THE UNIQUENESS FOR MINIMAL SURFACES IN S^3

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1. Introduction

In this paper, we shall show a couple of new uniqueness theorems for compact generalized minimal surfaces in the three dimensional open hemisphere.

At first, we remark on uniqueness results for minimal surfaces in $\mathbf{R}^3$. There are three basic theorems. The first theorem is due to Radó [9] and states that if a Jordan curve Γ has a one-to-one parallel or central projection onto a convex plane Jordan curve, then Γ spans a unique minimal disk. The second theorem is due to Nitsche [8] and states that if the total curvature of an analytic Jordan curve Γ does not exceed 4π, then Γ spans a unique minimal disk. The third uniqueness theorem states that if a C^2-Jordan curve Γ is sufficiently closed to a C^2-plane Jordan curve in the C^2-topology, then Γ spans a unique minimal disk (Tromba [11]). These three theorems were generalized by Meeks [6], which treats (not necessarily disk-type) compact minimal surfaces.

Now let Γ be a Jordan curve in the three dimensional open hemisphere H of S^3. Then the area-minimizing surface spanned by Γ in S^3 exists (Morrey [7, Theorem 9.4.3]) and is contained in H (Lawson [5]). Our main result is as follows. Let B be the 2-dimensional unit open disk, let $\varphi_0 : S^3 \to S^2$ be the projection of the Hopf fibering, and denote by $K(\Gamma)$ the convex hull of Γ.

Theorem 1.1. *Let $\Gamma \subset H$ be a Jordan curve such that there exists a mapping $\mathcal{X} \in C^0(\bar{B}, S^3) \cap H_2^1(B, S^3)$ whose restriction $\mathcal{X}|_{\partial B} : \partial B \to \Gamma$ is a homeomorphism. Assume that $\varphi_0|_\Gamma : \Gamma \to \tilde{\Gamma}_0$ is a one-to-one mapping of Γ onto a Jordan curve $\tilde{\Gamma}_0$ in S^2 and that $\varphi_0(K(\Gamma)^\circ) \subset \Omega_0$, where Ω_0 is one of the domains bounded by $\tilde{\Gamma}_0$ in S^2. Then Γ spans a unique generalized minimal surface in H. Moreover, the image of this minimal surface is the image of a section : $\bar{\Omega}_0 \to H$ of the Hopf fibering and its interior is an imbedded disk.*

We shall verify this result in §3 by proving an equivalent result:Theorem 3.2. The main tools of the proof are the maximum principle for minimal surfaces in S^3 (Theorem 2.12) and the convex hull theorem due to Lawson [5]. Besides Theorem 3.2, we will give a more concrete sufficient condition on Γ for the uniqueness (Theorem 3.4).

We should remark that Theorem 1.1 is an analogy of Radó's uniqueness theorem mentioned above. On the other hand, recently Sakaki [10] obtained an analogy in H of the third uniqueness theorem in $\mathbf{R}^3$. As for an analogous result for minimal surfaces in S^3 to the Nitsche's uniqueness theorem in $\mathbf{R}^3$, we will discuss in a forthcoming paper.

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2. Preliminaries, especially, the maximum principle

We set $S^3 = \{x \in \mathbf{R}^4; |x| = 1\}$. Let $\mathcal{R}$ be a Riemann surface with or without boundary. We denote the interior of $\mathcal{R}$ by $\mathcal{R}^\circ$ and the boundary of $\mathcal{R}$ by $\partial\mathcal{R}$.

DEFINITION 2.1. A mapping $\mathcal{X} = (\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3, \mathcal{X}^4) : \mathcal{R} \to S^3$ is called *a (generalized) minimal surface* if $\mathcal{X}$ satisfies the following conditions (i) and (ii).

(i) $\mathcal{X} \in C^0(\mathcal{R}, S^3) \cap C^2(\mathcal{R}^\circ, S^3)$ by regarding $\mathcal{R}$ as a 2-dimensional real differentiable manifold.

(ii) $\mathcal{X}$ is conformal and harmonic, i.e., for any local parameter $u^1 + \sqrt{-1}u^2$ of the Riemann surface $\mathcal{R}^\circ$,

$$\sum_{j=1}^{4}\left(\frac{\partial \mathcal{X}^j}{\partial u^1}\right)^2 = \sum_{j=1}^{4}\left(\frac{\partial \mathcal{X}^j}{\partial u^2}\right)^2, \quad \sum_{j=1}^{4}\frac{\partial \mathcal{X}^j}{\partial u^1}\cdot\frac{\partial \mathcal{X}^j}{\partial u^2} = 0,$$

and

$$\Delta \mathcal{X} = -2|\nabla \mathcal{X}|^2 \mathcal{X},$$

where

$$\Delta = \frac{\partial^2}{(\partial u^1)^2} + \frac{\partial^2}{(\partial u^2)^2} \quad \text{and} \quad |\nabla \mathcal{X}|^2 = \sum_{j=1}^{4}\left\{\left(\frac{\partial \mathcal{X}^j}{\partial u^1}\right)^2 + \left(\frac{\partial \mathcal{X}^j}{\partial u^2}\right)^2\right\}.$$

DEFINITION 2.2. Let $\mathcal{R}$ be a compact Riemann surface with boundary. A mapping $\mathcal{X} : \mathcal{R} \to S^3$ is called *a (generalized) minimal surface spanned by a Jordan curve* $\Gamma \subset S^3$ if $\mathcal{X}$ satisfies the following conditions (i) and (ii).

(i) $\mathcal{X}$ is a generalized minimal surface.

(ii) The restriction $\mathcal{X}|_{\partial\mathcal{R}}$ is a homeomorphism of $\partial\mathcal{R}$ onto Γ.

REMARK 2.3. Let B be the 2-dimensional unit open disk, and Γ be a Jordan curve in S^3. Assume that there exists a mapping $\mathcal{X} \in C^0(\bar{B}, S^3) \cap H_2^1(B, S^3)$ which satisfies the condition (ii) in Definition 2.2 for $\mathcal{R} = B$. Then Γ spans at least one generalized minimal surface $\mathcal{X}_1 : \bar{B} \to S^3$ (Morrey [7, p.389, Theorem 9.4.3]) which minimizes the area among all disk-type surfaces spanned by Γ. Moreover, if Γ is contained in the open hemisphere H of S^3, then the Morrey's solution is contained in H (Lawson [5]).

To make the discussion clear, we give the definitions of *branch points* and *branched immersions*. As for detailed properties of them, we refer to [4].

DEFINITION 2.4. Let M be a 2-dimensional differentiable manifold, N an n-dimensional differentiable manifold ($n \geq 2$), and $f : M \to N$ a C^1-mapping.

(i) We say that the mapping f has *a branch point (of order $m-1$) at $p \in M$* if there exist an integer $m \geq 2$, local coordinates (u^1, u^2) around p such that p corresponds to the origin $(0,0)$, and local coordinates $(x^1, \cdots, x^n)$ around $f(p)$ such that $f(p)$ corresponds to the origin $(0, \cdots, 0)$ which satisfy

$$x^1 + \sqrt{-1}x^2 = w^m + \sigma(w),$$

$$x^k = o(|w|^m) \text{ for } k = 3, \cdots, n,$$

$$\sigma(w) = o(|w|^m),$$

$$\frac{\partial \sigma}{\partial u^j}(w), \frac{\partial x^k}{\partial u^j}(w) = o(|w|^{m-1}) \text{ for } j = 1,2 \text{ and } k = 3, \cdots, n,$$

where $w = u^1 + \sqrt{-1}u^2$.

(ii) The mapping f is called *a branched immersion* if it is regular except for branch points.

REMARK 2.5. If $p \in M$ is a regular point of the mapping $f : M \to N$ in Definition 2.4, the condition (i) is satisfied for $m = 1$. For convenience, we mean a regular point by the expression branch point of order zero.

REMARK 2.6. A generalized minimal surface $\mathcal{X} : \mathcal{R} \to S^3$ is a branched immersion in $\mathcal{R}^\circ$ ([4, Proposition 2.4]).

Lemma 2.7. *Let $\mathcal{X} : \mathcal{R} \to S^3$ be a generalized minimal surface and p an interior point of $\mathcal{R}$. Let (y^1, y^2, y^3) be local coordinates around $\mathcal{X}(p) \in S^3$ such that $\mathcal{X}(p)$ corresponds to the origin $(0,0,0)$. Then, by rotating S^3, $\mathcal{X}$ is represented around p in the following manner. There exist a neighbourhood U of p and a C^1-diffeomorphism $\tau : U \to V$ of U onto a neighbourhood V of $w = u^1 + \sqrt{-1}u^2 = 0$ in $\mathbf{C}$ with $\tau(p) = 0$ such that τ is real analytic except the point p and such that $\mathcal{X}$ is represented in terms of w and an integer $m \geq 1$ as follows.*

$$y^1 + \sqrt{-1}y^2 = w^m, \ \ y^3 = \phi(w),$$

where

$$\phi \in C^2(V) \cap C^\omega(V \setminus \{0\}).$$

$$\phi(w) = 0(|w|^{m+1}),$$

$$(2-1) \qquad \frac{\partial \phi}{\partial u^j}(w) = 0(|w|^m), \ \frac{\partial^2 \phi}{\partial u^i \partial u^j}(w) = 0(|w|^{m-1}) \ for \ i,j = 1,2.$$

Moreover, if p is a regular point of $\mathcal{X}$, then $m = 1$ and $\phi \in C^\omega(V)$.

Proof. We can verify this lemma by using the same methods of the proofs of a series of lemmas [4, Lemma 1.3, Lemma 2.1, Lemma 2.2, Lemma 3.1]. Q.E.D.

REMARK 2.8. In Lemma 2.7 the diffeomorphism τ is not a conformal mapping in general, that is, w is not an isothermal parameter of the considered minimal surface in general.

REMARK 2.9. Let a branched immersion $f : M \to N$ have a branch point $p \in M$. Then for any sequence $\{p_n\}$ of regular points of f such that $p_n \to p$ $(n \to +\infty)$, the tangent space to $f(M)$ at $f(p_n)$ tends to a limit as $n \to \infty$ ([4, Lemma 3.1]). We call this limit the *tangent plane to $f(M)$ at p.* In Lemma 2.7 the tangent plane to $\mathcal{X}(\mathcal{R})$ at p is the (y^1, y^2)-plane, where the local coordinate neighbourhood is regarded as a domain of $\mathbf{R}^3$ via the coordinates.

The following *minimal surface equation* will be used to prove the *maximum principle* (Theorem 2.12) below.

Lemma 2.10. *Let D be a subdomain of $\{(y^1, y^2) \in \mathbf{R}^2; (y^1)^2 + (y^2)^2 \leq 1 - \varepsilon\}, 0 < \varepsilon < 1$, and ϕ be a function in $C^1(\bar{D}, (-\pi/2, \pi/2)) \cap C^2(D, (-\pi/2, \pi/2))$. Assume that the mapping $\tilde{\mathcal{X}} : \bar{D} \to S^3 \subset \mathbf{R}^4$ defined by*

$$(2-2) \qquad \begin{aligned} &\tilde{\mathcal{X}}(y^1, y^2) \\ &= (\sqrt{1-(y^1)^2-(y^2)^2}\cos\phi(y^1,y^2), y^1, y^2, \sqrt{1-(y^1)^2-(y^2)^2}\sin\phi(y^1,y^2)), \end{aligned}$$

is a minimal immersion. Then

$$(2-3) \qquad \begin{aligned} I(\phi) :=& \left\{\left(1-(y^1)^2-(y^2)^2\right)^2 {\phi_2}^2 + 1 - (y^1)^2\right\}\phi_{11} \\ &- 2\left\{\left(1-(y^1)^2-(y^2)^2\right)^2 \phi_1\phi_2 + y^1y^2\right\}\phi_{12} \\ &+ \left\{\left(1-(y^1)^2-(y^2)^2\right)^2 {\phi_1}^2 + 1 - (y^2)^2\right\}\phi_{22} \\ &+ 2(y^1\phi_1 + y^2\phi_2) \\ &\quad \times \left\{\left((y^1\phi_1 + y^2\phi_2)^2 - {\phi_1}^2 - {\phi_2}^2\right)\left(1-(y^1)^2-(y^2)^2\right) - 2\right\} \\ \equiv& \ 0 \qquad \text{in } D, \end{aligned}$$

where $\phi_1 = \partial\phi/\partial y^1, \phi_2 = \partial\phi/\partial y^2, \phi_{11} = \partial^2\phi/(\partial y^1)^2$, etc.

