PERIODIC RANK ONE GEODESICS IN HADAMARD SPACES

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INTRODUCTION

Let X be a locally compact Hadamard space [1]. Let ∂X be the ideal boundary of X and set $\overline{X} := X \cup \partial X$. The cone topology turns $\overline{X}$ into a compact Hausdorff space. There is a natural metric on ∂X, the *Tits metric* d_T, introduced by Misha Gromov [BGS]. If we consider ∂X together with the Tits metric and the *Tits topology* induced by it, then we indicate this by a subscript and write $\partial_T X$. Recall that the Tits topology is finer than the cone topology and that $\partial_T X$ is a CAT(1)-space. Typically, $\partial_T X$ is a rather wild noncompact topological space.

We say that a subset $F \subset X$ is a k-*flat* if F is convex and isometric to k-dimensional Euclidean space $\mathbb{R}^k$. A 1-flat is also called a *line*, a 2-flat a *flat plane*. A *flat halfplane* is a convex subset of X isometric to the subset $\{y \geq 0\}$ in the Euclidean (x, y)-plane. The boundary of a flat halfplane is a line, namely the line corresponding to the set $\{y = 0\}$. Many questions concerning the asymptotic geometry of X are related to the existence of flats or flat halfplanes.

Let Γ be a group of isometries of X. Recall that the induced action of Γ on ∂X and $\overline{X}$ respects the cone topology and the Tits metric, respectively. We say that a sequence (γ_n) in Γ converges to a point $\xi \in \partial X$ and write $\gamma_n \to \xi$ if $\gamma_n x \to \xi$ for one (and hence any) $x \in X$ [2]. We denote by $\Lambda = \Lambda(\Gamma)$ the *limit set* of Γ, that is, Λ is the set of all $\xi \in \partial X$ occurring as such a limit. By definition, $\Lambda \subset \partial X$ is closed and Γ-invariant.

We study the interplay between the action of Γ and the geometry of X. We need that Γ is sufficiently large. The weakest assumption ensuring that all the asymptotic geometry of X is related to Γ is $\Lambda = \partial X$. This is the

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[1] Our terminology is standard. For definitions and basic results we refer to [Ba],[BGS], or [E3].

[2] If not mentioned otherwise, we consider $\overline{X}$ and ∂X with the cone topology.

assumption occurring in our main conjectures below. It is known to hold under the following circumstances:
(1) We say that Γ is *uniform* or, more precisely, that *the action of* Γ *on* X *is uniform*, if there is a compact subset $K \subset X$ with $\Gamma K = X$. If Γ is uniform, then $\Lambda = \partial X$ by an elementary argument involving triangle comparison.
(2) If X is a Hadamard manifold and Γ is properly discontinuous with finite covolume, then the Liouville measure on the unit tangent bundle of X/Γ is finite. Hence, by the Poincaré recurrence theorem, all geodesics of X are nonwandering modulo Γ. The latter implies that $\Lambda = \partial X$.

We say that a line L in X has *rank one* if it does not bound a flat halfplane. This holds if and only if the endpoints of L have Tits distance $> \pi$. We say that a line L is Γ*-periodic* if there is a unit speed parametrization σ of L, an element $\gamma \in \Gamma$, and a number $\omega > 0$ such that $\gamma\sigma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$. A Γ-periodic line has rank one if and only if its endpoints have infinite Tits distance. The existence of γ-periodic lines of rank one is a central issue for algebraic properties of Γ, asymptotic properties of stochastic processes on X, and dynamical properties of the geodesic flow of X modulo Γ, see [Ba].

The main aim of the present paper is the proper formulation of the problems concerning rank rigidity and geodesic flow of Hadamard spaces.

CONJECTURE A (Closing Lemma). *If* Γ *is properly discontinuous,* $\Lambda = \partial X$*, and* $\operatorname{diam} \partial_T X > \pi$*, then* X *contains a* Γ*-periodic line of rank one.*

Conjecture A holds under the assumption that Γ satisfies the duality condition of Chen and Eberlein, compare Section 1 below. In particular, Conjecture A holds in the case where X is a Hadamard manifold and Γ is a properly discontinuous group of isometries of X with finite covolume. Conjecture A also holds under the assumption that X is a piecewise smooth complex of dimension two and Γ is a properly discontinuous group of isometries of X with compact quotient, see [BB1].

