Cone pseudodifferential operators in the edge symbolic calculus

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Abstract

We consider a class of parameter-dependent pseudodifferential operators on the model cone (with smooth compact cross section) of a wedge, which is shown to be a symmetric algebra. Its construction relies on a particular quantization of corresponding edge-degenerate symbols, based on the Mellin transform along the axial variable of the cone. These operator families are essential for the analysis of pseudodifferential operators on manifolds with edges. In particular, they serve as symbols for Laplace-Beltrami operators to wedge metrics and their parametrices if the parameter is treated as the covariable.

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Introduction

The present paper studies an algebra of parameter-dependent pseudodifferential operators on a manifold with conical singularities, where the parameters are involved as covariables in a specific degenerate way. Such operator families serve as an adequate symbol class for pseudodifferential operators on manifolds with edges. Also the study of resolvents of differential operators of Fuchs type leads to families of a similar form.

Our results fit into the frame of pseudodifferential calculi on manifolds with singularities, particularly with piecewise smooth geometry. They belong to the idea to reflect the stratification of such a space by a hierarchy of operator algebras with symbolic structure, and to organize an iterative procedure which starts from the calculus on a given space, say a cone, and constructs a next 'higher' calculus on a space with higher order singularities, say a wedge. It is well-known that, for instance, boundary value problems for pseudodifferential operators can be represented as operators along the boundary with operator-valued symbols, acting along $\mathbb{R}_+$, the inner normal. In this sense, not only Boutet de Monvel's algebra [1] (cf. also Schrohe, Schulze [7]) of boundary value problems with the transmission property is included in the context but also Vishik, Eskin's theory [3], [12], turned into a corresponding operator algebra, cf. Schulze [10]. The iteration of calculi leads to

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very complex analytic phenomena and it is still a serious problem to formulate manageable operator algebras for higher singularities such as of corner type or for boundary value problems with such singular boundaries.

The main objective of our paper is to develop an efficient new approach to the algebra of cone operator-valued edge symbols as originally introduced in [9]. One of the difficulties is that the edge covariables are involved in a degenerate form, i.e. multiplied by the axial variable of the cone (cf. [2], [10]). This edge-degenerate behaviour is a consequence of the wedge geometry, and can be observed, for instance, in the symbols of corresponding Laplace-Beltrami operators. In the new representation we can, in particular, avoid a number of extremely voluminous calculations in the precise analysis of operator-valued edge symbols by a new quantization of edge-degenerate interior symbols in which a part of the above inconvenient combinations of edge covariable and axial variable is dismissed. This relies on a form of the Mellin quantization for edge-degenerate pseudodifferential symbols from Gil, Schulze, Seiler [4].

We begin this paper with a short discussion of oscillatory integrals in the spirit of Kumano-go [6], here generalized to amplitude functions with values in Fréchet spaces. It is an important tool to obtain the composition result of the parameter-dependent cone pseudodifferential operators.

Section 2 provides the basic material about the Mellin calculus. We introduce there the class of holomorphic pseudodifferential operators (based on the Mellin transform) as well as the cone Sobolev spaces.

In Section 3 we discuss a class of parameter-dependent cone operators as used in the edge symbolic calculus, formulated in the old and new fashion. The main result of this section is Theorem 3.18, where we prove the equivalence of both representations. We also give a global Mellin quantization which plays an essential role in the proof above.

Some properties of our operator-valued edge symbols and some basic elements of the pseudodifferential calculus (e.g. composition) are treated in the last section.

Finally, a part of the results are postponed to the appendix in order to keep the exposition of this paper transparent.

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Basic notation

Let $\mathbb{R}_+ = \{r \in \mathbb{R} |\ r > 0\}$, $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real β set $\Gamma_\beta = \{z \in \mathbb{C} |\ \mathrm{Re}\ z = \beta\}$.

A *cut-off function* is a non-negative function $\sigma \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\sigma \equiv 1$ near $t = 0$. For functions φ, ψ we write $\varphi \prec \psi$ if $\varphi\psi = \varphi$ and $\mathrm{supp}\,\varphi \cap \mathrm{supp}\,(1-\psi) = \emptyset$. A function $\chi \in C^\infty(\mathbb{R}^q)$ is called an *excision function at* $\eta = \eta_0$ if $0 \le \chi \le 1$, $\chi(\eta) = 0$ on some neighborhood of η_0 and $\chi(\eta) = 1$ for large $|\eta - \eta_0|$.

