FUKUMOTO-FURUTA INVARIANTS OF PLUMBED HOMOLOGY 3–SPHERES

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Abstract. Recent major progress in study of the homology cobordism group of homology 3-spheres is related to the work of Fukumoto, Furuta, and Ue on their w–invariant. We identify the w–invariant with the $\bar{\mu}$–invariant of Neumann and Siebenmann for all plumbed homology spheres. This leads to proving that all Seifert fibered and some classes of plumbed homology spheres with non-trivial Rohlin invariant have infinite order in the homology cobordism group.

The homology cobordism group Θ^3 of oriented integral homology 3-spheres plays an outstanding role in the study of manifolds. One of its most spectacular applications is to the question of whether a closed topological manifold M of dimension five or higher is a polyhedron. It is shown in [9] and [10] that every such M is a polyhedron if and only if there exists a homology sphere with non-trivial Rohlin invariant which has second order in Θ^3. It is not known if such a homology sphere exists; in fact, it is not known whether Θ^3 has any torsion at all.

The group Θ^3 has been studied successfully in the last two decades by the methods of gauge theory. The main focus of such studies has been on Seifert fibered homology spheres $\Sigma(a_1,\dots,a_n)$ and other graph homology spheres. First, Fintushel and Stern [2] have shown that certain manifolds $\Sigma(a_1,\dots,a_n)$ have infinite order in Θ^3. Then Furuta [7] has found an infinite collection of Seifert fibered homology spheres of infinite order all of which are linearly independent in Θ^3, thus effectively establishing that Θ^3 is an infinitely generated abelian group.

Gauge theoretical methods have also given a boost to the study of Θ^3 via the $\bar{\mu}$–invariant of Neumann [11] and Siebenmann [17]. It has produced more examples of Seifert fibered homology spheres having infinite order in

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Θ^3, see [14] and [15]. Of particular importance in this study was the so called 10/8–condition on the signature of smooth spin manifolds established by Furuta [8]. Recently, Fukumoto and Furuta [5] extended this result to spin V–manifolds. Given an integral homology sphere Σ, they introduced a family of invariants $w(\Sigma, X, c)$ of Σ and a spin V–manifold (X, c) it bounds, and obtained some strong restrictions on the behavior of $w(\Sigma, X, c)$ whenever Σ has finite order in Θ^3.

In the special case of Seifert fibered homology sphere $\Sigma = \Sigma(a_1, \dots, a_n)$, Fukumoto, Furuta and Ue in [6] showed that $w(\Sigma, X, c) = -\bar{\mu}(\Sigma)$ for a specific choice of (X, c). Based on this result, they proved that $\bar{\mu}$ is a homology cobordism invariant for all $\Sigma(a_1, \dots, a_n)$ and their multiples provided one of Seifert invariants a_i is even, or otherwise their number does not exceed six.

We develop a general theory of the w–invariant for plumbed homology spheres and plumbed V–manifolds. Plumbed V–manifolds comprise a rather large class of V–manifolds and include, in particular, all V–manifolds studied in [3] and [6] in connection with plumbed homology spheres. Every plumbed homology sphere bounds a spin plumbed V–manifold.

Theorem 1. *Let Σ be a plumbed homology sphere then $w(\Sigma, X, c) = -\bar{\mu}(\Sigma)$ for every spin plumbed V–manifold (X, c) such that $\partial X = \Sigma$.*

Using this theorem, we study homology cobordism properties of Seifert fibered and plumbed homology spheres. We notice that the results of [6] and [15] can be combined to remove the above mentioned restriction on Seifert fibered homology spheres of [6], and prove the following general result.

Theorem 2. *Any Seifert fibered homology sphere $\Sigma(a_1, \dots, a_n)$ with non-vanishing $\bar{\mu}$–invariant has infinite order in the homology cobordism group.*

The modulo 2 reduction of the $\bar{\mu}$–invariant coincides with the Rohlin invariant. In particular, if the Rohlin invariant of $\Sigma(a_1, \dots, a_n)$ is non-trivial then $\Sigma(a_1, \dots, a_n)$ cannot have order two in Θ^3. Thus if one is to look for an element in Θ^3 solving positively the triangulation conjecture, all Seifert fibered homology spheres, as well as all homology spheres them homology cobordant, should be excluded from consideration.

The methods of [5] and [6] are further applied to study homology cobordism properties of general plumbed homology spheres. It is not clear if these techniques are strong enough to prove that $\bar{\mu}$ is a homology cobordism invariant for all of them. However, we use them to construct an infinite family of plumbed homology spheres which are not Seifert fibered but for which the conclusion of Theorem 2 is true. This is a generalization of the results of [3].

Theorem 3. *Let Σ be the splice of two Seifert fibered homology spheres $\Sigma(a_1, \dots, a_n)$ and $\Sigma(\alpha_1, \dots, \alpha_m)$ along fibers of degrees a_n and α_m, and let $p = a_1 \cdots a_{n-1} \cdot \alpha_1 \cdots \alpha_{m-1} - a_n \cdot \alpha_m$. Suppose that Σ has finite order in the homology cobordism group. If $p \geq 0$ then $\bar{\mu}(\Sigma) = 0$, if $p < 0$ then $\bar{\mu}(\Sigma)$ is either zero or one.*

Note that Σ is an algebraic link if and only if $p \leq 0$, and it is Seifert fibered if $p = 0$. Thus the theorem does not provide us with a definite conclusion about $\bar{\mu}(\Sigma)$ precisely in the case when Σ is a non-Seifert fibered algebraic links.

I wish to thank Dr. Furuta for useful discussions at Max-Planck-Institut für Mathematik in Bonn in 1998, and Dr. Fukumoto and Dr. Ue for sending me preprints [3], [4] and [6].

1. The w–invariants of Fukumoto and Furuta

In this section we review the w–invariants defined in [5] by using V–manifolds. We formulate Fukumoto-Furuta's extension of the smooth 10/8–theorem to spin V–manifolds and show how it is related to certain homology cobordism properties of the w–invariants.

1.1. **Closed V–manifolds.** The concept of V–manifold [13] will not be used in this paper in its full generality. We will only deal with compact 4-dimensional V–manifolds with isolated singular points. If X is such a V-manifold, each its singular point has a neighborhood isomorphic to the cone over a lens space or other spherical space form. The usual concepts of topology and differential geometry extend to V–manifolds after defining everything in an equivariant setting. In particular, we will be interested in a spin structure on W and the (complex) index of the associated Dirac operator $\mathcal{D}(X)$. Fukumoto and Furuta proved in [5] the following result about this index.

Theorem 4. *Let X be an oriented closed spin V–manifold with $b_1(X) = 0$, and let c be a spin structure on X. Then* $\operatorname{ind} \mathcal{D}(X)$ *is even, and either* $\operatorname{ind} \mathcal{D}(X) = 0$ *or* $b_-(X) < \operatorname{ind} \mathcal{D}(X) < b_+(X)$.

Note that if X is actually smooth, the statement of this theorem is equivalent to the famous 10/8–theorem of Furuta [8]. In this case, the index of Dirac operator is computed from the index theorem as

$$\operatorname{ind} \mathcal{D}(X) = -p_1(X)/24 = -\operatorname{sign}(X)/8, \tag{1}$$

which together with inequality $b_-(X) < \operatorname{ind} \mathcal{D}(X) < b_+(X)$ of the theorem implies that $b_2(X)/|\operatorname{sign}(X)| > 10/8$.

1.2. **Definition of w–invariants.** Let Σ be an oriented integral homology sphere, and let X be a compact oriented spin V–manifold with only isolated singularities such that $b_1(X) = 0$ and $\partial X = \Sigma$. Such a V–manifold always exists, moreover, it can be chosen to be smooth and simply connected. Let c be a spin structure on X. For every triple (Σ, X, c), the w–invariant is defined in [5] as

$$w(\Sigma, X, c) = \operatorname{ind} \mathcal{D}(X \cup Y) + \operatorname{sign}(Y)/8,$$

where Y is any smooth spin 4–manifold with $\partial Y = -\Sigma$. Here, $\mathcal{D}(X \cup Y)$ is Dirac operator associated with the spin structure on $X \cup Y$ induced by c and the spin structure on Y. Of course, if X is smooth we simply get $w(\Sigma, X, c) = -\operatorname{sign}(X)/8$.

