ON THE LODAY SYMBOL IN THE DELIGNE-BEILINSON COHOMOLOGY

by

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This note is thought as a complement to the volume on the Beilinson conjectures whose [EV] and [N] are two contributions. It gives an explicite formula for the Loday symbol in the Deligne-Beilinson cohomology. Thereby one obtains the proof of the "crucial lemma" 2.4 in [N],II, a formula for the evaluation of the Loday symbol on certain cycles. This formula was stated by A. Beilinson in [B], 7.0.2 and - together with very useful comments and the assumptions really necessary - [N], II, 2.4, however both times without proof. Note that the explicite description of the regulator map for Spec $\mathbb{Q}(\mu_N)$, where μ_N is the group of N-th roots of unity, given by A. Beilinson in [B], 7.1 relies on this crucial lemma.

Let $A_{\mathbb{C}}^{n+1}$ be the affine space of dimension $n + 1$ of coordinates X_i over the complex numbers $\mathbb{C}$. Let $\phi = 1 - X_0 \dots X_n$, $A = A_{\mathbb{C}}^{n+1} - (\phi = 0)$, $U = A - (X_0 = 0)$. Then

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$\phi|_U \in H^1_{\mathcal{D}}(U,(X_0 = 0);\ \mathbb{Z}(1))$, the group of invertible regular functions on U which are 1 on $(X_0 = 0)$ and $X_i \in H^1_{\mathcal{D}}(U,\mathbb{Z}(1))$, the group of invertible regular functions on U. One considers the cup product $\{\phi|_U,X_1,\dots,X_n\}$ in the Deligne-Beilinson cohomology group $H^{n+1}_{\mathcal{D}}(U,(X_0 = 0);\ \mathbb{Z}(n+1))$. As $H^{\bullet}_{\mathcal{D}}(U,(X_0 = 0);\cdot) \xleftarrow[\text{rest}]{} H^{\bullet}_{\mathcal{D}}(A,(X_0 = 0);\cdot)$ is an isomorphism, this defines an element $\text{rest}^{-1}\{\phi|_U,X_1,\dots,X_n\}$ in $H^{n+1}_{\mathcal{D}}(A,(X_0 = 0);\ \mathbb{Z}(n+1))$. This is the <u>Loday symbol in the Deligne-Beilinson cohomology</u>. In this article we give <u>explicite formuli</u> (<u>modulo torsion</u>) for the Loday symbol as a <u>Cech cocycle</u> (1.8), (2.3), (2.5)i.

Let $h : X \longrightarrow A$ be an algebraic morphism, with X smooth. This gives explicite formuli for $h^*\text{rest}^{-1}\{\phi|_U,X_1,\dots,X_n\}$ in $H^{n+1}_{\mathcal{D}}(X,S;\mathbb{Q}(n+1))$ if $h(S) \subset (X_0 = 0)$. If dimension $X \leq n$, then $H^{n+1}_{\mathcal{D}}(X,S;\mathbb{Q}(n+1)) = H^n(X,S;\mathbb{C}/\mathbb{Q}(n+1))$, the Betti cohomology group. Therefore we may evaluate $h^*\text{rest}^{-1}\{\phi_{|U},X_1,\dots,X_n\}$ along relative homology classes $[\gamma] \in H_n(X,S;\mathbb{Z})$. The previous explicite formuli give an expression (3.9) for this evaluation under certain assumptions on a representative γ of $[\gamma]$.

Our method consists of reducing the problem to the analytic Deligne cohomology (1.3), and there to define a substitute for the cup product if the functions X_i, $i \geq 1$ are not invertible (1.4), (1.5). As this definition makes sense for analytic varieties as well, we define in this way a sort of Loday symbol in the analytic case (1.6), (1.7), which is no longer unique (2.5) ii, (2.5) iii.

In §4 we weaken the condition on the dimension of the algebraic variety X by an assumption on the curvature of a sum of pull-backs of the Loday symbol. This allows to define it as the class of a global closed holomorphic n-form (4.2). We give in (4.4) and (4.5) the evaluation of this class along relative cycles with some assumptions which are milder than in (3.9).

Finally in (4.7) we explain the relationship with Bloch's regulator map $K_2(X)_{\mathbb{Q}} \longrightarrow H^2_{\mathcal{D}}(X,\mathbb{Q}(2))$ in any dimension.

I thank cordially M. Rapoport with whom I discussed several times on those points.

§1. Construction of a class x in $H_{\mathcal{D}}^{n+1}(A,Y;\mathbb{Q}(n+1))$

1.1 Let A be a smooth algebraic variety over $\mathbb{C}$, $Y+Z$ be a normal crossing divisor on A, where Z is defined by $X_1 \ldots X_m$, X_i being a global regular reduced function on A. We define the natural embeddings

$$\begin{array}{ccc} A-Y & \xrightarrow{\ i\ } & A \\ \uparrow & \nearrow_{\lambda} & \\ A-Y-Z & & \end{array}$$

Let ϕ be in $H_{\mathcal{D}}^1(A,Y+Z;\mathbb{Z}(1))$

$$= \ker \mathcal{O}(A)^* \longrightarrow \mathcal{O}(Y+Z)^*.$$

Define $U = A - Z$, $Y_U = Y \cap U$.

Then $\phi|_U$ lies in $H_{\mathcal{D}}^1(U,Y_U;\mathbb{Z}(1))$

$$= \operatorname{Ker} \mathcal{O}(U)^* \longrightarrow \mathcal{O}(Y_U)^*,$$

and X_i lies in $H_{\mathcal{D}}^1(U,\mathbb{Z}(1)) = \mathcal{O}(U)^*$. Choose $1 \leq n \leq m$. Then the cup product $\{\phi|_U, X_1, \ldots, X_n\}$ is defined as an element in $H_{\mathcal{D}}^{n+1}(U,Y_U;\mathbb{Z}(n+1))$. We construct in §1 a specific element $x \in H_{\mathcal{D}}^{n+1}(A,Y;\mathbb{Q}(n+1))$ from which we show in §2 that its restriction to U $x|_U \in H_{\mathcal{D}}^{n+1}(U,Y_U;\mathbb{Q}(n+1))$ is precisely $\{\phi|_U, X_1, \ldots, X_n\}_{\mathbb{Q}}$. In other words, we define a lifting of the cup product across Z.

1.2 Here we show that the problem is reduced to a problem in the analytic Deligne cohomology. Recall [E.V], 2.9 that

$$H^{q+1}_{\mathcal{D}}(A,Y;\mathbb{Z}(p+1))$$
$$= H^{q+1}(\bar{A},\mathrm{cone}[Rk_* i_! \mathbb{Z}(p+1) + F^{p+1}(\log\ (H+\bar{Y})(-\bar{Y}))$$
$$\longrightarrow \Omega^{\bullet}_{\bar{A}}(*H + \log \bar{Y})(-\bar{Y})][-1])$$

where $k : A \longrightarrow \bar{A}$ is a good compactification such that $H := \bar{A} - A$, $\bar{Y}$:= closure of Y in $\bar{A}$ and $H + \bar{Y}$ are divisors with normal crossings.
Forgetting the growth condition along H on the F^{p+1} part, one obtains a morphism in the analytic Deligne cohomology [E.V], 2.13:

$$H^{q+1}_{\mathcal{D},an}(A,Y;\mathbb{Z}(p+1))$$

$$= H^{q+1}(A,\mathrm{cone}[i_! \mathbb{Z}(p+1) + \Omega^{\geq p+1}_{A}(\log\ Y)(-Y)$$
$$\longrightarrow \Omega^{\bullet}_{A}(\log\ Y)(-Y)][-1])$$

$$= H^{q+1}(A,i_! \mathbb{Z}(p+1) + \Omega^{\leq p}_{A}(\log\ Y)(-Y)).$$

One obtains a commutative diagram of exact sequences

$$0 \longrightarrow \frac{H^q(A,Y;\mathbb{C})}{H^q(A,Y;\mathbb{Q}(p+1))+F^{p+1}H^q(A,Y;\mathbb{C})} \longrightarrow H^{q+1}_{\mathcal{D}}(A,Y;\mathbb{Q}(p+1)) \xrightarrow{\ d\ } H^{q+1}(A,Y;\mathbb{Q}(p+1))\cap F^{p+1}H^{q+1}(A,Y;\mathbb{C}) \longrightarrow 0$$

$$\Big\downarrow \qquad\qquad \Big\downarrow f_{p+1,q+1} \qquad\qquad \Big\|$$

$$0 \longrightarrow \frac{H^q(A,Y;\mathbb{C})}{H^q(A,Y;\mathbb{Q}(p+1))+H^q(A,\Omega_A^{\leq p+1}(\log Y)(-Y))} \longrightarrow H^{q+1}_{\mathcal{D},an}(A,Y;\mathbb{Q}(p+1)) \xrightarrow{\ d\ } H^{q+1}(A,Y;\mathbb{Q}(p+1) \cap H^{q+1}(A,\Omega_A^{\geq p+1}(\log Y)(-Y)) \longrightarrow 0$$

<u>Lemma</u> (see also [E.V], 2.13 and [B], 1.6.1)

-i- $f_{n+1,n+1}$ is injective. One has

$H^{n+1}_{\mathcal{D}}(A,Y;\mathbb{Q}(n+1)) = \{x \in H^{n+1}_{\mathcal{D}}(A,Y;\mathbb{Q}(n+1)$, such
    that $dx \in F^{n+1}H^{n+1}(A,Y;\mathbb{C})\}$

and Ker $d = H^n(A,Y;\mathbb{C}/\mathbb{Q}(n+1))$

-ii- $f_{p+1,q+1}$ is an isomorphism for $q < p$. One has then

$$H^{q+1}_{\mathcal{D}}(A,Y;\mathbb{Q}(p+1)) = H^q(A,Y;\mathbb{C}/\mathbb{Q}(p+1))$$

-iii- $f_{p+1,q+1}$ is an isomorphism for $\dim A < p + 1$.
One has then

$$H_{\mathfrak{D}}^{q+1}(A,Y;\mathbb{Q}(p+1)) = H^q(A,Y;\mathbb{C}/\mathbb{Q}(p+1))$$

<u>Proof.</u>

-i- One has $F^{n+1}H^n(A,Y;\mathbb{C}) = 0 = H^n(A,\Omega_A^{\geq n+1}(\log Y)(-Y))$ and $F^{n+1}H^{n+1}(A,Y;\mathbb{C}) = H^0(\bar{A},\Omega_{\bar{A}}^{n+1}(\log(H+\bar{Y})(-\bar{Y}))_{d\text{ closed}}$ is embedded in

$$H^{n+1}(A,\Omega_A^{\geq n+1}(\log Y)(-Y)) = H^0(A,\Omega_A^{n+1}(\log Y)(-Y))_{d\text{ closed}}.$$

One has

$\dfrac{H^n(A,Y;\mathbb{C})}{H^n(A,Y;\mathbb{Q}(n+1))} = H^n(A,Y;\mathbb{C}/\mathbb{Q}(n+1))$ as $H^{n+1}(A,Y;\mathbb{Q}(n+1))$ is torsion free.

ii,iii. In both cases the cohomology of F^{p+1} and $\Omega^{\geq p+1}$ appearing in the exact sequences vanish.