For $u \in C_0^\infty(\mathbb{R}^n)$, $v \in C_0^\infty(\mathbb{R}_+)$ the Fourier and Mellin transform respectively, are given by

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx, \quad \mathcal{M}v(z) = \int_0^\infty t^{z-1} v(t)\, dt.$$

These transforms can be extended to more general (distribution) spaces.

Let $U \subset \mathbb{R}^m$ be open, and set $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$. Then $S^\mu(U \times \mathbb{R}^n)$ consists of all $p \in C^\infty(U \times \mathbb{R}^n)$ with

$$\sup_{x \in K, \xi \in \mathbb{R}^n} \left\{ |D_\xi^\alpha D_x^\beta p(x,\xi)| \, \langle\xi\rangle^{|\alpha|-\mu} \right\} < \infty \tag{1}$$

for all $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^m$, and all compact sets $K \subset U$. This is a Fréchet space. Moreover, set $S^\mu(\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^n) = S^\mu(\mathbb{R} \times U \times \mathbb{R}^n)|_{\overline{\mathbb{R}}_+ \times U \times \mathbb{R}^n}$. This is a Fréchet space if we take as semi-norms the analogous expressions as in (1), where now K is a compact set in $\overline{\mathbb{R}}_+ \times U$.

For $\Lambda = \mathbb{R}^l$ or $\Lambda = \Gamma_\beta \times \mathbb{R}^l \cong \mathbb{R}^{1+l}$ we associate with a symbol $p \in S^\mu(U \times U \times \mathbb{R}^n \times \Lambda)$, $U \subset \mathbb{R}^n$, its parameter-dependent pseudodifferential operator $\mathrm{op}_x(p)(\lambda) : C_0^\infty(U) \to C^\infty(U)$ by

$$[\mathrm{op}_x(p)(\lambda)u](x) = \iint e^{i(x-x')\xi} p(x,x',\xi,\lambda) u(x') \, dx' đ\xi, \tag{2}$$

where $đ\xi = (2\pi)^{-n} \, d\xi$.

By pasting together (via partition of unity) on a smooth compact manifold X operators of the form (2) we obtain the space $L^\mu(X;\Lambda)$ of parameter-dependent pseudodifferential operators on X.

We denote by $X^\wedge$ the half-cylinder $\mathbb{R}_+ \times X$. With the symbols $P(t,t',\tau,\lambda) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu(X;\mathbb{R}_\tau \times \Lambda))$, where $n = \dim X$, we associate the corresponding operator $\mathrm{op}_t(P)(\lambda)$ viewing the parameter τ as a covariable. The space of all these operators is denoted by $L^\mu(X^\wedge;\Lambda)$. For $F(t,t',z,\lambda) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu(X;\Gamma_{\frac{1}{2}-\gamma} \times \Lambda))$ we also consider

$$[\mathrm{op}_M^\gamma(F)(\lambda)u](t) = \int_{\Gamma_{\frac{1}{2}-\gamma}} \int_0^\infty \left(\frac{t}{t'}\right)^{-z} F(t,t',z,\lambda) u(t') \frac{dt'}{t'} đz,$$

where $đz = (2\pi i)^{-1} dz$, and for t' fixed, $u(t')$ is viewed as a function in $C_0^\infty(X)$.

We further define

$$L^\mu(X^\wedge;\Lambda)_0 := \{P(\lambda) \in L^\mu(X^\wedge;\Lambda) \mid \sigma P(\lambda)\tilde{\sigma} = P(\lambda) \text{ for some cut-off functions } \sigma, \tilde{\sigma}\}.$$

The intersection $\bigcap_{\mu \in \mathbb{R}} L^\mu$ is denoted by $L^{-\infty}$ in all the cases.

Finally, for some Fréchet space E let $C_{b,F}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, E)$ be the space of smooth E-valued functions bounded with all their totally characteristic derivatives $(t\partial_t)^k \, (t'\partial_{t'})^l$.

1 Oscillatory integrals in Fréchet spaces

The purpose of this section is to generalize oscillatory integral technique to the case of amplitude functions with values in Fréchet spaces. Our approach is based on that of Kumano-go [5], [6] for the scalar case. As a minor modification we use the Mellin instead of the Fourier transform.

