Local action of the symmetric group and the twisted Yang-Baxter relation

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Introduction

The Yang-Baxter relation plays a central role in the Quantum Field Theory. The usual R-matrix describes the scattering of two "particles" (Fig.1). Here u and v are some characteristics of the particles which are called the spectral parameters. R-matrix ($R_{ij}^{kl}(u,v)$) is a meromorphic function on the space of spectral parameters. Now suppose that the spectral parameters change after scattering (Fig.2). New spectral parameters $\varphi(u,v)$, $\psi(u,v)$ are some meromorphic functions in the variables u and v. The factorization conditions impose a strong requirement on these functions (Fig.3). One can see from Fig.3 that

$$\varphi(u,\varphi(v,w)) = \varphi(\varphi(u,v),\varphi(\psi(u,v),w))$$

$$\varphi(\psi(u,\varphi(v,w)),\psi(v,w)) = \psi(\varphi(u,v),\varphi(\psi(u,v),w))$$

$$\psi(\psi(u,v),w) = \psi(\psi(u,\varphi(v,w)),\psi(v,w))$$

If we have the solution of these equations, then we can define the analogous Yang-Baxter relation, L-operators, Zamolodchikov algebra and so on. In §**1** we develop the mathematical formalism for φ and ψ and give nontrivial examples. In §**2** we introduce the twisted Yang-Baxter relation and corresponding algebraic structures. The most interesting examples of φ and ψ appear in the theory of factorization of matrix polynomials (§**3**) and and matrix θ-functions (§**4**). In §**5** we discus the generalized star-triangle relation and corresponding algebraic structures. In "physical" language it is the case when our particles are not elementary but are multiplets of m elementary particles.

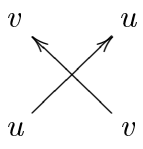

Figure 1

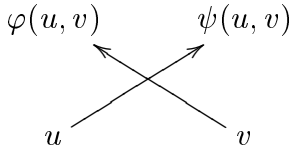

Figure 2

The examples of twisted R-matrices as well as the solutions of generalized star-triangle relation were found in [4] as intertwiners of cyclic representations and its tensor products of the algebra of monodromy matrices of the six-vertex model at root of unity [3]. These solutions are natural generalizations of the one from chiral

Potts model [1,2,3]. One can obtain other solutions of the twisted Yang-Baxter and star-triangle relations by calculating the intertwiners of the representations of the algebras of monodromy matrices at root of unity for other trigonometric and elliptic R-matrices.