CONJECTURE B (Diameter Rigidity). *If* X *is geodesically complete,* $\Lambda = \partial X$*, and* $\operatorname{diam} \partial_T X = \pi$*, then* X *is a symmetric space or a Euclidean building of rank at least two, or* X *is reducible.*

Note that $\operatorname{diam} \partial_T X = \pi$ holds if and only if all lines in X bound a flat halfplane. Geodesic completeness is a regularity assumption on X. It cannot be deleted without changing assumptions or assertion of Conjecture B.

If X is a symmetric space or a Euclidean building of rank at least two, then $\partial_T X$ is a spherical building and the Tits metric on $\partial_T X$ corresponds to its natural piecewise spherical metric as a building and hence has diameter π. Vice versa, Bernhard Leeb showed that if X is geodesically complete and

$\partial_T X$ is a spherical building, then X is a symmetric space or a Euclidean building [Le] [3]. On the other hand, the counter-examples in [BB2] show that a CAT(1)-space need not be a spherical join or building if its diameter is equal to π.

Conjecture B holds when X is a Hadamard manifold and Γ satisfies the duality condition; for a comprehensive proof, see Chapter IV in [Ba]. Jens Heber proved Conjecture B in the case of homogeneous Hadamard manifolds [He]. It is immediate from the results in [BB1, BB2] that Conjecture B also holds under the assumption that X is a piecewise smooth complex of dimension two or a piecewise Euclidean complex of dimension 3 and Γ is a properly discontinuous group of isometries of X with compact quotient.

Recall that a homogeneous Hadamard manifold can be represented as a simply connected solvable Lie group, S, with a left-invariant metric, see Proposition 1 in [Hei]. By Theorem 3.4 in [Wo], S does not contain lines of rank one which are periodic with respect to some isometry unless $\dim[S, S] = \dim S - 1$. On the other hand, a homogeneous Hadamard manifold admits a properly discontinuous group of isometries satisfying the duality condition if and only if it is a symmetric space, see Proposition 4.8 in [E2]. It follows that, in Conjecture A, the assumption that the action of Γ be properly discontinuous cannot be deleted without proper replacement. In fact, the assumption seems to be natural since the conjecture is obviously related to properties of the geodesic flow of the quotient X/Γ. For example, if the geodesic flow of X/Γ has a finite invariant measure which is positive on open subsets of the space of unit speed geodesics, then Γ satisfies the duality condition and X contains a Γ-periodic line of rank one.

The above conjectures have to be taken cum grano salis. Maybe there are some exotic counter-examples. However, except for inessentialities, the formulations seem to be correct. In this paper we prove two very elementary results related to them.

PROPOSITION 1. *Suppose that $\Lambda = \partial X$ and that, for each $\xi \in \partial_T X$, there is $\eta \in \partial_T X$ with $d(\xi, \eta) > \pi$. Then X contains a Γ-periodic line of rank one.*

Proposition 1 extends Lemma 7.2 from [BB1]. Note that the second assumption in Proposition 1 holds if $\operatorname{diam} \partial_T X > 2\pi$. Note also that we do not need here that Γ is properly discontinuous.

We say that a subset M of Λ is *minimal (with respect to Γ)* if M is non-empty, closed and Γ-invariant and does not contain a proper closed and Γ-invariant subset. A non-empty closed subset M of Λ is minimal if and only if the orbit $\Gamma\xi$ is dense in M for all $\xi \in M$. Zorn's Lemma together with

[3] Related geometric properties and characterizations of symmetric spaces and Euclidean buildings are discussed in [Br, CL, KL], to name a few.

the compactness of Λ ensures the existence of minimal subsets. Minimal subsets $M, N \subset \Lambda$ either coincide or are disjoint.

We say that the action of Γ on Λ is minimal if Λ is minimal. Conjecture A would follow if we could show that its assumptions imply that the action of Γ on Λ is minimal.

PROPOSITION 2. *Let X be geodesically complete and Γ be uniform.*

1. *If ∂X contains exactly one Γ-minimal subset, then the action of Γ on ∂X is topologically transitive.*
2. *If $|\partial X| \geq 3$ and if ∂X contains more than one Γ-minimal subset, then $(\partial X)^2$ contains a nonempty Γ-invariant subset P (with respect to the diagonal action of Γ) which does not intersect the diagonal such that, for all $x \in X$ and $(\xi, \eta) \in P$, there is a flat plane containing the sector (x, ξ, η).*

Proposition 2 is related to both of the above conjectures, its second part shows that, under the given assumptions, X contains many flat planes. Maybe the recent work of Alexander Lytchak [Ly] will result in more and better information on Conjecture B.