In the following let E be a Fréchet space.

Definition 1.1 Let $\mathcal{T}(\mathbb{R}_+ \times \mathbb{R}, E)$ be the space of all functions $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}, E)$ such that

$$(Su)(t_1, t_2) := u(e^{-t_1}, t_2) \ \in \ \mathcal{S}(\mathbb{R}_{t_1} \times \mathbb{R}_{t_2}).$$

A Fréchet topology on $\mathcal{T}(\mathbb{R}_+ \times \mathbb{R}, E)$ is defined by requiring the map $S : \mathcal{T}(\mathbb{R}_+ \times \mathbb{R}, E) \to \mathcal{S}(\mathbb{R}^2, E)$ to be a topological isomorphism. If $E = \mathbb{C}$ we suppress E from the notation.

Definition 1.2 (i) The space of *amplitude functions*, $\mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E)$, consists of all $h \in C^\infty(\mathbb{R}_+ \times \Gamma_0, E)$ such that for each continuous semi-norm p of E there exist reals $m = m_p$, $\mu = \mu_p$ such that

$$\sup\left\{ p\left((\partial_z^k (s\partial_s)^l h)(s, i\xi) \right) \langle \log s \rangle^{-m} \langle \xi \rangle^{-\mu} \mid (s,\xi) \in \mathbb{R}_+ \times \mathbb{R} \right\} < \infty \tag{3}$$

for all $k, l \in \mathbb{N}_0$.

(ii) By $\mathcal{A}(\mathbb{R}_+ \times \mathbb{C}, E)$ denote the space of *holomorphic amplitude functions*, i.e. all $h \in C^\infty(\mathbb{R}_+, \mathcal{O}(\mathbb{C}, E))$ such that for each continuous semi-norm p of E there exist reals $m = m_p$, $\mu = \mu_p$ such that

$$\sup\left\{ p\left((\partial_z^k (s\partial_s)^l h)(s, \delta + i\xi) \right) e^{-m\langle \log s \rangle} \langle \xi \rangle^{-\mu} \mid (s,\xi) \in \mathbb{R}_+ \times \mathbb{R},\ |\delta| \le j \right\} < \infty$$

for all $j, k, l \in \mathbb{N}_0$.

Remark 1.3 To verify that a function h is an amplitude it suffices to check the corresponding estimates for an arbitrary system $\{p_n \mid n \in \mathbb{N}\}$ of continuous semi-norms of E that determines the topology of E.

The following facts are immediate:

Lemma 1.4 (i) *If $h \in \mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E)$ then $\partial_z^k (s\partial_s)^l h \in \mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E)$;*

(ii) *let E_j be Fréchet spaces and the topology in E be the projective topology with respect to the linear maps $T_j : E \to E_j$. Then $h \in \mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E)$ if and only if $T_j(a) \in \mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E_j)$ for each j;*

(iii) *let E_0, E_1 be two Fréchet spaces and $(\cdot,\cdot) : E_0 \times E_1 \to E$ be a continuous and bilinear map. If $h_j \in \mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E_j)$, $j = 0, 1$, then $(h_0, h_1) \in \mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E)$;*

(iv) *let V be a closed subspace of E. For $e \in E$ let $[e] = e + V$. Then $h \in \mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E)$ implies that $[h] \in \mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E/V)$.*

Analogous statements are true for holomorphic amplitude functions.

Definition 1.5 A function $\chi_\varepsilon(s, z) :]0,1] \times \mathbb{R}_+ \times \Gamma_0 \to \mathbb{C}$ is called *regularizing*, if

(i) $\chi_\varepsilon \in \mathcal{T}(\mathbb{R}_+ \times \Gamma_0)$ for each ε;

(ii) $\sup\{|(\partial_z^k (s\partial_s)^l \chi_\varepsilon)(s,\xi)| \mid 0 < \varepsilon \le 1,\ (s,\xi) \in \mathbb{R}_+ \times \mathbb{R}\} < \infty$ for each $k, l \in \mathbb{N}_0$;

(iii) $\partial_z^k (s\partial_s)^l \chi_\varepsilon(s,\xi) \longrightarrow \begin{cases} 1, & k+l=0 \\ 0, & k+l\neq 0 \end{cases}$ pointwise on $\mathbb{R}_+ \times \mathbb{R}$ as ε tends to 0.

