CPT-2000/P.4076

Lecture on Hecke algebras[1]

O. Ogievetsky*) [2] & P. Pyatov**)

*) *Center of Theoretical Physics, Luminy, 13288 Marseille, France*

**) *Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia*
&
Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

[1]Based on lectures presented at the International School "Symmetries and Integrable Systems", Dubna, 8-11 June, 1999.

[2]On leave of absence from P. N. Lebedev Physical Institute, Theoretical Department, Leninsky pr. 53, 117924 Moscow, Russia

Contents

1 Introduction

This lecture is an introductive review of the structure and representation theory of Hecke algebras. To be more precise we shall discuss the A_n-type Hecke algebras which have found lastly a lot of applications in the theory of integrable systems and which are related to the representation theory of the A_n series of quantum groups (see, e.g., [ChP]). We have no intention to give an exhaustive account of the subject here, therefore some of important topics (like the R-matrix representations and the traces over Hecke algebras or the theory of Kazhdan-Lusztig polynomials) as well as most of the proofs will be omitted in our lecture. Instead we shall try to present the subject in a clear and memorizable way appealing to analogies with a familiar example of matrix algebra and developing simple mnemonics. For this purpose we use a recently developed approach due to R. Dipper, G. James, G. Murphy and others. The central role in this approach is played by a special commutative set of the so-called Jucys-Murphy elements which are used for construction of a full set of primitive idempotents (Young tableaux) and central idempotents (Young diagrams) in the Hecke algebra and, then, for developing the representation theory and for an explicit description of a correspondence between the Hecke algebra and the matrix algebras.

Before going on with the presentation let us emphasize an important fact: the A_n-type Hecke algebra $\mathcal{H}_{n+1}(q)$ in case if its parameter $q \in \mathbb{C}^\times$ is not a root of unity is isomorphic to the group algebra of the symmetric group $\mathbb{C}[\mathcal{S}_{n+1}]$. So the Hecke algebra just provides another language for description of the permutation symmetry and this language appears to be very useful in models where the permutation symmetry is realized in a nontrivial way, i.e., not by the standard permutation matrices.

We shall see that in the approach to Hecke algebras based on Jucys-Murphy elements, the main objects of the theory of symmetric groups, like Young diagrams and tableaux, or restrictions of representations of the symmetric group $\mathcal{S}_n$ to $\mathcal{S}_{n-1}$, appear in an intrinsic and natural way within the algebra itself.

Note also that the approach using the Jucys-Murphy elements which was originally developed for the A_n-type Hecke algebras has proved its usefulness in application to the Hecke algebras of other types and the Birman-Murakami-Wenzl algebras (for some history and references see [R]).

The lecture is aimed for graduate and postgraduate students; the reading assumes some basic knowledge of linear algebra and commutative algebra.

The lecture is organized as follows. In the next Section we recall some basic facts about the finite-dimensional semi-simple algebras. In Section 3 we give definitions of Coxeter groups, their Coxeter graphs and Hecke algebras. A Subsection with a proof of several formulations of semisimplicity of Hecke algebras is included. Section 4 contains additional information on the A_n-type Hecke algebras. An introduction to the theory of representations of symmetric groups is contained in Section 5, together with a proof of the hook formula which gives the dimensions of irreducible representations. In Section 6 the set of the Jucys-Murphy elements for the A_n-type Hecke algebas is introduced. In Section 7 eigenvalues of the Jucys-Murphy elements in different representations are calculated and the primitive idempotents in the Hecke algebras are constructed. Section 8 deals with the eigenvalues of central idempotents in different representations. There is a Subsection describing a Hecke algebra analogue of the tautological representation of the symmetric groups by permutation matrices and 1-dimensional representations of the Hecke algebra. Another Subsection contains generalities about a different system of primitive idempotents, called Young symmetrizers, for the symmetric group and the Hecke algebra. An isomorphism from the Hecke algebra to the direct product of matrix algebras is explicitly constructed in Section 9. This is equivalent to the description of the Hecke algebra representations. We apply the matrix structure to the construction of Temperley-Lieb algebras in Section 10. In Section 11 we reconsider the representation theory of the Hecke algebra from a slightly different point of view. An application to the construction of the Burau representation is given there. Section 12 illustrates results of Sections 9 and 11 on simple examples.

2 Algebraic background

In this Section we shall remind some basic results from the theory of semisimple finite dimensional algebras. Proofs and a more systematic introduction into the subject can be found in [DK].

Let $\mathcal{A}(K)$ be a finite-dimensional unital associative algebra over a field K. We shall consider a *regular* $\mathcal{A}$*-bimodule* $\mathcal{A}_{reg}$. As a linear space $\mathcal{A}_{reg}$ coincides with $\mathcal{A}$ so that one can identify elements of $\mathcal{A}_{reg}$ and $\mathcal{A}$. The structure of the left or right $\mathcal{A}$-action on $\mathcal{A}_{reg}$ is given by multiplication from the left or right hand side in $\mathcal{A}$, i.e., $\forall\ a \in \mathcal{A}\ ,\quad b \in \mathcal{A}_{reg}$ the left and right action of a on b is given, respectively, by

$$a \triangleright b := ab\ , \qquad b \triangleleft a := ba\ . \tag{2.1}$$

The representation of the algebra $\mathcal{A}$ given by its left action on $\mathcal{A}_{reg}$ is *faithful*, that is, the image of $\mathcal{A}$ in the algebra $\mathrm{End}(\mathcal{A}_{reg})$ of the endomorphisms of $\mathcal{A}_{reg}$ is isomorphic to $\mathcal{A}$ itself[3].

Clearly, investigation of an algebra is equivalent to the study of its regular representation. The left and right module structures of the regular bimodule carry the same information, so we shall discuss only the left module structure of $\mathcal{A}_{reg}$ in what follows.

Suppose $\mathcal{A}_{reg}$ decomposes into a direct sum of left $\mathcal{A}$-modules $\mathcal{M}_i$, $i = 1,\dots,s$,

$$\mathcal{A}_{reg} = \bigoplus_{i=1}^{s} \mathcal{M}_i\ . \tag{2.2}$$

The subspaces $\mathcal{M}_i \subset \mathcal{A}$ are *left ideals of* $\mathcal{A}$.

The corresponding decomposition of the unit element of $\mathcal{A}_{reg}$ ($\equiv$ of $\mathcal{A}$) is

$$1 = \sum_{i=1}^{s} e_i \qquad \text{where} \qquad e_i \in \mathcal{M}_i\ . \tag{2.3}$$

Consider the action of the element $e_i \in \mathcal{A}$ on the identity

$$e_i \equiv e_i \triangleright 1 = \sum_{j=1}^{s} e_i \triangleright e_j\ .$$

Since $e_i \triangleright e_j \equiv e_i e_j \in \mathcal{M}_j$ it follows that

$$e_i e_j = \delta_{ij} e_i\ . \tag{2.4}$$

So, the elements $\{e_i\}_{i=1}^{s}$ form the set of *mutually orthogonal idempotents* in $\mathcal{A}$. If, moreover, the module $\mathcal{M}_i$ is *indecomposable* then the corresponding idempotent e_i can not be further resolved into a sum of nontrivial mutually orthogonal idempotents. In this case one calls e_i the *primitive idempotent.*

[3] To see this, it is enough to consider the action of $\mathcal{A}$ on the identity element $1 \in \mathcal{A}_{reg}$.

The arguments used in the derivation of the formula (2.4) remain valid if one replaces the idempotent e_i by an arbitrary element $v_i \in \mathcal{M}_i$. Hence, one has $v_i e_j = \delta_{ij} v_i$ and therefore $\mathcal{M}_i = \mathcal{A} e_i$.

To sum up, there is a one-to-one correspondence between the decompositions of the regular module $\mathcal{A}_{reg}$ into a direct sum of (indecomposable) submodules and the resolutions of the unit element of the algebra $\mathcal{A}$ into a sum of mutually orthogonal (primitive) idempotents. The decomposition of $\mathcal{A}_{reg}$ into indecomposable submodules is unique in a sense that any two such decompositions contain the same number of components which under appropriate identification are pairwise isomorphic. Correspondingly, any two resolutions of the unit into a sum of primitive orthogonal idempotents contain the same number of terms which are related by some inner algebra automorphism.

The decomposition of the left regular module $\mathcal{A}_{reg}$ (or, equivalently, of the algebra $\mathcal{A}$) corresponding to a given resolution (2.3), (2.4) of the unit looks like

$$\mathcal{A} \ = \bigoplus_{i=1}^{s} \mathcal{A} e_i \ . \tag{2.5}$$

It is called the *left Peirce decomposition of the algebra* $\mathcal{A}$.

In the same way one can define the right Peirce decomposition $\mathcal{A} = \bigoplus_{i=1}^{s} e_i \mathcal{A}$. Applying then both left and right Peirce decompositions simultaneously one obtains the *(two-sided) Peirce decomposition of the algebra* $\mathcal{A}$

$$\mathcal{A} = \bigoplus_{i,j=1}^{s} e_i \mathcal{A} e_j \ . \tag{2.6}$$

The components $\mathcal{A}_{ij} := e_i \mathcal{A} e_j$ of the Peirce decomposition being in general neither left nor right ideals in $\mathcal{A}$ ($\equiv$ left or right submodules in $\mathcal{A}_{reg}$) provide a very convenient matrix interpretation for the elements of $\mathcal{A}$. Indeed, let a and b be any elements of $\mathcal{A}$ and let

$$a = \sum_{i,j=1}^{s} a_{ij} \ , \quad b = \sum_{i,j=1}^{s} b_{ij} \ , \quad \text{where} \quad a_{ij} := e_i a e_j \in \mathcal{A}_{ij} \ , \quad b_{ij} := e_i b e_j \in \mathcal{A}_{ij}$$

be their Peirce decomposition. Then, the components of the Peirce decomposition of their sum and product look, respectively, as

$$(a+b)_{ij} = a_{ij} + b_{ij} \ , \quad (ab)_{ij} = \sum_{k=1}^{s} a_{ik} b_{kj} \ ,$$

so that if one represents elements a, b of $\mathcal{A}$ by the matrices $||a_{ij}||_{i,j=1}^{s}$, $||b_{ij}||_{i,j=1}^{s}$ of their Peirce components, the addition and multiplication in $\mathcal{A}$ will be reproduced by the usual matrix addition and multiplication.

One also extracts useful information about the structure of an algebra considering *central resolutions* of its unit element, i.e. the resolutions (2.3), (2.4) with components e_i belonging to the center of the algebra. In this case one has

$$e_i \mathcal{A} e_j = 0 \ , \ \ \forall \, i \neq j \ , \qquad e_i \mathcal{A} e_i = e_i \mathcal{A} = \mathcal{A} e_i \ ,$$

and so the left, right and two-sided Peirce decompositions become identical and their components are (two-sided) ideals in $\mathcal{A}$.

Thus, there is a one-to-one correspondence between the central resolutions of the unit element of the algebra $\mathcal{A}$ and the decompositions of $\mathcal{A}$ into a direct sum of its ideals $\mathcal{A} = \oplus_{i=1}^{s} \mathcal{A}_{ii}$, or, equivalently, decompositions of $\mathcal{A}$ into a direct product of the subalgebras $\mathcal{A} \simeq \mathcal{A}_{11} \times \ldots \times \mathcal{A}_{ss}$.

We shall now illustrate and refine the general results described above on a basic example of the matrix algebra $M_n(K)$.

Recall that $M_n(K)$ is the algebra of $n \times n$ matrices with components in the field K. A linear basis in $M_n(K)$ is given by a set of n^2 *matrix units* $||e_{ij}||_{km} := \delta_{ik}\delta_{jm}$. The multiplication rules in this basis look like

$$e_{ij} \, e_{km} \ = \ \delta_{jk} \ e_{im} \ . \tag{2.7}$$

The diagonal matrix units e_{ii}, $i = 1, \ldots, n$, are mutually orthogonal primitive idempotents defining suitable resolution of the identity matrix I

$$I = \sum_{i=1}^{n} \ e_{ii} \ . \tag{2.8}$$

The corresponding left Peirce decomposition of the regular module is $M_n(K)_{reg} = \oplus_{i=1}^{n} \ V_i$, where $V_i := M_n(K) e_{ii}$ are *simple modules*. Note, the simplicity property of V_i strengthens the indecomposability condition from the general theory.

The basis in V_i is given by the elements e_{ki}, $k = 1, \ldots, n$, so that $\dim V_i = n$ and one can think of V_i as the i-th column of the matrix. In fact, the algebra $M_n(K)$ has only one nontrivial simple module V and all the modules V_i are isomorphic to V. Any $M_n(K)$-module is isomorphic to a direct sum of several copies of the simple module V, i.e., it is *semisimple*. In particular, $M_n(K)_{reg} \simeq nV$.

The components of the (two-sided) Peirce decomposition — $e_{ii}M_n(K)e_{jj}$ — are multiples of the matrix units

$$e_{ii}M_n(K)e_{jj} = Ke_{ij} \simeq K^1 \ .$$

The unit matrix does not admit central resolutions in the matrix algebra, and moreover, the algebra $M_n(K)$ does not have nontrivial (two-sided) ideals, i.e., it is *simple*. The center $Z(M_n(K))$ of $M_n(K)$ is spanned by the multiples of the unit matrix, $Z(M_n(K)) = KI \simeq K$.

In fact, the family of matrix algebras $M_n(K)$, $n = 1, 2, \ldots$, completes the list of finite-dimensional simple K-algebras in case when K is algebraically closed. Next by generality is the family of *semisimple algebras*, i.e., the algebras whose right regular module can be decomposed into a direct sum of simple modules. The general structure results for this case are refined in the Wedderburn-Artin theorem. Here we shall formulate it for a particular situation when $\mathcal{A}$ is the algebra over an algebraically closed field K.

Wedderburn-Artin theorem. Any semisimple algebra $\mathcal{A}$ over an algebraically closed field K is isomorphic to an algebra of the type $M_{n_1}(K) \times M_{n_2}(K) \times \ldots \times M_{n_s}(K)$ for some integer s and an unordered set of integers $\{n_1, \ldots, n_s\}$. Under this isomorphism the diagonal matrix units $\{e^{(i)}_{jj}\}_{j=1}^{n_i}$ (resp., the identity matrices $I^{(i)} = \sum_{j=1}^{n_i} e^{(i)}_{jj}$) from the subalgebras $M_{n_i}(K)$ correspond to the primitive idempotents (resp., central idempotents) in $\mathcal{A}$.
Any $\mathcal{A}$-module V is semisimple and it can be uniquely presented as $V = \bigoplus_{i=1}^{s} k_i V^{(i)}$, where $V^{(i)}$ are simple $M_{n_i}(K)$-modules, $\dim V^{(i)} = n_i$. In particular, simple $\mathcal{A}$-modules correspond to $V^{(i)}$, $i = 1, \ldots, s$, and the regular module decomposes as $\mathcal{A}_{reg} = \bigoplus_{i=1}^{s} n_i V^{(i)}$.

3 Coxeter groups and Hecke algebras

In this Section we give definitions of Coxeter groups and Hecke algebras. Subsection 3.2 deals with semisiplicity questions for Hecke algebras.

A detailed presentation of the theory of Coxeter groups can be found in [B, H]. A different definition, based on double cosets, of the Hecke algebra is given in [B], Ch.IV, § 2, Ex.22 (see also [K]). A connection between the two definitions was established by Iwahori [Iw] and for this reason the Hecke algebras which we will discuss here are called often the Iwahori-Hecke algebras.

3.1 Definitions

The *Coxeter group* is generated by the set of elements $\{s_i\}_{i\in\mathcal{I}}$ subject to following conditions.

– Each of the generators is a reflection:

$$(s_i)^2 = 1 \ , \quad \forall\ i \in \mathcal{I} \ . \tag{3.1}$$

– Consider pairs $i, j \in \mathcal{I}$ for which the element $s_i s_j$ is of finite order, i.e., there exists an integer $m_{ij} \geq 2$, such that

$$(s_i s_j)^{m_{ij}} = 1 \ . \tag{3.2}$$

Then, the system (3.1), (3.2) form a presentation of the Coxeter group by generators and defining relations (that is, there are no other relations on the set of generators $\{s_i\}$).

The second condition can be given a form

$$\underbrace{s_i s_j s_i \ \dots}_{m_{ij}\ \text{factors}} \ = \ \underbrace{s_j s_i s_j \ \dots}_{m_{ij}\ \text{factors}} \tag{3.3}$$

which in view of (3.1) is equivalent to (3.2).

As follows from the definition, the Coxeter group is given provided one fixes its *Coxeter matrix* $M = \|m_{ij}\|_{i,j\in\mathcal{I}}$ — the symmetric matrix with integer entries satisfying conditions $m_{ii} = 1$, $\forall\ i \in \mathcal{I}$; $m_{ij} \geq 2$, $\forall\ i \neq j \in \mathcal{I}$ (here one puts $m_{ij} = \infty$ if the corresponding pair $s_i s_j$ is of infinite order). We use notation $\mathcal{W}(M)$ for the Coxeter group corresponding to the Coxeter matrix M.

The set of data given by the Coxeter matrix is conveniently visualized with the help of the *Coxeter-Dynkin graph* — an edge-labeled graph, whose nodes are indexed by the generators of a Coxeter group and the nodes i and j are not connected if $m_{ij} = 2$ and connected by the line carrying label m_{ij} if $m_{ij} \geq 3$. In case if $m_{ij} = 3$ the label is usually omitted.

Figure 1 (at the end of the text) gives the list of the *connected* Coxeter-Dynkin graphs whose corresponding Coxeter group is finite. The Coxeter-Dynkin graph of an arbitrary finite Coxeter group is a disjoint union of the connected graphs presented there.

To each Coxeter matrix M one associates a $\mathbb{C}$-algebra $\mathcal{H}(M, q)$ depending on an invertible complex parameter $q \in \mathbb{C}^\times$ and called the *Hecke algebra.* It is generated by a set of elements $\{\sigma_i\}_{i \in \mathcal{I}}$ satisfying conditions[4]

$$(\sigma_i)^2 = 1 + (q - q^{-1})\, \sigma_i\ , \qquad \forall\ i \in \mathcal{I}\ ; \tag{3.4}$$

$$\underbrace{\sigma_i \sigma_j \sigma_i\ \dots}_{m_{ij}\ \text{factors}} = \underbrace{\sigma_j \sigma_i \sigma_j\ \dots}_{m_{ij}\ \text{factors}}\ , \qquad \forall\ i, j \in \mathcal{I}\ . \tag{3.5}$$

The condition (3.4) is called *the Hecke condition.* It states that all the generators σ_i of the Hecke algebra has the same minimal characteristic polynomial

$$P(\sigma_i) := (\sigma_i - q)(\sigma_i + q^{-1}) \equiv 0\ . \tag{3.6}$$

Note that for $q = 1$ the Hecke condition becomes $\sigma_i^2 = 1$. In this case the elements σ_i satisfy the relations (3.1) and (3.3) for the generating elements of the Coxeter group $\mathcal{W}(M)$, thus, the Hecke algebra $\mathcal{H}(M, 1)$ is isomorphic to a *group algebra* $\mathbb{C}[\mathcal{W}(M)]$.