Proof. Since a minimal immersion is a critical point of the area functional, we calculate the first variation of the area for any variation of surfaces with type (2-2) preserving the boundary values. Let ψ be any function in $C_0^2(\bar{D})$, and $\tilde{\mathcal{X}}^t$ a mapping defined by (2-2) in which we substitute $\phi + t\psi$ for ϕ. Denote by $A(t)$ the area of $\tilde{\mathcal{X}}^t$. By some calculation, we get

$$\begin{aligned} A'(0) = &-\iint_D \psi \cdot I(\phi)(1-(y^1)^2-(y^2)^2)^{-1} \\ &\times \left\{ \frac{1}{1-(y^1)^2-(y^2)^2} + {\phi_1}^2 + {\phi_2}^2 - (y^1\phi_1 + y^2\phi_2)^2 \right\}^{-3/2} dy^1 dy^2, \end{aligned} \tag{2-4}$$

where $I(\phi)$ is the expression defined in (2-3). If $\tilde{\mathcal{X}}$ is minimal, then $A'(0) = 0$ for all $\psi \in C_0^2(\bar{D})$. Hence, by (2-4), $I(\phi) \equiv 0$. Q.E.D.

DEFINITION 2.11. Let $\mathcal{X}_1 : \mathcal{R}_1 \to S^3$, $\mathcal{X}_2 : \mathcal{R}_2 \to S^3$ be two generalized minimal surfaces. Assume that $p = \mathcal{X}_1(\zeta_1) = \mathcal{X}_2(\zeta_2)$ for some points $\zeta_1 \in \mathcal{R}_1^\circ$, $\zeta_2 \in \mathcal{R}_2^\circ$. We say that $\mathcal{X}_1$ *locally lies one side of* $\mathcal{X}_2$ *near the point* p if there exist an open neighbourhood W of p in S^3, an imbedded 2-disk in $\overline{W}$ which contains p and divides W into two parts W_1 and W_2, such that each $\mathcal{X}_j(\mathcal{R}_j) \cap W$ is contained in $\overline{W_j}$, $j = 1, 2$.

Theorem 2.12 (Maximum Principle). *Let $\mathcal{X}_1 : \mathcal{R}_1 \to S^3$, $\mathcal{X}_2 : \mathcal{R}_2 \to S^3$ be two generalized minimal surfaces. Assume that $\mathcal{X}_1$ locally lies one side of $\mathcal{X}_2$ near a point $p = \mathcal{X}_1(\zeta_1) = \mathcal{X}_2(\zeta_2)$, $\zeta_1 \in \mathcal{R}_1^\circ$, $\zeta_2 \in \mathcal{R}_2^\circ$, and that at least one of these surfaces, say $\mathcal{X}_j$, has a branch point of order at most one at the point ζ_j. Then there exist neighbourhoods U_1, U_2 of ζ_1, ζ_2, respectively such that*

$$\mathcal{X}_1(U_1) = \mathcal{X}_2(U_2).$$

Proof. By rotating S^3, we may assume that p coincides with the point $(1,0,0,0)$. Set

$$H = \{(x^1, x^2, x^3, x^4) \in S^3; x^1 > 0\},$$

and define a mapping

$$\Phi : H \to \{(y^1, y^2, y^3) \in \mathbf{R}^3; (y^1)^2 + (y^2)^2 < 1 \text{ and } |y^3| < \pi/2\} =: W$$

by

$$\Phi(x^1, x^2, x^3, x^4) = (x^2, x^3, \tan^{-1}(x^4/x^1)).$$

Then (H, Φ) is a coordinate neighbourhood. Denote by (y^1, y^2, y^3) the local coordinate system in (H, Φ). Remark that the point p corresponds to the origin $(0,0,0)$, and

$$\Phi^{-1}(y^1, y^2, y^3) = (\sqrt{1-(y^1)^2-(y^2)^2}\cos y^3, y^1, y^2, \sqrt{1-(y^1)^2-(y^2)^2}\sin y^3).$$

Without loss of generality, we may assume that the surface $\mathcal{X}_2$ has a branch point of order at most one at the point ζ_2. From the assumption, the tangent planes to $\mathcal{X}_1(\mathcal{R}_1)$, $\mathcal{X}_2(\mathcal{R}_2)$ at ζ_1, ζ_2 (respectively) coincide. By rotating S^3, we may assume that these tangent planes are (y^1, y^2)-plane. By Lemma 2.7, there exist a neighbourhood $\tilde{U}_j$ of ζ_j in $\mathcal{R}_j$, an open disk V in $\mathbf{C}$ with center $w = 0$ and radius $\rho > 0$, and a C^1-diffeomorphism $\tau_j : \tilde{U}_j \to V$, such that

$$y_j^1(w) + \sqrt{-1}y_j^2(w) = w^{m_j},$$

$$y_j^3(w) = \phi_j(w),$$

where $y_j^k = y^k \circ \mathcal{X}_j \circ \tau_j{}^{-1}$ and ϕ_j, m_j are defined in the same manner as in Lemma 2.7 for $k = 1, 2, 3$, and $j = 1, 2$, and moreover $m_2 = 1$ or 2.

Let B_r be an open disk in (y^1, y^2)-plane in $W \subset \mathbf{R}^3$ with center zero and radius r, $1/\sqrt{2} > r > 0$. Let r be small so that

$$r^{1/m_j} < \rho, \ \ j = 1, 2,$$

and let γ be a simple closed arc in B_r which connects the origin $(0,0)$ and the boundary ∂B_r. Then in the domain $B_r \backslash \gamma$ we can take a single valued branch of $(y^1 + \sqrt{-1}y^2)^{1/m_j}$ which we denote by $\omega_j(y^1, y^2)$. Then $\tilde{\phi}_j := \phi_j \circ \omega_j$ is a real analytic function of y^1, y^2 in $B_r \backslash \gamma$. Let D be a simply-connected domain in $B_r \backslash \gamma$ whose boundary is a smooth regular curve containing the origin $(0,0)$. Now $\tilde{\phi}_j|_{\bar{D}}$ is of class C^1 by virtue of (2-1). From the assumption, by rotating S^3 if necessary, we may assume that

$$(2-5) \qquad \tilde{\phi}_1 \geq \tilde{\phi}_2 \ \text{ on } \ \bar{D}, \ \ \tilde{\phi}_1(0,0) = \tilde{\phi}_2(0,0) = 0.$$

Moreover, owing to (2-1),

$$(2-6) \qquad (\partial\tilde{\phi}_1/\partial\nu)(0,0) = (\partial\tilde{\phi}_2/\partial\nu)(0,0) = 0,$$

where $(\partial\tilde{\phi}_j/\partial\nu)(0,0)$ is the outer normal derivative of $\tilde{\phi}_j$ at $(0,0)$.

Consider minimal surfaces

$$\begin{aligned}\Phi^{-1}(\{(y^1, y^2, \tilde{\phi}_j(y^1, y^2)); (y^1, y^2) \in D\}) \\ = \{(\sqrt{1-(y^1)^2-(y^2)^2}\cos\tilde{\phi}_j, y^1, y^2, \sqrt{1-(y^1)^2-(y^2)^2}\sin\tilde{\phi}_j); (y^1, y^2) \in D\}.\end{aligned}$$

By Lemma 2.10,

$$I(\tilde{\phi}_1) = I(\tilde{\phi}_2) = 0.$$

Set

$$g = \tilde{\phi}_1, \quad h = \tilde{\phi}_2, \quad \text{and} \quad F = g - h. \tag{2-7}$$

Since $I(g) - I(h) = 0$, we obtain

$$\begin{aligned}
&\big\{\big(1-(y^1)^2-(y^2)^2\big)^2 {g_2}^2 + 1 - (y^1)^2\big\} F_{11} \\
&\quad - 2\big\{\big(1-(y^1)^2-(y^2)^2\big)^2 g_1 g_2 + y^1 y^2\big\} F_{12} \\
&\quad + \big\{\big(1-(y^1)^2-(y^2)^2\big)^2 {g_1}^2 + 1 - (y^2)^2\big\} F_{22} \\
&\quad + \big(1-(y^1)^2-(y^2)^2\big)^2 h_{11} \int_0^1 \frac{d}{dt}(h_2 + tF_2)^2 dt \\
&\quad - 2\big(1-(y^1)^2-(y^2)^2\big)^2 h_{12} \int_0^1 \frac{d}{dt}\{(h_1 + tF_1)(h_2 + tF_2)\} dt \\
&\quad + \big(1-(y^1)^2-(y^2)^2\big)^2 h_{22} \int_0^1 \frac{d}{dt}(h_1 + tF_1)^2 dt \\
&\quad + 2\big(1-(y^1)^2-(y^2)^2\big) \int_0^1 \frac{d}{dt}\Big[\{y^1(h_1 + tF_1) + y^2(h_2 + tF_2)\} \\
&\qquad \times \big[\{y^1(h_1 + tF_1) + y^2(h_2 + tF_2)\}^2 - (h_1 + tF_1)^2 - (h_2 + tF_2)^2\big]\Big] dt \\
&\quad - 4y^1 F_1 - 4y^2 F_2 \\
&\equiv 0,
\end{aligned} \tag{2-8}$$

where $g_1 = \partial g/\partial y^1$, $g_2 = \partial g/\partial y^2$, $h_1 = \partial h/\partial y^1, \cdots$, $F_{22} = \partial^2 F/(\partial y^2)^2$. Denote the left hand side of the equation (2-8) by $L(F)$. Then

$$L(F) = \sum_{1 \leq i,j \leq 2} a_{ij} F_{ij} + \sum_{j=1}^{2} b_j F_j = 0, \tag{2-9}$$

where

$$a_{11} = \big(1-(y^1)^2-(y^2)^2\big)^2 {g_2}^2 + 1 - (y^1)^2,$$

$$a_{12} = a_{21} = -\big\{\big(1-(y^1)^2-(y^2)^2\big)^2 g_1 g_2 + y^1 y^2\big\},$$

$$a_{22} = \big(1-(y^1)^2-(y^2)^2\big)^2 {g_1}^2 + 1 - (y^2)^2,$$

and b_j are continuous functions in D which are bounded on $\bar{D}$ by virtue of (2-1) and the condition $m_2 \leq 2$. Since $D \subset B_r$, $r < 1/\sqrt{2}$, we can see that the operator L is uniformly elliptic. Now by (2-5), (2-6), (2-7), and (2-9),

$$L(F) = 0 \text{ and } F \geq 0 \text{ on } \bar{D},$$

$$F(0,0) = 0, \ (\partial F/\partial \nu)(0,0) = 0.$$

Therefore, by virtue of the strong maximum principle for solutions of uniformly elliptic equations (c.f.[2, p.34 Lemma 3.4 and p.35 Theorem 3.5]),

$$F \equiv 0 \text{ on } \bar{D},$$

that is

$$\tilde{\phi}_1 \equiv \tilde{\phi}_2 \text{ on } \bar{D}.$$

By the arbitrariness of the arc γ, branches of $(y^1 + \sqrt{-1}y^2)^{1/m_j}$, and the domain D, we can conclude that $\mathcal{X}_1(U_1) = \mathcal{X}_2(U_2)$ for some neighbourhoods U_1, U_2 of ζ_1, ζ_2. Q.E.D.