The definition of $w(\Sigma, X, c)$ does not depend on the choice of Y. The reduction of $w(\Sigma, X, c)$ modulo 2 equals the Rohlin invariant of Σ for any choice of X and c. Moreover, $w(\Sigma, X, c)$ is additive with respect to connected sums of the triples (Σ, X, c). All these statements are verified in [5]. Together with Theorem 4 they imply the following important property of the w–invariants.

Theorem 5. *Let Σ be an oriented integral homology sphere which has finite order in the homology cobordism group Θ^3. Then, for any choice of X and c, either $w(\Sigma, X, c) = 0$ or*

$$-b_-(X) < w(\Sigma, X, c) < b_+(X).$$

For example, if Σ happens to bound a spin V–manifold X such that $b_+(X) \le 1$, $b_-(X) \le 1$ and $w(\Sigma, X, c) \ne 0$, then Σ cannot be of finite order in the homology cobordism group. The strength of Theorem 5 lies in the fact that many homology spheres Σ do bound spin V–manifolds X with small values of $b_\pm(X)$.

2. Calculus of plumbing graphs

This section is devoted to 3- and 4-dimensional manifolds obtained by plumbing D^2–bundles over 2-spheres, and to the $\bar\mu$–invariant of Neumann and Siebenmann.

2.1. **Plumbed manifolds.** Plumbing graph Γ is a graph with no cycles, each of whose vertices v_i carries an integer weight e_i, $i = 1, \dots, s$. Associate to each vertex v_i the D^2–bundle $Y(e_i)$ over S^2 with the Euler number e_i. If the vertex v_i has d_i edges connected to it in the graph Γ, choose d_i disjoint discs in the base of $Y(e_i)$ and call the disc bundle over the j–th disc B_{ij}, so that $B_{ij} = D^2 \times D^2$. When two vertices, v_i and v_k, are connected by an edge, identify B_{ij} with $B_{k\ell}$ by exchanging the base and fiber coordinates. This pasting operation is called plumbing, and the resulting smooth 4-manifold $P(\Gamma)$ is said to be obtained by plumbing according to Γ. Its boundary $M(\Gamma) = \partial P(\Gamma)$ is usually referred to as a plumbed 3–manifold.

The 4–manifold $P(\Gamma)$ is simply connected. The group $H_2(P(\Gamma))$ has a natural basis represented by the zero-sections of the plumbed bundles. We note that all these sections are embedded 2-spheres, and that they can be oriented in such a way that the intersection form of $P(\Gamma)$ will be given by the matrix $A(\Gamma) = (a_{ij})_{i,j=1,\dots,s}$ with the entries

$$a_{ij} = \begin{cases} e_i, & \text{if } i = j; \\ 1, & \text{if } i \text{ is connected to } j \text{ by an edge;} \\ 0, & \text{otherwise.} \end{cases}$$

Let $M = \partial P(\Gamma)$ be a 3-dimensional manifold obtained by plumbing according to a graph Γ, then M is an integral homology sphere if and only if the matrix $A(\Gamma)$ is unimodular, that is $\det A(\Gamma) = \pm 1$.

Different plumbing graphs can describe the same 3-dimensional manifold. In fact, plumbing graphs Γ_1 and Γ_2 describe the same 3-manifold if and only if Γ_1 can be obtained from Γ_2 by a sequence of moves described in [1] and in [17].

2.2. **Seifert manifolds and lens spaces.** Let M be a 3-manifold Seifert fibered over S^2 with Seifert invariants $(b;(a_1,b_1),\dots,(a_n,b_n))$, $a_i \geq 1$. It can be thought of as the boundary of the 4-manifold obtained by plumbing according to the star-shaped graph shown in Figure 1.

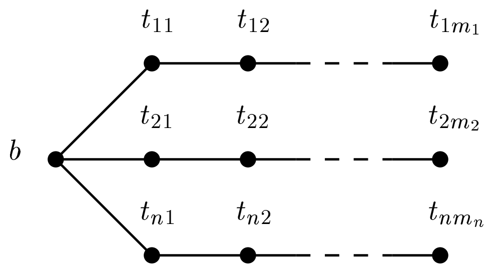


FIGURE 1

The integer weights t_{ij} in the graph are found from continued fractions $a_i/b_i = [t_{i1},\dots,t_{im_i}]$ where $i=1,\dots,n$ and we use the notation

$$[t_1,\dots,t_k] = t_1 - \cfrac{1}{t_2 - \cfrac{1}{\cdots - \cfrac{1}{t_k}}}$$

Different decompositions of a_i/b_i into continued fractions lead to plumbing graphs describing the same 3–manifold M but different 4–manifolds $P(\Gamma)$.

Seifert manifold M is an integral homology sphere if and only if the determinant of matrix $A(\Gamma)$ associated with the plumbing graph in Figure 1 is ± 1, which translates into the condition

$$a_1\cdots a_n \cdot \left(-b + \sum_{i=1}^{n} b_i/a_i\right) = \pm 1. \tag{2}$$

We fix an orientation on M by choosing $+1$ in (2). By considering (2) modulo a_i we see that it determines b_i modulo a_1 which is enough to determine the manifold M completely. We use the notation $\Sigma(a_1,\dots,a_n)$ for this manifold and call it a Seifert fibered homology sphere.

In general, the order of the first homology of $M(b;(a_1,b_1),\dots,(a_n,b_n))$ equals the absolute value of

$$a_1\cdots a_n\cdot\left(-b+\sum_{i=1}^{n} b_i/a_i\right),$$

unless the latter is equal to zero, in which case $H_1(M,\mathbb{Z})$ is infinite.

Lens spaces are a special case of Seifert fibered manifolds. Our orientation convention will be that a lens space $L(p,q)$ is obtained by $(-p/q)$–Dehn surgery on an unknot in S^3. Expanding $-p/q = [t_1,\dots,t_n]$ in a continuous fraction we get $L(p,q)$ as the boundary of the 4–manifold obtained by plumbing on the chain Γ shown in Figure 2. We will use notation $A(\Gamma)=\Delta(t_1,\dots,t_n)$.

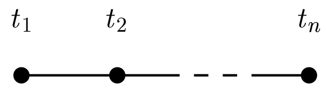


FIGURE 2

One can notice that this plumbing graph also represents lens space $L(p,q^*)$ with $-p/q^* = [t_n,\dots,t_1]$ where $qq^* = 1 \bmod p$. This reflects the fact that $L(p,q)$ and $L(p,q^*)$ are homeomorphic. Our convention will be that we will include S^3 but not $S^1\times S^2$ into lens spaces: these can both be obtained by chain plumbing, see for instance Figure 3. Thus our convention will be that all lens spaces are rational homology spheres.

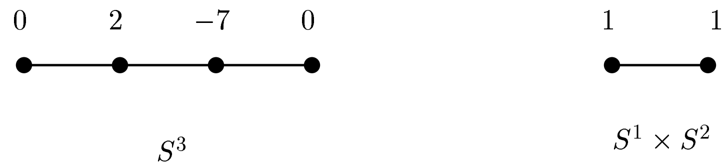
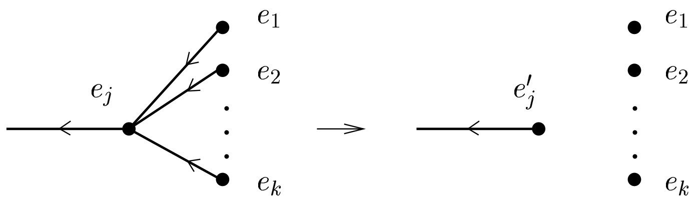


FIGURE 3

2.3. **Intersection forms of plumbed 4–manifolds.** There is a simple algorithm to diagonalize matrices of the form $A(\Gamma)$, see [1]. This algorithm is a graph realization of the usual Gauss diagonalization process for symmetric matrices. Let us allow trees weighted by arbitrary rational numbers. Given such a tree, pick a vertex and direct all its edges toward this vertex. Now, the tree can be simplified by performing operations of the two types shown in Figures 4 and 5 (we re-index vertices if necessary).



$$e'_j = e_j - \frac{1}{e_1} - \ldots - \frac{1}{e_k}$$

FIGURE 4

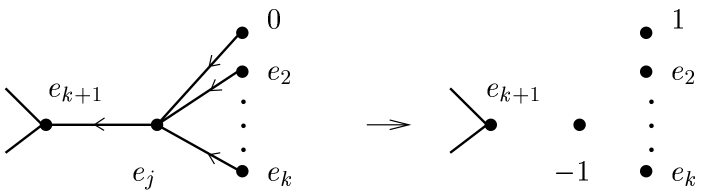


FIGURE 5

We end up with a finite collection of isolated points weighted by rational numbers $d_1, \ldots, d_s$. Then $D = \operatorname{diag}(d_1, \ldots, d_s)$ is a diagonalization of $A(\Gamma)$, that is, $A(\Gamma) = U^\top DU$ with $\det U = \pm 1$. In particular, $\det A(\Gamma) = \det D$ and $\operatorname{sign} A(\Gamma) = \operatorname{sign} D$.