Remark.

> A Coxeter group $\mathcal{W}$ is called *crystallographic* if it admits an irreducible representation in which
>
> – each element s_i is represented by a reflection in a hyperplane say, V_i,

[4] A group $\mathcal{B}(M)$ generated by the set of elements $\{\sigma_i\}_{i \in \mathcal{I}}$ with defining relations (3.5) only is called the braid group corresponding to the Coxeter matrix M.

– each product $s_i s_j$ is a rotation by the angle $\frac{2\pi}{m_{ij}}$, in the plane, orthogonal to $V_i \cap V_j$,

– the action of $\mathcal{W}$ preserves a lattice.

In particular, in a basis, formed by lattice elements, matrices of all $w \in \mathcal{W}$ have integer entries. Therefore, traces of matrices are integer numbers - this condition does not depend on a choice of a basis. The trace of a rotation $s_i s_j$ is $2\cos\frac{2\pi}{m_{ij}}$. It is an integer number if and only if $m_{ij} = 2, 3, 4, 6$. Therefore, the groups, corresponding to graphs H_3, H_4 and $I_2(p)$, $p = 5$ or $p \geq 7$, are not crystallographic. The other Coxeter groups are crystallographic, they appear in the theory of semisimple Lie groups, where they are called Weyl groups; they preserve root and weight lattices.

3.2 Semisimplicity

In what follows we restrict ourselves to the case of finite Coxeter groups. The corresponding group algebras (and, hence, the Hecke algebras $\mathcal{H}(M, 1)$) belong to the class of semisimple algebras due to the following theorem:

Maschke theorem. Let $\mathcal{G}$ be a finite group. Let K be a field whose characteristic does not divide the order of $\mathcal{G}$. Then the group algebra $K[\mathcal{G}]$ is semisimple. In particular, for a field K of characteristic 0, the group algebra $K[\mathcal{G}]$ is semisimple.

So, the statements of the Section 2 about semisimple algebras are applicable to the study of representations of the algebras $\mathcal{H}(M, 1)$. One can view the Hecke algebra $\mathcal{H}(M, q)$ as a deformation of the group algebra $\mathbb{C}[\mathcal{W}(M)] \simeq \mathcal{H}(M, 1)$. The group algebra is semisimple and thus does not admit nontrivial infinitesimal deformations. For a global deformation, the situation is more subtle — there are particular values of q for which the algebra $\mathcal{H}(M, q)$ looses semisimlicity. However for generic values of q we shall see that

$$\mathcal{H}(M, q) \cong \mathbb{C}[\mathcal{W}(M)] \ . \tag{3.7}$$

Thus, although the Hecke algebra looks as a deformation of the group algebra of the Coxeter group, this deformation of the algebra structure is trivial for almost all q.

To establish the isomorphism (3.7), we need another version of the definition of the Hecke algebra, equivalent to (3.4), (3.5). The defining relations (3.1), (3.2) (or (3.1), (3.3)) for the Coxeter group explicitly use the set of generators $S = \{s_i\}_{i\in\mathcal{I}}$. Therefore, it is more accurate to speak about a pair $(\mathcal{W}, S)$. This pair is called a *Coxeter system.* Since the group $\mathcal{W}$ is generated by elements in S, every element $w \in \mathcal{W}$ can be expressed as a product of some elements from S. The number $l(w)$ of terms in the shortest expression for w is called the *length* of w. Any expression for w (as a word in generators s_i) containing exactly $l(w)$ terms is called a *reduced* expression.

Now the Hecke algebra can be defined as an algebra which has, as a vector space, a basis T_w, one for each $w \in \mathcal{W}$; relations are

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ l(sw) = l(w) + 1 \ , \\ T_{sw} + (q - q^{-1})T_w & \text{otherwise} \ , \end{cases} \tag{3.8}$$

for $w \in \mathcal{W}$ and $s \in S$.

One more equivalent way to write relations for T_w's is

$$\begin{aligned} T_w T_{w'} &= T_{ww'} & &\text{if } \ l(ww') = l(w) + l(w') \ , \\ T_s^2 &= (q - q^{-1})T_s + 1 & &\text{if } \ s \in S \ . \end{aligned} \tag{3.9}$$

In fact, one can define more general Hecke algebras $\mathcal{H}(M, \{q_i\})$. Let $\{q_i\}_{i\in\mathcal{I}}$ be complex nonzero numbers. We assume that $q_i = q_j$ if the elements s_i and s_j are conjugate in $\mathcal{W}$. Then relations (3.9) in the general Hecke algebra can be replaced by

$$\begin{aligned} T_w T_{w'} &= T_{ww'} & &\text{if } \ l(ww') = l(w) + l(w') \ , \\ T_{s_i}^2 &= (q_i - q_i^{-1})T_{s_i} + 1 & &\text{for } \ s_i \in S \ . \end{aligned} \tag{3.10}$$

Assume that the Coxeter-Dynkin graph is connected. If all m_{ij} are either equal to 2 or odd numbers then all s_i are conjugate in $\mathcal{W}$: if vertices i and j are connected, we have $(s_j s_i)^{(1-m_{ij})/2} s_i (s_j s_i)^{(m_{ij}-1)/2} = s_j$, so s_i and s_j are conjugate. Since the graph is connected, we can move from any $s \in S$ to any other. Thus, relations (3.10) are reduced to (3.9) for the Coxeter groups with graphs A_n, D_n, E_n, H_n and $I_2(2k+1), k \geq 2$. In cases B_n, F_4, G_2 and $I_2(2k), k \geq 4$ the corresponding algebra (3.10) admits two independent parameters.

In the relations (3.10), one can consider all the noncoinciding q_i as independent indeterminates. Then, (3.10) define an algebra $\mathcal{H}_{gen}(M)$ over the ring of $K[q_i, q_i^{-1}]$ of Laurent polynomials in $\{q_i\}$ over an arbitrary field K. The algebra $\mathcal{H}_{gen}(M)$ is called the *generic* Hecke algebra of the Coxeter system $(\mathcal{W}, S)$. Thus, the algebra $\mathcal{H}(M, \{q_i\})$ is a *specialization* of the algebra $\mathcal{H}_{gen}(M)$ – the indeterminates take numerical values from $K \setminus \{0\}$.

We are now in a particular case of the following general situation.

Let K be a field, $\tilde{K}$ the ring of Laurent polynomials in some set of indeterminates $\{q_i\}$ over K, and LK the field of fractions of $\tilde{K}$.

Let $\mathcal{A}$ be an associative algebra with unity over the ring $\tilde{K}$. Assume that $\mathcal{A}$ is free as a $\tilde{K}$-module and let $\{e_\alpha\}$ be a basis of $\mathcal{A}$. Denote by $\Gamma^\gamma_{\alpha\beta}(q_i)$ the structure constants of $\mathcal{A}$,

$$e_\alpha e_\beta = \Gamma^\gamma_{\alpha\beta}(q_i) e_\gamma \ . \tag{3.11}$$

For any set of invertible elements $\{\bar{q}_i\}$ from K denote by f the specialization map, $f : \tilde{K} \setminus \{0\} \to K, q_i \mapsto \bar{q}_i$. We have then an associative algebra $\mathcal{A}_f$ over K; the structure constants of $\mathcal{A}_f$ are $\Gamma^\gamma_{\alpha\beta}(\bar{q}_i)$.

There are certain relations between the structures of $\mathcal{A}$ and $\mathcal{A}_f$. For simplicity, we shall consider only the case $char(K) = 0$. There is a useful characterization of semisimple algebras:

> Let $\mathcal{A}$ be an algebra over a field K, $char(K) = 0$. For any $x \in \mathcal{A}$ let L_x be the operator of the left multiplication by x, $L_x(a) = xa$. The bilinear form $(x, y) := \operatorname{Tr} L_x L_y$ is called the *trace form* on $\mathcal{A}$.
> Then the algebra $\mathcal{A}$ is semisimple iff the trace form is nondegenerate.
>
> In other words, let $\{e_\alpha\}$ be a basis of $\mathcal{A}$ and $T_{\alpha\beta} = \operatorname{Tr} L_{e_\alpha} L_{e_\beta}$ be the matrix of the trace form in the basis $\{e_\alpha\}$. The determinant $D = \det T_{\alpha\beta}$ is called the discriminant of $\mathcal{A}$ in the basis $\{e_\alpha\}$.
> Then $\mathcal{A}$ is semisimple iff $D \neq 0$.

Proposition. Let $char(K) = 0$. If $\mathcal{A}_f$ is semisimple over K then $\mathcal{A}$ is semisimple over LK.

Proof. In the basis (3.11), the discriminant $D(q_i)$ is a Laurent polynomial. Its specialization, $D(\bar{q}_i)$ is the discriminant of $\mathcal{A}_f$. By assumption, the algebra $\mathcal{A}_f$ is semisimple, therefore, $D(\bar{q}_i) \neq 0$. Then $D(q)$ is a rational function which is not 0 since its specialization at $q_i = \bar{q}_i$ is different from 0.

Thus, $\mathcal{A}$ is semisimple. The proof is finished.

Corollary 1. The generic Hecke algebra of a finite Coxeter group $\mathcal{W}$ is semisimple over $L\mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers.

Proof. Use the basis $\{T_w\}$, $w \in \mathcal{W}$ and apply the Proposition. The proof is finished.

For a semisimple algebra $\mathcal{A}$, dimensions of its irreducible representations are called the *numerical invariants* of $\mathcal{A}$. The following theorem is due to Tits (for details, see [CR]).

Deformation Theorem. Assume that the algebra $\mathcal{A}$ is semisimple over LK and the algebra $\mathcal{A}_f$ is semisimple over K. Then the numerical invariants of the algebras $\mathcal{A}$ and $\mathcal{A}_f$ coincide.

For generic Hecke algebras, $\Gamma^{\gamma}_{\alpha\beta}(q_i)$ are Laurent polynomials in the basis $\{T_w\}$. The function $D(q_i)$ is therefore a Laurent polynomial. Away from $q_i = 0$, the equation $D(q_i) = 0$ defines an algebraic variety $\mathcal{Z}$ in the space with parameters $\{q_i\}$ (in particular, when there is only one parameter q, $\mathcal{Z}$ consists of a finite number of points). Call a configuration of parameters $\{q_i\}$ *generic* if it does not belong to $\mathcal{Z} \cup (\cup_i \{q_i = 0\})$. We have then

Corollary 2. The specialization of the generic Hecke algebra is semisimple over $\mathbb{C}$ for generic $\{q_i\}$. Moreover, the numerical invariants of a semisimple specialization coincide with the numerical invariants of the group algebra $\mathbb{C}[\mathcal{W}(M)]$.

Next Sections are devoted to a particular example of the Coxeter group of the graph A_{n-1} and the corresponding Hecke algebra. We shall consider the case when q is not a root of unity. As we shall see, for such q, the Hecke algebra is semisimple.

In fact, the representation theory will show that the generic Hecke algebra of the type A_{n-1} is semisimple in a stronger sense: it is semisimple over the ring $\mathbb{Z}[q, q^{-1}, [2]_q^{-1}, [3]_q^{-1}, \dots, [n]_q^{-1}]$, where $[k]_q$ is a q-number, $[k]_q := (q^k - q^{-k})/(q - q^{-1})$ (see Sections 9 and 11; for other aspects of this semisimplicity statement, see [GS]).

In particular, consider structure constants modulo p. Then the Hecke algebra is semisimple over $F_p[q, q^{-1}, [2]_q^{-1}, \dots, [n]_q^{-1}]$, where F_p is a field of characteristic p. This has to be contrasted to the fact that the group algebra $F[\mathcal{W}]$ is not semisimple if p divides the order of $\mathcal{W}$. Thus, in characteristic $p \leq n$, the Hecke algebra deformation of the group algebra of the symmetric group is nontrivial.

4 Symmetric groups and A-type Hecke algebras

For the Coxeter-Dynkin diagram A_{n-1} the corresponding Coxeter group is the *symmetric group* $\mathcal{S}_n$ — the group of all permutations of the set of n elements $\{1, 2, \ldots, n\}$. It is generated by the set of elementary permutations $S = \{s_i\}_{i=1,\ldots,n-1}$, each s_i flips the objects i and $i+1$.

A comprehensive introduction to the theory of symmetric groups can be found, *e.g.* in [Ja, S].

The Hecke algebra of type A_{n-1} is usually denoted as $\mathcal{H}_n(q)$. The defining relations (3.4) and (3.5) in this particular case look like

$$\begin{aligned} \sigma_k \sigma_{k+1} \sigma_k &= \sigma_{k+1} \sigma_k \sigma_{k+1} \ , && \text{for } \ k = 1, \ldots, n-2 \ , && (4.1) \\ \sigma_k \sigma_j &= \sigma_j \sigma_k \ , && \text{if } \ |k - j| \geq 2 \ , && (4.2) \\ {\sigma_k}^2 &= 1 + (q - q^{-1}) \sigma_k \ , && \text{for } \ k = 1, \ldots, n-1 \ . && (4.3) \end{aligned}$$

The dimension of the algebra $\mathcal{H}_n(q)$ equals $n!$ — the number of different permutations of the set of n elements. A linear basis in the space of $\mathcal{H}_n(q)$ can be constructed as follows. Introduce elements σ_{ji} for $n \geq j \geq i \geq 1$

$$\sigma_{ji} = \begin{cases} 1 \text{ if } j = i; \\ \sigma_{j-1} \sigma_{j-2} \ldots \sigma_i \text{ if } j > i. \end{cases} \tag{4.4}$$

The linear basis is formed by elements

$$\sigma_{2 i_2} \sigma_{3 i_3} \ldots \sigma_{n i_n} \tag{4.5}$$

for some $i_2, i_3, \ldots, i_n$.

Remarks.

1. The length of a permutation $w \in \mathcal{S}_n$ is equal to its "disorder" — the cardinality of the set

$$\{(a, b) \mid a < b \ \text{ and } \ w(a) > w(b)\} \quad , \quad a, b \in \{1, 2, \ldots, n\} \ . \tag{4.6}$$

2. The words (4.5) with σ's replaced by s's give reduced expressions for the corresponding elements of $\mathcal{S}_n$. The reduced expression of the form (4.5) is called the *normal* reduced expression. To find the normal reduced expression for a permutation $\pi \in \mathcal{S}_n$, define

$$\nu_i = \#\{ \, j \mid j > \pi^{-1}(i) \; , \; \pi(j) < i \} \; . \tag{4.7}$$

Then

$$\pi = \sigma_{2,2-\nu_2}\sigma_{3,3-\nu_3} \dots \sigma_{n,n-\nu_n} \; . \tag{4.8}$$

5 Young diagrams and tableaux

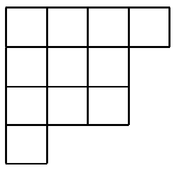
5.1 Partitions, diagrams, tableaux and Young graph

The central role in the theory of symmetric groups is played by partitions, Young diagrams and Young tableaux.

The *partition* λ of n is a sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \dots$, such that $\sum \lambda_j = n$, the notation is $\lambda \vdash n$.

The number of inequivalent irreducible representations of $\mathcal{S}_n$ (as well as the number of conjugacy classes of $\mathcal{S}_n$) is equal to the number of partitions of n. So, it is natural to label different irreducible representations of $\mathcal{S}_n$ by partitions $\lambda \vdash n$ — the representation corresponding to λ is denoted as V_λ.

A suitable graphical presentation for the partitions is provided by the Young diagrams. Given a partition λ one defines the *Young diagram* $[\lambda]$ as a set of nodes placed row by row into the lower-right quadrant, the j-th (from the top) row contains λ_j nodes. The definition should be clear from the example $\lambda = \{4, 3, 3, 1\} \vdash 11$:



(5.1)

Here each node is represented graphically by a box $\square$.

To each node placed in the i-th column and j-th row of the Young diagram one associates a number $c = q^{2(i-j)}$ called the *content* of the node. The nodes of the diagram above have the following contents:

$$\begin{array}{|c|c|c|c|}
\hline
1 & q^2 & q^4 & q^6 \\
\hline
q^{-2} & 1 & q^2 \\
\cline{1-3}
q^{-4} & q^{-2} & 1 \\
\cline{1-3}
q^{-6} \\
\cline{1-1}
\end{array} \tag{5.2}$$

By placing the numbers $1, 2, \dots, n$ into the nodes of the $[\lambda]$, one number into each node, we obtain a *Young tableau.* The node $\boxed{\text{j}}$ containing the number j is called the j-th node of the Young tableau.

The Young tableau is called the *standard Young tableau* if the numbers in each of its rows and columns are arranged in ascending order in the right and down directions, correspondingly.

Below two Young tableaux are drawn, the first one is standard, the second one is not:

$$\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 4 & 6 \\
\hline
3 & 7 & 8 \\
\cline{1-3}
5 & 10 & 11 \\
\cline{1-3}
9 \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|c|c|c|}
\hline
7 & 10 & 11 & 5 \\
\hline
3 & 1 & 8 \\
\cline{1-3}
4 & 2 & 6 \\
\cline{1-3}
9 \\
\cline{1-1}
\end{array} \tag{5.3}$$

Inductively, given the set of all standard Young tableaux containing $n-1$ nodes one constructs the standard Young tableaux with n nodes by adding n-th node in all allowed ways. Thus, each standard Young tableau represents a possible way of building the corresponding Young diagram step by step — by adding the j-th node at the step number j.

This procedure gives rise to a Young graph whose vertices are Young diagrams and edges show the allowed ways of adding the nodes. First few levels of the Young graph are shown on Figure 2 (circles on the Figure represent nodes). One should read this graph downwards, lines leading from one level to the next one show inclusions of Young diagrams. As it is clear from the inductive process, the number of paths leading from the upper level to a given Young diagram equals the number of the standard Young tableaux for this diagram. This number (it is drawn above each diagram on Figure 2) is known to be equal to the dimension of the corresponding representation of the symmetric group. More precisely, the set of standard Young tableaux related to a given Young diagram $[\lambda]$ labels basic vectors in the corresponding irreducible representation V_λ.