REMARK 2.13. In the situation of Theorem 2.12, ζ_j is either a regular point or a false branch point of $\mathcal{X}_j$, $j = 1, 2$.

3. Uniqueness theorems

In this section, we will prove a few uniqueness theorems for minimal surfaces in the three dimensional open hemisphere $H = \{x = (x^1, x^2, x^3, x^4) \in S^3; x^1 > 0\}$. From now on we assume that any Jordan curve $\Gamma \subset H$ in our discussion satisfies the assumption of Remark 2.3. Therefore, there exists at least one generalized minimal surface spanned by Γ in H. At first we define the *uniqueness* in our problem.

DEFINITION 3.1. Let $\Gamma \subset H$ be a Jordan curve. Fix three distinct points P_1, P_2, P_3 in Γ. We say that Γ *spans a unique generalized minimal surface in* H if the following conditions (i) and (ii) are satisfied.

(i) There exists a unique (up to conformal equivalence) compact Riemann surface $\mathcal{R}$ with boundary $\partial\mathcal{R}$ such that there is at least one generalized minimal surface $\mathcal{X} : \mathcal{R} \to H$ spanned by Γ.

(ii) Under the condition (i), fix three distinct points Q_1, Q_2, Q_3 in $\partial\mathcal{R}$. Then there exists a unique generalized minimal surface $\mathcal{X} : \mathcal{R} \to H$ spanned by Γ such that $\mathcal{X}(Q_j) = P_j$, $j = 1, 2, 3$.

Now let us recall the *Hopf fibering.* $S^1 = \{\zeta \in \mathbf{C}; |\zeta| = 1\}$ acts freely on $S^3 = \{(w, z) \in \mathbf{C}^2; |w|^2 + |z|^2 = 1\}$ on the right: $((w,z), e^{\sqrt{-1}\theta}) \in S^3 \times S^1 \to (we^{\sqrt{-1}\theta}, ze^{\sqrt{-1}\theta}) =: R_\theta((w,z)) \in S^3$. Let $\varphi_0 : S^3 \to S^3/S^1 = S^2$ be the canonical projection. Then $S^3(S^2, S^1, \varphi_0)$ is a principal fibre bundle and is called the *Hopf fibering.* Each fibre ${\varphi_0}^{-1}(y)$ $(y \in S^2)$ is a great circle of S^3 and is called a *Hopf circle.*

Next, we define a restricted Hopf fibering. Set

$$\begin{aligned}\tilde{S}^3 &:= S^3 - \{x = (x^1, x^2, x^3, x^4) \in S^3; x^1 = x^2 = 0\}\\ &= S^3 - \{(w,z) \in S^3 \subset \mathbf{C}^2; w = 0\}.\end{aligned}$$

Then S^1 acts freely on $\tilde{S}^3$ on the right. Let Σ be the 2-dimensional open hemisphere $\{(x^1, x^2, x^3, x^4) \in S^3; x^1 > 0 \text{ and } x^2 = 0\}$. We can define a *restricted Hopf fibering* $\tilde{S}^3(\Sigma, S^1, \varphi)$ by

$$\varphi : x \in \tilde{S}^3 \to \varphi(x) := \Sigma \cap \{R_\theta(x); e^{\sqrt{-1}\theta} \in S^1\} \in \Sigma.$$

Each fibre

$$\varphi^{-1}(y) = \{R_\theta(y); e^{\sqrt{-1}\theta} \in S^1\},\ \ y \in \Sigma,$$

is a Hopf circle.

Going back to our uniqueness problem, let $\Gamma \subset H$ be a Jordan curve. Denote by $K(\Gamma)$ the convex hull of Γ, and by $K(\Gamma)^\circ$ the interior of $K(\Gamma)$. (As for the definition of the *convex hull*, refer to [5, §3].)

Theorem 3.2. *Let $\Gamma \subset H$ be a Jordan curve. Assume that $\varphi|_\Gamma : \Gamma \to \tilde{\Gamma}$ is a one-to-one mapping of Γ onto a Jordan curve $\tilde{\Gamma}$ in Σ, and that*

$$\varphi(K(\Gamma)^\circ) \subset \Omega, \tag{3-1}$$

where Ω is the interior of $\tilde{\Gamma}$ in Σ. Then Γ spans a unique generalized minimal surface in H. Moreover, the image of this minimal surface is the image of a section $: \bar{\Omega} \to H$ of the restricted Hopf fibering and its interior is an imbedded disk.

Proof. Let $\mathcal{X} : \mathcal{R} \to H$ be any generalized minimal surface spanned by Γ. Since $\mathcal{X}(\mathcal{R}^\circ)$ is contained in $K(\Gamma)^\circ$ ([5, Theorem 1]),

$$\varphi(\mathcal{X}(\mathcal{R}^\circ)) \subset \Omega \tag{3-2}$$

by virtue of the assumption (3-1).

Set $B = \{\zeta = u^1 + \sqrt{-1}u^2 \in \mathbf{C}; |\zeta| < 1\}$. Let $\mathcal{X}_1 : \bar{B} \to H$ be the minimal surface spanned by Γ which was obtained by Morrey (c.f. Remark 2.3). Then $\mathcal{X}_1$ has no branch points in B ([3, Theorem 8.1 and 8.2]). Now let us prove that $\mathcal{X}_1(\bar{B})$ is the image of a section : $\bar{\Omega} \to H$. Namely, we will show that

$$\mathcal{X}_1(\bar{B}) = \{R_{h(q)}(q); q \in \bar{\Omega}\} \tag{3-3}$$

for some continuous function h of $\bar{\Omega}$ into $(-\pi/2, \pi/2)$. If not, owing to (3-2) and the injectivity of $\varphi|_\Gamma$, there exists a point $q \in \Omega$ such that $\varphi^{-1}(q) \cap \mathcal{X}_1(B)$ consists of at least two distinct points. Let

$$p_1,\ p_2 \in \varphi^{-1}(q) \cap \mathcal{X}_1(B),$$

$p_1 = R_{t_0}(p_2)$ $(0 < t_0 < \pi)$. Then

$$\mathcal{X}_1(B) \cap R_{t_0}(\mathcal{X}_1(B)) \ni p_1. \tag{3-4}$$

Set

$$T = \max\{t \in (-\pi, \pi); \mathcal{X}_1(B) \cap R_t(\mathcal{X}_1(B)) \neq \emptyset\}.$$

Such T exists and $T > 0$ because of the compactness of $\bar{B}$, the assumption $\mathcal{X}_1(\bar{B}) \subset H$, the injectivity of $\varphi|_\Gamma$, and (3-4). Let us take a point $p \in \mathcal{X}_1(B) \cap R_T(\mathcal{X}_1(B))$. Then $p = \mathcal{X}_1(\zeta_1) = R_T \circ \mathcal{X}_1(\zeta_2)$ for two distinct points ζ_1, $\zeta_2 \in B$. Now, by the definition of T, minimal surface $R_T \circ \mathcal{X}_1$ locally lies one side of $\mathcal{X}_1$ near the point p. Owing to Theorem 2.12, there exist neighbourhoods U_1, U_2 of ζ_1, ζ_2, respectively such that

$$\mathcal{X}_1(U_1) = R_T \circ \mathcal{X}_1(U_2).$$

Moreover, by the proof of Theorem 2.12 (or the unique continuation property of the analytic mapping $\mathcal{X}_1$), we see that $\mathcal{X}_1(U_1)$ is an open set of $\mathcal{X}_1(B)$. Therefore, the intersection $\mathcal{X}_1(B) \cap R_T \circ \mathcal{X}_1(B) \neq \emptyset$ is open and closed in $\mathcal{X}_1(B)$. Since $\mathcal{X}_1(B)$ is connected, $\mathcal{X}_1(B) \subset R_T \circ \mathcal{X}_1(B)$, which contradicts the fact that

$$\mathcal{X}_1(\partial B) \cap R_T \circ \mathcal{X}_1(\bar{B}) = \emptyset.$$

Hence we proved that $\mathcal{X}_1(\bar{B})$ is represented in the form (3-3), which implies also that $\mathcal{X}_1$ is injective and $\mathcal{X}_1|_B$ is an imbedding.

Next, we assume that $\mathcal{X}_2 : \mathcal{R} \to H$ is a generalized minimal surface in H spanned by Γ such that $\mathcal{X}_1(\bar{B}) \neq \mathcal{X}_2(\mathcal{R})$. Then there exists some t_1 $(0 < |t_1| < \pi)$ with

$$\mathcal{X}_1(B) \cap R_{t_1} \circ \mathcal{X}_2(\mathcal{R}^\circ) \neq \emptyset.$$

The same argument as above leads a contradiction, which proves the uniqueness of the image of minimal surfaces in H spanned by Γ, that is, $\mathcal{X}_1(\bar{B}) = \mathcal{X}_2(\mathcal{R})$. This implies also that $\mathcal{X}_2|_{\mathcal{R}^\circ}$ is an imbedding and that $\mathcal{R}^\circ = B$ (up to conformal equivalence), which are proved as follows. Suppose that $\mathcal{X}_2$ is not regular at a point $\zeta \in \mathcal{R}^\circ$. Because of the injectivity of $\mathcal{X}_2|_{\partial\mathcal{R}}$, $\mathcal{X}_2$ has no false branch points ([4, Theorem 6.3]), which implies that ζ is a true branch point of $\mathcal{X}_2$. Hence $\mathcal{X}_2$ has a transversal self-intersection near the point ζ ([3, Theorem 3.1 and remarks after the proof of Lemma 8.1]), which contradicts the fact that $\mathcal{X}_2(\mathcal{R}) = \mathcal{X}_1(\bar{B})$ is the image of a section : $\bar{\Omega} \to H$. Therefore $\mathcal{X}_2|_{\mathcal{R}^\circ}$ has no singular points and is an imbedding. Now well-defined function $f := {\mathcal{X}_2}^{-1} \circ \mathcal{X}_1 : \bar{B} \to \mathcal{R}$ is a homeomorphism which implies that $\mathcal{R}^\circ = B$ (up to conformal equivalence).

Let Q_1, Q_2, Q_3 be three distinct points in ∂B. Assume that *the three point condition*: $\mathcal{X}_1(Q_i) = \mathcal{X}_2(Q_i)$, $i = 1, 2, 3$. In view of relations $|\partial \mathcal{X}_j/\partial u^1|^2 = |\partial \mathcal{X}_j/\partial u^2|^2 > 0$ and $(\partial \mathcal{X}_j/\partial u^1) \cdot (\partial \mathcal{X}_j/\partial u^2) = 0$, $j = 1, 2$, the homeomorphism $f = {\mathcal{X}_2}^{-1} \circ \mathcal{X}_1 : \bar{B} \to \bar{B}$ is holomorphic or anti-holomorphic in B. By the three point condition, f is the identity mapping, which implies that $\mathcal{X}_1 \equiv \mathcal{X}_2$. Q.E.D.

Proof (of Theorem 1.1). The assumption of Theorem 1.1 is equivalent to that of Theorem 3.2, which can be verified as follows. The image $\varphi_0(H)$ is just the set $S^2 - \{\text{one point}\}$, which we denote by $\tilde{S}^2$. The mapping $f := \varphi \circ (\varphi_0|_H)^{-1} : \tilde{S}^2 \to \Sigma$ is well-defined and is a homeomorphism. Suppose that the assumption of Theorem 1.1 is satisfied. By the assumption $\varphi_0(K(\Gamma)^\circ) \subset \Omega_0$, we see $\partial(\varphi_0(K(\Gamma)^\circ)) \supset \tilde{\Gamma}_0$. From this fact and the simply-connectedness of $\varphi_0(K(\Gamma)^\circ)$, we know $\partial(\varphi_0(K(\Gamma)^\circ)) = \tilde{\Gamma}_0$ and $\Omega_0 = \varphi_0(K(\Gamma)^\circ) \subset \varphi_0(H) = \tilde{S}^2$. Therefore Ω_0 must be the interior of $\tilde{\Gamma}_0$ in $\tilde{S}^2$. Now the desired equivalence is trivial via the homeomorphism f. Hence Theorem 1.1 is derived from Theorem 3.2.
Q.E.D.