Example. For the tree Γ in Figure 6 we have $\det A(\Gamma) = \det D = 2 \cdot 3 \cdot (1/6) \cdot (-2) \cdot (-1/2) = 1$ and $\operatorname{sign} A(\Gamma) = \operatorname{sign} D = 3 - 2 = 1$.

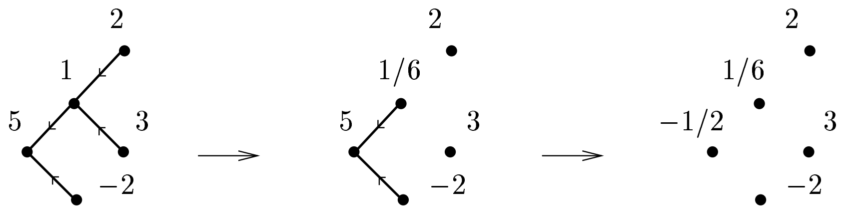


FIGURE 6

Example. The diagonalization of the matrix $A(\Gamma) = \Delta(t_1, \dots, t_n)$ corresponding to the tree in Figure 2 directed from right to left has entries

$$[t_n],\ [t_{n-1}, t_n], \dots, [t_2, \dots, t_{n-1}, t_n],\ [t_1, \dots, t_{n-1}, t_n].$$

The determinant of $\Delta(t_1, \dots, t_n)$ equals (up to a sign) the numerator of the reduced fraction $[t_1, \dots, t_{n-1}, t_n]$. We extend the concept of continued fraction by allowing division by zero in the sense of Figure 5.

2.4. **Graph links and splicing.** By a link $(\Sigma, K) = (\Sigma, k_1 \cup \ldots \cup k_m)$ we will mean a pair consisting of an oriented integral homology 3-sphere Σ and a collection K of disjoint oriented knots $S_1, \dots, S_m$ in Σ. Empty links are also allowed; they are just homology spheres. A link $(\Sigma(a_1, \dots, a_n), S_1 \cup \ldots \cup S_m)$ where the link components $S_1, \dots, S_m$ are fibers in $\Sigma(a_1, \dots, a_n)$ is called a Seifert link. Note that a general Seifert link may include singular fibers as well as some non-singular fibers.

Let (Σ, K) and (Σ', K') be links and choose components $S \subset K$ and $S' \subset K'$. Let $N(S)$ and $N(S')$ be tubular neighborhoods and $m, \ell \subset \partial N(S)$ and $m', \ell' \subset \partial N(S')$ be standard meridians and longitudes. The manifold

$$\Sigma'' = (\Sigma \setminus \operatorname{int} N(S)) \cup (\Sigma' \setminus \operatorname{int} N(S'))$$

obtained by pasting along boundaries by matching m to ℓ' and m' to ℓ, is an integral homology sphere. The link $(\Sigma'', (K \setminus S) \cup (K' \setminus S'))$ is called the splice of (Σ, K) and (Σ', K') along S and S'.

Any link which can be obtained from a finite number of Seifert links by splicing is called a graph link. Empty graph links are precisely plumbed, or graph, homology spheres, see [1]. Graph links can be conveniently represented by plumbing diagrams with arrowhead vertices attached to the boundary vertices corresponding to the link components.

Example. Figure 7 shows Seifert link (S^3, K) where K is the right handed trefoil, corresponding to a regular fiber in a Seifert fibration of S^3.

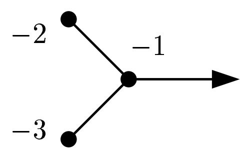


FIGURE 7

Example. Figure 8 shows Seifert link $(\Sigma(2,3,7), K)$ with K representing the singular fiber of degree 7.

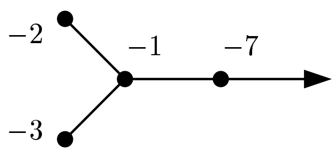


FIGURE 8

Splicing operation on graph links can be described in the language of plumbing graphs as follows. Suppose we are given two graph links represented by their plumbing diagrams Γ and Γ' as shown in Figure 9 with arrows attached to vertices weighted by e_n and e'_m respectively.

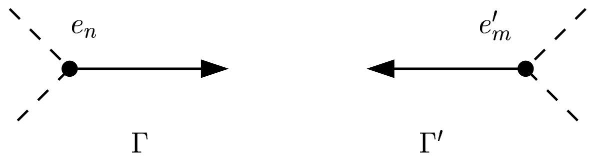


FIGURE 9

The plumbing diagram for the spliced link is shown in Figure 10. Here, $a = \det A(\Gamma_0)/\det A(\Gamma)$ where Γ_0 is the portion of the plumbing graph Γ obtained by removing the n–th vertex weighted by e_n and all adjacent edges, and similarly for a'.

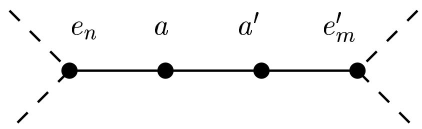


FIGURE 10

Example. The diagrams in Figure 11 show plumbed homology spheres obtained by splicing two copies of S^3 along left- and right-handed trefoils (on the left), and by splicing $\Sigma(2,3,7)$ and $\Sigma(2,3,11)$ along singular fibers of degrees 7 and 11 (on the right). Both diagrams were simplified by using the usual rules.

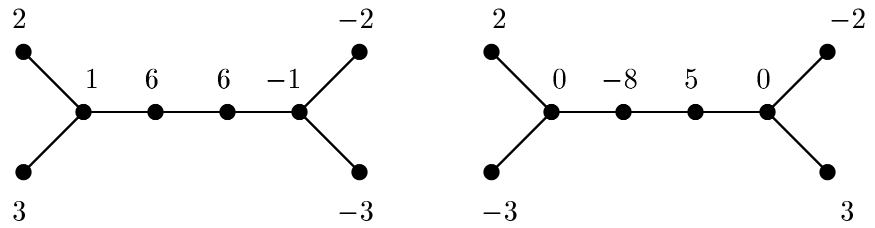


FIGURE 11

The above description of splicing in terms of plumbing graphs allows us to think of it as an operation on the corresponding plumbed 4-manifolds. This splicing operation in four dimensions can be described as follows. Let W and W' be plumbed 4-manifolds with boundaries $\partial W = \Sigma$ and $\partial W' = \Sigma'$, and let Σ'' be the splice of Σ and Σ' along knots S and S'. Represent Σ as the union $\Sigma = X \cup (T \times I) \cup N(S)$ where torus T is the boundary of a tubular neighborhood $N(S)$ of S, X is the knot exterior, and $I = [0,1]$. Let $\Sigma' = X' \cup (T \times I) \cup N(S')$ in a similar fashion. Then we can glue W to W' along $T \times I$ to get a manifold with boundary $\Sigma'' \cup S^3$, the S^3 component coming from gluing $N(S)$ with $N(S')$ by matching meridian to longitude and vice versa. Filling S^3 with 4-dimensional ball, we get the same plumbed 4-manifold as from the operation on plumbing graphs.