There is an explicit formula for this number, found in [FRT] (for additional references and an alternative proof see [V]).

Hook formula. The dimension of V_λ is given by

$$\dim V_\lambda = \frac{n!}{\prod\limits_{\alpha\in[\lambda]} h_\alpha}\ . \tag{5.4}$$

The product here is over all nodes α of $[\lambda]$. The *hook* of α is the set of nodes; it contains the node α itself together with the nodes which lie either under α in the same column or to the right of α in the same row. The *hook length* h_α of α is the number of elements in the hook of α.

The correspondence between the Young diagrams (Young tableaux) and the representations of symmetric groups, together with the general theory from Section 2, shows that central idempotents correspond to the Young diagrams while primitive idempotents correspond to the Young tableaux. This correspondence looks more natural and explicit on the level of Hecke algebras — and this is the subject of the next sections.

Remarks.

1. The Young graph has an important representation-theoretic meaning: the lines show the branching rule for the restriction $\mathcal{S}_n \downarrow \mathcal{S}_{n-1}$: a representation of $\mathcal{S}_n$ corresponding to a given diagram $[\lambda]$, decomposes into a sum of representations of $\mathcal{S}_{n-1}$ corresponding to those diagrams which are connected by a line with $[\lambda]$. Such *branching graph* for a chain of algebras is called sometimes a *Bratelli diagram.* In the considered case of the chain of the group algebras $\mathbb{C}[\mathcal{S}_n]$, the branching graph is *multiplicity free* which means that if a representation of $\mathcal{S}_{n-1}$ occurs in a given irreducible representation of $\mathcal{S}_n$, it occurs with multiplicity one.

2. The numbers drawn on the Young graph on Figure 2 enjoy the following property: a number at a vertex $[\lambda]$ of the Young graph is equal to the sum of numbers at those vertices of the previous level which are connected with $[\lambda]$. Such functions on Bratelli diagrams are called *generalised binomial coefficients* since the usual binomial coefficients satisfy this property for the Pascal triangle. Note, that the Pascal triangle is included in the Young graph, it corresponds to diagrams of the corner shape.

5.2 Hook formula

In this Subsection we will derive the hook formula (5.4).

Let $\lambda = \{\lambda_1, \ldots, \lambda_k\} \vdash n$ be a partition and $[\lambda]$ - the corresponding Young diagram. Denote by d_λ the number of standard tableaux of the shape λ.

Denote by h_i the hook lengths of the nodes in the first column of the diagram,

$$h_1 = \lambda_1 + k - 1 \ , \ h_2 = \lambda_2 + k - 2 \ , \ \ldots \ , \ h_k = \lambda_k \ .$$

We have $\sum h_j = n + \frac{k(k-1)}{2}$.

We shall first prove the Frobenius formula for d_λ.

Proposition 1.

$$d_\lambda = \frac{n!}{h_1! \ldots h_k!} \prod_{i<j} (h_i - h_j) \ . \tag{5.5}$$

Proof. The proof is by induction. For $n = 1$ the formula (5.5) clearly holds.

Assume that (5.5) holds for all diagrams with less than n nodes. For a diagram $[\lambda]$ with n nodes, the number d_λ is the sum of numbers for the diagrams at the $(n-1)$-st level in the Young graph, connected with $[\lambda]$.

The node n in $[\lambda]$ may appear as the end node in the j-th row of $[\lambda]$ if $h_j > h_{j+1} + 1$. If this is the case, denote by $[\lambda_j]$ the diagram with $n - 1$ nodes, the node n in the j-th row omitted. We have then $d_\lambda = \sum d_{\lambda_j}$.

The contribution d_{λ_j} in the sum from the diagram $[\lambda_j]$ is

$$\frac{(n-1)!}{h_1! \ldots h_{j-1}!(h_j - 1)! h_{j+1}! \ldots h_k!} \times$$
$$\prod_{a,b: a<b, a\neq j, b\neq j} (h_a - h_b) \prod_{a:a<j} (h_a - (h_j - 1)) \prod_{b:j<b} ((h_j - 1) - h_b) \ . \tag{5.6}$$

This can be rewritten as

$$\left(\frac{(n-1)!}{\prod h_j!}\prod_{i<j}(h_i-h_j)\right)\times h_j\prod_{c:c\neq j}\frac{h_j-h_c-1}{h_j-h_c}\ . \tag{5.7}$$

Collecting all diagrams $[\lambda_j]$ and comparing with (5.5) we conclude that the statement of the Proposition will be verified if the following formula holds:

$$\sum_j{}' h_j\prod_{c:c\neq j}\frac{h_j-h_c-1}{h_j-h_c}=n\ . \tag{5.8}$$

Here $h_1,\dots,h_k$ is the sequence of decreasing natural numbers and $\sum'$ means the summation over such j that $h_j>h_{j+1}+1$. However, if $h_j=h_{j+1}+1$ then the product $h_j\prod_{c:c\neq j}\frac{h_j-h_c-1}{h_j-h_c}$ vanishes, it contains the term $h_j-h_{j+1}-1$. Therefore we can replace the sum in (5.8) by the sum over all j.

The following Lemma completes the proof of the Proposition 1.

Lemma 1. Let $\{s_1,\dots,s_k\}$ be a set of pairwise different complex numbers. Then

$$\sum_j s_j\prod_{c:c\neq j}\frac{s_j-s_c-1}{s_j-s_c}=\sum_j s_j-\frac{k(k-1)}{2}\ . \tag{5.9}$$

Proof. Decompose the rational function $z^2\prod_i\frac{z-s_i-1}{z-s_i}$ in simple ratios. The polynomial part is $z^2-kz+\frac{k(k-1)}{2}-\sum_j s_j$, the residue at $z=s_j$ is $-s_j^2\prod_{i:i\neq j}\frac{s_j-s_i-1}{s_j-s_i}$. Therefore,

$$z^2\prod_i\frac{z-s_i-1}{z-s_i}=z^2-kz+\frac{k(k-1)}{2}-\sum_j s_j-\sum_j\frac{s_j^2}{z-s_j}\prod_{i:i\neq j}\frac{s_j-s_i-1}{s_j-s_i}\ . \tag{5.10}$$

Evaluating both sides at $z=0$ we arrive at the formula (5.9). The proof of the Lemma 1 is finished.

To deduce the hook formula (5.4) from (5.5), we shall use the following Lemma (for a different proof see [Ma]).

Lemma 2. Denote by $\tilde{h}_i$, $i = 1, \ldots, \lambda_1$ the hook lengths of the nodes in the first row of the diagram $[\lambda]$.

Define two sets of numbers: $A = \{\tilde{h}_2, \ldots, \tilde{h}_{\lambda_1}\}$ and $B = \{h_1 - h_2, \ldots, h_1 - h_k\}$. Then set $A \cap B$ is a permutation of the set $\{1, 2, \ldots, h_1 - 1\}$.

Proof. The numbers in the set A are pairwise different. Same for B. Also, the intersection $A \cap B$ is empty since otherwise we would have $h_1 = \tilde{h}_i + h_j$ for some i and j which is clearly impossible. Thus, $A \cap B$ is a set of $h_1 - 1$ pairwise distinct natural numbers, each is smaller than h_1. The assertion of the Lemma 2 follows.

Proposition 2. The number d_λ is given by (5.4).

Proof. Denote by (i, j) the node in the i-th row and j-th column, and by h_{ij} its hook length. By Lemma 2, we have

$$h_i! = \prod_{j:j>i} (h_i - h_j) \prod_s h_{is} \ . \qquad (5.11)$$

Therefore,

$$\prod_i h_i! = \prod_{i<j} (h_i - h_j) \prod_{i,j} h_{ij} \ . \qquad (5.12)$$

The last term in the l.h.s. of (5.12) is the product of hook lengths for all nodes. Substituting (5.12) in (5.5) we obtain the hook formula (5.4). The proof is finished.

Remark.

Traditionally, one derives the formula (5.5) from the Frobenius formula for the characters of the symmetric group.

Take a permutation $\pi \in \mathcal{S}_n$. Every permutation is a product of non-intersecting cycles. Assume that this product for π has μ_1 cycles of length 1, μ_2 cycles of length 2 etc. The set $\{\mu_i\}$ is called the *cycle structure* of π.

For a diagram $[\lambda]$, let V_λ be the corresponding representation of the symmetric group and χ_λ - its character, $\chi_\lambda(\pi) = \mathrm{Tr}_{V_\lambda} \pi$.

The Frobenius character formula states that

$$\chi_\lambda(\pi) = D_\lambda \left(\prod_{i<j} (x_i - x_j) \prod_j (x_1^j + \ldots + x_k^j)^{\mu_j} \right) |_{x_j = 0} \ . \qquad (5.13)$$

Here

$$D_\lambda = \frac{1}{h_1! \dots h_k!} \frac{\partial_1^h}{\partial x_1^{h_1}} \dots \frac{\partial_k^h}{\partial x_k^{h_k}} \tag{5.14}$$

and $f|_{x_j=0}$ means the value of a function f at the point 0.

The formula (5.5) is a particular case of (5.13) for $\pi =$ unity (the trace of the unity is the dimension of the space).

For a proof of the Frobenius character formula, see, *e.g.* [FH].

6 Jucys-Murphy elements

The *Jucys-Murphy elements* $K_j \in \mathcal{H}_n(q)$, $j = 1, \dots, n$ [Ju, M1] are defined inductively as

$$K_1 = 1 \ , \quad K_{j+1} = \sigma_j K_j \sigma_j \ . \tag{6.1}$$

As a consequence of (4.1), (4.2) all the elements K_j are mutually commuting (one doesn't need the relation (4.3) to check the commutativity; thus, the corresponding elements commute in the braid group already).

Moreover these elements form a maximal commutative subalgebra in $\mathcal{H}_n(q)$. This subalgebra, in the case of $\mathcal{S}_n$, is called the *Gelfand-Zetlin subalgebra* in [OV]. We will keep this notation for the Hecke algebra too.

It turns out that the set of symmetric functions in K_j span linearly the center $Z(\mathcal{H}_n(q))$ of $\mathcal{H}_n(q)$.

An important element of the center is the sum of all Jucys-Murphy elements

$$Z_n = \sum_{p=1}^{n} K_p \ . \tag{6.2}$$

Polynomials in Z_n generate the whole center $Z(\mathcal{H}_n(q))$ (see Theorem 2.14 of the ref. [DJ]).

One can construct another set of the Jucys-Murphy elements in $\mathcal{H}_n(q)$:

$$\tilde{K}_1 = 1 \ , \quad \tilde{K}_{j+1} = \sigma_{n-j} \tilde{K}_j \sigma_{n-j} \ . \tag{6.3}$$

The sets $\{K_j\}$ and $\{\tilde{K}_j\}$ are connected by an inner automorphism $\tilde{K}_j = w_0 K_j w_0^{-1}$, where

$$w_0 = \sigma_1(\sigma_2\sigma_1) \cdot \ldots \cdot (\sigma_{n-1} \dots \sigma_1) \tag{6.4}$$

is the unique longest element of $\mathcal{H}_n(q)$; the conjugation with w_0 reverts the numeration of σ_j, $w_0\sigma_j w_0^{-1} = \sigma_{n-j}$. Thus, w_0^2 belongs to the center and is, therefore, a symmetric function of K_j; in fact, $w_0^2 = \prod_{p=1}^{p=n} K_p$.

7 Primitive orthogonal idempotents

Due to the finite dimensionality of $\mathcal{H}_n(q)$ the elements K_j should satisfy certain polynomial identities, which (with some abuse of terminology) we call *characteristic identities.* It is remarkable that these identities carry the whole information about combinatorics of the Young tableaux. In this Section we discuss these relations in details (for proofs see [DJ, M2]).

The following rule works here:

> There is a one-to-one correspondence between the set of the standard Young tableaux with n nodes and a set of primitive orthogonal idempotents built of the Jucys-Murphy elements (they are polynomials in $K_1, \ldots, K_n$). We shall denote the idempotent corresponding to a tableau X by $P(X)$.
>
> Each idempotent from the set is the common eigenvector of the Jucys-Murphy elements $K_1, \ldots, K_n$ with respect to the left or right multiplication in $\mathcal{H}_n(q)$. The eigenvalue of the element K_j on the idempotent coincides with the content of the j-th node of the corresponding Young tableau.
>
> To construct the idempotents one needs to know the characteristic identities for the Jucys-Murphy elements $K_1, \ldots, K_n$. These characteristic identities are in a one-to-one correspondence with the standard Young tableaux containing $n - 1$ nodes, or, equivalently, with the set of the primitive idempotents of $\mathcal{H}_{n-1}(q)$. Thus, the building of the sets of characteristic identities and primitive idempotents is an inductive procedure.

We shall explain this inductive procedure on simplest examples. The general picture will become clear then.

We are going to construct the standard Young tableaux starting from the single node tableau $\boxed{1}$. The second node $\boxed{2}$ can be added in one of the two places marked by an asterisk on the picture

$$\begin{array}{cc} \boxed{1} & * \\ * & \end{array}$$

As the rule above says, the contents of these marked places form the set $\{q^2, q^{-2}\}$ of the eigenvalues of the operator K_2. The corresponding characteristic identity is

$$(K_2 - q^{-2})(K_2 - q^2) \equiv 0 \; ; \tag{7.1}$$

it is equivalent to the Hecke condition (4.3). The identity (7.1) gives rise to a pair of idempotents

$$\frac{K_2 - q^{-2}}{q^2 - q^{-2}} = \frac{\sigma_1 + q^{-1}}{[2]_q} \qquad \text{and} \qquad \frac{K_2 - q^2}{q^{-2} - q^2} = \frac{q - \sigma_1}{[2]_q} \; ,$$

where a standard notation for *q-numbers* $[n]_q := (q^n - q^{-n})/(q - q^{-1})$ is used.

According to the rule, the element K_2 takes the eigenvalue q^2 on the idempotent corresponding to the tableau $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ and the eigenvalue q^{-2} on the idempotent corresponding to the tableau $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$. Therefore the correspondence is

$$P(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}) \;=\; \frac{K_2 - q^{-2}}{q^2 - q^{-2}} \;=\; \frac{\sigma_1 + q^{-1}}{[2]_q} \; , \tag{7.2}$$

$$P(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}) \;=\; \frac{K_2 - q^2}{q^{-2} - q^2} \;=\; \frac{q - \sigma_1}{[2]_q} \; . \tag{7.3}$$

So, on this trivial example, one sees that in the numerators of the idempotents in the r.h.s. of (7.2), (7.3) one omits a term from the characteristic identity (7.1) corresponding to the place of the last node – here, the node $\boxed{2}$.

Denominators in the r.h.s of relations (7.2), (7.3) are needed to normalize the idempotents. They are calculated by replacing the Jucys-Murphy operators K_2 in the corresponding numerators with their eigenvalues.

The pair of idempotents (7.2), (7.3) is orthogonal (due to the characteristic identity (7.1)), primitive (in the algebra $\mathcal{H}_2(q)$) and complete (*i.e.*, their sum is equal to unity); they form the linear basis in the Gelfand-Zetlin subalgebra of $\mathcal{H}_2(q)$ (which in this particular case coincides with the whole algebra $\mathcal{H}_2(q)$).

We now proceed with the construction of three-nodes standard Young tableaux. The third node $\boxed{3}$ can be added to the two-nodes tableaux in four ways:

$$\begin{array}{ccc} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} & * \\ \;* \end{array} \qquad\qquad \begin{array}{cc} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} & * \\ * \end{array}$$

In accordance with the figures the element K_3 takes eigenvalues q^4 and q^{-2} on the eigenspace of K_2 with the eigenvalue q^2; K_3 takes the eigenvalues q^{-4} and q^2 on the eigenspace of K_2 with the eigenvalue q^{-2}. Thus, we have at this level two characteristic identities

$$P(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array})(K_3 - q^{-2})(K_3 - q^4) \sim (K_2 - q^{-2})(K_3 - q^{-2})(K_3 - q^4) \equiv 0, \quad (7.4)$$

$$P(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array})(K_3 - q^2)(K_3 - q^{-4}) \sim (K_2 - q^2)(K_3 - q^2)(K_3 - q^{-4}) \equiv 0. \quad (7.5)$$

Again, omitting each time the term corresponding to the place of the last node $\begin{array}{|c|}\hline 3 \\ \hline \end{array}$ we obtain the complete set of four primitive (in $\mathcal{H}_3(q)$) orthogonal idempotents:

$$P(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}) = P(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array})\frac{(K_3 - q^{-2})}{(q^4 - q^{-2})} = \frac{(K_2 - q^{-2})}{(q^2 - q^{-2})}\frac{(K_3 - q^{-2})}{(q^4 - q^{-2})}, \quad (7.6)$$

$$P(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}) = P(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array})\frac{(K_3 - q^4)}{(q^{-2} - q^4)} = \frac{(K_2 - q^{-2})}{(q^2 - q^{-2})}\frac{(K_3 - q^4)}{(q^{-2} - q^4)}, \quad (7.7)$$

$$P(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}) = P(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array})\frac{(K_3 - q^{-4})}{(q^2 - q^{-4})} = \frac{(K_2 - q^2)}{(q^{-2} - q^2)}\frac{(K_3 - q^{-4})}{(q^2 - q^{-4})}, \quad (7.8)$$

$$P(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}) = P(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array})\frac{(K_3 - q^2)}{(q^{-4} - q^2)} = \frac{(K_2 - q^2)}{(q^{-2} - q^2)}\frac{(K_3 - q^2)}{(q^{-4} - q^2)}. \quad (7.9)$$

Note that

$$P(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}) + P(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}) = P(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}) \quad (7.10)$$

and

$$P(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}) + P(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}) = P(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array})\ . \quad (7.11)$$

In terms of generators σ_1 and σ_2, the idempotents (7.6) and (7.9) can be expressed as

$$P(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}) \ = \ \frac{(\sigma_1 + q^{-1})(\sigma_2\sigma_1 + q^{-1}\sigma_2 + q^{-2})}{[2]_q[3]_q}\ , \quad (7.12)$$

$$P\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right) \;=\; \frac{(q-\sigma_1)(q^2-q\sigma_2+\sigma_2\sigma_1)}{[2]_q[3]_q}\;. \tag{7.13}$$

The idempotents (7.6)-(7.9) span the linear basis in the Gelfand-Zetlin subalgebra of $\mathcal{H}_3(q)$.