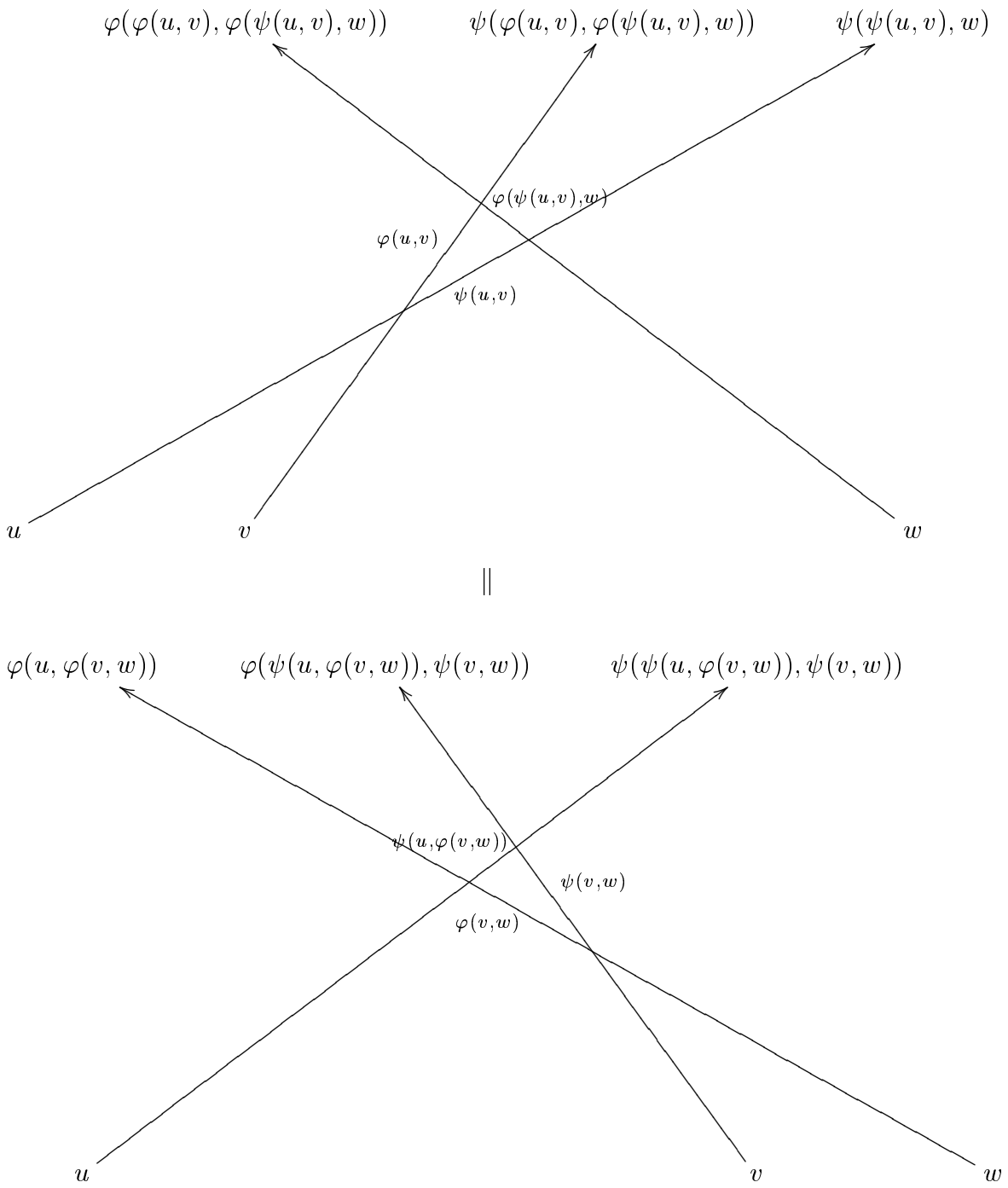

Figure 3

§1. Twisted transposition

Let U be a complex manifold, $\mu : U \times U \to U \times U$ be a birational automorphism of $U \times U$. We will use the notations: $\mu(u,v) = (\varphi(u,v), \psi(u,v))$ where $u, v \in U$. Here φ and ψ are meromorphic functions from $U \times U$ to U.

Let us introduce the following birational automorphisms of $U \times U \times U$: $\sigma_1 = \mu \times \mathrm{id}$ and $\sigma_2 = \mathrm{id} \times \mu$. We have: $\sigma_1(u,v,w) = (\varphi(u,v), \psi(u,v), w)$ and $\sigma_2(u,v,w) = (u, \varphi(v,w), \psi(v,w))$.

Definition *We call a map μ a twisted transposition if the automorphisms σ_1 and σ_2 satisfy the following relations:*

$$\sigma_1^2 = \sigma_2^2 = \mathrm{id}, \ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \tag{1}$$

If μ is a twisted transposition, then for each $N \in \mathbb{N}$ we have the birational action of the symmetric group S_N on the manifold U^N such that the transposition $(i, i+1)$ acts by automorphism $\sigma_i = \mathrm{id}^{i-1} \times \mu \times \mathrm{id}^{N-i-1}$. So we have $\sigma_i(u_1, \ldots, u_N) = (u_1, \ldots, \varphi(u_i, u_{i+1}), \psi(u_i, u_{i+1}), \ldots, u_N)$. It is clear that the relations (1) are equivalent to the defining relations in the group S_N: $\sigma_i^2 = e, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$.

It is easy to check that the relations (1) are equal to the following functional equations for φ and ψ:

$$\varphi(\varphi(u,v), \psi(u,v)) = u, \ \psi(\varphi(u,v), \psi(u,v)) = v$$

$$\varphi(u, \varphi(v,w)) = \varphi(\varphi(u,v), \varphi(\psi(u,v), w))$$

$$\varphi(\psi(u, \varphi(v,w)), \psi(v,w)) = \psi(\varphi(u,v), \varphi(\psi(u,v), w)) \tag{2}$$

$$\psi(\psi(u,v), w) = \psi(\psi(u, \varphi(v,w)), \psi(v,w))$$

Remarks 1. From (2) it follows that for each N the functions $\varphi(u_1, \varphi(u_2, \ldots, \varphi(u_N, w)\ldots)$ and $\psi(\ldots(\psi(w, u_1), u_2)\ldots, u_N)$ are invariant with respect to the action of the group S_N on the variables $u_1, \ldots, u_N$.

2. Informally one can consider μ as an infinite dimensional R-matrix in the space of functions. Namely, if we consider the space of meromorphic functions $\{f : U \times U \to \mathbb{C}\}$ as an "extended tensor square" of the space of meromorphic functions $\{f : U \to \mathbb{C}\}$, then the linear operator $R_\mu : f \to f \circ \mu$ (that is $R_\mu f(u,v) = f(\mu(u,v))$) satisfies the usual Yang-Baxter relation.

Examples 1. Let $q, q^{-1} : U \to U$ be birational automorphisms such that $qq^{-1} = q^{-1}q = \mathrm{id}$. Then

$$\mu(u,v) = (q(v), q^{-1}(u))$$

is a twisted transposition.

2. Let $U = \mathbb{C}$, then the following formula gives a twisted transposition:

$$\mu(u, v) = (1 - u + uv, \frac{uv}{1 - u + uv})$$

3. Let U be a finite dimensional associative algebra with unity $1 \in U$ and such that generic element $u \in U$ is invertible, for example $U = \mathrm{Mat}_m$. Then the following formula gives a twisted transposition:

$$\mu(u, v) = (1 - u + uv, (1 - u + uv)^{-1}uv)$$

For further examples see §**3,4**.