2.5. **The $\bar\mu$–invariant of plumbed homology spheres.** For any plumbed homology sphere $\Sigma = \partial P(\Gamma)$, there exists a unique homology class $w \in H_2(P(\Gamma),\mathbb{Z})$ satisfying the following two conditions. First, w is characteristic, that is (dot represents intersection number)

$$w \cdot x = x \cdot x \bmod 2 \quad \text{for all } x \in H_2(P(\Gamma),\mathbb{Z}), \tag{3}$$

and second, all the coordinates of w are either 0 or 1 in the natural basis of $H_2(P(\Gamma),\mathbb{Z})$. We call w the Wu class for $P(\Gamma)$. It is proved in [11] that the integer

$$\bar\mu(\Sigma) = \frac{1}{8}\,(\operatorname{sign} P(\Gamma) - w \cdot w) \tag{4}$$

only depends on Σ and not on Γ. We call it the $\bar\mu$-invariant. One can easily see that condition (3) implies that for no two adjacent vertices in Γ the corresponding coordinates of w are both equal to 1. Therefore, the class w can be chosen to be spherical, and $\bar\mu(\Sigma) = \mu(\Sigma) \bmod 2$ where $\mu(\Sigma)$ is the Rohlin invariant.

Another description of the $\bar{\mu}$–invariant is as follows. In the natural basis of $H_2(P(\Gamma),\mathbb{Z})$, the defining property (3) translates into the linear system

$$\sum_{i=1}^{s} a_{ij} w_j = a_{ii} \mod 2, \quad i = 1, \dots, s. \tag{5}$$

Due to the fact that $\det A(\Gamma) = 1 \mod 2$, this system has a unique solution $w = (w_1, \dots, w_s)$ over $\mathbb{Z}/2$. One can show that $w = \operatorname{diag} A(\Gamma)^{-1} \mod 2$, see for example [16]. Hence w_i can be computed as the mod 2 determinant of $A(\Gamma_i)$ where Γ_i is obtained from Γ by removing vertex v_i and all its adjacent edges. The subset $\mathcal{S}$ of vertices v_i of Γ such that $w_i = 1 \mod 2$ is called the Wu set of Γ. The $\bar{\mu}$–invariant is then given by the formula

$$\bar{\mu}(\Sigma) = \frac{1}{8}\left(\operatorname{sign} P(\Gamma) - \sum_{v_i \in \mathcal{S}} a_{ii}\right).$$

The $\bar{\mu}$–invariant is easy to compute. It is additive with respect to splicing, so one only needs to compute it for Seifert fibered homology spheres. Explicit formulas for $\bar{\mu}(\Sigma(a_1, \dots, a_n))$ can be found in [11].

All the above works as well for any plumbed $\mathbb{Z}/2$–homology sphere M thus defining $\bar{\mu}(M)$. It is no longer an integer although $8 \cdot \bar{\mu}(M)$ still is. For plumbed 3–manifolds M which are not $\mathbb{Z}/2$–homology spheres, one can define an invariant $\bar{\mu}(M, c)$ which will depend on a spin structure c on M, see [11] and [16].

3. Plumbed V–cobordisms

In this section we take a different point of view on plumbed 4-manifolds and think of them as cobordisms between plumbed 3-manifolds and disjoint unions of lens spaces. We describe such cobordisms by decorated plumbing graphs.

3.1. **Plumbed manifolds as V–cobordisms.** Given a 4-manifold $P(\Gamma)$ obtained by plumbing on a graph Γ let Γ' be a chain in Γ of the form shown in Figure 2. Plumbing on Γ' yields a submanifold $P(\Gamma')$ of $P(\Gamma)$ with boundary a lens space $L(p,q)$. The closure of $P(\Gamma) \setminus P(\Gamma')$ is a smooth compact 4-manifold X_0 with oriented boundary $-L(p,q) \cup \partial P(\Gamma)$, which can be thought of as a cobordism between $L(p,q)$ and $\partial P(\Gamma)$. Starting with several chains Γ' in Γ, one can establish a cobordism between $\partial P(\Gamma)$ and a disjoint union L of several lens spaces. The chains Γ' must be disjoint in the following sense: no two chains have a common vertex, and no edges of Γ have one endpoint on one chain and the other endpoint on the other.

Such cobordisms will be called plumbed V–cobordisms. A plumbed V–cobordism X_0 is always a smooth manifold; the name V–cobordism refers to the fact that each lens space on the boundary of X_0 can be capped off with a cone to make it into a V–manifold X. The latter will be referred to as a plumbed V–manifold.

Plumbed V–cobordisms can be conveniently represented by decorated plumbing graphs. These are plumbed graphs Γ with extra circles (not necessarily round) each of which encloses exactly one chain Γ'. The above condition on the chains to be disjoint translates into the condition that no edge of Γ should be intersected by more than one decorating circle.

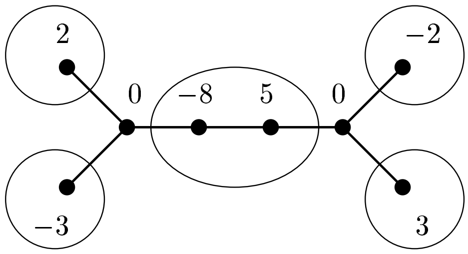


FIGURE 12

Example. Consider the plumbing diagram in Figure 12 with circles as shown in Figure 12. It describes a plumbed V–cobordism X_0 between $L = L(2,-1) \cup L(2,1) \cup L(41,5) \cup L(3,1) \cup L(3,-1)$ and Σ, where Σ is the splice of $\Sigma(2,3,7)$ and $\Sigma(2,3,11)$ along the singular fibers of degrees 7 and 11, compare with Figure 11. The only generators in the second homology of X_0 come from the vertices outside all of the circles, that is the vertices weighted by 0. Therefore, $H_2(W_0,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

3.2. **Rational intersection forms of plumbed V-cobordisms.** By the rational intersection form of a plumbed V–cobordism X_0 with $\partial X_0 = -L \cup \Sigma$ we mean the cup-product pairing

$$B : H^2(X_0,\mathbb{Q}) \otimes H^2(X_0,\mathbb{Q}) \to \mathbb{Q}.$$

The form B is non-degenerate over the rationals whenever $H_1(\Sigma,\mathbb{Q}) = 0$. The form B is also referred to as the rational intersection form of the plumbed V–manifold X, and by definition $\operatorname{sign} X = \operatorname{sign} X_0 = \operatorname{sign} B$.

Let X_0 be represented by a decorated plumbing graph Γ so that $P(\Gamma) = \bigcup P(\Gamma') \cup_L X_0$ where Γ' are disjoint chains inside the decorating circles. Since $H_1(L,\mathbb{Q}) = 0$ we have a splitting

$$A(\Gamma) = \bigoplus A(\Gamma') \,\oplus\, B. \tag{6}$$

In particular, $\operatorname{sign} A(\Gamma) = \sum \operatorname{sign} A(\Gamma') + \operatorname{sign} B$ and $\det A(\Gamma) = \prod \det A(\Gamma')\cdot \det B$. Let $L = \bigcup L(p_i,q_i)$ and suppose that Σ is an integral homology sphere. Then $\det A(\Gamma) = \pm 1$ and $\det B = \pm 1/p_1\cdots p_n$.

In practical terms, the intersection form $B = (b_{ij})$ can be calculated as follows. We call a vertex v and a decorating circle adjacent if the circle intersects an edge one of whose endpoints is v.

The generators in the second homology of X_0 correspond to the vertices of Γ outside of all the decorating circles. Given such a vertex v_i weighted by an integer e_i, we have

$$b_{ii} = e_i - \sum d_k, \quad d_k = \det A(\Gamma'_k)/\det A(\Gamma_k).$$

Here, the summation goes over all the circles adjacent to v_i, Γ_k is the chain inside the k–th circle, and Γ'_k is the portion of Γ_k obtained by removing the vertex inside Γ_k adjacent to v_i and removing all its adjacent edges. We agree that the determinant of an empty graph is equal to 1.

For any two generating vertices v_i and v_j connected by an edge inside Γ away from the decorating circles, $b_{ij} = 1$. If these two vertices are adjacent to the same decorating circle, we have

$$b_{ij} = \det A(\Gamma'_{ij})/\det A(\Gamma_{ij})$$

where Γ_{ij} is the chain inside the circle, and Γ'_{ij} is its portion obtained by removing all the vertices of Γ'_{ij} belonging to the shortest chain connecting v_i with v_j, and removing all their adjacent edges. Note that the sign of b_{ij}, $i \neq j$, can be reversed by changing orientations of the generators v_i or v_j. Since our plumbing graphs have no cycles, any choice of signs for b_{ij}, $i \neq j$, will do.

