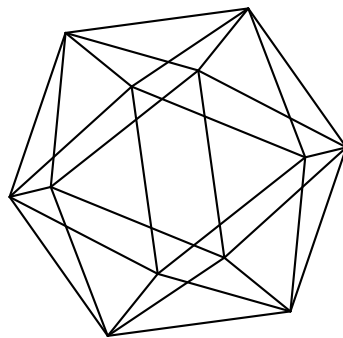


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ON TAU-FUNCTIONS FOR THE KdV HIERARCHY

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ABSTRACT. For an *arbitrary* solution to the KdV hierarchy, the generating series of logarithmic derivatives of the tau-function of the solution can be expressed by the basic matrix resolvent via algebraic manipulations [5, 7]. Based on this we develop in this paper two new formulae for the generating series by introducing the notion of a *pair* of wave functions of the solution. Applications to the Witten–Kontsevich tau-function, to the generalized Brézin–Gross–Witten (BGW) tau-function, as well as to a modular deformation of the generalized BGW tau-function which we call the Lamé tau-function are also given.

CONTENTS

1.	Introduction	1
2.	Review of the matrix resolvent approach to the KdV hierarchy	10
3.	From a pair of wave functions to the n -point functions	14
4.	Generating series of the generalized BGW correlators	18
5.	Generating series of the Lamé partial correlation functions	23
6.	Further remarks	25
	References	28

1. INTRODUCTION

To make this article accessible also to non-specialists, we recall the definitions of all of the main objects studied (KdV hierarchy, tau-function, matrix resolvents, wave functions, . . .), in some cases with definitions slightly different from the most standard ones. Experts can simply skip this material.

1.1. The Korteweg–de Vries hierarchy. The Korteweg–de Vries (KdV) equation

$$u_t = u u_x + \frac{1}{12} u_{xxx}, \tag{1}$$

discovered in the study of water waves in the 19th century, extends to a *hierarchy* of PDEs,

$$\frac{\partial u}{\partial t_k} = \frac{u^k}{k!} u_x + Q_k(u, u_x, u_{xx}, \dots) \quad (k \geq 0), \tag{2}$$

where u is now a function of the multivariable $\mathbf{t} = (t_0 = x, t_1 = t, t_2, t_3, \dots)$ and the Q_k are specific polynomials with $Q_0 = 0$ and $Q_1 = u_{xxx}/12$. We recall a simple construction of this hierarchy. Denote by \mathcal{A} the polynomial algebra $\mathbb{Q}[u_0, u_1, u_2, \dots]$ with the grading $\deg u_i = i$, and by ∂ the derivation on \mathcal{A} that satisfies $\partial(u_i) = u_{i+1}$. For arbitrary elements Q_0, Q_1, Q_2, \dots in \mathcal{A} we can define uniquely a family of derivations D_k ($k \geq 0$) on \mathcal{A} by

$$D_0 = \partial, \quad [D_k, D_0] = 0, \quad D_k(u_0) = \frac{u_0^k}{k!} u_1 + Q_k. \tag{3}$$

[†]Deceased on March 19, 2019.

It turns out that if we require that $Q_k \in \mathcal{A}^{\geq 2}$ with $Q_0 = 0$ and $Q_1 = u_3/12$ and that all D_k commute with D_1 , then the polynomials Q_k exist and are unique. The first few values are

$$\begin{aligned} Q_0 &= 0, & Q_1 &= \frac{1}{12}u_3, & Q_2 &= \frac{1}{12}(2u_1u_2 + u_0u_3) + \frac{1}{240}u_5, \\ Q_3 &= \frac{1}{24}(u_1^3 + 4u_0u_1u_2 + u_0^2u_3) + \frac{1}{240}(5u_2u_3 + 3u_1u_4 + u_0u_5) + \frac{1}{6720}u_7, \end{aligned}$$

and all the D_k commute. (See [13, 34]. A new proof of the existence and pairwise commutativity is given in Section 2.) We call these unique derivations D_k the KdV derivations. The formal system (3) called the abstract KdV hierarchy leads to the compatible system of PDEs (2) if $u = u(\mathbf{t})$ and we identify u_i and D_k with $\partial_x^i(u)$ and $\partial/\partial t_k$, respectively.

Let V be a ring of functions of x closed under ∂_x . (Usually V will be $\mathbb{C}[[x]]$ or $\mathbb{C}((x))$.) For given initial data $f(x) \in V$, one can solve the KdV hierarchy (2) to get a unique solution $u = u(\mathbf{t})$ in $V[[\mathbf{t}_{>0}]]$ with $u(x, 0, 0, \dots) = f(x)$. This gives a 1-1 correspondence:

$$\{\text{solution } u \text{ of (2) in } V[[\mathbf{t}_{>0}]]\} \longleftrightarrow \{\text{initial data } f\} = V. \quad (4)$$

Here are three cases of particular interest.

Example 1. $f(x) = x$. The corresponding unique solution u in $\mathbb{C}[[x, \mathbf{t}_{>0}]]$ governs [39, 33] the integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{p_1} \cdots \psi_n^{p_n}, \quad (5)$$

where $\overline{\mathcal{M}}_{g,n}$ denotes the Deligne–Mumford moduli space of stable algebraic curves of genus g with n distinct marked points and ψ_j denotes the so-called ψ -class (see Section 4 for the precise definition). This solution is often called the Witten–Kontsevich solution, denoted by u_{WK} .

Example 2. $f(x) = \frac{C}{(x-1)^2}$, $C \in \mathbb{C}$. The corresponding solution u in $\mathbb{C}[[x, \mathbf{t}_{>0}]]$ will be called the generalized BGW solution [35, 1], denoted by $u_{\Theta(C)}$. For $C = 1/8$, $u_{\Theta(C)}$ governs [36] the integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{p_1} \cdots \psi_n^{p_n}, \quad (6)$$

called the n -point Θ -class intersection numbers. Here $\Theta_{g,n}$ denotes the Norbury Θ -class [36] whose precise definition will be recalled in Section 4. We call $u_{\Theta(1/8)}$ the BGW–Norbury solution.

Example 3. $f(x) = C \wp(x; \tau)$, $C \in \mathbb{C}$, $\tau \in \mathfrak{H} = \text{upper half plane}$. Here we can take V to be the ring $\mathbb{C}[g_2, g_3, \wp, \wp'] / (\wp'^2 - 4\wp^3 + g_2\wp + g_3)$ with the usual meaning of g_2 and g_3 . The corresponding unique solution u in $V[[\mathbf{t}_{>0}]]$ is a modular deformation of $u_{\Theta(C)}$, denoted by u_{elliptic} and discussed in more detail in Section 5. We call u_{elliptic} a Lamé solution of the KdV hierarchy.

1.2. The tau-function and the matrix resolvent approach. Let $(D_k)_{k \geq 0}$ be the KdV derivations defined above. By a *tau-structure* for the abstract KdV hierarchy we mean a collection of elements $\Omega_{p,q} \in \mathcal{A}$ ($p, q \geq 0$) satisfying

$$\Omega_{0,0} = u_0, \quad \Omega_{p,q} = \Omega_{q,p}, \quad D_r(\Omega_{p,q}) = D_q(\Omega_{p,r}), \quad \forall p, q, r \geq 0. \quad (7)$$

(A more general abstract tau-structure allows $\Omega_{0,0}$ to be some other element of \mathcal{A} with non-degenerate 0th-order term, but this will not be studied in this paper.) Since the D_k commute, the elements

$$\Omega_{p_1, \dots, p_n} := D_{p_1} \cdots D_{p_{n-2}}(\Omega_{p_{n-1}, p_n}) \quad (n \geq 3, p_1, \dots, p_n \geq 0) \quad (8)$$

are symmetric in their indices. One can show that the tau-structure exists and is unique up to replacing $\Omega_{p,q}$ by $\Omega_{p,q} + c_{p,q}$, where $c_{p,q} = c_{q,p}$ are constants. By the *canonical* tau-structure we will mean the unique choice for which all $\Omega_{p,q}$ vanish when all $u_i = 0$, the first few values being

$$\Omega_{0,1} = \frac{u_0^2}{2} + \frac{u_2}{12}, \quad \Omega_{0,2} = \frac{u_0^3}{6} + \frac{u_1^2}{24} + \frac{u_0u_2}{12} + \frac{u_4}{240}, \quad \Omega_{1,1} = \frac{u_0^3}{3} + \frac{u_1^2}{24} + \frac{u_0u_2}{6} + \frac{u_4}{144}, \quad \dots$$

If $u = u(\mathbf{t})$ is a solution of (2) and we write $\Omega_{p,q}(\mathbf{t})$ as the image of $\Omega_{p,q}$ under $u_i \mapsto \partial_x^i(u)$, then (7) implies that there exists a function $\tau = \tau(\mathbf{t})$, a so-called *tau-function* of the solution u , such that

$$\Omega_{p,q}(\mathbf{t}) = \frac{\partial^2 \log \tau(\mathbf{t})}{\partial t_p \partial t_q} \quad (p, q \geq 0). \quad (9)$$

The symmetry in (8) is then obvious, since the image $\Omega_{p_1, \dots, p_n}(\mathbf{t})$ of Ω_{p_1, \dots, p_n} under $u_i \mapsto \partial_x^i(u)$ is

$$\Omega_{p_1, \dots, p_n}(\mathbf{t}) = \frac{\partial^n \log \tau(\mathbf{t})}{\partial t_{p_1} \cdots \partial t_{p_n}} \quad (n \geq 2, p_1, \dots, p_n \geq 0). \quad (10)$$

Denote $\Omega_p(\mathbf{t}) := \partial_{t_p}(\log \tau(\mathbf{t}))$. These logarithmic derivatives are called *n-point correlation functions* of u and the specializations $\Omega_{p_1, \dots, p_n}(x) := \Omega_{p_1, \dots, p_n}(x, \mathbf{0})$ are called *n-point partial correlation functions* of u , while the evaluation of $\Omega_{p_1, \dots, p_n}(x)$ at a particular value of x will be called an *n-point correlator* of u . If we choose the canonical tau-structure as defined above, we will call τ *the tau-function* of u , although it is still defined up to multiplying the exponential of an arbitrary linear function of \mathbf{t} . This definition of the tau-function τ is known to agree with other literature ([12, 25]).

The key fact for this paper is that the canonical tau-structure for the abstract KdV hierarchy can be given by an explicit generating series using the so-called matrix resolvent (MR) approach [5, 7]. This goes as follows. Let \mathcal{L} (“matrix Lax operator”) be the operator $\partial + \Lambda(\lambda) + q$, where $\Lambda(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & 1 \\ -2u_0 & 0 \end{pmatrix}$. Let $\mathcal{S} = \text{sl}_2(\mathbb{C})((\lambda^{-1}))$ be the space of $\text{sl}_2(\mathbb{C})$ -valued formal Laurent series in λ^{-1} . The *principal grading* on $\mathcal{A} \otimes \mathcal{S}$ is defined by $\deg E = 1$, $\deg F = -1$, $\deg \lambda = 2$, $\deg u_i = 0$, where $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are the Weyl generators. The **basic matrix resolvent** of \mathcal{L} is defined as the unique element $R(\lambda) = R(\lambda; u_0, u_1, \dots) \in \mathcal{S} \otimes \mathcal{A}$ satisfying

$$[\mathcal{L}, R(\lambda)] = 0, \quad R(\lambda) = \Lambda(\lambda) + \text{l.o.t.}, \quad \text{Tr } R(\lambda)^2 = 2\lambda, \quad (11)$$

where “l.o.t.” means lower order terms with respect to the principal gradation. The proof of existence and uniqueness of $R(\lambda)$ can be found in [7] or in Section 2 of the present paper. The upper right entry $R(\lambda)_{12}$ of the basic matrix resolvent which we denote by $b(\lambda)$ plays an important role. The following proposition, originally proved in [5], will be re-proved in Section 2 in a more straightforward way.

Proposition 1 ([5, 7, 43]). (i) *The differential polynomials $\Omega_{p,q} \in \mathcal{A}$ defined by the generating series*

$$\frac{\text{tr}(R(\lambda)R(\mu))}{(\lambda - \mu)^2} - \frac{\lambda + \mu}{(\lambda - \mu)^2} = \sum_{p,q \geq 0} \frac{(2p+1)!!(2q+1)!!}{\lambda^{p+1}\mu^{q+1}} \Omega_{p,q} \quad (12)$$

form a tau-structure for the abstract KdV hierarchy, vanishing at $u_0 = u_1 = \dots = 0$.

(ii) *For any integer $n \geq 3$ and the corresponding Ω_{p_1, \dots, p_n} as defined in (8) we have*

$$\sum_{p_1, \dots, p_n \geq 0} \Omega_{p_1, \dots, p_n} \prod_{j=1}^n \frac{(2p_j+1)!!}{\lambda_j^{p_j+1}} = - \sum_{\sigma \in S_n/C_n} \frac{\text{tr}(R(\lambda_{\sigma(1)}) \cdots R(\lambda_{\sigma(n)}))}{\prod_{i=1}^n (\lambda_{\sigma(i+1)} - \lambda_{\sigma(i)})}, \quad (13)$$

where S_n denotes the symmetry group, C_n the cyclic group, and it is understood that $\sigma(n+1) = \sigma(1)$.

Just as with $\Omega_{p,q}$ and $\Omega_{p,q}(\mathbf{t})$, we can evaluate the abstract basic matrix resolvent $R(\lambda)$ for a solution $u(\mathbf{t})$ of the KdV hierarchy by substituting $\partial_x^i(u)$ for u_i . The resulting matrix-valued function $R(\lambda, \mathbf{t})$ is called the *basic matrix resolvent* of u . Denote for short $R(\lambda, x) = R(\lambda, x, \mathbf{0})$.

Corollary 1 ([5]). *For any $n \geq 2$, the following formulas hold true:*

$$\sum_{p_1, \dots, p_n} \Omega_{p_1, \dots, p_n}(\mathbf{t}) \prod_{j=1}^n \frac{(2p_j+1)!!}{\lambda_j^{p_j+1}} = - \sum_{\sigma \in S_n/C_n} \frac{\text{tr } R(\lambda_{\sigma(1)}, \mathbf{t}) \cdots R(\lambda_{\sigma(n)}, \mathbf{t})}{\prod_{i=1}^n (\lambda_{\sigma(i+1)} - \lambda_{\sigma(i)})} - \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2} \delta_{n2}, \quad (14)$$

$$\sum_{p_1, \dots, p_n} \Omega_{p_1, \dots, p_n}(x) \prod_{j=1}^n \frac{(2p_j+1)!!}{\lambda_j^{p_j+1}} = - \sum_{\sigma \in S_n/C_n} \frac{\text{tr } R(\lambda_{\sigma(1)}, x) \cdots R(\lambda_{\sigma(n)}, x)}{\prod_{i=1}^n (\lambda_{\sigma(i+1)} - \lambda_{\sigma(i)})} - \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2} \delta_{n2}. \quad (15)$$

We note that $R(\lambda, x)$ is equal to $R(\lambda)$ with u_i replaced by $\partial_x^i(f)$, where f is the initial data of the solution. Therefore, formula (15) produces all higher order logarithmic derivatives of the tau-function of u at $\mathbf{t}_{>0} = \mathbf{0}$ with the knowledge of the initial data f .

1.3. From wave functions to n -point correlation functions. A key ingredient for us (and for an upcoming paper [23]) will be a pair (ψ, ψ^*) , consisting of a wave function ψ and the dual wave function ψ^* associated with ψ . We explain this briefly here and in detail in Section 3.

We start with the time-independent case. Let $f(x)$ be an element of V , and L the linear Schrödinger operator $\partial_x^2 + 2f(x)$. By a *wave function* of f we will mean an element $\psi = \psi(z, x)$ in the module $\tilde{V}((z^{-1})) e^{xz}$ satisfying the equation $L(\psi) = z^2\psi$ of the form $\psi = (1 + \phi_1(x)/z + \phi_2(x)/z^2 + \dots) e^{xz}$, where \tilde{V} is any ring with $V \subseteq \partial_x(\tilde{V}) \subseteq \tilde{V}$. The *dual wave function*¹ ψ^* of f associated with ψ is then defined as the unique element in $\tilde{V}((z^{-1})) e^{-xz}$ satisfying $L(\psi^*) = z^2\psi^*$ of the form $\psi^* = (1 + \phi_1^*(x)/z + \phi_2^*(x)/z^2 + \dots) e^{-xz}$ for which $\partial_x^i(\psi)\psi^*$ has residue 0 at $z = \infty$ for all $i \geq 0$. The pair (ψ, ψ^*) consisting of a wave function ψ of f and the unique dual wave function ψ^* associated with ψ is called a pair of wave functions of f . Given f , the wave function ψ of f is unique up to multiplication by an arbitrary element in $1 + z^{-1}\mathbb{C}((z^{-1}))$, but $\psi\psi^*$ is unique and coincides with $b(z^2, x)$, the upper right entry of the matrix $R(z^2, x)$ defined above.