The inductive procedure of constructing the characteristic identities and the orthogonal idempotents described here generalizes straightforwardly to the general case of Young tableaux with n nodes. We finish examples with the four-nodes Young diagrams.

The full set of characteristic identities is

$$P\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right) \;\cdot\; (K_4-q^{-2})(K_4-q^6)\equiv 0\;, \tag{7.14}$$

$$P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right) \;\cdot\; (K_4-q^{-4})(K_4-1)(K_4-q^4)\equiv 0\;, \tag{7.15}$$

$$P\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right) \;\cdot\; (K_4-q^{-4})(K_4-1)(K_4-q^4)\equiv 0\;, \tag{7.16}$$

$$P\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right) \;\cdot\; (K_4-q^{2})(K_4-q^{-6})\equiv 0\;. \tag{7.17}$$

Some examples of corresponding idempotents are (we list only those idempotents which are proportional to $P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right)$; they are quadratic in K_4)

$$P\left(\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\right) \;=\; P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right)\cdot\frac{(K_4-q^{-4})}{(q^4-q^{-4})}\frac{(K_4-1)}{(q^4-1)}\;, \tag{7.18}$$

$$P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\right) \;=\; P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right)\cdot\frac{(K_4-q^{-4})}{(1-q^{-4})}\frac{(K_4-q^4)}{(1-q^4)}\;, \tag{7.19}$$

$$P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\right) \;=\; P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right)\cdot\frac{(K_4-1)}{(q^{-4}-1)}\frac{(K_4-q^4)}{(q^{-4}-q^4)}\;. \tag{7.20}$$

We leave the formulas for other idempotents in $\mathcal{H}_4(q)$ as an exercise for a reader.

To sum up, the general recipe to write down relations between the Jucys-Murphy elements in $\mathcal{H}_n(q)$ is

> Take a Young tableau X with $(n-1)$ nodes. Let $P(X)$ be the corresponding primitive idempotent in the algebra $\mathcal{H}_{n-1}(q)$.
> Denote by $\alpha_1, \dots, \alpha_\mu$ all allowed places where the n-th node can be added to X. Denote by f_X a polynomial $f_X(t) := \prod_{a=1}^{\mu}(t - c(\alpha_a))$, where $c(\alpha_a)$ is the content of the node α_a. Then the identity
>
> $$P(X) f_X(K_n) = 0 \tag{7.21}$$
>
> holds in (the Gelfand-Zetlin subalgebra of) the algebra $\mathcal{H}_n(q)$.
> Moreover, to obtain the set of all relations in the Gelfand-Zetlin subalgebra of $\mathcal{H}_n(q)$, one adds to the relations in the Gelfand-Zetlin subalgebra of $\mathcal{H}_{n-1}(q)$ (which are known from the previous step) the union over all Young tableaux with $(n-1)$ nodes of equations (7.21).

Expressions for idempotents follow easily from equations (7.21). The important statement is that they exhaust the full list of idempotents.

> To construct primitive idempotents in the algebra $\mathcal{H}_n(q)$, denote by $f_{(X,b)}$ a polynomial $f_{(X,b)}(t) := \prod_{a=1;\ a\neq b}^{\mu}(t - c(\alpha_b))$. Then a primitive idempotent corresponding to the Young tableau obtained from X by adding the n-th node at the place α_b is
>
> $$P(X) \frac{f_{(X,b)}(K_n)}{f_{(X,b)}(c(\alpha_b))} \ . \tag{7.22}$$
>
> The constructed idempotents are labeled by Young tableaux with n nodes. They form the complete set of primitive orthogonal idempotents in $\mathcal{H}_n(q)$.

The inductive procedure of building idempotents implies important "divisibility" and "completness" properties of the idempotents $P(X)$.

Assume that an idempotent P admits further resolution, that is, it can be decomposed into a sum of idempotents, $P = \sum_i P_i$. We have $PP_i = P_i$

for all i. We say then that the idempotent P divides P_i and we write $P \,|\, P_i$. For standard tableaux X and Y we write $X \,|\, Y$ if the tableau Y contains the tableau X. For instance, a standard tableau $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ contains standard tableaux $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$ and $\begin{array}{|c|}\hline 1 \\ \hline \end{array}$. By the construction of idempotents $P(X)$ we have

$$X \,|\, Y \quad \Longrightarrow \quad P(X) \,|\, P(Y) \ . \tag{7.23}$$

For a standard tableau X with m nodes, let $\mathcal{N}_n(X)$ be a set of all standard tableaux Y with n nodes such that $X \,|\, Y$. For $n \geq m$ the set $\mathcal{N}_n(X)$ is not empty and, by (7.23), $P(X) \,|\, P(Y)$ for all $Y \in \mathcal{N}_n$. From the inductive procedure of building the idempotents it follows that the set of idempotents $P(Y)$, $Y \in \mathcal{N}_n$ gives a complete decomposition of the idempotent $P(X)$ in the algebra $\mathcal{H}_n(q)$ (simple formulas of this sort are (7.10) and (7.11)),

$$P(X) = \sum_{Y \in \mathcal{N}_n(X)} P(Y) \ . \tag{7.24}$$

Remark.

For a Young tableau X let X^T be a tableau obtained from X by transposition with respect to the main diagonal. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array}^{\;T} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} 4 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \ .$$

As follows from the inductive procedure of building the idempotents, the idempotent $P(X^T)$ can be obtained from the idempotent $P(X)$ by substitution $q \to -q^{-1}$,

$$P(X^T) = P(X)|_{q \to -q^{-1}} \ . \tag{7.25}$$

Note, that replacement $q \mapsto -q^{-1}$, $\sigma_i \mapsto \sigma_i$ defines an isomorphism $\mathcal{H}_q(n) \to \mathcal{H}_{-q^{-1}}(n)$ for all n.

There is another relation between the idempotents $P(X^T)$ and $P(X)$. A map $\tau : \sigma_i \mapsto -\sigma_i^{-1}$ preserves the Hecke algebra relations and thus can be extended to an automorphism of the Hecke algebra $\mathcal{H}_n(q)$. One has

$$\tau(P(X)) = P(X^T) \ . \tag{7.26}$$

The behaviour of idempotents under the map τ is well defined on the classical level (*i.e.* for the symmetric group) and it refines the relation $V_\lambda \otimes V_{1^n} \simeq V_{\lambda^T}$. In contrast, the relation (7.25) has no classical counterpart: it doesn't make much sense to say "replace 1 by -1".

The replacement $q \mapsto -q$, $\sigma_i \mapsto -\sigma_i$ defines an isomorphism $\mathcal{H}_q(n) \to \mathcal{H}_{-q}(n)$ for all n. This replacement clearly does not affect idempotents.

Finally, there is an antiautomorphism $\kappa : \mathcal{H}_n(q) \to \mathcal{H}_n(q)$, which is trivial on the Hecke generators, $\sigma_i \mapsto \sigma_i$. From the definition of K_j it is clear that the idempotents are stable under κ. However, when an idempotent is written explicitly in terms of σ_i's, as, for instance, in (7.12) or (7.13), the application of κ produces a different expression.

8 Central idempotents

8.1 Characteristic polynomials

Now we shall say few words about Young central idempotents (block idempotents in the notation of [DJ]). As it is mentioned already in Sections 5 and 6, the central idempotents in the algebra $\mathcal{H}_n(q)$ are labeled by the Young diagrams $[\lambda]$, for $\lambda \vdash n$, and they can be suitably expressed as polynomials in the element Z_n. A characteristic polynomial for Z_n is given exclusively in terms of the combinatorics of the Young diagrams.

The element Z_n satisfies the characteristic identity

$$\prod_{\lambda \vdash n} (Z_n - c(\lambda)) \equiv 0 \ , \tag{8.1}$$

where the product is taken over all partitions λ of n, and each coefficient $c(\lambda)$ is equal to the sum of the contents of all the nodes of the corresponding Young diagram $[\lambda]$.
The spectrum of Z_n is simple, that is, for different diagrams $[\lambda]$ the corresponding coefficients $c(\lambda)$, as Laurent polynomials in q, are different.

For an operator with a simple spectrum, it is straightforward to write its spectral decomposition.

The Young central idempotent $P([\lambda_0])$ (below we write $P([\lambda])$ for the central idempotent corresponding to a Young diagram $[\lambda]$) is an eigenvector of Z_n with eigenvalue $c(\lambda_0)$:

$$P([\lambda_0]) = \prod_{\lambda \vdash n, \lambda \neq \lambda_0} \frac{Z_n - c(\lambda)}{c(\lambda_0) - c(\lambda)} \ . \tag{8.2}$$

Each Young central idempotent $P([\lambda])$ is the sum of the orthogonal primitive idempotents corresponding to the tableaux of the shape of the diagram $[\lambda]$.

One example is given for illustration purposes here:

$$P(\ydiagram{2,1}) \ = \ P(\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}) + P(\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}) \ = \ \frac{(Z_3 - q^{-2}[3]_q)}{([3]_q - q^{-2}[3]_q)} \frac{(Z_3 - q^{2}[3]_q)}{([3]_q - q^{2}[3]_q)} \ .$$

Remark.

In the "classical limit" (that is, for $q \to 1$) the content $c = q^{2(i-j)}$ of each node tends to 1. Writing $q = 1 + \epsilon + o(\epsilon)$, $\epsilon \to 0$, one has $c = 1 + (i-j) \cdot 2\epsilon + o(\epsilon)$. The quantity $(i-j)$ is the "classical" content of the node. The classical analogue, z_n, of the operator Z_n counts the sum of the classical contents of a diagram. It might happen that the values of z_n on two different diagrams coincide while the operator Z_n distinguishes them. Here is the minimal example:

$$\ydiagram{4,1,1} \qquad\qquad \ydiagram{3,3} \tag{8.3}$$

The value of z_6 on each diagram is 3; the values of Z_6 are $q^{-4} + q^{-2} + 1 + q^2 + q^4 + q^6$ and $q^{-2} + 2 + 2q^2 + q^4$ respectively. The example (8.3) again shows the use of Hecke algebras: quantization removes degeneration.

This example also shows the particular role of the element Z_n: not each symmetric function of K_j's distinguishes representations. For example, the value of the product of all K_j's is q to the power twice the sum of the classical contents of a diagram; therefore, it doesn't distinguish the diagrams (8.3).

8.2 One-dimensional and Burau representations

Traditionally the central idempotents corresponding to the single row and single column Young diagrams are called the *q-symmetrizer* and *q-antisymmetrizer*, respectively. Further on we shall use a shorthand notation for the corresponding one dimensional Hecke algebra modules: $S(n) := V_{\{n\}}$ and $A(n) := V_{\{1^n\}}$. The q-symmetrizer and q-antisymmetrizer are then denoted by $P_{S(n)}$ and $P_{A(n)}$. Besides the expressions for these idempotents as polynomials in Jucys-Murphy elements one can write explicit formulas for them in terms of the linear basis $\{T_\pi\}_{\pi\in\mathcal{S}_n}$ (for its definition see Eqs.(4.5)–(4.8)):

$$P_{S(n)} = \frac{q^{\frac{-n(n-1)}{2}}}{[n]_q!} \sum_{\pi\in\mathcal{S}_n} q^{l(\pi)} T_\pi \ , \tag{8.4}$$

where $[n]_q! := [1]_q[2]_q \dots [n]_q$. By the Remark in Section 7, for the q-antisymmetrizer we have $P_{A(n)} = P_{S(n)}|_{q\to -q^{-1}}$.

Relations (7.2), (7.3), (7.12) and (7.13) are the simplest examples of (8.4) and the analogous expression for $P_{A(n)}$.

For the symmetric group $\mathcal{S}_n$, the representation corresponding to the symmetrizer, is the trivial one, $S(n) : s_i \mapsto 1$ for all the elementary permutations s_i. In the representation, corresponding to the antisymmetrizer, one has $A(n) : s_i \mapsto -1$ for all the elementary permutations s_i. For a permutation $\pi \in \mathcal{S}_n$, the number $\mathrm{sgn}\,(\pi) := \pi|_{A(n)} = (-1)^{l(\pi)}$ is called the *parity* of π. Even permutations (that is, permutations of parity $+1$ or, equivalently, permutations which belong to the kernel of A) form the *alternating* group — $\mathcal{A}_n$, a normal subgroup of index 2 in the symmetric group $\mathcal{S}_n$.

To any permutation $\pi \in \mathcal{S}_n$ of the set $\mathcal{J} = \{1, 2, \dots, n\}$ one associates an operator $\tilde{\pi}$ in a vector space with a basis e_α labelled by the set $\mathcal{J}$,

$$\tilde{\pi} : e_i \mapsto e_{\pi(i)} \ . \tag{8.5}$$

The parity of a permutation π is the determinant of the corresponding operator $\tilde{\pi}$.

The two one-dimensional representations of the Hecke algebra $\mathcal{H}_n(q)$, corresponding to the q-symmetrizer and the q-antisymmetrizer, are, respectively[5] ,

$$S(n): \qquad \sigma_i \mapsto q, \qquad\qquad i = 1, \ldots, n-1 \ , \tag{8.6}$$
$$A(n): \qquad \sigma_i \mapsto -q^{-1}, \qquad i = 1, \ldots, n-1 \ . \tag{8.7}$$

The n-dimensional representation (8.5) of $\mathcal{S}_n$ has an analogue for the Hecke algebra $\mathcal{H}_n(q)$ as well, the operators $\tilde{\sigma}_i$ act as

$$\tilde{\sigma}_i : \begin{cases} e_j & \mapsto \quad q e_j \quad \text{if } j \neq i, i+1 \ , \\ e_i & \mapsto \quad q^{-1} e_{i+1} \ , \\ e_{i+1} & \mapsto \quad q e_i + (q - q^{-1}) e_{i+1} \ . \end{cases} \tag{8.8}$$

This representation, as a representation of the braid group $\mathcal{B}_n$, is called the *Burau representation* [Bu].

8.3 Young symmetrizers

In classical textbooks on symmetric groups, the presentation is usually based on another system of primitive idempotents, called *Young symmetrizers*.

For the symmetric group $\mathcal{S}_n$, there are two subgroups, $\mathcal{U}$ and $\mathcal{V}$, associated to a standard Young tableau X of shape $\lambda \vdash n$: the group $\mathcal{U}$ consists of permutations preserving rows of X, the group $\mathcal{V}$ consists of permutations preserving columns of X. Define two elements in the group algebra:

$$u(X) = \sum_{g \in \mathcal{U}} g \ , \ v(X) = \sum_{g \in \mathcal{V}} (-1)^{\operatorname{sgn}(g)} g \ . \tag{8.9}$$

The Young symmetrizers are defined by

$$E(X) = \frac{\dim V_\lambda}{n!} u(X) v(X) \ . \tag{8.10}$$

[5]Note a consequence of the fact that the idempotent $P_{S(n)}$ corresponds to the representation $S(n)$: the value of $P_{S(n)}$ in the representation $S(n)$ is equal to 1, therefore,

$$\sum_{\pi \in \mathcal{S}_n} q^{2l(\pi)} = q^{\frac{n(n-1)}{2}} [n]_q! \ .$$

Elements $E(X)$, when X runs through the set of all standard tableaux, form a complete system of primitive idempotents.

Example. For $\mathcal{S}_3$,

$$E\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = \frac{1}{3}(1+s_1)(1-s_1s_2s_1) = \frac{1}{3}(1+s_1-s_2s_1-s_1s_2s_1) \ ,$$

$$E\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = \frac{1}{3}(1+s_1s_2s_1)(1-s_1) = \frac{1}{3}(1-s_1-s_1s_2+s_1s_2s_1) \ .$$

The system (8.10) of idempotents can be generalized to Hecke algebras (see [Gy]).

For each diagram $[\lambda]$ we shall briefly describe one representative of the system of idempotents, belonging to the matrix summand corresponding to $[\lambda]$. Define *row* and *column* standard tableaux $[\lambda]_r$ and $[\lambda]_c$. They are obtained by filling the duagram $[\lambda]$ by numbers $1, \ldots, n$, in the increasing order, row by row, or column by column, respectively.

One associates a Hecke subalgebra h_i to each row i of $[\lambda]_r$: h_i is generated by

$$\sigma_{\lambda_1+\ldots+\lambda_{i-1}+1}, \ldots, \sigma_{\lambda_1+\ldots\lambda_i-1} \ .$$

Classically, this is a group algebra of the subgroup of permutations of the i-th row. Let S_i be the q-symmetrizer of h_i and

$$u_q(\lambda) = \prod S_i \ . \tag{8.11}$$

Similarly, one defines Hecke subalgebras h'_j by the columns of $[\lambda]_c$. Let A_j be the q-antisymmetrizer of h'_j and

$$v_q(\lambda) = \prod A_j \ . \tag{8.12}$$

Finally, let $w \in \mathcal{S}_n$ be the permutation transforming $[\lambda]_r$ to $[\lambda]_c$.

Then, up to a factor, the q-analogue of the Young symmetrizer is given by

$$E_q(\lambda) = u_q(\lambda)T_w^{-1}v_q(\lambda)T_w \ . \tag{8.13}$$

Example.

$$E_q\left(\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = \frac{1}{[3]_q}(q^{-1}+\sigma_1)\sigma_2^{-1}(q-\sigma_1)\sigma_2 \ . \tag{8.14}$$

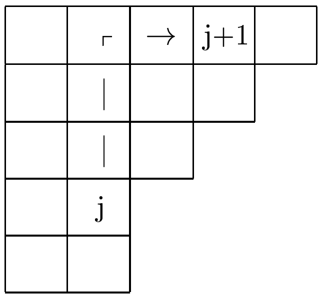
9 Matrix structure

The amount of idempotents constructed in Section 7 is sufficient to establish the semisimplicity of $\mathcal{H}_n(q)$: the number of idempotents is the same as for the group algebra $\mathbb{C}[\mathcal{S}_n]$. Moreover, the construction from Section 7 shows exactly where the semisimplicity might be lost: it happens when a denominator needed to normalize an idempotent, vanishes. An accurate look gives exceptional values of q: these are those q for which $[n]_q! = 0$. So, the Hecke algebra $\mathcal{H}_n(q)$ is semisimple if $q^{2i} \neq 1$, $i = 2, \ldots, n$.

The idempotents $P(X)$ for standard Young tableaux X of a given shape $[\lambda]$ play the role of the diagonal matrix units e_{ii} from Section 2.