§2. Twisted Yang-Baxter relation and corresponding algebraic structures

Let V be a n-dimensional vector space. For each $u \in U$ we denote by $V(u)$ a vector space canonically isomorphic to V. Let R be a meromorphic function from $U \times U$ to $End(V \otimes V)$. We will consider $R(u,v)$ as a linear operator

$$R(u,v) : V(u) \otimes V(v) \to V(\varphi(u,v)) \otimes V(\psi(u,v))$$

Definition *We call R a twisted R-matrix (with respect to the twisted transposition μ) if it satisfies the following properties:*

1. The composition

$$V(u) \otimes V(v) \to V(\varphi(u,v)) \otimes V(\psi(u,v)) \to V(u) \otimes V(v)$$

is equal to the identity, that is $R(\varphi(u,v),\psi(u,v))R(u,v) = 1$.

2. The following diagram is commutative:

$$\begin{array}{ccc}
V(\varphi(u,v)) \otimes V(\psi(u,v)) \otimes V(w) & \xrightarrow{1 \otimes R(\psi(u,v),w)} & V(\varphi(u,v)) \otimes V(\varphi(\psi(u,v)),w)) \otimes V(\psi(\psi(u,v),w)) \\
\uparrow{\scriptstyle R(u,v)\otimes 1} & & \downarrow{\scriptstyle R(\varphi(u,v),\varphi(\psi(u,v),w))\otimes 1} \\
V(u) \otimes V(v) \otimes V(w) & & \widetilde{V} \\
\downarrow{\scriptstyle 1\otimes R(v,w)} & & \uparrow{\scriptstyle 1\otimes R(\psi(u,\varphi(v,w)),\psi(v,w))} \\
V(u) \otimes V(\varphi(v,w)) \otimes V(\psi(v,w)) & \xrightarrow{R(u,\varphi(v,w))\otimes 1} & V(\varphi(u,\varphi(v,w))) \otimes V(\psi(u,\varphi(v,w))) \otimes V(\psi(v,w))
\end{array}$$

Here $\widetilde{V} = V(\varphi(u,\varphi(v,w))) \otimes V(\psi(\varphi(u,v),\varphi(\psi(u,v),w))) \otimes V(\psi(\psi(u,v),w))$.

In other words,

$$R^{12}(\varphi(u,v),\varphi(\psi(u,v),w)))R^{23}(\psi(u,v),w)R^{12}(u,v) = R^{23}(\psi(u,\varphi(v,w)),\psi(v,w))R^{12}(u,\varphi(v,w))R^{23}(v,w) \qquad (3)$$

Here $R^{12} = R \otimes 1$ and $R^{23} = 1 \otimes R$ are linear operators in $V \otimes V \otimes V$.

We call (3) twisted Yang-Baxter relation.

Let $\{x_i, i = 1 \ldots, n\}$ be a basis of the linear space V, $\{x_i(u)\}$ be the corresponding basis of the linear space $V(u)$. It is clear that the following two linear operators are twisted R-matrices for each μ:

$$x_i(u) \otimes x_j(v) \to x_i(\varphi(u,v)) \otimes x_j(\psi(u,v))$$

$$x_i(u) \otimes x_j(v) \to x_j(\varphi(u,v)) \otimes x_i(\psi(u,v))$$

Let $R_{ij}^{\alpha\beta}(u,v)$ be a matrix element of twisted R-matrix R, which means

$$R(u,v) : x_i(u) \otimes x_j(v) \to R_{ij}^{\alpha\beta}(u,v) x_\beta(\varphi(u,v)) \otimes x_\alpha(\psi(u,v))$$

We sum over repeating indices.

Definition *We call the twisted Zamolodchikov algebra associated with R an associative algebra Z_R with generators $\{x_i(u); i = 1 \dots, n, u \in U\}$ and defining relations*

$$x_i(u)x_j(v) = R_{ij}^{\alpha\beta}(u,v) x_\beta(\varphi(u,v)) x_\alpha(\psi(u,v))$$

It is clear that the twisted Zamolodchikov algebra satisfies the usual property: for generic $u_1, \dots, u_N \in U$ the elements $\{x_{i_1}(u_1) \dots x_{i_N}(u_N); 1 \leqslant i_1, \dots, i_N \leqslant n\}$ are linear independent.

Let W be a vector space, L be a meromorphic function $L : U \to End(V \otimes W)$. We will consider $L(u)$ as a linear operator $L(u) : V(u) \otimes W \to W \otimes V(u)$.

