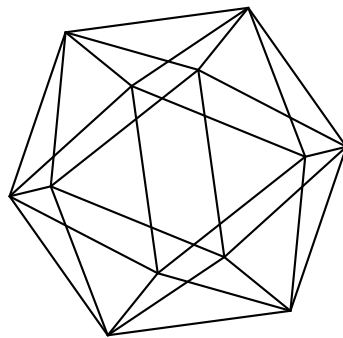


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A LARGE SIEVE INEQUALITY FOR QUARTIC CHARACTERS

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ABSTRACT. In this paper, we prove a large sieve inequality for quartic Dirichlet characters. The result is analogous to large sieve inequalities for the quadratic and cubic Dirichlet characters.

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1. INTRODUCTION

The large sieve was an idea originated by J. V. Linnik [16] in 1941 while studying the distribution of quadratic non-residues. Refinements and extensions in various directions of this idea were made by many [1–9, 11, 12, 16, 18, 20–27]. Large sieve results for Dirichlet characters with a fixed order are particularly useful in analytic number theory. We refer the readers to [10], Section 7, for some early large sieve-type results on general r -th order characters. Let $(a_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of complex numbers, D. R. Heath-Brown's quadratic large sieve [12, Theorem 1] states that for any $\varepsilon > 0$,

$$(1.1) \quad \sum_{m \leq M}^* \left| \sum_{n \leq N}^* a_n \left(\frac{n}{m} \right) \right|^2 \ll_{\varepsilon} (MN)^{\varepsilon} (M+N) \sum_{n \leq N}^* |a_n|^2,$$

where the asterisks indicate that m, n run over positive odd square-free integers and $(\frac{\cdot}{m})$ is the Jacobi symbol.

Similar to (1.1), Heath-Brown also established the following large sieve inequality involving the cubic symbols [13, Theorem 2]:

$$(1.2) \quad \sum_{\substack{m \in \mathbb{Z}[\omega] \\ N(m) \leq M}}^* \left| \sum_{\substack{n \in \mathbb{Z}[\omega] \\ N(n) \leq N}}^* a_n \left(\frac{n}{m} \right)_3 \right|^2 \ll \left(M + N + (MN)^{2/3} \right) (MN)^{\varepsilon} \sum_{N(n) \leq N} |a_n|^2,$$

where the asterisks indicate that m, n run over square-free elements of $\mathbb{Z}[\omega]$, $\omega = \exp(2\pi i/3)$ that are congruent to 1 modulo 3 and $(\frac{\cdot}{m})_3$ is the cubic residue symbol.

Using (1.2), S. Baier and M. P. Young [3, Theorem 1.4] proved the following large sieve inequality for cubic Dirichlet characters:

$$\sum_{Q < q \leq 2Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* \left| \sum_{M < m \leq 2M}^* a_m \chi(m) \right|^2 \ll (QM)^{\varepsilon} \min \left\{ Q^{5/3} + M, Q^{4/3} + Q^{1/2}M, Q^{11/9} + Q^{2/3}M, Q + Q^{1/3}M^{5/3} + M^{12/5} \right\} \sum_{M < m \leq 2M}^* |a_m|^2,$$

where the star on the sum over χ restricts the sum to primitive characters and the asterisks attached to the sum over m indicates that m runs over square-free integers.

It is our goal in this paper to prove a large sieve inequality for quartic Dirichlet characters. First we prove the following theorem involving the quartic symbols.

Theorem 1.1. *Let M, N be positive integers, and let $(a_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of complex numbers, where n runs over $\mathbb{Z}[i]$. Then we have*

$$(1.3) \quad \sum_{\substack{m \in \mathbb{Z}[i] \\ N(m) \leq M}}^* \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N(n) \leq N}}^* a_n \left(\frac{n}{m}\right)_4 \right|^2 \ll_{\varepsilon} \left(M + N + (MN)^{3/4} \right) (MN)^{\varepsilon} \sum_{N(n) \leq N} |a_n|^2,$$

for any $\varepsilon > 0$, where the asterisks indicate that m and n run over square-free elements of $\mathbb{Z}[i]$ that are congruent to 1 modulo $(1+i)^3$ and $(\frac{\cdot}{m})_4$ is the quartic residue symbol.

Next we shall establish the following large sieve inequality for quartic Dirichlet characters.

Theorem 1.2. *Let $(a_m)_{m \in \mathbb{N}}$ be an arbitrary sequence of complex numbers. Then*

$$(1.4) \quad \sum_{Q < q \leq 2Q} \sum_{\substack{\chi \pmod{q} \\ \chi^4 = \chi_0, \chi^2 \neq \chi_0}}^* \left| \sum_{M < m \leq 2M}^* a_m \chi(m) \right|^2 \ll (QM)^{\varepsilon} \min \left\{ Q^{7/4} + M, Q^{11/8} + Q^{1/2}M, Q^{5/4} + Q^{2/3}M, Q + Q^{1/2}M + M^{17/7} \right\} \sum_{M < m \leq 2M}^* |a_m|^2,$$

where the star on the sum over χ restricts the sum to primitive characters and the asterisks attached to the sum over m indicates that m runs over square-free integers.

Following the techniques of [12, 13], Theorem 1.1 is proved via recursive uses of the Poisson summation formula. Theorem 1.2 follows, after some transformations, from Theorem 1.1. We note that (1.3) is used in the (6.18). Mark that the characters involved in the second line of (6.18) are actually quadratic, since they are squares of the quartic symbol. Therefore, it is conceivable that the bounds in (1.4) can be improved if a large sieve inequality for quadratic characters in $\mathbb{Z}[i]$ is available.

Finally, we wish to mention that it is highly conceivable that these theorems will find applications in the study of families of L -functions involving quartic characters, analogous to those results in [3] and [17].

1.3. Notations. The following notations and conventions are used throughout the paper.

$$e(z) = \exp(2\pi iz) = e^{2\pi iz}.$$

$$\tilde{e}(z) = \exp(2\pi i(z + \bar{z})).$$

$f = O(g)$ or $f \ll g$ means $|f| \leq cg$ for some unspecified positive constant c .

2. PRELIMINARIES

2.1. Quartic symbol and the quartic Gauss sum. The symbol $(\frac{\cdot}{n})_4$ is the quartic residue symbol in the ring $\mathbb{Z}[i]$. For a prime $\pi \in \mathbb{Z}[i]$ with $N(\pi) \neq 2$, the quartic character is defined for $a \in \mathbb{Z}[i]$, $(a, \pi) = 1$ by $(\frac{a}{\pi})_4 \equiv a^{(N(\pi)-1)/4} \pmod{\pi}$, with $(\frac{a}{\pi})_4 \in \{\pm 1, \pm i\}$. When $\pi|a$, it is defined that $(\frac{a}{\pi})_4 = 0$. Then the quartic character can be extended to composite n with $(N(n), 2) = 1$ multiplicatively.

Note that in $\mathbb{Z}[i]$, every ideal coprime to 2 has a unique generator congruent to 1 modulo $(1+i)^3$ ([15, Lemma 7, page 121]). Such a generator is called primary. Recall that the quartic reciprocity law states that for two primary primes $m, n \in \mathbb{Z}[i]$,

$$\left(\frac{m}{n}\right)_4 = \left(\frac{n}{m}\right)_4 (-1)^{((N(n)-1)/4)((N(m)-1)/4)}.$$

Observe that a non-unit $n = a + bi$ in $\mathbb{Z}[i]$ with $a, b \in \mathbb{Z}$ is congruent to 1 modulo $(1 + i)^3$ if and only if $a \equiv 1 \pmod{4}, b \equiv 0 \pmod{4}$ or $a \equiv 3 \pmod{4}, b \equiv 2 \pmod{4}$ by Lemma 6 on page 121 of [15].

For $n \in \mathbb{Z}[i], n \equiv 1 \pmod{(1 + i)^3}$, we set

$$g(r, n) = \sum_{x \pmod{n}} \left(\frac{x}{n}\right)_4 \tilde{e}\left(\frac{rx}{n}\right),$$

where here and after

$$(2.1) \quad \tilde{e}(z) = \exp(2\pi i(z + \bar{z})).$$

The quartic Gauss sum $g(n)$ is then defined to be $g(n) = g(1, n)$.

For $\gcd(s, n) = 1$, we have

$$g(rs, n) = \overline{\left(\frac{s}{n}\right)_4} g(r, n).$$

It's easy to see that the above equality in fact holds for any s when $\left(\frac{\cdot}{n}\right)_4$ is a primitive character.

It's well-known that for square-free n 's,

$$|g(n)| = \sqrt{N(n)}.$$

Suppose $n \equiv \pm 1 \pmod{(1 + i)^3}$ with no rational prime divisor, so $(n, \bar{n}) = 1$. Let χ_n be a multiplicative character on $\mathbb{Z}[i]/(n)$, we define

$$(2.2) \quad \tau(\chi_n) = \sum_{1 \leq x \leq N(n)} \chi_n(x) e\left(\frac{x}{N(n)}\right).$$

Now we specify χ_n to be $\left(\frac{\cdot}{n}\right)_4$. On writing $x = y\bar{n} + \bar{y}n$, where y varies over a set of representatives in $\mathbb{Z}[i] \pmod{n}$, with \bar{n} being the complex conjugate of n , it's easy to see that

$$\tau(\chi_n) = \sum_{y \pmod{n}} \left(\frac{y\bar{n}}{n}\right)_4 e\left(\frac{y}{n} + \frac{\bar{y}}{n}\right) = \left(\frac{\bar{n}}{n}\right)_4 g(n).$$

It follows that for $(n_1, n_2) = 1$,

$$\tau(\chi_{n_1 n_2}) = \left(\frac{N(n_2)}{n_1}\right)_4 \left(\frac{N(n_1)}{n_2}\right)_4 \tau(\chi_{n_1}) \tau(\chi_{n_2}),$$

and that if n is square-free

$$|\tau(\chi_n)| = \sqrt{N(n)}.$$

Similarly, we have for n square-free

$$|\tau(\chi_n^2)| = \sqrt{N(n)}.$$

2.2. Primitive quartic Dirichlet characters. The classification of all the primitive cubic characters of conductor q coprime to 3 is given in [3]. Similarly, one can give a classification of all the primitive quartic characters of conductor q coprime to 2. Every such character is of the form $m \rightarrow \left(\frac{m}{n}\right)_4$ for some $n \in \mathbb{Z}[i]$, with $n \equiv 1 \pmod{(1 + i)^3}$, n square-free and not divisible by any rational primes and $N(n) = q$.