In this Section we shall build analogues of the off-diagonal matrix units e_{ik}, $i \neq k$, in the Hecke algebra, thus, establishing explicitly an isomorphism from $\mathcal{H}_n(q)$ to a direct product of matrix algebras. As we shall see, a very convenient tool for this is provided by commutation relations between σ_i's and the projectors $P(X_\lambda)$.

Consider a pair of successive, say, j-th and $(j+1)$-th, nodes of a Young tableau. Take a hook containing these two nodes. We are interested in a part $L_{j,j+1}$ of this hook which starts from the node j and ends at the node $(j+1)$, as it is shown on a figure below.



(9.1)

Denote by $\ell_{j,j+1}$ *the length* of $L_{j,j+1}$, that is, 1 plus the number of nodes in $L_{j,j+1}$, the nodes j and $(j+1)$ excluded. We need an "oriented" length: $\ell_{j,j+1}$ is positive if $L_{j,j+1}$ is oriented left and down (that is, to arrive from the node j to the node $(j+1)$ one has to go left and then down — or counter-clockwise), and negative if $L_{j,j+1}$ is oriented up and right (clockwise). On the picture (9.1), the length $\ell_{j,j+1}$ is negative.

For example, for a tableau

1	2	4	6
3	7	8	
5	10	11	
9			

(9.2)

one has: $\ell_{5,6} = -5$, $\ell_{89} = 4$ *etc.*

There is another way to calculate the length: if $c(j) = q^{2\alpha}$ and $c(j+1) = q^{2\beta}$ are, respectively, the contents of j-th and $(j+1)$-th nodes, then $\ell_{j,j+1} = \alpha - \beta$.

When we need to specify a tableau X for which the length is calculated, we write $\ell_{j,j+1}(X)$.

Let X_λ be a tableau of the shape $\lambda \vdash n$. Denote by $X_\lambda s_j$ a tableau obtained from X_λ by permuting j-th and $(j+1)$-th nodes.

Note that

$$\ell_{j,j+1}(X_\lambda) = -\ell_{j,j+1}(X_\lambda s_j) \ . \tag{9.3}$$

As in Section 7 we use notation $P(X_\lambda)$ for a primitive idempotent corresponding to a standard tableau X_λ and in addition we put $P(X_\lambda) = 0$ for any nonstandard tableau X_λ.

Given any $j = 1, \ldots, n-1$ and a pair of tableaux X_λ and $X_\lambda s_j$ such that at least one of them is standard the following relations are satisfied in $\mathcal{H}_n(q)$ (see [M2]):

$$P(X_\lambda)\left(\sigma_j + \frac{q^{-\ell_{j,j+1}}}{[\ell_{j,j+1}]_q}\right) = \left(\sigma_j - \frac{q^{\ell_{j,j+1}}}{[\ell_{j,j+1}]_q}\right) P(X_\lambda s_j) \ =: \ P(X_\lambda | X_\lambda s_j) \tag{9.4}$$ [6]

Note: in the relations (9.4) the length $\ell_{j,j+1}$ is calculated for the tableau X_λ (not $X_\lambda s_j$), $\ell_{j,j+1} = \ell_{j,j+1}(X_\lambda)$.

Interchanging $X_\lambda \leftrightarrow X_\lambda s_j$ and using relation (9.3), one can rewrite for-

[6]The notation *expression* $=: A$ means that the quantity A is defined to be the expression in the left hand side.

mulas (9.4) in the following equivalent form:

$$\Big(\sigma_j + \frac{q^{-\ell_{j,j+1}}}{[\ell_{j,j+1}]_q}\Big)\, P(X_\lambda) = P(X_\lambda s_j)\,\Big(\sigma_j - \frac{q^{\ell_{j,j+1}}}{[\ell_{j,j+1}]_q}\Big)\ , \tag{9.5}$$

where, again, the length $\ell_{j,j+1}$ is calculated for the tableau X_λ. The set of relations (9.4) for *all* the pairs $\{X_\lambda, X_\lambda s_j\}$ with at least one tableau X_λ, or $X_\lambda s_j$ being a standard one is equivalent to the union of sets of relations (9.4) and (9.5) for standard X_λ only.

For the rest of this section we shall reserve a notation $c(X_\lambda|j)$ for the content of the j-th node of the tableau X_λ. One can give an alternative expression for the numerical coefficients in the above formulas in terms of contents of the nodes:

$$\frac{q^{-\ell_{j,j+1}}}{[\ell_{j,j+1}]_q} \;=\; (q-q^{-1})\frac{c(X_\lambda|j+1)}{c(X_\lambda|j)-c(X_\lambda|j+1)}\ , \tag{9.6}$$

$$-\frac{q^{\ell_{j,j+1}}}{[\ell_{j,j+1}]_q} \;=\; (q-q^{-1})\frac{c(X_\lambda s_j|j+1)}{c(X_\lambda s_j|j)-c(X_\lambda s_j|j+1)}\,. \tag{9.7}$$

Let us comment on the meaning of the relations (9.4).

For the standard tableaux X_λ and $X_\lambda s_j$ the elements $P(X_\lambda)$ and $P(X_\lambda s_j)$ are idempotents like the diagonal matrix units, say, e_{ii} and e_{kk}, in the matrix algebra (see Section 2). The equation (9.4) states that the element $P(X_\lambda|X_\lambda s_j)$ behaves as $P(X_\lambda)$ from the left and, at the same time, as $P(X_\lambda s_j)$ from the right. Thus, the elements $P(X_\lambda|X_\lambda s_j)$ have the nature of (non-normalized) off-diagonal matrix units e_{ik} and e_{ki}, $i \neq k$. In particular, analogues of relations $e_{ik}e_{ki} = e_{ii}$ and $e_{ki}e_{ik} = e_{kk}$ (see (2.7)) hold

$$P(X_\lambda|X_\lambda s_j)\ P(X_\lambda s_j|X_\lambda) \;=\; \left(1-\frac{1}{[\ell_{j,j+1}]_q^2}\right) P(X_\lambda)\ , \tag{9.8}$$

$$P(X_\lambda s_j|X_\lambda)\ P(X_\lambda|X_\lambda s_j) \;=\; \left(1-\frac{1}{[\ell_{j,j+1}]_q^2}\right) P(X_\lambda s_j)\ . \tag{9.9}$$

These relations follow by application of the eqs. (9.4), (9.5) and an equality

$$(\sigma + \frac{q^{-\ell}}{[\ell]_q})(\sigma - \frac{q^{\ell}}{[\ell]_q}) = 1 - \frac{1}{{[\ell]_q}^2} = \frac{[\ell+1]_q[\ell-1]_q}{{[\ell]_q}^2}\ , \tag{9.10}$$

which is satisfied for any element σ enjoying the Hecke condition (4.3), and any natural l.

A direct inspection shows that one of the tableaux, X_λ or $X_\lambda s_j$, in the relation (9.4) is not standard if and only if the nodes j and $(j+1)$ are neighbors, *i.e.*, they are situated in the tableau X_λ (or $X_\lambda s_j$) as $\begin{array}{|c|c|}\hline \mathrm{j} & \mathrm{j{+}1} \\ \hline\end{array}$ or $\begin{array}{|c|}\hline \mathrm{j} \\ \hline \mathrm{j{+}1} \\ \hline\end{array}$. We have $\ell_{j,j+1} = \pm 1$ in this case.

Therefore, when both tableaux X_λ and $X_\lambda s_j$ are standard, the numerical coefficients in the right hand side of the eqs. (9.8) and (9.9) are always different from zero ($\ell_{j,j+1} \neq \pm 1$).

When one of the tableaux, X_λ or $X_\lambda s_j$, is not standard, the relation (9.4) does not produce new matrix units (both $P(X_\lambda|X_\lambda s_j)$ and $P(X_\lambda s_j|X_\lambda)$ vanish); it just guarantees the closure of the action of the generators σ_j, $j = 1, \ldots, n-1$, on the set of idempotents labeled by the standard tableaux.

We found the (non-normalised) off-diagonal matrix units $P(X_\lambda|X_\lambda s_j)$ corresponding to pairs of nearest standard Young tableaux X_λ and $X_\lambda s_j$, *i.e.* the tableaux related by the elementary permutations s_j.

We are now going to define all other matrix units, for an arbitrary permutation $\pi \in \mathcal{S}_n$. The relations from which the correctness of the definition will follow are collected in a proposition below.

Proposition. Let σ and σ' denote any conseqtive pair of generators of the algebra $\mathcal{H}_n(q)$ (*i.e.*, σ_j and σ_{j+1} for $j = 1, \ldots, n-2$). Define elements $\sigma(\alpha, \beta)$ and $\sigma'(\alpha, \beta)$ as

$$\sigma(\alpha, \beta) := \sigma + (q - q^{-1}) \frac{\beta}{\alpha - \beta} \tag{9.11}$$

and the same for $\sigma'(\alpha, \beta)$.
The following relations hold

$$\sigma(\alpha, \beta)\sigma(\beta, \alpha) = f(\alpha, \beta) f(\beta, \alpha) \ , \quad \text{where} \quad f(\alpha, \beta) := \frac{q\alpha - q^{-1}\beta}{\alpha - \beta} \ , \tag{9.12}$$

$$\sigma(\alpha, \beta)\sigma'(\alpha, \gamma)\sigma(\beta, \gamma) = \sigma'(\beta, \gamma)\sigma(\alpha, \gamma)\sigma'(\alpha, \beta) \ . \tag{9.13}$$

The operation of constructing the operator $\sigma(\alpha, \beta)$ from the operator σ is called the "baxterisation" of the Hecke element σ.

The relation (9.12) is a different parametrization of (9.10) (take $\alpha = q^{2a}$, $\beta = q^{2b}$ in (9.12) and then put $a - b = \ell$). The relation (9.12) is called the "unitarity condition" for $\sigma(\alpha, \beta)$.

The equation (9.13) is a version of the "Yang-Baxter equation with a spectral parameter" (the braid relation of the type (4.1) is often called the Yang-Baxter equation).

Now, observe that each side of relation (9.4) is a product of two factors, one of them is a primitive idempotent $P(X)$ and the other one is a "baxterized" element $\sigma_j(\alpha, \beta)$ whose arguments α and β are chosen to be just the contents of the j-th and $(j+1)$-th nodes of the idempotent Young tableau X (c.f. with (9.6), (9.7) and (9.11)).

Consider a pair X_λ and Y_λ of standard Young tableaux of the same shape $\lambda \vdash n$. Then there exists a permutation $\pi \in \mathcal{S}_n$ converting X_λ into Y_λ : $Y_\lambda = X_\lambda \pi$. Let

$$\pi = s_{i_1} s_{i_2} \ldots s_{i_k} \tag{9.14}$$

be some expression for π in terms of the elementary permutations s_i, $i = 1, \ldots, n-1$. Then one can find an appropriate monomial

$$\sigma_{i_1}(\alpha_1, \beta_1)\sigma_{i_2}(\alpha_2, \beta_2) \ldots \sigma_{i_k}(\alpha_k, \beta_k)$$

in the "baxterized" elements such that a repeated application of the relation (9.4) results in

$$
\begin{aligned}
P_{i_1,i_2,\ldots,i_n}(X_\lambda|Y_\lambda) \;&:=\; P(X_\lambda)\Big(\sigma_{i_1}(\alpha_1,\beta_1)\sigma_{i_2}(\alpha_2,\beta_2)\ldots\sigma_{i_k}(\alpha_k,\beta_k)\Big) \\
&= \;\Big(\sigma_{i_1}(\beta_1,\alpha_1)\sigma_{i_2}(\beta_2,\alpha_2)\ldots\sigma_{i_k}(\beta_k,\alpha_k)\Big)P(Y_\lambda)\ . \qquad (9.15)
\end{aligned}
$$

Here the parameters α_i, β_i of baxterized elements are chosen as follows:

$$
\begin{array}{ll}
\alpha_1 = c(X_\lambda|i_1)\ , & \beta_1 = c(X_\lambda|i_1+1)\ , \\
\alpha_2 = c(X_\lambda s_{i_1}|i_2)\ , & \beta_2 = c(X_\lambda s_{i_1}|i_2+1)\ , \\
\ldots & \ldots \\
\alpha_k = c(X_\lambda s_{i_1}s_{i_2}\ldots s_{i_{k-1}}|i_k)\ , & \beta_k = c(X_\lambda s_{i_1}s_{i_2}\ldots s_{i_{k-1}}|i_k+1)\ .
\end{array}
\qquad (9.16)
$$

There are many different expressions for π in terms of elementary permutations. The corresponding quantities of the type (9.15) are proportional. To prove this it is enough to check the basic relations, that is, to consider permutations $\pi_1 := s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $\pi_2 := s_i s_j = s_j s_i$, $|i-j| > 1$ and $\pi_3 := s_i s_i = id$.

For the permutation π_2 one clearly has $P_{i,j}(X_\lambda|X_\lambda\pi_2) = P_{j,i}(X_\lambda|X_\lambda\pi_2)$. Fot π_1 the identity $P_{i,i+1,i}(X_\lambda|X_\lambda\pi_1) = P_{i+1,i,i+1}(X_\lambda|X_\lambda\pi_1)$ follows from the Yang-Baxter equation with spectral parameter (9.13)[7]. For π_3 the two corresponding equalities $P_{id}(X_\lambda|X_\lambda) \equiv P(X_\lambda)$ and $P_{i,i}(X_\lambda|X_\lambda)$ are related by a numerical factor due to the eq. (9.12).

Note that proportional results appear only when we use π_3. Therefore, if two expressions (9.14) are obtained by relations of types π_1 and π_2 only, the corresponding results (9.15) coincide.

At this point it becomes clear how to establish the matrix structure inside the Hecke algebra. To do this one considers a restricted set of formulae (9.15) and (9.16) where both X_λ and Y_λ are standard Young tableaux and chooses

[7]A. Isaev pointed out that this consistency follows easily from an observation [Is] that elements $A_i = \sigma_i K_i - K_i\sigma_i$ satisfy the braid relations, $A_iA_{i+1}A_i = A_{i+1}A_iA_{i+1}$. On an idempotent $P(X_\lambda)$, one can replace the operator K_i by a content and one obtains, up to a factor, exactly the combination

$$\sigma_j + (q-q^{-1})\frac{c(X_\lambda|j+1)}{c(X_\lambda|j) - c(X_\lambda|j+1)}\ .$$

a *reduced* expression (9.14) for the permutation $\pi : X_\lambda \mapsto Y_\lambda$, *i.e.* the one containing exactly $l(\pi)$ terms. Then, the expressions $P_{\dots}(X_\lambda|Y_\lambda)$ given by the formula (9.15) for different reduced expressions of π appear to be equal to one and the same quantity, which is natural to denote as $P(X_\lambda|Y_\lambda)$. The elements $P(X_\lambda|X_\lambda s_j)$ given in the basic relations (9.4) are particular representatives of the family defined above. The elements $P(X_\lambda|Y_\lambda)$ are non-normalized off-diagonal matrix units. To obtain matrix units satisfying exactly the matrix algebra relations (2.7) one should perform rescalings of the form

$$P_{\text{norm}}(X_\lambda|Y_\lambda) := \prod_{p=1}^{k} f^{-1}(\alpha_p, \beta_p)\ P(X_\lambda|Y_\lambda)\ , \tag{9.17}$$

where the set of parameters α_p, β_p, $p = 1, \dots, k$ is taken from (9.16) for some reduced expression (9.14) of the permutation $\pi : X_\lambda \mapsto Y_\lambda$; the function $f(\alpha, \beta)$ is defined in (9.13). Note, that all factors $f(\alpha_p, \beta_p)$ in this formula are different from the zero and infinity for generic values of q and if X_λ and Y_λ are chosen to be standard and the reduced expression for the permutation π is used.

The matrix $M_\lambda := ||P_{\text{norm}}(X_\lambda|Y_\lambda)||$ (note that $P_{\text{norm}}(X_\lambda|X_\lambda) = P(X_\lambda)$) whose rows and columns are labeled, respectively, by standard Young tableaux X_λ and Y_λ of the same shape $\lambda \vdash n$, gives a realisation of the matrix algebra $M_{\dim V_\lambda}(\mathbb{C}) \subset \mathcal{H}_n(q)$, the value of $\dim V_\lambda$ is given by the hook formula (5.4). The elements

$$P_{\text{norm}}(X_\lambda|Y_\lambda) \subset P(X_\lambda)\ \mathcal{H}_n(q)\ P(Y_\lambda)$$

are components of the two-sided Peirce decomposition. They form a linear basis in an ideal generated by the central idempotent $P([\lambda])$. Finally,

$$\mathcal{H}_n(q) = \bigoplus_{\lambda \vdash n} M_{\dim V_\lambda}(\mathbb{C})\ .$$

Now we can construct irreducible representations of the Hecke algebra $\mathcal{H}_n(q)$. Indeed the elements of each column (row) of the matrix M_λ can serve as basic vectors for the irreducible left (right) module V_λ, the action of $\mathcal{H}_n(q)$ is given by a multiplication from the left (right). An action of elementary generators σ_i on the module V_λ can be obtained with the use of relations (9.4). Instead of following this way, we shall describe a somewhat different method in Section 11.

10 Application: Temperley–Lieb algebras

After the matrix structure of some semisimple algebra is understood, it is straightforward to see what are the possible factor-algebras of the algebra in question, or, in other words, which additional relations one can impose on the generators. The factor-algebras are in a one-to-one correspondence with subsets of the set of matrix summands of the original algebra, so the possible additional relations kill the rest of the matrix summands (that is, those summands which do not belong to a chosen subset). To factor out a matrix summand, say M, it is enough to choose a nonzero element $v \in M$ and to impose $v = 0$: the matrix algebra is simple, so the ideal, generated by v is the whole matrix summand.

For instance, given a natural number k, we can keep in a factor-algebra of $\mathcal{H}_n(q)$ only those matrix summands which correspond to the Young diagrams with less than $k+1$ rows – denote this factor-algebra by $\mathcal{H}_{n,k}(q)$. The algebras $\mathcal{H}_{n,k}$ are studied in the book [Mar].

To this end, if $n = k+1$ we should forbid only one diagram, corresponding to the q-antisymmetrizer. This matrix summand is one-dimensional, there is only one standard tableau of the shape of a single column, with numbers $1, 2, \ldots, k+1$ in increasing order. Denote this tableau by X_a.

Thus, for $n = k+1$, the additional relation is $P_{A(k+1)} = 0$.