We now proceed to the time-dependent case. Let $L = \partial^2 + 2u_0$ be a linear operator on \mathcal{A} , called the Lax operator for the KdV hierarchy, and introduce a sequence of differential operators A_k defined by $A_k = \frac{1}{(2k+1)!!} (L^{\frac{2k+1}{2}})_+$ ($k \geq 0$). Here $(\cdot)_+$ means taking the differential part of a pseudo-differential operator ([13]). Let $u = u(\mathbf{t})$ be an arbitrary solution to the KdV hierarchy (2). By a *wave function* of u we will mean an element $\psi = \psi(z, \mathbf{t})$ in $\tilde{V}[[\mathbf{t}_{>0}]]((z^{-1})) e^{\sum_{k=0}^{\infty} t_k z^{2k+1}/(2k+1)!!}$ of the form $\psi = (1 + \phi_1(\mathbf{t})/z + \phi_2(\mathbf{t})/z^2 + \dots) e^{\sum_{k=0}^{\infty} t_k z^{2k+1}/(2k+1)!!}$ satisfying the equations

$$L(\psi) = z^2\psi, \quad \partial_{t_k}(\psi) = A_k(\psi). \quad (16)$$

Define the *dual wave function* $\psi^* = \psi^*(z, \mathbf{t})$ of u associated with ψ as the unique element in $\tilde{V}[[\mathbf{t}_{>0}]]((z^{-1})) e^{-\sum_{k=0}^{\infty} t_k z^{2k+1}/(2k+1)!!}$ of the form $\psi^* = (1 + \phi_1^*(\mathbf{t})/z + \phi_2^*(\mathbf{t})/z^2 + \dots) e^{-\sum_{k=0}^{\infty} t_k z^{2k+1}/(2k+1)!!}$ satisfying

$$L(\psi^*) = z^2\psi^*, \quad -\partial_{t_k}(\psi^*) = A_k(\psi^*) \quad (17)$$

and that $\partial_x^i(\psi(z, \mathbf{t}))\psi^*(z, \mathbf{t})$ has residue 0 at $z = \infty$ for all $i \geq 0$. We say that a wave function ψ of u and the dual wave function ψ^* of u associated with ψ form a *pair of wave functions* of u . The existence of a pair of wave functions of u , which is known, will be shown in Section 3. Given a solution $u = u(\mathbf{t})$, the wave function ψ of u is unique only up to multiplication by an arbitrary element in $1 + z^{-1}\mathbb{C}((z^{-1}))$, but the product $\psi\psi^*$ is unique and coincides with $b(z^2, \mathbf{t})$, the upper right entry of $R(z^2, \mathbf{t})$, where ψ^* is the dual wave function of u associated with ψ .

Let $u = u(\mathbf{t})$ be an arbitrary solution of (2) in $V[[\mathbf{t}_{>0}]]$. The basic matrix resolvent $R(z^2, \mathbf{t})$ of u can be expressed in terms of a pair of wave functions of u (Lemma 8, Section 3). This enables us to give a new formula for the generating series of n -point correlation functions $\Omega_{p_1, \dots, p_n}(\mathbf{t})$ of u . Let $(\psi(z, \mathbf{t}), \psi^*(z, \mathbf{t}))$ be a pair of wave functions of u . Define

$$D(z, w, \mathbf{t}) := \frac{\psi(z, \mathbf{t})\psi_x^*(w, \mathbf{t}) - \psi^*(w, \mathbf{t})\psi_x(z, \mathbf{t})}{w^2 - z^2}. \quad (18)$$

Theorem 1. *For any fixed $n \geq 2$ the following formula holds true:*

$$\sum_{p_1, \dots, p_n} \Omega_{p_1, \dots, p_n}(\mathbf{t}) \prod_{j=1}^n \frac{(2p_j + 1)!!}{z_j^{2p_j+2}} = - \sum_{\sigma \in S_n/C_n} \prod_{i=1}^n D(z_{\sigma(i)}, z_{\sigma(i+1)}, \mathbf{t}) - \frac{\delta_{n2}}{(z_1 - z_2)^2}. \quad (19)$$

¹A dual wave function satisfies $L^*\psi^* = \lambda\psi^*$, where L^* is the formal adjoint operator of L . For the KdV hierarchy, $L^* = L$ and a dual wave function ψ^* is also a wave function.

We note that under the gauge freedom of the second type $\psi(z, \mathbf{t}) \mapsto g(z)\psi(z, \mathbf{t}) = \tilde{\psi}(z, \mathbf{t})$, the dual wave function ψ^* associated with ψ is mapped to $\psi^*(z, x)/g(z) = \tilde{\psi}^*(z, x)$. Therefore we have

$$\tilde{D}(z, w, \mathbf{t}) := \frac{\tilde{\psi}(z, \mathbf{t}) \tilde{\psi}_x^*(w, \mathbf{t}) - \tilde{\psi}^*(w, \mathbf{t}) \tilde{\psi}_x(z, \mathbf{t})}{w^2 - z^2} = \frac{g(z)}{g(w)} D(z, w, \mathbf{t}).$$

However, products of the factors of the form $g(z)/g(w)$ cancel in each sum of the right hand side of the formula (19) and therefore remain unchanged under the gauge freedom of the second type. This argument agrees with the fact that the n -point correlation functions $\Omega_{p_1, \dots, p_n}(\mathbf{t})$ of u (for $n \geq 2$) are the evaluations of certain elements of \mathcal{A} with u_i replaced by $\partial_x^i(u)$.

For an element $f(x) \in V$ and a solution $u(\mathbf{t}) \in V[[\mathbf{t}_{>0}]]$ to the KdV hierarchy related via the 1-1 correspondence (4), we have $\psi(z, x, \mathbf{0}) \equiv \psi(z, x)$. For a given f , one can find an explicit recursion for solving $\psi(z, x)$ (see Section 3). However, we do not know an efficient way of solving $\psi(z, \mathbf{t})$. This is because the recursive procedure of solving $\psi(z, \mathbf{t})$ requires the knowledge of $u(\mathbf{t})$, but this requires first solving the KdV hierarchy. However, due to (19), the knowledge of $\psi(z, x)$ gives rise to a construction of the logarithm of the tau-function $\tau(\mathbf{t})$ of u (since $u = \partial_x^2(\log \tau)$, (19) also constructs u) in a simple way. More precisely, if we specialize (19) to $\mathbf{t} = (x, \mathbf{0})$, then it gives a formula $\Omega_{p_1, \dots, p_n}(x)$ for all $n \geq 2$ and $p_i \geq 0$ and hence, in view of (10), gives the entire Taylor series of $\log \tau(\mathbf{t})$ (which, we recall, is only defined up to the addition of a linear term). This is important because in our concrete computations for Examples 1–3 of Section 1.1, we can compute $\psi(z, x)$ and $\psi^*(z, x)$, and hence the function $D(z, w, x)$, explicitly.

We now formulate a theorem that is equivalent to Theorem 1 but involves a function $K(z, w, \mathbf{t})$ which, unlike $D(z, w, \mathbf{t})$, depends only on the solution $u(\mathbf{t})$ (i.e. it is independent of the choice of a pair of wave functions of $u(\mathbf{t})$). The function $K(z, w, \mathbf{t})$ is defined by

$$K(z, w, \mathbf{t}) := \frac{b(z^2, \mathbf{t}) b_x(w^2, \mathbf{t}) - b(w^2, \mathbf{t}) b_x(z^2, \mathbf{t})}{2(w^2 - z^2)} - \frac{w b(z^2, \mathbf{t}) + z b(w^2, \mathbf{t})}{w^2 - z^2}. \quad (20)$$

Theorem 2. *For any fixed $n \geq 2$ the following formula holds true:*

$$\sum_{p_1, \dots, p_n} \Omega_{p_1, \dots, p_n}(\mathbf{t}) \prod_{j=1}^n \frac{(2p_j + 1)!!}{z_j^{2p_j + 2}} = -\frac{1}{\prod_{i=1}^n b(z_i^2, \mathbf{t})} \sum_{\sigma \in S_n/C_n} \prod_{i=1}^n K(z_{\sigma(i)}, z_{\sigma(i+1)}, \mathbf{t}) - \frac{\delta_{n2}}{(z_1 - z_2)^2}. \quad (21)$$

Remark 1. Both the formula (19) and the formula (21) are generalized to the Toda lattice hierarchy in [41]. For example, we have the following theorem.

Theorem ([41]). *Fix $k \geq 2$ an integer. For an arbitrary solution (v, w) to the Toda lattice hierarchy associated with the Lax operator $L_{\text{Toda}} = \Lambda + v(n) + w(n)\Lambda^{-1}$, let $\psi_1(\lambda, n, \mathbf{t}), \psi_2(\lambda, n, \mathbf{t})$ be a pair of wave functions of the solution (v, w) . Here Λ denotes the shift operator $\Lambda : f(n) \mapsto f(n+1)$. Define*

$$D(\lambda, \mu, n, \mathbf{t}) = \frac{\psi_1(\lambda, n, \mathbf{t}) \psi_2(\mu, n-1, \mathbf{t}) - \psi_1(\lambda, n-1, \mathbf{t}) \psi_2(\mu, n, \mathbf{t})}{\lambda - \mu}.$$

Then the following formula holds true:

$$\sum_{i_1, \dots, i_k \geq 0} \frac{\Omega_{i_1, \dots, i_k}^{\text{Toda}}(n, \mathbf{t})}{\lambda_1^{i_1+2} \dots \lambda_k^{i_k+2}} = (-1)^{k-1} \frac{e^{kq(n-1, \mathbf{t})}}{\prod_{j=1}^k \lambda_j} \sum_{\sigma \in S_k/C_k} \prod_{j=1}^k D(\lambda_{\sigma(j)}, \lambda_{\sigma(j+1)}, n, \mathbf{t}) - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}, \quad (22)$$

where $\Omega_{i_1, \dots, i_k}^{\text{Toda}}(n, \mathbf{t})$ denote the k -point correlation functions of (v, w) , and the $q(n, \mathbf{t})$ is defined via

$$w(n, \mathbf{t}) = e^{q(n-1, \mathbf{t}) - q(n, \mathbf{t})}, \quad \frac{\partial q(n, \mathbf{t})}{\partial t_i} = -S_i(n, \mathbf{t}), \quad i \geq 0.$$

See [41] for the precise definitions of the functions $S_i(n, \mathbf{t})$, $\Omega_{i_1, \dots, i_k}^{\text{Toda}}(n, \mathbf{t})$, and of a pair of wave functions for the Toda lattice hierarchy.

We observe that all the following three functions

$$K(z, w, \mathbf{t}) - \frac{b(w^2, \mathbf{t})}{z-w}, \quad K(z, w, \mathbf{t}) - \frac{b(z^2, \mathbf{t})}{z-w}, \quad K(z, w, \mathbf{t}) - \frac{1}{2} \frac{b(z^2, \mathbf{t}) + b(w^2, \mathbf{t})}{z-w}$$

belong to the ring $V[[\mathbf{t}_{>0}]][[z^{-1}, w^{-1}]]$. We also remark that if we define K as the right hand side of (20) with $b(z^2, \mathbf{t})$ and $b(w^2, \mathbf{t})$ replaced by $b(z^2)$ and $b(w^2)$, respectively, then it easily follows that the three functions $K - b(z^2)/(z-w)$, $K - b(w^2)/(z-w)$ and $K - \frac{1}{2}(b(z^2) + b(w^2))/(z-w)$ all belong to $\mathcal{A}[[z^{-1}, w^{-1}]]$. Moreover, the identity (21) is then also true in the **abstract sense** (similarly to Proposition 1). We have for example

$$\begin{aligned} K - \frac{1}{2} \frac{b(z^2) + b(w^2)}{z-w} &= \frac{u_0}{2} \left(\frac{1}{zw^2} - \frac{1}{z^2w} \right) - \frac{u_1}{2} \frac{1}{z^2w^2} + \frac{6u_0^2 + u_2}{8} \left(\frac{1}{zw^4} - \frac{1}{z^2w^3} + \frac{1}{z^3w^2} - \frac{1}{z^4w} \right) \\ &\quad - \frac{12u_0u_1 + u_3}{8} \left(\frac{1}{z^2w^4} + \frac{1}{z^4w^2} \right) + \dots \end{aligned}$$

Proposition 2. *Let $\psi(z, \mathbf{t})$ be any wave function of $u(\mathbf{t})$. The functions K and D satisfy the relation*

$$K(z, w, \mathbf{t}) = b(z^2, \mathbf{t}) \frac{\psi(w, \mathbf{t})}{\psi(z, \mathbf{t})} D(z, w, \mathbf{t}) = b(w^2, \mathbf{t}) \frac{\psi^*(z, \mathbf{t})}{\psi^*(w, \mathbf{t})} D(z, w, \mathbf{t}) = \psi^*(z, \mathbf{t}) \psi(w, \mathbf{t}) D(z, w, \mathbf{t}). \quad (23)$$

The proof of Proposition 2 and Theorem 2 will be given in Section 3.

Let us now consider the case $V = \mathbb{C}[[x]]$, namely, consider arbitrary solutions of the KdV hierarchy in the ring $\mathbb{C}[[t_0, t_1, t_2, \dots]]$, where we recall that $t_0 = x$. Define $D(z, w)$ and $K(z, w)$ as follows:

$$D(z, w) := D(z, w, \mathbf{0}) = \frac{\psi(z, \mathbf{0}) \psi_x^*(w, \mathbf{0}) - \psi^*(w, \mathbf{0}) \psi_x(z, \mathbf{0})}{w^2 - z^2}, \quad K(z, w) := K(z, w, \mathbf{0}). \quad (24)$$

Then Proposition 2 immediately implies the following corollary.

Corollary 2. *For $V = \mathbb{C}[[x]]$, we have*

$$D(z, w) - \frac{1}{z-w} \in \mathbb{C}[[z^{-1}, w^{-1}]], \quad (25)$$

$$K(z, w) - \frac{b(z^2, \mathbf{0})}{z-w}, \quad K(z, w) - \frac{b(w^2, \mathbf{0})}{z-w}, \quad K(z, w) - \frac{1}{2} \frac{b(z^2, \mathbf{0}) + b(w^2, \mathbf{0})}{z-w} \in \mathbb{C}[[z^{-1}, w^{-1}]]. \quad (26)$$

Note that for this case, pairs of wave functions of the solution correspond to certain points of the Sato Grassmannian. In such correspondences, the coefficients of $D(z, w) - 1/(z-w)$ coincide with the affine coordinates [27] of the point of the Sato Grassmannian. Based on this observation we find that for $V = \mathbb{C}[[x]]$ the identity (19) evaluated at $\mathbf{t} = \mathbf{0}$ is equivalent to a formula obtained by J. Zhou [44] from the fermionic picture (see the Theorem 5.3 of [44]). The affine coordinates can alternatively be used to compute the Plücker coordinates in the expansion of tau-function (as opposed to its logarithm) (cf. [3, 27, 37, 42, 44]). Geometric meaning (in Sato Grassmannian) of the coefficients of the power series in (26) remains an interesting open question, which we plan to do in a subsequent publication.