3. STRATEGY FOR THE PROOF OF THEOREM 1.1

Our proof of Theorem 1.1 uses the ideas in [12, 13]. We first estimate

$$\sum_1 = \sum_{\substack{m \in \mathbb{Z}[i] \\ M < N(m) \leq 2M}}^* \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}}^* a_n \left(\frac{n}{m}\right)_4 \right|^2.$$

We further simplify notation by supposing that the coefficients a_n are supported on such integers $n \in \mathbb{Z}[i]$ satisfying $N < N(n) \leq 2N$. We begin by defining the norm

$$\mathcal{B}_1(M, N) = \sup \left\{ \sum_1 : \sum_n |a_n|^2 = 1 \right\}.$$

Therefore, we need to show

$$\mathcal{B}_1(M, N) \ll_\varepsilon (MN)^\varepsilon (M + N + (MN)^{3/4}).$$

Introducing a smooth weight function, we have

$$\sum_1 \ll \sum_m \exp\left(-2\pi \frac{N(m)}{M}\right) \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}}^* a_n \left(\frac{n}{m}\right)_4 \right|^2,$$

the sum being over all $m \in \mathbb{Z}[i]$ for which $m \equiv 1 \pmod{(1+i)^3}$. If we now expand the above expression we obtain sums of the form

$$(3.1) \quad \sum_m \exp\left(-2\pi \frac{N(m)}{M}\right) \left(\frac{n_1}{m}\right)_4 \overline{\left(\frac{n_2}{m}\right)_4}.$$

We note the following analogue of Lemma 2 of Heath-Brown and Patterson [14]. As the proof is similar, we omit it here.

Lemma 3.1. *Let χ be a character of modulus $f \neq 1$, not necessarily primitive. Then, for $w \leq 1, \varepsilon > 0$,*

$$\theta(w, \chi) = \sum_{\substack{a \equiv 1 \pmod{(1+i)^3} \\ (a, f) = 1}} \chi(a) e^{-2\pi N(a)w} \ll E(\chi) w^{-1} + N(f)^{1/2+\varepsilon},$$

where $E(\chi) = 1$ if χ is principal, 0 otherwise. The implied constant depends only on ε .

Lemma 3.1 implies that each of these sums in (3.1) are $O(N(n_1 n_2)^{1/2+\varepsilon})$, provided that the character involved is non-principal. Since n_1 and n_2 are square-free, $\left(\frac{n_1}{m}\right)_4 \overline{\left(\frac{n_2}{m}\right)_4}$ is principle only if $n_1 = n_2$. It follows that

$$\sum_1 \ll_\varepsilon N^\varepsilon \left(M \sum_n |a_n|^2 + N \sum_{n_1, n_2} |a_{n_1} a_{n_2}| \right) \ll_\varepsilon N^\varepsilon (M + N^2) \sum_n |a_n|^2.$$

We therefore have

$$(3.2) \quad \mathcal{B}_1(M, N) \ll_\varepsilon N^\varepsilon (M + N^2).$$

This will be the starting point for an iterative bound for $\mathcal{B}_1(M, N)$.

Similar to the proof of [12, Lemma 1], using the duality principle (see for example, [19, Chap. 9]) and the quartic reciprocity law by considering the case for $n = a + bi$ with $a \equiv 1 \pmod{4}, b \equiv 0 \pmod{4}$ or $a \equiv 3 \pmod{4}, b \equiv 2 \pmod{4}$ (and similarly for m), we can establish the following lemma.

Lemma 3.2. *We have $\mathcal{B}_1(M, N) \leq 2\mathcal{B}_1(N, M)$. Moreover, there exist coefficients a'_n, a''_n with $|a'_n| = |a''_n| = |a_n|$ such that*

$$\begin{aligned} \sum_{\substack{m \in \mathbb{Z}[i] \\ M < N(m) \leq 2M}}^* \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}}^* a_n \left(\frac{n}{m}\right)_4 \right|^2 &\leq 2 \sum_{\substack{m \in \mathbb{Z}[i] \\ M < N(m) \leq 2M}}^* \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}}^* a'_n \left(\frac{m}{n}\right)_4 \right|^2 \\ &\leq 4 \sum_{\substack{m \in \mathbb{Z}[i] \\ M < N(m) \leq 2M}}^* \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}}^* a''_n \left(\frac{n}{m}\right)_4 \right|^2. \end{aligned}$$

Our next lemma is a trivial modification of Lemma 9 of [12], which shows that the norm $\mathcal{B}_1(M, N)$ is essentially increasing.

Lemma 3.3. *There is an absolute constant $C \geq 1$ as follows. Let $M_1, N \gg 1$ and $M_2 \geq CM_1 \log(2M_1N)$. Then*

$$\mathcal{B}_1(M_1, N) \ll \mathcal{B}_1(M_2, N).$$

Similarly, if $M, N_1 \gg 1$ and $N_2 \geq CN_1 \log(2N_1M)$. Then

$$\mathcal{B}_1(M, N_1) \ll \mathcal{B}_1(M, N_2).$$

Next, we define

$$\mathcal{B}_2(M, N) = \sup \left\{ \sum_2 : \sum_n |a_n|^2 = 1 \right\},$$

where

$$(3.3) \quad \sum_2 = \sum_{\substack{m \in \mathbb{Z}[i] \\ M < N(m) \leq 2M}} \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}}^* a_n \left(\frac{m}{n} \right)_4 \right|^2,$$

the summation over m running over all integers of $\mathbb{Z}[i]$ in the relevant range.

It follows directly from Lemma 3.2 that

$$(3.4) \quad \mathcal{B}_1(M, N) \leq 2\mathcal{B}_2(M, N).$$

For the other direction, we have the following.

Lemma 3.4. *There exist $X, Y \gg 1$ such that $XY^3 \ll M$ and*

$$\mathcal{B}_2(M, N) \ll (\log M)^3 M^{1/2} X^{-1/2} Y^{-3/2} \min(Y\mathcal{B}_1(X, N), X\mathcal{B}_1(Y, N)).$$

Proof. To handle \sum_2 we write each of the integers m occurring in the outer summation of (3.3) in the form $m = ab^2c^3d$, where $a, b, c \equiv 1 \pmod{(1+i)^3}$ are square-free, and d is a product of a unit, a power of $1+i$, and a fourth power (so that d can be written as $d = u(1+i)^j e^4$ where u is a unit, $0 \leq j \leq 3$ and $e \in \mathbb{Z}[i]$). We split the available ranges for a, b, c and d into sets $X < N(a) \leq 2X, Y < N(b) \leq 2Y, Z < N(c) \leq 2Z$ and $W < N(d) \leq 2W$, where X, Y, Z and W are powers of 2. There will therefore be $O(\log^3 M)$ possible quadruples X, Y, Z, W . We may now write

$$\sum_2 \ll \sum_{X, Y, Z, W} \sum_2(X, Y, Z, W)$$

accordingly, so that

$$\sum_2 \ll (\log^3 M) \sum_2(X, Y, Z, W)$$

for some quadruple X, Y, Z, W . However,

$$\sum_2(X, Y, Z, W) \leq \sum_{b, c, d} \sum_{a \in \mathbb{Z}[i]}^* \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}}^* a_n \left(\frac{b^2c^3d}{n} \right)_4 \left(\frac{a}{n} \right)_4 \right|^2,$$

where $X' = X'(b, c, d) = M/N(b^2c^3d)$. It is easy to see that $X \ll X' \ll X$, and hence by Lemma 3.2

$$\sum_2(X, Y, Z, W) \ll \sum_{b, c, d} \mathcal{B}_1(X', N) \sum_n |a_n|^2 \ll YZW^{1/4} \max\{\mathcal{B}_1(X', N) : X \ll X' \ll X\} \sum_n |a_n|^2,$$

since there are $O(W^{1/4})$ possible integers d .