Example 1.6 Let $\chi \in \mathcal{T}(\mathbb{R}_+ \times \mathbb{R})$ with $\chi(1,0) = 1$, and set $\chi_\varepsilon(s, i\xi) = \chi(s^\varepsilon, \varepsilon\xi)$. Then χ_ε is regularizing in the sense of Definition 1.5.

Definition 1.7 A function $\chi_\varepsilon(s,z) :]0,1] \times \mathbb{R}_+ \times \mathbb{C} \to \mathbb{C}$ will be called *holomorphically regularizing*, if

(i) $(\varepsilon, s, i\xi) \mapsto \chi_\varepsilon(s, \delta + i\xi)$ is regularizing in the sense of Definition 1.5 for each $\delta \in \mathbb{R}$;

(ii) $z \mapsto \chi_\varepsilon(s,z)$ is entire, and $\xi \mapsto \chi_\varepsilon(s, \delta+i\xi) \in \mathcal{S}(\mathbb{R})$ uniformly for δ in compact intervals;

(iii) for each ε there is a compact set $K_\varepsilon \subset \mathbb{R}_+$ such that $\chi_\varepsilon(s,z) = 0$ whenever $s \notin K_\varepsilon$.

Example 1.8 Let $\varphi \in C_0^\infty(\mathbb{R}_+)$ with $\varphi(1) = \mathcal{M}\varphi(0) = 1$. Here, $\mathcal{M}$ is the Mellin transform. The function $\chi_\varepsilon(s,z) = \varphi(s^\varepsilon)\mathcal{M}\varphi(\varepsilon z)$ is regularizing in the sense of Definition 1.7.

Theorem 1.9 *Let $h \in \mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E)$ and let χ_ε be regularizing. Then the limit*

$$\mathrm{Os}[h] = \iint_0^\infty s^{i\xi} h(s,i\xi)\, \frac{ds}{s} đ\xi := \lim_{\varepsilon\to 0} \iint_0^\infty s^{i\xi} \chi_\varepsilon(s,i\xi) h(s,i\xi)\, \frac{ds}{s} đ\xi$$

exists in E and is independent of the choice of χ_ε. The same holds for $h \in \mathcal{A}(\mathbb{R}_+ \times \mathbb{C}, E)$ with χ_ε being holomorphically regularizing. In particular, both definitions of the oscillatory integral coincide on $\mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E) \cap \mathcal{A}(\mathbb{R}_+ \times \mathbb{C}, E)$.

This result essentially follows from integration by parts.

Remark 1.10 Let $\{p_n \mid n \in \mathbb{N}\}$ be a fixed system of continuous semi-norms that determines the topology of E, and $\bar\mu = (\mu_n)$, $\bar m = (m_n)$ be fixed sequences of reals. If we denote by $\mathcal{A}^{\bar\mu,\bar m}(\mathbb{R}_+ \times \Gamma_0, E)$ the space of all amplitude functions h that satisfy the estimates (3) for p_n with $m = m_n$ and $\mu = \mu_n$, then $\mathcal{A}^{\bar\mu,\bar m}(\mathbb{R}_+ \times \Gamma_0, E)$ carries a natural Fréchet topology. The map $h \mapsto \mathrm{Os}[h] : \mathcal{A}^{\bar\mu,\bar m}(\mathbb{R}_+ \times \Gamma_0, E) \to E$ is linear and continuous. Moreover,

$$\mathcal{A}(\mathbb{R}_+ \times \Gamma_0, E) = \cup_{\bar\mu,\bar m} \mathcal{A}^{\bar\mu,\bar m}(\mathbb{R}_+ \times \Gamma_0, E).$$

Here, the union is taken over all sequences $\bar\mu$, $\bar m$. Analogous statements hold for holomorphic amplitude functions.

2 Mellin pseudodifferential operators

Pseudodifferential operators based on the Mellin transform constitute a special ingredient of the cone algebra (cf. [2] and [10]). They are adequate for the treatment of differential operators of Fuchs type which are the natural ones on manifolds with conical singularities. Such differential operators are roughly speaking polynomials in the totally characteristic derivative $-t\partial_t$. For this reason microlocalization by means of the Mellin transform $\mathcal{M}_{t\to z}$ is natural since $-t\partial_t$ corresponds in the Mellin image to multiplication by the complex variable z. For an introductory exposition of the cone calculus we refer the reader to the chapters 7 and 8 of [2].