1.4. Applications. We now discuss each of the three examples from Section 1.1 in turn.

Example 1: Intersection numbers of ψ -classes. An explicit formula for the generating series of (5) in terms of the basic matrix resolvent was derived in [5]; see also (128) and (129). A particular pair of wave functions $(\psi(z, x), \psi^*(z, x))$ of $f = x$ were given in [5]:

$$\psi(z, x) = \frac{\sqrt{z}}{(z^2 - 2x)^{\frac{1}{4}}} e^{\frac{1}{3}z^3 - \frac{1}{3}(z^2 - 2x)^{\frac{3}{2}}} \sum_{k \geq 0} \frac{(-1)^k}{288^k} \frac{(6k)!}{(3k)!(2k)!} (z^2 - 2x)^{-\frac{3k}{2}}, \quad (27)$$

$$\psi^*(z, x) = \frac{\sqrt{z}}{(z^2 - 2x)^{\frac{1}{4}}} e^{-\frac{z^3}{3} + \frac{1}{3}(z^2 - 2x)^{\frac{3}{2}}} \sum_{k \geq 0} \frac{1}{288^k} \frac{(6k)!}{(3k)!(2k)!} (z^2 - 2x)^{-\frac{3k}{2}}. \quad (28)$$

They can be viewed as the asymptotic expansion of the following two analytic functions, respectively, as $z \rightarrow \infty$ within appropriate sectors [9]:

$$\sqrt{2\pi z} e^{\frac{z^3}{3}} 2^{\frac{1}{3}} \text{Ai}(\xi), \quad e^{\frac{\pi i}{6}} \sqrt{2\pi z} e^{-\frac{z^3}{3}} 2^{\frac{1}{3}} \text{Ai}(\omega\xi),$$

where $\xi = 2^{-\frac{2}{3}}(z^2 - 2x)$, $\omega = e^{2\pi i/3}$. Restricting the formal series (27)–(28) to $x = 0$ we obtain

$$\psi(z, 0) = \sum_{k=0}^{\infty} \frac{(-1)^k}{288^k} \frac{(6k)!}{(3k)!(2k)!} z^{-3k}, \quad \psi^*(z, 0) = \psi(-z, 0), \quad (29)$$

$$\psi_x(z, 0) = \sum_{k=0}^{\infty} \frac{1+6k}{1-6k} \frac{(-1)^k}{288^k} \frac{(6k)!}{(3k)!(2k)!} z^{-3k+1}, \quad \psi_x^*(z, 0) = \psi_x(-z, 0). \quad (30)$$

These series appeared in the Faber–Zagier formula [28] on relations in the tautological ring of the moduli space of curves. Now consider the function $D(z, w) = D(z, w, \mathbf{0})$. According to (25) we know that this function has a pole only at $z = w$. The first few terms for $D(z, w) - 1/(z - w)$ are given by

$$\begin{aligned} D(z, w) - \frac{1}{z - w} &= \frac{5}{24} z^{-1} w^{-3} - \frac{7}{24} z^{-2} w^{-2} + \frac{5}{24} z^{-3} w^{-1} - \frac{455}{1152} z^{-2} w^{-5} \\ &+ \frac{385}{1152} z^{-3} w^{-4} - \frac{385}{1152} z^{-4} w^{-3} + \frac{455}{1152} z^{-5} w^{-2} + \dots \end{aligned} \quad (31)$$

Theorem 1 then implies the following slightly simplified version of a formula of J. Zhou:

Theorem (Zhou [44]). *For $n \geq 2$, the generating series of the n -point intersection numbers of ψ -classes has the expression*

$$\sum_{g \geq 0} \sum_{p_1, \dots, p_n \geq 0} \int_{\mathcal{M}_{g,n}} \psi_1^{p_1} \cdots \psi_n^{p_n} \prod_{j=1}^n \frac{(2p_j + 1)!!}{\lambda_j^{p_j+1}} = - \sum_{\sigma \in S_n/C_n} \prod_{i=1}^n D(z_{\sigma(i)}, z_{\sigma(i+1)}) - \frac{\delta_{n2}}{(z_1 - z_2)^2}. \quad (32)$$

We observe that an explicit formula for the coefficients in (31) was obtained by J. Zhou [42] by solving Virasoro constraints using the *fermionic* method. Later the explicit formula was re-proved by F. Balogh and one of the authors of the present paper [3] using the Sato Grassmannian approach [37].

The Laurent series of the corresponding function $K(z, w)$ begins

$$K(z, w) - \frac{1}{2} \frac{b(z^2, \mathbf{0}) + b(w^2, \mathbf{0})}{z - w} = -\frac{1}{2} \frac{1}{z^2 w^2} + \frac{5}{16} \left(\frac{1}{z w^6} - \frac{1}{z^2 w^5} + \frac{1}{z^3 w^4} - \frac{1}{z^4 w^3} + \frac{1}{z^5 w^2} - \frac{1}{z^6 w} \right) + \dots \quad (33)$$

Example 2: Theta-classes. For the BGW-Norbury solution $u_{\Theta(1/8)}$ introduced in Example 2 (with $C = 1/8$), denote by $\Omega_{p_1, \dots, p_n}^{\Theta(1/8)}(\mathbf{t})$ the n -point correlation functions of $u_{\Theta(1/8)}$. According to Norbury [36] $\Omega_{p_1, \dots, p_n}^{\Theta(1/8)}(\mathbf{t})$ for $n \geq 2$ are related to the Θ -class intersection numbers by

$$\Omega_{p_1, \dots, p_n}^{\Theta(1/8)}(\mathbf{0}) = \int_{\mathcal{M}_{g,n}} \Theta_{g,n} \psi_1^{p_1} \cdots \psi_n^{p_n} \quad (g = p_1 + \cdots + p_n + 1). \quad (34)$$

(The expression on the right vanishes for $g \neq p_1 + \cdots + p_n + 1$.) For any $n \geq 1$, denote by $F(\lambda_1, \dots, \lambda_n)$ the following generating series of the n -point Θ -class intersection numbers

$$F(\lambda_1, \dots, \lambda_n) = \sum_{p_1, \dots, p_n} \prod_{j=1}^n \frac{(2p_j + 1)!!}{\lambda_j^{p_j+1}} \int_{\mathcal{M}_{1+p_1+\dots+p_n,n}} \Theta_{1+p_1+\dots+p_n,n} \psi_1^{p_1} \cdots \psi_n^{p_n}. \quad (35)$$

Using the relationship (34) and Corollary 1 we will prove:

Theorem 3. For fixed $n \geq 2$, the generating series $F(\lambda_1, \dots, \lambda_n)$ has the expression

$$F(\lambda_1, \dots, \lambda_n) = - \sum_{\sigma \in S_n/C_n} \frac{\text{tr}(M(\lambda_{\sigma(1)}) \cdots M(\lambda_{\sigma(n)}))}{\prod_{i=1}^n (\lambda_{\sigma(i+1)} - \lambda_{\sigma(i)})} - \delta_{n,2} \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2}, \quad (36)$$

where $M(\lambda) = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} + \sum_{k \geq 0} \left[\frac{(2k-1)!!}{2^k} \right]^3 \begin{pmatrix} k & 1 \\ -\frac{8k^3+12k^2+4k+1}{8(k+1)} & -k \end{pmatrix} \frac{\lambda^{-k}}{k!}$.

The string type equation (see in Section 4) relates the 2-point Θ -class intersection numbers to 1-point intersection numbers, from which we deduce:

Corollary 3. The 1-point Θ -class intersection numbers have the expression

$$\int_{\overline{\mathcal{M}}_{g,1}} \Theta_{g,1} \psi_1^{g-1} = \frac{(2g-1)!!^2}{8^g g! (2g-1)}, \quad \forall g \geq 1. \quad (37)$$

For a general C in Example 2, let $\tau_{\Theta(C)}(\mathbf{t})$ be the tau-function of $u_{\Theta(C)}$, $R(\lambda, \mathbf{t})$ the basic matrix resolvent of $u_{\Theta(C)}$, and $\Omega_{p_1, \dots, p_n}^{\Theta(C)}(\mathbf{t})$ the n -point correlation functions of $u_{\Theta(C)}$. We will call $\Omega_{p_1, \dots, p_n}^{\Theta(C)}(\mathbf{0})$ the n -point generalized BGW correlators. For $n \geq 2$, $\Omega_{p_1, \dots, p_n}^{\Theta(C)}(\mathbf{0})$ are uniquely determined by $u_{\Theta(C)}$ and so by f , where $f = \frac{C}{(x-1)^2}$. To determine $\Omega_p^{\Theta(C)}(\mathbf{0})$, we impose a string-type equation for $\tau_{\Theta(C)}$

$$\sum_{i \geq 0} (1+2i) t_i \frac{\partial \log \tau_{\Theta(C)}}{\partial t_i} + C = \frac{\partial \log \tau_{\Theta(C)}}{\partial t_0}. \quad (38)$$

Since $\sum_{i \geq 0} (1+2i) t_i \frac{\partial}{\partial t_i}$ gives the infinitesimal scaling symmetry of the KdV hierarchy and since eq. (38) implies that $u_{\Theta(C)}(x, \mathbf{0}) = C/(x-1)^2$, the tau-function $\tau_{\Theta(C)}$ of $u_{\Theta(C)}$ satisfying the additional constraint (38) exists. (From this way, $\tau_{\Theta(C)}$ is determined up to a multiplicative constant.) A matrix model for $\tau_{\Theta(C)}$ was obtained by A. Alexandrov [1]. The function $\tau_{\Theta(C)}$ is called the *generalized BGW tau-function*. As before denote $R(\lambda, x) = R(\lambda, x, \mathbf{0})$, $b(\lambda, x) = R(\lambda, x)_{12}$.

Theorem 4. For any fixed $C \in \mathbb{C}$, the formal power series $b(\lambda, x)$ of λ^{-1} only depends on the variable $\zeta := \lambda(x-1)^2$. Explicitly, the matrix-valued formal series $R = R(\lambda, x)$ has the explicit expression

$$R = \begin{pmatrix} -\frac{C\lambda^{\frac{1}{2}}}{\zeta^{\frac{3}{2}}} G_{\frac{3}{2}}(\zeta) & G_{\frac{1}{2}}(\zeta) \\ \frac{\lambda}{\zeta} \left((\zeta - 2C) G_{\frac{1}{2}}(\zeta) - \frac{3C}{\zeta} G_{\frac{3}{2}}(\zeta) - \frac{6C(C+1)}{\zeta^2} G_{\frac{5}{2}}(\zeta) \right) & \frac{C\lambda^{\frac{1}{2}}}{\zeta^{\frac{3}{2}}} G_{\frac{3}{2}}(\zeta) \end{pmatrix}, \quad (39)$$

where $\zeta^{3/2} := \lambda^{3/2} (x-1)^3$ and we have set

$$\Delta := 1 - 8C, \quad G_\alpha(\zeta) := {}_3F_0\left(\alpha, \alpha + \frac{\sqrt{\Delta}}{2}, \alpha - \frac{\sqrt{\Delta}}{2}; ; \frac{1}{\zeta}\right). \quad (40)$$

The proof is in Section 4. It is interesting to mention that if $-C$ is a triangular number, i.e. $C = -\frac{1}{2}p(p+1)$ ($p \geq 0$) then $b(\lambda, x)$ truncates to a polynomial of λ^{-1} .

Corollary 4. The 1-point generalized BGW correlators have the explicit expression

$$\Omega_p^{\Theta(C)}(\mathbf{0}) = \frac{1}{(p+1)!(1+2p)} \prod_{i=0}^p \left(C + \frac{i(i+1)}{2} \right), \quad j \geq 0. \quad (41)$$

The above identities (36)–(37), (39)–(41) and the proofs were reported by one of the authors in [40].

We proceed to applying Theorem 1 to the computation of generalized BGW correlators.

Theorem 5. Let $\alpha = \sqrt{\frac{1}{4} - 2C}$, and for $k \geq 0$ define $a_k(\alpha) \in \mathbb{Q}[\alpha^2] = \mathbb{Q}[C]$ by

$$a_k(\alpha) = \frac{(\alpha - k + \frac{1}{2})_{2k}}{2^k k!} = \frac{(-1)^k}{k!} \prod_{j=1}^k \left(C + \binom{j}{2} \right) = (-1)^k (2k-1) \Omega_{k-1}^{\Theta(C)}(\mathbf{0}).$$

Then the two functions

$$\psi(z, x) = \sum_{k \geq 0} \frac{a_k(\alpha)}{z^k (1-x)^k} e^{zx}, \quad \psi^*(z, x) = \psi(-z, x) \quad (42)$$

form a pair of wave functions of $f = C/(x-1)^2$. Moreover, if $D_{\Theta(C)}(z, w) = D(z, w)$ is the corresponding function as defined in (24), then for any $n \geq 2$ we have the generating function

$$\sum_{p_1, \dots, p_n} \Omega_{p_1, \dots, p_n}^{\Theta(C)}(\mathbf{0}) \prod_{j=1}^n \frac{(2p_j + 1)!!}{z_j^{2p_j+2}} = - \sum_{\sigma \in S_n/C_n} \prod_{i=1}^n D_{\Theta(C)}(z_{\sigma(i)}, z_{\sigma(i+1)}) - \frac{\delta_{n2}}{(z_1 - z_2)^2}. \quad (43)$$

According to equation (25) we know that $D(z, w)$ has an expansion of the form

$$D_{\Theta(C)}(z, w) = \frac{1}{z-w} + \sum_{m, n \geq 0} \frac{A_{mn}(\alpha)}{z^m (-w)^n} \quad (44)$$

with coefficients $A_{mn}(\alpha) \in \mathbb{Q}[\alpha]$. These coefficients are given explicitly by the following proposition.

Proposition 3. The coefficients A_{mn} can be given either by the recursion and boundary conditions

$$A_{m, n+1}(\alpha) - A_{m+1, n}(\alpha) = \frac{m-n}{m+n} a_m(\alpha) a_n(\alpha), \quad A_{m, 0}(\alpha) = A_{0, n}(\alpha) = 0 \quad (45)$$

or else explicitly by any of the three formulas

$$A_{mn}(\alpha) = \sum_{\substack{r \geq m, s \geq 0 \\ r+s=m+n-1}} \frac{r-s}{r+s} a_r(\alpha) a_s(\alpha) = \sum_{\substack{r \geq n, s \geq 0 \\ r+s=m+n-1}} \frac{r-s}{r+s} a_r(\alpha) a_s(\alpha) \quad (46)$$

$$= 2 \frac{m! n! a_m(\alpha) a_n(\alpha)}{(m+n-1)(m+n-1)!} \sum_{k \in \mathbb{Z}} (-1)^k \frac{\binom{m+n-1}{m+k} \binom{m+n-1}{n+k}}{\alpha - k - \frac{1}{2}}. \quad (47)$$

Observe that the summation in (47) is finite (since the k th term vanishes unless $|k+1/2| < m, n$) and is even in α (as one sees by sending α to $-\alpha$ and k to $-k-1$), and that either formula (46) or (47) implies that $A_{mn}(\alpha)/a_m(\alpha)$ for fixed m and n is a polynomial of degree $n-1$ in α^2 .

The Laurent series of the corresponding function $K(z, w)$ (defined in (24)) begins

$$K(z, w) - \frac{1}{2} \frac{b(z^2, \mathbf{0}) + b(w^2, \mathbf{0})}{z-w} = \frac{C}{2} \left(\frac{1}{zw^2} - \frac{1}{z^2w} \right) - \frac{C}{z^2w^2} + \frac{3C(1+C)}{4} \left(\frac{1}{zw^4} - \frac{1}{z^2w^3} + \frac{1}{z^3w^2} - \frac{1}{z^4w} \right) + \dots \quad (48)$$

We also note that, similarly as above, the pair of wave functions ψ, ψ^* of the above theorem can be viewed as the asymptotic expansions of the following two analytic functions, respectively, as z goes to ∞ within a certain sector of the complex z -plane:

$$\sqrt{\frac{2\xi}{\pi}} e^z K_\alpha(\xi), \quad \sqrt{2\pi\xi} e^{-z} I_\alpha(\xi), \quad (49)$$

where I_α and K_α denote the modified Bessel functions and $\xi = z(1-x)$. Finally we note that the function $D_{\Theta(C)}(z, w)$ in terms of these analytic functions reads

$$D_{\Theta(C)}(z, w) = 2 e^{z-w} \sqrt{zw} \frac{K_\alpha(z) w I'_\alpha(w) - I_\alpha(w) z K'_\alpha(z)}{z^2 - w^2}, \quad (50)$$

which can be recognized as a Bessel kernel (cf. [4, 38]).

Example 3: Lamé solutions. Let τ_{elliptic} be the tau-function of the Lamé solution $u_{\text{elliptic}}(\mathbf{t})$ introduced above, which as we will see gives an elliptic generalization of $\tau_{\Theta(C)}$. Denote by $R(\lambda, \mathbf{t})$ the basic matrix resolvent of u_{elliptic} , and denote $R(\lambda, x) = R(\lambda, x, \mathbf{0})$ and $b(\lambda, x) = R(\lambda, x)_{12}$. If we restrict to the particular case when $-C$ is a triangular number, i.e. $C = -\frac{p(p+1)}{2}$ for some integer $p \geq 0$, then $u_{\text{elliptic}}(\mathbf{t})$ is a special *finite-gap* solution [19, 16] of the KdV hierarchy. Denote by

$$y^2 = S_p(\lambda) \quad (51)$$

the corresponding *spectral curve* [19, 16], where $S_p(\lambda)$ is a polynomial of λ of degree $1+2p$ with leading coefficient 1. The first few S_p are $S_0 = \lambda$, $S_1 = \lambda^3 - \frac{g_2}{4}\lambda - \frac{g_3}{4}$, $S_2 = (\lambda^2 - 3g_2)(\lambda^3 - \frac{9}{4}g_2\lambda + \frac{27}{4}g_3)$. The following theorem will be deduced from results of [19, 16, 18] in Section 5.

Theorem 6. *In the case that $C = -\frac{p(p+1)}{2}$ ($p \geq 0$), the product $\sqrt{\frac{S_p(\lambda)}{\lambda}} \phi(\lambda, x)$ is a degree p polynomial in λ with leading coefficient 1 and a degree p polynomial in $\wp = \wp(x; \tau)$. For $p = 1$,*

$$b = 2\sqrt{\lambda} \frac{\lambda - \wp}{\sqrt{4\lambda^3 - g_2\lambda - g_3}};$$

For $p = 2$,

$$b = \frac{1}{2}\sqrt{\lambda} \frac{4\lambda^2 - 12\wp\lambda + 36\wp^2 - 9g_2}{\sqrt{(\lambda^2 - 3g_2)(4\lambda^3 - 9g_2\lambda + 27g_3)}};$$

For $p = 3$,

$$b = \frac{4\lambda^3 - 24\wp\lambda^2 + 60(3\wp^2 - g_2)\lambda - 900\wp^3 + 225g_2\wp + 225g_3}{\sqrt{16\lambda^6 - 504g_2\lambda^4 + 2376g_3\lambda^3 + 4185g_2^2\lambda^2 - 36450g_2g_3\lambda - 3375g_2^3 + 91125g_3^2}}.$$

Organization of the paper. In Section 2 we review the MR approach. In Section 3 we prove Theorem 1. In Section 4 we prove Theorems 3, 4, 5, and compute explicitly some generalized BGW correlators. In Section 5 we compute Lamé correlators. Further remarks are given in Section 6.

