Medvedev degrees of 2-dimensional subshifts of finite type

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Abstract

In this paper we apply some fundamental concepts and results from recursion theory in order to obtain an apparently new counterexample in symbolic dynamics. Two sets X and Y are said to be *Medvedev equivalent* if there exist partial recursive functionals from X into Y and vice versa. The *Medvedev degree* of X is the equivalence class of X under Medvedev equivalence. There is an extensive recursion-theoretic literature on the lattice of Medvedev degrees of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$. This lattice is known as $\mathcal{P}_s$. We prove that $\mathcal{P}_s$ consists precisely of the Medvedev degrees of 2-dimensional subshifts of finite type. We use this result to obtain an infinite collection of 2-dimensional subshifts of finite type which are, in a certain sense, mutually incompatible.

Definition 1. Let A be a finite set of symbols. The *full 2-dimensional shift* on A is the dynamical system consisting of the natural action of $\mathbb{Z}^2$ on the compact set $A^{\mathbb{Z}^2}$. A *2-dimensional subshift* is a nonempty closed set $X \subseteq A^{\mathbb{Z}^2}$ which is invariant under the action of $\mathbb{Z}^2$. A 2-dimensional subshift X is said to be *of finite type* if it is defined by a finite set of forbidden configurations. An interesting paper on 2-dimensional subshifts of finite type is Mozes [19]. A standard reference for the 1-dimensional case is the book of Lind/Marcus [17], which also includes an appendix [17, §13.10] on the 2-dimensional case.

Remark 1. In the study of 2-dimensional subshifts of finite type, it has been useful to note that they are essentially the same thing as *tiling problems* in the sense of Wang [38].

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On the one hand, if F is a finite set of Wang tiles, let T_F be the set of tilings of the plane by F. Identifying T_F as a subset of $F^{\mathbb{Z}^2}$, it is clear that T_F is a 2-dimensional subshift of finite type. Conversely, given a 2-dimensional subshift of finite type, X, it is easy to construct a finite set of Wang tiles, F, such that T_F is recursively isomorphic to X. Namely, let r be a positive integer which is greater than or equal to the diameters of all of the forbidden configurations defining X, and let $F \subseteq A^{\{1,\ldots,r\}^2}$ be the set of $r \times r$ configurations which do not contain any of the forbidden configurations. Our isomorphism of X onto T_F associates to each point $x \in X$ a tiling $t \in T_F$ defined by $t(m,n)(i,j) = x(m+i, n+j)$ for all $(m,n) \in \mathbb{Z}^2$ and $(i,j) \in \{1,\ldots,r\}^2$. The adjacency rules for $\tau_1, \tau_2 \in F$ are: (a) τ_1 is allowed to occur immediately to the left of τ_2 if and only if $\tau_1(i+1,j) = \tau_2(i,j)$ for all $1 \leq i \leq r-1$ and $1 \leq j \leq r$, and (b) τ_1 is allowed to occur immediately below τ_2 if and only if $\tau_1(i,j+1) = \tau_2(i,j)$ for all $1 \leq i \leq r$ and $1 \leq j \leq r-1$.

Mozes [19, 20] and recently Hochman/Meyerovitch [15] have used the tiling methods of Wang [38] and Robinson [23] to construct 2-dimensional subshifts of finite type with interesting dynamical properties.

Definition 2. If X is a 2-dimensional subshift, the *shift operators* $S_1, S_2 : X \to X$ are defined by $S_1(x)(m,n) = x(m+1,n)$ and $S_2(x)(m,n) = x(m,n+1)$ for all $x \in X$ and $(m,n) \in \mathbb{Z}^2$. If X and Y are 2-dimensional subshifts, a *shift morphism* of X into Y is a continuous function $f : X \to Y$ which commutes with the shift operators, i.e., $f(S_1(x)) = S_1(f(x))$ and $f(S_2(x)) = S_2(f(x))$ for all $x \in X$. It follows that f commutes with the action of $\mathbb{Z}^2$ on X and Y. A *shift isomorphism* of X onto Y is a shift morphism of X one-to-one onto Y, i.e., a homeomorphism of X onto Y which commutes with S_1 and S_2. We say that X and Y are *shift isomorphic* if there exists a shift isomorphism of X onto Y.