The assumptions of Theorem 3.2 are satisfied by Jordan curves which are sufficiently near to G-convex ([5, §3]) Jordan curves in Σ. However, for these Jordan curves, the uniqueness is already known([10]). Theorem 3.4 below gives many examples of Jordan curves which are more distant from Jordan curves in the geodesic hypersphere of S^3 but satisfy the assumptions of Theorem 3.2.

In Theorem 3.4 and its proof, $\sin^{-1}$ means the inverse function of the function $\sin : [0, \pi/2] \to [0, 1]$.

DEFINITION 3.3. Let $\Gamma \subset H$ be a Jordan curve such that $\varphi|_\Gamma : \Gamma \to \tilde{\Gamma}$ is a one-to-one mapping of Γ onto a Jordan curve $\tilde{\Gamma}$ in Σ. Then the *height function* of Γ is the continuous function $h : \tilde{\Gamma} \to (-\pi/2, \pi/2)$ which is defined in the following manner.

$$R_{h(q)}(q) = (\varphi|_\Gamma)^{-1}(q) \text{ for any } q \in \tilde{\Gamma}.$$

Theorem 3.4. *Let $\Gamma \subset H$ be a Jordan curve, and*

$$\tilde{\Gamma} = \{x \in H; x^1 = \sqrt{1-c^2} \text{ and } x^2 = 0\}$$

be a circle in Σ with radius c $(0 < c < 1)$. Assume that $\varphi|_\Gamma : \Gamma \to \tilde{\Gamma}$ is bijective and for the height function h of Γ,

$$(3-5) \qquad |h(q_1) - h(q_2)| < \sin^{-1}\Big(\frac{\sin(\mathrm{dist}(q_1, q_2)/2)}{c}\Big)$$

for any q_1, $q_2 \in \tilde{\Gamma}$ $(q_1 \neq q_2)$, where $\mathrm{dist}(q_1, q_2)$ is the geodesic distance between q_1 and q_2. Then Γ spans a unique generalized minimal surface in H. This minimal surface has the property mentioned in the last sentence of Theorem 3.2.

Proof. Let Ω be the interior of $\tilde{\Gamma}$ in Σ. We shall prove that Γ satisfies the condition (3-1) in Theorem 3.2. Let $\sigma : H \to \mathbf{R}^3$ be the homeomorphism which is defined as follows.

$$\sigma(x) = \Big(\frac{x^2}{x^1}, \frac{x^3}{x^1}, \frac{x^4}{x^1}\Big), \quad x = (x^1, x^2, x^3, x^4) \in H \subset \mathbf{R}^4.$$

Then each geodesic in H is mapped to a straight line in $\mathbf{R}^3$ and also the inverse statement is true. Therefore, a subset A of H is G-convex if and only if $\sigma(A) \subset \mathbf{R}^3$ is convex. Hence, $\sigma(K(\Gamma))$ is the convex hull of $\sigma(\Gamma)$.

Let p be any point of $K(\Gamma)^\circ$. Our purpose is to prove that $q := \varphi(p)$ is contained in Ω, which is equivalent to the inequality

$$(3-6) \qquad x^1(q) > \sqrt{1-c^2},$$

where $x^1(q)$ is the first coordinate of q $(\in \mathbf{R}^4)$.

There exist (not necessarily distinct) four points p_1, p_2, p_3, $p_4 \in \Gamma \cap \partial K(\Gamma)$ and non-negative numbers λ_1, λ_2, λ_3, λ_4 with

$$(3-7) \qquad \sum_{j=1}^{4} \lambda_j = 1,$$

such that

$$(3-8) \qquad \sigma(p) = \sum_{j=1}^{4} \lambda_j \sigma(p_j)$$

(c.f.[1, p.9]). Let $q_j = \varphi(p_j)$, $j = 1, \cdots, 4$. Since $q_j \in \tilde{\Gamma}$, it is represented as follows.

$$q_j = (\sqrt{1-c^2},\ \ 0,\ \ c\cdot\cos\beta_j,\ \ c\cdot\sin\beta_j).$$

Set $\alpha_j = h(q_j)$. Then

$$p_j = (\sqrt{1-c^2}\cos\alpha_j,\ \ \sqrt{1-c^2}\sin\alpha_j,\ \ c\cdot\cos(\alpha_j+\beta_j),\ \ c\cdot\sin(\alpha_j+\beta_j)),$$

$$(3-9) \qquad \sigma(p_j) = \frac{1}{\sqrt{1-c^2}\cos\alpha_j}(\sqrt{1-c^2}\sin\alpha_j,\ \ c\cdot\cos(\alpha_j+\beta_j),\ \ c\cdot\sin(\alpha_j+\beta_j)).$$

Set

$$(3-10) \qquad \sigma(p) = (y^1, y^2, y^3).$$

Then

$$p = (1+|y|^2)^{-1/2}(1, y^1, y^2, y^3), \ \ |y|^2 = \sum_{j=1}^{3}(y^j)^2,$$

$$(3-11) \qquad q = (1+|y|^2)^{-1/2}(1+\sqrt{-1}y^1, y^2+\sqrt{-1}y^3)e^{\sqrt{-1}\theta} \in \mathbf{C}\times\mathbf{C},$$

where $\theta \in (-\pi/2, \pi/2)$ is the uniquely determined number so that

$$(3-12) \qquad (1+\sqrt{-1}y^1)e^{\sqrt{-1}\theta} \text{ is real and positive.}$$

From (3-8), (3-9), (3-10), and (3-11), we get

$$\begin{aligned} x^1(q) &= (1+|y|^2)^{-1/2}(\cos\theta - y^1\sin\theta) \\ (3-13) \qquad &= (1+|y|^2)^{-1/2}\Big(\cos\theta - \sin\theta\sum_{j=1}^{4}\lambda_j\tan\alpha_j\Big). \end{aligned}$$

By (3-8), (3-9), (3-10), and (3-12),

$$(3-14) \qquad 0 = y^1\cos\theta + \sin\theta = \sin\theta + \cos\theta\sum_{j=1}^{4}\lambda_j\tan\alpha_j.$$

Therefore,

$$\Big(\cos\theta - \sin\sum_{j=1}^{4}\lambda_j\tan\alpha_j\Big)^2$$

$$= \cos^2\theta - 2\sin\theta\cos\theta\sum_{j=1}^{4}\lambda_j\tan\alpha_j + \sin^2\theta\Big(\sum_{j=1}^{4}\lambda_j\tan\alpha_j\Big)^2$$

$$= \cos^2\theta + \sin^2\theta + \cos^2\theta\Big(\sum_{j=1}^{4}\lambda_j\tan\alpha_j\Big)^2 + \sin^2\theta\Big(\sum_{j=1}^{4}\lambda_j\tan\alpha_j\Big)^2$$

$$(3-15) \qquad = 1 + \Big(\sum_{j=1}^{4}\lambda_j\tan\alpha_j\Big)^2.$$

On the other hand, owing to (3-7), (3-8), (3-9), and (3-10),

$$1+|y|^2 = 1 + \Big(\sum_{j=1}^{4}\lambda_j\tan\alpha_j\Big)^2 + \frac{c^2}{1-c^2}\Big(\sum_{j=1}^{4}\lambda_j\frac{\cos(\alpha_j+\beta_j)}{\cos\alpha_j}\Big)^2$$

$$+ \frac{c^2}{1-c^2}\Big(\sum_{j=1}^{4}\lambda_j\frac{\sin(\alpha_j+\beta_j)}{\cos\alpha_j}\Big)^2$$

$$= \frac{c^2}{1-c^2}\sum_{1\le j,k\le 4}\lambda_j\lambda_k(1+\tan\alpha_j\tan\alpha_k)$$

$$\times\Big\{\frac{1-c^2}{c^2} + \cos(\beta_j-\beta_k) - \tan(\alpha_j-\alpha_k)\sin(\beta_j-\beta_k)\Big\}$$

$$(3-16) \qquad \le \frac{c^2}{1-c^2}\sum_{1\le j,k\le 4}\lambda_j\lambda_k(1+\tan\alpha_j\tan\alpha_k)$$

$$\times\Big\{\frac{1}{c^2} - 1 + \cos(\beta_j-\beta_k) + |\tan(\alpha_j-\alpha_k)\sin(\beta_j-\beta_k)|\Big\},$$

where we used the property $1+\tan\alpha_j\tan\alpha_k > 0$ which is verified as follows. By virtue of the assumption (3-5) and the remark before Definition 3.3,

$$|\alpha_j - \alpha_k| = |h(q_j) - h(q_k)| < \pi/2.$$

Moreover $|\alpha_j| = |h(q_j)| < \pi/2$, $j = 1, \cdots, 4$. Therefore,

$$1 + \tan\alpha_j\tan\alpha_k = \frac{\cos(\alpha_j-\alpha_k)}{\cos\alpha_j\cos\alpha_k} > 0.$$

Remark that

$$\operatorname{dist}(q_j, q_k) = 2\sin^{-1}\left(c\left|\sin\frac{\beta_j - \beta_k}{2}\right|\right). \tag{3-17}$$

If $\beta_j - \beta_k \neq n\pi$ ($n \in \mathbf{Z}$), then, by (3-17) and (3-5),

(3 − 18)

$$\begin{aligned}
C :=& -1 + \cos(\beta_j - \beta_k) + |\tan(\alpha_j - \alpha_k)\sin(\beta_j - \beta_k)| \\
=&2\left(\sin\frac{\beta_j - \beta_k}{2}\right)^2\left\{-1 + |\tan(\alpha_j - \alpha_k)|\left|\tan\frac{\beta_j - \beta_k}{2}\right|^{-1}\right\} \\
=&2\left(\sin\frac{\beta_j - \beta_k}{2}\right)^2\left\{-1 + |\tan(\alpha_j - \alpha_k)|\left|\tan\left(\sin^{-1}\left(\frac{\sin(\operatorname{dist}(q_j, q_k)/2)}{c}\right)\right)\right|^{-1}\right\} \\
<&0.
\end{aligned}$$

If $\beta_j - \beta_k = n\pi$, it is trivial that

$$C \leq 0. \tag{3-19}$$

From (3-7), (3-16), (3-18), and (3-19), we get

$$1 + |y|^2 \leq (1 - c^2)^{-1}\left\{1 + \left(\sum_{j=1}^{4}\lambda_j \tan\alpha_j\right)^2\right\}. \tag{3-20}$$

In view of (3-13), (3-15), and (3-20), we see that

$$x^1(q) \geq \sqrt{1 - c^2}.$$

Moreover, $x^1(q) = \sqrt{1 - c^2}$ if and only if $\beta_j - \beta_k = 2n_{jk}\pi$ ($n_{jk} \in \mathbf{Z}$) for all $j,k \in \{1, 2, 3, 4\}$, which implies that $p = p_1 = p_2 = p_3 = p_4$ and hence p is in $\partial K(\Gamma)$, which contradicts the choice of p. Therefore, the inequality (3-6) holds.

Now we have proved that Γ satisfies all assumptions of Theorem 3.2, which implies the desired conclusion. Q.E.D.