Assume now that $n \geq k+1$. Denote by $P_{A(k+1)}\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ the q-antisymmetrizer of the subalgebra $\mathcal{H}_{k+1}(q) \subset \mathcal{H}_n(q)$, generated by $\sigma_1, \ldots, \sigma_k$. Let $\mathcal{N}_a$ be the set of all standard tableaux X for which $X_a \,|\, X$, that is, X contains the tableau X_a in the first column. By (7.24), we have $\sum_{X \in \mathcal{N}_a} P(X) = P_{A(k+1)}\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$. Since in the factor-algebra $\mathcal{H}_{n,k}(q)$ we have $P(X) = 0$ for all X with more than k rows, we must have

$$P_{A(k+1)}\{\sigma_1, \sigma_2, \ldots, \sigma_k\} = 0 \; . \tag{10.18}$$

On the other hand, consider a matrix summand M_λ, corresponding to a diagram λ with more than $k+1$ rows. This matrix summand contains idempotents for all the standard tableaux of the shape λ. Among these tableaux there is a tableau X such that $X_a \,|\, X$. By (7.23), the idempotent $P(X)$ is equal to $P_{A(k+1)}\{\sigma_1, \sigma_2, \ldots, \sigma_k\}\, P(X)$. Therefore, the condition $P_{A(k+1)}\{\sigma_1, \sigma_2, \ldots, \sigma_k\} = 0$ implies $P(X) = 0$ which factors out the matrix summand M_λ.

Conclusion: to obtain the algebra $\mathcal{H}_{n,k}(q)$, one has to add to the Hecke algebra relations (4.1)-(4.3) one more relation (10.18).

Note that for each string $\{\sigma_j, \ldots, \sigma_{j+k-1}\}$ of generators, the q-antisymmetrizer corresponding to the subalgebra $\mathcal{H}_{k+1}(q) \subset \mathcal{H}_n(q)$ generated by the elements $\sigma_j, \ldots, \sigma_{j+k-1}$ – denote it $P_{A(k+1)}\{\sigma_j, \ldots, \sigma_{j+k-1}\}$ – vanishes in the factor-algebra $\mathcal{H}_{n,k}(q)$. Indeed, for an element $x = \sigma_1\sigma_2 \ldots \sigma_{k+1}$ one has $x\sigma_i = \sigma_{i+1}x$ for $i = 1, 2, \ldots, k$. Therefore, the conjugation with x moves the string $\{\sigma_1, \ldots, \sigma_k\}$ to $\{\sigma_2, \ldots, \sigma_{k+1}\}$. Composing such conjugations, one can move the string $\{\sigma_1, \ldots, \sigma_k\}$ to $\{\sigma_j, \ldots, \sigma_{j+k-1}\}$. Thus, (10.18) implies $P_{A(k+1)}\{\sigma_j, \ldots, \sigma_{j+k-1}\} = 0$.

The important case is $k = 2$. The additional relations then involve only pairs of neighboring generators σ_j and σ_{j+1}:

$$P_{A(3)}\{\sigma_j, \sigma_{j+1}\} = 0 \;, \qquad \text{for} \quad j = 1, \ldots, n-2 \;. \tag{10.19}$$

The q-antisymmetrizer $P_{A(3)}\{\sigma_1, \sigma_2\}$ is given by (7.9), or (7.13). Substituting $\sigma_1 \mapsto \sigma_j$ and $\sigma_2 \mapsto \sigma_{j+1}$ in the expression for $P_{A(3)}\{\sigma_1, \sigma_2\}$ one obtains the expression for any q-antisymmetrizer $P_{A(3)}\{\sigma_j, \sigma_{j+1}\}$. The factor-algebra (10.19) is usually written in terms of the q-antisymmetrizers $P_{A(2)}\{\sigma_i\} = (q - \sigma_i)/[2]_q =: a_i$ (as in (7.3)). The relations (10.19) then take a form

$$a_i a_{i\pm 1} a_i = \frac{1}{[2]_q^2} a_i \;, \quad i = 1, \ldots, n-1. \tag{10.20}$$

Together with the conditions

$$a_i^2 = a_i \;, \quad \text{for} \quad i = 1, \ldots, n-1 \;, \tag{10.21}$$
$$a_i a_j = a_j a_i \;, \quad \text{for} \quad |i - j| \geq 2 \;, \tag{10.22}$$

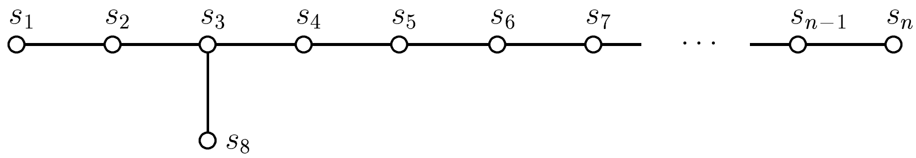
which are implied by the Hecke algebra relations[8], the eqs. (10.20) define the *Temperley-Lieb algebra* [TL]. Its dimension equals the Catalan number $\dfrac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$.

Remark.

One can define the Temperley-Lieb algebra for any Coxeter graph Γ. Let $\mathcal{H}(\Gamma)$ be the Hecke algebra for Γ. Consider an edge l of the graph

[8]Note that the braid relations (4.1) for $\sigma_i = q - [2]_q a_i$ follow from (10.20) and (10.21).

whose vertices are labelled by generators σ and σ'. Let $\mathfrak{h}_l$ be a subalgebra generated by σ and σ'. We have $\sigma^2 = 1 + (q - q^{-1})\sigma$ and $\sigma'^2 = 1 + (q' - q'^{-1})\sigma'$. The algebra $\mathfrak{h}_l$ has a representation $A(\mathfrak{h}_l)$: $\sigma \mapsto -q^{-1}$, $\sigma' \mapsto -q'^{-1}$. Let $P_{A(\mathfrak{h}_l)} \in \mathfrak{h}_l \subset \mathcal{H}(\Gamma)$ be the corresponding idempotent. Then the factor-algebra of $\mathcal{H}(\Gamma)$ by the relations $P_{A(\mathfrak{h}_l)} = 0$ for all edges l is called the Temperley-Lieb algebra $\mathcal{TL}(\Gamma)$ of the graph Γ. It is known that $\mathcal{TL}(\Gamma)$ can be finite-dimensional even if the Hecke algebra $\mathcal{H}(\Gamma)$ is infinite-dimensional. Example: the graph E_n, with $n > 8$:



Graphs Γ for which $\mathcal{TL}(\Gamma)$ has finite dimension were classified in [G]. In the simply-laced case the classification results in three families, A_n, D_n $(n > 3)$ and E_n $(n > 5)$.

11 Representations

In this Section we shall build explicitly representations V_λ. The basis of the representation V_λ will be labeled by standard Young tableaux X of the shape $[\lambda]$. In the representation V_λ corresponding to $[\lambda]$ the idempotents $P(X)$ become diagonal matrices with 1 on the place corresponding to the tableau X and 0 on other places.

11.1 Construction

As we stressed earlier, the whole information about representations (or the matrix structure) of the algebra $\mathcal{H}_n(q)$ is contained in the relations (9.4).

After formulas (9.4) are written we can take a different point of view. It turns out that one can forget that primitive idempotents $P(X_\lambda)$ in these formulas are built from the generators σ_i. More precisely, consider a set of generators $\mathcal{X}_\lambda$ labeled by standard Young tableaux X_λ of the shape $[\lambda]$.

Proposition. The relations

$$\Big(\sigma_j + \frac{q^{-\ell_{j,j+1}}}{[\ell_{j,j+1}]_q}\Big)\,\mathcal{X}_\lambda = \mathcal{X}_\lambda s_j\,\Big(\sigma_j - \frac{q^{\ell_{j,j+1}}}{[\ell_{j,j+1}]_q}\Big) \tag{11.1}$$

are compatible with the Hecke algebra relations on the generators σ_i. Here the element $\mathcal{X}_\lambda s_j$ corresponds to a tableau $X_\lambda s_j$ in case if $X_\lambda s_j$ is standard and we put $\mathcal{X}_\lambda s_j = 0$ otherwise; the length $\ell_{j,j+1}$ is calculated for the tableau X_λ; as for relations (9.4), the nontrivial relations among (11.1) are only those for which either X_λ or $X_\lambda s_j$ is standard.

In other words, moving elements $\mathcal{X}_\lambda$ from the right to the left according to the rule (11.1) preserves the set of the relations (4.1), (4.2) and (4.3).

To verify this statement one has to write down the relation, similar to (9.15), which follows from (11.1):

$$\begin{aligned}&\mathcal{X}_\lambda\Big(\sigma_{i_1}(\alpha_1,\beta_1)\sigma_{i_2}(\alpha_2,\beta_2)\dots\sigma_{i_k}(\alpha_k,\beta_k)\Big)\\ &= \Big(\sigma_{i_1}(\beta_1,\alpha_1)\sigma_{i_2}(\beta_2,\alpha_2)\dots\sigma_{i_k}(\beta_k,\alpha_k)\Big)\mathcal{Y}_\lambda\ .\end{aligned} \tag{11.2}$$

We have to check that if one applies a relation from the Hecke algebra to transform the left hand side of the equality (11.2), the corresponding right hand side gets transformed by a Hecke algebra relation as well.

It is straightforward for the relations (4.2) and (4.3).

As for the relation (4.1), the statement follows from an observation that if $\sigma(\alpha,\beta)$ satisfies the Yang-Baxter equation with the spectral parameter, then $\sigma(\beta,\alpha)$ satisfies the Yang-Baxter equation with the spectral parameter as well. Indeed, if $\sigma(\alpha,\beta)$ satisfies (9.13) then its inverse $\sigma(\alpha,\beta)^{-1}$ also satisfies (9.13). It is left to notice that $\sigma(\alpha,\beta)^{-1}$ is proportional to $\sigma(\beta,\alpha)$ due to (9.12).

Note, that we have checked not the relation (4.1) itself but its analogue with the spectral parameter. However, one can see directly that the Yang-Baxter equation for a Hecke generator σ is completely equivalent to the Yang-Baxter equation with the spectral parameter for $\sigma(\alpha,\beta)$.

Using (11.1) we shall build representations of the Hecke algebra in a way similar to the way of building the representation of canonical commutation relations $p_i x^j - x^j p_i = \delta_i^j$. Recall that using the canonical commutation relations, one moves all momenta p_i to the right. Then, one introduces a "vacuum state" $|\rangle$ annihilated by momenta, $p_i|\rangle = 0$, and obtains a representation on the space of polynomials in x^j. The vacuum state is a one-dimensional representation of the algebra of momenta.

In order to proceed in the same manner in the Hecke algebra case we shall need to sort the relations (11.1) into three parts.

Denote by $\mathcal{J}_0$ the set of those relations (11.1) for which both tableaux, X_λ and $X_\lambda s_j$ are standard; by $\mathcal{J}_+$ those relations for which X_λ is standard while $X_\lambda s_j$ is not; by $\mathcal{J}_-$ those relations for which X_λ is not standard while $X_\lambda s_j$ is. In what follows we shall use only two subsets – $\mathcal{J}_0$ and $\mathcal{J}_+$ – for construction of the simple left $\mathcal{H}_n(q)$-modules V_λ [9].

Now we are prepared to build representations.

To build a representation of the Hecke algebra, it is enough to write the action of generators σ_i.

Consider an expression of the form $a\mathcal{X}_\lambda b$, where a and b are elements of the Hecke algebra, $a, b \in \mathcal{H}_n(q)$. We use relations $\mathcal{J}_0 \bigcup \mathcal{J}_+$ to move all generators σ_j to the right. We need then a vacuum state — a one-dimensional representation of $\mathcal{H}_n(q)$ (in a one-dimensional representations all elements of the algebra become numbers). There are two one-dimensional representations, (8.6) and (8.7), of the Hecke algebra $\mathcal{H}_n(q)$. Both of them serve equally well for our purposes and we take a representation given by the q-symmetrizer. Thus all the σ_i should be evaluated as $\sigma_i = q$ on the right. In other words, our vacuum state is defined by $\sigma_i|\rangle = q|\rangle$. The construction of the representation is finished.

To sum up, the construction of irreducible representations V_λ is given as follows.

We interpret $\mathcal{X}_\lambda$ as basis vectors of the vector space V_λ. Note that to construct the representation in V_λ, we move σ's through one element $\mathcal{X}$. After this move we set σ_j to q. This amounts to substituting $\sigma_j \mapsto q$ in the right hand sides of relations $\mathcal{J}_0 \bigcup \mathcal{J}_+$. So we can immediately write down the matrices of σ_j in the basis $\{\mathcal{X}_\lambda\}$:

$$\sigma_j : \mathcal{X}_\lambda \mapsto -\frac{q^{-\ell_{j,j+1}}}{[\ell_{j,j+1}]_q}\,\mathcal{X}_\lambda + \frac{[\ell_{j,j+1}-1]_q}{[\ell_{j,j+1}]_q}\,\mathcal{X}_\lambda s_j \ . \tag{11.3}$$

[9] Alternatively, to construct the simple right $\mathcal{H}_n(q)$-modules V_λ, we use $\mathcal{J}_0$ and $\mathcal{J}_-$.

Here, as before, one sets $\mathcal{X}_\lambda s_j$ to 0 if the tableau X_λ is not standard; and, as before, $\ell_{j,j+1}$ is the length, calculated for the tableau X_λ, $\ell_{j,j+1} = \ell_{j,j+1}(X_\lambda)$.

Note that in the basis $\{\mathcal{X}_\lambda\}$ of standard tableaux, the matrix of the element $\sigma_1 \in \mathcal{H}_q(n)$ is diagonal for arbitrary n and any partition λ of n.

A short comment on the meaning of the procedure for constructing the representations. In the formulation of the Proposition in this Section, the Young tableaux $\mathcal{X}_\lambda$ are treated as independent symbols on which the generators σ_i can be multiplied from left and from the right. In other words, we start with a tautological $\mathcal{H}_n(q)$-bimodule $\mathcal{H}_n(q) \otimes_{\mathbb{C}} V_\lambda \otimes_{\mathbb{C}} \mathcal{H}_n(q)$, and then impose the relations $\mathcal{J}_0 \bigcup \mathcal{J}_+$. These relations allow to translate the left component $\mathcal{H}_n(q)$ of the space $\mathcal{H}_n(q) \otimes_{\mathbb{C}} V_\lambda \otimes_{\mathbb{C}} \mathcal{H}_n(q)$ to the right of V_λ and, thus, we obtain an $\mathcal{H}_n(q)$-bimodule structure on the factor space $V_\lambda \otimes_{\mathbb{C}} \mathcal{H}_n(q)$. Denote this bimodule by W_λ. Relations $\mathcal{J}_0 \bigcup \mathcal{J}_+$ ensure that $Z_n \mathcal{X}_\lambda = c(\lambda)\mathcal{X}_\lambda$, where $c(\lambda)$ is the sum of contents of all nodes of the diagram λ. Moreover, for any two standard tableaux X_λ and $Y_{\lambda'}$ we have $P(X_\lambda)\mathcal{Y}_{\lambda'} = \mathcal{Y}_{\lambda'}$ if $X_\lambda = Y_{\lambda'}$ and 0 otherwise.

Finally, we build a tensor product, over $\mathcal{H}_n(q)$, of the bi-module W_λ and the 1-dimensional left module $S(n)$,

$$V_\lambda := W_\lambda \bigotimes_{\mathcal{H}_n(q)} S(n) . \tag{11.4}$$

The result is a left $\mathcal{H}_n(q)$-module V_λ with a linear basis given by all standard tableaux X_λ.

Remarks.

1. Formulas (11.3) show that for representation V_λ for any $\lambda \vdash n$, the entries of matrices of the generators σ_i in the basis of X_λ are rational functions of q; coefficients of numerators and denominators of these functions are integer numbers. Therefore, the entries of matrices corresponding to words (4.5) are rational functions with integer coefficients as well. In particular, for $q = 1$, the entries become rational numbers. Therefore, all representations of the symmetric group $\mathcal{S}_n$ are defined over the field $\mathbb{Q}$ of rational numbers.

   Note that the Frobenius character formula shows that the value of any character χ_λ on any permutation π is an integer number.

2. Under the operation of transposition of Young tableaux, the oriented lengths change the sign and the matrices of σ_j's undergo the transformation $q \mapsto -q^{-1}$ (compare with (7.25) and (7.26)).

11.2 Affine Hecke algebra

One may ask, if the Burau representation (8.8) can be obtained in the spirit of the Proposition in this Section. The answer is positive. Consider a set of variables $\{y_i, i = 1, \ldots, n\}$. The following relations are compatible with the Hecke algebra relations:

$$\begin{aligned} \sigma_i y_j &= y_j \sigma_i \ , \qquad\qquad \text{if } j \neq i, i+1 \ , \\ \sigma_i y_i &= y_{i+1}(\sigma_i - (q - q^{-1})) \ , \\ \sigma_i y_{i+1} &= y_i \sigma_i + (q - q^{-1}) y_{i+1} \ . \end{aligned} \tag{11.5}$$

Using these relations to move all σ's to the right and then evaluating on the vacuum state S, one obtains precisely the Burau representation (8.8).

The Burau representation has a 1-dimensional subrepresentation with the basic vector $\zeta_n = y_1 + y_2 + \ldots + y_n$. The quotient representation is also irreducible, it is isomorphic to the representation V_λ corresponding to the partition $\lambda = \{n-1, 1\} \vdash n$.

Formulas (11.5) together with the Hecke algebra relations (4.1)–(4.3) define an algebra with the set of the Hecke generators σ_j extended by free generators y_i ("free" means that one does not specify any relations between y_i's). One may further restirct this algebra[10] by demanding the commutativity for y_i's: $y_i y_j = y_j y_i$ for all i and j. One demands also that y_i are invertible, that

[10]In a similar way, one may consider an algebra with the Hecke generators σ_j and free variables $\mathcal{X}_\lambda$, subjected to the relations (11.1), and ask what are possible relations for elements $\mathcal{X}_\lambda$. For instance, to analyze possible quadratic relations, use (11.1) to move σ_j through the product $\mathcal{X}\mathcal{Y}$

$$\sigma_j \mathcal{X}\mathcal{Y} = -\frac{q^{-\ell_X}}{[\ell_X]_q}\mathcal{X}\mathcal{Y} - \frac{[\ell_X + \ell_Y]_q}{[\ell_X]_q[\ell_Y]_q}(s_j\mathcal{X})\mathcal{Y} + (s_j\mathcal{X})(s_j\mathcal{Y})(\sigma_j - \frac{q^{\ell_Y}}{[\ell_Y]_q}) \ . \tag{11.6}$$

Here ℓ_X (resp., ℓ_Y) is the length $\ell_{j,j+1}$ calculated for the tableau X (resp., Y).