In the same way we have

$$\begin{aligned}
\sum_2(X, Y, Z, W) &\leq \sum_{a,b,d} \sum_{\substack{c \in \mathbb{Z}[i] \\ Z' < N(c) \leq 2Z'}}^* \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}}^* a_n \left(\frac{ab^2d}{n} \right)_4 \left(\frac{c^3}{n} \right)_4 \right|^2 \\
&= \sum_{a,b,d} \sum_{\substack{c \in \mathbb{Z}[i] \\ Z' < N(c) \leq 2Z'}}^* \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}}^* \bar{a}_n \overline{\left(\frac{ab^2d}{n} \right)_4} \overline{\left(\frac{c^3}{n} \right)_4} \right|^2 \\
&= \sum_{a,b,d} \sum_{\substack{c \in \mathbb{Z}[i] \\ Z' < N(c) \leq 2Z'}}^* \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}}^* \bar{a}_n \overline{\left(\frac{ab^2d}{n} \right)_4} \left(\frac{c}{n} \right)_4 \right|^2 \\
&\ll \sum_{a,b,d} \mathcal{B}_1(Z', N) \sum_n |a_n|^2 \\
&\ll XYW^{1/4} \max \{ \mathcal{B}_1(Z', N) : Z \ll Z' \ll Z \} \sum_n |a_n|^2,
\end{aligned}$$

where $Z' = Z'(a, b, d) = M/N(ab^2d)$. As $Y \ll M^{1/2}X^{-1/2}Z^{-3/2}W^{-1/2}$, we see that

$$\mathcal{B}_2(M, N) \ll (\log M)^3 M^{1/2} X^{-1/2} Z^{-3/2} W^{-1/4} \min(Z\mathcal{B}_1(X, N), X\mathcal{B}_1(Z, N)).$$

The assertion of the lemma now follows on replacing Z by Y above. \square

As in [12], we introduce an infinitely differentiable weight function $W : \mathbb{R} \rightarrow \mathbb{R}$, defined by

$$(3.5) \quad W(x) = \begin{cases} \exp\left(\frac{-1}{(2x-1)(5-2x)}\right), & \text{if } \frac{1}{2} < x < \frac{5}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

We now have

$$(3.6) \quad \sum_2 \ll \sum_{m \in \mathbb{Z}[i]} W\left(\frac{N(m)}{M}\right) \left| \sum_{\substack{n \in \mathbb{Z}[i] \\ N < N(n) \leq 2N}} a_n \left(\frac{m}{n}\right)_4 \right|^2,$$

where we recall that we can drop the conditions on a_n on the inner sum above of the right-hand side expression by supposing that the coefficients a_n are supported on square-free integers $n \equiv 1 \pmod{(1+i)^3} \in \mathbb{Z}[i]$ lying in the range $N < N(n) \leq 2N$.

Expanding the sum on the right-hand side of (3.6), we obtain

$$\sum_2 \ll \sum_{n_1, n_2} a_{n_1} \bar{a}_{n_2} \sum_{m \in \mathbb{Z}[i]} W\left(\frac{N(m)}{M}\right) \left(\frac{m}{n_1}\right)_4 \overline{\left(\frac{m}{n_2}\right)_4}.$$

We set

$$\sum_3 = \sum_3(M, N) = \sum_{(n_1, n_2)=1} a_{n_1} \bar{a}_{n_2} \sum_{m \in \mathbb{Z}[i]} W\left(\frac{N(m)}{M}\right) \left(\frac{m}{n_1}\right)_4 \overline{\left(\frac{m}{n_2}\right)_4}$$

and define

$$\mathcal{B}_3(M, N) = \sup \left\{ \sum_3 : \sum_n |a_n|^2 = 1 \right\}.$$

Similar to [12, Lemma 7; 13], we have the following

Lemma 3.5. *Let $\varepsilon > 0$ be given. Then there exist positive integers $\Delta_2 \geq \Delta_1$ such that*

$$\mathcal{B}_2(M, N) \ll_{\varepsilon} N^{\varepsilon} \mathcal{B}_3\left(\frac{M}{\Delta_1}, \frac{N}{\Delta_2}\right).$$

We complete the chain of relations amongst the various norms by giving the following estimate for $\mathcal{B}_3(M, N)$ in terms of $\mathcal{B}_2(M, N)$.

Lemma 3.6. *Let $N \geq 1$. Then for any $\varepsilon > 0$ we have*

$$\mathcal{B}_3(M, N) \ll_{\varepsilon} MN^{4\varepsilon-1} \max\{\mathcal{B}_2(K, N) : K \leq N^2/M\} + M^{-1}N^{3+4\varepsilon} \sum_{K > N^2/M} K^{-2-\varepsilon} \mathcal{B}_2(K, N),$$

where K runs over powers of 2.

This bound uses the Poisson summation formula and is the key in the proof of Theorem 1.1. Note that it does not cover the case in which $N = 1/2$, say, for which we have the trivial bound

$$(3.7) \quad \mathcal{B}_3(M, N) \ll_{\varepsilon} M, \quad (N \leq 1).$$

Section 4 will be devoted to the proof of Lemma 3.6.

4. PROOF OF LEMMA 3.6

Our proof of Lemma 3.6 requires the application of the Poisson summation formula. We shall write

$$\chi(m) = \left(\frac{m}{n_1}\right)_4 \overline{\left(\frac{m}{n_2}\right)_4},$$

which is a primitive character (on the group $(\mathbb{Z}[i]/(n_1n_2))^{\times}$) to modulus $q = n_1n_2$, provided that n_1, n_2 and 2 are pair-wise coprime and n_1 and n_2 are square-free.

Lemma 4.1. *With the above notations we have*

$$\sum_{m \in \mathbb{Z}[i]} W\left(\frac{N(m)}{M}\right) \chi(m) = \frac{\chi(-\sqrt{-4})g(n_1)\overline{g(n_2)}M}{N(q)} \left(\frac{n_2}{n_1}\right)_4 \overline{\left(\frac{n_1}{n_2}\right)_4} \left(\frac{-1}{n_2}\right)_4 \sum_{m \in \mathbb{Z}[i]} \widetilde{W}\left(\sqrt{\frac{N(k)M}{N(q)}}\right) \overline{\chi}(k),$$

where

$$\widetilde{W}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(N(x+yi)) \tilde{e}\left(\frac{t(x+yi)}{\sqrt{-4}}\right) dx dy,$$

for non-negative t . Here $\tilde{e}(z)$ is defined in (2.1) and $g(n)$ is the Gauss sums defined in Section 2.1.

Proof. This lemma is analogous to Lemma 10 in [13] and the proof is very similar. The differences include we need to start with the Poisson summation formula for $\mathbb{Z}[i]$, which takes the form.

$$\sum_{j \in \mathbb{Z}[i]} f(j) = \sum_{k \in \mathbb{Z}[i]} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x+yi) \tilde{e}\left(\frac{k(x+yi)}{\sqrt{-4}}\right) dx dy.$$

We omit the details of the rest of proof as it simply goes along the same lines as the proof of Lemma 10 in [13]. \square

Our next result will be used to separate the variables in a function of a product, which is Lemma 12 of [12].

Lemma 4.2. *Let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ be an infinitely differentiable function whose derivatives satisfy the bound*

$$\rho(k)(x) \ll_{k,A} |x|^{-A}$$

for $|x| \geq 1$, for any positive constant A . Let

$$\rho_+(s) = \int_0^{\infty} \rho(x)x^{s-1} dx, \quad \rho_-(s) = \int_0^{\infty} \rho(-x)x^{s-1} dx.$$

Then $\rho_+(s)$ and $\rho_-(s)$ are holomorphic in $\Re(s) = \sigma > 0$, and satisfy

$$\rho_+(s), \rho_-(s) \ll_{A,\sigma} |s|^{-A},$$

in that same domain, for any positive constant A . Moreover if $\sigma > 0$ we have

$$\rho(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \rho_+(s)x^{-s} ds \quad \text{and} \quad \rho(-x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \rho_-(s)x^{-s} ds$$

for any positive x .

We are now ready to prove Lemma 3.6.

Proof of Lemma 3.6. In the notation of Lemma 4.1 we have

$$\sum_3(M, N) = \sum_{(n_1, n_2)=1} a_{n_1} \bar{a}_{n_2} \sum_{m \in \mathbb{Z}[i]} W\left(\frac{N(m)}{M}\right) \chi(m).$$

We proceed to evaluate the inner sum using Lemma 4.1, whence

$$(4.1) \quad \sum_3(M, N) = M \sum_{k \in \mathbb{Z}[i]} \sum_{(n_1, n_2)=1} c_{n_1} \bar{c}_{n_2} \left(\frac{n_2}{n_1}\right)_4 \overline{\left(\frac{n_1}{n_2}\right)_4} \left(\frac{-1}{n_2}\right)_4 \widetilde{W}\left(\sqrt{\frac{N(k)M}{N(n_1 n_2)}}\right) \bar{\chi}(k),$$

where

$$c_n = a_n \left(\frac{-\sqrt{-4}}{n}\right)_4 \frac{g(n)}{N(n)}.$$

Note by the law of quartic reciprocity, we have

$$\left(\frac{n_2}{n_1}\right)_4 \overline{\left(\frac{n_1}{n_2}\right)_4} = (-1)^{((N(n_1)-1)/4)((N(n_2)-1)/4)}.$$

Now we let

$$S_1 = \{n \in \mathbb{Z}[i] : N < N(n) \leq 2N, n \text{ square-free}, n = a + bi, a, b \in \mathbb{Z}, a \equiv 1 \pmod{4}, b \equiv 0 \pmod{4}\},$$

and

$$S_2 = \{n \in \mathbb{Z}[i] : N < N(n) \leq 2N, n \text{ square-free}, n = a + bi, a, b \in \mathbb{Z}, a \equiv 3 \pmod{4}, b \equiv 2 \pmod{4}\}.$$

We can then recast the inner sum in (4.1) as

$$\begin{aligned} & \sum_{(n_1, n_2)=1} \cdots \\ &= \sum_{\substack{(n_1, n_2)=1 \\ n_1 \in S_1, n_2 \in S_1}} \cdots + \sum_{\substack{(n_1, n_2)=1 \\ n_1 \in S_1, n_2 \in S_2}} \cdots + \sum_{\substack{(n_1, n_2)=1 \\ n_1 \in S_2, n_2 \in S_1}} \cdots + \sum_{\substack{(n_1, n_2)=1 \\ n_1 \in S_2, n_2 \in S_2}} \cdots - 2 \sum_{\substack{(n_1, n_2)=1 \\ n_1 \in S_2, n_2 \in S_2}} \cdots \\ &= \sum_{(n_1, n_2)=1} c_{n_1} \bar{c}_{n_2} \left(\frac{-1}{n_2}\right)_4 \widetilde{W}\left(\sqrt{\frac{N(k)M}{N(n_1 n_2)}}\right) \bar{\chi}(k) - 2 \sum_{(n_1, n_2)=1} c'_{n_1} \bar{c}'_{n_2} \left(\frac{-1}{n_2}\right)_4 \widetilde{W}\left(\sqrt{\frac{N(k)M}{N(n_1 n_2)}}\right) \bar{\chi}(k), \end{aligned}$$

where we let $c'_n = c_n$ if $n \in S_2$ and 0 otherwise. Due to similarities, it suffices to estimate

$$M \sum_{k \in \mathbb{Z}[i]} \sum_{(n_1, n_2)=1} c_{n_1} \bar{c}_{n_2} \left(\frac{-1}{n_2}\right)_4 \widetilde{W}\left(\sqrt{\frac{N(k)M}{N(n_1 n_2)}}\right) \bar{\chi}(k),$$