2.1 Weighted cone Sobolev spaces

For a compact smooth manifold X we construct the Sobolev spaces on $X^\wedge = \mathbb{R}_+ \times X$ in a way that they satisfy a number of natural conditions, in particular, for smoothness $s \in \mathbb{R}$ they are subspaces of $H^s_{loc}(X^\wedge)$ and contain $H^s_{comp}(X^\wedge)$ as a subspace. The cone Sobolev spaces can be understood as a suitable modification of the usual spaces H^s, defined with help of the Mellin transform in the axial direction $t \in \mathbb{R}_+$.

First of all, for $s, \gamma \in \mathbb{R}$ define $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ as the closure of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to the norm

$$\|u\|^2_{\mathcal{H}^{s,\gamma}} = \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \int_{\mathbb{R}^n} (1 + |z|^2 + |\xi|^2)^s |(\mathcal{M}_{t\to z}\mathcal{F}_{x\to\xi}u)(z,\xi)|^2 \,|dz| d\xi.$$

Definition 2.1 For $s, \gamma, \varrho \in \mathbb{R}$ we introduce the cone Sobolev spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$ as the closure of $C_0^\infty(X^\wedge)$ with respect to the norms

$$\|u\|^2_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \sum_{j=1}^{N} \|(\varphi_j u) \circ (1 \times \theta_j)^{-1}\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)} \quad \text{and}$$

$$\|u\|^2_{\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho} = \|\sigma u\|^2_{\mathcal{H}^{s,\gamma}(X^\wedge)} + \sum_{j=1}^{N} \|[(1-\sigma)\varphi_j u] \circ \kappa_j^{-1}\|^2_{H^{s,\varrho}(\mathbb{R}^{1+n})},$$

respectively. Here, $\{\varphi_j\}_{j=1,\dots,N}$ is a partition of unity subordinate to the covering $X = \bigcup_{j=1}^N U_j$, $\theta_j : U_j \to V_j \subset \mathbb{R}^n$ are charts, and

$$\kappa_j(t,x) = t\tilde{\kappa}_j(x) : \mathbb{R}_+ \times U_j \to \mathbb{R}^{1+n}$$

for diffeomorphisms $\tilde{\kappa}_j : U_j \to \tilde{V}_j \subset S^n$, with S^n being the unit sphere in $\mathbb{R}^{1+n}$. Moreover, $H^{s,\varrho}(\mathbb{R}^{1+n}_y) = \langle y\rangle^{-\varrho} H^s(\mathbb{R}^{1+n}_y)$ are the usual weighted Sobolev spaces on $\mathbb{R}^{1+n}$ and $\sigma \in C_0^\infty(\overline{\mathbb{R}}_+)$ is a cut-off function. In case $\varrho = 0$ we suppress it from the notation. Note that other localization data yield equivalent norms.

Remark 2.2 The following elementary properties are valid:

(i) $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$ are Hilbert spaces. In particular, $\mathcal{H}^{0,0}(X^\wedge) = \mathcal{K}^{0,0}(X^\wedge) = t^{-\frac{n}{2}} L^2(X^\wedge)$, where $X^\wedge$ is equipped with the product metric;

(ii) $t^{-\mu}\mathcal{H}^{s,\gamma}(X^\wedge) = \mathcal{H}^{s,\gamma-\mu}(X^\wedge)$ and $t^{-\mu}\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho = \mathcal{K}^{s,\gamma-\mu}(X^\wedge)^{\varrho+\mu}$;

(iii) $H^s_{comp}(X^\wedge) \hookrightarrow \mathcal{H}^{s,\gamma}(X^\wedge) \hookrightarrow H^s_{loc}(X^\wedge)$, and the same is true for $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$;

(iv) the scalar product in $t^{-\frac{n}{2}}L^2(X^\wedge)$ extends to non-degenerate sesquilinear pairings $\mathcal{H}^{s,\gamma}(X^\wedge) \times \mathcal{H}^{-s,-\gamma}(X^\wedge) \to \mathbb{C}$ and $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho \times \mathcal{K}^{-s,-\gamma}(X^\wedge)^{-\varrho} \to \mathbb{C}$, respectively;