1.3 <u>Corollary.</u> In order to construct an element $x \in H_{\mathfrak{D}}^{n+1}(A,Y;\mathbb{Q}(n+1))$, it is enough to construct it as an element of $H_{\mathfrak{D},an}^{n+1}(A,Y;\mathbb{Q}(n+1))$ and to verify that its curvature dx is algebraic, that is in $F^{n+1}H^{n+1}(A,Y;\mathbb{C})$.

<u>Therefore in</u> (1.4), (1.5), (1.6), (1.7), <u>we assume only</u> A, $Y+Z$ <u>to be analytic</u>, X_i <u>to be global holomorphic on</u> A, ϕ <u>to be global holomorphc invertible on</u> A <u>such that</u> $\phi_{|Y\cup Z} = 1$.

1.4 Consider $\phi : A \longrightarrow \mathbb{C}^*$, with $\phi(Y \cup Z) = 1$. Let $\mathscr{A}_0 \cup \mathscr{A}_1$ be an analytic open cover of $\mathbb{C}^*$ such that $1 \in \mathscr{A}_1 - \mathscr{A}_0$, $\log \phi|_{\phi^{-1}(\mathscr{A}_i)}$ is single valued and $\log \phi|_{\phi^{-1}(\mathscr{A}_1) \cap (Y \cup Z)} = 0$. One has

$$\log \phi|_{\phi^{-1}(\mathscr{A}_i)} \in H^0(\phi^{-1}(\mathscr{A}_i), \mathcal{O}_A(-Y - Z)).$$

Then for any refinement $(A_i)_{i \in I}$ of $\phi^{-1}(\mathscr{A}_i)$, with map $\sigma : I \longrightarrow \{0,1\}$, one has

$$\alpha) \quad \log_i \phi := \log \phi|_{A_i \subset \phi^{-1}(\mathscr{A}_{\sigma(i)})} \in H^0(A_i, \mathcal{O}_{A_i}(-Y-Z))$$

$$\beta) \quad z^{n-1}_{i_0 i_1} := (\delta \log \phi)_{i_0 i_1} = \log_{i_1} \phi - \log_{i_0} \phi \in H^0(A_{i_0 i_1}, \lambda_! \mathbb{Z}(1))$$

and $(\delta z^{n-1}) = 0$.

Take such a refinement with

γ) if $A_{i_0 \ldots i_k} \cap (Y \cup Z) = \phi$,

$$\log_{i_0 \ldots i_k} X_k \in H^0(A_{i_0 \ldots i_k}, \mathcal{O}_A).$$

Define

$$g_{i_0 \dots i_k} = \log_{i_0 \dots i_k} X_k \quad \text{if } A_{i_0 \dots i_k} \cap (Y \cup Z) = \phi$$
$$0 \qquad\qquad \text{if } A_{i_0 \dots i_k} \cap (Y \cup Z) \neq \phi .$$

One has

$$g_{i_0 \dots i_k} \in H^0(A_{i_0 \dots i_k}, \mathcal{O}_A(-Y-Z)) .$$

We want to construct

$$\overline{x} \in H^{n+1}(A, \lambda_! \mathbb{Z}(n+1) \to \Omega_A^{\leq n-1}(\log(Y+Z))(-Y-Z) \to \Omega_A^n(\log Y)(-Y))$$

as a cocycle $\overline{x} = (x^{-1}, x^0, \dots, x^n)$ in the Cech complex

$(\mathcal{C}^\bullet(A_i, \lambda_! \mathbb{Z}(n+1) \longrightarrow \Omega_A^{\leq n-1}(\log (Y+Z)(-Y-Z) \longrightarrow \Omega_A^n(\log Y)(-Y)),$
$(-1)^\bullet \delta + d)$:

$$x^{-1} \in \mathcal{C}^{n+1}(\lambda_! \mathbb{Z}(n+1))$$
$$x^0 \in \mathcal{C}^n(\mathcal{O}_A(-Y-Z))$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$x^n \in \mathcal{C}^0(\Omega_A^n(\log Y)(-Y))$$

with $(-1)^{n+1}\delta x^j + dx^{j-1} = 0$

1.5 The condition 1.4, α implies that

$x_i^n := \log_i \phi \frac{dX_1}{X_1} \wedge \ldots \wedge \frac{dX_n}{X_n}$ is in $H^0(A_i, \Omega_A^n(\log Y)(-Y))$. This defines x_i^n.

We have to resolve the equation

$$(dx^{n-1})_{i_0 i_1} = (-1)^n (\delta x^n)_{i_0 i_1} = (-1)^n z^{n-1}_{i_0 i_1} \frac{dX_1}{X_1} \wedge \ldots \wedge \frac{dX_n}{X_n} .$$

Define

$$x^{n-1}_{i_0 i_0} = (-1)^n z^{n-1}_{i_0 i_1} \, g_{i_0 i_n} \frac{dX_2}{X_2} \wedge \ldots \wedge \frac{dX_n}{X_n}$$

$$\in H^0(A_{i_0 i_1}, \Omega_A^{n-1}(\log(Y+Z))(-Y-Z)).$$

Assume by induction that we may define for $1 \leq \ell \leq k$

$$z^{n-\ell}_{i_0 \ldots i_\ell} \in H^0(A_{i_0 \ldots i_\ell}, \lambda_! \mathbb{Z}(\ell))$$

with $(\delta z^{n-\ell}) = 0$

$$x^{n-\ell}_{i_0 \ldots i_\ell} = (-1)^{\ell n} z^{n-\ell}_{i_0 \ldots i_\ell} \, g_{i_0 \ldots i_\ell} \frac{dX_{\ell+1}}{X_{\ell+1}} \wedge \ldots \wedge \frac{dX_n}{X_n}$$

$$dx^{n-\ell} = (-1)^n \delta x^{n-\ell+1} \qquad \ell \leq k$$

Define

$$z^{n-(k+1)}_{i_0\ldots i_{k+1}} := \delta(z^{n-k}_{i_0\ldots i_k} g_{i_0\ldots i_k}).$$

If for all $\ell \in \{0,\ldots,k+1\}$,

$$A_{i_0\ldots\hat{i}_\ell\ldots i_{k+1}} \cap (Y \cup Z) \neq \phi, \text{ then } z^{n-(k+1)}_{i_0\ldots i_{k+1}} = 0$$

(especially if $A_{i_0\ldots i_{k+1}} \cap (Y \cup Z) \neq \phi$).

Otherwise $A_{i_1\ldots i_{k+1}} \cap (Y \cup Z) = \phi$ (say).

Then

$$z^{n-(k+1)}_{i_0\ldots i_{k+1}} = \sum_{\ell=1}^{k+1} (-1)^\ell z^{n-k}_{i_0\ldots\hat{i}_\ell\ldots i_{k+1}} (g_{i_0\ldots\hat{i}_\ell\ldots i_{k+1}} - g_{i_1\ldots i_{k+1}})$$

$$+ (\delta z^{n-k})_{i_0\ldots i_{k+1}} g_{i_1\ldots i_{k+1}}.$$

If $z_{i_0\ldots\hat{i}_\ell\ldots i_{k+1}} \neq 0$, then $A_{i_0\ldots\hat{i}_\ell\ldots i_{k+1}} \cap (Y \cup Z) = \phi$,
therefore $g_{i_0\ldots\hat{i}_\ell\ldots i_{k+1}} - g_{i_1\ldots i_{k+1}} \in \mathbb{Z}(1)$.

Therefore one has

$$z^{n-(k+1)}_{i_0\ldots i_{k+1}} \in H^0(A_{i_0\ldots i_{k+1}}, \lambda_! \mathbb{Z}(k+1)).$$

We may define

$$x^{n-(k+1)}_{i_0\ldots i_{k+1}} = (-1)^{(k+1)\cdot n} z^{n-(k+1)}_{i_0\ldots i_{k+1}}\ g_{i_0\ldots i_{k+1}} \frac{dX_{k+2}}{X_{k+2}} \wedge\ldots\wedge \frac{dX_n}{X_n}$$

$$\in H^0(A_{i_0\ldots i_{k+1}}, \Omega_A^{n-(k+1)}(\log(Y+Z))(-Y-Z))$$

with $dx^{n-(k+1)} = (-1)^n \delta x^{n-k}$ if $k < n$.

