I_4 any sheaf of which is a mathematical instanton bundle with spectrum $(0,0,0,0)$, and the Ein component $N(0,0,0,2)$ – see [4], [5], [8], [14]. The rationality of $N(0,0,0,2)$ is proved in [8] and reproved in [28] by another method. It is also shown in [8] that $N(0,0,0,2) \setminus N(0,0,0,2)^{\mathrm{nc}} \neq \emptyset$.

(i.5) $M(0,5)$ has three irreducible components, according to a recent result of C. Almeida, M. Jardim, A. Tikhomirov and S. Tikhomirov [1]. The first one is the 37-dimensional rational instanton component I_5 [7], [24], [18], a general sheaf of which is a mathematical instanton bundle. The next one is the 40-dimensional Ein component $N(0,0,2,3)$ – see [9], [10, Theorem 4.7], [14], and it coincides with the component Q_2 of $M(0,5)$ introduced by Ellingsrud and Strømme (we use the notation from Section 1). This component is stably rational by Theorem 12. (A weaker statement about unirationality of $N(0,0,2,3) = Q_2$ was mentioned in Section 1.) The third one is a 37-dimensional component M_b described as the closure in $M(0,5)$ of the set $\{[E] \in M(0,5) \mid E$ is a cohomology bundle of a monad of the type $0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0\}$.

(i.6) $M(0,6)$ contains the instanton component I_6 of dimension 45 (see [25]) and at least one more component of dimension ≥ 45 which contains a (possibly open) locally closed subset $M_6 = \{[E] \in M(0,6) \mid E$ is the cohomology bundle of a monad $0 \to 2\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to 8\mathcal{O}_{\mathbb{P}^3} \to 2\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0\}$ – see [14, Table 5.3, $c_2 = 6$, (2,i)], where $\dim M_6 = 45$ by Barth's formula [4, p. 216]. However, $M(0,6)$ does not contain Ein components, since there are no integer solutions for a, b, c satisfying the conditions $b \geq a \geq 0,\ c > a + b,\ c^2 - a^2 - b^2 = 6$ – see [19, Section 2].

(i.7) $M(0,7)$ contains at least four irreducible components. They are: the instanton component I_7 of dimension 53 [24], the two Ein components $N(0,0,3,4)$ and $N(0,1,1,3)$ of dimensions 65 and 55, respectively, and a component of dimension ≥ 53 which contains a (possibly open) locally closed subset $M_7 = \{[E] \in M(0,7) \mid E$ is the cohomology bundle of a monad $0 \to 2\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to 8\mathcal{O}_{\mathbb{P}^3} \to 2\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0\}$ – see [14, Table 5.3, case $c_2 = 7$, (2,i)], where $\dim M_7 = 53$ by Barth's formula [loc. cit.]. Here the Ein components $N(0,0,3,4)$ and $N(0,1,1,3)$ are stably rational by Theorem 12, and there are no other Ein components in $M(0,7)$ by [19, Section 2].

(i.8) $M(0,8)$ contains at least three irreducible components. They are: the instanton component I_8 of dimension 61 [25], the Ein component $N(0,0,1,3)$ of dimension 62, and a component of dimension ≥ 61 which contains a (possibly open) locally closed subset $M_8 = \{[E] \in M(0,8) \mid E$ is the cohomology bundle of a monad $0 \to 2\mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to 12\mathcal{O}_{\mathbb{P}^3} \to 4\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0\}$ – see [14, Table 5.3, case $c_2 = 8$, (2,i)], where $\dim M_8 = 61$ by Barth's formula. Here the Ein component $N(0,0,1,3)$ is stably rational by Theorem 12, and there are no other Ein components in $M(0,8)$ by [19, Section 2].

We complete, using [19, Section 2], the list of all by now known irreducible components of $M(0,n)$ for $9 \le n \le 20$. For these values of n, besides the instanton components I_n of dimension $8n-3$, the rationality or stable rationality of which is unknown, and also Ein components, the only known irreducible components are two more components - see [1, Main Theorem 1]. They are: the 76-dimensional component of $M(0,10)$, a general bundle of which is a cohomology sheaf of a monad $0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(1) \to 0$, and the 135-dimensional component of $M(0,17)$, a general bundle of which is a cohomology sheaf of a monad $0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(4) \oplus \mathcal{O}_{\mathbb{P}^3}(1) \to 0$.