1. PROOF OF PROPOSITION 1

The concept of duality was introduced by Chen and Eberlein, see [CE]. It is related to the nonwandering set of the geodesic flow of the quotient X/Γ.

1.1. DEFINITION. We say that a pair of points $\xi, \eta \in \partial X$ is *dual (with respect to Γ)* if there exists a sequence (γ_n) in Γ with $\gamma_n \to \xi$ and $\gamma_n^{-1} \to \eta$.

The set of dual pairs $(\xi, \eta) \in (\partial X)^2$ is denoted $\mathcal{D}$. By the definition of the limit set Λ we have $\mathcal{D} \subset \Lambda^2$. The *duality condition* of Chen and Eberlein requires that $\mathcal{D} = (\partial X)^2$, see [CE].

1.2. LEMMA. *The set $\mathcal{D}$ of dual pairs has the following properties:*

(D1) *For each $\xi \in \Lambda$, the set $\mathcal{D}_\xi = \{\eta \in \partial X \mid (\xi, \eta) \in \mathcal{D}\}$ is nonempty.*
(D2) *$i(\mathcal{D}) = \mathcal{D}$, where $i : (\partial X)^2 \to (\partial X)^2$ is the canonical involution, $i(\xi, \eta) = (\eta, \xi)$.*
(D3) *$\mathcal{D}$ is invariant under the Γ^2-action on $(\partial X)^2$.*
(D4) *$\mathcal{D}$ is closed in the geometric topology.*

Proof. (D2) is clear and (D4) is immediate from the definition of the cone topology. Let (γ_n) be a sequence in Γ converging to $\xi \in \Lambda$. Let x be a point in X. Now $\overline{X}$ is compact in the cone topology. Hence passing to a subsequence if necessary, $\gamma_n^{-1} x$ converges to some point $y \in \overline{X}$. If $y \in X$, then there is a constant $C \geq 0$ such that $d(\gamma_n^{-1} x, y) \leq C$ for all n. But then

$$d(x, \gamma_n y) = d(\gamma_n^{-1} x, y) \leq C$$

for all n, contradicting $\gamma_n y \to \xi$. Hence $y \in \partial X$, hence (D1).

Let $(\xi, \eta) \in \mathcal{D}$ and (γ_n) be a sequence in Γ with $\gamma_n \to \xi$ and $\gamma_n^{-1} \to \eta$. Let $\gamma \in \Gamma$. Set $\tilde{\gamma}_n := \gamma_n \gamma^{-1}$. Then $\tilde{\gamma}_n \to \xi$ and $\tilde{\gamma}_n^{-1} \to \gamma\eta$, hence (D3). □

1.3. DEFINITION. We say that a pair $(\xi, \eta) \in (\partial X)^2$ is *visible* if there is a geodesic $\sigma \in X$ with end points $\sigma_+ = \xi$ and $\sigma_- = \eta$.

Note that the set $\mathcal{V} \subset (\partial X)^2$ of visible pairs does not meet the diagonal $\Delta := \{(\xi, \xi) \mid \xi \in \partial X\}$. Following Eberlein and O'Neill [EO] we say that X is a *visibility space* if $\mathcal{V} = (\partial X)^2 \setminus \Delta$. In today's language, an important result of Eberlein says that a locally compact Hadamard space X with uniform isometry group is a visibility space if and only if it is hyperbolic in the sense of Gromov, see [E1].

1.4. LEMMA. *The set $\mathcal{V}$ of visible pairs has the following properties:*

(V1) *If X is geodesically complete, then $\mathcal{V}_\xi = \{\eta \in \partial X \mid (\xi, \eta) \in \mathcal{V}\}$ is nonempty for all $\xi \in \partial X$.*
(V2) $i(\mathcal{V}) = \mathcal{V}$.
(V3) *$\mathcal{V}$ is invariant under the diagonal action of Γ on $(\partial X)^2$.*

1.5. LEMMA. *Let $\xi \in \Lambda$ and $\eta \in \mathcal{V}_\xi$. Then $\mathcal{D}_\xi \subset \overline{\Gamma\eta}$, the closure of the Γ-orbit of η.*