If $k = n$

$$x^{-1}_{i_0\ldots i_{n+1}} = (-1)^{(n+1)n} z^{-1}_{i_0\ldots i_{n+1}}.$$

1.6 <u>Proposition.</u> The Cech cocycle $\bar{x} = (x^{-1}, x^0, \ldots, x^n)$ constructed in (1.5) defines a cohomology class

$$\bar{x} \in H^{n+1}(A, \lambda_! \mathbb{Z}(n+1)) \longrightarrow \Omega_A^{\leq n-1}(\log\ (Y+Z))(-Y-Z)$$
$$\downarrow$$
$$\Omega_A^n(\log Y)(-Y)).$$

1.7 Let Z_ℓ be a smooth component of Z. We consider the morphism of restriction

$$\begin{array}{ccc} i_! \mathbb{Z}(n+1) & \longrightarrow & \Omega_A^{\leq n}(\log Y)(-Y) \\ \downarrow \frac{1}{2i\pi} & & \downarrow \text{restriction}_\ell \\ i|_{Z_\ell !} \mathbb{Z}(n) & \longrightarrow & \Omega_{Z_\ell}^{\leq n-1}(\log Y)(-Y) \end{array}$$

whose kernel contains

$$\lambda_! \mathbb{Z}(n+1) \longrightarrow \Omega_A^{\leq n-1}(\log\ (Y+Z))(-Y-Z) \longrightarrow \Omega_A^n(\log\ Y)(-Y)\ ,$$

and whose cohomology reads

$$H^{n+1}_{\mathfrak{D},an}(A,Y;\mathbb{Q}(n+1)) \xrightarrow[\text{restriction}_{\Omega}]{} H^{n+1}_{\mathfrak{D},an}(Z_{\Omega},Y;\mathbb{Q}(n)).$$

<u>Theorem.</u> There is a class

$x \in H^{n+1}_{\mathfrak{D},an}(A,Y;\mathbb{Q}(n+1))$, such that $\text{restriction}_{\Omega}\ x = 0$ and such that

$$dx = \frac{d\phi}{\phi} \wedge \frac{dX_1}{X_1} \wedge \dots \wedge \frac{dX_n}{X_n} \in H^{n+1}(A,Y;\mathbb{Q}(n+1)) \cap H^0(A,\Omega_A^{n+1}(\log\ Y)(-Y))_{d\ \text{closed}}\ .$$

<u>Proof.</u> Define x as the image of $\bar{x}$ via

$$\begin{array}{l} H^{n+1}(A,\lambda_! \mathbb{Z}(n+1) \longrightarrow \Omega_A^{\leq n-1}(\log\ (Y+Z))(-Y-Z) \longrightarrow \Omega_A^n(\log\ Y)(-Y)) \\ \qquad\qquad \downarrow \\ H^{n+1}_{\mathfrak{D},an}(A,Y;\mathbb{Q}(n+1)) \end{array}$$

given by the same cocycle. One has $dx = dx_i^n$.

1.8 Go back to the algebraic situation described in 1.1.

Then $dx = \frac{d\phi}{\phi} \wedge \frac{dX_1}{X_1} \wedge \dots \wedge \frac{dX_n}{X_n} \in F^{n+1}H^{n+1}(A,Y;\mathbb{C})$.

We obtain by 1.2i

<u>Theorem.</u> The class x of 1.7 is in

$$H_{\mathfrak{D}}^{n+1}(A,Y;\mathbb{Q}(n+1)) \quad \text{and} \quad dx = \frac{d\phi}{\phi} \wedge \frac{dX_1}{X_1} \wedge \dots \wedge \frac{dX_n}{X_n} .$$

§2. Restriction of x to U.

2.1. In this paragraph, we want to show that the restriction to U of the class x constructed in 1.8 is

$$y := \{\phi_{|U}, X_1, \ldots, X_n\} \in H^{n+1}_{\mathcal{D}}(U, Y_U; \mathbb{Q}(n+1)).$$

As $dy = \frac{d\phi}{\phi} \wedge \frac{dX_1}{X_1} \wedge \ldots \wedge \frac{dX_n}{X_n}$ [E.V], (3.7), we have by (1.2)i:

Lemma. $(x_{|U} - y) \in H^n(U, Y_U; \mathbb{C}/\mathbb{Q}(n+1))$.
Therefore we may assume, as in (1.4), (1.5), (1.6) and (1.7) that A - and therefore U - are only analytic manifolds.

2.2 We take a refinement U_j of $X_j \cap U$ such that $\log X_{i|U_j} := \log_j X_i$ is single valued, that is $\log_j X_i \in H^0(U_j, \mathcal{O}_U)$ for $i \leq n$. Define $\mu = i|_U : U - Y_U \longrightarrow U$. Define y as a cocycle $y = (y^{-1}, y^0, \ldots, y^n)$ in the Cech complex $(\mathcal{C}^{\bullet}(U_j, \mu_! \mathbb{Z}(n+1) \longrightarrow \Omega^{\leq n}_U(\log Y_U)(-Y_U)), (-1)^{\bullet}\delta + d)$ with

$y^{-1} \in \mathcal{C}^{n+1}(\mu_! \mathbb{Z}(n+1))$

$y^0 \in \mathcal{C}^n(\mathcal{O}_U(-Y_U))$

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$y^n \in \mathcal{C}^0(\Omega^n_U(\log Y)(-Y))$

with $(-1)^{n+1}\delta y^j + dy^{j-1} = 0.$

One has [E.V.] (3.2):

$$y_j^n = \log_j \phi \frac{dX_1}{X_1} \wedge \dots \wedge \frac{dX_n}{X_n}$$

$$y_{j_0 j_1}^{n-1} = (-1)^n z_{j_0 j_1}^{n-1} \log_{j_1} X_1 \frac{dX_2}{X_2} \wedge \dots \wedge \frac{dX_n}{X_n}$$

.

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$$y_{j_0 \dots j_k}^{n-k} = (-1)^{kn} z_{j_0 \dots j_k}^{n-k} \log_{j_k} X_k \frac{dX_{k+1}}{X_{k+1}} \wedge \dots \wedge \frac{dX_n}{X_n}$$

$$y_{j_0 j_{n+1}}^{-1} = (-1)^{(n+1)n} z_{j_0 \dots j_{n+1}}^{-1}$$

with $z_{j_0 j_1}^{n-1} = z_{j_0 j_1}^{n-1} = (\delta \log \phi)_{j_0 j_1} \in H^0(U_{j_0 j_1}, \mu_! \mathbb{Z}(1))$

$$z_{j_0 \dots j_k}^{n-k} = \delta(z_{j_0 \dots j_{k-1}}^{n-k+1} \log_{j_{k-1}} X_{k-1})$$

$$\in H^0(U_{j_0 \dots j_k}, \mu_! \mathbb{Z}(k))$$

Therefore one has

$$x^n - y^n = 0$$

and for $1 \leq k \leq n$:

$$(x^{n-k} - y^{n-k})_{i_0 \dots i_k} = (-1)^{n-k}(z_{i_0 \dots i_k}^{n-k} g_{i_0 \dots i_k} - z_{i_0 \dots i_k}^{n-k} \log_{i_k} X_k$$

$) \cdot$

$$\frac{dX_{k+1}}{X_{k+1}} \wedge \dots \wedge \frac{dX_n}{X_n}$$

and

$$x^{-1} - y^{-1} = (-1)^{(n+1)n}(z^{-1} - z^{-1}).$$

2.3 Define

$$N^{n-1}_{i_0 i_1} = z^{n-1}_{i_0 i_1}\, g_{i_0 i_1} - z^{n-1}_{i_0 i_1}\, \log_{i_1} X_1$$

$$= z^{n-1}_{i_0 i_1}(g_{i_0 i_1} - \log_{i_1} X_1) \in H^0(U_{i_0 i_1}, \mu_! \mathbb{Z}(2))$$

$$(\delta N^{n-1}) = z^{n-2} - z^{n-2}.$$

Define

$$r^{n-2}_{i_0 i_1} = (-1)^n N^{n-1}_{i_0 i_1}\, \log_{i_1} X_2 \frac{dX_3}{X_3} \wedge \ldots \wedge \frac{dX_n}{X_n}$$

$$\in H^0(U_{i_0 i_1}, \Omega^{n-2}_U(\log Y_U)(-Y_U)).$$

One has

$$x^{n-1} - y^{n-1} - dr^{n-2} = 0$$

Define by induction $1 \leq \ell \leq k$:

$$N^{n-\ell}_{i_0 \ldots i_\ell} \in H^0(U_{i_0 \ldots i_\ell}, \mu_! \mathbb{Z}(\ell+1))$$

with $\delta N^{n-\ell} = z^{n-\ell-1} - Z^{n-\ell-1}$

$$r^{n-\ell-1}_{i_0\ldots i_\ell} = (-1)^{\ell n} N^{n-\ell}_{i_0\ldots i_\ell} \log_{i_\ell} X_{\ell+1} \frac{dX_{\ell+2}}{X_{\ell+2}} \wedge \ldots \wedge \frac{dX_n}{X_n}$$

$$\in H^0(U_{i_0\ldots i_\ell}, \Omega_U^{n-(\ell+1)}(\log Y_U)(-Y_U))$$

such that

$$x^{n-\ell} - y^{n-\ell} - ((-1)^n \delta r^{n-\ell} + dr^{n-(\ell+1)}) = 0 \quad \ell < k .$$

Define

$$N^{n-k}_{i_0\ldots i_k} = z^{n-k}_{i_0\ldots i_k} g_{i_0\ldots i_k} - Z^{n-k}_{i_0\ldots i_k} \log_{i_k} X_k$$

$$- \delta(N^{n-k+1}_{i_0\ldots i_{k-1}} \log_{i_{k-1}} X_k)_{i_0\ldots i_k}.$$

One has

$$\delta N^{n-k} = z^{n-k-1} - Z^{n-k-1}$$

and

$$N^{n-k}_{i_0\ldots i_k} = z^{n-k}_{i_o\ldots i_k}(g_{i_0\ldots i_k} - \log_{i_k} X_k)$$

$$- (-1)^{k-1} N^{n-k+1}_{i_0\ldots i_{k-1}} (\delta \log X_k)_{i_{k-1} i_k}$$

$$\in H^0(U_{i_0\ldots i_k}, \mu_! \mathbb{Z}(k+1)).$$

Define

$$r^{n-k-1}_{i_0\ldots i_k} = (-1)^{kn} N^{n-k}_{i_0\ldots i_k} \log_{i_k} X_{k+1} \frac{dX_{k+2}}{X_{k+2}} \wedge \ldots \wedge \frac{dX_n}{X_n}$$

$$\in H^0(U_{i_0\ldots i_k}, \Omega_U^{n-(k+1)}(\log Y_U)(-Y_U))$$

then

$$x^{n-k} - y^{n-k} - ((-1)^n \delta r^{n-k} - dr^{n-k-1}) = 0$$

Therefore one has

$x-y - ((-1)^n\delta + d)r = 0$, and $x - y$ is a coboundary.

<u>Proposition.</u> One has

$x|_U = y$ in $H^{n+1}_{\mathcal{D},an}(U, Y_U; \mathbb{Z}(n+1))$ and

$x|_U = y$ in $H^{n+1}_{\mathcal{D}}(U, Y_U; \mathbb{Q}(n+1))$.