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Note added. After this work has been finished we learned that the identities (36)–(37), (39)–(41) were independently proved by M. Bertola and G. Ruzza (see the recent arXiv preprint [8]) through a different way, where the matrix-resolvent approach is applied to an isomonodromic system whose isomonodromic tau-function is identified in [8] with the (generalized) BGW tau-function.

2. REVIEW OF THE MATRIX RESOLVENT APPROACH TO THE KDV HIERARCHY

We continue with more details about the MR approach to the study of tau-functions.

2.1. Fundamental lemma. Recall from Introduction some of our notations $\mathcal{A} = \mathbb{Q}[u_0, u_1, u_2, \dots]$, $\Lambda(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & 0 \\ -2u_0 & 0 \end{pmatrix}$, $\mathcal{L} = \partial + \Lambda + q$, and $\mathcal{S} = \mathfrak{sl}_2(\mathbb{C})((\lambda^{-1}))$. Denote by $(\cdot|\cdot)$ the *normalized* Cartan–Killing form. We have $(A|B) = \text{Tr } AB$, $\forall A, B \in \mathfrak{sl}_2(\mathbb{C})$. This bilinear form naturally extends to \mathcal{S} . According to B. Kostant [32] and V. Kac [30], the space \mathcal{S} admits the decomposition

$$\mathcal{S} = \text{Ker ad}_\Lambda \oplus \text{Im ad}_\Lambda, \quad \text{Ker ad}_\Lambda \perp \text{Im ad}_\Lambda, \quad (52)$$

$$\text{Ker ad}_\Lambda = \left\{ \sum_{\ell \leq m} c_\ell \Lambda_{1+2\ell} \mid m \in \mathbb{Z}, c_\ell \in \mathbb{C}, c_m \neq 0 \right\}, \quad (53)$$

where $\Lambda_{1+2\ell} := \Lambda \lambda^\ell$. It is easy to see that $\deg \Lambda_{1+2\ell} = 1 + 2\ell$ (recall that \deg denotes the principal gradation) and that $(\Lambda|\Lambda) = 2\lambda$.

Lemma 1 (Drinfeld–Sokolov [15]). *There exists a unique pair (U, H) such that*

$$U \in \mathcal{A} \otimes (\text{Im ad}_\Lambda)^{<0}, \quad H \in \mathcal{A} \otimes (\text{Ker ad}_\Lambda)^{<0}, \quad (54)$$

$$e^{-\text{ad}_U} \mathcal{L} = \partial + \Lambda + H. \quad (55)$$

The proof, using (52)–(53) and the mathematical induction, can be found in [15, 7].

2.2. Matrix resolvent recursive relations. Let us prove that equations in (11) has a unique solution R . The first equation of (11) reads $\partial(R) + [\Lambda + q, R] = 0$. Write

$$R = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \quad (56)$$

Then it follows that

$$a = \frac{1}{2} \partial(b), \quad (57)$$

$$c = (\lambda - 2u_0)b - \frac{1}{2} \partial^2(b), \quad (58)$$

$$\partial^3(b) - 4(\lambda - 2u_0)\partial(b) + 4u_1b = 0. \quad (59)$$

Substituting (57)–(58) into the 3rd equation of (11) we find

$$b \partial^2(b) - \frac{1}{2} (\partial(b))^2 - 2(\lambda - 2u_0)b^2 = -2\lambda. \quad (60)$$

Applying ∂ to both sides of this equation yields (59). This shows compatibility between the 1st and the 3rd equations in (11). The 2nd equation of (11) is also compatible with the 3rd one. This gives the existence of R . Let us now prove the uniqueness. According to the 2nd equation in (11) we have

$$b = \sum_{k \geq -1} \frac{b_k}{\lambda^{k+1}}, \quad b_k \in \mathcal{A}, \quad b_{-1} = 1. \quad (61)$$

Then (59) and (60) imply that b_k satisfies the following recursive relations: for $k \geq 0$,

$$\partial(b_k) = 2u_0 \partial(b_{k-1}) + u_1 b_{k-1} + \frac{1}{4} \partial^3(b_{k-1}), \quad (62)$$

$$b_k = \sum_{\substack{k_1, k_2 \geq -1 \\ k_1 + k_2 = k-2}} \left[\left(\frac{b_{k_1} \partial^2(b_{k_2})}{4} - \frac{\partial(b_{k_1}) \partial(b_{k_2})}{8} \right) + u_0 b_{k_1} b_{k_2} \right] - \frac{1}{2} \sum_{\substack{k_1 \geq 0, k_2 \geq -1 \\ k_1 + k_2 = k-2}} b_{k_1} b_{k_2+1}, \quad (63)$$

which will be referred as the *matrix resolvent recursive relations*. (One recognizes that (62) is the Lenard–Magri recursion.) Formula (63) uniquely determines b_0, b_1, b_2, \dots with the first few being

$$b_0 = u_0, \quad b_1 = \frac{3}{2} u_0^2 + \frac{1}{4} u_2. \quad (64)$$

We note that the formal series $b = b(\lambda)$ determined here coincides with b in the Introduction.

Lemma 2. *The basic matrix resolvent can be expressed as $R = e^{\text{ad}_U}(\Lambda)$.*

Proof. Applying e^{ad_U} to $[\partial + \Lambda + H, \Lambda] \equiv 0$ gives $[\mathcal{L}, e^{\text{ad}_U}(\Lambda)] = 0$. Since $\deg U < 0$, the matrix $R := e^{\text{ad}_U}(\Lambda)$ satisfies the 2nd equation in (11). Finally, $\text{Tr } R^2 = (\Lambda|\Lambda) = 2\lambda$. The lemma is proved. \square

Definition 1. Define a family of derivations D_k ($k \geq 0$) by means of (3) with

$$Q_k := \frac{\partial(b_k)}{(2k+1)!!} - \frac{u_0^k}{k!} u_1. \quad (65)$$

2.3. From matrix resolvent to tau-structure. The basic matrix resolvent R has the expression

$$R = \begin{pmatrix} \frac{1}{2} \sum_{i \geq -1} \partial(b_i) \lambda^{-1-i} & \sum_{i \geq -1} b_i \lambda^{-1-i} \\ \sum_{i \geq -1} \lambda^{-1-i} \left((\lambda - 2u) b_i - \frac{\partial^2(b_i)}{2} \right) & -\frac{1}{2} \sum_{i \geq -1} \partial(b_i) \lambda^{-1-i} \end{pmatrix} \quad (66)$$

where we recall that $b_k \in \mathcal{A}$ ($k \geq -1$) are uniquely determined by (63) with $b_{-1} = 1$. Introduce

$$V_k := -\frac{1}{(2k+1)!!} (\lambda^k R(\lambda))_+ + \frac{1}{(2k+1)!!} \begin{pmatrix} 0 & 0 \\ b_k & 0 \end{pmatrix}, \quad k \geq 0. \quad (67)$$

Lemma 3. *The derivations D_k ($k \geq 0$) defined in Definition 1 satisfy*

$$[D_k - V_k, \mathcal{L}] = 0, \quad k \geq 0. \quad (68)$$

Proof. Since D_k commutes with $D_0 = \partial$, equation (68) can be written equivalently as

$$D_k(q) = [V_k, \mathcal{L}].$$

Noting that

$$V_k = \frac{1}{(1+2k)!!} \begin{pmatrix} -\frac{1}{2} \sum_{i=-1}^{k-1} \partial(b_i) \lambda^{k-1-i} & -\sum_{i=-1}^{k-1} b_i \lambda^{k-1-i} \\ \sum_{i=-1}^{k-1} \lambda^{k-1-i} \left(\frac{\partial^2(b_i)}{2} - (\lambda - 2u_0) b_i \right) & \frac{1}{2} \sum_{i=-1}^{k-1} \partial(b_i) \lambda^{k-1-i} \end{pmatrix}$$

we have by a direct calculation that

$$[V_k, \mathcal{L}] = [V_k, \partial + \Lambda + q] = \frac{1}{(1+2k)!!} \begin{pmatrix} 0 & 0 \\ -2\partial(b_k) & 0 \end{pmatrix}.$$

Here (62) has been used for the simplification. The lemma is proved. \square

Definition 2. Define $P_k = e^{\text{ad}_U}(\Lambda_{1+2k})$, $k = 0, 1, 2, 3, \dots$

Lemma 4. *The following formula holds true*

$$D_k(P_\ell) = [V_k, P_\ell], \quad \forall k, \ell \geq 0. \quad (69)$$

Proof. Applying $e^{-\text{ad}_U}$ to the both sides of (68) yields

$$[D_k + S_k, \partial + \Lambda + H] = 0$$

where $S_k := \sum_{i \geq 0} \frac{(-1)^i}{(i+1)!} \text{ad}_U^i(D_k(U)) - e^{-\text{ad}_U}(V_k)$. Clearly, S_k belongs to $\mathcal{A}_u \otimes \mathcal{S}$. Decompose $S_k = S_k^{\text{Ker}} + S_k^{\text{Im}}$ with $S_k^{\text{Ker}} \in \mathcal{A} \otimes \text{Ker}_{\text{ad}_\Lambda}$, $S_k^{\text{Im}} \in \mathcal{A} \otimes \text{Im}_{\text{ad}_\Lambda}$. We have

$$\partial(S_k^{\text{Ker}}) = D_k(H), \quad (70)$$

$$\partial(S_k^{\text{Im}}) = [S_k^{\text{Im}}, \Lambda + H]. \quad (71)$$

Equation (71) implies that S_k^{Im} must vanish. Otherwise, write $S_k^{\text{Im}} = \sum_{j \leq m} S_k^{\text{Im}, [j]}$ for some m with $S_k^{\text{Im}, [m]}$ is not zero. Then since $\deg H < 0$, the highest (principal) degree equation of (71) reads $[S_k^{\text{Im}, [m]}, \Lambda] = 0$. This implies that $S_k^{\text{Im}, [m]} = 0$, which produces a contradiction. Therefore S_k belongs to $\mathcal{A} \otimes \text{Ker}_{\text{ad}_\Lambda}$. This implies the identity

$$[D_k + S_k, \Lambda_{1+2\ell}] = 0.$$

Applying e^{ad_U} to this identity yields (69). \square

Let us proceed to the tau-structure. We first verify that $\Omega_{p,q}$ are well-defined from eq. (12). Indeed,

$$R(\mu) = R(\lambda) + R'(\lambda)(\mu - \lambda) + (\mu - \lambda)^2 \partial_\lambda \left(\frac{R(\lambda) - R(\mu)}{\lambda - \mu} \right),$$

so we have

$$\text{RHS of (12)} = \frac{2\lambda}{(\lambda - \mu)^2} - \frac{\text{tr}(R(\lambda)R'(\lambda))}{\lambda - \mu} + \text{tr}\left(R(\lambda)\partial_\lambda\left(\frac{R(\lambda) - R(\mu)}{\lambda - \mu}\right)\right) - \frac{\lambda + \mu}{(\lambda - \mu)^2}.$$

It is obvious from the property $\text{tr} R(\lambda)^2 = 2\lambda$ that $\text{tr}(R(\lambda)R'(\lambda)) = 1$. Therefore RHS of (12) = $\text{tr}\left(R(\lambda)\partial_\lambda\left(\frac{R(\lambda) - R(\mu)}{\lambda - \mu}\right)\right) \in \mathcal{A}[[\lambda^{-1}, \mu^{-1}]] \lambda^{-1}\mu^{-1}$. This finishes the verification.

Introduce $\nabla(\lambda) = -\sum_{k \geq 0} \frac{(1+2k)!!}{\lambda^{1+k}} D_k$.

Lemma 5. *The following identity holds true:*

$$\nabla(\lambda)(R(\mu)) = \frac{[R(\lambda), R(\mu)]}{\lambda - \mu} - [Q(\lambda), R(\mu)], \quad Q(\lambda) := \begin{pmatrix} 0 & 0 \\ b(\lambda) & 0 \end{pmatrix}. \quad (72)$$

Proof. Using equation (69) with $\ell = 0$ we have

$$\begin{aligned} \nabla(\lambda)(R(\mu)) &= -\sum_{k \geq 0} \frac{1}{\lambda^{1+k}} \left[-(\mu^k R(\mu))_+ + \begin{pmatrix} 0 & 0 \\ b_k & 0 \end{pmatrix}, R(\mu) \right] \\ &= \frac{1}{2\pi i} \oint_{|\mu| < |\rho| < |\lambda|} d\rho \frac{[R(\rho), R(\mu)]}{(\lambda - \rho)(\rho - \mu)} - \left[\begin{pmatrix} 0 & 0 \\ b(\lambda) - 1 & 0 \end{pmatrix}, R(\mu) \right] \\ &= \frac{[R(\lambda), R(\mu)]}{\lambda - \mu} - [\text{Coef}_{\lambda^1} R(\lambda), R(\mu)] - \left[\begin{pmatrix} 0 & 0 \\ b(\lambda) - 1 & 0 \end{pmatrix}, R(\mu) \right]. \end{aligned}$$

The lemma is proved. \square

Proof of Proposition 1. The fact that $\Omega_{0,0} = u$ can be obtained from a straightforward residue computation. Since the RHS of (12) is invariant with respect to $\lambda \leftrightarrow \mu$, we have $\Omega_{p,q} = \Omega_{q,p}$. Applying $\nabla(\nu)$ to the both sides of (12) we obtain

$$\begin{aligned} \sum_{p,q,r \geq 0} D_r(\Omega_{p,q}) \frac{(1+2p)!!(1+2q)!!(1+2r)!!}{\lambda^{p+1}\mu^{q+1}\nu^{r+1}} &= \frac{\text{Tr}[R(\nu), R(\lambda)]R(\mu)}{(\lambda - \mu)^2(\nu - \lambda)} + \frac{\text{Tr} R(\lambda) [R(\nu), R(\mu)]}{(\lambda - \mu)^2(\nu - \mu)} \\ &\quad - \frac{\text{Tr}[Q(\nu), R(\lambda)]R(\mu)}{(\lambda - \mu)^2} - \frac{\text{Tr} R(\lambda) [Q(\nu), R(\mu)]}{(\lambda - \mu)^2} \\ &= -\frac{\text{Tr}[R(\nu), R(\lambda)]R(\mu)}{(\lambda - \mu)(\mu - \nu)(\nu - \lambda)}. \end{aligned}$$

The RHS of this identity is invariant with respect to the permutations of λ, μ, ν , so is the LHS. This implies that $D_r(\Omega_{p,q}) = D_p(\Omega_{q,r}) = D_q(\Omega_{r,p})$. For $n = 3$, the statement is already proven in the above proof of Proposition 1. Formula (13) can be proved by induction; details can be found in [5, 7] and are omitted here. The proposition is proved. \square

Corollary 5. *The D_k ($k \geq 0$) (defined in Definition 1) commute pairwise. Moreover, we have $Q_0 = 0$, $Q_1 = u_3/12$, and $Q_k \in \mathcal{A}^{\geq 2}$.*

Proof. We have for all $k, \ell \geq 0$,

$$D_k D_\ell(\Omega_{0;0}) = D_k D_0(\Omega_{\ell;0}) = D_0 D_k(\Omega_{\ell;0}) = D_0 D_0(\Omega_{k;\ell}) = D_\ell D_k(\Omega_{0;0}).$$

Therefore $[D_k, D_\ell] = 0$. The fact that $Q_0 = 0$ and $Q_1 = u_3/12$ can be easily seen from (64) and (65). Define $B_k = b_k|_{u_i=0, i \geq 1}$. Then $B_{-1} = 1$. Equation (63) implies that

$$B_k = \sum_{\substack{k_1, k_2 \geq -1 \\ k_1 + k_2 = k-2}} u_0 B_{k_1} B_{k_2} - \frac{1}{2} \sum_{\substack{k_1 \geq 0, k_2 \geq -1 \\ k_1 + k_2 = k-2}} B_{k_1} B_{k_2+1}, \quad k \geq 0.$$

It follows from this recursion that $B_k = \frac{(1+2k)!!}{(k+1)!} u_0^{k+1}$ for all $k \geq 0$, which implies $Q_k \in \mathcal{A}^{\geq 2}$. The corollary is proved. \square

Corollary 5 and the uniqueness statement proved in [34] imply that the D_k defined in Definition 1 coincide with the ones defined in Section 1.1. Our definition of tau-function (see Section 1.2) of any solution also coincides with the one in the literature [12, 25], which can be seen by observing that

$$\Omega_{p,q}^{[0]} = \frac{u^{p+q+1}}{p! q! (p+q+1)},$$

where $\Omega_{p,q}^{[0]}$ denotes the degree zero part of $\Omega_{p,q}$.