Definition 3. Recall that two sets X and Y are *Medvedev equivalent* (see Rogers [24, §13.7] and Medvedev [18]) if there exist partial recursive functionals which map X into Y and Y into X. Also, X and Y are *recursively homeomorphic* if there exists a partial recursive functional which maps X one-to-one onto Y whose inverse is a partial recursive functional which maps Y one-to-one onto X. Finally, X and Y are *Muchnik equivalent* (see Muchnik [21]) if (a) each point in X is carried by some partial recursive functional to some point in Y, and (b) each point in Y is carried by some partial recursive functional to some point in X. Obviously recursive homeomorphism implies Medvedev equivalence, and Medvedev equivalence implies Muchnik equivalence. However, the converses do not hold. The equivalence classes under Medvedev equivalence and Muchnik equivalence are known as *Medvedev degrees* and *Muchnik degrees* respectively.

Remark 2. We shall now show that the Medvedev degree, Muchnik degree, and recursive homeomorphism type of a 2-dimensional subshift X depend only on the shift isomorphism type of X.

Theorem 1. *All shift morphisms $f : X \to Y$ are given by partial recursive functionals. If X and Y are shift isomorphic, then X and Y are recursively homeomorphic, hence Medvedev equivalent, hence Muchnik equivalent.*

Proof. Let A and B be finite sets of symbols such that X and Y are subshifts of the full 2-dimensional shifts $A^{\mathbb{Z}^2}$ and $B^{\mathbb{Z}^2}$ respectively. Let $f : X \to Y$ be a shift morphism. By the Curtis/Hedlund/Lyndon Theorem (see Boyle [7] or Lind/Marcus [17]) f is a *block code*, i.e., we can find an integer $r \geq 0$ and a projection $\pi : A^{\{-r,\ldots,r\}^2} \to B$ such that

$$f(x)(m,n) = \pi(S_1^m S_2^n(x) \restriction \{-r,\ldots,r\}^2)$$

for all $x \in X$ and $(m,n) \in \mathbb{Z}^2$. In particular f is given by a partial recursive functional. If moreover f is a shift isomorphism of X onto Y, it follows that f is a recursive homeomorphism of X onto Y. This completes the proof. $\square$

Definition 4. A closed set P is said to be Π^0_1 (see Rogers [24, Chapter 15]) if it is *effectively closed*, i.e., P is the complement of the union of a recursive sequence of basic open neighborhoods. Let $\mathcal{P}_s$ (respectively $\mathcal{P}_w$) be the lattice of Medvedev degrees (respectively Muchnik degrees) of nonempty Π^0_1 sets $P \subseteq \{0,1\}^{\mathbb{N}}$. It is known that the lattices $\mathcal{P}_s$ and $\mathcal{P}_w$ are distributive. There is a lattice homomorphism of $\mathcal{P}_s$ onto $\mathcal{P}_w$ obtained by mapping the Medvedev degree of P to the Muchnik degree of P. The lattices $\mathcal{P}_s$ and $\mathcal{P}_w$ are mathematically rich and have been studied extensively. See Alfeld [1], Binns [2, 3, 4], Binns/Simpson [5], Cenzer/Hinman [9], Cole/Simpson [11], and Simpson [26, 27, 29, 30, 31, 32, 33, 34, 35, 36].

Remark 3. An open problem is to characterize the recursive homeomorphism types of 2-dimensional subshifts of finite type. Obviously every 2-dimensional subshift of finite type is a Π^0_1 subset of $A^{\mathbb{Z}^2}$ and is therefore recursively homeomorphic to, hence of the same Medvedev degree and Muchnik degree as, a Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$. We shall now prove a theorem in the opposite direction. Namely, every Π^0_1 subset of $\{0,1\}^{\mathbb{N}}$ is of the same Medvedev degree as, hence of the same Muchnik degree as, some 2-dimensional subshift of finite type. This characterization of the Medvedev degrees and Muchnik degrees of 2-dimensional subshifts of finite type appears to be new.

Theorem 2. *Given a nonempty Π^0_1 set $P \subseteq \{0,1\}^{\mathbb{N}}$, we can find a 2-dimensional subshift of finite type which is of the same Medvedev degree as P.*

Proof. Our proof uses the tiling constructions of Robinson [23] and Hanf [14] and Myers [22].

Let F be a finite set of Wang tiles, and let $\tau \in F$ be a distinguished tile in F. We write $T_F^\tau = \{t \in T_F \mid t(0,0) = \tau\}$. The tilings in T_F^τ are said to be *origin-constrained*. Given a Π^0_1 set $P \subseteq \{0,1\}^{\mathbb{N}}$, Hanf [14] shows how to construct an origin-constrained tiling system F, τ and a projection $\pi : F \to \{0,1\}$ such that

$$P = \{\langle \pi(t(n,0)) \mid n \in \mathbb{N}\rangle \mid t \in T_F^\tau\}.$$

Namely, F and τ are such that each origin-constrained tiling $t \in T_F^\tau$ describes a non-halting run of a particular deterministic Turing machine M starting with

$$\langle \pi(t(n,0)) \mid n \in \mathbb{N}\rangle \in \{0,1\}^{\mathbb{N}}$$

inscribed on the right half of the Turing machine tape. The distinguished tile τ represents the initial internal state of M. Since M is deterministic, t is uniformly recursive relative to $\langle \pi(t(n,0)) \mid n \in \mathbb{N} \rangle$. It follows that T_F^τ and P are recursively homeomorphic.