Definition *We call L an L-operator for twisted R-matrix R if the following diagram is commutative:*

$$\begin{array}{ccc}
V(u) \otimes W \otimes V(v) & \xrightarrow{L(u)\otimes 1} & W \otimes V(u) \otimes V(v) \\
\uparrow{\scriptstyle 1\otimes L(v)} & & \downarrow{\scriptstyle 1\otimes R(u,v)} \\
V(u) \otimes V(v) \otimes W & & W \otimes V(\varphi(u,v)) \otimes V(\psi(u,v)) \\
\downarrow{\scriptstyle R(u,v)\otimes 1} & & \uparrow{\scriptstyle L(\varphi(u,v))\otimes 1} \\
V(\varphi(u,v)) \otimes V(\psi(u,v)) \otimes W & \xrightarrow{1\otimes L(\psi(u,v))} & V(\varphi(u,v)) \otimes W \otimes V(\psi(u,v))
\end{array}$$

Which is

$$R(u,v)L^1(u)L^2(v) = L^1(\varphi(u,v))L^2(\psi(u,v))R(u,v) \tag{4}$$

where $L^1 = L \otimes 1$ and $L^2 = 1 \otimes L$.

Let $\{L_i^j(u); i,j = 1, \dots, n\}$ be the elements in $End(W)$ such that $L(u) : x_i(u) \otimes W \to (L_i^\alpha(u)W) \otimes x_\alpha(u)$. We can consider $L(u)$ as a $n \times n$ matrix with matrix elements $(L_i^j(u))$.

Definition *The algebra of monodromy matrices M_R for the twisted R-matrix R is an associative algebra with generators $\{L_i^j(u); i,j = 1, \dots, n, u \in U\}$ and defining relations (4).*

It is clear that representations of the algebra M_R in a vector space W coincide with L-operators $L(u) \in End(V \otimes W)$.

One can introduce a coproduct Δ in the algebra M_R by the usual way:

$$\Delta L_i^j(u) = L_i^\alpha(u) \otimes L_\alpha^j(u)$$

In the language of L-operators, if we have two L-operators $L_1(u) : V(u) \otimes W_1 \to W_1 \otimes V(u)$ and $L_2(u) : V(u) \otimes W_2 \to W_2 \otimes V(u)$, then we can construct an L-operator in the space $W_1 \otimes W_2$ as the composition:

$$V(u) \otimes W_1 \otimes W_2 \to W_1 \otimes V(u) \otimes W_2 \to W_1 \otimes W_2 \otimes V(u)$$

Definition *Q-operator is the element of the algebra M_R given by the formula:*

$$Q(u) = \mathrm{tr} L(u) = L^i_i(u)$$

From relations (4) it follows that Q-operators satisfy the following relations:

$$Q(u)Q(v) = Q(\varphi(u,v))Q(\psi(u,v))$$

Remark One can also define the dynamical twisted Yang-Baxter relation.

§3. Local action of the symmetric group and factorization of matrix polynomials

For the general theory of matrix polynomials and factorization see [7]. For our purposes we state the results, which may be well known to the experts.

We denote by $S(a)$ the set of eigenvalues of a matrix $a \in Mat_m$. More generally, we denote by $S(a_1, \dots, a_d)$, $a_1, \dots, a_d \in Mat_m$, the set of roots of the polynomial $f(t) = \det(t^d - a_1 t^{d-1} + \dots + (-1)^d a_d)$. We will consider polynomials with generic coefficients only, so $\#S(a_1, \dots, a_d) = md$.

Proposition 1. *Let*

$$t^d - a_1 t^{d-1} + \dots + (-1)^d a_d = (t - b_1) \dots (t - b_d) \tag{5}$$

for generic matrices $a_1, \dots, a_d \in Mat_m$, then $S(b_i) \cap S(b_j) = \emptyset$ for $i \neq j$ and $S(b_1) \cup \dots \cup S(b_d) = S(a_1, \dots, a_d)$. For each decomposition $S(a_1, \dots, a_d) = A_1 \cup \dots \cup A_d$, such that $\#A_i = m$, $A_i \cap A_j = \emptyset$ $(i \neq j)$ there exists a unique the factorization (5) with $S(b_i) = A_i$.

Proof The first statement follows from the equation $\det(t^d - a_1 t^{d-1} + \dots + (-1)^d a_d) = \det(t - b_1) \dots \det(t - b_d)$.

On the other hand, if we know eigenvalues of $b_1, \dots, b_d$ then we can calculate eigenvectors of them. For $\lambda \in S(b_d)$ the corresponding eigenvector is a vector v_λ, such that $(\lambda^d - a_1 \lambda^{d-1} + \dots + (-1)^d a_d) v_\lambda = 0$. If we know all eigenvectors of b_d, then we can calculate all eigenvectors of b_{d-1} similarly and so on. This implies the uniqueness. By our construction of $b_1, \dots, b_d$ the determinants of the matrix polynomials in the right hand side and the left hand side of (5) have the same set of roots. Moreover, for each root λ the operators represented by these matrix polynomials have the same kernel if we set $t = \lambda$. It implies that these polynomials are equal.