2.4 Consider the morphisms

rest: $H_{\mathcal{D}}^{n+1}(A,Y;\mathbb{Q}(n+1)) \longrightarrow H_{\mathcal{D}}^{n+1}(U,Y_U;\mathbb{Q}(n+1))$

(respectively, if A is analytic

$$\text{rest}^{an} : H_{\mathcal{D},an}^{n+1}(A,Y;\mathbb{Z}(n+1)) \longrightarrow H_{\mathcal{D},an}^{n+1}(U,Y_U;\mathbb{Z}(n+1))$$

and

$$U : H_{\mathcal{D}}^{1}(A,Y+Z;\mathbb{Z}(1)) \longrightarrow H_{\mathcal{D}}^{n+1}(U,Y_U;\mathbb{Q}(n+1))$$

(respectively, if A is analytic

$$U^{an} : H_{\mathcal{D},an}^{1}(A,Y+Z;\mathbb{Z}(1)) \longrightarrow H_{\mathcal{D}}^{n+1}(U,Y_U;\mathbb{Z}(n+1))$$

defined by

$$U\phi = \{\phi|_U, X_1, \ldots, X_n\}.$$

Then (1.7), (1.8) and (2.3) prove the

<u>Theorem</u>

image $U \subset$ image rest

(respectively image $U^{an} \subset$ image rest^{an}).

2.5 <u>Remarks</u>

-i- <u>The universal situation</u>

Consider

$$B := \mathbb{A}_{\mathbb{C}}^{n+1} - (\Psi = 0), \quad \Psi = 1 - Y_0 \dots Y_n$$

where Y_i are the coordinates. Then one has [N], (2.1):

$$H_{\mathcal{D}}^{n+1}(B,(Y_0 = 0);\mathbb{Z}(n+1)) \xrightarrow{\text{rest}} H_{\mathcal{D}}^{n+1}(B - \bigcup_1^n (Y_i = 0),(Y_0=0);\mathbb{Z}(n+1))$$

is an isomorphism. Take A as in (1.1). Then $(1 - \phi)/X_1 \dots X_n \in H^0(A,\mathcal{O}(-Y))$. Define $X_0 := (1-\phi)/X_1 \dots X_n$. One defines a morphism

$$h_\phi : A \longrightarrow B$$
$$X_i \longleftarrow Y_i \quad 0 \le i \le n$$

with $h_\phi^* \Psi = \phi$.

Then

$$h_\phi^* \text{rest}^{-1}\{\Psi|_{B - \bigcup_1^n (Y_i = 0)}, Y_1,\dots,Y_n\} = x'$$

is in $H_{\mathcal{D}}^{n+1}(A,Y;\mathbb{Q}(n+1))$, of restriction

$$\text{rest } x' = h_\phi^*\{\Psi|_{B - \bigcup_1^n (Y_i = 0)}, Y_1,\dots,Y_n\}$$

$$= \{\phi|_U, X_1,\dots,X_n\}.$$

In (1.5), we have given explicite formuli for x as a Cech cocycle. This applies for

$$\text{rest}^{-1}\{\Psi|_{B - \overset{n}{\underset{1}{\cup}} (Y_i = 0)}, Y_1, \ldots, Y_n\},$$

and therefore by pull-back for x'. Of course we could have worked directly on B, the universal case.

-ii- If A is only analytic, there is no universal situation. One observes the following: [N], (2.1) and (1.2) imply that

$H^n(B,(Y_0 = 0); \mathbb{C}/\mathbb{Q}(n+1))$
$= H^n(B - \overset{n}{\underset{1}{\cup}}(Y_i = 0), (Y_0 = 0); \mathbb{C}/\mathbb{Q}(n+1))$, and therefore that

$H^{n+1}_{\mathcal{D},an}(B,(Y_0=0);\mathbb{Q}(n+1))$ injects into

$$H^{n+1}_{\mathcal{D},an}(B - \overset{n}{\underset{1}{\cup}}(Y_i = 0), (Y_0 = 0); \mathbb{Q}(n+1)).$$

The class x of (1.5) is then uniquely defined by (2.3):

$$x|_{B- \overset{n}{\underset{1}{\cup}} (Y_i = 0)} = y \quad \text{in}$$

$$H^{n+1}_{\mathcal{D},an}(B - \overset{n}{\underset{1}{\cup}} (Y_i = 0), \quad (Y_0 = 0); \mathbb{Q}(n+1)).$$

-iii- More generally, whenever $H^n(A,Y;\mathbb{C}/\mathbb{Q}(n+1))$ injects into $H^n(U,Y_U;\mathbb{C}/\mathbb{Q}(n+1))$, then rest^{an} is injective (modulo torsion) via (1.2). Therefore in this case x constructed in (1.5) is uniquely defined by $x|_U$ via (2.3).

§3 Pull-back of x to X and formula [N], II.2.4

3.1 Let X be a smooth algebraic variety over $\mathbb{C}$ of dimension $\leq n$, equiped with a morphism

$$h : X \longrightarrow A$$

where now A is the universal situation described in (2.5),i, with coordinates X_i, and with $\phi = 1 - X_0 \ldots X_n$.

Define $h^* X_i = a_i \in H^0(X, \mathcal{O}_X)$ for $i \geq 1$

$$h^* \phi = f \in H^1_{\mathcal{D}}(X, S + T; \mathbb{Z}(1))$$

where T is defined by

$$t := a_1 \ldots a_n \quad \text{and} \quad S \text{ is a divisor contained in } h^* Y.$$

Define

$$\begin{array}{ccc} X - S & \xrightarrow{j} & X \\ \uparrow & \overset{\nu}{\nearrow} & \\ X-S-T & & \end{array}$$

One has

$$h^* \text{rest}^{-1} \{\phi|_U, X_1, \ldots, X_n\} \in H^{n+1}_{\mathcal{D}}(X, S; \mathbb{Q}(n+1))$$
$$= H^n(X, S; \mathbb{C}/\mathbb{Q}(n+1)) \quad (1.2)\text{iii}.$$

As S is not necessarily a normal crossing divisor, we will explain this more precisely (3.2), (3.3), (3 4), (3.5), (3.6). Then we want to evaluate this class along relative homology classes $[\gamma] \in H_n(X,S;\mathbb{Z})$. (3.4)

3.2 We assume in (3.2), (3.3), (3.4) that X is smooth analytic, T is a divisor defined by $a_1 \ldots a_n = t = 0$, $a_i \in H^0(X,\mathcal{O}_X)$, and S is a divisor.

We define subcomplexes $\Omega^{\bullet}_{X,S+T}$ and $\Omega^{\bullet}_{X,S}$ of the holomorphic de Rham complex $\Omega^{\bullet}_X$ by: for each open set U $\Omega^i_{X,S}(U) = \{\omega \in \Omega^i_X(U),\ \omega_{|S\cap U} = 0\}$, $\Omega^i_{X,S+T}(U) = \{\omega \in \Omega^i_{X,S}(U),$ $\omega|_{a_j=0} = 0$ for any $1 \leq j \leq n\}$.
The sheaves $\Omega^i_{X,S}$ and $\Omega^i_{X,S+T}$ are coherent. As $\Omega^0_{X,S} = \mathcal{O}_X(-S)$, one has a natural inclusion

$$j_!\mathbb{C} \xrightarrow{\text{incl}} \Omega^{\bullet}_{X,S}$$

which defines a map in cohomology

$$H^{\bullet}(X,S;\mathbb{C}) \xrightarrow{\text{incl}} H^{\bullet}(X,\Omega^{\bullet}_{X,S}).$$

If S is a divisor with normal crossings, then $\Omega^{\bullet}_{X,S}$ is the complex $\Omega^{\bullet}_X(\log S)(-S)$, and incl is a quasi isomorphism. In general we construct a "splitting" of incl.

<u>Lemma.</u> There is a morphism p in $D^b(X)$

$$p : \Omega^{\bullet}_{X,S} \longrightarrow j_!\mathbb{C}$$

such that $p \circ \mathrm{incl}$ is an isomorphism.

<u>Proof.</u> Let $\sigma : \tilde{X} \longrightarrow X$ be an embedded resolution of S. This means $\sigma^{-1}S = \tilde{S}$ is a divisor with normal crossings, σ is proper and $\sigma|_{X-S}$ is an isomorphism.

Consider

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sigma} & X \\ {\scriptstyle \tilde{j}}\nwarrow & & \nearrow{\scriptstyle j} \\ & X-S & \end{array}$$

One has $\sigma^*\Omega^i_{X,S} \subset \Omega^i_{\tilde{X}}(\log \tilde{S})(-\tilde{S})$,

and $\sigma^{-1}j_!\mathbb{C} \longrightarrow \tilde{j}_!\mathbb{C}$. Therefore one has a diagram in $D^b(X)$

$$\begin{array}{ccc} \Omega^{\bullet}_{X,S} & \xrightarrow{\sigma^*} & R\sigma_*\Omega^{\bullet}_{\tilde{X}}(\log \tilde{S})(-\tilde{S}) \\ {\scriptstyle \mathrm{incl}}\uparrow & & \uparrow{\scriptstyle R\sigma_*\widetilde{\mathrm{incl}}} \\ j_!\mathbb{C} & \xrightarrow{\sigma^{-1}} & R\sigma_*\tilde{j}_!\mathbb{C}. \end{array}$$

As σ is proper and $\tilde{j}$ is exact one has $R\sigma_*\tilde{j}_! = R\sigma_!\tilde{j}_! = R(\sigma \circ \tilde{j})_! = j_!$ in $D^b(X)$, and σ^{-1} is an isomorphism in $D^b(X)$. As $\widetilde{\mathrm{incl}}$ is a quasi isomorphism $R\sigma_*\,\widetilde{\mathrm{incl}}$ is an isomorphism in $D^b(X)$.

Define

$$p = (\sigma^{-1})^{-1} \circ (R\sigma_* \, \widetilde{\mathrm{incl}})^{-1} \circ \sigma^*$$

3.3 Define

$$K^\bullet = j_!\mathbb{Q}(n+1) \longrightarrow \Omega^\bullet_{X,S}$$

and

$$K'^\bullet = \nu_!\mathbb{Q}(n+1) \longrightarrow \Omega^{\leq n-1}_{X,S+T} \longrightarrow \Omega^n_{X,S}$$

which is a subcomplex of $K^\bullet$. One has:

$$\begin{array}{ccc} j_!\mathbb{Q}(n+1) & \longrightarrow & j_!\mathbb{C} \\ \downarrow \mathrm{incl} & & \\ K^\bullet & & \\ \downarrow p & & \\ j_!\mathbb{Q}(n+1) & \longrightarrow & j_!\mathbb{C} \end{array}$$

with: $p \circ \mathrm{incl}$ is an isomorphism (3.2).