3. FROM A PAIR OF WAVE FUNCTIONS TO THE n -POINT FUNCTIONS

Denote by $L = \partial^2 + 2u_0 : \mathcal{A} \rightarrow \mathcal{A}$ the Lax operator for the KdV hierarchy. Let us first briefly recall the pseudo-differential operator approach [13] to the study of the KdV equation. Notations will be the same as in Section 1.3. Denote $A_k = \frac{1}{(2k+1)!!} (L^{\frac{2k+1}{2}})_+$ ($k \geq 0$). We recall that A_k is a differential operator with coefficients in \mathcal{A} . Define a family of derivations D_k by $D_k(u_0) := [A_k, L/2]$. It is straightforward to show (by using the properties of pseudo-differential operators) that these D_k commute pairwise with $D_0 = \partial$, $D_1(u_0) = u_0 u_1 + u_3/12$ and $D_k(u_0) = u_0^k u_1/k! + \dots$, where the dots denotes terms in $\mathcal{A}^{\geq 2}$. So these D_k coincide with the ones defined in Section 1.1, and the KdV hierarchy can be written equivalently as

$$\frac{\partial L}{\partial t_k} = [A_k, L] \quad (73)$$

with ∂ replaced by ∂_x and u_i by $\partial_x^i(u)$. In the rest of this section, we give a way of constructing a pair of wave functions of an arbitrary solution, and apply it as well as Corollary 1 to prove Theorem 1.

3.1. Dressing operators and pairs of wave functions. Similarly as in Section 1.3, we start with the *time-independent* case. Namely, consider $L = \partial^2 + 2f(x)$ with $f(x) \in V$. Here we recall that V is a ring of functions of x closed under ∂_x . A pseudo-differential operator Φ of the form $\Phi = \sum_{m \geq 0} \phi_m(x) \partial_x^{-m}$ with $\phi_0(x) \equiv 1$ is called a *dressing operator* of L if

$$\Phi(x) \circ \partial_x^2 \circ \Phi(x)^{-1} = L.$$

Here $\phi_m(x)$ ($m \geq 1$) will live in a certain ring \tilde{V} satisfying $V \subseteq \partial_x(\tilde{V}) \subseteq \tilde{V}$.

If $\Phi(x)$ exists then the freedom of $\Phi(x)$ is given by an operator of the form $\sum_{m \geq 0} g_m \partial_x^{-m}$ for arbitrary constants g_m with $g_0 = 1$ through the right composition

$$\Phi(x) \mapsto \Phi(x) \circ \sum_{m \geq 0} g_m \partial_x^{-m}. \quad (74)$$

Define for any $p(x) \in \tilde{V}$ and for all $m \in \mathbb{Z}$, $\partial_x^{-m}(p(x) e^{xz}) := \sum_{l \geq 0} \binom{-m}{l} \partial_x^l(p(x)) z^{-m-l} e^{xz}$. If $\Phi = \Phi(x) = \sum_{m \geq 0} \phi_m(x) \partial_x^{-m}$ is a dressing operator of L , then it is straightforward to check that $\psi := \Phi(x)(e^{xz})$ is in $\tilde{V}((z^{-1})) e^{xz}$ and is a wave function of f (cf. Section 1.3). Explicitly, we have $\psi = \sum_{m \geq 0} \phi_m(x) z^{-m} e^{xz}$. On the contrary, if $\psi = \sum_{m \geq 0} \phi_m(x) z^{-m} e^{xz}$ is a wave function of f , then $\Phi := \sum_{m \geq 0} \phi_m(x) \partial_x^{-m}$ is a dressing operator of L . This gives the 1-1 correspondence between the dressing operators of L and the wave functions of f . Also, the freedom of Φ generated by the right composition (74) matches to $\psi \mapsto g(z) \psi$ with $g(z) = \sum_{m \geq 0} g_m z^{-m}$.

Let us give a proof of the existence of a wave function. Write

$$\psi(z, x) = e^{\beta(z, x)} e^{xz}, \quad \beta(z, x) := \sum_{j \geq 1} \beta_j(x) z^{-j}. \quad (75)$$

The defining equation $L(\psi) = z^2 \psi$ for the wave function ψ written in terms of β is given by

$$\beta_{xx} + \beta_x^2 + 2z\beta_x + 2f = 0. \quad (76)$$

Substituting the expression (75) of β in this equation we obtain

$$\beta'_j(x) = -\frac{1}{2} \beta''_{j-1}(x) - \frac{1}{2} \sum_{\substack{j_1, j_2 \geq 1 \\ j_1 + j_2 = j-1}} \beta'_{j_1}(x) \beta'_{j_2}(x) - f \delta_{j,1}, \quad (77)$$

where $\beta_0(x)$ is defined to as the 0-function and $' = d/dx$. Clearly, the recursion (77) uniquely determines $\beta'_j(x)$ for all $j \geq 1$ and $\beta'_j(x) \in V$. Since $V \subseteq \partial_x(\tilde{V})$, we conclude that the solution ψ of the form (75) to the equation $L(\psi) = z^2\psi$ exists with $\beta_j(x) \in \tilde{V}$. Expanding the exponential $e^{\beta(z,x)}$ into Taylor series of z^{-1} we obtain finally the required form $\psi = \sum_{m \geq 0} \phi_m(x) z^{-m} e^{xz}$ with $\phi_0(x) \equiv 1$ and $\phi_m(x) \in \tilde{V}$ for a wave function.

Now let ψ be a wave function of f and Φ the (unique) dressing operator of L associated to ψ . Define for any $p(x) \in \tilde{V}$ and for all $m \in \mathbb{Z}$, $\partial_x^{-m}(p(x) e^{-xz}) := \sum_{l=0}^{\infty} \binom{-m}{l} \partial_x^l(p(x)) (-z)^{-m-l} e^{-xz}$. Define a particular element ψ^* in $\tilde{V}((z^{-1})) e^{-xz}$ by

$$\psi^* = \psi^*(z, x) := (\Phi^{-1})^*(e^{-xz}), \quad (78)$$

where $(\Phi^{-1})^*$ denotes the formal adjoint operator of Φ^{-1} . For any $y_1 = a_1 e^{xz} \in \tilde{V}((z^{-1})) e^{xz}$ and $\tilde{y}_1 = b_1 e^{xz} \in \tilde{V}((z^{-1})) e^{-xz}$, define $y_1 \tilde{y}_1 := a_1 b_1 \in \tilde{V}((z^{-1}))$.

Lemma 6. *The elements ψ and ψ^* form a pair of wave functions of f .*

Proof. It is easy to check that ψ^* is indeed of the form $\psi^* = (1 + \phi_1^*(x)/z + \phi_2^*(x)/z^2 + \dots) e^{-xz}$ with $\phi^*(x) \in \tilde{V}$ and satisfies $L(\psi^*) = z^2\psi^*$. We are left to show that for all $i \geq 0$,

$$\text{res}_{z=\infty} \partial_x^i(\psi(z, x)) \psi^*(z, x) dz = 0.$$

Note that for any two pseudo-differential operators P, Q we have

$$\text{res}_{z=\infty} P(e^{xz}) Q(e^{-xz}) dz = \text{res}_{\partial_x} P \circ Q^*. \quad (79)$$

Taking $P = \partial_x^i \circ \Phi$ and $Q = \Phi^{-1}$ in this identity we find that

$$\text{res}_{z=\infty} \partial_x^i(\psi(z, x)) \psi^*(z, x) dz = \text{res}_{z=\infty} \partial_x^i \circ \Phi(e^{xz}) (\Phi^{-1})^*(e^{-xz}) dz = \text{res}_{\partial_x} \partial_x^i \circ \Phi \circ \Phi^{-1} = 0.$$

The lemma is proved. \square

Hence we have proved the existence of a pair of wave functions of f .

Observe that the above proof is revertible. Namely, the zero-residue condition ensures the uniqueness of the dual wave function associated with ψ . It must be the ψ^* defined by (78).

Lemma 7. *We have $\psi_x \psi^* - \psi \psi_x^* = 2z$ and $\psi(z, x) \psi^*(z, x) = b(z^2, x)$, with b as in (56).*

Proof. Denote $W = \psi_x \psi^* - \psi \psi_x^*$. We have $W_x = 0$. It follows that W must have the form $W = 2z + \sum_{k \geq 0} s_k z^{-k}$ where s_k are constants. For any $i \geq -1$,

$$\begin{aligned} \text{res}_{z=\infty} z^i W dz &= \text{res}_{z=\infty} \left(\partial_x \circ \Phi \circ \partial_x^i(e^{xz}) (\Phi^{-1})^*(e^{-xz}) - (-1)^i \Phi(e^{xz}) \partial_x \circ (\Phi^{-1})^* \circ \partial_x^i(e^{-xz}) \right) dz \\ &= \text{res}_{\partial_x} \left(\partial_x \circ \Phi \circ \partial_x^i \circ \Phi^{-1} + \Phi \circ \partial_x^i \circ \Phi^{-1} \circ \partial_x \right) = \text{res}_{\partial_x} \left(\partial_x \circ L^{\frac{i}{2}} + L^{\frac{i}{2}} \circ \partial_x \right) = 0. \end{aligned}$$

The last equality uses the fact that L is a differential operator and that $(L^*)^{1/2} = -L^{1/2}$. Therefore all s_k vanish. The first identity is proved. To prove the second equality we observe that both the LHS and the RHS satisfies the same differential equation (60) (with u_0 replaced by f and ∂ by ∂_x). Moreover, they are both formal power series in z^{-1} with the leading term 1. Then the uniqueness statement implies the second identity. The lemma is proved. \square

We note that the identity $\psi(z, x) \psi^*(z, x) = b(z^2, x)$ can be used as an alternative and possibly useful criterion for a pair of wave functions. We should also notice that for an arbitrary solution $u(\mathbf{t})$ there is a particular subclass of pairs of wave functions of $u(\mathbf{t})$ with the additional property $\psi(-z, x) = \psi^*(z, x)$, which was often assumed in the literature.

Let us now proceed to the *time-dependent* case, i.e. to prove the existence of a pair of wave functions of an arbitrary solution of the KdV hierarchy in the ring $V[[\mathbf{t}_{>0}]]$. We recall that for the operators $L = \partial^2 + 2u_0$ and A_k , once a solution $u = u(\mathbf{t})$ is taken, we will replace ∂ by ∂_x and u_i by $\partial_x^i(u)$.

Proposition 4. *Let $u = u(\mathbf{t})$ be a solution to the KdV hierarchy (2) in the ring $V[[\mathbf{t}_{>0}]]$. There exists a pseudo-differential operator Φ of the form*

$$\Phi = \Phi(\mathbf{t}) = \sum_{m \geq 0} \phi_m(\mathbf{t}) \partial_x^{-m}, \quad \phi_0(\mathbf{t}) \equiv 1 \quad (80)$$

such that

$$L = \Phi \circ \partial_x^2 \circ \Phi^{-1}, \quad (81)$$

$$\partial_{t_k}(\Phi) = -\frac{1}{(2k+1)!!} (\Phi \circ \partial_x^{2k+1} \circ \Phi^{-1})_- \circ \Phi, \quad \forall k \geq 0. \quad (82)$$

Here, $t_0 = x$ as usual and $\phi_m(\mathbf{t}) \in \tilde{V}[[\mathbf{t}_{>0}]]$ ($m \geq 1$).

Proof. Let $f(x) := u(x, \mathbf{0})$ be the initial data of the solution. According to the above time-independent theory, we can take a pseudo-differential operator $\Phi(x)$ of the form $\Phi(x) = \sum_{m=0}^{\infty} \phi_m(x) \partial_x^{-m}$ with $\phi_0(x) \equiv 1$ and $\phi_m(x) \in \tilde{V}$ ($m \geq 1$), such that $\partial_x^2 + 2f(x) = \Phi(x) \circ \partial_x^2 \circ \Phi(x)^{-1}$.

Now consider the initial value problem for $\Phi(\mathbf{t})$ given by equations (81) and (82) with the initial data $\Phi(x, \mathbf{0}) = \Phi(x)$. First let us check the compatibility between (81) and the $k = 0$ equation of (82). We notice that $(\Phi \circ \partial_x \circ \Phi^{-1})_- = (L^{1/2})_- = L^{1/2} - (L^{1/2})_+ = \Phi \circ \partial_x \circ \Phi^{-1} - \partial_x$. Therefore,

$$\partial_{t_0}(\Phi) = -\Phi \circ \partial_x + \partial_x \circ \Phi = \partial_x(\Phi).$$

This is compatible with our convention $t_0 = x$. Secondly, we will prove the compatibility between (81) and the equation (82) with $k \geq 1$. Indeed, we have

$$\begin{aligned} \frac{\partial L}{\partial t_k} &= \partial_{t_k}(\Phi) \circ \partial_x^2 \circ \Phi^{-1} - \Phi \circ \partial_x^2 \circ \Phi^{-1} \circ \partial_{t_k}(\Phi) \circ \Phi^{-1} \\ &= -\frac{1}{(2k+1)!!} \left(L^{\frac{2k+1}{2}} \right)_- \circ L + \frac{1}{(2k+1)!!} L \circ \left(L^{\frac{2k+1}{2}} \right)_- = [A_k, L]. \end{aligned}$$

This is true as u is a solution of the KdV hierarchy (cf. (73)). Finally, we check the compatibility between all the equations of (82). Indeed,

$$\begin{aligned} &\partial_{t_\ell} \partial_{t_k}(\Phi) - \partial_{t_k} \partial_{t_\ell}(\Phi) \\ &= \frac{\left(\left[\left(L^{\frac{2\ell+1}{2}} \right)_-, L^{\frac{2k+1}{2}} \right] - \left[\left(L^{\frac{2k+1}{2}} \right)_-, L^{\frac{2\ell+1}{2}} \right] + \left[\left(L^{\frac{2k+1}{2}} \right)_-, \left(L^{\frac{2\ell+1}{2}} \right)_- \right] \right)_-}{(2k+1)!! (2\ell+1)!!} \circ \Phi \\ &= \frac{\left(\left[\left(L^{\frac{2\ell+1}{2}} \right)_-, L^{\frac{2k+1}{2}} \right] - \left[L^{\frac{2k+1}{2}}, \left(L^{\frac{2\ell+1}{2}} \right)_+ \right] \right)_-}{(2k+1)!! (2\ell+1)!!} \circ \Phi = \frac{\left(\left[L^{\frac{2\ell+1}{2}}, L^{\frac{2k+1}{2}} \right] \right)_-}{(2k+1)!! (2\ell+1)!!} \circ \Phi = 0. \end{aligned}$$

The proposition is proved. \square

We call $\Phi(\mathbf{t})$ in Proposition 4 a *dressing operator* of $L = \partial_x^2 + 2u(\mathbf{t})$. It is clear from the proof of Proposition 4 that the freedom of the dressing operator $\Phi(\mathbf{t})$ is characterized by that of $\Phi(x)$, i.e., by a sequence of arbitrary constants g_1, g_2, \dots through

$$\Phi \mapsto \Phi \circ \sum_{m=0}^{\infty} g_m \partial_x^{-m}, \quad g_0 = 1.$$

Similarly as in the time-independent case we will use the dressing operator to prove the existence of a pair of wave functions of u . As in (80), write $\Phi(\mathbf{t}) = \sum_{m \geq 0} \phi_m(\mathbf{t}) \partial_x^{-m}$ with $\phi_0(x) = 1$. Put $Q = e^{\sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)!} t_k}$. Define for any $p(\mathbf{t}) \in \tilde{V}[[\mathbf{t}_{>0}]]$ and for all $m \in \mathbb{Z}$, $\partial_x^{-m} \left(p(\mathbf{t}) e^{\sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)!} t_k} \right) := \sum_{l \geq 0} \binom{-m}{l} \partial_x^l (p(\mathbf{t})) z^{-m-l} e^{\sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)!} t_k}$. Then one can check that the following element

$$\psi(z, \mathbf{t}) := \Phi(\mathbf{t})(Q) = \left(1 + \frac{\phi_1(\mathbf{t})}{z} + \frac{\phi_2(\mathbf{t})}{z^2} + \dots \right) e^{\sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)!} t_k} \quad (83)$$

is a wave function of u . For any $p(\mathbf{t}) \in \tilde{V}[[\mathbf{t}_{>0}]]$ and $m \in \mathbb{Z}$, define $\partial_x^{-m} \left(p(\mathbf{t}) e^{-\sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)!} t_k} \right) := \sum_{l \geq 0} \binom{-m}{l} \partial_x^l (p(\mathbf{t})) (-z)^{-m-l} e^{-\sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)!} t_k}$. For any $y_1 = a_1 e^{xz} \in \tilde{V}[[\mathbf{t}_{>0}]]((z^{-1})) e^{\sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)!} t_k}$, $\tilde{y}_1 = b_1 e^{xz} \in \tilde{V}[[\mathbf{t}_{>0}]]((z^{-1})) e^{-\sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)!} t_k}$, define $y_1 \tilde{y}_1 := a_1 b_1 \in \tilde{V}[[\mathbf{t}_{>0}]]((z^{-1}))$. One can then verify that the following element

$$\psi^*(z, \mathbf{t}) := (\Phi(\mathbf{t})^{-1})^*(Q) = \left(1 + \frac{\phi_1^*(\mathbf{t})}{z} + \frac{\phi_2^*(\mathbf{t})}{z^2} + \dots \right) e^{-\sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)!} t_k} \quad (84)$$

is the dual wave function of u associated with ψ . Hence we have proved the existence of a pair of wave functions of u .