Eqn. (11.6) has to be compatible with possible relations. A consistent choice is $\mathcal{X}_\lambda \mathcal{Y}_\lambda = \mathcal{X}_\lambda$ if $\mathcal{X}_\lambda$ coincides with $\mathcal{Y}_\lambda$ and $\mathcal{X}_\lambda \mathcal{Y}_\lambda = 0$ otherwise, which is, of course, in accordance with relations (9.4) where the role of $\mathcal{X}_\lambda$ is played by the idempotents $P(X_\lambda)$.

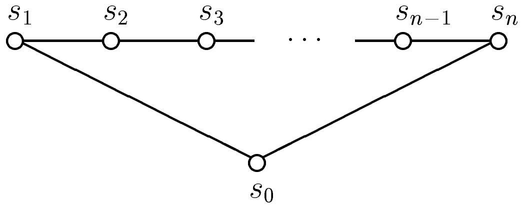
is, y_i^{-1} exists for each i. This algebra is called the *affine* Hecke algebra and denoted $\hat{\mathcal{H}}_n(q)$ (see [ChP], Sec. 12.3, and references therein). Clearly $\mathcal{H}_n(q)$ is contained in $\hat{\mathcal{H}}_n(q)$ as a subalgebra generated by elements σ_j, $j = 1, \ldots, n-1$. The last two lines in (11.5) can be rewritten as $\sigma_i y_i \sigma_i = y_{i+1}$. A comparison with (6.1) shows that there is a homomorphism (called the *evaluation map*), identical on σ_i's, $\hat{\mathcal{H}}_n(q) \to \mathcal{H}_n(q)$: $y_i \mapsto J_i$. The image of the invariant vector ζ_n under this map is the central idempotent Z_n.

Note that the affine Hecke algebra is not the same as the Hecke algebra of the affine Coxeter graph. We shall end this section by a brief account of interplay between these objects. We shall do it on the "classical level", that is, for $q = 1$, on the simplest example of the symmetric group $\mathcal{S}_3$ – the Coxeter group of the graph A_2.

We briefly remind the necessary notions.

The affine Coxeter graph $\hat{A}_n$ for the A_n is

$\hat{A}_n$ $(n>1)$



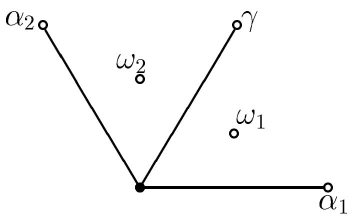
(11.7)

For $n = 2$, the Coxeter group $\mathcal{W}_a$ for this graph has three generators s_0, s_1 and s_2 with $s_i^2 = e$ and

$$s_1 s_2 s_1 = s_2 s_1 s_2 \ , \ \ s_2 s_0 s_2 = s_0 s_2 s_0 \ , \ \ s_0 s_1 s_0 = s_1 s_0 s_1 \ . \tag{11.8}$$

The subset $\{s_1, s_2\}$ generates the Coxeter group $\mathcal{W} \simeq \mathcal{S}_3$.

With the graph A_2 one associates two lattices on the euclidean plane E^2, the root lattice Q, with a basis $\{\alpha_1, \alpha_2\}$, and the weight lattice P, with a basis $\{\omega_1, \omega_2\}$, as shown on the picture:



In an orthonormal basis e_x and e_y on the plane, $\alpha_1 = \sqrt{2}e_x$, $\alpha_2 = -\frac{1}{\sqrt{2}}e_x + \frac{\sqrt{3}}{\sqrt{2}}e_y$, $\omega_1 = \frac{1}{\sqrt{2}}e_x + \frac{1}{\sqrt{6}}e_y$ and $\omega_2 = \frac{\sqrt{2}}{\sqrt{3}}e_y$.

The basis $\{\alpha_1, \alpha_2\}$ is dual to the basis $\{\omega_1, \omega_2\}$, $(\alpha_i, \omega_j) = \delta_{ij}$. The matrix $(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is called the Cartan matrix, the matrix $(\omega_i, \omega_j) = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is its inverse.

The vector $\gamma = \alpha_1 + \alpha_2$ is called the highest root. For A_2 one has $\gamma = \omega_1 + \omega_2$.

The group $\mathcal{W}$ acts on the plane E^2, the action of the generators s_i, $i = 1, 2$, is given by reflections in the roots α_i, $s_i(v) : v \mapsto v - (\alpha_i, v)\alpha_i$. This action preserves lattices Q and P.

If a group $\mathcal{G}$ acts on a lattice L, one can form a semi-direct product, $\mathcal{G} \ltimes L$. This is a group generated by $\mathcal{G}$ and translations on vectors from L. A "normal form" of elements in the semidirect product $\mathcal{G} \ltimes L$: any element $x \in \mathcal{G} \ltimes L$ can be uniquely written as $x = h\overline{x}$, where h is a lattice element and $\overline{x} \in \mathcal{G}$.

If $\mathcal{G}$ acts on two lattices, L_1 and L_2, then any homomorphism $L_1 \to L_2$ induces a homomorphism $\mathcal{G} \ltimes L_1 \to \mathcal{G} \ltimes L_2$.

Assume that a group $\mathcal{G}$ acts on a lattice L and $\mathcal{G} \ltimes L$ is the corresponding semi-direct product. Denote by l_i basis vectors of L and by ℓ_i – translations on vectors l_i. For any integer N denote by NL a lattice with basis vectors $l_i^{(N)} = Nl_i$. The lattice NL is a sublattice in L. On the other hand, the lattices L and NL are isomorphic, the isomorphism is given by $l_i \mapsto l_i^{(N)}$. This induces a homomorphism, denoted by ι_N, from the group $\mathcal{W} \ltimes L$ to itself, $\iota_N : \mathcal{W} \ltimes L \to \mathcal{W} \ltimes L$, given by $t_i \mapsto t_i^N$.

In our situation, we can form two semi-direct products, $\mathcal{W} \ltimes Q$ and $\mathcal{W} \ltimes P$.

The group $\mathcal{W} \ltimes Q$.

Denote by t_1 and t_2 translations on vectors α_1 and α_2 respectively. The relations in the group $\mathcal{W} \ltimes Q$ are given by the action of the Coxeter group $\mathcal{W}$ on the lattice Q:

$$s_1 t_1 s_1 = t_1^{-1} \ , \ s_2 t_1 s_2 = t_1 t_2 \ , \ s_1 t_2 s_1 = t_1 t_2 \ , \ s_2 t_2 s_2 = t_2^{-1} \ , \tag{11.9}$$

together with the relation $t_1 t_2 = t_2 t_1$.

Denote $t = t_1 t_2$ (a translation on the vector γ). Then the set of defining relations is

$$s_1 t s_1 t = t s_1 t s_1 \ , \ \ (s_1 t s_1)(s_2 t s_2) = t \ , \ \ s_1 s_2 s_1 t = t^{-1} s_1 s_2 s_1 \ . \tag{11.10}$$

The groups $\mathcal{W}_a$ and $\mathcal{W} \ltimes Q$.

It is straightforward to see that the map $s_0 \mapsto s_1 s_2 s_1 t$ is an isomorphism between $\mathcal{W}_a$ and $\mathcal{W} \ltimes Q$.

The group $\mathcal{W} \ltimes P$.

Denote by u_1 and u_2 translations on vectors ω_1 and ω_2 respectively. The relations in the group $\mathcal{W} \ltimes P$ are given by the action of the Coxeter group $\mathcal{W}$ on the lattice P:

$$s_1 u_1 s_1 = u_1^{-1} u_2 \ , \ \ s_1 u_2 s_1 = u_2 \ , \ \ s_2 u_1 s_2 = u_1 \ , \ \ s_2 u_2 s_2 = u_1 u_2^{-1} \ , \tag{11.11}$$

together with the relation $u_1 u_2 = u_2 u_1$.

Denote $u = s_1 u_1 s_1 = u_1^{-1} u_2$ (a translation on the vector $\omega_2 - \omega_1$, orthogonal to γ). Then all the relations are

$$s_1 u s_1 u = u s_1 u s_1 \ , \ \ s_2 s_1 u s_1 s_2 = s_1 u s_1 \ , \ \ u (s_1 u s_1)(s_2 u s_2) = e \ . \tag{11.12}$$

Here and below, e is the unit element of a group.

There is a relation between the group $\mathcal{W} \ltimes P$ and the group $\mathcal{AW}$ – the classical counterpart of the affine Hecke algebra.

The group $\mathcal{AW}$.

The relations, defining the group $\mathcal{AW}$ are (see (11.5)):

$$\begin{array}{lll} s_1 y_1 = y_2 s_1 \ , & s_1 y_2 = y_2 s_1 \ , & s_1 y_3 = y_3 s_1 \ , \\ s_2 y_1 = y_1 s_2 \ , & s_2 y_2 = y_3 s_2 \ , & s_2 y_3 = y_2 s_2 \ , \end{array} \tag{11.13}$$

and $y_i y_j = y_j y_i$, $i, j = 1, 2, 3$. The group $\mathcal{AW}$ is a semi-direct product $\mathcal{W} \ltimes \mathbb{Z}^3$, the action on $\mathbb{Z}^3$ is given by tautological permutations of basis elements by $\mathcal{W} \simeq \mathcal{S}_3$.

Denote $y = y_1$. Then the set of defining relations is

$$s_2 y = y s_2 \ , \ \ s_1 y s_1 y = y s_1 y s_1 \ . \tag{11.14}$$

The group $\overline{\mathcal{AW}}$.

We shall use the quotient $\overline{\mathcal{AW}}$ of $\mathcal{AW}$ by the relation $\zeta_3 = e$, where ζ_3 is the central element in the group $\mathcal{AW}$, $\zeta_3 = y_1 y_2 y_3 = y(s_1 y s_1)(s_2 s_1 y s_1 s_2)$. The image of y in $\overline{\mathcal{AW}}$ denote by Y. Then the relations are

$$s_2 Y = Y s_2 \ , \ \ s_1 Y s_1 Y = Y s_1 Y s_1 \ , \ \ Y(s_1 Y s_1)(s_2 s_1 Y s_1 s_2) = e \ . \tag{11.15}$$

Denote by π the projection $\mathcal{AW} \to \overline{\mathcal{AW}}$, $\pi : y \mapsto Y$.

The groups $\mathcal{W} \ltimes P$ and $\overline{\mathcal{AW}}$.

The map $u \mapsto s_1 Y s_1$ establishes an isomorphism between $\mathcal{W} \ltimes P$ and $\overline{\mathcal{AW}}$.

The groups $\mathcal{W} \ltimes Q$ and $\mathcal{W} \ltimes P$.

The lattice Q is a sublattice, with index 3, of the lattice P, $\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = -\omega_1 + 2\omega_2$. Denote by α the corresponding map between the semidirect products, $\alpha : \mathcal{W} \ltimes Q \to \mathcal{W} \ltimes P$,

$$\alpha(t_1) = u_1^2 u_2^{-1} \ , \ \ \alpha(t_2) = u_1^{-1} u_2^2 \ . \tag{11.16}$$

In terms of translations t and u one has: $\alpha : t \mapsto s_1 u^2 s_1 u$.

Similarly, the lattice $3P$, isomorphic to P, is a sublattice of Q with index 3, $3\omega_1 = \alpha_1 + \gamma$, $3\omega_2 = \alpha_2 + \gamma$. Denote by β the corresponding maps between the semidirect products, $\beta : \mathcal{W} \ltimes P \to \mathcal{W} \ltimes Q$,

$$\beta(u_1) = t_1^2 t_2 \ , \ \ \beta(u_2) = t_1 t_2^2 \ . \tag{11.17}$$

In terms of translations t and u one has: $\beta : u \mapsto s_2 t^{-1} s_2 s_1 t s_1$. For a map $\overline{\mathcal{AW}} \simeq \mathcal{W} \ltimes P \xrightarrow{\beta} \mathcal{W} \ltimes Q \simeq \mathcal{W}_a$ one obtains $Y_1 \mapsto (s_1 s_2 s_0)^2$, $Y_2 \mapsto (s_2 s_0 s_1)^2$ and $Y_3 \mapsto (s_0 s_1 s_2)^2$ where Y_i are imeages of y_i in $\overline{\mathcal{AW}}$.

One has $\alpha \circ \beta = \iota_3$ and $\beta \circ \alpha = \iota_3$.

Although there is a certain similarity between maps α and β, they have different properties.

The image of the map β consists of elements (in the normal form) $t_1^i t_2^j w$, where $w \in \mathcal{W}$ and $i+j \equiv 0 \pmod 3$. This subgroup is not normal in $\mathcal{W} \ltimes Q$. Indeed, $(t_1^m t_2^n)^{-1} s_1 (t_1^m t_2^n) = t_1^{-2m+n} s_1$ and $(t_1^m t_2^n)^{-1} s_2 (t_1^m t_2^n) = t_2^{m-2n} s_2$ which implies that the only normal subgroup in $\mathcal{W} \ltimes Q$ containing $\mathcal{W}$ is the whole group $\mathcal{W} \ltimes Q$.

The image of the map α consists of elements (in the normal form) $u_1^i u_2^j w$, where $w \in \mathcal{W}$ and $i-j \equiv 0 \pmod 3$. One can verify that it is, on the contrary, the normal subgroup of $\mathcal{W} \ltimes P$. It is a unique nontrivial normal subgroup in $\mathcal{W} \ltimes P$ containing $\mathcal{W}$.

Centers of 3-fold rotations in the group $\mathcal{W} \ltimes Q$ form the lattice P: a transformation $t_1^i t_2^j s_1 s_2$ is a counter-clockwise rotation around a point $(-j+i)\omega_1 + j\omega_2$. Centers of 3-fold rotations in the group $\mathcal{W} \ltimes P$ form the lattice $\frac{1}{3}Q$: a transformation $u_1^i u_2^j s_1 s_2$ is a counter-clockwise rotation around a point $\frac{i}{3}\alpha_1 + \frac{i+j}{3}\alpha_2$. Pick a 3-fold rotation in $\mathcal{W} \ltimes P$ around a point which does not belong to P. This rotation generates the cyclic group $\mathbb{Z}_3$. We have then an isomorphism $\mathcal{W} \ltimes P \simeq \mathbb{Z}_3 \ltimes (\mathcal{W} \ltimes Q)$.

Note that the lattice parts in the normal form of elements in such $\mathbb{Z}_3$ form a set of representatives of the quotient P/Q. For instance, take a counter-clockwise rotation $u_1 s_1 s_2$ around the point $\frac{1}{3}\gamma$. Its square is $u_2 s_2 s_1$; the lattice parts of elements $\{e, u_1 s_1 s_2, u_2 s_2 s_1\}$ are $\{0, u_1, u_2\}$. These are the only weights ω satisfying condition $0 \leq (\omega, \alpha) \leq 1$ for all positive roots – that is, for vectors α_1, α_2 and γ. In the general theory, weights satisfying this condition are called minuscule.

The groups $\mathcal{W} \ltimes Q$ and $\mathcal{AW}$.

There is a map $\psi : \mathcal{W} \ltimes Q \to \mathcal{AW}$, given by

$$\psi(t_1) = y_1 y_2^{-1} \ , \ \psi(t_2) = y_2 y_3^{-1} \ . \tag{11.18}$$

In terms of elements t and y one has: $\psi : t \mapsto y_1 y_3^{-1} = y s_2 s_1 y^{-1} s_1 s_2$. For the composition $\mathcal{W}_a \xrightarrow{\sim} \mathcal{W} \ltimes Q \xrightarrow{\psi} \mathcal{AW}$ one has: $s_0 \mapsto s_1 s_2 s_1 y_1 y_3^{-1}$.

The map ψ is an embedding. The image of ψ is a normal subgroup $(\mathcal{AW})_0$ consisting of elements consisting of elements $y_1^i y_2^j y_3^k w$, $w \in \mathcal{W}$, with $i+j+k=0$.

Denote by Z the subgroup in $\mathcal{AW}$ generated by the central element ζ_3. The subgroups $(\mathcal{AW})_0$ and Z are normal in $\mathcal{AW}$. The subgroup Z is "almost" complementary to $(\mathcal{AW})_0$. Take the pullback $\tilde{Z}$ (with respect to π) of the subgroup $\mathbb{Z}_3 \subset \mathcal{W} \ltimes P$, which we just considered. The subgroup $\tilde{Z}$ is isomorphic to the infinite cyclic group $\mathbb{Z}$, the subgroup Z is the infinite cyclic subgroup in $\tilde{Z}$, with index 3. We have an isomorphism $\mathcal{AW} \simeq \tilde{Z} \ltimes (\mathcal{W} \ltimes Q)$.

The map α is now a composition, $\alpha = \pi \circ \psi$.

Resumé.

All the essential maps are π, ψ and β. They are shown on the picture.

$$\begin{tikzcd}
& \mathcal{AW} \arrow[dl, "\pi"'] & \\
\overline{\mathcal{AW}} \arrow[rr, "\beta"'] & & \mathcal{W}_a \arrow[ul, "\psi"']
\end{tikzcd}$$

For the compositions one has: $\pi \circ \psi \circ \beta = \iota_3$ and $\beta \circ \pi \circ \psi = \iota_3$. The maps β and ψ have trivial kernels. The composition $\psi \circ \beta \circ \pi$ is $y \mapsto y^3 \zeta_3^{-1}$, its kernel equals the kernel of π, it is the subgroup Z.

All maps on the picture commute with ι_N.

One can describe maps between the groups (not the group algebras) $\mathcal{AW}$, $\overline{\mathcal{AW}} \simeq \mathcal{W} \ltimes P$ and $\mathcal{W}_a \simeq \mathcal{W} \ltimes \mathcal{Q}$. To analyze the maps, it is useful to explore the normal form of an element in the semidirect product $\mathcal{G} \ltimes L$.

We shall be interested in maps ϕ satisfying two conditions: ϕ is over $\mathcal{W} = \mathcal{S}_3$, which means that ϕ preserves elements s_1 and s_2; the image of ϕ is not inside $\mathcal{W} = \mathcal{S}_3$ – this condition is imposed only because the relations (11.10) are satisfied for $t = s_1 s_2$ or $s_2 s_1$ so we exclude maps $t \mapsto s_1 s_2$ or $s_2 s_1$.

The details of calculations below is an easy exercise.

Any map $\mathcal{AW} \to \mathcal{AW}$ has the form: $y \mapsto y^i \zeta_3^j$.