Note that $k = 0$ may be omitted if $N \geq 1$, since then $N(n_1 n_2) > 1$ and $\chi(0) = 0$, the character being non-trivial. We may now apply Lemma 4.2 to the function $\rho(x) = \widetilde{W}(x)$, which satisfies the necessary conditions of the lemma, as one sees by repeated integration by parts. We decompose the available k into

sets for which $K < N(k) \leq 2K$, where K runs over powers of 2, and use $\sigma = \varepsilon$ for $K \leq N^2/M$, and $\sigma = 4 + \varepsilon$ otherwise. This gives

$$\sum_3 \ll_\varepsilon M \sum_K (KM)^{-\sigma/2} \int_{-\infty}^{\infty} |\rho_+(\sigma + it)| |S(\sigma + it)| dt,$$

where

$$S(s) = \sum_{K < N(k) \leq 2K} \left| \sum_{(n_1, n_2)=1} b_{n_1} b'_{n_2} \left(\frac{-1}{n_2} \right)_4 \bar{\chi}(k) \right|, \quad \text{with } b_n = c_n N(n)^{s/2}, \quad b'_n = \bar{c}_n N(n)^{s/2}.$$

We use the Möbius function to detect the coprimality condition in the inner sum of $S(s)$, giving

$$\begin{aligned} S(s) &\ll \sum_d \sum_{K < N(k) \leq 2K} \left| \sum_{d|(n_1, n_2)} b_{n_1} b'_{n_2} \left(\frac{-1}{n_2} \right)_4 \bar{\chi}(k) \right| \\ &= \sum_d \sum_{K < N(k) \leq 2K} \left| \sum_{d|n} b_n \overline{\left(\frac{k}{n} \right)}_4 \right| \left| \sum_{d|n} b'_n \left(\frac{-k}{n} \right)_4 \right| \leq S_1^{1/2} S_2^{1/2}, \end{aligned}$$

by Cauchy's inequality, where

$$S_1 = \sum_d \sum_{K < N(k) \leq 2K} \left| \sum_{d|n} b_n \overline{\left(\frac{k}{n} \right)}_4 \right|^2$$

and satisfies the bound

$$S_1 \leq \sum_d \mathcal{B}_2(K, N) \sum_{d|n} |b_n|^2 \leq \mathcal{B}_2(K, N) \sum_n d(n) |a_n|^2 N(n)^{\sigma-1} \ll_\varepsilon N^{\varepsilon+\sigma-1} \mathcal{B}_2(K, N).$$

S_2 can be treated similarly. It follows then

$$S(s) \ll_\varepsilon N^{\varepsilon+\sigma-1} \mathcal{B}_2(K, N),$$

and since

$$\int_{-\infty}^{\infty} |\rho_+(\sigma + it)| dt \ll_\varepsilon 1,$$

we infer that

$$\sum_3 \ll_\varepsilon M \sum_K (KM)^{-\sigma/2} N^{\varepsilon+\sigma-1} \mathcal{B}_2(K, N).$$

This completes the proof of Lemma 3.6. □

5. THE RECURSIVE ESTIMATE AND THE PROOF OF THEOREM 1.1

Lemmas 3.4, 3.5 and 3.6 allow us to estimate $\mathcal{B}_1(M, N)$ recursively, as follows.

Lemma 5.1. *Suppose that $3/2 < \xi \leq 2$, and that*

$$(5.1) \quad \mathcal{B}_1(M, N) \ll_\varepsilon (MN)^\varepsilon \left(M + N^\xi + (MN)^{3/4} \right).$$

for any $\varepsilon > 0$. Then

$$\mathcal{B}_1(M, N) \ll_\varepsilon (MN)^\varepsilon \left(M + N^{(9\xi-6)/(4\xi-1)} + (MN)^{3/4} \right).$$

for any $\varepsilon > 0$.

Proof. By the symmetry expressed in Lemma 3.2 the hypothesis (5.1) yields

$$\mathcal{B}_1(M, N) \ll_{\varepsilon} \left(M^{\xi} + N + (MN)^{3/4} \right) (MN)^{\varepsilon}.$$

It follows from (3.2) that the above estimation is valid with $\xi = 2$. We now feed this into Lemma 3.4, whence

$$(5.2) \quad \mathcal{B}_2(M, N) \ll_{\varepsilon} (MN)^{2\varepsilon} M^{1/2} X^{-1/2} Y^{-3/2} \min(Yf(X, N), Xf(Y, N)),$$

where

$$f(Z, N) = Z^{\xi} + N + (ZN)^{3/4}.$$

If $X \geq Y$ we bound the minimum in (5.2) by $Yf(X, N)$, whence

$$\mathcal{B}_2(M, N) \ll_{\varepsilon} (MN)^{2\varepsilon} M^{1/2} X^{-1/2} Y^{-3/2} \left(YX^{\xi} + YN + Y(XN)^{3/4} \right).$$

Here we have

$$M^{1/2} X^{-1/2} Y^{-3/2} YX^{\xi} \ll M^{\xi} Y^{1-3\xi}$$

since $X \ll MY^{-3}$. On recalling that $\xi > 3/2 > 1/3$ and $Y \gg 1$ we see that this is $O(M^{\xi})$. Moreover

$$M^{1/2} X^{-1/2} Y^{-3/2} YN = M^{1/2} X^{-1/2} Y^{-1/2} N \ll M^{1/2} N.$$

Finally

$$M^{1/2} X^{-1/2} Y^{-3/2} Y(XN)^{3/4} = M^{1/2} X^{1/4} Y^{-1/2} N^{3/4} \ll M^{3/4} N^{3/4} \ll M^{1/2} N + M^{3/2} \leq M^{1/2} N + M^{\xi},$$

since $\xi > 3/2$. It follows that

$$(5.3) \quad \mathcal{B}_2(M, N) \ll_{\varepsilon} (MN)^{2\varepsilon} \left(M^{1/2} N + M^{\xi} \right)$$

when $X \geq Y$.

In the alternative case we bound the minimum in (5.2) by $Xf(Y, N)$, whence

$$\mathcal{B}_2(M, N) \ll_{\varepsilon} (MN)^{2\varepsilon} M^{1/2} X^{-1/2} Y^{-3/2} \left(XY^{\xi} + XN + X(YN)^{3/4} \right).$$

Here

$$M^{1/2} X^{-1/2} Y^{-3/2} XY^{\xi} \ll M^{1/2} X^{1/2} Y^{1/2} \ll M \ll M^{\xi}$$

since $\xi \leq 2$ and $XY \ll M$. Moreover

$$M^{1/2} X^{-1/2} Y^{-3/2} XN = M^{1/2} X^{1/2} Y^{-3/2} N \ll M^{1/2} N$$

since we are now supposing that $Y \geq X$. Finally

$$M^{1/2} X^{-1/2} Y^{-3/2} X(YN)^{3/4} = M^{1/2} X^{1/2} Y^{-3/4} N^{3/4} \ll M^{1/2} Y^{-1/4} N^{3/4} \ll M^{1/2} N^{3/4} \ll M^{1/2} N + M^{\xi},$$

as before. It follows that (5.3) holds when $Y \geq X$ too. It will be convenient to observe that (5.3) still holds when $M < 1/2$, since then $\mathcal{B}_2(M, N) = 0$.

We are now ready to use (5.3) (with a new value for ε) in Lemma 3.6, to obtain a bound for $\mathcal{B}_3(M, N)$. We readily see that

$$\max \{ \mathcal{B}_2(K, N) : K \leq N^2/M \} \ll_{\varepsilon} N^{\varepsilon} \left(M^{-1/2} N^2 + M^{-\xi} N^{2\xi} \right)$$

and

$$\sum_{K > N^2/M} K^{-2-\varepsilon} \mathcal{B}_2(K, N) \ll_{\varepsilon} N^{\varepsilon} \left(M^{3/2} N^{-2} + M^{2-\xi} N^{2\xi-4} \right).$$

Thus, if $N \geq 1$, we will have

$$\mathcal{B}_3(M, N) \ll_{\varepsilon} N^{5\varepsilon} \left(M^{1/2} N + M^{1-\xi} N^{2\xi-1} \right).$$

When this is used in Lemma 3.5 we find that when $N/\Delta_2 \geq 1$,

$$\begin{aligned} \mathcal{B}_3\left(\frac{M}{\Delta_1}, \frac{N}{\Delta_2}\right) &\ll_{\varepsilon} N^{5\varepsilon} \left(M^{1/2}N + M^{1-\xi}N^{2\xi-1}\Delta_1^{\xi-1}\Delta_2^{1-2\xi}\right) \leq N^{5\varepsilon} \left(M^{1/2}N + M^{1-\xi}N^{2\xi-1}\Delta_2^{-\xi}\right) \\ &\leq N^{5\varepsilon} \left(M^{1/2}N + M^{1-\xi}N^{2\xi-1}\right). \end{aligned}$$

Note that when $M \geq N$, we have $M^{1/2}N \leq (MN)^{3/4}$ and when $M \leq N$, we have $(MN)^{3/4} \leq M^{1-\xi}N^{2\xi-1}$ since $\xi > 3/2$. Thus we conclude that

$$\mathcal{B}_2(M, N) \ll_{\varepsilon} N^{6\varepsilon} \left((MN)^{3/4} + M^{1-\xi}N^{2\xi-1}\right),$$

provided that $N/\Delta_2 \geq 1$. In the alternative case (3.7) applies, whence

$$\mathcal{B}_2(M, N) \ll_{\varepsilon} (MN)^{6\varepsilon} \left(M + (MN)^{3/4} + M^{1-\xi}N^{2\xi-1}\right),$$