REMARK 3.5. Set $Q = (\sqrt{1 - c^2}, 0, 0, 0)$. For any two points $q_1, q_2 \in \tilde{\Gamma}$ ($q_1 \neq q_2$), denote by $\operatorname{angle}(q_1, q_2)$ the angle of $\angle q_1 Q q_2 (\in (0, \pi])$. Then the assumption (3-5) is equivalent to the following condition.

$$|h(q_1) - h(q_2)| < \operatorname{angle}(q_1, q_2)/2.$$

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YANG–MILLS CONNECTIONS OF HOMOGENEOUS BUNDLES

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YANG–MILLS CONNECTIONS OF HOMOGENEOUS BUNDLES

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Dedicated to Professor Shingo Murakami on his 60th birthday

0. Introduction

Let (M, g) be a compact riemannian manifold and P a principal fiber bundle over M with compact structure group K. A functional $\mathcal{F}_{\mathrm{YM}}$ which maps a connection ∇ to the square integral $\int_M |R^\nabla|^2 v_g$ of the norm of the curvature tensor of ∇ is called the Yang–Mills functional. A Yang–Mills connection is by definition a critical point of the functional $\mathcal{F}_{\mathrm{YM}}$. Therefore there is some possibility that so called the direct method and the heat equation method can be applied to construct a Yang–Mills connection of P.

When the manifold M is an algebraic manifold and the group K is a unitary group, there is a strong relationship between the notion of stable vector bundles and Yang–Mills connections ([K]), and Donaldson shows the existence of a Yang–Mills connection by the heat equation method ([D]).

In this paper we consider homogeneous bundles as simple examples in order to see in what situations the direct method and the heat equation method can be applied to the existence problem. Let the riemannian manifold M be expressed as a homogeneous space G/H and the principal fiber bundle P as $G \times_\rho K$ using a Lie group homomorphism $\rho : H \to K$. The space $\mathcal{C}_G$ of all G–invariant connections forms a finite dimensional vector space. Corresponding to the direct method, we will get the following

Theorem 1. *Assume that the Lie group H is connected. The function $\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G$ is proper if and only if one of the following conditions holds.* (1) *The fundamental group $\pi_1(M)$ of M is finite.* (2) *The Lie algebra $\mathfrak{k}$ of the structure group K has no trivial factor as H-module.*

This means that if (1) or (2) holds, then any minimizing sequence for the function $\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G$ has a convergent subsequence to a Yang–Mills connection. But if neither (1) nor (2) holds, a minimizing sequence may diverge to "∞".

However, even if neither (1) nor (2) holds, we can find a Yang–Mills connection by the heat equation – an ordinary differential equation in our case – method.

Theorem 2. *The heat equation with a G–invariant connection ∇_0 as the initial data has a solution ∇_t which is a bounded curve in the space $\mathcal{C}_G$. In particular, the bundle P admits a Yang–Mills connection.*

As a particular case of Theorem 2, we will see what happens in the case of homogeneous complex situations. Finally, we will prove Mountain–Pass Lemma for the function $\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G$. Remark that, when we consider Einstein's equation the corresponding statement to Theorem 2 does not hold, i.e., the solution diverges in general ([WZ, Introduction]).

This work was done while the author was staying in Max–Planck–Institut für Mathematik, to which he is grateful for the hospitality.

1. Properness

We will prove Theorem 1 in this section. Let M be a compact homogeneous riemannian manifold G/H, where G is a compact Lie group and H is a closed subgroup. Denote by $\boldsymbol{g}$, $\boldsymbol{h}$ the Lie algebra of the Lie group G, H respectively. Fix a bi-invariant inner product $\langle\ ,\ \rangle$ on $\boldsymbol{g}$ and denote by $\boldsymbol{m}$ the orthogonal complement of $\boldsymbol{h}$ in $\boldsymbol{g}$. The riemannian metric of the space M is represented by an H–invariant inner product g on $\boldsymbol{m}$. Define a principal fiber bundle $P = G \times_\rho K$ using a compact Lie group K and a homomorphism $\rho : H \to K$. The Lie algebra of K is denoted by $\boldsymbol{k}$ and is endowed with a bi-invariant inner product $\langle\ ,\ \rangle$. The differential : $\boldsymbol{h} \to \boldsymbol{k}$ of the Lie group homomorphism ρ is denoted by the same symbol ρ. The space $\boldsymbol{k}$ becomes an H–module and an H^0–module via ρ, where H^0 is the identity component of H. For basic facts about Lie groups, refer to [H].

As usual, we denote by $\boldsymbol{g}'$ the semi-simple part of the Lie algebra $\boldsymbol{g}$ and by $\boldsymbol{z}(\boldsymbol{g})$ its center. Let $\boldsymbol{m}'$ be the projection image from $\boldsymbol{g}'$ to $\boldsymbol{m}$. The vector space $\boldsymbol{m}$ decomposes as H–module :

$$\boldsymbol{m} = \boldsymbol{m}' \oplus \boldsymbol{m} \cap \boldsymbol{z}(\boldsymbol{g}), \tag{1.1}$$

which corresponds to the decomposition of the universal covering of M into a compact manifold and a vector space. Therefore the fundamental group $\pi_1(M)$ is finite if and only if $\boldsymbol{m} \cap \boldsymbol{z}(\boldsymbol{g})$ vanishes. When the Lie algebra $\boldsymbol{k}$ decomposes into the semi-simple part and the center, the function $\mathcal{F}_{\mathrm{YM}}$ correspondingly decomposes. These facts reduce the proof of Theorem 1 to the following propositions.

Proposition 1.1. *Assume that the Lie group H is connected. If the space $\boldsymbol{k}$ has a trivial factor as H–module and the space $\boldsymbol{m} \cap \boldsymbol{z}(\boldsymbol{g})$ does not vanish, then the function $\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G$ is not proper.*

Proposition 1.2. *If one of the following conditions holds, then the function $\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G$ is proper.*

(1) *The space $\boldsymbol{k}$ has no trivial factor as H^0-module.*

(2) *The Lie algebra $\boldsymbol{k}$ is commutative and the space $\boldsymbol{m} \cap \boldsymbol{z}(\boldsymbol{g})$ vanishes.*

(3) *The Lie algebra $\boldsymbol{k}$ is semi-simple and the space $\boldsymbol{m} \cap \boldsymbol{z}(\boldsymbol{g})$ vanishes.*

We will give proofs of these propositions in this section. The following lemma is fundamental.

Lemma 1.3 ([KN, Chapter II Theorem 11.7]). *The space $\mathcal{C}_G$ is canonically identified with the space of all H-homomorphisms $\mathrm{Hom}_H(\boldsymbol{m}, \boldsymbol{k})$, and the curvature tensor $R^A \in \mathrm{Hom}_H(\wedge^2 \boldsymbol{m}, \boldsymbol{k})$ of an element $A \in \mathrm{Hom}_H(\boldsymbol{m}, \boldsymbol{k})$ is given by*

$$R^A(v, w) = [A(v), A(w)] - A([v, w]_{\boldsymbol{m}}) - \rho([v, w]_{\boldsymbol{h}}), \tag{1.2}$$

where $(\ )_{\boldsymbol{h}}$ and $(\ )_{\boldsymbol{m}}$ denote the components with respect to the decomposition $\boldsymbol{g} = \boldsymbol{h} \oplus \boldsymbol{m}$.

From now on an element of the space $\mathrm{Hom}_H(\boldsymbol{m}, \boldsymbol{k})$ is identified with a connection of P, and so the function $\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G$ is regarded as

$$\mathcal{F}_{\mathrm{YM}}(A) = \mathrm{Vol}(M) \times |R^A|^2. \tag{1.3}$$

Since the properness of the function $\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G$ is independent of the choice of inner products of $\boldsymbol{m}$, we may assume that the inner product g is the restriction of $\langle\ ,\ \rangle$ in this section.

Proof (of Proposition 1.1). The assumption implies that there are non-zero elements X in a trivial factor of the H-module $\boldsymbol{k}$ and v_0 in $\boldsymbol{m} \cap \boldsymbol{z}(\boldsymbol{g})$. Then we can define an element A in $\mathrm{Hom}_H(\boldsymbol{m}, \boldsymbol{k})$ by $A(v) = \langle v, v_0 \rangle X$, which satisfies $R^{\lambda A} = R^0$ for any real number λ by formula (1.2). Q.E.D.

We decompose $\boldsymbol{m}$ as H^0-module into the trivial factor $\boldsymbol{m}_0$ and the sum $\boldsymbol{m}_1$ of the irreducible factors. Then we have inclusions :

$$\boldsymbol{m}_1 \subset \boldsymbol{m}' \quad \text{and} \quad \boldsymbol{m} \cap \boldsymbol{z}(\boldsymbol{g}) \subset \boldsymbol{m}_0. \tag{1.4}$$

Lemma 1.4. *There exist positive constants c_1, c_2 and c_3 such that for any $A \in \mathrm{Hom}_H(\boldsymbol{m}, \boldsymbol{k})$ it holds that*

$$|R^A| \geq c_1 \big|A|\boldsymbol{m}_1\big|^2 - c_2 \big|A|\boldsymbol{m}'\big| - c_3. \tag{1.5}$$

Proof. We set $[A \wedge A](v,w) = [A(v), A(w)]$ and observe that if $[A \wedge A] = 0$, then $A(\boldsymbol{m}_1) = 0$. In fact

$$
\begin{aligned}
0 &= \langle [A(\boldsymbol{m}), A(\boldsymbol{m})], \rho(\boldsymbol{h})\rangle = \langle A(\boldsymbol{m}), [\rho(\boldsymbol{h}), A(\boldsymbol{m})]\rangle \\
&= \langle A(\boldsymbol{m}), A([\boldsymbol{h}, \boldsymbol{m}])\rangle = \langle A(\boldsymbol{m}), A(\boldsymbol{m}_1)\rangle .
\end{aligned} \tag{1.6}
$$

Therefore if we set $c_1 = \inf\{|[A \wedge A]|; A \in \mathrm{Hom}_H(\boldsymbol{m}, \boldsymbol{k}), |A| = 1\}$, then $c_1 > 0$. For the second term $A([v,w]_{\boldsymbol{m}})$ of formula (1.2), it depends only on $A|\boldsymbol{m}'$. Q.E.D.

Proof (of Proposition 1.2 (1)). Since the space $\boldsymbol{k}$ has no trivial factor as H^0–module, the space $\mathrm{Hom}_H(\boldsymbol{m}, \boldsymbol{k})$ coincides with $\mathrm{Hom}_H(\boldsymbol{m}_1, \boldsymbol{k})$. Thus A, $A|\boldsymbol{m}'$ and $A|\boldsymbol{m}_1$ coincide in Lemma 1.4. Q.E.D.

Proof (of Proposition 1.2 (2)). Let A be any element of $\mathrm{Hom}_H(\boldsymbol{m}, \boldsymbol{k})$. Since the Lie algebra $\boldsymbol{k}$ is commutative, the first term $[A \wedge A]$ of formula (1.2) vanishes. And since $\boldsymbol{k}$ is trivial as H^0–module, $A(\boldsymbol{m}_1) = 0$. On the other hand, since $\boldsymbol{m} \cap \boldsymbol{z}(\boldsymbol{g}) = 0$, it holds that $\boldsymbol{m} = \boldsymbol{m}' = [\boldsymbol{m}, \boldsymbol{m}]_{\boldsymbol{m}} + \boldsymbol{m}_1$. Therefore if $A \neq 0$, then the second term $A([\boldsymbol{m}, \boldsymbol{m}]_{\boldsymbol{m}}) \neq 0$. Thus we can define a positive number c_1 by $\inf\{|A|[\boldsymbol{m}, \boldsymbol{m}]_{\boldsymbol{m}}|; A \in \mathrm{Hom}_H(\boldsymbol{m}, \boldsymbol{k}), |A| = 1\}$ and setting c_2 the norm of the third term, we get $|R^A| \geq c_1|A| - c_2$. Q.E.D.