Example. The rational intersection form of the plumbed V–cobordism described by the decorated plumbing graph in Figure 12 is given by

$$B = \begin{pmatrix} -11/246 & -1/41 \\ -1/41 & -7/246 \end{pmatrix}.$$

Example. An important problem in calculations with plumbing graphs is to identify the extra lens space that arises from splicing, see Figure 13 where the lens space in question is circled.

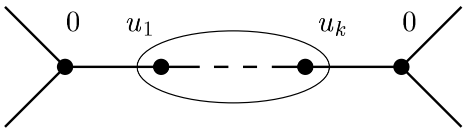


FIGURE 13

This problem can be solved as follows. In the canonical meridian-longitude bases, the splicing is given by the formula

$$\begin{pmatrix} m \\ \ell \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m' \\ \ell' \end{pmatrix}.$$

To find the parameters of the lens space, one should re-write this gluing operation in the base-fiber bases. Suppose that we splice Seifert links $(\Sigma(a_1, \dots, a_n), S)$ and $(\Sigma(\alpha_1, \dots, \alpha_m), S')$ along Seifert fibers S and S' of degrees a_n and α_m, respectively. Then

$$\begin{pmatrix} m \\ \ell \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ -a_1 \cdots a_{n-1} & d_n \end{pmatrix} \begin{pmatrix} x \\ h \end{pmatrix}, \quad \begin{pmatrix} m' \\ \ell' \end{pmatrix} = \begin{pmatrix} \alpha_m & \beta_m \\ -\alpha_1 \cdots \alpha_{m-1} & \delta_m \end{pmatrix} \begin{pmatrix} x' \\ h' \end{pmatrix},$$

where

$$d_n = a_1 \cdots a_{n-1} \cdot \sum_{i=1}^{n-1} b_i/a_i \quad \text{and} \quad \delta_m = \alpha_1 \cdots \alpha_{m-1} \cdot \sum_{j=1}^{m-1} \beta_j/\alpha_j.$$

Therefore,

$$\begin{pmatrix} x \\ h \end{pmatrix} = \begin{pmatrix} q & -p^* \\ -p & -q^* \end{pmatrix} \begin{pmatrix} x' \\ h' \end{pmatrix},$$

where

$$p = a_1 \cdots a_{n-1} \cdot \alpha_1 \cdots \alpha_{m-1} - a_n \cdot \alpha_m,$$

$$q = -a_1 \cdots a_{n-1} \cdot \alpha_1 \cdots a_{m-1} \cdot \sum_{i=1}^{n-1} b_i/a_i - b_n \cdot \alpha_m,$$

$$q^* = -a_1 \cdots a_{n-1} \cdot \alpha_1 \cdots a_{m-1} \cdot \sum_{j=1}^{m-1} \beta_j/\alpha_j - \beta_m \cdot \alpha_n,$$

and p^* is determined from equation $pp^* + qq^* = 1$ as

$$p^* = -a_1 \cdots a_{n-1} \cdot \alpha_1 \cdots \alpha_{m-1} \sum_{i=1}^{n-1} b_i/a_i \sum_{j=1}^{m-1} \alpha_j/\beta_j + b_n\beta_m.$$

The resulting lens space is $L(p, q)$ with the above parameters p and q. Note that it only depends on the two spliced Seifert links and not on the rest of the manifold.

3.3. **Spin structures on plumbed V–cobordisms.** There exists a simple algorithm which decides whether a given plumbed V–cobordism X_0 admits a spin structure. If such a spin structure exists it has to be unique because $H_1(X_0, \mathbb{Z}) = 0$.

Given a V–cobordism X_0 represented by a decorated plumbing diagram Γ, consider the vertices outside all of the circles. They form a free basis in $H_2(X_0, \mathbb{Z})$. Enumerate these vertices from 1 to n and let $w_i = \det A(\Gamma_i) \bmod 2$ where Γ_i is obtained from Γ by removing the i–th vertex and all adjacent edges, $i = 1, \dots, n$.

Proposition 6. *The V–cobordism X_0 admits a spin structure if and only if $w_i = 0 \bmod 2$ for all $i = 1, \dots, n$.*

Proof. We work in the above specified basis. The V–cobordism X_0 is spin if and only if its integral intersection matrix on $H_2(X_0, \mathbb{Z})$ is even. The latter is the inverse of the rational intersection matrix B. The i–th diagonal element of B^{-1} equals $\det B_i / \det B$, where B_i is the minor obtained from B by removing the i–th row and i–th column.

To compute the determinants, let us introduce the following notation. The chains circled in the graph Γ will be denoted by Γ'_j. For every j, the lens space obtained by plumbing according to Γ'_j will be called $L(p_j, q_j)$, so that $p_j = \pm \det A(\Gamma'_j)$. The determinant of B can be calculated using the splitting (6) as

$$\det B = \frac{\det A(\Gamma)}{\prod_j \det A(\Gamma'_j)} = \pm \frac{1}{\prod_j p_j},$$

where we take into account that $\det A(\Gamma) = \pm 1$ since Σ is an integral homology sphere. Similarly,

$$\det B_i = \frac{\det A(\Gamma_i)}{\prod_j \det A(\Gamma'_j)}.$$

Therefore, $w_i = \det B_i / \det B = \det A(\Gamma_i) \mod 2$, and the result follows. □

Example. The plumbed V–cobordism X_0 shown in Figure 12 is spin. This can be seen in two different ways. One is to find its rational intersection form B and its inverse,

$$B^{-1} = \begin{pmatrix} -42 & 36 \\ 36 & -66 \end{pmatrix},$$

and note that B^{-1} is even. The other is to apply Proposition 6 as follows. Enumerate the vertices generating $H_2(X_0, \mathbb{Z})$ from left to right. Removing vertex number 1 together with its adjacent edges leaves us with the graph Γ_1 shown in Figure 14.

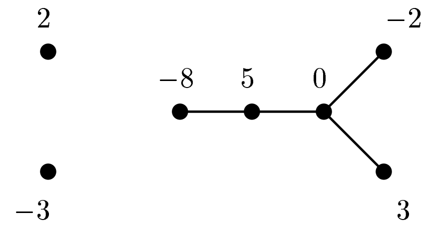


FIGURE 14

The determinant of $A(\Gamma_1)$ is the product of the determinants of the connected components of Γ_1. The one vertex component in the upper left corner has even determinant, therefore, $w_1 = 0 \mod 2$. Similarly, $w_2 = 0 \mod 2$, so that the V–cobordism X_0 is spin.

Let X_0 be a V–cobordism represented by a decorated plumbing diagram Γ, and v_j a vertex outside of all the decorating circles. We say that v_j is in the Wu set of X_0 if $w_j = 1 \bmod 2$. Thus X_0 is spin if and only if its Wu set is empty. Proposition 6 implies that the Wu set of X_0 is calculated by the same rules as the Wu set of (undecorated) graph Γ, see Section 2.5. Therefore, one can simply say that the Wu set of X_0 consists of the vertices of Γ which belong to the Wu set of Γ and sit outside all of the decorating circles. This observation implies the following result.

Corollary. *Every plumbed homology sphere bounds a spin plumbed V–cobordism.*

Proof. Let Σ be an integral homology 3-sphere obtained by plumbing according to a graph Γ. Since the Wu set of Γ contains no adjacent vertices, we can draw circles around all vertices v_i with $w_i = 1$ so that no edge will be intersected by two circles. An extra precaution should be taken not to get a zero-weighted vertex inside any of the circles. This can be easily avoided by blowing up the corresponding vertices. The graph Γ with so drawn circles represents a spin plumbed V–cobordism W_0 with boundary Σ. Note that this of course is not the most economical method of constructing a spin cobordism in terms of its second Betti number – one can do much better by including as many vertices inside each circle as possible making sure to encircle all the vertices in the Wu set. □

3.4. **Spin V–cobordisms for Seifert fibered homology spheres.** Let $\Sigma(a_1, \dots, a_n)$ be a Seifert fibered homology sphere such that one of the invariants, say a_1, is even. Using the freedom allowed by the equation (2), one can choose Seifert invariants $(b; (a_1, b_1), \dots, (a_n, b_n))$ so that b_1 is odd, b is even and all b_i with $i = 2, \dots, n$, are even. Each a_i/b_i admits then a continued fraction expansion $[t_{i1}, \dots, t_{im_i}]$ with all t_{ij} even, so that the manifold obtained by plumbing on the graph in Figure 1 is spin. We can make it into a spin V–cobordism X_0 by decorating as shown in Figure 15.