In view of Lemma 3.3 and (3.4) we may now deduce that

$$\mathcal{B}_1(M, N) \leq \mathcal{B}_1(M', N) \ll \mathcal{B}_2(M', N) \ll_{\varepsilon} (M'N)^{6\varepsilon} \left(M' + (M'N)^{3/4} + M'^{1-\xi}N^{2\xi-1}\right),$$

for any $M' \geq CM \log(2MN)$. Note that when $M^{4\xi-1} \leq N^{8\xi-7}$, we have

$$(MN)^{3/4} \leq M^{1-\xi}N^{2\xi-1}.$$

We shall now choose

$$M' = C \max \left\{ M, N^{(8\xi-7)/(4\xi-1)} \right\} \log(2MN),$$

so that when $M \geq N^{(8\xi-7)/(4\xi-1)}$, we have

$$M' + (M'N)^{3/4} + M'^{1-\xi}N^{2\xi-1} \ll (MN)^{\varepsilon} \left(M + (MN)^{3/4}\right),$$

while when $M \leq N^{(8\xi-7)/(4\xi-1)}$, we have

$$\begin{aligned} M' + (M'N)^{3/4} + M'^{1-\xi}N^{2\xi-1} &\ll (MN)^{\varepsilon} \left(N^{(8\xi-7)/(4\xi-1)} + N^{(8\xi-7)(1-\xi)/(4\xi-1)}N^{2\xi-1}\right) \\ &\ll (MN)^{\varepsilon} N^{(9\xi-6)/(4\xi-1)}. \end{aligned}$$

We then deduce that

$$\mathcal{B}_1(M, N) \ll_{\varepsilon} (MN)^{20\varepsilon} \left(M + (MN)^{3/4} + N^{(9\xi-6)/(4\xi-1)}\right).$$

Lemma 5.1 now follows. \square

We now proceed to prove Theorem 1.1.

Proof of Theorem 1.1. We note that it follows from (3.2) that the estimation given in Lemma 5.1 is valid with $\xi = 2$. We further observe that

$$\frac{9\xi - 6}{4\xi - 1} < \xi,$$

for $\xi > 3/2$. Therefore, the infimum of the fraction above is $3/2$. We therefore arrive at the following bound

$$\mathcal{B}_1(M, N) \ll_{\varepsilon} (MN)^{\varepsilon} \left(M + N^{3/2} + (MN)^{3/4}\right),$$

for any $\varepsilon > 0$. Using Lemma 3.2 we then have

$$\begin{aligned} \mathcal{B}_1(M, N) &\ll_{\varepsilon} (MN)^{\varepsilon} \min \left\{ M + N^{3/2} + (MN)^{3/4}, N + M^{3/2} + (MN)^{3/4} \right\} \\ &\ll_{\varepsilon} (MN)^{\varepsilon} \left(M + N + (MN)^{3/4}\right), \end{aligned}$$

where the last estimation follows since when $N \leq M$, $N^{3/2} = N^{3/4}N^{3/4} \leq (MN)^{3/4}$ and similarly when $N \geq M$, $M^{3/2} \leq (MN)^{3/4}$. This establishes Theorem 1.1. \square

6. THE QUARTIC LARGE SIEVE FOR DIRICHLET CHARACTERS

We now proceed to prove Theorem 1.2. It is easy to reduce the expression on the left-hand side of (1.4) to a sum of similar expressions with the additional summation conditions $(q, 2) = 1$ and $(m, 2) = 1$ included. Thus it suffices to estimate

$$(6.1) \quad \sum_{\substack{Q < q \leq 2Q \\ (q, 2) = 1}} \sum_{\substack{\chi \pmod{q} \\ \chi^4 = \chi_0, \chi^2 \neq \chi_0}}^* \left| \sum_{\substack{M < m \leq 2M \\ (m, 2) = 1}}^* a_m \chi(m) \right|^2 = \sum'_{\substack{n \in \mathbb{Z}[i] \\ Q < N(n) \leq 2Q \\ n \equiv 1 \pmod{(1+i)^3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m, 2) = 1}}^* a_m \chi_n(m) \right|^2 \\ = \frac{1}{2} \sum'_{\substack{n \in \mathbb{Z}[i] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{(1+i)^3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m, 2) = 1}}^* a_m \chi_n(m) \right|^2,$$

where the apostrophe indicates that n is square-free and has no rational prime divisor and $\chi_n(m) = \left(\frac{m}{n}\right)_4$ is the quartic residue symbol. We shall use this notation for all $n \in \mathbb{Z}[i]$ and $m \in \mathbb{Z}$.

6.1. Definition of certain norms. In the following, we shall estimate the expression in the last line of (6.1). We begin by defining a norm corresponding to the double sum in the last line of (6.1) by

$$B_1(Q, M) := \sup_{(a_m)} \|a_m\|^{-2} \sum'_{\substack{n \in \mathbb{Z}[i] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{(1+i)^3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m, 2) = 1}}^* a_m \chi_m(n) \right|^2,$$

where

$$\|a_m\|^2 = \sum_m |a_m|^2,$$

and where by convention we suppose that (a_m) is not identically zero.

We further define a norm $B_2(Q, M)$ in the same way as $B_1(Q, M)$ except removing the condition that n has no rational prime divisor. Similarly, we define a norm $B_3(Q, M)$ by further removing the condition that n is square-free.

We now use the function $W(x)$ defined in (3.5) to see that

$$B_3(Q, M) \ll \sup_{(a_m)} \|a_m\|^{-2} \sum_{n \in \mathbb{Z}[i]} W\left(\frac{N(n)}{Q}\right) \left| \sum_{\substack{M < m \leq 2M \\ (m, 2) = 1}}^* a_m \chi_m(n) \right|^2.$$

Expanding the sum on the right-hand side, we obtain

$$\sum_{\substack{M < m_1 \leq 2M \\ M < m_2 \leq 2M \\ (m_1 m_2, 2) = 1}}^* a_{m_1} \bar{a}_{m_2} \sum_{n \in \mathbb{Z}[i]} W\left(\frac{N(n)}{Q}\right) \chi_{m_1}(n) \bar{\chi}_{m_2}(n).$$

As in [12], it turns out that we may restrict our attention to the case in which m_1 and m_2 are coprime. We define another norm $B_4(Q, M)$ corresponding to the above sum with the restriction $(m_1, m_2) = 1$ included by

$$B_4(Q, M) := \sup_{(a_m)} \|a_m\|^{-2} \sum_{\substack{M < m_1, m_2 \leq 2M \\ (m_1 m_2, 2) = 1 \\ (m_1, m_2) = 1}}^* a_{m_1} \bar{a}_{m_2} \sum_{n \in \mathbb{Z}[i]} W\left(\frac{N(n)}{Q}\right) \chi_{m_1}(n) \bar{\chi}_{m_2}(n).$$

Moreover, we define a norm $C_1(M, Q)$ dual to $B_1(Q, M)$ by

$$C_1(M, Q) := \sup_{(b_n)} \|b_n\|^{-2} \sum_{\substack{M < m \leq 2M \\ (m, 2) = 1}}^* \left| \sum'_{\substack{n \in \mathbb{Z}[i] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{(1+i)^3}}} b_n \chi_n(m) \right|^2.$$

By the duality principle, we have

$$(6.2) \quad B_1(Q, M) = C_1(M, Q).$$

Furthermore, we define a norm $C_2(M, Q)$ by extending the summation over m in the definition of $C_1(M, Q)$ to all integers m with $M < m \leq 2M$. Trivially, we have

$$(6.3) \quad C_1(M, Q) \leq C_2(M, Q).$$

6.2. Comparison of the norms. For the proof of our Theorem 1.2, we need the following lemma on the norms defined in the previous section.

Lemma 6.3. *Let $Q, M \geq 1$ and C be a sufficiently large positive constant. Then we have the following inequalities:*

$$(6.4) \quad C_2(M, Q) \ll (QM)^\varepsilon \left(M + Q^{7/4} \right);$$

$$(6.5) \quad C_2(M, Q) \ll M^\varepsilon Q^{1-1/v} \sum_{j=0}^{v-1} C_2(2^j M^v, Q)^{1/v}, \quad \text{for each fixed positive integer } v;$$

$$(6.6) \quad B_1(Q_1, M) \ll B_1(Q_2, M), \quad \text{if } Q_1, M \geq 1 \text{ and } Q_2 \geq CQ_1 \log(2Q_1 M);$$

$$(6.7) \quad B_2(Q, M) \ll (\log 2Q)^3 Q^{1/2} X^{-1/2} B_1(XQ^\varepsilon, M), \quad \text{for some } X \text{ with } 1 \leq X \leq Q;$$

$$(6.8) \quad B_3(Q, M) \ll (\log 2Q)^3 Q^{1/2} X^{-1/2} B_2(XQ^\varepsilon, M), \quad \text{for some } X \text{ with } 1 \leq X \leq Q;$$

$$(6.9) \quad B_3(Q, M) \ll M^\varepsilon B_4 \left(\frac{Q}{\Delta_1}, \frac{M}{\Delta_2} \right), \quad \text{for some } \Delta_1, \Delta_2 \in \mathbb{N} \text{ with } \Delta_2^2 \geq \Delta_1;$$

$$(6.10) \quad B_4(Q, M) \ll Q + QM^{\varepsilon-2} \max \{ B_3(K, M) : K \leq M^4 Q^{-1} \} + Q^{-1} M^{6+\varepsilon} \sum_{K > M^4/Q} K^{-2-\varepsilon} B_3(K, M),$$

where the sum over K in (6.10) runs over powers of 2.