Proof. Let $\zeta \in \mathcal{D}_\xi$ and (γ_n) be a sequence in Γ with $\gamma_n \to \xi$ and $\gamma_n^{-1} \to \zeta$. Let σ be a geodesic in X from ξ to η and x be a point on σ. Let σ_n be the geodesic from $\gamma_n x$ to η and $\tau_n : [0, d_n] \to X$ be the geodesic segment from $\gamma_n x$ to x. Let $T > 0$ be given. Then $d_n > T$ for all sufficiently large n. Since $\gamma_n x \to \xi$, we have $d(\sigma_n(T), \tau_n(T)) \to 0$ as $n \to \infty$. Now $\gamma_n^{-1}\sigma_n$ is a geodesic starting in x and $d(\gamma_n^{-1}(\sigma_n(T)), \gamma_n^{-1}(\tau_n(T))) \to 0$. Since $\gamma_n^{-1} x \to \zeta$ and $T > 0$ is arbitrary, we conclude that $\gamma_n^{-1}\eta \to \zeta$. □

1.6. COROLLARY. *Let $\xi \in \Lambda$ and $\eta \in \mathcal{V}_\xi$. Assume that η is contained in a minimal subset M of Λ. Then $\mathcal{D}_\xi = M$.*

Proof. By Lemma 1.5 we have $\mathcal{D}_\xi \subset \overline{\Gamma}_\eta$. Since M is closed and Γ-invariant, we conclude that $\overline{\Gamma}\eta \subset M$. Since M is minimal and $\mathcal{D}_\xi$ is closed and invariant under Γ, we get $\mathcal{D}_\xi = M$ as claimed. □

The main results relating rank one geodesics, Tits metric and isometries are as follows, see Section III.3 in [Ba].

1.7. LEMMA. *Let $\xi, \eta \in \partial X$. Then:*

1. *If there is a line L in X with endpoints ξ and η, then $d_T(\xi, \eta) \geq \pi$ with equality if and only if L bounds a flat half plane.*
2. *If $d_T(\xi, \eta) > \pi$, then there is a line in L with endpoints ξ and η.*

1.8. LEMMA. *Let $\xi, \eta \in \partial X$ be points with $d_T(\xi,\eta) > \pi$. Let (γ_n) be a sequence of isometries of X with $\gamma_n \to \xi$ and $\gamma_n^{-1} \to \eta$. Then for all n sufficiently large, γ_n is an axial isometry of X. If L_n is an axis of γ_n, then up to passing to a subsequence, L_n converges to a line from η to ξ.*

The relevance of Γ-periodic geodesics or lines, respectively, is due to their hyperbolic behaviour at infinity:

1.9. LEMMA. *Suppose L is an oriented line of rank one, shifted by an axial isometry γ of X. Let ξ and η be the end points of L. Then for all neighborhoods U of ξ and V of η in $\overline{X}$ we have:*

1. *There is an $n \geq 0$ such that*
$$\gamma^k(\overline{X} \setminus V) \subset U \quad \textit{and} \quad \gamma^{-k}(\overline{X} \setminus U) \subset V \quad \textit{whenever } k \geq n.$$
2. *For all $\alpha \in \partial X \setminus U$ and $\beta \in \partial X \setminus V$ there are lines in X with endpoints α and ξ and η and β, respectively. Furthermore, there is a compact set $K \subset X$ such that any such line intersects K.*

It is now immediate that the endpoints of a Γ-periodic geodesic of rank one have infinite Tits distance to any other point in ∂X.

1.10. PROPOSITION. *Suppose that $\Lambda = \partial X$. Then the following are equivalent:*

1. *X contains a Γ-periodic rank one geodesic.*
2. *$\partial_T X$ contains isolated points.*
3. *For all $\xi \in \partial X$ there is $\eta \in \partial X$ with $d_T(\xi,\eta) > \pi$.*
4. *There are points $\xi, \eta \in \partial X$ with $d_T(\xi,\eta) > \pi$,*
   where η is contained in some minimal subset of ∂X,

Proof. Suppose that L is a Γ-periodic line in X. Then the endpoints of L have infinite Tits distance to any other point of $\partial_T X$, hence $(1) \Rightarrow (2)$. The implications $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are clear. Now assume (4) and let M be a minimal set containing η. Since $d_T(\xi,\eta) > \pi$, there is a line in X with endpoints ξ and η, that is $(\xi,\eta) \in \mathcal{V}$. Hence $\mathcal{D}_\xi = M$, by Lemma 1.6. Now Lemma 1.8 applies and shows that X contains a Γ-periodic geodesic of rank one. Hence $(4) \Rightarrow (1)$. ☐

2. PROOF OF PROPOSITION 2

We first discuss a strengthened version of Assertion (1) of Proposition 2.