Any other map between the groups $\mathcal{W}_a$, $\mathcal{AW}$ and $\overline{\mathcal{AW}}$ is a composition of a map from the picture and the map ι_N, for instance,

— any map $\mathcal{AW} \to \mathcal{W} \ltimes Q$ is a composition $\mathcal{AW} \xrightarrow{\pi} \overline{\mathcal{AW}} \xrightarrow{\beta} \mathcal{W} \ltimes Q \xrightarrow{\iota_N} \mathcal{W} \ltimes Q$,

— the only maps $\overline{\mathcal{W}} \to \overline{\mathcal{W}}$ are maps ι_N,

etc.

12 Examples

Let us illustrate the statements from Sections 9 and 11 on simple examples of diagrams with 3 and 4 nodes.

The 1-dimensional representations of the Hecke algebras $\mathcal{H}_n(q)$ are (8.6) and (8.7).

The Hecke algebra $\mathcal{H}_3(q)$ has one more representation, corresponding to the diagram $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array}$. Consider a set of standard Young tableaux corresponding to $\lambda = \{2,1\} \vdash 3$:

$$X_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \quad X_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,.$$

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be the corresponding generators in the set of relations $\mathcal{J}_0 \bigcup \mathcal{J}_+$ for $\lambda = \{2,1\}$. Explicitly,

$$(\sigma_1 - q)\ \mathcal{X}_1 \ = \ 0\,, \tag{12.1}$$

$$(\sigma_1 + q^{-1})\ \mathcal{X}_2 \ = \ 0\,, \tag{12.2}$$

$$(\sigma_2 + \frac{q^{-2}}{[2]_q})\ \mathcal{X}_1 \ = \ \mathcal{X}_2\ (\sigma_2 - \frac{q^2}{[2]_q})\,, \tag{12.3}$$

$$(\sigma_2 - \frac{q^2}{[2]_q})\ \mathcal{X}_2 \ = \ \mathcal{X}_1\ (\sigma_2 + \frac{q^{-2}}{[2]_q})\,. \tag{12.4}$$

Relations $P(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array})\mathcal{X}_i = 0$, $P(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array})\mathcal{X}_i = 0$ and $Z_3\mathcal{X}_i = [3]_q\mathcal{X}_i$, $i = 1,2$, can be verified directly as well as relations $P(X_i)\mathcal{X}_j = \delta_{ij}\mathcal{X}_j$, $i,j = 1,2$.

The construction from Section 11 results in the following operators for the elements σ_1 and σ_2 in the basis $\{\mathcal{X}_1, \mathcal{X}_2\}$:

$$\sigma_1 \mapsto \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}\,, \tag{12.5}$$

$$\sigma_2 \mapsto \begin{pmatrix} -\dfrac{q^{-2}}{[2]_q} & \dfrac{[3]_q}{[2]_q} \\ \dfrac{1}{[2]_q} & \dfrac{q^2}{[2]_q} \end{pmatrix} . \tag{12.6}$$

The last example: the Hecke algebra $\mathcal{H}_4(q)$.

The representation corresponding to the diagram $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$ is 2-dimensional. To construct it, note that there is a surjective homomorphism $\mathcal{H}_4(q) \to \mathcal{H}_3(q)$, given by $\sigma_1 \mapsto \sigma_1'$, $\sigma_2 \mapsto \sigma_2'$ and $\sigma_3 \mapsto \sigma_1'$, where $\{\sigma_1, \sigma_2, \sigma_3\}$ are the Hecke generators of $\mathcal{H}_4(q)$ and $\{\sigma_1', \sigma_2'\}$ are the Hecke generators of $\mathcal{H}_3(q)$. Take now the 2-dimensional representation of $\mathcal{H}_3(q)$ which was just described.

The idempotents corresponding to two representations for partitions $\{2,1,1\} \vdash 4$ and $\{3,1\} \vdash 4$ are related by the formula (7.25), so it is enough to construct only one of them.

Consider a partition $\lambda = \{2,1,1\} \vdash 4$. Consider a set of standard Young tableaux corresponding to

$$X_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} , \quad X_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} , \quad X_3 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} . \tag{12.7}$$

Let $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ denote the corresponding generators in the set of relations $\mathcal{J}_0 \bigcup \mathcal{J}_+$ for $\lambda = \{2,1,1\}$. Explicitly,

$$(\sigma_1 - q)\ \mathcal{X}_1 \ = \ 0 \ , \tag{12.8}$$

$$(\sigma_1 + q^{-1})\ \mathcal{X}_{2,3} \ = \ 0 \ , \tag{12.9}$$

$$(\sigma_2 + q^{-1})\ \mathcal{X}_3 \ = \ 0 \ , \tag{12.10}$$

$$(\sigma_3 + q^{-1})\ \mathcal{X}_1 \ = \ 0 \ , \tag{12.11}$$

$$([2]_q\sigma_2 + q^{-2})\ \mathcal{X}_1 \ = \ \mathcal{X}_2\ ([2]_q\sigma_2 - q^2) \ , \tag{12.12}$$

$$([2]_q\sigma_2 - q^2)\ \mathcal{X}_2 \ = \ \mathcal{X}_1\ ([2]_q\sigma_2 + q^{-2}) \ , \tag{12.13}$$

$$([3]_q\sigma_3 + q^{-3})\ \mathcal{X}_2 \ = \ \mathcal{X}_3\ ([3]_q\sigma_3 - q^3) \ , \tag{12.14}$$

$$([3]_q\sigma_3 - q^3)\ \mathcal{X}_3 \ = \ \mathcal{X}_2\ ([3]_q\sigma_3 + q^{-3}) \ . \tag{12.15}$$

A straightforward calculation shows that $Z_4\mathcal{X}_i = (q^2 + 1 + q^{-2} + q^{-4})\mathcal{X}_i$, $i = 1, 2, 3$, as expected (calculate for $\mathcal{X}_1$; then use (12.12) and (12.14)).

Applying the construction from Section 11, we obtain the following operators in the representation space:

$$\sigma_1 \mapsto \begin{pmatrix} q & 0 & 0 \\ 0 & -q^{-1} & 0 \\ 0 & 0 & -q^{-1} \end{pmatrix} , \tag{12.16}$$

$$\sigma_2 \mapsto \begin{pmatrix} -\frac{q^{-2}}{[2]_q} & \frac{[3]_q}{[2]_q} & 0 \\ \frac{1}{[2]_q} & \frac{q^2}{[2]_q} & 0 \\ 0 & 0 & -q^{-1} \end{pmatrix} , \tag{12.17}$$

$$\sigma_3 \mapsto \begin{pmatrix} -q^{-1} & 0 & 0 \\ 0 & -\frac{q^{-3}}{[3]_q} & \frac{[4]_q}{[3]_q} \\ 0 & \frac{[2]_q}{[3]_q} & \frac{q^3}{[3]_q} \end{pmatrix} . \tag{12.18}$$

One can directly verify that these matrices indeed satisfy all relations in the Hecke algebra.

Finally, we shall establish the matrix structure directly inside the Hecke algebra on the same example (12.7). More precisely, we shall see that the relations (12.8)-(12.15), where the elements $\mathcal{X}_i$ are to be treated as primitive idempotents $P(X_i) \in \mathcal{H}_4(q)$ (c.f. with (11.1) and (9.4)), provide an isomorphism between the $\mathcal{H}_4(q)$-biideal generated by $P(X_i)$, $i = 1, 2, 3$, and the matrix algebra $M_3(\mathbb{C})$.

We have relations (12.8)-(12.15) in which $\mathcal{X}_i$ are substituted by $P(X_i)$. Additionally, we have the rest of relations (11.1) belonging to the subset $\mathcal{J}_-$:

$$0 \ = \ P(X_1) \ (\sigma_1 - q) \ , \tag{12.19}$$

$$0 \ = \ P(X_{2,3}) \ (\sigma_1 + q^{-1}) \ , \tag{12.20}$$

$$0 \ = \ P(X_3) \ (\sigma_2 + q^{-1}) \ , \tag{12.21}$$

$$0 \;=\; P(X_1)\ (\sigma_3 + q^{-1})\ . \tag{12.22}$$

The generators $P(X_i)$ are the diagonal matrix units e_{ii}. The off-diagonal matrix units e_{12}, e_{21}, e_{23} and e_{32}, are given by the formulas (12.12)–(12.15). After proper normalization one has

$$P(X_1|X_2) \;=\; \frac{1}{[3]_q} P(X_1)([2]_q\sigma_2 + q^{-2}) \;\equiv\; \frac{1}{[3]_q}([2]_q\sigma_2 - q^2)P(X_2)\,, \tag{12.23}$$

$$P(X_2|X_1) \;=\; ([2]_q\sigma_2 + q^{-2})P(X_1) \;\equiv\; P(X_2)([2]_q\sigma_2 - q^2)\,, \tag{12.24}$$

$$P(X_2|X_3) \;=\; \frac{1}{[4]_q} P(X_2)([3]_q\sigma_3 + q^{-3}) \;\equiv\; \frac{1}{[4]_q}([3]_q\sigma_3 - q^3)P(X_3)\,, \tag{12.25}$$

$$P(X_3|X_2) \;=\; \frac{1}{[2]_q}([3]_q\sigma_3 + q^{-3})P(X_2) \;\equiv\; \frac{1}{[2]_q} P(X_3)([3]_q\sigma_3 - q^3)\,. \tag{12.26}$$

Finally, introduce analogues of matrix units e_{13} and e_{31}:

$$P(X_1|X_3) = \frac{1}{[3]_q[4]_q} P(X_1)([2]_q\sigma_2 + q^{-2})([3]_q\sigma_3 + q^{-3})\,, \tag{12.27}$$

$$P(X_3|X_1) = \frac{1}{[2]_q} P(X_3)([3]_q\sigma_3 - q^3)([2]_q\sigma_2 - q^2)\,, \tag{12.28}$$

Alltogether these elements form the linear basis in the biideal. The correspondence of these basic elements with the set of $M_3(\mathbb{C})$ units is as follows

$$\begin{pmatrix} P(X_1) & P(X_1|X_2) & P(X_1|X_3) \\ P(X_2|X_1) & P(X_2) & P(X_2|X_3) \\ P(X_3|X_1) & P(X_3|X_2) & P(X_3) \end{pmatrix}\,. \tag{12.29}$$

Acknowledgements

This work was supported in parts by the CNRS grant PICS No.608 and the RFBR grant No. 98-01-22033.

We are indebted to A. Isaev for enlightning explanations about the baxterisation and to E. Kovalchuk for valuable help in preparation of the manuscript.

References

[B] Bourbaki, N.: *'Groupes et algèbres de Lie, Chapitres IV, V, VI'*, Paris, Hermann, 1968.

[Bu] Burau, W.: *'Über Zopfgruppen und gleichsinnig verdrillte Verkettungen'*, Abh. Math. Semin. Hamburg Univ. **11** (1935) 179-186.

[ChP] Chari, V. and Pressley, A.: *'A guide to quantum groups'*, Cambridge University Press, 1994.

[CR] Curtis, C. and Reiner, I.: *'Methods of representation theory'*, Volume II, John Wiley & Sons, 1987.

[DK] Drozd, Yu. A. and Kirichenko, V. V.: *'Finite dimensional algebras'*, Springer-Verlag, 1994. An original Russian edition by Publisher of Kiev State University "Vishcha shkola", Kiev 1980.

[DJ] Dipper, R. and James, G.: *'Blocks and idempotents of Hecke algebras of general linear groups'*, Proc. London Math. Soc. (3) **54** (1987) 57-82.

[FRT] Frame, J.S., Robinson, G. de B., Thrull, R.M.: *'The hook graphs of the symmetric group'*, Canad. J. Math. **6** (1954) 316-324.

[FH] Fulton, W., Harris, J.:*'Representation Theory'*, Springer-Verlag, 1991.

[GS] Gerstenhaber, M. and Schaps, M.: *'Hecke algebras, $U_q sl_n$, and the Donald-Flanigan conjecture for $\mathcal{S}_n$'*, Trans. Amer. Math. Soc. **349** (1997) 3353-3371.

[G] Graham, J. J.: *'Modular representations of Hecke algebras and related algebras'*, PhD thesis, University of Sydney, 1995.

[Gy] Gyoja, A.: *'A q-analogue of Young symmetrizer'*, Osaka J. Math. **23** (1986) 841-852.

[H] Humphreys, J.E.: *'Reflection Groups and Coxeter Groups'*, Cambridge University Press, 1990.

[Is] Isaev, A. *'R-matrix approach to differential calculus on quantum groups'*, Phys. Part. Nucl. **28** (1997) 267-294.

[Iw] Iwahori, N.: *'On the structure of the Hecke ring of a Chevalley group over a finite field'*, J. Fac. Sci. Univ. Tokyo, Sect. 1A Math., **10** (part 2) (1964) 215-236.

[Ja] James, G.D.: *'The representation theory of symmetric groups'*, Lect. Notes in Math. **682**, Springer, Berlin, 1978.

[Ju] Jucys, A. A.: *'Symmetric polynomials and the center of the symmetric group ring'*, Rep. Math. Phys. (1) **5** (1974) 107-112.

[K] Krieg, A.: *'Hecke algebras'*, Memoirs of the AMS, **87**, No. 435, 1990.

[Ma] Macdonald, I. G.: *'Symmetric functions and Hall polynomials'*, Clarendon Press, 1979.

[Mar] Martin, P.: *'Potts models and related problems in statistical mechanics'*, World Scientific, 1991.

[M1] Murphy, G. E.: *'A new construction of Young's seminormal representation of the symmetric groups'*, J. Algebra **69** (1981) 287-297.

[M2] Murphy, G. E.: *'On the representation theory of the symmetric groups and associated Hecke algebras'*, J. Algebra **152** (1992) 492-513.

[OV] Okounkov, A. and Vershik, A.: *'A new approach to representation theory of symmetric groups'*, Selecta Math. **2** (1996) 581-605.

[R] Ram, A.: *'Seminormal representations of Weyl groups and Iwahori-Hecke algebras'*, Proc. of London Math. Soc. **75** (1997) 99-133.

[S] Sagan, B.E.: *'The symmetric group: representations, combinatorial algorithms, and symmetric functions'*, Wadsworth & Brooks/Cole math. ser., 1991.

[TL] Temperley, N. and Lieb, E.: *'Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular plane lattices: some exact results for the "percolation" problem'*, Proc. Roy. Soc. London A **322** (1971) 251-280.

[V] Vershik, A.: *'Hook formula and related identities'*, Zapiski LOMI **172** (1989) 3-20.

Figure 1.

Connected Coxeter graphs of finite Coxeter groups.

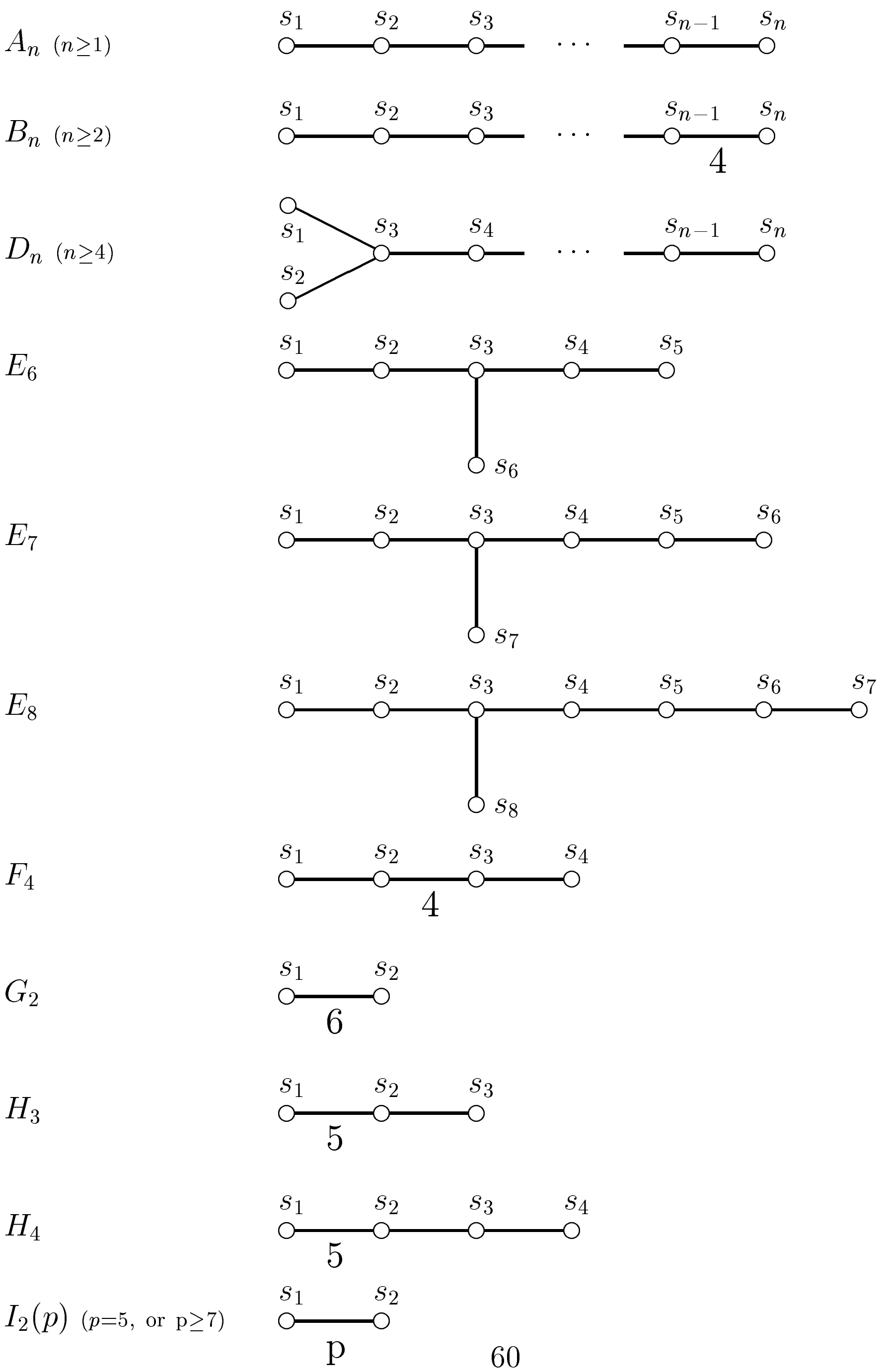


Figure 2.
Graph of Young diagrams.