To prove the case of semi-simple Lie algebra $\boldsymbol{k}$, we introduce the following usual notations. For a reductive Lie algebra $\boldsymbol{j}$, $\boldsymbol{t}(\boldsymbol{j})$ denotes a Cartan subalgebra. When $\boldsymbol{j}$ is semi–simple and endowed with a bi-invariant inner product $\langle\ ,\ \rangle$, we denote by $\Delta(\boldsymbol{j})$ its root system as a subset of $\boldsymbol{t}(\boldsymbol{j})$ and characterize root vectors $X_\alpha \in \boldsymbol{j}$ for $\alpha \in \Delta(\boldsymbol{j})$ by (1) $[u, X_\alpha] = \langle u, \alpha\rangle X_{-\alpha}$ for all $u \in \boldsymbol{t}(\boldsymbol{j})$ and (2) $[X_\alpha, X_{-\alpha}] = \alpha$. The following lemma will be proved later.

Lemma 1.5. *Let $\boldsymbol{k}$ be a compact semi-simple Lie algebra. For an element (w_0, w_1, w_2) of $\boldsymbol{k}^3$ we define an element (u_0, u_1, u_2) of $\boldsymbol{k}^3$ by*

$$
u_0 = [w_1, w_2] - w_0, \quad u_1 = [w_2, w_0] - w_1, \quad u_2 = [w_0, w_1] - w_2, \tag{1.7}
$$

then this map : $\boldsymbol{k}^3 \to \boldsymbol{k}^3$ is proper.

Note that $\boldsymbol{m}_0$ becomes a subalgebra of $\boldsymbol{g}$, i.e., $[\boldsymbol{m}_0, \boldsymbol{m}_0] \subset \boldsymbol{m}_0$, and $[\boldsymbol{m}_0, \boldsymbol{m}_1]$ is contained in $\boldsymbol{m}_1$. Since Cartan subalgebras $\boldsymbol{t}(\boldsymbol{h})$ and $\boldsymbol{t}(\boldsymbol{m}_0)$ commute, there is a Cartan subalgebra $\boldsymbol{t}(\boldsymbol{g})$ which contains $\boldsymbol{t}(\boldsymbol{h})$ and $\boldsymbol{t}(\boldsymbol{m}_0)$. The space $\boldsymbol{t}(\boldsymbol{g})$ decomposes into the center $\boldsymbol{z}(\boldsymbol{g})$ and a Cartan subalgebra $\boldsymbol{t}(\boldsymbol{g}')$. It admits also an orthogonal decomposition :

$$
\boldsymbol{t}(\boldsymbol{g}) = \boldsymbol{t}(\boldsymbol{h}) \oplus \boldsymbol{t}(\boldsymbol{m}_0) \oplus \boldsymbol{t}(\boldsymbol{g}) \cap \boldsymbol{m}_1 . \tag{1.8}
$$

We denote by $\boldsymbol{t}(\boldsymbol{g}')_0$ the image of the orthogonal projection from $\boldsymbol{t}(\boldsymbol{g}')$ to $\boldsymbol{t}(\boldsymbol{m}_0)$.

Lemma 1.6. *Denoting by $(\boldsymbol{m}_0)'$ the semi-simple part of $\boldsymbol{m}_0$, we get*

$$t(\boldsymbol{g}')_0 + (\boldsymbol{m}_0)' = \boldsymbol{m}' \cap \boldsymbol{m}_0. \tag{1.9}$$

Proof. It is clear that the left hand side is contained in the right hand side. Let v be an element of the right hand side which is orthogonal to the left hand side. Then v is an element of the center $\boldsymbol{z}(\boldsymbol{m}_0)$, and is orthogonal to $\boldsymbol{t}(\boldsymbol{g}')$. Therefore we see that $v \in \boldsymbol{z}(\boldsymbol{g})$ and so by (1.1) we conclude that $v = 0$. Q.E.D.

We rewrite Proposition 1.2 (3) as follows in order to use it in section 2.

Proposition 1.7. *If the Lie algebra $\boldsymbol{k}$ is semi-simple, then $|A|\boldsymbol{m}'|$ is estimated from above by using $|R^A|$.*

Proof. First remark that, by Lemma 1.5, if we take $v_0, v_1, v_2 \in \boldsymbol{m}$ with $[v_0, v_1]_{\boldsymbol{m}} = v_2$, $[v_1, v_2]_{\boldsymbol{m}} = v_0$, $[v_2, v_0]_{\boldsymbol{m}} = v_1$, then $|A(v_i)|$'s are estimated by using $|R^A|$. Therefore we can get an estimation of $A|(\boldsymbol{m}_0)'$ because the space $(\boldsymbol{m}_0)'$ is spanned by its roots and root vectors. Next we decompose a root $\alpha \in \Delta(\boldsymbol{g}')$ by (1.8) and denote by α_0, α_1 the $\boldsymbol{t}(\boldsymbol{m}_0)$, $\boldsymbol{t}(\boldsymbol{g}) \cap \boldsymbol{m}_1$–component, respectively, and set $\alpha' = \alpha_0 + \alpha_1$. The vector α' is the $\boldsymbol{m}$–component of α, and belongs to $\boldsymbol{m}'$.

Now assume that $\alpha_0 \neq 0$. Then

$$\begin{aligned} \pm|\alpha_0|^2 \cdot X_{\pm\alpha} &= [\alpha_0, X_{\pm\alpha}] = [\alpha_0, (X_{\pm\alpha})_{\boldsymbol{h}}] + [\alpha_0, (X_{\pm\alpha})_{\boldsymbol{m}}] \\ &= [\alpha_0, (X_{\pm\alpha})_{\boldsymbol{m}}] \in \boldsymbol{m}, \end{aligned} \tag{1.10}$$

and so $X_{\pm\alpha} \in \boldsymbol{m}$. Setting $v_0 = |\alpha'|^{-2}\alpha'$, $v_1 = |\alpha'|^{-1}X_\alpha$, $v_2 = |\alpha'|^{-1}X_{-\alpha}$, we can get an estimation of $A(\alpha')$ by the previous remark. Moreover, since $A(\mathrm{Ad}_h\alpha') = \mathrm{Ad}_{\rho(h)}A(\alpha')$ for $h \in H^0$, we get an estimation of $|A(\alpha_0)|$ by

$$|A(\alpha_0)| = \left|A\left(\int_{H^0}(\mathrm{Ad}_h\alpha')dh\right)\right| \leq \int_{H^0}|A(\mathrm{Ad}_h\alpha')|dh = |A(\alpha')|, \tag{1.11}$$

where dh is the Haar measure of H^0. Since the space $\boldsymbol{t}(\boldsymbol{g}')_0$ is spanned by such α_0's, we get an estimation of $A|\boldsymbol{t}(\boldsymbol{g}')_0$. Combining with the estimation of $A|(\boldsymbol{m}_0)'$, we get an estimation of $A|(\boldsymbol{m}' \cap \boldsymbol{m}_0)$ by Lemma 1.6. Finally, using Lemma 1.4 and the inequality : $|A|\boldsymbol{m}'| \leq |A|(\boldsymbol{m}' \cap \boldsymbol{m}_0)| + |A|\boldsymbol{m}_1|$ from (1.3), we get an estimation of $A|\boldsymbol{m}_1$, therefore of $A|\boldsymbol{m}'$. Q.E.D.

Proof (of Lemma 1.5). We set $c = \max\{|u_0|, |u_1|, |u_2|\}$ and $\ell = |w_0|$, and show that ℓ is bounded from above by using c. In the following, c_i's mean positive constants which depend only on c and do not depend on ℓ. At first we see that

$$\begin{aligned} |w_1|^2 - c|w_1| &\leq \langle w_1, w_1 + u_1\rangle = \langle w_1, [w_2, w_0]\rangle \\ &= \langle [w_1, w_2], w_0\rangle = \langle w_0 + u_0, w_0\rangle \leq \ell^2 + c\,\ell, \end{aligned} \tag{1.12}$$

and so $|w_1| \leq \ell + c$. By the same way we see also that $|w_2| \leq \ell + c$.

We choose a Cartan subalgebra $\boldsymbol{t}(\boldsymbol{k})$ containing w_0 and a linear order $\succ$ of $\boldsymbol{t}(\boldsymbol{k})$ so that if $\langle w_0, \alpha\rangle > 0$, then $\alpha \succ 0$. Denote by $\Pi = \{\alpha_i\}$ the fundamental root system. Since Π is basis of $\boldsymbol{t}(\boldsymbol{k})$, whose pattern is independent of the choice of orders, it holds that

$$\left|\sum_{\alpha_i \in \Pi} x_i \alpha_i\right| \geq c_1 \sum_{\alpha_i \in \Pi} |x_i| \quad \text{for any } (x_i). \tag{1.13}$$

We set $w_1 = z + \sum a_\alpha X_\alpha$, where z is an element of $\boldsymbol{t}(\boldsymbol{k})$ and the summation is taken for roots $\alpha \in \Delta(\boldsymbol{k})$. Then we see that

$$[w_0, w_1] = \sum a_\alpha \langle \alpha, w_0\rangle X_{-\alpha}, \tag{1.14}$$

$$w_2 = [w_0, w_1] - u_2 = \sum a_\alpha \langle \alpha, w_0\rangle X_{-\alpha} - u_2, \tag{1.15}$$

$$[w_0, w_2] = -\sum a_\alpha \langle \alpha, w_0\rangle^2 X_\alpha - [w_0, u_2]. \tag{1.16}$$

And so,

$$u_1 = [w_2, w_0] - w_1 = \sum a_\alpha \langle \alpha, w_0\rangle^2 X_\alpha + [w_0, u_2] - z - \sum a_\alpha X_\alpha, \tag{1.17}$$

$$\sum a_\alpha (\langle \alpha, w_0\rangle^2 - 1) X_\alpha = z + u_1 - [w_0, u_2]. \tag{1.18}$$

Since $\{X_\alpha; \alpha \in \Delta(\boldsymbol{k})\}$ are orthogonal, it follows that

$$|a_\alpha||\langle \alpha, w_0\rangle^2 - 1| \leq |X_\alpha|^{-1}\big((\ell + c) + c + c_2 c \ell\big) \leq c_3(\ell + 1). \tag{1.19}$$

Therefore, if $\langle \alpha, w_0\rangle^2 \geq 2$, then

$$|a_\alpha| \langle \alpha, w_0\rangle^2 \leq 2c_3(\ell + 1). \tag{1.20}$$

And since

$$\begin{aligned} [w_1, w_2]_{\boldsymbol{t}(\boldsymbol{k})} &= [w_1, \sum a_\alpha \langle \alpha, w_0\rangle X_{-\alpha} - u_2]_{\boldsymbol{t}(\boldsymbol{k})} \\ &= [z + \sum a_\alpha X_\alpha, \sum a_\alpha \langle \alpha, w_0\rangle X_{-\alpha}]_{\boldsymbol{t}(\boldsymbol{k})} - [w_1, u_2]_{\boldsymbol{t}(\boldsymbol{k})} \\ &= \sum (a_\alpha)^2 \langle \alpha, w_0\rangle \cdot \alpha - [w_1, u_2]_{\boldsymbol{t}(\boldsymbol{k})}, \end{aligned} \tag{1.21}$$

we get

$$
(1.22)\qquad \begin{aligned} w_0 &= [w_1, w_2]_{t(k)} - (u_0)_{t(k)} \\ &= \sum (a_\alpha)^2 \langle \alpha, w_0 \rangle \cdot \alpha - [w_1, u_2]_{t(k)} - (u_0)_{t(k)}. \end{aligned}
$$

Now for a positive number ε, we define a subset Π_ε of the fundamental root system Π by

$$
(1.23)\qquad \Pi_\varepsilon = \{\alpha_i \in \Pi ; \langle \alpha_i, w_0 \rangle < \varepsilon |w_0||\alpha_i|\}.
$$