We see immediately that $H_2(X_0, \mathbb{Z}) = \mathbb{Z}$. The rational intersection form of X_0 is $b - \sum b_i/a_i = -1/a_1 \cdots a_n$, so that it is negative definite, with $b_+ = 0$ and $b_- = 1$.

Let $\Sigma(a_1, \dots, a_n)$ be a Seifert fibered homology sphere with all invariants a_i odd. For any choice of Seifert invariants $(0; (a_1, b_1), \dots, (a_n, b_n))$, consider the standard plumbing graph Γ as in Figure 1 (we choose $b = 0$ for the sake of simplicity). The central vertex of Γ weighted by 0 is in the Wu set for

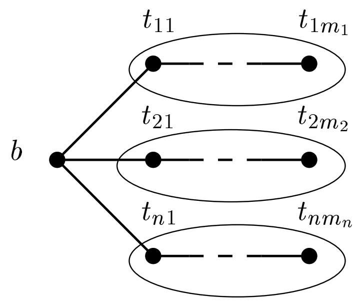


FIGURE 15

any choice of Seifert invariants. Blow Γ up $(n-1)$ times as shown in Figure 16. The 0-framed vertices adjacent to the central vertex are not in the Wu set of the blown up graph, so if we decorate it as shown in Figure 16 we will get a spin plumbed V–cobordism X_0.

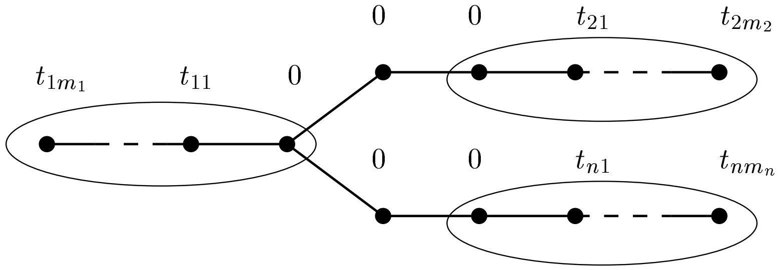


FIGURE 16

Proposition 7. *The signature of the rational intersection form B of spin plumbed V–cobordism X_0 shown in Figure 16 equals* $\sum_{i=1}^n \operatorname{sign}(a_i/b_i) - 1$.

Proof. Let us consider the decorated plumbing graph in Figure 17 and the associated rational intersection form B'. The (undecorated) plumbing graphs in Figures 16 and 17 represent the same plumbed homology sphere, and one can check that their intersection forms have equal signatures. Splitting these forms over the rationals, we get

$$\sum_{i=1}^n \operatorname{sign}\Delta(0, t_{i1}, \dots, t_{im_i}) + \operatorname{sign} B = \sum_{i=1}^n \operatorname{sign}\Delta(t_{i1}, \dots, t_{im_i}) + \operatorname{sign} B'.$$

Since $\operatorname{sign}\Delta(t_{i1}, \dots, t_{im_i}) = \operatorname{sign}\Delta(0, t_{i1}, \dots, t_{im_i}) + \operatorname{sign}(a_i/b_i)$ by the diagonalization algorithm, and $\operatorname{sign} B' = -1$, we are finished. $\square$

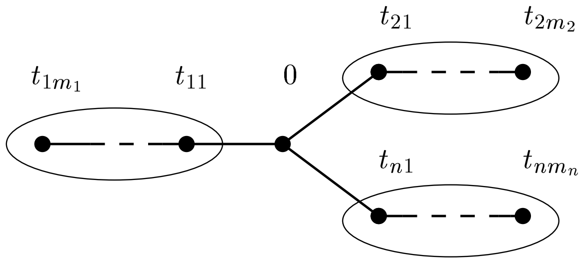


FIGURE 17

3.5. **Splicing plumbed V–cobordisms.** Given two plumbed V–cobordisms described by decorated plumbing graphs, we splice them as (undecorated) plumbing graphs as shown in Figure 10, and keep the decorations on both components unchanged. Thus in the gluing region the spliced graph may look as shown in Figure 18.

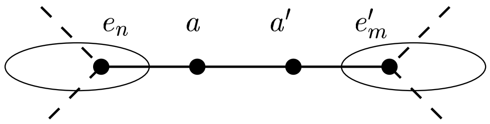


FIGURE 18

Note that the two new vertices, framed in Figure 18 by a and a', do not belong to the Wu set of the spliced graph, as by removing either of them together with the two adjacent edges, we split the graph in two connected components with determinants 0 and ± 1. The latter follows easily from the formulas for a and a' given in Section 2.4. Therefore, the splice of two spin plumbed V–cobordisms is again a spin plumbed V–cobordism.

4. THE w–INVARIANTS FOR SPIN PLUMBED V–COBORDISMS

An explicit formula for the w–invariant given in [6] is used to prove that, for any spin plumbed V–manifold (X, c) with boundary Σ, we have $w(\Sigma, X, c) = -\bar{\mu}(\Sigma)$. This implies in particular that given a plumbed integral homology sphere Σ, the integer $w(\Sigma, X, c)$ is independent of the choice of a spin plumbed V–cobordism (X, c).

4.1. **A formula for the w–invariants.** Let X be a compact oriented spin V–manifold with $b_1(X) = 0$, $\partial X = \Sigma$, and with only isolated singularities which are the cones over lens spaces, $L(p_i, q_i)$. Then ind $\mathcal{D}(X \cup Y)$ and therefore $w(\Sigma, X, c)$ can be calculated explicitly by applying the V–index theorem. Such a calculation in [6] provides us with the formula

$$w(\Sigma, X, c) = -\frac{1}{8}\left(\operatorname{sign}(X) + \sum \sigma(q_i, p_i, \varepsilon_i)\right), \tag{7}$$

where $\operatorname{sign}(X)$ stands for the signature of the rational intersection form of X, and functions $\sigma(q, p, \varepsilon)$ for any relatively prime integers p and q and $\varepsilon = \pm 1$ are defined by the formula

$$\sigma(q, p, \varepsilon) = \frac{1}{p} \sum_{k=1}^{|p|-1} \left(\cot\left(\frac{\pi k}{p}\right)\cot\left(\frac{\pi k q}{p}\right) + 2\varepsilon^k \csc\left(\frac{\pi k}{p}\right)\csc\left(\frac{\pi k q}{p}\right)\right).$$

For a spin plumbed V–manifold X, the parameter ε can be calculated as follows. Let X be described by a decorated plumbing graph Γ and let $L(p, q)$ be the lens space obtained by plumbing on a chain Γ' inside one of the decorating circles. Suppose that the chain looks like the one in Figure 2, with $-p/q = [t_1, \dots, t_n]$. The Wu set of the graph Γ is uniquely determined; in particular, we know which of the vertices of Γ' belong to this Wu set. We set $\varepsilon = 1$ if the vertex of Γ' weighted by t_1 is in the Wu set, and $\varepsilon = -1$ otherwise. In particular, if p is odd, $\varepsilon = (-1)^{q-1}$.

The paper [6] gives us the formula $\varepsilon = (-1)^{w(m)-1}$ where $L(p, q)$ is thought of as $(-p/q)$-surgery on an unknot K in S^3 with meridian m, and with the spin structure given by $w \in H^1(S^3 \setminus K, \mathbb{Z}/2)$. The latter obviously agrees with the spin structure described by plumbing on a chain, so the consistency of two formulas for ε will follow from the next lemma.

Lemma 8. *The intersection of the Wu set of Γ with the set of vertices of Γ' is a Wu set of Γ'.*

Proof. If p is odd then $L(p, q)$ is a $\mathbb{Z}/2$–homology sphere. Therefore, the intersection form $A(\Gamma)$ of $P(\Gamma)$ splits over $\mathbb{Z}/2$ in two summands, one of which is $A(\Gamma')$, and hence the Wu set also splits. If p is even, blow up Γ near Γ' as shown in Figure 19.