Since the proofs of (6.5)-(6.10) are essentially the same as those of (31)-(36) of Lemma 4.1 in [13], we omit the proofs here.

We note that it follows from (6.2)-(6.5), we have

$$(6.11) \quad B_1(Q, M) \ll (QM)^\varepsilon \left(Q^{1-1/v} M + Q^{1+3/(4v)} \right)$$

for any $v \in \mathbb{N}$.

6.4. **Estimating C_2 .** In this section we prove (6.4). Recall $C_2(M, Q)$ is the norm of the sum

$$(6.12) \quad \sum_{M < m \leq 2M} \left| \sum'_{\substack{n \in \mathbb{Z}[i] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{(1+i)^3}}} b_n \chi_n(m) \right|^2,$$

where the apostrophe indicates that n is square-free and has no rational prime divisor.

The sum in (6.12) is obviously bounded by

$$(6.13) \quad \ll \sum_{m \in \mathbb{Z}} W\left(\frac{m}{M}\right) \left| \sum'_n b_n \chi_n(m) \right|^2,$$

where the weight function W is defined as in (3.5). Expanding out the sum in (6.13) we get

$$\sum'_{\substack{n_1, n_2 \in \mathbb{Z}[i] \\ Q < N(n_1), N(n_2) \leq 2Q \\ n_1, n_2 \equiv \pm 1 \pmod{(1+i)^3}}} b_{n_1} \bar{b}_{n_2} \sum_{m \in \mathbb{Z}} W\left(\frac{m}{M}\right) \chi_{n_1} \bar{\chi}_{n_2}(m).$$

Now we extract the greatest common divisor Δ of n_1 and n_2 , getting

$$\sum'_{\substack{N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{(1+i)^3}}} \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[i] \\ \frac{Q}{N(\Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(\Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{(1+i)^3} \\ (n_1, n_2) = 1 \\ (n_1 n_2, \Delta \bar{\Delta}) = 1}} b_{n_1 \Delta} \bar{b}_{n_2 \Delta} \sum_{\substack{m \in \mathbb{Z} \\ (m, N(\Delta)) = 1}} W\left(\frac{m}{M}\right) \chi_{n_1} \bar{\chi}_{n_2}(m).$$

Now we write $\delta = (n_1, \bar{n}_2)$ and change variables via $n_1 \rightarrow \delta n_1$, $n_2 \rightarrow \bar{\delta} n_2$ to get

$$(6.14) \quad \sum'_{\substack{N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{(1+i)^3}}} \sum'_{\substack{N(\delta) \leq \frac{2Q}{N(\Delta)} \\ \delta \equiv \pm 1 \pmod{(1+i)^3} \\ (N(\delta), N(\Delta)) = 1}} \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[i] \\ \frac{Q}{N(\delta \Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(\delta \Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{(1+i)^3} \\ (N(n_1), N(n_2)) = 1 \\ (n_1 n_2, \delta \bar{\delta} \Delta \bar{\Delta}) = 1}} b_{n_1 \Delta \delta} \bar{b}_{n_2 \Delta \bar{\delta}} \\ \times \sum_{\substack{m \in \mathbb{Z} \\ (m, N(\Delta)) = 1}} W\left(\frac{m}{M}\right) \chi_{n_1} \bar{\chi}_{n_2} \bar{\chi}_\delta^2(m),$$

where we use that for $m \in \mathbb{Z}$, $\chi_\delta \bar{\chi}_{\bar{\delta}}(m) = \chi_\delta^2(m) = \bar{\chi}_\delta^2(m)$. Next we remove the coprimality condition in the sum over m by the Möbius function, getting

$$\sum_{\substack{m \in \mathbb{Z} \\ (m, N(\Delta)) = 1}} W\left(\frac{m}{M}\right) \chi_{n_1} \bar{\chi}_{n_2} \bar{\chi}_\delta^2(m) = \sum_{l|N(\Delta)} \mu(l) \chi_{n_1} \bar{\chi}_{n_2} \bar{\chi}_\delta^2(l) \sum_{m \in \mathbb{Z}} W\left(\frac{m}{M/l}\right) \chi_{n_1} \bar{\chi}_{n_2} \bar{\chi}_\delta^2(m),$$

which by the Poisson summation formula is

$$(6.15) \quad \sum_{l|N(\Delta)} \mu(l) \chi_{n_1} \bar{\chi}_{n_2} \bar{\chi}_\delta^2(l) \frac{M}{lN(n_1 n_2 \delta)} \sum_{h \in \mathbb{Z}} \widehat{W}\left(\frac{hM}{lN(n_1 n_2 \delta)}\right) \\ \times \sum_{r \pmod{N(n_1 n_2 \delta)}} \chi_{n_1} \bar{\chi}_{n_2} \bar{\chi}_\delta^2(r) e\left(\frac{rh}{N(n_1 n_2 \delta)}\right),$$

where

$$\widehat{W}(x) = \int_{-\infty}^{\infty} W(y)e(-xy)dy.$$

When $h = 0$, the expression in (6.15) vanishes unless $n_1 = n_2 = \delta = 1$. Hence, the contribution of $h = 0$ to (6.14) is

$$(6.16) \quad \ll MQ^\varepsilon \sum'_{\substack{Q < N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{(1+i)^3}}} |b_\Delta|^2 \ll MQ^\varepsilon \|b_n\|^2.$$

In the sequel, we assume that $h \neq 0$. The sum over r in (6.15) can be computed by writing $r = r_1N(n_2\delta) + r_2N(n_1\delta) + r_3N(n_1n_2)$ to get

$$(6.17) \quad \begin{aligned} & \sum_{r \pmod{N(n_1n_2\delta)}} \chi_{n_1} \bar{\chi}_{n_2} \bar{\chi}_\delta^2(r) e\left(\frac{rh}{N(n_1n_2\delta)}\right) \\ &= \sum_{r_1 \pmod{N(n_1)}} \chi_{n_1}(r_1N(n_2\delta)) e\left(\frac{r_1h}{N(n_1)}\right) \sum_{r_2 \pmod{N(n_2)}} \bar{\chi}_{n_2}(r_2N(n_1\delta)) e\left(\frac{r_2h}{N(n_2)}\right) \\ & \quad \times \sum_{r_3 \pmod{N(\delta)}} \bar{\chi}_\delta^2(r_3N(n_1n_2)) e\left(\frac{r_3h}{N(\delta)}\right) \\ &= \bar{\chi}_{n_1}(h) \chi_{n_2} \chi_\delta^2(-h) \chi_{n_1}(N(n_2\delta)) \bar{\chi}_{n_2}(N(n_1\delta)) \bar{\chi}_\delta^2(N(n_1n_2)) \tau(\chi_{n_1}) \overline{\tau(\chi_{n_2})} \overline{\tau(\chi_\delta^2)}, \end{aligned}$$

where $\tau(\chi)$ is defined as in (2.2).

Using quartic reciprocity and the identity

$$\left(\frac{m}{n}\right)_4 = \overline{\left(\frac{\bar{m}}{\bar{n}}\right)}_4$$

following from the definition of the quartic residue symbol, we get the identities

$$\begin{aligned} \chi_n(N(m)) \bar{\chi}_m(N(n)) &= \left(\frac{N(m)}{n}\right)_4 \overline{\left(\frac{N(n)}{m}\right)}_4 = \left(\frac{m}{\bar{n}}\right)_4 = \chi_n^2(m) \\ \chi_n(N(m)) \bar{\chi}_m^2(N(n)) &= \left(\frac{N(m)}{n}\right)_4 \left(\frac{N(n)}{m}\right)_4^2 = \left(\frac{N(m)}{\bar{n}}\right)_4 = \chi_{\bar{n}}(N(m)) \end{aligned}$$

and

$$\bar{\chi}_n(N(m)) \bar{\chi}_m^2(N(n)) = \overline{\left(\frac{N(m)}{n}\right)}_4 \overline{\left(\frac{N(n)}{m}\right)}_4^2 = \left(\frac{N(m)}{n}\right)_4 = \chi_n(N(m)),$$

valid for all $m, n \in \mathbb{Z}[i]$ with $m, n \equiv \pm 1 \pmod{(1+i)^3}$. We use them to simplify the last line of (6.17), obtaining

$$\bar{\chi}_{n_1}(h) \chi_{n_2} \chi_\delta^2(-h) \left(\frac{n_2^2 N(\delta)}{n_1}\right)_4 \left(\frac{N(\delta)}{n_2}\right)_4 \tau(\chi_{n_1}) \overline{\tau(\chi_{n_2})} \overline{\tau(\chi_\delta^2)}.$$

Now, changing $n_2 \rightarrow \bar{n}_2$, the contribution of $h \neq 0$ to the sum in (6.14) takes the form

$$S_W(M, Q) = M \sum'_{\substack{N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{(1+i)^3}}} \sum'_{\substack{N(\delta) \leq \frac{2Q}{N(\Delta)} \\ \delta \equiv \pm 1 \pmod{(1+i)^3} \\ (N(\delta), N(\Delta))=1}} \frac{\overline{\tau(\chi_\delta)}}{N(\delta)} \sum_{l|N(\Delta)} \frac{\mu(l)}{l} \overline{\chi_\delta^2}(l) \sum_{h \neq 0} \chi_\delta^2(h) \\ \times \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[i] \\ \frac{Q}{N(\delta\Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(\delta\Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{(1+i)^3} \\ (N(n_1), N(n_2))=1 \\ (n_1 n_2, \delta \bar{\delta} \Delta \bar{\Delta})=1}} \widehat{W} \left(\frac{hM}{lN(n_1 n_2 \delta)} \right) c_{\Delta, \delta, l, h, n_1} c'_{\Delta, \delta, l, h, n_2} \left(\frac{n_1}{n_2} \right)_4^2,$$

where

$$c_{\Delta, \delta, l, h, n} := \chi_n(l) \overline{\chi_n}(h) \left(\frac{N(\delta)}{\bar{n}} \right)_4 \frac{\tau(\chi_n)}{N(n)} b_{n\Delta\delta} \quad \text{and} \quad c'_{\Delta, \delta, l, h, n} := \chi_n(l) \overline{\chi_n}(h) \left(\frac{N(\delta)}{\bar{n}} \right)_4 \frac{\tau(\chi_n)}{N(n)} \bar{b}_{\bar{n}\Delta\bar{\delta}}.$$