Now, starting with F and τ as above, Myers [22] shows how to construct a finite set of tiles F' and a projection $\pi' : F' \to F$ with the following properties:

1. F' is a *Robinson set of tiles.*

   (Unexplained terms are defined in Myers' paper [22].)

2. For each tiling $t' \in T_{F'}$, the π'-images of the *center rows* of the *finite boards* of t' are *synchronized.*

3. For each tiling $t' \in T_{F'}$, the result of *piecing together* the π'-images of the center rows of the finite boards of t' is $\langle t(n,0) \mid n \in \mathbb{Z} \rangle$ for some $t \in T_F^\tau$. It follows that t is uniformly recursive relative to t'.

4. Given an origin-constrained tiling $t \in T_F^\tau$, we can find a tiling $t' \in T_{F'}$ such that $\langle t(n,0) \mid n \in \mathbb{Z} \rangle$ is the result of piecing together the π'-images of the center rows of the finite boards of t'. Moreover t' is uniformly recursive relative to t.

Properties 3 and 4 imply that $T_{F'}$ is Medvedev equivalent to T_F^τ. Therefore, $T_{F'}$ is Medvedev equivalent to P. Since $T_{F'}$ is a 2-dimensional subshift of finite type, our theorem is proved. $\square$

Remark 4. Actually Hanf [14] and Myers [22] deal only with Π_1^0 sets $P \subseteq \{0,1\}^{\mathbb{N}}$ of the form

$$P = S(I,J) = \{p \in \{0,1\}^{\mathbb{N}} \mid p \text{ separates } I, J\}$$

where I and J are recursively enumerable subsets of $\mathbb{N}$. However, their constructions work just as well for arbitrary Π_1^0 sets $P \subseteq \{0,1\}^{\mathbb{N}}$.

Definition 5. Let $\mathcal{S}^2$ be the set of all 2-dimensional subshifts. For $X, Y \in \mathcal{S}^2$ we write $X \geq Y$ if there exists a shift morphism $f : X \to Y$. Obviously $\geq$ is transitive and reflexive. We write $X \equiv Y$ if $X \geq Y$ and $Y \geq X$. Obviously $\equiv$ is an equivalence relation on $\mathcal{S}^2$. Obviously $X \equiv Y$ whenever X and Y are shift isomorphic, but the converse does not hold. Let $\mathcal{S}^2/{\equiv}$ be the set of equivalence classes of $\mathcal{S}^2$ modulo $\equiv$. There is an obvious partial ordering of $\mathcal{S}^2/{\equiv}$ induced by $\geq$. It is easy to verify that, under this partial ordering, $\mathcal{S}^2/{\equiv}$ is a distributive lattice. Let $\mathcal{S}^2_{\text{fin}}$ be the subset of $\mathcal{S}^2$ consisting of the 2-dimensional subshifts of finite type. It is easy to verify that $\mathcal{S}^2_{\text{fin}}/{\equiv}$ is a sublattice of $\mathcal{S}^2/{\equiv}$.

Theorem 3. *There is a natural lattice homomorphism of $\mathcal{S}^2_{\text{fin}}/{\equiv}$ onto $\mathcal{P}_s$. Namely, for each $X \in \mathcal{S}^2_{\text{fin}}$ we map the $\equiv$-equivalence class of X to the Medvedev degree of X.*

Proof. The greatest lower bound and least upper bound operations in each of the lattices $\mathcal{S}^2/{\equiv}$, $\mathcal{S}^2_{\text{fin}}/{\equiv}$, $\mathcal{P}_s$, $\mathcal{P}_w$ are given by disjoint union and Cartesian product respectively. In particular, our mapping of $\mathcal{S}^2_{\text{fin}}/{\equiv}$ to $\mathcal{P}_s$ is a lattice homomorphism. By Theorem 2 this homomorphism is onto $\mathcal{P}_s$. $\square$

Remark 5. A consequence of Theorem 3 is that there is a natural lattice homomorphism of $\mathcal{S}^2_{\text{fin}}/{\equiv}$ onto $\mathcal{P}_w$, obtained by mapping the $\equiv$-equivalence class of $X \in \mathcal{S}^2_{\text{fin}}$ to the Muchnik degree of X. In particular we have a new invariant, the Muchnik degree, associated to (shift isomorphism types of) 2-dimensional subshifts of finite type. Obviously this invariant is qualitatively different from other such invariants which have been considered previously in the symbolic dynamics literature. Moreover, this new invariant is of clear interest, inasmuch as $\mathcal{P}_w$ is known to contain many specific, natural Muchnik degrees which are closely related to significant topics in the foundations of mathematics. Among these topics are algorithmic randomness [32, 36], almost everywhere domination [35], reverse mathematics [28, 32], the hyperarithmetical hierarchy [11], and Kolmogorov complexity [16].