2.1. PROPOSITION. *Assume that $\Lambda = \partial X$ and that ∂X contains exactly one minimal set, M. Let $\xi \in M$ and $\eta \in \mathcal{V}_\xi$. Then $\mathcal{D}_\xi = \partial X$ and the orbit $\Gamma\eta$ is dense in ∂X.*

Proof. Let $\zeta \in \partial X$. Then $\mathcal{D}_\zeta$ is closed and Γ-invariant. Hence $\mathcal{D}_\zeta$ contains a minimal set. Since M is the unique minimal subset in ∂X, we get $M \subset \mathcal{D}_\zeta$. Hence, vice versa, ζ is dual to any point from M, in particular to ξ. This is the first assertion. The second is immediate from Lemma 1.5. ☐

2.2. LEMMA. *Suppose that $|\partial X| \geq 3$ and that $(\partial X)^2 \setminus \Delta$ contains a minimal subset P with*

$$\epsilon = \epsilon(P) := \inf\{\angle_x(\xi, \eta) \mid x \in X \text{ and } (\xi, \eta) \in P\} = \pi.$$

Then $(\partial X)^2 \setminus \Delta$ also contains a minimal subset P' with $\epsilon(P') < \pi$.

Proof. For $\xi, \eta \in \partial X$, set $\angle(\xi, \eta) := \sup_{x \in X} \angle_x(\xi, \eta)$. Recall that the Tits metric is the length metric associated to the angle metric $\angle = \angle(\xi, \eta)$.

Now let $P \subset (\partial X)^2$ with $\epsilon = \epsilon(P) = \pi$. Then for all $x \in X$ and $(\xi, \eta) \in P$ the geodesic rays $x\xi$, $x\eta$ in X form a geodesic line because, by assumption, $\angle_x(\xi, \eta) = \pi$. Varying x we obtain parallel geodesic lines. Therefore, X metrically splits as $X = X' \times \mathbb{R}$ where ξ and η are the ends of the lines $x' \times \mathbb{R}$, $x' \in X'$. Hence the metric splitting $X = X'' \times \mathbb{R}^k$ with k maximal is nontrivial, that is, $k \geq 1$. It is also Γ-invariant.

Suppose first that $\partial X'' = \emptyset$. Then $k \geq 2$ since $|\partial X| \geq 3$. Moreover, $\partial X = \partial(x'' \times \mathbb{R}^k)$ for any $x'' \in X''$. The assertion of the lemma is now obvious.

Otherwise the boundary at infinity ∂X is the spherical join (in the Tits metric) $A * B$ of two non-empty Γ-invariant closed subsets $A, B \subset \partial X$, where $\partial(X'' \times r) = A$ and $\partial(x'' \times \mathbb{R}^k) = B$ for all $x'' \in X''$ and $r \in \mathbb{R}^k$. We have $\angle(a, b) = \pi/2$ for all $a \in A$, $b \in B$. Now $A \times B$ is a closed and Γ-invariant subset of $(\partial X)^2 \setminus \Delta$. Any minimal $P' \subset A \times B$ satisfies the requirement asserted in the lemma. ☐

2.3. PROPOSITION. *Suppose that Γ is uniform and that $(\partial X)^2 \setminus \Delta$ contains a minimal subset P with $\epsilon(P) < \pi$. Then all triples (x, ξ, η) with $x \in X$ and $(\xi, \eta) \in P$ span a flat sector bounded by the rays $x\xi$ and $x\eta$.*

Proof. Set $\delta = \inf\{\angle(\xi, \eta) \mid (\xi, \eta) \in P\}$. Since P does not intersect the diagonal and $\angle$ is lower semicontinuous in the cone topology, we have $\delta > 0$. Invoking the minimality of P we conclude that $\angle(\xi, \eta) = \delta$ for all pairs $(\xi, \eta) \in P$. Therefore, $\epsilon \leq \angle_x(\xi, \eta) \leq \delta$ for all $x \in X$ and $(\xi, \eta) \in P$, by the definition of ϵ and the Tits metric, respectively.