Proposition 2. *Let $a_1, a_2 \in Mat_m$ be generic matrices. Then there exists a unique pair of matrices $b_1, b_2 \in Mat_m$ such that $(t - a_1)(t - a_2) = (t - b_1)(t - b_2)$ and $S(b_1) = S(a_2), S(b_2) = S(a_1)$. We have $b_1 = a_1 + \Lambda^{-1}, b_2 = a_2 - \Lambda^{-1}$ where $a_2 \Lambda - \Lambda a_1 = 1$.*

Proof If $S(b_2) \cap S(a_2) \neq \emptyset$, then $\det(a_2 - b_2) = 0$, because a_2 and b_2 have a common eigenvector. Otherwise, we can put $\Lambda = (a_2 - b_2)^{-1}$.

Let $U = Mat_m$. From the propositions 1 and 2 it follows that the formula $\mu(a_1, a_2) = (b_1, b_2)$ gives a twisted transposition, where $b_1 + b_2 = a_1 + a_2$, $b_1 b_2 = a_1 a_2, S(b_1) = S(a_2), S(b_2) = S(a_1)$. We have $\mu(a_1, a_2) = (a_1 + \Lambda^{-1}, a_2 - \Lambda^{-1})$, where Λ is the solution of the linear matrix equation $a_2 \Lambda - \Lambda a_1 = 1$.

Let $\overline{U} = \overline{Mat_m}$ be the set of $m \times m$ matrices with different eigenvalues and fixed order of eigenvalues. The proposition 1 gives an action of the symmetric group S_{mN} on the space $\overline{U}^N$ by birational automorphisms. By definition, for $\sigma \in S_{mN}$, $b_1, \dots, b_N \in \overline{U}$ we have $\sigma(b_1, \dots, b_N) = (b'_1, \dots, b'_N)$ where $(t - b_1) \dots (t - b_N) = (t - b'_1) \dots (t - b'_N)$ and $\overline{S}(b'_i) = \sigma \overline{S}(b_i)$, $\overline{S}$ stands for the ordered set of eigenvalues.

This action is local in the following sense. The transposition $(i, i+1)$ for $\alpha m < i < (\alpha+1)m$ acts only inside the $\alpha+1$-th factor of $\bar{U}^N$ and the transposition $(\alpha m, \alpha m+1)$ acts only inside the product of the α-th and the $\alpha+1$-th factors. We have also the twisted transposition $\mu : \overline{U} \times \overline{U} \to \overline{U} \times \overline{U}$ in this case which is the action of the element $(1, m+1)(2, m+2)\ldots(m-1, 2m-1) \in S_{2m}$.

Remark Let U be the set of matrix polynomials of the form $at + b$, where $a, b \in Mat_m, a = (a_{ij}), b = (b_{ij})$ and $a_{ij} = 0$ for $i < j$, $b_{ij} = 0$ for $i > j$. It is possible to define the twisted transposition μ such that for $\mu(f(t), g(t)) = (f_1(t), g_1(t))$ we have $f(t)g(t) = f_1(t)g_1(t)$, $\det f(t)$ and $\det g_1(t)$ have the same set of roots and the first coefficients of $f(t)$ and $g_1(t)$ have the same diagonal elements. In [4] we found the solutions of the corresponding twisted Yang-Baxter relation (for $m = 2$), which is generalization of the R-matrix from chiral Potts model.

§4. Local action of the symmetric group and factorization on matrix θ-functions

Let $\Gamma \subset \mathbb{C}$ be the lattice generated by 1 and τ where $\mathrm{Im}\tau > 0$. We have $\Gamma = \{\alpha + \beta\tau; \alpha, \beta \in \mathbb{Z}\}$. Let $\varepsilon \in \mathbb{C}$ be primitive root of unity of degree m. Let $\gamma_1, \gamma_2 \in Mat_m$ be $m \times m$ matrices such that $\gamma_1^m = \gamma_2^m = 1, \gamma_2\gamma_1 = \varepsilon\gamma_1\gamma_2$. We have $\gamma_1 v_\alpha = \varepsilon^\alpha v_\alpha, \gamma_2 v_\alpha = v_{\alpha+1}$ in some basis $\{v_\alpha; \alpha \in \mathbb{Z}/m\mathbb{Z}\}$ of $\mathbb{C}^m$. Let us assume that $\{v_1, \dots, v_m\}$ is the standard basis of $\mathbb{C}^m$.

We denote by $M\Theta_{n,m,c}(\Gamma)$ for $n, m \in \mathbb{N}, c \in \mathbb{C}$ the space of everywhere holomorphic functions $f : \mathbb{C} \to Mat_m$, which satisfy the following equations:

$$f(z + \frac{1}{m}) = \gamma_1^{-1} f(z) \gamma_1$$

$$f(z + \frac{1}{m}\tau) = e^{-2\pi i(mnz - c)} \gamma_2^{-1} f(z) \gamma_2 \tag{6}$$

Proposition 3. $\dim M\Theta_{n,m,c}(\Gamma) = m^2 n$ *and for each element* $f \in M\Theta_{n,m,c}(\Gamma)$ *the equation* $\det f(z) = 0$ *has exactly* mn *zeros modulo* $\frac{1}{m}\Gamma$. *The sum of these zeros is equal to* $mc + \frac{mn}{2}$ *modulo* Γ.