3.2. Proof of Theorem 1. Let $u = u(\mathbf{t})$ be an arbitrary solution to the KdV hierarchy (2) and (ψ, ψ^*) a pair of wave functions of u . We first prove a useful lemma.

Lemma 8. *Define*

$$\Psi(z, \mathbf{t}) = \begin{pmatrix} \psi(z, \mathbf{t}) & \psi^*(z, \mathbf{t}) \\ -\psi_x(z, \mathbf{t}) & -\psi_x^*(z, \mathbf{t}) \end{pmatrix}. \quad (85)$$

Then we have

$$\det \Psi(z, \mathbf{t}) \equiv 2z \quad (86)$$

and

$$b(\lambda, \mathbf{t}) = \psi(z, \mathbf{t}) \psi^*(z, \mathbf{t}), \quad (87)$$

where $\lambda = z^2$. Moreover, the matrix resolvent $R(\lambda, \mathbf{t})$ of u is given by

$$R(\lambda, \mathbf{t}) \equiv -\Psi(z, \mathbf{t}) \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \Psi^{-1}(z, \mathbf{t}). \quad (88)$$

Proof. The proof for (86) and (87) is almost identical with that for Lemma 7, so we omit the details. Let us now prove (88). We have

$$-\Psi(z, \mathbf{t}) \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \Psi^{-1}(z, \mathbf{t}) = \begin{pmatrix} \frac{1}{2}(\psi\psi^*)_x & \psi\psi^* \\ -\psi_x\psi_x^* & -\frac{1}{2}(\psi\psi^*)_x \end{pmatrix}. \quad (89)$$

So (87) shows the (1,2)-entry identity of (88). The (1,1)-entry identity and (2,2)-entry identity are also true due to (57). It remains to show

$$-\psi_x\psi_x^* = (\lambda - 2u)b - \frac{1}{2}\partial_x^2(b).$$

Indeed,

$$\text{RHS} = (\lambda - 2u)\psi\psi^* - \frac{1}{2}(\psi_{xx}\psi + 2\psi_x\psi_x^* + \psi\psi_{xx}^*) = \text{LHS}.$$

The lemma is proved. \square

According to equation (88) we can write the basic matrix resolvent $R(z^2, \mathbf{t})$ of u in terms of ψ, ψ^* as

$$R(z^2, \mathbf{t}) = z + \begin{pmatrix} \psi \\ -\psi_x \end{pmatrix} \begin{pmatrix} \psi_x^* & \psi^* \end{pmatrix}. \quad (90)$$

For simplicity, we will often denote $R(\lambda) = R(\lambda, \mathbf{t})$. Through a direct calculation we have

$$\mathrm{tr} R(z_1^2) R(z_2^2) = (\psi_x^*(z_1) \psi(z_2) - \psi^*(z_1) \psi_x(z_2)) (\psi_x^*(z_2) \psi(z_1) - \psi^*(z_2) \psi_x(z_1)) - 2z_1 z_2. \quad (91)$$

Hence

$$\sum_{p_1, p_2} \Omega_{p_1, p_2}(\mathbf{t}) \frac{(2p_1 + 1)!! (2p_2 + 1)!!}{z_1^{2p_1+2} z_2^{2p_2+2}} = -D(z_1, z_2; \mathbf{t}) D(z_2, z_1; \mathbf{t}) - \frac{1}{(z_1 - z_2)^2}. \quad (92)$$

This shows formula (19) for $n = 2$. For $n \geq 3$, we are going to use a formula given in [20]:

$$\sum_{\sigma \in S_n / C_n} \frac{\mathrm{tr} R(\lambda_{\sigma(1)}) \cdots R(\lambda_{\sigma(n)})}{\prod_{i=1}^n (\lambda_{\sigma(i+1)} - \lambda_{\sigma(i)})} = \sum_{\sigma \in S_{n-2}} \frac{\langle R_n(\lambda_k), \mathrm{ad}_{R_n(\lambda_{\sigma(1)})} \cdots \mathrm{ad}_{R_n(\lambda_{\sigma(n-2)})} R_n(\lambda_{k-1}) \rangle}{(\lambda_{\sigma(n-1)} - \lambda_{n-2})(\lambda_n - \lambda_{n-1})(\lambda_{\sigma(1)} - \lambda_n) \prod_{i=1}^{n-3} (\lambda_{\sigma(i+1)} - \lambda_{\sigma(i)})},$$

where $\mathrm{ad}_a b := [a, b]$ and $\langle a, b \rangle := \mathrm{tr} ab$. Here we have abbreviated $R_i(\lambda) = R_i(\lambda, \mathbf{t})$. This formula tells that the contributions to the n -point generating series coming from the term z in (90) are zero. Therefore the theorem is proved by using the same argument as in [22]. The theorem is proved. \square

3.3. Proof of Theorem 2. We will first prove Proposition 2. It follows easily from Lemma 8 that the four functions $\psi, \psi^*, \psi_x, \psi_x^*$ satisfy the following three relations:

$$\begin{aligned} \psi(z, \mathbf{t}) \psi^*(z, \mathbf{t}) &= b(z^2, \mathbf{t}), \\ \psi_x(z, \mathbf{t}) \psi^*(z, \mathbf{t}) - \psi(z, \mathbf{t}) \psi_x^*(z, \mathbf{t}) &= 2z, \\ \psi_x(z, \mathbf{t}) \psi^*(z, \mathbf{t}) + \psi(z, \mathbf{t}) \psi_x^*(z, \mathbf{t}) &= b_x(z^2, \mathbf{t}). \end{aligned}$$

Solving this system we obtain

$$\psi^*(z, \mathbf{t}) = \frac{b(z^2, \mathbf{t})}{\psi(z, \mathbf{t})}, \quad \psi_x(z, \mathbf{t}) = \psi(z, \mathbf{t}) \frac{b_x(z^2, \mathbf{t}) + 2z}{2b(z^2, \mathbf{t})}, \quad \psi_x^*(z, \mathbf{t}) = \frac{b_x(z^2, \mathbf{t}) - 2z}{2\psi(z, \mathbf{t})}.$$

Substituting these expressions into (18), we obtain the first equality of (23). Proposition 2 is then proved. Theorem 2 follows easily from Theorem 1 and the first equality of (23), as the factors of the form $\psi(z, \mathbf{t})/\psi(w, \mathbf{t})$ cancel in each product of the sum of the right hand side of (19). \square

As the remark given in the introduction (right after Theorem 2), the abstract version of Theorem 2 follows immediately. We note that it is not difficult to give a direct proof of the abstract version of Theorem 2 by using the definition of K (cf. equation 20), Lemma 5 and Proposition 1; we leave it as an exercise for interested readers.

4. GENERATING SERIES OF THE GENERALIZED BGW CORRELATORS

Denote by $\pi : \overline{\mathcal{M}}_{g, n+1} \rightarrow \overline{\mathcal{M}}_{g, n}$ the forgetful map forgetting the last marked point, and $\rho : \overline{\mathcal{M}}_{g-1, n+2} \rightarrow \overline{\mathcal{M}}_{g, n}$ and $\phi_{h, I} : \overline{\mathcal{M}}_{h, |I|+1} \times \overline{\mathcal{M}}_{g-h, |J|+1} \rightarrow \overline{\mathcal{M}}_{g, n}$, $I \sqcup J = \{1, \dots, n\}$ the gluing maps. Norbury [36] introduced a collection of cohomology classes $\{\Theta_{g, n} \in H^*(\overline{\mathcal{M}}_{g, n})\}_{2g-2+n>0}$ satisfying

$$\text{i) } \Theta_{g, n} \text{ is of pure degree,} \quad (93)$$

$$\text{ii) } \rho^* \Theta_{g, n} = \Theta_{g-1, n+2}, \quad \phi_{h, I}^* \Theta_{g, n} = \pi_1^* \Theta_{h, |I|+1} \cdot \pi_2^* \Theta_{g-h, |J|+1}, \quad (94)$$

$$\text{iii) } \Theta_{g, n+1} = \psi_{n+1} \cdot \pi^* \Theta_{g, n}, \quad (95)$$

$$\text{iv) } \Theta_{1, 1} = 3\psi_1. \quad (96)$$

Norbury proved that such $\Theta_{g,n}$ exists and it must satisfy $\Theta_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$. Define $Z_\Theta = Z_\Theta(\mathbf{t})$ as the following generating series of intersection numbers (called the partition function)

$$Z_\Theta(\mathbf{t}) = \exp\left(\sum_{g,n \geq 0} \frac{1}{n!} \sum_{p_1, \dots, p_n \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{p_1} \dots \psi_n^{p_n} t_{p_1} \dots t_{p_n}\right). \quad (97)$$

Define $u_\Theta = u_\Theta(\mathbf{t}) := \frac{\partial^2 \log Z_\Theta(\mathbf{t})}{\partial t_0^2}$. The integrals $\int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{p_1} \dots \psi_n^{p_n}$, called the n -point Θ -class intersection numbers (aka the n -point Θ -class correlators), are independent of choice of $\Theta_{g,n}$. They vanish unless $p_1 + \dots + p_n = g - 1$. Therefore u_Θ belongs to $\mathbb{Q}[[t_0, t_1, t_2, \dots]]$.

Norbury's Theorem ([36]). *The formal power series u_Θ is a solution of the KdV hierarchy (2) with*

$$u_\Theta(t_0 = x, \mathbf{0}) = \frac{1}{8(x-1)^2}. \quad (98)$$

Moreover, Z_Θ is the tau-function of the solution u_Θ satisfying the string type equation

$$\sum_{i \geq 0} (1+2i) t_i \frac{\partial Z_\Theta}{\partial t_i} + \frac{1}{8} Z_\Theta = \frac{\partial Z_\Theta}{\partial t_0}. \quad (99)$$

Norbury's Theorem tells that $u_\Theta = u_{\Theta(1/8)}$. The latter is defined in Example 2 of Introduction. So $\tau_{\Theta(1/8)}$ and Z_Θ can only differ by the exponential of a linear function. Noting that equation (99) coincides with equation (38) with $C = 1/8$, so the linear function can only be a constant, which can easily be normalized as zero. We conclude that $Z_\Theta = \tau_{\Theta(1/8)}$. The latter is often called the BGW tau-function [10, 29, 1, 14, 35], originally studied in matrix models. The goal of this section is to prove Theorems 3, 4, 5, and give some explicit computations.

4.1. The essential second kind topological ODE of A_1 -type. The topological ODE of \mathfrak{g} -type with \mathfrak{g} being a simple Lie algebra was introduced and studied in [6, 7], which will be called the first kind topological ODE. To prove Theorem 4, let us introduce an ODE associated to $\mathfrak{sl}_2(\mathbb{C})$

$$2\zeta^3 \rho''' + 3\zeta^2 \rho'' - 2\zeta(\zeta - 2C)\rho' - 2C\rho = 0, \quad ' := \frac{d}{d\zeta} \quad (100)$$

where C is an arbitrary parameter. It will be used for computing the Θ -class intersection numbers in full genera ($C = 1/8$) and the generalized BGW correlators. We call (100) the *essential second kind topological ODE of A_1 -type*. See Section 6.1 for more details.

Proposition 5. *For any fixed $C \in \mathbb{C}$, there exists a unique series ρ in $\mathbb{C}[[z^{-1}]]$ satisfying equation (100) as well as the initial condition*

$$\rho(\infty) = 1. \quad (101)$$

Moreover, $\rho \in \mathbb{Q}[C][[\zeta^{-1}]]$ and it satisfies the following nonlinear ODE

$$\zeta \rho'^2 + \left(1 - \frac{2C}{\zeta}\right) \rho^2 - \rho \rho' - 2\zeta \rho \rho'' = 1. \quad (102)$$

Proof. Write

$$\rho = \sum_{k \geq -1} (2k+1)! \frac{\rho_k}{\zeta^{k+1}} \quad (103)$$

with $\rho_{-1} = 1$. Substituting this expression in (100) we obtain

$$\rho_k = \frac{C + \frac{k(k+1)}{2}}{k+1} \rho_{k-1}, \quad k \geq 0. \quad (104)$$

This implies the uniqueness statement of the proposition and $\rho \in \mathbb{Q}[C][[\zeta^{-1}]]$. To show (102), note that

$$\frac{d}{d\zeta} \left(\zeta \rho'^2 + \left(1 - \frac{2C}{\zeta}\right) \rho^2 - \rho \rho' - 2\zeta \rho \rho'' \right) = 0.$$

Therefore $\zeta \rho'^2 + (1 - 2C/\zeta) \rho^2 - \rho \rho' - 2\zeta \rho \rho'' \equiv C_1$ where C_1 is a constant independent of ζ . The fact that $C_1 \equiv 1$ can be deduced from (103). The proposition is proved. \square

We now apply Proposition 5 to derive the explicit expression of the basic matrix resolvent of the solution of the KdV hierarchy characterized by the initial data as in Example 2.

Proof of Theorem 4. Let $\rho = \rho(\zeta)$ be the unique element in Proposition 5. According to Proposition 5, $\rho = \rho(\zeta)$ satisfies equations (100), (101), (102). Define $\tilde{b}(\lambda, x) = \rho(\lambda(x-1)^2)$. Then it is easy to check that $\tilde{b}(\lambda, x)$ satisfies (59) and (60) with u replaced by $\frac{C}{(x-1)^2}$ and ∂ by ∂_x , and it has the form (61). Hence the uniqueness statement in the definition of the basic matrix resolvent (see Section 2.2) implies $b(\lambda, x) = \tilde{b}(\lambda, x)$. The above equation (104) yields an explicit expression for ρ :

$$\rho = 1 + \sum_{k \geq 0} \frac{(2k+1)!! \prod_{i=0}^k (C + \frac{i(i+1)}{2})}{(k+1)! \zeta^{k+1}} = {}_3F_0\left(\frac{1}{2}, \frac{1}{2} + \alpha, \frac{1}{2} - \alpha; \frac{1}{\zeta}\right). \quad (105)$$

(Recall that $2\alpha = \sqrt{1 - 8C}$.) This gives the $(1, 2)$ -entry of R . Other entries can be obtained by using (57) and (58). \square

As already pointed out after Theorem 4, if $-C$ is a triangular number, i.e., $C = -\frac{p(p+1)}{2}$ for some $p \in \mathbb{N}$, then the formal series $\rho = \rho(\zeta)$ truncates to the polynomial $\rho = {}_3F_0(\frac{1}{2}, 1+p, -p; \frac{1}{\zeta})$ in ζ^{-1} . This corresponds to the rational limit of the p -soliton solution to the KdV hierarchy.

For $n \geq 2$, using Corollary 1 one immediately obtains an explicit formula of the generating series of the n -point generalized BGW correlators (any C) in terms of the matrix R given in Theorem 4.

Proof of Theorem 3. Following from Norbury's Theorem, Corollary 1, Theorem 4 with $C = 1/8$. \square

Proof of Corollary 4. Note that equation (38) implies

$$(1 + 2p) \Omega_p^{\Theta(C)}(0) = \Omega_{0,p}^{\Theta(C)}(0), \quad p \geq 0.$$

Formula (41) is then a consequence of (105) (as it is an easy exercise that $\Omega_{0,k} = b_k/(2k+1)!!$). \square

Proof of Corollary 3. According to Norbury's Theorem, formula (37) is a special case of (41). \square

4.2. Proof of Theorem 5 and Proposition 3. It is straightforward to check that the function $W := \sqrt{2\xi/\pi} e^z K_\alpha(\xi)$ with $\xi = z(1-x)$ satisfies the differential equation $L(W) = z^2 W$, where $L = \partial_x^2 + 2C/(1-x)^2$. Moreover, as $z \rightarrow \infty$ within an appropriate sector, the asymptotic behavior of W coincides with ψ defined in (42). Similarly, the analytic function $W^* := \sqrt{2\pi\xi} e^{-z} I_\alpha(\xi)$ satisfies $L(W^*) = z^2 W^*$ and has asymptotics coinciding with ψ^* . It remains to show the zero-residue condition. This can be proved by noticing that

$$\begin{aligned} \psi(z, x) \psi^*(z, x) &= \sum_{k \geq 0} \frac{a_k(\alpha)}{z^k (1-x)^k} \sum_{\ell \geq 0} (-1)^\ell \frac{a_\ell(\alpha)}{z^\ell (1-x)^\ell} \\ &= {}_3F_0\left(\frac{1}{2}, \frac{1}{2} + \alpha, \frac{1}{2} - \alpha; \frac{1}{z^2(1-x)^2}\right) = b(z^2, x). \end{aligned} \quad (106)$$

This proves that ψ and ψ^* are a pair of wave functions as claimed, and equation (43) then follows from Theorem 1 with $\mathbf{t} = 0$. This completes the proof of Theorem 5.