(v) the embedding $\mathcal{H}^{s,\gamma}(X^\wedge) \hookrightarrow \mathcal{H}^{s',\gamma}(X^\wedge)$ is continuous if $s \geq s'$;

(vi) the embedding $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho \hookrightarrow \mathcal{K}^{s',\gamma'}(X^\wedge)^{\varrho'}$ is continuous if $s \geq s'$, $\gamma \geq \gamma'$, $\varrho \geq \varrho'$, and compact if $s > s'$, $\gamma > \gamma'$, $\varrho > \varrho'$.

Proposition 2.3 *Let $s, \gamma \in \mathbb{R}$. Then there exist $\varrho, \varrho' \in \mathbb{R}$ such that*

$$\mathcal{H}^{s,\gamma}(X^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \qquad \mathcal{K}^{s,\gamma}(X^\wedge)^{\varrho'} \hookrightarrow \mathcal{H}^{s,\gamma}(X^\wedge).$$

PROOF: Follows from Lemmas 3.1.20 and 4.2.2 in [7]. □

Definition 2.4 Using the pairing from Remark 2.2(iv), we associate to every operator $A \in \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\varrho'})$ the formal adjoint $A^* \in \mathcal{L}(\mathcal{K}^{-s',-\gamma'}(X^\wedge)^{-\varrho'}, \mathcal{K}^{-s,-\gamma}(X^\wedge)^{-\varrho})$ that satisfies

$$(Au, v) = (u, A^*v) \quad \text{for all} \quad u, v \in C_0^\infty(X^\wedge).$$

Analogously, for every operator $A \in \mathcal{L}(\mathcal{H}^{s,\gamma}(X^\wedge), \mathcal{H}^{s',\gamma'}(X^\wedge))$ we also define the formal adjoint $A^* \in \mathcal{L}(\mathcal{H}^{-s',-\gamma'}(X^\wedge), \mathcal{H}^{-s,-\gamma}(X^\wedge))$.

Next we introduce a family of isomorphisms on the cone Sobolev spaces defined above.

Definition 2.5 For each $\lambda > 0$ define the linear map $\kappa_\lambda : C_0^\infty(X^\wedge) \to C_0^\infty(X^\wedge)$ by

$$(\kappa_\lambda u)(t, x) := \lambda^{\frac{n+1}{2}} u(\lambda t, x),$$

where $n = \dim X$. These mappings extend by continuity to linear operators on $\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$ and $\mathcal{H}^{s,\gamma}(X^\wedge)$ for all $s, \gamma, \varrho \in \mathbb{R}$, and the set $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is a (*strongly continuous*) *group* of isomorphisms, that means,

(i) $\kappa_\lambda \kappa_\varrho = \kappa_{\lambda\varrho}$ for all $\lambda, \varrho > 0$;

(ii) for each $u \in \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$ the function $\lambda \mapsto \kappa_\lambda u : \mathbb{R}_+ \to \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho$ is continuous, and analogously for $u \in \mathcal{H}^{s,\gamma}(X^\wedge)$.

2.2 Mellin operators

Before we pass to the parameter-dependent version of the cone algebra we recall some definitions and properties of the Mellin pseudodifferential operators with operator-valued symbols.

Definition 2.6 For $\gamma \in \mathbb{R}$ let $\mathcal{T}_\gamma(\mathbb{R}_+, C^\infty(X))$ denote the space of all functions $u \in C^\infty(\mathbb{R}_+, C^\infty(X))$ such that

$$(S_\gamma u)(t) := e^{(\gamma - \frac{1}{2})t} u(e^{-t}) \in \mathcal{S}(\mathbb{R}_t, C^\infty(X)).$$

The topology is that inherited from $\mathcal{S}(\mathbb{R}, C^\infty(X))$ via the isomorphism S_γ. We view $\mathcal{T}_\gamma(\mathbb{R}_+, C^\infty(X))$ as a subspace of $C^\infty(X^\wedge)$.