Proof For $m = 1$ we have the usual θ-functions $\Theta_{n,c}(\Gamma) = M\Theta_{n,1,c}(\Gamma)$ and all these statements are well known in this case ([8]). One have a basis $\{\theta_\alpha(z); \alpha \in \mathbb{Z}/n\mathbb{Z}\}$ in the space $\Theta_{n,c}(\Gamma)$ such that $\theta_\alpha(z + \frac{1}{n}) = e^{2\pi i \frac{\alpha}{n}} \theta_\alpha(z), \theta_\alpha(z + \frac{1}{n}\tau) = e^{-2\pi i(z - \frac{n-1}{2n}\tau - \frac{1}{n}c)} \theta_{\alpha+1}(z)$ ([8]). From (6) it follows that $f(z+1) = f(z)$ and $f(z+\tau) = e^{-2\pi i(m^2 nz - c_1)} f(z)$ for some $c_1 \in \mathbb{C}$. So the matrix elements of $f(z)$ are θ-functions from the space $\Theta_{m^2 n, c_1}(\Gamma)$. We have decomposition $f(z) = \sum_\alpha \varphi_\alpha \theta_\alpha(z)$, where $\varphi_\alpha \in Mat_m$ are constant matrices, $\{\theta_\alpha\}$ is a basis in the space $\Theta_{m^2 n, c_1}(\Gamma)$. Substituting this decomposition in (6) one can calculate the dimension of the space $M\Theta_{n,m,c}(\Gamma)$. We have also $\det f(z + \frac{1}{m}) = \det f(z)$ and $\det f(z + \frac{1}{m}\tau) = e^{-2\pi i(m^2 nz - mc)} \det f(z)$. From this follows the statement about zeros of the equation $\det f(z) = 0$.

Proposition 4. *For generic complex numbers* $\lambda_1, \dots, \lambda_{mn}$ *such that* $\lambda_1 + \dots + \lambda_{mn} \equiv mc + \frac{mn}{2}$ *mod*$\frac{1}{m}\Gamma$ *and nonzero vectors* $v_1, \dots, v_{mn} \in \mathbb{C}^m$ *there exist a unique up to proportionality element* $f(z) \in M\Theta_{n,m,c}(\Gamma)$ *such that* $\det f(\lambda_\alpha) = 0, f(\lambda_\alpha) v_\alpha = 0$ *for* $1 \leqslant \alpha \leqslant mn$.

Proof Considering the decomposition $f(z) = \sum_\alpha \varphi_\alpha \theta_\alpha(z)$ one has the system of linear equations $\{\sum_\alpha \theta_\alpha(\lambda_\beta) \varphi_\alpha v_\beta = 0; \beta = 1, \dots, mn\}$ for matrix elements of $\{\varphi_\alpha\}$. One can see that this system defines $\{\varphi_\alpha\}$ uniquely up to proportionality for generic $\lambda_1, \dots, \lambda_{mn}, v_1, \dots, v_{mn}$.

We denote by $S(f)$ the set of zeros of the equation $\det f(z) = 0$ modulo $\frac{1}{m}\Gamma$.

Proposition 5. *Let* $f(z) \in M\Theta_{n,m,c}(\Gamma)$ *be generic element and we have factorization* $f(z) = f_1(z) \dots f_n(z)$ *where* $f_\alpha(z) \in M\Theta_{1,m,c_\alpha}(\Gamma)$, $c_1 + \dots + c_n = c$. *Then* $S(f_\alpha) \cap S(f_\beta) = \emptyset$ *for* $\alpha \neq \beta$ *and* $S(f) = S(f_1) \cup \dots \cup S(f_n)$. *For each decomposition* $S(f) = A_1 \cup \dots \cup A_n$ *such that* $A_\alpha \cap A_\beta = \emptyset$ *for* $\alpha \neq \beta$ *and* $\#A_\alpha = m$ *there exists*

a unique factorization $f(z) = f_1(z)\dots f_n(z)$ *up to proportionality of* f_α *such that* $S(f_\alpha) = A_\alpha$ *for* $\alpha = 1, \dots, n$.

Proof is similar to the proof of proposition 1, we just change polynomials by θ-functions.

Let U_c be the projectivisation of the linear space $M\Theta_{1,m,c}(\Gamma)$ and $U = \bigcup_{c\in\mathbb{C}} U_c$. We have the following twisted transposition $\mu : U \times U \to U \times U$. By definition $\mu(f,g) = (f_1, g_1)$, where $f(z)g(z) = f_1(z)g_1(z)$ and $S(f_1) = S(g), S(g_1) = S(f)$.