<u>Corollary.</u> There are morphisms

$$\begin{array}{ccc} H^{\bullet-1}(X,S;\mathbb{C}/\mathbb{Q}(n+1)) & \xrightarrow{\mathrm{incl}} & H^\bullet(X,K^\bullet) \\ & & \downarrow p \\ & & H^{\bullet-1}(X,S;\mathbb{C}/\mathbb{Q}(n+1)) \end{array}$$

with: $p \circ \text{incl}$ is an isomorphism.

3.4 Let $\bar{z}$ be a cohomology class in

$$\frac{H^0(X,\Omega^n_{X,S})}{H^{n-1}(X,\nu_!\mathbb{Q}(n+1) \longrightarrow \Omega^{\leq n-1}_{X,S+T})} \subset H^{n+1}(X,K'^{\bullet})$$

of representative $\omega \in H^0(X,\Omega^n_{X,S})$.
Its image z in $H^{n+1}(X,K^{\bullet})$ lies in

$$\frac{H^0(X,\Omega^n_{X,S})}{H^{n-1}(X,j_!\mathbb{Q}(n+1) \longrightarrow \Omega^{\leq n-1}_{X,S})} \subset H^{n+1}(X,K^{\bullet})$$

and is of representative ω. Then for any n-chain γ with $\partial\gamma \subset S$ representing the homology class $[\gamma] \in H_n(X,S;\mathbb{Z})$ one has $\langle[\gamma],pz\rangle = \int_\gamma \omega$ modulo $\mathbb{Q}(n+1)$.

3.5 Remark

If X is affine, then one has $H^{n+1}(X,j_!\mathbb{Q}(n+1)) = 0$ by [BBD], 6.2.1. On the other hand, the sheaves $\Omega^i_{X,S}$ being coherent, they don't here higher cohomology. This implies

$$H^{n+1}(X,K'^{\bullet}) = \frac{H^0(X,\Omega^n_{X,S})}{H^{n-1}(X,S;\mathbb{Q}(n+1)) + dH^0(X,\Omega^{\leq n-1}_{X,S+T})}$$

and

$$H^{n+1}(X,K^{\bullet}) = \frac{H^0(X,\Omega^n_{X,S})}{H^{n-1}(X,S;\mathbb{Q}(n+1)) + dH^0(X,\Omega^{\leq n-1}_{X,S})}.$$

As $H^0(X,\Omega^{n-1}_{X,S+T})$ injects in $H^0(X,\Omega^{n-1}_{X,S})$ the map $H^{n+1}(X,K'^{\bullet}) \longrightarrow H^{n+1}(X,K^{\bullet})$ is surjective. One is then always in the situation of (3.4).

3.6 We go back to the situation (3.1). One has morphisms

$$h^*\Omega^i_A(\log\ (Y+Z))(-Y-Z) \longrightarrow \Omega^i_{X,S+T}$$
$$h^*\Omega^i_A(\log\ Y)(-Y) \longrightarrow \Omega^i_{X,S}$$
$$h^{-1}\lambda_!\mathbb{Q}(n+1) \longrightarrow \nu_!\mathbb{Q}(n+1)$$
$$h^{-1}i_!\mathbb{Q}(n+1) \longrightarrow j_!\mathbb{Q}(n+1)\ .$$

Therefore one has morphisms in $D^b(A)$:

$$\begin{array}{ccccc} \lambda_!\mathbb{Q}(n+1) & \longrightarrow & \Omega^{\leq n-1}_A(\log(Y+Z))(-Y-Z) & \longrightarrow & \Omega^n_A(\log\ Y)(-Y) \\ \downarrow h^* & & & & \\ Rh_*K'^{\bullet} & & & & \end{array}$$

and

$$\begin{array}{ccc} i_!\mathbb{Q}(n+1) & \longrightarrow & \Omega^{\leq n}_A(\log\ Y)(-Y) \\ \downarrow & & \\ Rh_*K^{\bullet}\ . & & \end{array}$$

This proves the

<u>Lemma</u>. One has commutative diagrams

$$
\begin{array}{ccc}
H^{n+1}_{\mathcal{D}}(A,Y;\mathbb{Q}(n+1)) & \longrightarrow & H^{n}(X,S;\mathbb{C}/\mathbb{Q}(n+1)) \\
{\scriptstyle 1.2.i}\downarrow & & \uparrow{\scriptstyle p} \\
H^{n+1}_{\mathcal{D},an}(A,Y;\mathbb{Q}(n+1)) & \xrightarrow{\quad h^{*}\quad} & H^{n+1}(X,K^{\cdot}) \\
\uparrow & & \uparrow \\
 & & H^{n+1}(X,K'^{\cdot}) \\
 & & \uparrow{\scriptstyle h^{*}}
\end{array}
$$

$$H^{n+1}(A,\lambda_{!}\mathbb{Q}(n+1) \longrightarrow \Omega_A^{\leq n-1}\ (\log\ (Y+Z))(-Y-Z) \longrightarrow \Omega_A^{n}(\log\ Y)(-Y))$$

3.7 Consider the open cover $h^{-1}A_j$ of X (1.4). Then $h^*\overline{x}$ is represented by the cocycle

$$h^*\overline{x} = (h^{-1}x^{-1}, h^*x^0, \ldots, h^*x^n) \quad \text{in}$$
$$(\mathcal{C}^{n+1}(h^{-1}A_i, K'^{\cdot}),\ (-1)^{n+1}\delta + d)$$

with

$$h^{-1}x^{-1} = (-1)^{(n+1)n} z^{-1}$$

$$h^*x^{n-k} = (-1)^{kn} z^{n-k}_{i_0\ldots i_k} h^* g_{i_0\ldots i_k} \frac{da_{k+1}}{a_{k+1}} \wedge \ldots \wedge \frac{da_n}{a_n} \qquad 1 \leq k \leq n$$

.

.

.

$$h^*x^n = \log_i f \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} \quad \text{with} \quad \log_i f = h^*\log_i \phi .$$

Define for simplicity

$$G_{i_0\ldots i_k} = h^* g_{i_0\ldots i_k} \in H^0(h^{-1}A_{i_0\ldots i_k}, \mathcal{O}_X(-S-T)).$$

3.8 Let X_j be a refinement of $h^{-1}A_j$ such that another determination $\ell n_j f$ of $\log_i f$ on X_j exists with

$$\ell n_j f \in H^0(X_j, t\mathcal{O}_X(-S)).$$

Observe that this implies
if $X_j \cap (S \cup T) \neq \phi$, then

$$\ell n_j f = \log_j f, \text{ and therefore } (\ell n_{i_1} f - \ell n_{i_0} f) \in H^0(X_{i_0 i_1}, \upsilon_! \mathbb{Z}(1)).$$

Define the element

$u = (u^{-1}, u^0, \ldots u^n)$ in
$(\mathscr{C}^{n+1}(X_j, K'^{\bullet}), (-1)^{n+1}\delta + d)$ by:

$$u^{-1} = (-1)^{(n+1)\cdot n} z^{-1}$$

$$u^{n-k} = (-1)^{kn} z^{n-k}_{i_0\ldots i_k} G_{i_0\ldots i_k} \frac{da_{k+1}}{a_{k+1}} \wedge \ldots \wedge \frac{da_n}{a_n}$$

. $\qquad 1 \leq k \leq n$

.

.

$$u^n = \ell n_i f \ \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n}$$

with $Z^{n-1}_{i_0 i_1} = (\delta \ell n f)_{i_0 i_1}$

$$Z^{n-k}_{i_0 \ldots i_k} = \delta (Z^{n-k+1}_{i_0 \ldots i_{k-1}} G_{i_0 \ldots i_{k-1}})_{i_0 \ldots i_k} .$$

As in (1.5), the condition
$(\ell n_{i_1} f - \ell n_{i_0} f) \in H^0(X_{i_0 i_1}, \nu_! \mathbb{Z}(1))$ implies that
$Z^{n-k}_{i_0 \ldots i_k} \in H^0(X_{i_0 \ldots i_k}, \nu_! \mathbb{Z}(k))$ and that u is a Cech cocycle, defining a cohomology class u in $H^{n+1}(X, K'^{\cdot})$

Proposition One has

$$h^* \bar{x} = u \quad \text{in} \quad H^{n+1}(X, K'^{\cdot}).$$

Proof. Choose a refinement X_j' of X_j such that if $X'_{i_0 \ldots i_k} \cap (S \cup T) = \phi$, then $\log_{i_0 \ldots i_k} a_{k+1}$ is single valued on $X'_{i_0 \ldots i_k}$, that is in $H^0(X'_{i_0 \ldots i_k}, \mathcal{O}_X)$.
Define

$$h_{i_0 \ldots i_k} = \begin{cases} \log_{i_0 \ldots i_k} a_{k+1} & \text{if } X'_{i_0 \ldots i_k} \cap (S \cup T) = \phi \\ 0 & \text{if } X'_{i_0 \ldots i_k} \cap (S \cup T) \neq \phi . \end{cases}$$

In this refinement X_j' one has

$h^* x^n - u^n = (\log_i f - \ell n_i f) \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n}$. Define
$N_i^n = (\log_i f - \ell n_i f) \in H^0(X_i', \nu_! \mathbb{Z}(1))$.
One has $(\delta N^n)_{i_0 i_1} = Z^{n-1}_{i_0 i_1} - Z^{n-1}_{i_0 i_1}$.