The number ε will be fixed later independently of ℓ. In the following, the constants c_i are independent also of ε. Put

$$
(1.24)\qquad S = \Delta(k) \cap \sum_{\alpha_i \in \Pi_\varepsilon} \mathbf{Z}\alpha_i
$$

An element β of $\Delta(k) - S$ can be represented as $\sum m_i \alpha_i$ $(\alpha_i \in \Pi)$, where all m_i are non-negative or all are non-positive. And so

$$
(1.25)\qquad \begin{aligned} |\langle \beta, w_0 \rangle| &= \sum |m_i \langle w_0, \alpha_i \rangle| \\ &\geq \langle w_0, \alpha_i \rangle \quad \text{for some } \alpha_i \in \Pi - \Pi_\varepsilon \\ &\geq c_4 \varepsilon\, \ell. \end{aligned}
$$

Therefore if $\ell \geq 1$ and $\sqrt{2}(c_4\varepsilon)^{-1}$, then, by (1.20), we see that

$$
(1.26)\qquad \begin{aligned} (a_\beta)^2 |\langle \beta, w_0 \rangle| &\leq \big(2c_3(\ell+1)\big)^2 |\langle \beta, w_0 \rangle|^{-3} \\ &\leq \big(2c_3(\ell+1)\big)^2 (c_4 \varepsilon\, \ell)^{-3} \\ &\leq c_5 \varepsilon^{-3}. \end{aligned}
$$

We represent $\sum_{\alpha \in S} (a_\alpha)^2 \langle \alpha, w_0 \rangle \cdot \alpha$ as $\sum s_i \alpha_i$ $(\alpha_i \in \Pi_\varepsilon)$. Since

$$
(1.27)\qquad \sum_{\alpha \in S} (a_\alpha)^2 \langle \alpha, w_0 \rangle \cdot \alpha = \sum_{\alpha \in S,\, \alpha \succ 0} \big((a_\alpha)^2 + (a_{-\alpha})^2\big) \langle \alpha, w_0 \rangle \cdot \alpha,
$$

all s_i are nonnegative, and so

$$
(1.28)\qquad \begin{aligned} \langle w_0, \sum s_i \alpha_i \rangle &= \sum s_i \langle w_0, \alpha_i \rangle < \sum s_i |w_0||\alpha_i| \varepsilon \\ &\leq c_6 \varepsilon\, \ell \sum s_i. \end{aligned}
$$

On the other hand, from (1.13), (1.22) and (1.26), if ℓ is greater than 1 and $\sqrt{2}(c_4\varepsilon)^{-1}$, then

$$
\begin{aligned}
c_1 \sum s_i &\leq |\sum s_i\alpha_i| \\
&= \Big|w_0 - \sum_{\beta\notin S}(a_\beta)^2\langle\beta, w_0\rangle\cdot\beta + [w_1, u_2]_{t(k)} + (u_0)_{t(k)}\Big| \\
&\leq \ell + c_7\varepsilon^{-3} + c_8\ell + c_9.
\end{aligned}
\tag{1.29}
$$

Combining it with (1.28),

$$
\langle w_0, \sum s_i\alpha_i\rangle \leq c_{10}\varepsilon\,\ell^2 \quad + \text{(polynomial of } \ell \text{ of order 1)}. \tag{1.30}
$$

Therefore, again using (1.22) and (1.26), we see that if $\ell \geq 1$ and $\sqrt{2}(c_4\varepsilon)^{-1}$, then

$$
\begin{aligned}
\ell^2 &= \langle w_0, w_0\rangle \\
&= \langle w_0, \sum_{\beta\notin S}(a_\beta)^2\langle\beta, w_0\rangle\cdot\beta + \sum s_i\alpha_i - [w_1, u_2]_{t(k)} - (u_0)_{t(k)}\rangle \\
&\leq c_{10}\varepsilon\,\ell^2 + c|[w_0, w_1]| \quad + \text{(polynomial of } \ell \text{ of order 1)} \\
&= c_{10}\varepsilon\,\ell^2 + c|w_2 + u_2| \quad + \text{(polynomial of } \ell \text{ of order 1)} \\
&\leq c_{10}\varepsilon\,\ell^2 \quad + \text{(polynomial of } \ell \text{ of order 1)}.
\end{aligned}
\tag{1.31}
$$

Thus choosing ε so that $c_{10}\varepsilon < 1/2$, we get the desired estimation of ℓ. Q.E.D.

2. Gradient Flow

We consider the heat equation for the functional $\mathcal{F}_{\mathrm{YM}}$ with respect to the L_2 inner product, which becomes

$$
\frac{d}{dt}\nabla_t = -(\mathrm{grad}\,\mathcal{F}_{\mathrm{YM}})_{\nabla_t} = (\nabla_t)^k(R^{\nabla_t})_{ki}. \tag{2.1}
$$

If we choose $\nabla_0 \in \mathcal{C}_G$ as the initial data of this equation, then the solution ∇_t is a curve in $\mathcal{C}_G$ and coincides with the solution of the ordinary differential equation defined by the vector field $-\mathrm{grad}(\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G)$. As is easily computed from formula (1.2), the equation is given by (up to constant multiplication of time variable t),

$$
\begin{aligned}
\frac{d}{dt}A_i &= \sum[A_j, [A_j, A_i]] - \sum C_j{}^k{}_i[A_j, A_k] - \sum C_j{}^s{}_i[A_j, \rho_s] \\
&\quad + (1/2)\sum C_j{}^i{}_k[A_j, A_k] - (1/2)\sum C_j{}^i{}_k C_j{}^l{}_k A_l - (1/2)\sum C_j{}^i{}_k C_j{}^s{}_k \rho_s,
\end{aligned}
\tag{2.2}
$$

where we take orthonormal basis $\{v_i\}$ of $\boldsymbol{m}$ with respect to g and basis $\{v_p\}$ of $\boldsymbol{h}$, and set

$$
\begin{aligned}
A_i &= A(v_i), \quad \rho_s = \rho(v_s), \\
[v_i, v_j] &= \sum C_i{}^k{}_j v_k + \sum C_i{}^s{}_j v_s.
\end{aligned} \tag{2.3}
$$

All the summations are taken for j, k, l, s, which appear twice in the terms.

We will prove Theorem 2 for equation (2.2). Denote by $A(t)$ the solution. At first, by Proposition 1.7, the norm of $A(t)|\boldsymbol{m}'$ is estimated from above by using $|R^{A(0)}|$. Therefore, denoting by $(\boldsymbol{m}')^{\perp}$ the orthogonal compliment of $\boldsymbol{m}'$ in $\boldsymbol{m}$ with respect to g, it is sufficient to prove that $A(t)|(\boldsymbol{m}')^{\perp}$ is bounded. To show it, we choose an arbitrary unit vector v_0 in $(\boldsymbol{m}')^{\perp}$, choose orthonormal basis $\{v_i; 0 \le i < \dim \boldsymbol{m}\}$ of $\boldsymbol{m}$ containing v_0, and prove that $A_0(t)$ is bounded. For $A_0(t)$, equation (2.2) is simplified as

$$
\frac{d}{dt} A_0 = \sum [A_j, [A_j, A_0]] - \sum C_j{}^k{}_0 [A_j, A_k]. \tag{2.4}
$$

In fact the structure constants $C_j{}^0{}_k$ vanish in equation (2.2) because $[\boldsymbol{m}, \boldsymbol{m}]_{\boldsymbol{m}} \subset \boldsymbol{m}'$. Moreover, since the inner product g is H-invariant and v_0 is orthogonal to $\boldsymbol{m}'$, the vector v_0 is an element of $\boldsymbol{m}_0$, and as the remark following Lemma 1.5, $[\boldsymbol{m}_0, \boldsymbol{m}] \subset \boldsymbol{m}$, which implies that also the structure constants $C_j{}^s{}_0$ vanish.

Next, as we see from equation (2.1) or (2.2), when the Lie algebra $\boldsymbol{k}$ decomposes as $\boldsymbol{k}' \oplus \boldsymbol{z}(\boldsymbol{k})$, the solution also decomposes, and the $\boldsymbol{z}(\boldsymbol{k})$-component of $A_0(t)$ is constant from equation (2.4). Therefore we may assume that *the Lie algebra $\boldsymbol{k}$ is semi-simple.* Moreover, since the equations do not depend on the choice of inner products on $\boldsymbol{k}$, we may assume also that the root vectors X_α of $\boldsymbol{k}$ are unit.

Now we define a function L on the vector space $\boldsymbol{k}$ as follows. Let 2δ be the sum of all positive roots of $\boldsymbol{k}$. We represent 2δ as $2\delta = \sum n_i \alpha_i$. Let $\{\omega_i; 1 \le i \le r\}$ be the fundamental weight system of $\boldsymbol{k}$, and set $\xi_i = (n_i)^{-1}\omega_i$. For $w \in \boldsymbol{k}$, we define

$$
L(w) = \max\{\langle \mathrm{Ad}_\gamma w, \xi_i \rangle; 1 \le i \le r,\ \gamma \in K\}. \tag{2.5}
$$

Lemma 2.1. *For $w \in \boldsymbol{k}$, the value $L(w)$ is realized by $\gamma \in K$ such that* $\mathrm{Ad}_\gamma w$ *belongs to the positive Weyl chamber $\overline{W}$. In particular L is a norm of $\boldsymbol{k}$.*

Proof. From the assumption, for any $X \in \boldsymbol{k}$,

$$
0 = \langle [X, \mathrm{Ad}_\gamma w], \xi_i \rangle = \langle X, [\mathrm{Ad}_\gamma w, \xi_i] \rangle. \tag{2.6}
$$

Therefore $\mathrm{Ad}_\gamma w$ and ξ_i belong to the same abelian subalgebra of $\boldsymbol{k}$. Since all Cartan subalgebras are conjugate, we may assume that $\mathrm{Ad}_\gamma w \in \boldsymbol{t}(\boldsymbol{k})$. If $\langle \mathrm{Ad}_\gamma w, \alpha_j \rangle < 0$ for some

$\alpha_j \in \Pi$, then, taking $\eta \in K$ which gives the reflection with respect to α_j, for any ξ_k we see that

$$
\begin{aligned}
\langle \mathrm{Ad}_{\eta\gamma} w, \xi_k \rangle &= \langle \mathrm{Ad}_\eta(\mathrm{Ad}_\gamma w), \xi_k \rangle \\
&= \big\langle \mathrm{Ad}_\gamma w - 2|\alpha_j|^{-2} \langle \alpha_j, \mathrm{Ad}_\gamma w \rangle \cdot \alpha_j, \xi_k \big\rangle \\
&= \langle \mathrm{Ad}_\gamma w, \xi_k \rangle - 2|\alpha_j|^{-2} \langle \alpha_j, \mathrm{Ad}_\gamma w \rangle (n_j)^{-1} \delta_{jk} \\
&\geq \langle \mathrm{Ad}_\gamma w, \xi_k \rangle .
\end{aligned} \tag{2.7}
$$

That is, when $\mathrm{Ad}_\gamma w$ is mapped into $\overline{W}$ by the Weyl group, the value $L(w)$ is still realized. Q.E.D.