Let $\overline{U}$ be the set of elements f from U with fixed order on $S(f)$. For $f \in \overline{U}$ let $\overline{S}(f)$ be the set $S(f)$ with corresponding order. We have a local action of the symmetric group S_{mN} on the space $\overline{U}^N$. By definition, for $\sigma \in S_{mN}$ we have $\sigma(f_1, \dots, f_N) = (f_1^\sigma, \dots, f_N^\sigma)$ where $f_1(z)\dots f_N(z) = f_1^\sigma(z)\dots f_N^\sigma(z)$ and $\overline{S}(f_\alpha^\sigma) = \sigma\overline{S}(f_\alpha)$ for $1 \leqslant \alpha \leqslant N$.

Remark It is possible to construct twisted R-matrices for this twisted transposition μ as intertwiners of tensor products of cyclic representations of the algebra of monodromy matrices for elliptic Belavin R-matrix [5] at the point of finite order (see also [6]). It will be the subject of another paper.

§5. Generalization of the star-triangle relation

Let U be a complex manifold with an action of the symmetric group S_m by birational automorphisms. Let g_i be a birational automorphism of U corresponding to the transposition $(i, i+1)$, where $1 \leqslant i < m$. We assume also that the group S_{2m} acts on the manifold $U \times U$ by birational automorphisms, such that the transposition $(i, i+1)$ for $i < m$ acts by $g_i \times \mathrm{id}$ and the transposition $(i, i+1)$ for $i > m$ acts by $\mathrm{id} \times g_{i-m}$. Let f be the birational automorphism of $U \times U$ corresponding to the transposition $(m, m+1)$. We will use the notations $f(u, v) = (\lambda(u, v), \mu(u, v))$. We have: $g_i^2 = f^2 = \mathrm{id}$, $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, $f g_m^l f = g_m^l f g_m^l$, $f g_1^r f = g_1^r f g_1^r$, where $g_m^l = g_m \times \mathrm{id}$ and $g_1^r = \mathrm{id} \times g_1$.

It is clear that we have the local action of the symmetric group S_{mN} on the manifold U^N for each N.

Let V be a n-dimensional vector space. For each $u \in U$ we denote by $V(u)$ a vector space canonically isomorphic to V. Let G_i $(1 \leqslant i < m)$ be a meromorphic function from U to $End(V)$ and F be a meromorphic function from $U \times U$ to $End(V \otimes V)$. We will consider $G_i(u)$ and $F(u, v)$ as linear operators

$$G_i(u) : V(u) \to V(g_i(u))$$

$$F(u, v) : V(u) \otimes V(v) \to V(\lambda(u, v)) \otimes V(\mu(u, v))$$

Definition *We call G_i, F a twisted GF-matrices if the following properties hold:*

1. The compositions

$$V(u) \to V(g_i(u)) \to V(u)$$

$$V(u) \otimes V(v) \to V(\lambda(u, v)) \otimes V(\mu(u, v)) \to V(u) \otimes V(v)$$

are equal to the identity.

2. The following diagrams are commutative:

$$\begin{array}{ccc}
V(g_i(u)) & \xrightarrow{G_{i+1}(g_i(u))} & V(g_{i+1}g_i(u)) \\
\uparrow{\scriptstyle G_i(u)} & & \downarrow{\scriptstyle G_i(g_{i+1}g_i(u))} \\
V(u) & & V(g_i g_{i+1} g_i(u)) \\
\downarrow{\scriptstyle G_{i+1}(u)} & & \uparrow{\scriptstyle G_{i+1}(g_i g_{i+1}(u))} \\
V(g_{i+1}(u)) & \xrightarrow{G_i(g_{i+1}(u))} & V(g_i g_{i+1}(u))
\end{array}$$

$$\begin{array}{ccc}
V(\lambda(u,v))\otimes V(\mu(u,v)) & \xrightarrow{G_m(\lambda(u,v))\otimes 1} & V(g_m(\lambda(u,v)))\otimes V(\mu(u,v)) \\
\uparrow{\scriptstyle F(u,v)} & & \downarrow{\scriptstyle F(g_m(\lambda(u,v))),\mu(u,v))} \\
V(u)\otimes V(v) & & V(\lambda(g_m(\lambda(u,v)),\mu(u,v)))\otimes V(\mu(g_m(\lambda(u,v)),\mu(u,v))) \\
\downarrow{\scriptstyle G_m(u)\otimes 1} & & \uparrow{\scriptstyle G_m(\lambda(g_m(u),v))\otimes 1} \\
V(g_m(u))\otimes V(v) & \xrightarrow{F(g_m(u),v)} & V(\lambda(g_m(u),v))\otimes V(\mu(g_m(u),v))
\end{array}$$

$$\begin{array}{ccc}
V(\lambda(u,v))\otimes V(\mu(u,v)) & \xrightarrow{1\otimes G_1(\mu(u,v))} & V(\lambda(u,v))\otimes V(g_1(\mu(u,v))) \\
\uparrow{\scriptstyle F(u,v)} & & \downarrow{\scriptstyle F(\lambda(u,v)),g_1(\mu(u,v)))} \\
V(u)\otimes V(v) & & V(\lambda(\lambda(u,v),g_1(\mu(u,v))))\otimes V(\mu(\lambda(u,v),g_1(\mu(u,v)))) \\
\downarrow{\scriptstyle 1\otimes G_1(v)} & & \uparrow{\scriptstyle 1\otimes G_1(\mu(u,g_1(v)))} \\
V(u)\otimes V(g_1(v)) & \xrightarrow{F(u,g_1(v))} & V(\lambda(u,g_1(v)))\otimes V(\mu(u,g_1(v)))
\end{array}$$