Choose sequences $(y_n, \xi_n, \eta_n) \in X \times P$ with $\angle_{y_n}(\xi_n, \eta_n) \to \epsilon$. Since X is geodesically complete, there is a sequence (x_n) in X with $y_n \in x_n\xi_n$, and $d(x_n, y_n) = n$. In particular,

$$\epsilon \leq \angle_z(\xi_n, \eta_n) \leq \angle_{y_n}(\xi_n, \eta_n) \to \epsilon$$

for any $z \in x_n y_n$. Hence $\limsup \angle_{z_n}(x_n, \eta_n) \geq \pi - \epsilon$ for any sequence (z_n) with z_n in the interior of the segment $x_n y_n$.

Now Γ is uniform and P is Γ-invariant. Hence up to passing to a subsequence we can assume that $x_n \to x \in X$ and $(\xi_n, \eta_n) \to (\xi, \eta) \in P$.

Let z be any point on the ray $x\xi$ different from x and choose a sequence (z_n) as above with $\lim z_n = z$. By the upper semicontinuity of angles we conclude that $\angle_z(x, \eta) \geq \pi - \epsilon$. On the other hand, by the definition of ϵ we also have $\angle_x(z, \eta) = \angle_x(\xi, \eta) \geq \epsilon$. Now the sum of the angles in the triangle (x, z, η) is $\leq \pi$, hence $\angle_x(z, \eta) = \epsilon$ and $\angle_z(x, \eta) = \pi - \epsilon$. Since z was an arbitrary point on $x\xi$, we conclude that (x, ξ, η) spans a flat sector and hence that $\angle_z(\xi, \eta) = \epsilon$ for all $z \in x\xi$. On the other hand,

$$\delta = \angle(\xi, \eta) = \lim_{z \to \xi} \angle_z(\xi, \eta) = \epsilon,$$

where $z \in x\xi$ as above. Hence $\angle_x(\xi, \eta) = \delta$ for all x in X and (ξ, η) in P. $\square$

Proof of Assertion (2) of Proposition 2. By Lemma 2.2 we can assume that $\epsilon(P) < \pi$. Let $x \in X$ and $(\xi, \eta) \in P$. We use that X is geodesically complete: let $\sigma : \mathbb{R} \to X$ be a unit speed geodesic with $\sigma(0) = x$ and $\sigma(\infty) = \xi$. Let $\tau_n : \mathbb{R} \to X$ be a unit speed geodesic with $\tau_n(0) = \sigma(-n)$ and $\tau_n(\infty) = \eta$. It is immediate from Proposition 2.3 that τ_n bounds a flat halfplane E_n which contains $\sigma([-n, \infty))$. As $n \to \infty$, a subsequence of the E_n will converge to a flat plane as asserted. $\square$

We finish this section by discussing a strong attraction property of minimal subsets if they are unique. We say that a subset $A \subset \partial X$ is Γ-*collapsible* if there is a point ξ in ∂X such that, for each neighborhood U of ξ in ∂X, there is $\gamma \in \Gamma$ with $\gamma A \subset U$. If A is finite, $A = \{a_1, \dots, a_n\}$, then A is Γ-collapsible if and only if the closure of the orbit of $(a_1, \dots, a_n)$ under the diagonal action of Γ on $(\partial X)^n$ intersects the diagonal $\Delta_n := \{(\xi, \dots, \xi) \mid \xi \in \partial X\}$.

If A is not Γ-collapsible, then we say that A is Γ-*separated*. For example, if there are two distinct minimal sets $M, M' \subset \partial X$ for the action of Γ, then any set $A = \{\xi, \xi'\}$ with $\xi \in M$ and $\xi' \in M'$ is Γ-separated.

2.4. LEMMA. *Let $n \geq 2$ be infimal with the property that there exists a Γ-separated $A \subset \partial X$ of cardinality n. Then either $n = 2$ or $n = \infty$.*

Proof. Assume that there is a Γ-separated $A \subset \partial X$ of cardinality $n \in \mathbb{N}$ with $n \geq 2$, $A = \{a_1, \dots, a_n\}$. Let $a := (a_1, \dots, a_n) \in (\partial X)^n$. Then $\overline{\Gamma a} \subset (\partial X)^n$ is compact and invariant under the (diagonal) action of Γ. Thus there exists a minimal set $P \subset \overline{\Gamma a}$ for the action of Γ.