Define

$$r_i^{n-1} = N_i^n h_i \frac{da_2}{a_2} \wedge \dots \wedge \frac{da_n}{a_n} \in H^0(X_i^!, \Omega^{n-1}_{X,S+T}).$$

One has

$$h^* x^n - u^n = dr_i^{n-1}.$$

Define by induction for $1 \leq \ell < k$

$$N^{n-\ell}_{i_0 \dots i_\ell} = (z^{n-\ell}_{i_0 \dots i_\ell} - Z^{n-\ell}_{i_0 \dots i_\ell}) G_{i_0 \dots i_\ell}$$
$$- \delta(N^{n-\ell+1}_{i_0 \dots i_{\ell-1}} h_{i_0 \dots i_{\ell-1}})_{i_0 \dots i_\ell}$$

$$\in H^0(X^!_{i_0 \dots i_\ell}, \nu_! \mathbb{Z}(\ell+1))$$

with $(\delta N^{n-\ell}) = z^{n-\ell-1} - Z^{n-\ell-1}$

and $$r^{n-\ell-1}_{i_0 \dots i_\ell} = (-1)^{\ell n} N^{n-\ell}_{i_0 \dots i_\ell} h_{i_0 \dots i_\ell} \frac{da_{\ell+2}}{a_{\ell+2}} \wedge \dots \wedge \frac{da_n}{a_n}$$

$$\in H^0(X^!_{i_0 \dots i_\ell}, \Omega^{n-(\ell+1)}_{X,S+T})$$

with

$$(h^* x^{n-\ell} - u^{n-\ell}) - [(-1)^n \delta r^{n-1} + dr^{n-(\ell+1)}] = 0.$$

Define

$$N^{n-k}_{i_0\ldots i_k} = (z^{n-k}_{i_0\ldots i_k} - Z^{n-k}_{i_0\ldots i_k}) G_{i_0\ldots i_k}$$
$$- \delta(N^{n-k+1}_{i_0\ldots i_{k-1}} h_{i_0\ldots i_{k-1}})_{i_0\ldots i_k}$$

One has

$$\delta N^{n-k} = z^{n-k-1} - Z^{n-k-1} .$$

If $X^!_{i_0\ldots \hat{i}_\ell \ldots i_k} \cap (SUT) \neq \phi$ for all $\ell \in \{0,\ldots,k\}$, then $N^{n-k}_{i_0\ldots i_k} = 0$. Especially if $X^!_{i_0\ldots i_k} \cap (SUT) \neq \phi$. Otherwise $X^!_{i_1\ldots i_k} \cap (SUT) = \phi$ (say). Then

$$N^{n-k}_{i_0\ldots i_k} = (z^{n-k}_{i_0\ldots i_k} - Z^{n-k}_{i_0\ldots i_k})(G_{i_0\ldots i_k} - h_{i_1\ldots i_k})$$

$$- \sum_{\ell=1}^{k} (-1)^{\ell} N^{n-k+1}_{i_0\ldots \hat{i}_\ell \ldots i_k} (h_{i_0\ldots \hat{i}_\ell \ldots i_k} - h_{i_1\ldots i_k}) .$$

If $(z^{n-k} - Z^{n-k})_{i_0\ldots i_k} \neq 0$, then $X^!_{i_0\ldots i_k} \cap (SUT) = \phi$, and $(G_{i_0\ldots i_k} - h_{i_1\ldots i_k}) \in \mathbb{Z}(1)$.

If $N^{n-k+1}_{i_0\ldots \hat{i}_\ell \ldots i_k} \neq 0$, then $X^!_{i_0\ldots \hat{i}_\ell \ldots i_k} \cap (SUT) = \phi$, and $(h_{i_0\ldots \hat{i}_\ell \ldots i_k} - h_{i_1\ldots i_k}) \in \mathbb{Z}(1))$. Therefore

$$N^{n-k}_{i_0\ldots i_k} \in H^0(X^!_{i_0\ldots i_k}, \nu_! \mathbb{Z}(k+1)).$$

Define

$$r^{n-k-1}_{i_0\ldots i_k} = (-1)^{kn} N^{n-k}_{i_0\ldots i_k} h_{i_0\ldots i_k} \frac{da_{k+2}}{a_{k+2}} \wedge\ldots\wedge \frac{da_n}{a_n} .$$

One has

$$(h^*x^{n-k} - u^{n-k}) - [(-1)^n\delta r^{n-k} + dr^{n-k-1}] = 0.$$

Therefore $(h^*\bar{x} - u) - [(-1)^n\delta + d]r = 0$, and $(h^*\bar{x} - u)$ is a coboundary in $\mathscr{C}^{\bullet}(K'^{\bullet})$.

3.9. Let γ be an n-chain with support $\gamma \subset \mathscr{U}$, $\mathscr{U}$ open analytic, $\partial\gamma \subset S$, of homology class $[\gamma] \in H_n(X,S;\mathbb{Z})$ such that:

there is a determination ℓnf of $\log f$ on $\mathscr{U}$ with

$$\ell nf \in H^0(\mathscr{U}, t\mathcal{O}_X(-S)).$$

By 3.8, one has

$$h^*\bar{x} = \text{class of } \omega = \ell nf \frac{da_1}{a_1} \wedge\ldots\wedge \frac{da_n}{a_n}$$

in $H^{n+1}(\mathscr{U}, K'^{\bullet})$.

By (3.4), one obtains

Theorem (see [B], 7.0.2 and [N], II, (2.4)):

$$<[\gamma], ph^* x> = \int_\gamma \ell n f \frac{da_1}{a_1} \wedge \ldots \wedge \frac{da_n}{a_n} \text{ modulo } \mathbb{Q}(n+1).$$

3.10 Remark The condition X affine of [N], II, (2.4) does not appear in (3.9). This is just because the assumption on the existence of ℓnf is sufficient to assure that ph^*x is represented by a global n-form on $\mathcal{U}$ (via (3.8)).

3.11 Comment

The formula 3.9 depends on the existence of a representative γ of the homology class $[\gamma] \in H_n(X,S;\mathbb{Z})$ along which there is a single valued determination of $\log f$ which vanishes on support $\gamma \cap S$ and support $\gamma \cap (a_i = 0)$ for $1 \leq i \leq n$. So it is not valid in general. In §4 we weaken the assumptions on dimension X and on γ in order to write a slightly more general formula in the case $n = 1$.

§4 Other formuli on X and relationship with Bloch's regulator map

4.1 Let X be a smooth affine variety over $\mathbb{C}$ equiped with morphisms $h^{\alpha} : X \longrightarrow A$, $\alpha = 1,\ldots,N$, where A is the universal situtation as in (3.1). We define

$$h^{\alpha *}\phi = f^{\alpha} \in H^{1}_{\mathcal{D}}(X,S+T^{\alpha};\mathbb{Z}(1))$$
$$h^{\alpha *}X_i = a_i^{\alpha} \in H^0(X,\mathcal{O}_X)$$

where $t^{\alpha} := a_1^{\alpha} \ldots a_n^{\alpha}$ defines T^{α} and S is a divisor contained in $\bigcap_1^N h^{\alpha-1}Y$. This defines

$$u := \sum_1^N h^{\alpha *}\mathrm{rest}^{-1}\{\phi|_U, X_1,\ldots,X_n\} \in H^{n+1}_{\mathcal{D}}(X,S;\mathbb{Q}(n+1)).$$

Define $j : X-S \longrightarrow X$.

Recall (3.6) that we have defined

$$h^{\alpha *} : (i_!\mathbb{Q}(n+1) \longrightarrow \Omega_A^{\leq n}(\log Y)(-Y)) \longrightarrow Rh_*^{\alpha}(j_!\mathbb{Q}(n+1) \longrightarrow \Omega^{\leq n}_{X,S})$$

in $D^b(A)$.

This defines

$$\bar{u} := \sum_{1}^{N} h^{\alpha *}\mathrm{rest}^{-1}\{\phi|_U, X_1,\ldots,X_n\} \quad \text{as a class in}$$

$$H^{n+1}(X, j_!\mathbb{Q}(n+1) \longrightarrow \Omega^{\leq n}_{X,S}).$$

Lemma. The natural morphism

$$H^{n+1}(X,K^{\bullet}) \longrightarrow H^{n+1}(X,j_!\mathbb{Q}(n+1)) \longrightarrow \Omega^{\leq n}_{X,S})$$

is injective. The class $\bar{u}$ lies in $H^{n+1}(X,K^{\bullet})$ if and only if

$$d\bar{u} = \sum_{1}^{N} \frac{df^{\alpha}}{f^{\alpha}} \wedge \frac{da_1^{\alpha}}{f_1^{\alpha}} \wedge \ldots \wedge \frac{da_n^{\alpha}}{a_n^{\alpha}} = 0.$$

Proof. The kernel of
$H^{n+1}(X,K^{\bullet}) \longrightarrow H^{n+1}(X,j_!\mathbb{Q}(n+1) \longrightarrow \Omega^{\leq n}_{X,S})$
comes from $H^{n+1}(X,\Omega^{\geq n+1}_{X,S}[-1]) = 0$, and $\bar{u} \in H^{n+1}(X,K^{\bullet})$ if and only if it maps to 0 under

$$d : H^{n+1}(X,j_!\mathbb{Q}(n+1)) \longrightarrow \Omega^{\leq n}_{X,S})$$
$$\downarrow$$
$$H^{n+1}(X,\Omega^{\geq n+1}_{X,S}) = H^{0}(X,\Omega^{n+1}_{X,S})_{d \text{ closed}}.$$

One has

$$d\bar{u} = \sum_{1}^{N} h^{\alpha *} \frac{d\phi}{\phi} \wedge \frac{dX_1}{X_1} \wedge \ldots \wedge \frac{dX_n}{X_n}$$

$$= \sum_{1}^{N} \frac{df^\alpha}{f^\alpha} \wedge \frac{da_1^\alpha}{a_1^\alpha} \wedge \ldots \wedge \frac{da_n^\alpha}{a_n^\alpha}.$$

4.2 <u>Corollary</u> There is $\omega \in H^0(X,\Omega^n_{X,S})_{d\text{ closed}}$ representing u via the composed morphism

$$\begin{array}{ccc} H^0(X,\Omega^n_{X,S})_{d\text{ closed}} & \longrightarrow & H^{n+1}(X,K^\bullet) \\ & & \downarrow p\ (3.2) \\ & & H^n(X,S;\mathbb{C}/\mathbb{Q}(n+1)) \\ & & \downarrow\ (1.2) \\ & & H^{n+1}_{\mathcal{D}}(X,S;(\mathbb{Q}(n+1)) \end{array}$$

if $du = d\bar{u} = \sum_{1}^{N} \frac{df^\alpha}{f^\alpha} \wedge \frac{da_1^\alpha}{a_1^\alpha} \wedge \ldots \wedge \frac{da_n^\alpha}{a_n^\alpha} = 0.$

<u>Proof.</u> One has the exact sequence

$$\begin{array}{ccc} 0 \longrightarrow \dfrac{H^n(X,\Omega^\bullet_{X,S})}{H^n(X,S;\mathbb{Q}(n+1))} & \longrightarrow & H^{n+1}(X,K^\bullet) \\ & & \downarrow d \\ & & H^{n+1}(X,S;\mathbb{Q}(n+1)) \cap H^0(X,\Omega^{n+1}_{X,S})_{d\text{ closed}} \\ & & \downarrow \\ & & 0 \end{array}$$

Therefore $\bar{u} \in \dfrac{H^n(X,\Omega^{\bullet}_{X,S})}{H^n(X,S;\mathbb{Q}(n+1)}$.