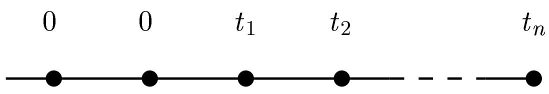


Figure 19

Let Γ'' be the chain with vertices framed by $0, t_1, \dots, t_n$. Since $\det A(\Gamma'')$ is odd we can apply the argument above and conclude that the restriction of the Wu set of Γ on Γ'' is a Wu set of Γ''. To complete the proof we only need to check that the restriction of the Wu set of Γ'' on Γ' is a Wu set of Γ', which is immediate. $\square$

4.2. **The σ–function.** The function $\sigma(q,p,\varepsilon)$ has the following properties, proved in [6], which in practice allow to calculate it without actually referring to its definition:

(1) $\sigma(q+cp,p,\varepsilon) = \sigma(q,p,(-1)^c\varepsilon)$;
(2) $\sigma(-q,p,\varepsilon) = \sigma(q,-p,\varepsilon) = -\sigma(q,p,\varepsilon)$;
(3) If $p+q$ is odd then $\sigma(p,q,-1) + \sigma(q,p,-1) = \operatorname{sign}(pq)$, in particular, $\sigma(1,p,-1) = -\operatorname{sign} p$ for any even p;
(4) For any p, q such that $|p| > |q|$ and $p+q$ is odd, there exists a continued fraction decomposition $p/q = [t_1, \dots, t_n]$ with all t_i even and $|t_i| \geq 2$. Then $\sigma(q,p,-1) = -\sum_{i=1}^n \operatorname{sign} t_i$.

We will show that in fact $\sigma(q,p,\varepsilon)$ is nothing else but the $\bar\mu$–invariant of $L(p,q)$. To be more precise, if p is odd then $L(p,q)$ is a $\mathbb{Z}/2$–homology sphere, so the $\bar\mu$–invariant is defined as in Section 2.5 and only depends on $L(p,q)$ and not on the choice of a plumbing graph. The only difference is that $\bar\mu(L(p,q))$ is not an integer although $8 \cdot \bar\mu(L(p,q)$ still is. If p is even, there are two different Wu classes because of the fact that $H^1(L(p,q),\mathbb{Z}/2) = \mathbb{Z}/2$. Suppose that $L(p,q)$ is obtained by plumbing on a chain Γ' like the one shown in Figure 2 so that $-p/q = [t_1, \dots, t_n]$. Since $q = \pm[t_2, \dots, t_n]$ is odd, the Wu set of $A(\Gamma')$ is uniquely determined by whether the first vertex of Γ', weighted by t_1, is in the Wu set. We say that $v = 1$ if it is in the Wu set and $v = 0$ otherwise. Fixing v allows to define the $\bar\mu$–invariants $\bar\mu(L(p,q),v)$ as before.

Proposition 9. *If p is odd then $\sigma(q,p,(-1)^{q-1}) = 8\,\bar\mu(L(p,q))$. If p is even, $\sigma(q,p,(-1)^{v-1}) = 8\,\bar\mu(L(p,q),v)$.*

Proof. First we note that we may assume without loss of generality that $|p| > |q|$. Suppose that p is odd and q is even. We find a continued fraction decomposition $p/q = [t_1, \dots, t_n]$ as in Property (4) and evaluate $\sigma(q,p,-1) = -\sum_{i=1}^n \operatorname{sign} t_i = -\operatorname{sign}\Delta(t_1, \dots, t_n)$ (the latter equality is true because all $|t_i|$ are greater than or equal to 2). Since the Wu set is empty, $8\,\bar\mu(L(p,q)) = -\operatorname{sign}\Delta(t_1,\dots,t_n)$.

If both p and q are odd, we use Property (1) to get $\sigma(q,p,1) = \sigma(q-p,p,-1)$ with even $q-p$. By the previous argument, $\sigma(q-p,p,-1) = 8\,\bar\mu(L(p,q-p))$. Let $-p/(q-p) = [t_1,\dots,t_n]$ with all t_i even, then $-p/q = [0,1,t_1,\dots,t_n]$ with Wu set consisting of just one vertex weighted by 0. Since $\operatorname{sign}\Delta(t_1,\dots,t_n) = \operatorname{sign}\Delta(0,1,t_1,\dots,t_n)$ by the diagonalization algorithm, we see that $\bar\mu(L(p,q-p)) = \bar\mu(L(p,q))$, which completes the proof in this case.

If p is even, one should consider two separate cases, $v=0$ and $v=1$. The first is treated as the case of odd p and even q, with all even t_i which provides for $v=0$. The case $v=1$ is completely similar to the case of odd p and q, keeping in mind that adding two vertices framed by 0 and 1 does not change the Wu set of $\Delta(t_1,\dots,t_n)$. □

It should be mentioned that Proposition 9 may provide for easy proofs of various number-theoretical properties of $\sigma(q,p,\varepsilon)$ proved in [6].

4.3. **Proof of Theorem 1.** Let X be a spin plumbed V–manifold described by a decorated plumbing graph Γ with chains Γ'_i, $i=1,\dots,n$, inside the decorating circles, so that $\partial P(\Gamma) = \Sigma$ and $\partial P(\Gamma'_i) = L(p_i,q_i)$. Let w_i be the Wu classes of $A(\Gamma'_i)$, $i=1,\dots,n$, and w the Wu class of $A(\Gamma)$. Since X is spin, $w = w_1+\ldots+w_n$. Then

$$
\begin{aligned}
w(\Sigma,X,c) &= -(\operatorname{sign}X_0 + \textstyle\sum \sigma(q_i,p_i,\varepsilon_i))/8, && \text{by formula (7)},\\
&= -(\operatorname{sign}X_0 + \textstyle\sum(\operatorname{sign}P(\Gamma'_i) - w_i\cdot w_i))/8, && \text{by Proposition 9},\\
&= -(\operatorname{sign}X_0 + \textstyle\sum \operatorname{sign}P(\Gamma'_i) - w\cdot w)/8, && \text{since } X_0 \text{ is spin},\\
&= -(\operatorname{sign}P(\Gamma) - w\cdot w)/8, && \text{by formula (6)},\\
&= -\bar\mu(\Sigma).
\end{aligned}
$$

In particular we see that $w(\Sigma,X,c)$ is independent of the choice of (X,c) as long as we stay in the class of spin plumbed V–manifolds.

5. Homology cobordism properties of the $\bar\mu$–invariant

Using Theorem 1, we can reformulate Theorem 5 of Fukumoto and Furuta [5] as follows.

Theorem 10. *Let Σ be a plumbed homology sphere bounding a spin plumbed V–manifold X. If Σ has finite order in the homology cobordism group then either $\bar\mu(\Sigma) = 0$ or $-b_-(X) < -\bar\mu(\Sigma) < b_+(X)$.*

In this section, we apply this result to study the homology cobordism properties of Seifert fibered and plumbed homology spheres. We generalize results obtained in this direction in [3], [5], and [6].

5.1. **Proof of Theorem 2.** Let $\Sigma = \Sigma(a_1, \dots, a_n)$ be a Seifert fibered homology sphere with one of the a_i even. Let X be the plumbed V–cobordism shown in Figure 15. It is spin and it has $b_-(X) = 1$ and $b_+(X) = 0$. By Theorem 10, if Σ has finite order in the homology cobordism group, $\bar\mu(\Sigma)$ must vanish.

Let $\Sigma = \Sigma(a_1, \dots, a_n)$ be a Seifert fibered homology sphere with all the a_i odd, and suppose it has finite order in the homology cobordism group. Choose invariants b_i so that all a_i/b_i but one are positive. Then the plumbed spin V–cobordism X shown in Figure 16 has $b_+(X) + b_-(X) = n - 1$ and $b_+(X) - b_-(X) = n - 3$, see Proposition 7. Therefore, $b_+(X) = n - 2$ and $b_-(X) = 1$, and Theorem 10 implies that

$$0 \le -\bar\mu(\Sigma) < n - 2. \tag{8}$$

Now choose a different set of invariants (a_i, b_i) so that all a_i/b_i but one are negative. The above argument with V–cobordism in Figure 16 shows that

$$1 - n < -\bar\mu(\Sigma) \le 0. \tag{9}$$

These inequalities together prove that $\bar\mu(\Sigma)$ vanishes.