For Proposition 3 we first observe that the recursion and boundary conditions for $A_{mn}(\alpha)$ given in (45) are obviously equivalent to either of the closed formulas in (46), so it suffices to prove the former. For convenience we write simply a_k and A_{mn} for $a_k(\alpha)$ and $A_{mn}(\alpha)$ and set

$$A(z) := \psi(z, 0) = \sum_{k=0}^{\infty} \frac{a_k}{z^k}, \quad B(z) := \psi_x(z, 0) = z + \sum_{k=0}^{\infty} \frac{b_k}{z^k}$$

with $b_k = a_{k+1} + ka_k$. (Avoid from confusion with the notations a_k, b_k in other sections.) One checks easily that $(A(z), B(z))$ satisfies the first-order differential system

$$\theta_z \begin{pmatrix} A(z) \\ B(z) \end{pmatrix} = \begin{pmatrix} z & -1 \\ 2C - z^2 & 1 + z \end{pmatrix} \begin{pmatrix} A(z) \\ B(z) \end{pmatrix}, \quad (107)$$

where $2C = \frac{1}{4} - \alpha^2$ and $\theta_z := z \frac{d}{dz}$. (This implies a second-order differential equation for A equivalent to the equation $LW = z^2W$ given above.) Setting $y = -w$, we find the two generating functions

$$\sum_{m,n \geq 0} \frac{(m+n)(A_{m+1,n} - A_{m,n+1})}{z^m y^n} = (\theta_z + \theta_y) \left(\frac{A(z)B(y) - B(z)A(y)}{z+y} \right)$$

(this follows from the definition of the A_{mn} and the fact that the Euler operator $\theta_z + \theta_y$ annihilates the degree zero function $\frac{z-y}{z+y}$) and

$$\sum_{m,n \geq 0} \frac{(n-m)a_m a_n}{z^m y^n} = (\theta_z - \theta_y)(A(z)A(y)).$$

Using the differential equations (107) we find that the right hand sides of both of these formulas equal $(z-y)A(z)A(y) + A(z)B(y) - B(z)A(y)$, proving equation (45). For (47), we notice that the first of equations (46) shows that the polynomial $A_{mn}(\alpha)$ is divisible by $a_m(\alpha)$ (because each a_r with $r \geq m$ is divisible by a_m), and since $a_n(\alpha)$ has only simple zeros, this shows that the quotient $\tilde{A}_{mn} := A_{mn}/a_m a_n$ has only simple poles at half-integral values of α . Since \tilde{A}_{mn} is also small at infinity, it has a partial fraction decomposition as $\sum_k c_{mn}(k)/(\alpha - k - \frac{1}{2})$ for some coefficients $c_{mn}(k)$. That these coefficients have the values given in (47) can be proved by comparing residues, using either the recursion (45) or one of the closed formulas (46), together with a simple binomial coefficient identity. The details are left to the reader. \square

4.3. Computations.

4.3.1. *Some Θ -class intersection numbers.* According to Norbury's Theorem, we have

$$\Omega_{p_1, \dots, p_n}^{\Theta(1/8)}(0) = \sum_{g=0}^{\infty} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{p_1} \dots \psi_n^{p_n}.$$

Here $n \geq 1$. Note that the degree-dimension matching reads

$$p_1 + \dots + p_n + 2g - 2 + n = 3g - 3 + n \quad \Leftrightarrow \quad p_1 + \dots + p_n = g - 1.$$

So, actually, $\Omega_{p_1, \dots, p_n}^{\Theta(1/8)}(0) = \int_{\overline{\mathcal{M}}_{1+p_1+\dots+p_n,n}} \Theta_{1+p_1+\dots+p_n,n} \psi_1^{p_1} \dots \psi_n^{p_n}$ as we have already given in (34).

We are going to compute some Θ -class intersection numbers of the form $\int_{\overline{\mathcal{M}}_{1+nb,n}} \Theta_{1+nb,n} \psi_1^b \dots \psi_n^b$, denoted by $\Omega_{b^n}^{\Theta(1/8)}(0)$ for short. Using an algorithm designed in [20] and using Theorem 3, one can compute these Θ -class intersection numbers in relatively high genera. For example we have

$$\begin{aligned} \Omega_{1^{11}}^{\Theta(1/8)}(0) &= \frac{3727154672771403705644393643825}{67108864}, \\ \Omega_{2^8}^{\Theta(1/8)}(0) &= \frac{10497097022517857530944569189202112366380675}{36028797018963968}, \\ \Omega_{3^6}^{\Theta(1/8)}(0) &= \frac{291143373745168297982109927833062542748609508458550221626925092375}{604462909807314587353088}. \end{aligned}$$

Several more Θ -class correlators are in Table 1. One observes that $\Omega_{0^n}^{\Theta(1/8)}(0) = \frac{(n-1)!}{8}$.

	$b = 0$	$b = 1$	$b = 2$	$b = 3$
$n = 1$	$\frac{1}{8}$	$\frac{3}{128}$	$\frac{15}{1024}$	$\frac{525}{32768}$
$n = 2$	$\frac{1}{8}$	$\frac{63}{512}$	$\frac{125565}{131072}$	$\frac{178066035}{8388608}$
$n = 3$	$\frac{1}{4}$	$\frac{7221}{2048}$	$\frac{8160299505}{8388608}$	$\frac{5357097499513095}{4294967296}$
$n = 4$	$\frac{3}{4}$	$\frac{4825971}{16384}$	$\frac{6118287865593075}{1073741824}$	$\frac{3673662570422147820860595}{4398046511104}$
$n = 5$	3	$\frac{3540311739}{65536}$	$\frac{2089963670900974355205}{17179869184}$	$\frac{7614423907504732590945890803999875}{2251799813685248}$
$n = 6$	15	$\frac{1209901485555}{65536}$	$\frac{31867458860062839143669852025}{4398046511104}$	$\frac{32942281960173069977596091564715863342175375}{576460752303423488}$

TABLE 1. $\Omega_{b^n}^{\Theta(1/8)}$, $b = 0, 1, 2, 3$.

The algorithm designed in [20] also produces explicit full genera formulas for the Θ -class correlators:

$$\begin{aligned} \sum_{p \geq 0} \frac{(1+2p)!!}{\lambda^{p+1}} \Omega_{0,p}^{\Theta(1/8)}(0) &= b - 1 \\ -512 \cdot 3!! \cdot 5!! \sum_{i \geq 0} \frac{(1+2p)!!}{\lambda^{p+1}} \Omega_{1,2,p}^{\Theta(1/8)}(0) &= -1024 \lambda^3 a + 256 \lambda^2 (b - a) + \lambda (848b - 616a - 128c) \\ &\quad + 3321b - 2187a - 432c \\ 1024 \cdot 3!!^3 \sum_{p \geq 0} \frac{(1+2p)!!}{\lambda^{p+1}} \Omega_{1,1,1,p}^{\Theta(1/8)}(0) &= 2048 \lambda^4 b + \lambda^3 (256b - 2048c) - \lambda^2 (1536a + 96b + 768c) \\ &\quad + \lambda (1728a - 8820b - 864c) + 26352a - 46989b + 4284c \end{aligned}$$

In the above formulas, $a = \frac{1}{8\lambda} {}_3F_0\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{\lambda}\right)$, $b = {}_3F_0\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{\lambda}\right)$, and

$$c = \left(\lambda - \frac{1}{4}\right) {}_3F_0\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{\lambda}\right) - \frac{3}{8\lambda} {}_3F_0\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{\lambda}\right) - \frac{27}{32\lambda^2} {}_3F_0\left(\frac{5}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{\lambda}\right).$$

4.3.2. *Correlators with parameter C .* We list some correlators of the form $\Omega_{b^n}^{\Theta(C)}(0)$ in Tables 2–3.

	$b = 0$	$b = 1$
$n = 1$	C	$\frac{1}{6}C(C+1)$
$n = 2$	C	$\frac{1}{6}C(C+1)(2C+5)$
$n = 3$	$2C$	$\frac{1}{3}C(C+1)(2C+7)(3C+10)$
$n = 4$	$6C$	$C(C+1)(22C^3 + 292C^2 + 1320C + 1925)$
$n = 5$	$24C$	$C(C+1)(364C^4 + 8028C^3 + 69089C^2 + 261625C + 350350)$
$n = 6$	$120C$	$C(C+1)(8160C^5 + 272480C^4 + 3843730C^3 + 27340910C^2 + 93831500C + 119119000)$

TABLE 2. $\Omega_{b^n}^{\Theta(C)}(0)$, $b = 0, 1$.

$n = 1$	$\frac{1}{30}C(C+1)(C+3)$
$n = 2$	$\frac{1}{60}C(C+1)(C+3)(3C^2+38C+126)$
$n = 3$	$\frac{1}{60}C(C+1)(C+3)(15C^4+550C^3+8011C^2+52521C+126126)$
$n = 4$	$\frac{1}{120}C(C+1)(C+3)(285C^6+21120C^5+701455C^4+12823420C^3+131525532C^2+698301072C+1466593128)$
$n = 5$	$\frac{1}{30}C(C+1)(C+3)(1035C^8+131220C^7+7929500C^6+286890460C^5+6581287505C^4+95511193020C^3+838324176858C^2+3995785717308C+7792009289064)$
$n = 6$	$\frac{1}{72}C(C+1)(C+3)(49329C^{10}+9683190C^9+941488056C^8+57517664804C^7+2370818604241C^6+67214920642718C^5+1301029520426886C^4+16691838842700000C^3+133819015248860760C^2+596987475819494760C+1110408075747354384)$

TABLE 3. $\Omega_{2n}^{\Theta(C)}(0)$.

We list first few explicit formulas for certain correlators in full genera:

$$\sum_{i \geq 0} \frac{(1+2p)!!}{\lambda^{p+1}} \Omega_{0,p}^{\Theta(C)}(0) = b - 1$$

$$3!! \sum_{i \geq 0} \frac{(1+2p)!!}{\lambda^{p+1}} \Omega_{1,p}^{\Theta(C)}(0) = \lambda(2b-3) + c - Cb$$

$$5!! \sum_{i \geq 0} \frac{(1+2p)!!}{\lambda^{i+1}} \Omega_{2,p}^{\Theta(C)}(0) = \lambda^2(3b-5) + 2\lambda(c-Cb) + 2Ca - \frac{C(C+3)}{2}b + Cc$$

$$- \frac{3!!^2}{2} \sum_{p \geq 0} \frac{(1+2p)!!}{\lambda^{i+1}} \Omega_{1,1,p}^{\Theta(C)}(0) = -\lambda^2 a + 2C\lambda(b-a) - C(C+3)a + C(2C+3)b - Cc$$

$$- 2 \cdot 5!!^2 \sum_{i \geq 0} \frac{(1+2p)!!}{\lambda^{p+1}} \Omega_{2,2,p}^{\Theta(C)}(0) = -4\lambda^4 a + 8C\lambda^3(b-a) + 4C\lambda^2((8C+9)b - c - (4C+6)a) \\ + C\lambda((36C^2+216C+180)b - (12C^2+132C+120)a - 24(C+1)c) \\ + 12C(3C^3+49C^2+151C+105)b - 18C(5+7C+2C^2)c - 9C(C^3+30C^2+99C+70)a$$

with $a = \frac{C}{\lambda}G_{\frac{3}{2}}(\lambda)$, $b = G_{\frac{1}{2}}(\lambda)$, and $c = (\lambda-2C)G_{\frac{1}{2}}(\lambda) - \frac{3C}{\lambda}G_{\frac{3}{2}}(\lambda) - \frac{6C(C+1)}{\lambda^2}G_{\frac{5}{2}}(\lambda)$, where we recall that $G_{\alpha}(\zeta)$ is defined in (40).

5. GENERATING SERIES OF THE LAMÉ PARTIAL CORRELATION FUNCTIONS

In this section, we study the Lamé tau-function τ_{elliptic} .

5.1. An explicit recursion for the basic matrix resolvent. Recall that the Lamé solution u_{elliptic} of the KdV hierarchy is the unique solution in the ring $V[[\mathbf{t}_{>0}]]$, where $V = \mathbb{C}[g_2, g_3, \wp, \wp'] / (\wp'^2 - 4\wp^3 + g_2\wp + g_3)$, having the initial data $f = C\wp(x; \tau)$. Let $R(\lambda, x)$ and $b(\lambda, x)$ denote the functions as in Section 1.4 (Example 3). Replacing ∂ with ∂_x and with u with $f = C\wp(x; \tau)$ in (60) we obtain the following nonlinear ODE for $b(\lambda, x)$

$$b_{xx}b - \frac{1}{2}b_x^2 - 2(\lambda - 2C\wp(x; \tau))b^2 = -2\lambda. \quad (108)$$

We note again that $b = b(\lambda, x)$ is the unique power series solution in λ^{-1} , having the form $b = \sum_{k \geq -1} b_k / \lambda^{k+1}$ with $b_{-1} = 1$, to equation (108). Denote $X = \wp(x; \tau)$, $Y = \wp'(x; \tau)$. Recall that

$$Y^2 = 4X^3 - g_2X - g_3 = 4(X - e_1)(X - e_2)(X - e_3)$$

with $g_2 = 60G_4$ and $g_3 = 140G_6$, where $G_{2k} = \sum_{(m,n) \in \mathbb{Z}^2 - (0,0)} \frac{1}{(m+n\tau)^{2k}}$, $k \geq 2$. In the (X, λ) -coordinates, equation (108) has the equivalent expression

$$\left(X^3 - \frac{g_2}{4}X - \frac{g_3}{4}\right)(2bb'' - b'^2) + \left(3X^2 - \frac{g_2}{4}\right)bb' + (2CX - \lambda)b^2 + \lambda = 0. \quad (109)$$

Here $' = \frac{\partial}{\partial X}$. It follows that b_k must have the form $b_k = P_{k+1}$ with $P_0 = 1$ and P_m ($m \geq 1$) being polynomials in X of degree m whose coefficients are polynomials of g_2, g_3, C . Equation (109) uniquely determines P_k , $k \geq 1$ in the following recursive way:

$$\begin{aligned} P_k &= \sum_{i=0}^{k-1} \left[CX P_i P_{k-i-1} + \left(\frac{3}{2}X^2 - g_2\right) P_i P'_{k-i-1} + \left(X^3 - \frac{1}{4}g_2X - \frac{1}{4}g_3\right) \left(P_i P''_{k-i-1} - \frac{1}{2}P'_i P'_{k-i-1}\right) \right] \\ &\quad - \frac{1}{2} \sum_{i=1}^{k-1} P_i P_{k-i}. \end{aligned} \quad (110)$$

We do not have a closed expression of b for a general value of C .