We reduced to the case that the Lie algebra $\boldsymbol{k}$ is semi-simple in order to use the following

Lemma 2.2. *Let $\boldsymbol{k}$ be semi-simple. There exists a positive number ε with the following property. Let w be a unit vector in the positive Weyl chamber $\overline{W}$. If $\langle w, \alpha_i \rangle < \varepsilon$, then there is ξ_j such that $\langle w, \xi_j \rangle > \langle w, \xi_i \rangle$.*

Proof. Set $w = \sum x_k \alpha_k$. Then all x_k are positive and $\langle w, \xi_k \rangle = (n_k)^{-1} \langle w, \omega_k \rangle = (n_k)^{-1} x_k$. Assume that $\langle w, \xi_j \rangle \leq \langle w, \xi_i \rangle$ for all ξ_j. Then, since $\langle \alpha_j, \alpha_i \rangle \leq 0$ for $j \neq i$,

$$
\begin{aligned}
\langle w, \alpha_i \rangle &= \sum_j \langle \alpha_j, \alpha_i \rangle x_j \\
&\geq \langle \alpha_i, \alpha_i \rangle x_i + \sum_{j \neq i} \langle \alpha_j, \alpha_i \rangle n_j (n_i)^{-1} x_i \\
&= (n_i)^{-1} \sum_j \langle n_j \alpha_j, \alpha_i \rangle x_i \\
&= (n_i)^{-1} \langle 2\delta, \alpha_i \rangle x_i > 0,
\end{aligned} \tag{2.8}
$$

because 2δ belongs to the open positive Weyl chamber W. Thus the conclusion follows from the continuity. Q.E.D.

Proof (of Theorem 2). It is sufficient to prove that $L(A_0(t))$ is bounded. Since $A_0(t)$ is real analytic, $L(A_0(t))$ is continuous and, by Lemma 2.1, piecewisely represented as

$$
L(A_0(t)) = \langle \mathrm{Ad}_{\gamma(t)} A_0(t), \xi_1 \rangle, \qquad \mathrm{Ad}_{\gamma(t)} A_0(t) \in \overline{W}, \tag{2.9}
$$

where $\gamma(t)$ is a real analytic curve of K and ξ_1 is taken by renumbering of suffix. We may assume that $\gamma(t) = 1$ at a time $t = t_0$ by changing the Cartan subalgebra $\boldsymbol{t}(\boldsymbol{k})$ if necessary.

We set $A_j = u_j + \sum_\alpha x_j^\alpha X_\alpha$, where $u_j \in \boldsymbol{t}(\boldsymbol{k})$ and $\alpha \in \Delta(\boldsymbol{k})$. At the time $t = t_0$, we see that

$$\frac{d}{dt}L(A_0) = \langle \frac{d}{dt}A_0, \xi_1\rangle + \langle [\frac{d}{dt}\gamma, A_0], \xi_1\rangle = \langle \frac{d}{dt}A_0, \xi_1\rangle. \tag{2.10}$$

Thus assigning (2.4), the last expression

$$\begin{aligned}
&= -\sum_{j\neq 0}\langle [A_0, A_j], [\xi_1, A_j]\rangle - \sum_{j,k\neq 0} C_j{}^k{}_0\langle [\xi_1, A_j], A_k\rangle \\
&= -\sum_{j\neq 0}\langle \sum_{\alpha\in\Delta(k)} x_j^\alpha\langle\alpha, A_0\rangle X_{-\alpha}, \sum_{\alpha\in\Delta(k)} x_j^\alpha\langle\alpha,\xi_1\rangle X_{-\alpha}\rangle \\
&\quad - \sum_{j,k\neq 0} C_j{}^k{}_0\langle \sum_{\alpha\in\Delta(k)} x_j^\alpha\langle\alpha,\xi_1\rangle X_{-\alpha}, u_k + \sum_{\alpha\in\Delta(k)} x_k^\alpha X_\alpha\rangle \\
&= -\Big\{ \sum_{j\neq 0,\,\alpha\in\Delta(k)} \langle\alpha,\xi_1\rangle\langle\alpha, A_0\rangle (x_j^\alpha)^2 + \sum_{j,k\neq 0,\,\alpha\in\Delta(k)} C_j{}^k{}_0\langle\alpha,\xi_1\rangle x_j^\alpha x_k^{-\alpha}\Big\} \\
&= -\sum_{\alpha\in\Delta(k),\,\alpha\succ 0}\Big\{\sum_{j\neq 0}\langle\alpha,\xi_1\rangle\langle\alpha,A_0\rangle(x_j^\alpha)^2 + \sum_{k\neq 0}\langle\alpha,\xi_1\rangle\langle\alpha,A_0\rangle(x_k^{-\alpha})^2 \\
&\qquad\qquad + \sum_{j,k\neq 0}(C_j{}^k{}_0 - C_k{}^j{}_0)\langle\alpha,\xi_1\rangle x_j^\alpha x_k^{-\alpha}\Big\}.
\end{aligned}$$

This summation is taken only for positive roots $\alpha \in \Delta(\boldsymbol{k})$ such that $\langle\alpha,\xi_1\rangle \neq 0$. If we represent such α as $\sum m_i\alpha_i$, then $m_1 \geq 1$ and all $m_i \geq 0$, and so $\langle\alpha,\xi_1\rangle \geq |\omega_1|^{-1}$. Therefore by Lemma 2.2, it holds that $\langle\alpha, A_0\rangle \geq \varepsilon|A_0|$. In fact, if $|\langle\alpha, A_0\rangle| < \varepsilon|A_0|$, then $|\langle\alpha_1, A_0\rangle| < \varepsilon|A_0|$ and so $\langle A_0,\xi_i\rangle > \langle A_0,\xi_1\rangle$ for some i, which contradicts the maximality of $\langle A_0,\xi_1\rangle$. We regard the last expression as a quadratic form of (x_j^α) and $(x_k^{-\alpha})$, and see that, if $L(A_0)$ is sufficiently large, and so is $|A_0|$, then the coefficients of $(x_j^\alpha)^2$ and $(x_k^{-\alpha})^2$ are sufficiently greater than that of $x_j^\alpha x_k^{-\alpha}$, which implies the non-positivity of the last expression. Q.E.D.

REMARK 2.3. From the boundedness of $A(t)$, we see that any subsequence of $A(t)$ has a subsequence which converges to a Yang–Mills connection. It seems to the author that $A(t)$ itself converges. At least it is clear that if the closure of the set $\{A(t); t \in \mathbf{R}^+\}$ contains an isolated Yang–Mills connection, then $A(t)$ converges. Here we mean by isolated to be isolated modulo the action of the normalizer group $N_K(\rho(H))$ of $\rho(H)$ in K.

3. Appendix

At first, we consider the relation between equation (2.2) and holomorphic vector bundles. Let M be an algebraic manifold and P a principal $U(r)$–bundle. Take the complexification $GL(r, \mathbf{C})$ of the compact Lie group $U(r)$, and complexify P to a principal $GL(r, \mathbf{C})$–bundle $P^{\mathbf{C}}$. There is a one-to-one correspondence between holomorphic structures $\bar{\partial}$ of $P^{\mathbf{C}}$ and connections ∇ of P whose curvature tensor R^{∇} are of type (1,1). Kobayashi shows that if the corresponding connection ∇ to a holomorphic structure $\bar{\partial}$ is a Yang–Mills connection, then $\bar{\partial}$ is semi-stable ([K]). Conversely, the following hold. Let $\bar{\partial}_0$ be a holomorphic structure and ∇_0 the corresponding connection. Heat equation (2.1) with initial data ∇_0 has a unique solution ∇_t, whose curvature tensors are of type (1,1). Let $\bar{\partial}_t$ be the corresponding holomorphic structure. Then all $\bar{\partial}_t$ are conjugate to $\bar{\partial}_0$ under automorphisms of $P^{\mathbf{C}}$. Moreover, if $\bar{\partial}_0$ is stable, then both ∇_t and $\bar{\partial}_t$ converge, and $\lim \bar{\partial}_t$ is conjugate to $\bar{\partial}_0$ ([D]). In our homogeneous situation, we get

Corollary 3.1. *Let M and P be as above with homogeneous assumption. Let $\bar{\partial}_0$ be an invariant holomorphic structure. Then the solution $\bar{\partial}_t$ has a convergent subsequence. But if $\bar{\partial}_0$ is not semi-stable, then the limit of $\bar{\partial}_t$ is not conjugate to $\bar{\partial}_0$.*

Next we consider so called

Mountain–Pass Lemma. *Let S be a manifold and f a function of S. If there are relative minima $x_1, x_2 \in S$ of f which are not contained in a connected component of the critical point set, then there exists an unstable critical point $x_3 \in S$.*

Theorem 3.2. *Mountain–Pass Lemma holds for the space $S = \mathcal{C}_G$ and the function $f = \mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G$.*

Example 3.3 Assume that G is semi-simple and set $H = \{\mathrm{id}\}$, $K = G$ and $\rho = \mathrm{id}$. Then the space $\mathcal{C}_G$ is identified with $\mathrm{End}_{\mathbf{R}}(g)$, and $\mathcal{F}_{\mathrm{YM}}(A) = 0$ if and only if A is a Lie algebra homomorphism. Therefore $A = 0$ and $A = \mathrm{id}$ are critical points of $\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G$, and belong to different connected components. Thus we can conclude, by Mountain–Pass Lemma, that there exists another unstable Yang–Mills connection in $\mathcal{C}_G$. When the riemannian metric g on G is bi-invariant, it is easy to get such an unstable Yang–Mills connection, say $A = (1/2)\mathrm{id}$. However it is not clear to see the existence of such a connection for a general left invariant metric on G without our Theorem.

As the above example, if the space S is a vector space and if the function f is (by Theorem 1) proper, then Mountain–Pass Lemma holds by [C, (VI 6.1)]. For general case, i.e., when the fundamental group $\pi_1(M)$ may be infinite, we use the next lemma. Let V be a finite dimensional vector space, $\bar{S}$ a closed convex domain of V and f a smooth function on V. A point x in $\bar{S}$ is said to be *critical in* $\bar{S}$ if and only if one of the following conditions is satisfied. (1) x is an interior point of $\bar{S}$ and is critical for f. (2) x is a boundary point of $\bar{S}$ and it holds that $(df)_x(y-x) \geq 0$ for all $y \in \bar{S}$. The following is a finite dimensional version of Struwe's Mountain–Pass Lemma, where Palais–Smale condition is equivalent to the properness.

Lemma 3.4 ([S, Chapter II Theorem 1.13]). *If the function $f|\bar{S}$ is proper, then Mountain–Pass Lemma holds replacing critical by critical in $\bar{S}$.*

Proof (of Theorem 3.2). Let $\{v_i; 1 \leq i \leq k\}$ be orthonormal basis of $(\boldsymbol{m}')^{\perp}$ and $\{v_i; k < i \leq n\}$ that of $\boldsymbol{m}'$. We regard the vector space $V = \mathrm{Hom}_H(\boldsymbol{m}, \boldsymbol{k})$ as a subspace of $\boldsymbol{k}^n = \{(A_1, \ldots, A_k, A_{k+1}, \ldots, A_n)\}$, where $A_i = A(v_i)$. Using the decomposition : $\boldsymbol{k} = \boldsymbol{k}' \oplus \boldsymbol{z}(\boldsymbol{k})$ and the function L on $\boldsymbol{k}'$ defined by (2.5), we set

$$\bar{S} = \{(A_i) \in V;\ L((A_i)_{\boldsymbol{k}'}) \leq c \text{ and } |(A_i)_{\boldsymbol{z}(\boldsymbol{k})}| \leq c \text{ for } 1 \leq i \leq k\}, \tag{3.1}$$

where c is a sufficiently large constant. Then $\bar{S}$ is a closed convex domain of V and by Proposition 1.7 the function $\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G$ is proper on $\bar{S}$. If $A \in \partial\bar{S}$ is critical in $\bar{S}$, then, by definition, $(d\mathcal{F}_{\mathrm{YM}}|\mathcal{C}_G)_A(B-A) \geq 0$ for all $B \in \bar{S}$. But Proof of Theorem 2 implies the opposite inequality, provided that c is sufficiently large. Thus $A \in \bar{S}$ is critical in $\bar{S}$ if and only if A is critical in the usual sense, and so the proof reduces to Lemma 3.4. Q.E.D.

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