As X is affine, one has

$$H^n(X,\Omega^{\bullet}_{X,S}) = H^0(X,\Omega^n_{X,S})_{d\ \text{closed}}/dH^0(X,\Omega^{n-1}_{X,S}).$$

4.3 Let γ be an n-chain on X with $\partial\gamma \subset S$, of homology class $[\gamma] \in H_n(X,S;\mathbb{Z})$. One has

$$<[\gamma],u> = \int_{\gamma}\omega \quad \text{modulo} \quad \mathbb{Q}(n+1).$$

4.4 We assume now $n = 1$ in (4.4) and (4.5). Given $[\gamma]$ as in 4.3, then is a representative γ of $[\gamma]$ as a chain as in [N], II, 2.4:

$\gamma = \gamma_0 + \sum_{i\geq 1} \gamma_i$ with $\partial\gamma_0 = \phi$, $\partial\gamma_i \neq \phi \subset S$ for $i \geq 1$. We first compute $<[\gamma_0],u>$.

<u>Proposition.</u> Let $p_0 \in$ support γ_0 be a point such that $\log f^{\alpha}$ is single valued along $\gamma_0 - p_0$, and vanishes along $t^{\alpha} = 0$ and S, for $\alpha = 1,\ldots,N$.

1) Assume $p_0 \notin \bigcup_1^N T^{\alpha}$. Then if $p_0 \notin S$ or if p_0 is an isolated point of $S \cap$ support γ_0, one has

$$<[\gamma_0],u> = \int_{\gamma_0} \sum_{\alpha} \log^{\alpha} \frac{da_1^{\alpha}}{a_1^{\alpha}} - \sum_{\alpha} \log a_1^{\alpha}(p_0) \int_{\gamma_0} \frac{df^{\alpha}}{f^{\alpha}} \quad \text{modulo } \mathbb{Q}(2).$$

2) If $p_0 \in S$ is not isolated in $S \cap$ support γ_0, or if $p_0 \in \bigcap_1^N T^\alpha$ is not isolated in $\bigcap_1^N T^\alpha \cap$ support γ_0, one has

$$<[\gamma_0],u> = \int_{\gamma_0} \sum_\alpha \log f^\alpha \frac{da_1^\alpha}{a_1^\alpha} \text{ modulo } \mathbb{Q}(2).$$

3) If $\log f^\alpha$ is single valued along γ_0 and vanishes along $t^\alpha = 0$ and S for $\alpha = 1,\dots,N$, one has

$$<[\gamma_0],u> = \int_{\gamma_0} \sum_\alpha \log f^\alpha \frac{da_1^\alpha}{a_1^\alpha} \text{ modulo } \mathbb{Q}(2).$$

Proof. In 1) and 2), there are an open set $\mathcal{U}$ containing γ_0, I a segment in $\mathcal{U}$ with $p_0 = I \cap$ support γ_0, and a determination $\ell n_1 f^\alpha$ on $\mathcal{U}_1 = \mathcal{U} - I$ with $\ell n_1 f^\alpha \in H^0(\mathcal{U}_1, t^\alpha \mathcal{O}_X(-S))$. For any $\epsilon > 0$, define an open set $\mathcal{U}_{0\epsilon}$ containing p_0 such that:

(*) is fulfilled in case 1)

(**) is fulfilled in case 2)

with

(*) $\log a_1^\alpha$ is single valued along $\mathcal{U}_{0\epsilon} \cap$ support γ_0 and verifies

$$\sup_{x,y\in \mathcal{U}_{0\epsilon}\cap \text{support } \gamma_0} |\log a_1^\alpha(x) - \log a_1^\alpha(y)| < \epsilon$$

(**) $\mathcal{U}_{0\epsilon} \cap \text{support } \gamma_0 \subset S$ or $\bigcap_1^N T^\alpha$

(As support $\gamma_0 \cap S$ (or support $\gamma_0 \cap \bigcap_1^N T^\alpha$) is compact, the condition 2) says that a subsegment of γ_0 centered at p_0 is contained in S (or in $\bigcap_1^N T^\alpha$). Therefore one may realize (**)).

Let $\mathcal{V}_\epsilon = \mathcal{U}_1 \cup \mathcal{U}_{0\epsilon}$. Take a common refinement of the covers $\mathcal{U}_1 \cup \mathcal{U}_{0\epsilon}$ and $\mathcal{V}_\epsilon \cap h^{\alpha-1}A_i$ of $\mathcal{V}_\epsilon$. By (3.8), $\bar{u}|_{\mathcal{V}\epsilon}$ is represented by the Cech cocycle in this cover

$$(u^{-1},u^0,u^{-1})\in\mathcal{C}^2(\mathcal{V}_\epsilon,j_!\mathbb{Q}(2))\times\mathcal{C}^1(\mathcal{V}_\epsilon,\mathcal{O}_X(-S))\times\mathcal{C}^0(\mathcal{V}_\epsilon,\Omega^1_{X,S,d\text{ closed}})$$

with

$$u^{-1} = \sum_\alpha z^\alpha_{i_0i_1i_2},\quad u^0 = -\sum_\alpha z^\alpha_{i_0i_1}G^\alpha_{i_0i_1},\quad u^1 = \sum_\alpha \ell n_i f^\alpha \frac{da^\alpha_1}{a^\alpha_1}$$

with

$$G^\alpha_{i_0i_1} = h^{\alpha *}g_{i_0i_1},\quad z^\alpha_{i_0i_1} = (\delta\ \ell n\ f^\alpha)_{i_0i_1},$$

$$z^\alpha_{i_0i_1i_2} = \delta(z^\alpha_{i_0i_1}\ G^\alpha_{i_0i_1})_{i_0i_1i_2}\ .$$

By (4.2) there is a refinement $(\mathcal{V}_i)_{i=0,\ldots,\ell}$ of the open cover, there are

$$\omega \in H^0(X,\Omega^1_{X,S})_{d\text{ closed}},\quad s \in \mathcal{C}^1(\mathcal{V}_i, j_!\mathbb{Q}(2))$$

and $r \in \mathcal{C}^0(\mathcal{V}_i, \mathcal{O}_X(-S))$ with

$$u^{-1} = -\delta s,\ u^0 = -\delta r + s,\quad u^1 = \omega + dr.$$

Following the orientation of γ_0, take an order $\mathcal{V}_i$ with

$p_0 \in \mathcal{V}_0 - \bigcup_{i\geq 1} \mathcal{V}_i$

$p_1 \in \mathcal{V}_0 \cap \mathcal{V}_1 \cap \gamma_0$

$p_\ell \in \mathcal{V}_{\ell-1} \cap \mathcal{V}_\ell \cap \gamma_0$

$p_{\ell+1} \in \mathcal{V}_\ell \cap \mathcal{V}_0 \cap \gamma_0$

One has

$\int_{\gamma_0} \omega = F - R_\epsilon$ with

$$F = \int_{p_{\ell+1}}^{p_1} \sum_\alpha \ell n_0\ f^\alpha \frac{da_1^\alpha}{a_1^\alpha} + \int_{p_1}^{p_{\ell+1}} \sum_\alpha \ell n_1\ f^\alpha \frac{da_1^\alpha}{a_1^\alpha}$$

$$R_\epsilon = \int_{p_{\ell+1}}^{p_1} dr_0 + \int_{p_1}^{p_2} dr_1 + \ldots + \int_{p_\ell}^{p_{\ell+1}} dr_\ell$$

$$= r_0\Big|_{p_{\ell+1}}^{p_1} + r_1\Big|_{p_1}^{p_2} + \ldots + r_\ell\Big|_{p_\ell}^{p_{\ell+1}} \text{ (Stokes)}$$

$$= \sum_\alpha [Z_{10}^\alpha G_{10}^\alpha(p_1) + Z_{21}^\alpha G_{21}^\alpha(p_2) + \ldots + Z_{\ell,\ell-1}^\alpha G_{\ell,\ell-1}^\alpha(p_\ell)$$

$$+ Z_{0\ell}^\alpha G_{0\ell}^\alpha(p_{\ell+1})] \text{ modulo } \mathbb{Q}(2).$$

One has

$$Z_{21}^\alpha = \ldots = Z_{\ell,\ell-1}^\alpha = 0.$$

In 1), $G_{10}^\alpha(p_1)$ and $G_{0\ell}^\alpha(p_{\ell+1})$ are two determinations of $\log a_1^\alpha$ by (1.4), γ. Therefore one has

$$R_\epsilon = \sum_\alpha Z_{10}^\alpha \log a_1^\alpha(p_1) + Z_{0\ell}^\alpha \log a_1^\alpha(p_{\ell+1}) \quad \text{modulo } \mathbb{Q}(2).$$

As Z_{10}^α and $Z_{0\ell}^\alpha$ donot depend on ϵ, one has

$$\Big|\sum_\alpha Z_{10}^\alpha(\log a_1^\alpha(p_1) - \log a_1^\alpha(p_0)) + Z_{0\ell}^\alpha(\log a_1^\alpha(p_{\ell+1}) - \log a_1^\alpha(p_0))\Big|$$

$$\leq \text{constant}.\ \epsilon \text{ by } (*).$$

Therefore R_ϵ tends to

$$R = \sum_\alpha (Z_{10}^\alpha + Z_{0\ell}^\alpha)\log a_1^\alpha(p_0) = \sum_\alpha \log a_1^\alpha(p_0) \int_{\gamma_0} \frac{df^\alpha}{f^\alpha}$$

as ϵ tends to zero.

In 2), R_ϵ does not depend on ϵ, and
$G^\alpha_{10}(p_1) = G^\alpha_{0\ell}(p_{\ell+1}) = 0$ by (**) and (1.4) γ. This proves the cases 1) and 2).

In case 3), consider an open set $\mathcal{U}$ containing γ_0 such that a determination $\ell n f^\alpha$ of $\log f^\alpha$ exists and is single valued on $\mathcal{U}$ with

$$\ell n f^\alpha \in H^0(\mathcal{U}, t^\alpha \mathcal{O}_X(-S)).$$

Then take a common refinement of $\mathcal{U} \cap h^{\alpha-1}A_i$,
, and a refinement $(\mathcal{V}_i)_{i=0,\ldots,\ell}$ of it with ω, s, r as before, and p_i as before.