5.2. Proof of Theorem 6. Define $R^{\text{SP}}(\lambda, x) := \sqrt{\frac{S_p(\lambda)}{\lambda}} R(\lambda, x)$. We have

$$\begin{aligned} [\mathcal{L}, R^{\text{SP}}] &= 0, \\ \det R^{\text{SP}} &= -S_p(\lambda). \end{aligned}$$

One identifies R^{SP} with the matrix resolvent in [18]. This shows that $\sqrt{\frac{S_p(\lambda)}{\lambda}} b(\lambda, x)$ is a polynomial in λ of degree p with leading coefficient 1. The fact that $\sqrt{\frac{S_p(\lambda)}{\lambda}} b(\lambda, x)$ is also a degree p polynomial of $X = \wp$ can be deduced from homogeneity. Indeed, if we introduce a gradation by $\text{wt } \lambda = 2$, $\text{wt } X = 2$, $\text{wt } g_2 = 4$, $\text{wt } g_3 = 6$, then we find that $b(\lambda, x)$ is homogeneous of degree 0. Note that $S_p(\lambda)$ is homogeneous of degree $4p + 2$, i.e. $\text{wt } S_p(\lambda) = 4p + 2$. The theorem is proved. \square

5.3. Computations. The first few partial correlation functions of u_{elliptic} are

$$\begin{aligned} \Omega_{0,0}(x) &= C \wp(x), \quad \Omega_{0,0,0}(x) = C \wp'(x), \quad \Omega_{0,0,0,0}(x) = 6C \wp(x)^2 - \frac{Cg_2}{2}, \\ \Omega_{1,1}(x) &= C \left(\frac{(C+1)(2C+5)}{6} \wp(x)^3 - \frac{(C+1)g_2}{8} \wp(x) - \frac{C+2}{24} g_3 \right), \\ \Omega_{1,1,1}(x) &= \frac{C(C+1)}{24} \wp'(x) \left(4(2C+7)(3C+10) \wp(x)^3 - (11C+28)g_2 \wp(x) - 2(C+5)g_3 \right). \end{aligned}$$

Considering the Laurent series expansion of $\wp(x)$ at $x = 0$, we obtain

$$\begin{aligned} \Omega_{0,0}(x) &= C \left(\frac{1}{x^2} + \frac{g_2}{20} x^2 + \frac{g_3}{28} x^4 + \frac{g_2^2}{1200} x^6 + \frac{3g_2g_3}{6160} x^8 + \frac{49g_2^3 + 750g_3^2}{7644000} x^{10} + \dots \right), \\ \Omega_{0,0,0}(x) &= C \left(\frac{2}{x^3} - \frac{g_2}{10} x - \frac{g_3}{7} x^3 - \frac{g_2^2}{200} x^5 - \frac{3g_2g_3}{770} x^7 - \frac{49g_2^3 + 750g_3^2}{764400} x^9 - \dots \right), \\ \Omega_{1,1}(x) &= \frac{C(C+1)(2C+5)}{6x^6} + \frac{C^2(C+1)}{20x^2} g_2 + \frac{C(6C^2+14C+1)}{168} g_3 + \frac{C(C+1)(8C+5)}{2400} g_2^2 x^2 + \dots, \\ \Omega_{1,1,1}(x) &= \frac{C(C+1)(2C+7)(3C+10)}{6x^9} + \frac{C(C+1)(9C^2+34C+35)}{60x^5} g_2 \\ &\quad + \frac{C(C+1)(18C^2+109C+140)}{168x^3} g_3 + \frac{C(C+1)(24C^2+109C+140)}{2400x} g_2^2 + \dots. \end{aligned}$$

Since each of these expressions is modular of some weight with respect to the action $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, $x \mapsto \frac{x}{c\tau+d}$ of the full modular group, we call these Lamé partial correlation functions *modular deformations*

of the generalized BGW partial correlation functions (after a shift of x by 1). The partial correlation functions also allow us to consider other analytic in τ aspects. For example, using

$$\wp(x) = \frac{\pi^2}{3}(2 + 3 \cot^2(\pi x)) + 16 q^2 \pi^2 \sin^2(\pi x) + 4 q^4 (5 + 4 \cos(2\pi x)) \sin^2(\pi x) + \dots$$

one can obtain the $q \rightarrow 0$ expansion ($q = e^{\pi i \tau}$) of the Lamé partial correlation functions. The above modular deformations could be interpreted in an alternative way. Namely, we first switch on the periods $(2\omega, 2\omega')$ of the Weierstrass \wp -function. (Previously we have taken $\omega = \frac{1}{2}$ and $\omega' = \frac{\tau}{2}$). It is easy to see that the form of the partial correlation functions do not change when the initial data is given by $f(x) \equiv C \wp(x; 2\omega, 2\omega')$. Denote $\tau = \frac{\omega'}{\omega}$, $q = e^{\pi i \tau}$. Then we have as $\omega \rightarrow \infty$,

$$\wp(x) = \frac{1}{x^2} + \frac{\pi^4 x^2 (1 + 240 q^2 + 2160 q^4 + \dots)}{240 \omega^4} + \frac{\pi^6 x^4 (1 - 504 q^2 - 16632 q^4 + \dots)}{6048 \omega^6} + O(\omega^{-7}).$$

Hence the modular deformation of a generalized BGW partial correlation function can also be viewed as an $\omega \rightarrow \infty$ asymptotic.

Remark 2. For $C = 1/8$, we expect the existence of a deformation of the Norbury class such that the partition function is equal to the Lamé tau-function. For other values of C , we also expect the existence of cohomology classes on $\overline{\mathcal{M}}_{g,n}$ giving rise to the Lamé tau-function.

Using Corollary 1 we can obtain formulas like

$$\begin{aligned} -\frac{3!!^2}{2} \sum_{i \geq 0} \frac{(1+2i)!!}{\lambda^{i+1}} \Omega_{1,1,i}(x) &= -\lambda^2 a + C \lambda (\wp'(x) b - 2\wp(x) a) \\ &+ \frac{C}{4} (g_2 - 4(C+3)\wp(x)^2) a + \frac{C(2C+3)}{2} \wp(x) \wp'(x) b - \frac{C}{2} \wp'(x) c, \end{aligned} \quad (111)$$

where a, b, c are the components of $R(\lambda, x)$.

6. FURTHER REMARKS

6.1. The second kind topological ODE of A_1 -type. Consider the solution of the KdV hierarchy with initial data $f = C/(1-x)^2$. Let $R = R(\lambda, x)$ denote the basic matrix resolvent of this solution evaluated at $\mathbf{t}_{>0} = \mathbf{0}$ and $t_0 = x$. Define $M(\lambda, x) = \frac{1}{\lambda^{1/2}} \lambda^{\frac{\sigma_3}{4}} R(\lambda, x) \lambda^{-\frac{\sigma_3}{4}}$. According to Theorem 4 we have $M(\lambda, x) = M(\zeta)$ with $\zeta = \lambda(x-1)^2$. Moreover, $M = M(\zeta)$ satisfies

$$\frac{dM}{d\zeta} + \frac{1}{2} \left[\begin{pmatrix} 0 & 1/\sqrt{\zeta} \\ 1/\sqrt{\zeta} - 2C/\zeta & 0 \end{pmatrix}, M \right] = 0. \quad (112)$$

We call (112) the second kind topological ODE of A_1 -type. Note that as $\zeta \rightarrow \infty$, we have

$$M(\zeta) \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + O(1/\sqrt{\zeta}).$$

Equation (112) together with this boundary condition uniquely determines M . This M further satisfies $\det M(\zeta) = -1$. The second kind topological ODE (112) can be written equivalently as

$$\frac{dM}{dz} + \left[\begin{pmatrix} 0 & 1 \\ 1 - \frac{2C}{z^2} & 0 \end{pmatrix}, M \right] = 0, \quad (113)$$

where $\zeta = z^2$. More details about the second kind topological ODE and its generalization to an arbitrary simple Lie algebra will be given in a subsequent publication [24].

6.2. M -bispectrality. Let us consider $V = \mathbb{C}((x))$, the ring of Laurent series of x . Let $u = u(\mathbf{t})$ be a solution of the KdV hierarchy (2) in $V[[\mathbf{t}_{>0}]]$, $f(x) = u(x, \mathbf{0})$ the initial value. Let us recall a notion of bispectral solutions [22] defined by using $R(\lambda, x)$. Denote $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Definition 3. The solution u is called M -bispectral if there exist three scalar functions $g(\lambda) \neq 0$, $g_0(\lambda) \neq 0$, $h(\lambda, x)$, and an $\mathfrak{sl}_2(\mathbb{C})$ -valued function $\tilde{R}(h)$ such that

$$R(\lambda, x) = g(\lambda) g_0(\lambda)^{\sigma_3} \tilde{R}(h(\lambda, x)) g_0(\lambda)^{-\sigma_3}. \quad (114)$$

Theorem 7. *The solution u of the KdV hierarchy is M -bispectral iff*

$$\text{either } f(x) = (x - A)C, \quad \text{or } f(x) = \frac{C}{(x - A)^2} + B, \quad (115)$$

where A, B, C are constants.

Proof. Denote by $a(\lambda, x), b(\lambda, x), c(\lambda, x)$ the entries of $R(\lambda, x)$. They must have the form

$$a(\lambda, x) = g(\lambda) \rho_1(h(\lambda, x)), \quad (116)$$

$$b(\lambda, x) = q(\lambda) g(\lambda) \rho_2(h(\lambda, x)), \quad (117)$$

$$c(\lambda, x) = q(\lambda)^{-1} g(\lambda) \rho_3(h(\lambda, x)) \quad (118)$$

for some functions $g(\lambda) \neq 0$, $q(\lambda) = g_0(\lambda)^2 \neq 0$, $\rho_1(h), \rho_2(h) \neq 0$, $\rho_3(h)$. Substituting these expressions in equation (57) we obtain

$$\frac{1}{2} q(\lambda) g(\lambda) \rho_2'(h(\lambda, x)) h_x = g(\lambda) \rho_1(h(\lambda, x)).$$

This implies existence of three functions of one-variable W_1, W_2, P satisfying

$$h(\lambda, x) = P(x W_1(\lambda) + W_2(\lambda)). \quad (119)$$

Hence we can assume that

$$h(\lambda, x) = x W_1(\lambda) + W_2(\lambda), \quad W_1(\lambda) \neq 0. \quad (120)$$

It further implies that

$$\frac{1}{2} q(\lambda) W_1(\lambda) = \frac{\rho_1(h(\lambda, x))}{\rho_2'(h(\lambda, x))} = C_0,$$

where C_0 is a constant. Using equations (118) and (58) we have

$$(\lambda - 2f(x)) \rho_2(h(\lambda, x)) - \frac{W_1(\lambda)^2}{2} \rho_2''(h(\lambda, x)) = \frac{W_1(\lambda)^2}{4C_0^2} \rho_3(h(\lambda, x)).$$

Therefore

$$\lambda - 2f(x) = \frac{W_1(\lambda)^2}{2} \frac{\rho_2''(h(\lambda, x))}{\rho_2(h(\lambda, x))} + \frac{W_1(\lambda)^2}{4C_0^2} \frac{\rho_3(h(\lambda, x))}{\rho_2(h(\lambda, x))}. \quad (121)$$

Differentiating both sides w.r.t. λ we obtain

$$1 = \partial_\lambda \left(\frac{W_1(\lambda)^2}{2} \frac{\rho_2''(h(\lambda, x))}{\rho_2(h(\lambda, x))} + \frac{W_1(\lambda)^2}{4C_0^2} \frac{\rho_3(h(\lambda, x))}{\rho_2(h(\lambda, x))} \right). \quad (122)$$

Denote

$$G(h) := \frac{1}{2} \frac{\rho_2''(h)}{\rho_2(h)} + \frac{1}{4C_0^2} \frac{\rho_3(h)}{\rho_2(h)}.$$

We have

$$1 = \partial_\lambda (W_1(\lambda)^2 G(h(\lambda, x))) = 2W_1 W_1' G + (x W_1' + W_2') W_1^2 G'.$$

Differentiating both sides with respect to x we find

$$0 = 3W_1^2 W_1' G' + (x W_1' + W_2') W_1^3 G''.$$

This leads to two possibilities:

$$\text{i) } W_1' = 0 : \quad W_2' G'' = 0, \quad (123)$$

$$\text{ii) } W_1' \neq 0 : \quad \left(G'' \neq 0 \text{ and } x = -\frac{3W_1' G' + W_2' W_1 G''}{W_1' W_1 G''} \right) \quad \text{or} \quad (G'' = G' = 0). \quad (124)$$

For the case i), we have $W_1 \equiv C_1$ for some non-zero constant C_1 . So

$$1 = W_2' G' C_1^2.$$

This implies that $W_2' \neq 0$ and so $G'' \equiv 0$. Hence $G(h) = C_2 h + C_4$ and $W_2 = \frac{\lambda}{C_2 C_1^2} + C_3$ for some constants C_2, C_3, C_4 . Substituting these expressions in (121) we find

$$u = -\frac{1}{2} C_1^3 C_2 x - \frac{1}{2} C_1^2 C_2 C_3 - \frac{1}{2} C_1^2 C_4.$$

For the case ii), we have

$$1 = -3 \left(\frac{G'}{G''} \right)', \quad \text{or} \quad G = \text{Const.}$$

So

$$G(h) = \frac{C_1}{(h + C_2)^2} + C_3$$

for some constants C_1, C_2, C_3 . Then we find that

$$W_1 = \left(\frac{\lambda + 2C_3 C_4}{C_3} \right)^{\frac{1}{2}}, \quad (125)$$

$$W_2 = C_5 (\lambda + 2C_3 C_4)^{\frac{1}{2}} - C_2, \quad (126)$$

$$f = -\frac{C_1}{2(x + \sqrt{C_3} C_5)^2} - C_3 C_4 \quad (127)$$

for some constants C_4, C_5 . We have proved that u must have the form in (115).

We are left to prove the solvability of existence of g_0, g, \tilde{R}, h when f is taken to be one of (115). For $f(x) = (x - A)C$, we can take for instance

$$g_0 = 1, \quad g = \frac{\sqrt{\lambda}}{2^{\frac{1}{3}} C^{\frac{1}{3}}}, \quad W_1 = -2^{\frac{1}{3}} C^{\frac{1}{3}}, \quad W_2 = \frac{\lambda}{2^{\frac{2}{3}} C^{\frac{2}{3}}},$$

and we have

$$b = \frac{\sqrt{\lambda}}{2^{\frac{1}{3}} C^{\frac{1}{3}}} \sum_{g=0}^{\infty} \frac{(6g-1)!!}{96^g g!} \zeta^{-3g-\frac{1}{2}}, \quad \zeta := -(x-A) 2^{\frac{1}{3}} C^{\frac{1}{3}} + \frac{\lambda}{2^{\frac{2}{3}} C^{\frac{2}{3}}}.$$

For $f(x) = \frac{C}{(x-A)^2} + B$, we can take for instance

$$g_0 = \frac{1}{(\lambda - 2B)^{\frac{1}{4}}}, \quad g = \sqrt{\lambda}, \quad W_1 = \sqrt{\lambda - 2B}, \quad W_2 = -A \sqrt{\lambda - 2B},$$

and we have

$$b = \frac{\sqrt{\lambda}}{\sqrt{\lambda - 2B}} {}_3F_0 \left(\frac{1}{2}, \frac{1 + \sqrt{1 - 8C}}{2}, \frac{1 - \sqrt{1 - 8C}}{2}; ; \frac{1}{(x-A)\sqrt{\lambda - 2B}} \right).$$

The theorem is proved. \square

Remark 3. For the first case of (115), when (A, C) is taken as $(0, 1)$, the above theorem recovers the following formula of [5, 7, 43]:

$$\sum_{g, p_1, \dots, p_n \geq 0} \int_{\overline{\mathcal{M}}_{g, n}} \psi_1^{p_1} \cdots \psi_n^{p_n} \prod_{j=1}^n \frac{(2p_j + 1)!!}{\lambda_j^{p_j+1}} = - \sum_{\sigma \in S_n / C_n} \frac{\text{tr } M(\lambda_{\sigma(1)}) \cdots M(\lambda_{\sigma(n)})}{\prod_{i=1}^n (\lambda_{\sigma(i+1)} - \lambda_{\sigma(i)})} - \delta_{n2} \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2}, \quad (128)$$

where $M(\lambda)$ is a 2 by 2 matrix given explicitly by

$$M(\lambda) = \begin{pmatrix} \frac{1}{2} \sum_{g=1}^{\infty} \frac{(6g-5)!!}{24^{g-1} (g-1)!} \lambda^{-3g+2} & \sum_{g=0}^{\infty} \frac{(6g-1)!!}{24^g g!} \lambda^{-3g} \\ \sum_{g=0}^{\infty} \frac{1+6g}{1-6g} \frac{(6g-1)!!}{24^g g!} \lambda^{-3g+1} & -\frac{1}{2} \sum_{g=1}^{\infty} \frac{(6g-5)!!}{24^{g-1} (g-1)!} \lambda^{-3g+2} \end{pmatrix}. \quad (129)$$

We note that $\lambda^{-\frac{1}{2}} M(\lambda)$ satisfies an ODE in λ , *the first kind topological ODE of A_1 -type* [6, 7].

Remark 4. For the first case of (115), when $(A, C) = (q, 1)$, the corresponding KdV tau-function gives the partition function of stable quasi-map invariants [11] of a point.²

Remark 5. For the second case of (115), when (A, B, C) is take as $(1, 0, \frac{1}{8})$, the above theorem recovers the formula (39).

The following lemma due to Norbury has a similar flavour with Theorem 7.

Lemma 9 ([36]). *A collection of cohomology classes $\{\Theta_{g, n} \in H^*(\overline{\mathcal{M}}_{g, n})\}_{2g-2+n>0}$ satisfying the properties i) and ii) (see equations (93) and (94)) must satisfy*

$$\text{either } \deg \Theta_{g, n} \equiv 0, \quad \text{or } \deg \Theta_{g, n} \equiv 2g - 2 + n.$$

Proof Denote $d(g, n) = \deg \Theta_{g, n}$. Using the first equation of (94) we obtain

$$d(g, n) = d(g - 1, n + 2)$$

which implies the existence of a function H of one variable such that $d(g, n) = H(2g - 2 + n)$. Now using the second equation of (94) we find

$$H(x + y) = H(x) + H(y).$$

Therefore there exists an integer constant Q such that $d(g, n) = (2g - 2 + n)Q$. Noting that

$$0 \leq d(g, n) \leq 3g - 3 + n,$$

we have either $Q = 0$ or $Q = 1$. □

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