There is another way to prove the theorem for Seifert fibered homology spheres $\Sigma = \Sigma(a_1, \dots, a_n)$ with all a_i odd. It uses inequality (8) and the result of [15] which says that if an algebraic link has finite order in the homology cobordism group, its $\bar\mu$–invariant is non-negative. Since $\Sigma(a_1, \dots, a_n)$ is an algebraic link, we conclude that $-\bar\mu(\Sigma) \le 0$, which completes the proof.

5.2. **Proof of Theorem 3.** Let us denote Seifert invariants of $\Sigma(a_1, \dots, a_n)$ and $\Sigma(\alpha_1, \dots, \alpha_m)$ respectively by (a_i, b_i) and (α_j, β_j). The splicing construction provides us with the plumbing graph Γ shown in Figure 20, which we decorate as indicated. One may assume without loss of generality that the V–cobordism X we obtain is spin: if all the a_i and α_j are odd this is automatic; if some of them are even certain precautions should be made in choosing b_i and β_j.

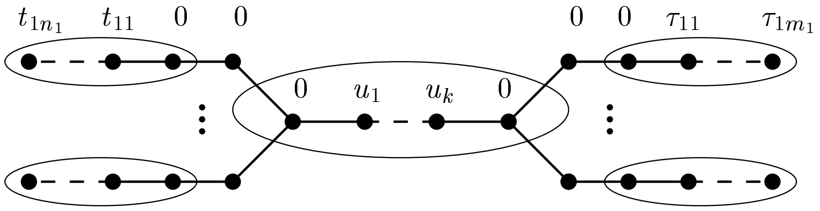


FIGURE 20

We use here the notations $a_i/b_i = [t_{i1}, \dots, t_{in_i}]$, $\alpha_j/\beta_j = [\tau_{j1}, \dots, \tau_{jm_j}]$, $p/q = [u_1, \dots, u_k]$ and $p/q^* = [u_k, \dots, u_1]$, where the parameters p, q and q^* were computed in Section 3.2.

Let us denote the rational intersection form of X by B, and consider another plumbed V–cobordism X' given by the plumbing graph Γ' decorated as shown in Figure 21.

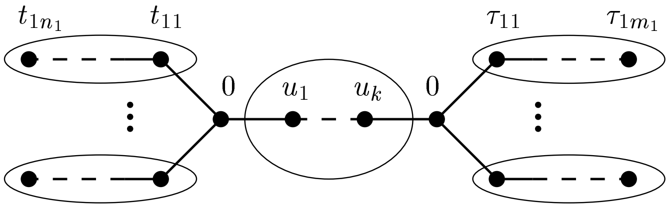


FIGURE 21

Let B' be the rational intersection matrix of X'. One can easily see that the matrices $A(\Gamma)$ and $A(\Gamma')$ have the same signature. Decomposing these two matrices over the rationals provides us with the identity

$$\begin{aligned}\operatorname{sign} B + \sum_{i=1}^{n-1} \operatorname{sign} \Delta(0, t_{i1}, \dots, t_{in_i}) + \sum_{j=1}^{m-1} \operatorname{sign} \Delta(0, \tau_{j1}, \dots, \tau_{jm_j}) \\ + \operatorname{sign} \Delta(0, u_1, \dots, u_k, 0) = \operatorname{sign} B' + \sum_{i=1}^{n-1} \operatorname{sign} \Delta(t_{i1}, \dots, t_{in_i}) \\ + \sum_{j=1}^{m-1} \operatorname{sign} \Delta(\tau_{j1}, \dots, \tau_{jm_j}) + \operatorname{sign} \Delta(u_1, \dots, u_k).\end{aligned}$$

By diagonalization algorithm, $\operatorname{sign} \Delta(0, t_{i1}, \dots, t_{in_i}) - \operatorname{sign} \Delta(t_{i1}, \dots, t_{in_i}) = -\operatorname{sign}(a_i/b_i)$ and similarly for α_j/β_j. The difference $\operatorname{sign} \Delta(0, u_1, \dots, u_k, 0)$

$-\operatorname{sign}\Delta(u_1,\dots,u_k)$ is computed as follows. Decompose the integral intersection matrix $\Delta(0,u_1,\dots,u_k,0)$ over the rationals in the direct sum of $\Delta(u_1,\dots,u_k)$ and the rational intersection matrix U of the decorated plumbing graph shown in Figure 22.

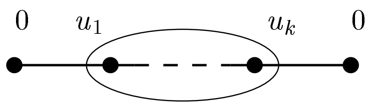


FIGURE 22

Then $\operatorname{sign}\Delta(0,u_1,\dots,u_k,0)-\operatorname{sign}\Delta(u_1,\dots,u_k)=\operatorname{sign}U$. The two-by-two matrices B' and U are computed by the usual rules as

$$B'=\begin{pmatrix}-\sum b_i/a_i-q/p & -1/p\\ -1/p & -\sum\beta_j/\alpha_j-q^*/p\end{pmatrix},\quad U=\begin{pmatrix}-q/p & -1/p\\ -1/p & -q^*/p\end{pmatrix},$$

and the resulting formula for $\operatorname{sign}B$ is

$$\operatorname{sign}B=\operatorname{sign}B'-\operatorname{sign}U+\sum_{i=1}^{n-1}\operatorname{sign}(a_i/b_i)+\sum_{j=1}^{m-1}\operatorname{sign}(\alpha_j/\beta_j).$$

After simplification using explicit formulas from Section 3.2, we obtain

$$B'=\begin{pmatrix}\alpha_m/(a_1\cdots a_{n-1}\cdot p) & -1/p\\ -1/p & a_n/(\alpha_1\cdots\alpha_{m-1}\cdot p)\end{pmatrix}$$

and $\det B'=-1/(a_1\cdot a_{n-1}\cdot\alpha_1\cdot\alpha_{m-1}\cdot p)$. Thus $\operatorname{sign}B'=0$ if $p>0$, and $\operatorname{sign}B'=-2$ if $p<0$ (the latter could be seen alternatively from the fact that, in the case when p is negative, Σ is an algebraic link and therefore bounds a negative definite plumbed manifold). Concerning matrix U, we compute

$$\det U=-p^*/p$$
$$=\frac{a_1\cdots a_{n-1}\cdot\alpha_1\cdots\alpha_{m-1}\sum b_i/a_i\sum\alpha_j/\beta_j-a_n\cdot\alpha_m(b_n/a_n)(\beta_m/\alpha_m)}{a_1\cdots a_{n-1}\cdot\alpha_1\cdots\alpha_{m-1}-a_n\cdot\alpha_m},$$

where the summation extends over $i = 1, \dots, n-1$ and $j = 1, \dots, m-1$. Replacing $\sum b_i/a_i$ by $\sum b_i/a_i + k$ at the expense of changing b_n/a_n to $b_n/a_n - k$, and doing the same with the other set of invariants, we see that $\det U$ is positive at least for large $|k|$. If k is large positive, $\operatorname{sign} U = -2$; if k is negative and large in absolute value, $\operatorname{sign} U = 2$.

Suppose that Σ has finite order in the homology cobordism group, and that p is positive. Then $\operatorname{sign} B' = 0$. If we choose b_i/a_i, $i = 1, \dots, n-1$, and α_j/β_j, $j = 1, \dots, m-1$, to be large positive then $\operatorname{sign} B = n + m - 4$. Since the rank of B is $n + m - 2$, we conclude that $b_+ = n + m - 3$, $b_- = 1$, and

$$-1 < -\bar{\mu}(\Sigma) < n + m - 3.$$

If we choose b_i/a_i, $i = 1, \dots, n-1$, and α_j/β_j, $j = 1, \dots, m-1$, to be negative and of large absolute value then $\operatorname{sign} B = -n - m + 4$ so that $b_+ = 1$, $b_- = n + m - 3$, and

$$3 - n - m < -\bar{\mu}(\Sigma) < 1.$$

These two sets of inequalities on $\bar{\mu}(\Sigma)$ imply that $\bar{\mu}(\Sigma) = 0$.

Suppose now that p is negative then $\operatorname{sign} B' = -2$ and arguing as above we get inequalities

$$-2 < -\bar{\mu}(\Sigma) < n + m - 4 \quad \text{and} \quad 2 - n - m < -\bar{\mu}(\Sigma) \leq 0,$$

which imply that $\bar{\mu}(\Sigma)$ is either zero or one. This completes the proof.

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