It is clear that we have also twisted transposition μ in this situation. By definition μ is the birational automorphism of $U\times U$ corresponding to the element $(1,m+1)(2,m+2)\dots(m-1,2m-1)\in S_{2m}$. We have

$$\mu=\prod_{1\leqslant\alpha\leqslant m} g^l_{m-\alpha+1}\dots g^l_{m-1} f g^r_{m-1}\dots g^r_\alpha$$

where $g^l_\alpha=g_\alpha\times\mathrm{id}$, $g^r_\alpha=\mathrm{id}\times g_\alpha$ and the product is ordered from $\alpha=1$ to $\alpha=m$. One can also construct the corresponding twisted R-matrix as a product of m^2 factors:

$$R=\prod_{1\leqslant\alpha\leqslant m}(G_{m-\alpha+1}\otimes 1)\dots(G_{m-1}\otimes 1)F(1\otimes G_{m-1})\dots(1\otimes G_\alpha) \qquad (7)$$

Here we omit arguments of R, G_α and F.

Let W be a vector space, L be a meromorphic function from U to $End(V\otimes W)$. We will consider $L(u)$ as a linear operator $L(u): V(u)\otimes W\to W\otimes V(u)$.

Definition *We call L an L-operator for twisted GF-matrices G_i, F if the following diagrams are commutative:*

$$\begin{array}{ccc}
V(u)\otimes W & \xrightarrow{L(u)} & W\otimes V(u) \\
\downarrow{\scriptstyle G_i(u)\otimes 1} & & \downarrow{\scriptstyle 1\otimes G_i(u)} \\
V(g_i(u))\otimes W & \xrightarrow{L(g_i(u))} & W\otimes V(g_i(u))
\end{array}$$

$$\begin{array}{ccc}
V(u)\otimes W\otimes V(v) & \xrightarrow{L(u)\otimes 1} & W\otimes V(u)\otimes V(v) \\
\uparrow{\scriptstyle 1\otimes L(v)} & & \downarrow{\scriptstyle 1\otimes F(u,v)} \\
V(u)\otimes V(v)\otimes W & & W\otimes V(\lambda(u,v))\otimes V(\mu(u,v)) \\
\downarrow{\scriptstyle F(u,v)\otimes 1} & & \uparrow{\scriptstyle L(\lambda(u,v))\otimes 1} \\
V(\lambda(u,v))\otimes V(\mu(u,v))\otimes W & \xrightarrow{1\otimes L(\mu(u,v))} & V(\lambda(u,v))\otimes W\otimes V(\mu(u,v))
\end{array}$$

That is

$$G_i(u)L(u) = L(g_i(u))G_i(u), \text{ for } 1 \leqslant i < m$$
$$F(u,v)L^1(u)L^2(v) = L^1(\lambda(u,v))L^2(\mu(u,v))F(u,v) \tag{8}$$

We can consider $L(u)$ as a $n \times n$ matrix $L(u) = (L_i^j(u))$, where $L_i^j(u)$ are the elements in $End(W)$ such that

$$L(u) : x_i(u) \otimes W \to (L_i^j(u)W) \otimes x_j(u)$$

Here $\{x_i\}$ is a basis in V and $\{x_i(u)\}$ is the corresponding basis in $V(u)$.

Definition *The algebra of monodromy matrices M_{GF} for the twisted GF-matrices G_i, F is an associative algebra with the generators $\{L_i^j(u); i,j = 1\ldots n, u \in U\}$ and the defining relations (8).*

It is clear that the representations of the algebra M_{GF} in the space W coincide with the L-operators $L(u) \in End(V \otimes W)$.

Since the relations (4) follow from the relations (8), one has a homomorphism $M_{GF} \to M_R$, which is identity on the generators, where R is given by (7).

It is clear that the algebra M_{GF} has the usual coproduct $\Delta : L_i^j(u) \to L_i^\alpha(u) \otimes L_\alpha^j(u)$.

Remarks 1. One can also construct the Zamolodchikov algebra for GF-matrices.

2. For $Q(u) = \mathrm{tr}L(u) = L_i^i(u)$ we have the following relations:

$$Q(u) = Q(g_i(u)) \text{ for } 1 \leqslant i < m$$
$$Q(u)Q(v) = Q(\lambda(u,v))Q(\mu(u,v))$$

3. It is possible to construct twisted GF-matrices for the elliptic local action of the symmetric group defined in §4 as intertwiners of tensor products of cyclic representations of the algebra of monodromy matrices for elliptic Belavin R-matrix ([5]) at the point of finite order. It will be the subject of another paper.

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