For each pair (i, j), $1 \leq i < j \leq n$, the projection $p_{ij} : (\partial X)^n \to (\partial X)^2$, $p_{ij}(\xi_1, \dots, \xi_n) = (\xi_i, \xi_j)$ is continuous and Γ-equivariant. Hence the sets $P_{ij} = p_{ij}(P) \subset (\partial X)^2$ are minimal. Thus $P_{ij} \cap \Delta_2 = \emptyset$ for some pair $i < j$. Otherwise $P_{ij} \subset \Delta_2$ for all $i < j$ which would imply that $P \subset$

Δ_n in contradiction to the assumption that A is Γ-separated, that is, that $\overline{\Gamma a} \cap \Delta_n = \emptyset$. Hence the claim. ☐

2.5. LEMMA. *Assume that ∂X contains a Γ-separated set A with two elements. Then $(\partial X)^2$ contains a nonempty minimal subset P which does not intersect the diagonal $\Delta = \Delta_2$.*

Proof. Let $A = \{a_1, a_2)\}$ and set $a = (a_1, a_2)$. Then the set $Q = \overline{\Gamma a} \subset (\partial X)^2$ is compact and Γ-invariant and does not intersect the diagonal. Hence Q contains a minimal subset P and P does not intersect the diagonal. ☐

2.6. LEMMA. *Assume that ∂X contains exactly one minimal set, M. Then for any $\xi \in M$, any neighborhood U of ξ in ∂X, and any set $A \subset \partial X$ which is compact with respect to the Tits metric, there exist $\gamma \in \Gamma$ with $\gamma A \subset U$.*

Proof. Assume first that A is a finite subset of ∂X. Let B be the set of all $\xi \in \partial X$ such that for every neighborhood U of ξ there exists $\gamma \in \Gamma$ with $\gamma A \subset U$. Then B is closed and Γ-invariant. By Lemma 2.5, B is not empty. Since M is the unique minimal subset of ∂X, we have $M \subset B$.

Suppose now, more generally, that $A \subset \partial X$ is compact in the Tits topology. For ξ and U as in the assertion, choose a smaller neighborhood $V \subset U$ of ξ and a number $\epsilon > 0$ such that the balls of radius ϵ about points $\eta \in V$ in the Tits metric are contained in U. Now if A is compact in the Tits metric, it can be covered by a finite number of balls of radius ϵ in that metric. ☐

2.7. PROPOSITION. *Assume that X is geodesically complete and that ∂X contains exactly one minimal set, M. Suppose M contains a point which has a compact neighborhood in the Tits metric. Then X contains a Γ-periodic line of rank one.*

Proof. For $\eta \in \partial X$, denote by $B(\eta, \rho)$ the closed ball of radius ρ about η in the Tits metric. Under the assumptions of the proposition, there is a point $\eta \in M$ and a radius $\rho > 0$ such that $B(\eta, \rho)$ is compact in the Tits metric.

Let L be a line in X ending in η and let ξ be the other end point of L. Let x be a point on L. For $\zeta \in \partial X$, let σ_ζ be the unit speed geodesic ray from x to ζ. For $\epsilon, T > 0$ set

$$V(\epsilon, T) := \{\zeta \in \partial X \mid \angle_{(x,T)}(\eta, \zeta) < \epsilon\},$$

where $\angle_{(x,T)}(\eta, \zeta)$ is the angle of the Euclidean comparison triangle of $(x, \sigma_\eta(T), \sigma_\zeta(T))$ corresponding to the vertex x. Then $V(\epsilon, T)$ is a neighborhood of η in the cone topology. Fix $\epsilon \in (0, \rho)$. By Lemma 2.6, there exist $\gamma \in \Gamma$ with $\gamma(B(\eta, \rho)) \subset V(\epsilon, T)$. Then

$$\begin{aligned} d_T\big(\gamma^{-1}(\xi), \eta\big) &= d_T\big(\gamma^{-1}(\xi), B(\eta, \rho)\big) + \rho \\ &= d_T\big(\xi, \gamma(B(\eta, \rho))\big) + \rho \\ &> \pi - \epsilon + \rho > \pi. \end{aligned}$$

Hence Proposition 1.10 applies and shows that X contains a Γ-periodic line of rank one. □

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