Remark 6. We shall now present an application which is stated purely in terms of 2-dimensional subshifts, with no reference to Medvedev degrees or Muchnik degrees. Namely, we shall construct an infinite collection of 2-dimensional subshifts of finite type which are, in a certain sense, mutually incompatible.

Definition 6. If X and Y are 2-dimensional subshifts on k and l symbols respectively, let $X+Y$ and $X \times Y$ be the disjoint union and Cartesian product of X and Y. These are 2-dimensional subshifts on $k+l$ and kl symbols respectively. If $X = (X, S_1, S_2)$ is a 2-dimensional subshift on k symbols, and if a, b, c, d are integers such that $ad - bc \neq 0$, let $X[a,b,c,d] = (X, S_1^a S_2^b, S_1^c S_2^d)$. This is a 2-dimensional subshift on $k^{|ad-bc|}$ symbols.

Definition 7. If $\mathcal{U}$ is a collection of 2-dimensional subshifts, let $\text{cl}(\mathcal{U})$ be the closure of $\mathcal{U}$ under the operations of Definition 6. In other words, $\text{cl}(\mathcal{U})$ is the smallest collection of 2-dimensional subshifts with the following properties:

1. For all $X \in U$, $X \in \text{cl}(\mathcal{U})$.
2. For all $X \in \text{cl}(\mathcal{U})$ and $Y \in \text{cl}(\mathcal{U})$, $X + Y \in \text{cl}(\mathcal{U})$ and $X \times Y \in \text{cl}(\mathcal{U})$.
3. For all $X \in \text{cl}(\mathcal{U})$ and all $a, b, c, d \in \mathbb{Z}$ with $ad - bc \neq 0$, $X[a,b,c,d] \in \text{cl}(\mathcal{U})$.

Theorem 4. *We can find an infinite collection of 2-dimensional subshifts of finite type, $\mathcal{W}$, such that for all partitions of $\mathcal{W}$ into two subcollections, $\mathcal{U}$ and $\mathcal{V}$, there is no shift morphism of X into Y for any $X \in \text{cl}(\mathcal{U})$ and $Y \in \text{cl}(\mathcal{V})$.*

Proof. By Binns/Simpson [5] let P_i, $i = 1, 2, \ldots$, be nonempty Π^0_1 subsets of $2^{\mathbb{N}}$ whose Medvedev degrees are *independent*, i.e., $P_{i_i} + \cdots + P_{i_m}$ is not Medvedev reducible to $P_{j_1} \times \cdots \times P_{j_n}$ provided $\{i_1, \ldots, i_m\} \cap \{j_1, \ldots, j_n\} = \emptyset$. By Theorem 2, for each $i = 1, 2, \ldots$ let X_i be a 2-dimensional subshift of finite type which is Medvedev equivalent to P_i. Let $\mathcal{W}$ be the collection X_i, $i = 1, 2, \ldots$, and let

$\mathcal{U}, \mathcal{V}$ be a partition of $\mathcal{W}$. By induction on $X \in \mathrm{cl}(\mathcal{U})$ and $Y \in \mathrm{cl}(\mathcal{V})$ we can easily show that neither of X and Y is Medvedev reducible to the other. Hence by Theorem 1 there is no shift morphism of X into Y or vice versa. Q.E.D. $\square$

Remark 7. In this paper we have dealt only with 2-dimensional subshifts. What about the 1-dimensional case? Clearly Theorem 1 holds in this case. On the other hand, Theorems 2 and 3 fail, because all 1-dimensional subshifts of finite type contain periodic points and are therefore of Medvedev degree zero and of Muchnik degree zero. An open problem is to characterize the Medvedev degrees, Muchnik degrees, and recursive homeomorphism types of 1-dimensional Π^0_1 subshifts. In this direction Cenzer/Dashti/King [8] have constructed a 1-dimensional Π^0_1 subshift which is of nonzero Muchnik degree, hence of nonzero Medvedev degree. We do not know whether Theorems 2, 3, and 4 hold for 1-dimensional Π^0_1 subshifts. We do not know whether Theorem 4 holds for 1-dimensional subshifts of finite type, or for 1-dimensional subshifts in general.

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