Below there are listed the Ein components of $M(0,n)$ for $9 \le n \le 20$. Their rationality or stable rationality follows from Theorem 12 and Remark 15, and their dimensions are given by (3).

$n = 9$: $N(0,0,0,3)$ rational of dimension 69, $N(0,0,4,5)$ stably rational of dimension 96;

$n = 10$: no Ein components;

$n = 11$: $N(0,0,5,6)$ stably rational of dimension 133, $N(0,1,2,4)$ stably rational of dimension 98;

$n = 12$: $N(0,0,2,4)$ stably rational of dimension 104;

$n = 13$: $N(0,0,6,7)$ stably rational of dimension 176;

$n = 14$: $N(0,1,1,4)$ rational of dimension 117;

$n = 15$: $N(0,0,1,4)$ stably rational of dimension 123, $N(0,0,7,8)$ stably rational of dimension 225, $N(0,1,3,5)$ stably rational of dimension 152;

$n = 16$: $N(0,0,0,4)$ rational of dimension 129, $N(0,0,3,5)$ stably rational of dimension 158;

$n = 17$: $N(0,0,8,9)$ stably rational of dimension 280, $N(0,2,2,5)$ stably rational of dimension 170;

$n = 18$: no Ein components;

$n = 19$: $N(0,0,9,10)$ stably rational of dimension 341, $N(0,1,4,6)$ stably rational of dimension 218;

$n = 20$: $N(0,0,4,6)$ stably rational of dimension 224, $N(0,1,2,5)$ rational of dimension 187.

(ii) $e = -1$. The scheme $M(-1,n)$ is known to be nonempty only for $n = 2m$, $m \ge 1$ [13]. Moreover, Hartshorne in [13] produced a family H_m of bundles with minimal spectrum from $M(-1,2m)$, using the Serre construction similar to that of 'tHooft instanton bundles from I_m. (For the notion of spectrum see [13, Section 7].) Hartshorne showed that, for each m, the family H_m is contained in a unique irreducible $(16m-5)$-dimensional component of $M(-1,2m)$ which is smooth along H_m. Denote this component by Y_{2m}.

Now observe the spaces $M(-1,2m)$ for $m = 1,2,3$.

(ii.1) $M(-1,2) = Y_2$ is an irreducible rational variety of dimension 11 [15].

(ii.2) $M(-1,4)$ has two irreducible components: the rational component Y_4 of dimension 27, and the rational component M of dimension 28 which consists of bundles with maximal spectrum [2].

(ii.3) $M(-1,6)$ has at least three irreducible components: the component Y_3 of the expected dimension 43; the Ein component $N(-1,0,0,2)$ which, by Theorem 12, is a rational variety of the expected dimension 43; the Ein component $N(-1,0,2,3)$

which, by Theorem 12, is a stably rational variety of dimension 50. Note that these two Ein components differ by the spectra of bundles therein (see [26]). Besides, as it follows from [19], there are no other Ein components in $M(-1,6)$.

We complete the list of all known irreducible components of $M(-1,n)$ for $8 \leq n \leq 20$, n even. Besides the components Y_n of dimension $8n-5$, the rationality or stable rationality of which is unknown, these are Ein components of $M(-1,n)$. (As above, here [19, Section 2], Theorem 12, Remark 15, and (3) are used.)

$n=8: N(-1,0,3,4)$ stably rational of dimension 78, $N(-1,1,1,3)$ stably rational of dimension 67;

$n=10: N(-1,0,1,3)$ rational of dimension 80, $N(-1,0,4,5)$ stably rational of dimension 112;

$n=12: N(-1,0,0,3)$ rational of dimension 93, $N(-1,0,5,6)$ stably rational of dimension 152, $N(-1,1,2,4)$ stably rational of dimension 116;

$n=14: N(-1,0,2,4)$ rational of dimension 128, $N(-1,0,6,7)$ stably rational of dimension 198;

$n=16: N(-1,0,7,8)$ stably rational of dimension 250, $N(-1,1,1,4)$ stably rational of dimension 143, $N(-1,1,3,5)$ stably rational of dimension 176;

$n=18: N(-1,0,1,4)$ rational of dimension 154, $N(-1,0,3,5)$ rational of dimension 188, $N(-1,0,8,9)$ stably rational of dimension 308, $N(-1,2,2,5)$ stably rational of dimension 197;

$n=20: N(-1,0,0,4)$ rational of dimension 165, $N(-1,0,9,10)$ stably rational of dimension 372, $N(-1,1,4,6)$ stably rational of dimension 248.

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