We now estimate the sum over n_1 and n_2 directly using (1.3). We denote the inner sum in the definition of $S_W(M, Q)$ above to be $U(\Delta, \delta, l, h)$ so that

$$U(\Delta, \delta, l, h) = \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[i] \\ \frac{Q}{N(\delta\Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(\delta\Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{(1+i)^3} \\ (N(n_1), N(n_2))=1 \\ (n_1 n_2, \delta \bar{\delta} \Delta \bar{\Delta})=1}} \widehat{W} \left(\frac{hM}{lN(n_1 n_2 \delta)} \right) c_{\Delta, \delta, l, h, n_1} c'_{\Delta, \delta, l, h, n_2} \left(\frac{n_1}{n_2} \right)_4^2.$$

To separate the variables n_1, n_2 , we remove the coprimality condition $(N(n_1), N(n_2)) = 1$ in the standard way using the Möbius function to obtain

$$U(\Delta, \delta, l, h) = \sum_{\substack{e' \in \mathbb{Z} \\ e' \equiv 1 \pmod{4}}} \mu(|e'|) \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[i] \\ \frac{Q}{N(\delta\Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(\delta\Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{(1+i)^3} \\ e'|N(n_1), e'|N(n_2) \\ (n_1 n_2, \delta \bar{\delta} \Delta \bar{\Delta})=1}} \widehat{W} \left(\frac{hM}{lN(n_1 n_2 \delta)} \right) c_{\Delta, \delta, l, h, n_1} c'_{\Delta, \delta, l, h, n_2} \left(\frac{n_1}{n_2} \right)_4^2.$$

Due to the presence of $\left(\frac{n_1}{n_2}\right)_4$ and the restrictions that n_1, n_2 are square-free and have no rational prime divisors, we can recast the above as

$$U(\Delta, \delta, l, h) = \sum'_{\substack{e \in \mathbb{Z}[i] \\ e \equiv 1 \pmod{(1+i)^3} \\ (N(e), N(\delta\Delta))=1}} \mu(N(e)) \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[i] \\ \frac{Q}{N(\delta\Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(\delta\Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{(1+i)^3} \\ e|n_1, \bar{e}|n_2 \\ (n_1 n_2, \delta \bar{\delta} \Delta \bar{\Delta})=1}} \widehat{W} \left(\frac{hM}{lN(n_1 n_2 \delta)} \right) c_{\Delta, \delta, l, h, n_1} c'_{\Delta, \delta, l, h, n_2} \left(\frac{n_1}{n_2} \right)_4^2 \\ = \sum'_{\substack{e \in \mathbb{Z}[i] \\ e \equiv 1 \pmod{(1+i)^3} \\ N(e) \leq \frac{2Q}{N(\delta\Delta)} \\ (N(e), N(\delta\Delta))=1}} \mu(N(e)) \chi_{N(e)}(l) \overline{\chi_{N(e)}}(h) \left(\frac{N(\delta)}{N(e)} \right)_4 \frac{\tau(\chi_e)}{N(e)} \frac{\tau(\chi_{\bar{e}})}{N(\bar{e})} \left(\frac{e}{\bar{e}} \right)_4^2 \\ \times \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[i] \\ \frac{Q}{N(e\delta\Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(e\delta\Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{(1+i)^3} \\ (n_1 n_2, e\bar{e}\delta\bar{\delta}\Delta\bar{\Delta})=1}} \widehat{W} \left(\frac{hM}{lN(n_1 n_2 e\bar{e}\delta)} \right) c_{\Delta, \delta, l, h, e, n_1} c'_{\Delta, \delta, l, h, e, n_2} \left(\frac{n_1}{n_2} \right)_4^2.$$

where

$$c_{\Delta,\delta,l,h,e,n} := \chi_n(l)\bar{\chi}_n(h) \left(\frac{N(\delta)}{\bar{n}}\right)_4 \left(\frac{N(e)}{n}\right)_4 \left(\frac{N(n)}{e}\right)_4 \left(\frac{n_1}{\bar{e}}\right)_4^2 \frac{\tau(\chi_n)}{N(n)} b_{ne\Delta\delta}$$

and

$$c'_{\Delta,\delta,l,h,e,n} := \chi_n(l)\bar{\chi}_n(h) \left(\frac{N(\delta)}{\bar{n}}\right)_4 \left(\frac{N(e)}{n}\right)_4 \left(\frac{N(n)}{\bar{e}}\right)_4 \left(\frac{e}{n_2}\right)_4^2 \frac{\tau(\chi_n)}{N(n)} \bar{b}_{ne\Delta\delta}.$$

Next observe that we may freely truncate the sum over h for

$$|h| \leq \frac{Q^2 l}{N(\delta)N(\Delta)^2 M} (QM)^\varepsilon =: H$$

since \widehat{W} has rapid decay. More precisely, if we let $S_W(M, Q) = S'_W(M, Q) + E$ where $S'_W(M, Q)$ is the contribution to $S_W(M, Q)$ from $0 < |h| \leq H$, then $E \ll (MQ)^{-100} \|b\|^2$.

It remains to bound $S'_W(M, Q)$ and we have

$$\begin{aligned} S'_W(M, Q) &\ll M \sum'_{\substack{N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{(1+i)^3}}} \sum'_{\substack{N(\delta) \leq \frac{2Q}{N(\Delta)} \\ \delta \equiv \pm 1 \pmod{(1+i)^3} \\ (N(\delta), N(\Delta))=1}} \frac{1}{(N(\delta))^{1/2}} \sum_{l|N(\Delta)} \frac{1}{l} \\ &\quad \times \sum'_{\substack{e \in \mathbb{Z}[i] \\ e \equiv 1 \pmod{(1+i)^3} \\ N(e) \leq \frac{2Q}{N(\delta\Delta)} \\ (N(e), N(\delta\Delta))=1}} \frac{1}{N(e)} \sum_{0 < |h| \leq H} |U'(\Delta, \delta, l, e, h)|, \end{aligned}$$

where

$$U'(\Delta, \delta, l, e, h) = \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[i] \\ \frac{Q}{N(e\delta\Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(e\delta\Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{(1+i)^3} \\ (n_1 n_2, e\bar{e}\delta\bar{\delta}\Delta\bar{\Delta})=1}} \widehat{W} \left(\frac{hM}{lN(n_1 n_2 e\bar{e}\delta)} \right) c_{\Delta,\delta,l,h,e,n_1} c'_{\Delta,\delta,l,h,e,n_2} \left(\frac{n_1}{n_2} \right)_4^2.$$

We now remove the weight \widehat{W} in $U'(\Delta, \delta, l, e, h)$ by applying Lemma 4.2 to the function $\rho(x) = \widehat{W}(x)$, which satisfies the conditions of that lemma, as one sees by repeated integration by parts. We may assume $h > 0$ here, since the contribution of the negative h 's can be treated similarly and satisfies the same bound. We use $\sigma = \varepsilon$ to see that

$$U'(\Delta, \delta, l, e, h) \ll_\varepsilon \left(\frac{lN(e\bar{e}\Delta)}{hM} \right)^\varepsilon \int_{-\infty}^{\infty} |\rho_+(\varepsilon + it)| |V(\varepsilon + it)| dt,$$

where

$$V(s) = \left| \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[i] \\ \frac{Q}{N(e\delta\Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(e\delta\Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{(1+i)^3} \\ (n_1 n_2, e\bar{e}\delta\bar{\delta}\Delta\bar{\Delta})=1}} d_{n_1} d'_{n_2} \left(\frac{n_1}{n_2} \right)_4^2 \right|,$$

with

$$d_n = c_{\Delta,\delta,l,h,e,n} N(n)^s \quad \text{and} \quad d'_n = c'_{\Delta,\delta,l,h,e,n} N(n)^s.$$

Note that d_{n_1} and d'_{n_2} depend on $\Delta, \delta, l, h, e, n$ and s , and

$$|d_n| \ll \left(\frac{N(e\delta\Delta)}{Q} \right)^{1/2-\varepsilon} |b_{ne\Delta\delta}|, \quad |d'_n| \ll \left(\frac{N(e\delta\Delta)}{Q} \right)^{1/2-\varepsilon} |b_{\bar{n}e\Delta\bar{\delta}}|.$$