One as

$$\int_{\gamma_0} \omega = F - R$$

with

$$F = \int_{\gamma_0} \sum_\alpha \ell n f^\alpha \ \frac{da^\alpha_1}{a^a_1}$$

$$R = \sum_{\alpha}[Z^{\alpha}_{10}G^{\alpha}_{10}(p_0)+\ldots+Z^{\alpha}_{0\ell}G^{\alpha}_{0\ell}(p_{\ell+1})] \text{ modulo } \mathbb{Q}(2).$$

As $Z^{\alpha}_{j,j-1} = Z^{\alpha}_{0\ell} = 0$, one obtains 3).

4.5 Take γ_1 with $\partial\gamma_1 \neq \phi \subset S$. Let $p_0 \in$ support $\gamma_1 \cap S$. If for all $\alpha = 1,\ldots,N$ there is a single valued determination of $\log f^{\alpha}$ along $\gamma_1 - p_0$ which vanishes along $t^{\alpha} = 0$ and S, then $\log f^{\alpha}$ is single valued along γ_1 as well.

<u>Proposition.</u> Let $p_0 \in$ support $\gamma_1 - S$ be a point such that $\log f^{\alpha}$ is simple valued along $\gamma_1 - p_0$, and vanishes along $t^{\alpha} = 0$ and S for $\alpha = 1,\ldots,N$.

1) Assume $p_0 \notin \bigcup_1^N T^{\alpha}$. Then one has

$$\langle[\gamma_1],u\rangle = \int_{\gamma_1} \sum_{\alpha} \log f^{\alpha} \frac{da^{\alpha}_1}{a^{\alpha}_1} - \sum_{\alpha} \log a^{\alpha}_1(p_0) \int_{\gamma_1} \frac{df^{\alpha}}{f^{\alpha}} \quad \text{modulo } \mathbb{Q}(2)$$

2) If $p_0 \in \bigcap_1^N T^{\alpha}$ and is not isolated in $\bigcap_1^N T^{\alpha} \cap$ support γ_1, then one has

$$\langle[\gamma_1],u\rangle = \int_{\gamma_1} \sum_{\alpha} \log f^{\alpha} \frac{da^{\alpha}_1}{a^{\alpha}_1} \text{ modulo } \mathbb{Q}(2).$$

3) If $\log f^{\alpha}$ is single valued along γ_1 and vanishes along $t^{\alpha} = 0$ and S for $\alpha = 1,\ldots,N$ then one has

$$<[\gamma_1],u> = \int_{\gamma_1} \sum_{\alpha} \log f^{\alpha} \frac{da_1^{\alpha}}{a_1^{\alpha}} \text{ modulo } \mathbb{Q}(2).$$

<u>Proof.</u> For 1,2,3 define $(\mathcal{V}_i)_{i=0,\ldots,\ell}$ as in the proof of (4.4), 1 and (4.4),2. Write

$$\partial\gamma_1 = \{s_0,\ldots,s_k\} \subset S.$$

One has to take

$p_0 \in \mathcal{V}_0 - \bigcup_{i\geq 1} \mathcal{V}_i$

$p_1 \in \mathcal{V}_0 \cap \mathcal{V}_1 \cap \gamma_1$

$p_{\ell_1} \in \mathcal{V}_{\ell_1-1} \cap \mathcal{V}_{\ell_1} \cap \gamma_1$

$s_1 \in \mathcal{V}_{\ell_1}$

$s_2 \in \mathcal{V}_{\ell_1+1}$

$p_{\ell_1+2} \in \mathcal{V}_{\ell_1+1} \cap \mathcal{V}_{\ell_1+2} \cap \gamma_1$

$p_{\ell_1+3} \in \mathscr{V}_{\ell_1+2} \cap \mathscr{V}_{\ell_1+3} \cap \gamma_1$

.

.

.

$p_{\ell_1+\ell_2} \in \mathscr{V}_{\ell_1+\ell_2-1} \cap \mathscr{V}_{\ell_2} \cap \gamma_1$

$s_3 \in \mathscr{V}_{\ell_2}$

.

.

.

$p_\ell \in \mathscr{V}_{\ell-1} \cap \mathscr{V}_\ell \cap \gamma_1$

$p_{\ell+1} \in \mathscr{V}_\ell \cap \mathscr{V}_0 \cap \gamma_1.$

Note that the corresponding R is defined by

$$R = \int_{p_{\ell+1}}^{p_1} dr_0 + \int_{p_1}^{p_2} dr_1 + \ldots + \int_{p_{\ell_1}}^{s_1} dr_{\ell_1}$$

$$+ \int_{s_2}^{p_{\ell_1+2}} dr_{\ell_1+1} + \int_{p_{\ell_1+2}}^{p_{\ell_1+3}} dr_{\ell_1+2} + \ldots + \int_{p_\ell}^{p_{\ell+1}} dr_\ell$$

As $r \in \mathscr{C}^1(\mathcal{O}_X(-S))$, one has

$$R = (r_0 - r_1)(p_1) + (r_1 - r_2)(p_2) + \ldots + (r_{\ell_1-1} - r_{\ell_1})(p_{\ell_1})$$
$$+ (r_{\ell_1+1} - r_{\ell_1+2})(p_{\ell_1+2}) + \ldots + (r_\ell - r_0)(p_{\ell+1}).$$

One concludes as in (4.4).

4.6 Let now X be a smooth affine variety over $\mathbb{C}$. Let $f_0^\alpha,\dots,f_n^\alpha$ be global invertible algebraic function on X, for $\alpha = 1,\dots,N$. We consider the cup product $u = \sum_1^N \{f_0^\alpha,\dots,f_n^\alpha\} \in H_{\mathcal{D}}^{n+1}(X,\mathbb{Q}(n+1))$. Assuming $du = \sum_1^N \frac{df_0^\alpha}{f_0^\alpha} \wedge\dots\wedge \frac{df_n^\alpha}{f_n^\alpha} = 0$, we have ((1.2), i, with $Y = \phi$):

$u \in H^n(X,\mathbb{C}/\mathbb{Q}(n+1))$.

Now, X being affine, we have as in (4.2):

$$H^n(X,\mathbb{C}/\mathbb{Q}(n+1)) = \frac{H^0(X,\Omega_X^n)_{d\ \text{closed}}}{H^n(X,\mathbb{Q}(n+1))+dH^0(X,\Omega_X^{n-1})}$$

and if $\omega \in H^0(X,\Omega_X^n)_{d\ \text{closed}}$ represents u, one has: for any $[\gamma] \in H_n(X,\mathbb{Z})$ of representative γ:

$$<[\gamma],u> = \int_\gamma \omega \quad \text{modulo} \quad \mathbb{Q}(n+1).$$

4.7 Take $n = 1$, and X no longer affine. As explained by R. Hain in his talk at the Max-Planck-Institut, fall 1987, one has Bloch's regular map

$$r : K_2(X)_{\mathbb{Q}} \longrightarrow H_{\mathcal{D}}^2(X,\mathbb{Q}(2))$$

This is defined as follows. Let $x = \prod_{1}^{N} \{f_0^\alpha, f_1^\alpha\}$ be in $K_2(\mathbb{C}(X))$. Let U be a affine subset of X such that $f_i^\alpha \in \mathcal{O}(U)^*$. Then the any product

$$\sum_{1}^{N} f_0^\alpha \cup f_1^\alpha \text{ lies in } H^2_{\mathcal{D}}(U,\mathbb{Q}(2)) \subset \varinjlim_{\substack{V \text{ Zariski} \\ \text{open in } X}} H^2_{\mathcal{D}}(V,\mathbb{Q}(2)).$$

The existence of the dilogarithm function tells us that

$$\sum_{1}^{N} f_0^\alpha \cup f_1^\alpha \in \varinjlim_{\substack{V \text{ Zariski} \\ \text{open in } X}} H^2_{\mathcal{D}}(V,\mathbb{Q}(2))$$

does not depend on the decomposition choosen of x as symbols $\{f_0^\alpha, f_1^\alpha\}$. The existence of a Gersten-Quillen resolution for $H^2_{\mathcal{D}}(2)_{\mathbb{Q}}$ tells us that if $x \in K_2(X) \subset K_2(\mathbb{C}(x))$, then $r(x) := \sum_{1}^{N} f_0^\alpha \cup f_1^\alpha$ lies in $H^2_{\mathcal{D}}(X,\mathbb{Q}(2)) \subset \varinjlim_{V} H^2_{\mathcal{D}}(V,\mathbb{Q}(2))$.

Assume $dr(x) = 0$.

Proposition. Let $[\gamma] \in H_1(U,\mathbb{Z})$, of representative γ. Let $p_0 \in$ support γ such that $\log f_0^\alpha$ is single valued along $\gamma - p_0$. Then

$$<[\gamma],r(x)> = \int_\gamma \sum_\alpha \log f_0^\alpha \frac{df_1^\alpha}{f_1^\alpha}$$

$$- \sum_\alpha \log f_1^\alpha(p_0) \int_\gamma \frac{df_0^\alpha}{f_0^\alpha} \quad \text{modulo} \quad \mathbb{Q}(2).$$

If X is a curve, this is true modulo $\mathbb{Z}(2)$.

The <u>proof</u> is word by word the same as in (4.4)1), where one replaces $G^\alpha_{i_0 i_1}$ by $\log_{i_1} f_1^\alpha$. If X is a curve, then

$$H^2_{\mathcal{D}}(U,\mathbb{Z}(2)) = H^1(U,\mathbb{C})/H^1(U,\mathbb{Z}(2))$$

$$= H^1(U,\mathbb{C}/\mathbb{Z}(2)).$$

REFERENCES

[BBD] A. BEILINSON, J. BERNSTEIN, P. DELIGNE: Faisceaux pervers, Astérisque no100.

[B] A. BEILINSON: Higher regulators and values of L functions, Sovremenie Problemi Mathematici 24, Moscow, Viniti (1984), Translation in J. Soviet Math. 30, 2036-2060 (1985).

The two following artiles appeared in "Beilinson conjectures", Perspectives in Mathematics, Academic Press; editors: M. Rapoport, N. Schappacher, P. Schneider, (1988).

[EV] H. ESNAULT, E. VIEHWEG: Deligne-Beilinson cohomology.

[N] J. NEUKIRCH: The Beilinson conjecture for algebraic number fields.