Now, using the Cauchy-Schwarz inequality and the estimate (1.3) upon noting that this estimate remains valid if the summation conditions $m, n \equiv 1 \pmod{(1+i)^3}$ therein are replaced by $m, n \equiv \pm 1 \pmod{(1+i)^3}$ and $\left(\frac{n}{m}\right)_4$ replaced by $\left(\frac{n_1}{n_2}\right)_4^2$, we bound $V(\varepsilon + it)$ by

$$\begin{aligned}
(6.18) \quad & V(\varepsilon + it) \\
& \ll \left(\sum'_{\substack{n_1 \in \mathbb{Z}[i] \\ \frac{Q}{N(e\delta\Delta)} < N(n_1) \leq \frac{2Q}{N(e\delta\Delta)} \\ n_1 \equiv \pm 1 \pmod{(1+i)^3} \\ (n_1, e\bar{e}\delta\bar{\delta}\Delta\bar{\Delta})=1}} |d_{n_1}|^2 \right)^{1/2} \left(\sum'_{\substack{n_1 \in \mathbb{Z}[i] \\ \frac{Q}{N(e\delta\Delta)} < N(n_1) \leq \frac{2Q}{N(e\delta\Delta)} \\ n_1 \equiv \pm 1 \pmod{(1+i)^3} \\ (n_1, e\bar{e}\delta\bar{\delta}\Delta\bar{\Delta})=1}} \left| \sum'_{\substack{n_2 \in \mathbb{Z}[i] \\ \frac{Q}{N(e\delta\Delta)} < N(n_2) \leq \frac{2Q}{N(e\delta\Delta)} \\ n_2 \equiv \pm 1 \pmod{(1+i)^3} \\ (n_2, e\bar{e}\delta\bar{\delta}\Delta\bar{\Delta})=1}} d'_{n_2} \left(\frac{n_1}{n_2}\right)_4^2 \right) \right|^{1/2} \\
& \ll (QM)^{4\varepsilon} \left(\frac{N(e\delta\Delta)}{Q}\right)^{\frac{1}{4}-2\varepsilon} \sum'_{\substack{Q/N(e) < N(n) \leq 2Q/N(e) \\ n \equiv \pm 1 \pmod{(1+i)^3} \\ (N(n), N(e))=1}} |b_{en}|^2.
\end{aligned}$$

Since

$$\int_{-\infty}^{\infty} |\rho_+(\sigma + it)| dt \ll_{\varepsilon} 1,$$

we deduce that

$$\begin{aligned}
(6.19) \quad & S'_W(M, Q) \ll M(QM)^{7\varepsilon} \sum'_{\substack{N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{(1+i)^3}}} \sum'_{\substack{N(\delta) \leq \frac{2Q}{N(\Delta)} \\ \delta \equiv \pm 1 \pmod{(1+i)^3} \\ (N(\delta), N(\Delta))=1}} \frac{1}{(N(\delta))^{1/2}} \sum_{l|N(\Delta)} \frac{1}{l} \sum'_{\substack{e \in \mathbb{Z}[i] \\ e \equiv 1 \pmod{(1+i)^3} \\ N(e) \leq \frac{2Q}{N(\delta\Delta)} \\ (N(e), N(\delta\Delta))=1}} \frac{1}{N(e)} \\
& \times \sum_{0 < |h| \leq H} \left(\frac{N(e\delta\Delta)}{Q}\right)^{\frac{1}{4}-2\varepsilon} \sum'_{\substack{Q/N(e) < N(n) \leq 2Q/N(e) \\ n \equiv \pm 1 \pmod{(1+i)^3} \\ (N(n), N(e))=1}} |b_{en}|^2 \\
& \ll Q^{7/4+2\varepsilon} (QM)^{8\varepsilon} \sum'_{\substack{N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{(1+i)^3}}} \frac{1}{N(\Delta)^{7/4+2\varepsilon}} \sum'_{\substack{N(\delta) \leq \frac{2Q}{N(\Delta)} \\ \delta \equiv \pm 1 \pmod{(1+i)^3} \\ (N(\delta), N(\Delta))=1}} \frac{1}{(N(\delta))^{5/4+2\varepsilon}} \sum_{l|N(\Delta)} 1 \\
& \times \sum'_{\substack{e \in \mathbb{Z}[i] \\ e \equiv 1 \pmod{(1+i)^3} \\ N(e) \leq \frac{2Q}{N(\delta\Delta)} \\ (N(e), N(\delta\Delta))=1}} \frac{1}{N(e)^{3/4+2\varepsilon}} \sum'_{\substack{Q/N(e) < N(n) \leq 2Q/N(e) \\ n \equiv \pm 1 \pmod{(1+i)^3} \\ (N(n), N(e))=1}} |b_{en}|^2 \\
& \ll Q^{7/4+2\varepsilon} (QM)^{8\varepsilon} \sum'_{\substack{Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{(1+i)^3}}} |b_n|^2 \sum'_{\substack{e \in \mathbb{Z}[i] \\ e \equiv 1 \pmod{(1+i)^3} \\ e|n}} \frac{1}{N(e)^{3/4+2\varepsilon}} \\
& \ll Q^{7/4+2\varepsilon} (QM)^{9\varepsilon} \sum'_{\substack{Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{(1+i)^3}}} |b_n|^2.
\end{aligned}$$

Combining (6.16) and (6.19), we deduce that (6.14) and hence (6.12) is bounded by

$$\ll (QM)^\varepsilon \left(M + Q^{7/4}\right) \|b_n\|^2$$

which implies the desired bound (6.4).

7. COMPLETION OF THE PROOF OF THEOREM 1.2

We start with (6.11) with any $v \geq 2$ (as one checks easily that $v = 1$ does not lead to any improvement) as an initial estimate. From (6.7) and (6.11), it follows that

$$B_2(Q, M) \ll (QM)^\varepsilon Q^{1/2} X^{-1/2} (X^{1+3/(4v)} + X^{1-1/v} M)$$

for a suitable X with $1 \ll X \ll Q$. The worst case is $X = Q$ which shows $B_2(Q, M)$ also satisfies (6.11). Repeating the argument, we have

$$B_3(Q, M) \ll (QM)^\varepsilon \left(Q^{1+3/(4v)} + Q^{1-1/v} M \right).$$

Combining this with (6.10), we obtain

$$\begin{aligned} B_4(Q, M) &\ll Q + (QM)^{9\varepsilon} Q M^{-2} \max \left\{ K^{1+3/(4v)} + K^{1-1/v} M : K \leq M^4 Q^{-1} \right\} \\ &\quad + (QM)^{9\varepsilon} M^6 Q^{-1} \sum_{K \geq M^4/Q} K^{-2-\varepsilon} (K^{1+3/(4v)} + K^{1-1/v} M) \\ &\ll Q + (QM)^{10\varepsilon} (Q^{-3/(4v)} M^{2+3/v} + Q^{1/v} M^{3-4/v}), \end{aligned}$$

where the sum over K runs over powers of 2. From this and (6.9), we deduce that

$$B_3(Q, M) \ll \frac{Q}{\Delta_1} + (QM)^\varepsilon \left(\left(\frac{Q}{\Delta_1} \right)^{-3/(4v)} \left(\frac{M}{\Delta_2} \right)^{2+3/v} + \left(\frac{Q}{\Delta_1} \right)^{1/v} \left(\frac{M}{\Delta_2} \right)^{3-4/v} \right)$$

for some positive integers Δ_1, Δ_2 with $\Delta_2^2 \geq \Delta_1$. From this and the trivial bound

$$B_1(Q, M) \ll B_3(Q, M),$$

we deduce that

$$(7.1) \quad B_1(Q, M) \ll Q + (QM)^\varepsilon \left(Q^{-3/(4v)} M^{2+3/v} + Q^{1/v} M^{3-4/v} \right).$$

Combining (7.1) with (6.6), we deduce that

$$(7.2) \quad B_1(Q, M) \ll (\tilde{Q}M)^\varepsilon \left(\tilde{Q} + \tilde{Q}^{-3/(4v)} M^{2+3/v} + \tilde{Q}^{1/v} M^{3-4/v} \right)$$

if $\tilde{Q} \geq CQ \log(2QM)$. We choose $\tilde{Q} := \max(Q^{1+\varepsilon}, M^{4-4v/7})$. Then (7.2) implies that

$$(7.3) \quad B_1(Q, M) \ll (QM)^\varepsilon \left(Q + Q^{1/v} M^{3-4/v} + M^{17/7} \right).$$

It's easy to see that the choice $v = 2$ is optimal and a further cycle in the above process does not lead to an improvement of our result. Combining (6.11) with $v = 1, 2, 3$ and (7.3) with $v = 2$, we obtain our final estimate

$$(7.4) \quad B_1(Q, M) \ll (QM)^\varepsilon \min \left\{ Q^{\frac{7}{4}} + M, Q^{\frac{11}{8}} + Q^{\frac{1}{2}} M, Q^{\frac{5}{4}} + Q^{\frac{2}{3}} M, Q + Q^{\frac{1}{2}} M + M^{\frac{17}{7}} \right\}.$$

which together with (6.1) (noting that the last expression in (6.1) is $\ll B_1(Q, M)$ by the law of quartic reciprocity) implies Theorem 1.2. \square

Calculating the right-hand side of (1.4) for various ranges of Q and M , we obtain that it is bounded by

$$\ll (QM)^\varepsilon \|a_m\|^2 \cdot \begin{cases} M & \text{if } Q \leq M^{4/7}, \\ Q^{7/4} & \text{if } M^{4/7} < Q \leq M^{4/5}, \\ Q^{1/2}M & \text{if } M^{4/5} < Q \leq M^{8/7}, \\ Q^{11/8} & \text{if } M^{8/7} < Q \leq M^{24/17}, \\ Q^{2/3}M & \text{if } M^{24/17} < Q \leq M^{12/7}, \\ Q^{5/4} & \text{if } M^{12/7} < Q \leq M^{68/35}, \\ M^{17/7} & \text{if } M^{68/35} < Q \leq M^{17/7}, \\ Q & \text{if } M^{17/7} < Q. \end{cases}$$

For convenience, we enclose a table displaying the estimates for $B_1(Q, M)$ that we get for various ranges. This table should be read as follows. If the fractions α and β are the $(n-1)$ -th and n -th entries, respectively, in the first row, and the term T is the n -th entry in the second row, then the estimate $B_1(Q, M) \ll (QM)^\varepsilon T$ holds in the range $M^\alpha < Q \leq M^\beta$.

Range	4/7	4/5	8/7	24/17	12/7	68/35	17/7	∞
Bounds	M	$Q^{7/4}$	$Q^{1/2}$	$Q^{11/8}$	$Q^{2/3}$	$Q^{5/4}$	$M^{17/7}$	Q

It can be easily checked that (6.11) with $v \geq 3$ does not lead to an improvement of (7.4).

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