To each symbol $h \in C^\infty_{b,F}(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu(X;\Gamma_{\frac{1}{2}-\gamma}))$ we associate a continuous Mellin pseudodifferential operator

$$\mathrm{op}^\gamma_M(h) : \mathcal{T}_\gamma(\mathbb{R}_+, C^\infty(X)) \to \mathcal{T}_\gamma(\mathbb{R}_+, C^\infty(X)),$$

which is defined by

$$[\mathrm{op}^\gamma_M(h)u](t) = \iint_0^\infty \left(\frac{t}{t'}\right)^{-(\frac{1}{2}-\gamma+i\tau)} h(t,t',\frac{1}{2}-\gamma+i\tau)u(t')\,\frac{dt'}{t'}đ\tau.$$

Here, the integrand is viewed as an amplitude function in $\mathcal{A}(\mathbb{R}_+ \times \Gamma_0, C^\infty(X))$ for each fixed $t > 0$.

Remark 2.7 Let $h \in C^\infty_{b,F}(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu(X;\Gamma_{\frac{1}{2}-\gamma}))$. One can check (as we shall do it later on in a similar situation, cf. Lemma 4.1) that

$$a(s,i\xi) := (\,(t,z) \mapsto h(t,st,z+i\xi)\,)$$

belongs to $\mathcal{A}(\mathbb{R}_+ \times \Gamma_0, C^\infty_{b,F}(\mathbb{R}_+, L^\mu(X;\Gamma_{\frac{1}{2}-\gamma}))$, i.e. the function a is an amplitude function with values in $C^\infty_{b,F}(\mathbb{R}_+, L^\mu(X;\Gamma_{\frac{1}{2}-\gamma}))$. Hence the oscillatory integral

$$h_L(t,z) = \iint_0^\infty s^{i\xi}h(t,st,z+i\xi)\,\frac{ds}{s}đ\xi$$

converges in $C^\infty_{b,F}(\mathbb{R}_+, L^\mu(X;\Gamma_{\frac{1}{2}-\gamma}))$. Actually, h_L is the left-symbol satisfying $\mathrm{op}^\gamma_M(h) = \mathrm{op}^\gamma_M(h_L)$. Analogously, we get the existence of the corresponding right-symbol h_R.

Proposition 2.8 *Each $h \in C^\infty_{b,F}(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu(X;\Gamma_{\frac{1}{2}-\gamma}))$ induces a continuous operator*

$$\mathrm{op}^\gamma_M(h) : \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge)$$

for each $s \in \mathbb{R}$. Moreover, we have continuity of the mapping

$$h \mapsto \mathrm{op}^\gamma_M(h) : C^\infty_{b,F}(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu(X;\Gamma_{\frac{1}{2}-\gamma})) \to \mathcal{L}(\mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge), \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge)).$$

Proof: By order reduction and conjugation with t^γ, the proof can be reduced to the case $\mathrm{op}^0_M(h) : L^2(X^\wedge) \to L^2(X^\wedge)$ and $\mu = 0$. Since $L^2(X^\wedge) = L^2(\mathbb{R}_+, L^2(X))$, the result follows from an easy extension of the usual Calderón-Vaillancourt Theorem (in the formulation for Mellin pseudodifferential operators) to the operator-valued case. □

Definition 2.9 For $\mu \in \mathbb{R}\cup\{-\infty\}$ we denote by $M^\mu_{\mathcal{O}}(X;\mathbb{R}^q)$ the subspace of all holomorphic functions $h(\cdot,\eta) : \mathbb{C} \to L^\mu(X;\mathbb{R}^q)$ such that

$$h(\beta+i\tau,\eta) \in L^\mu(X;\mathbb{R}_\tau \times \mathbb{R}^q_\eta)$$

uniformly for β in compact intervals. This is a Fréchet space with the following system of semi-norms:

$$\sup_{|\beta|\le N} p_N(h(\beta+i\tau,\eta)), \qquad N \in \mathbb{N},$$

where $\{p_N\}_{N\in\mathbb{N}}$ is a system of semi-norms in $L^\mu(X;\mathbb{R}\times\mathbb{R}^q)$. In particular, in the case $\dim X = 0$ we replace $L^\mu(X;\mathbb{R}^q)$ by $S^\mu(\mathbb{R}^q)$ and write $M^\mu_{\mathcal{O}}(\mathbb{R}^q)$. The class without the parameter $\eta\in\mathbb{R}^q$ will be denoted by $M^\mu_{\mathcal{O}}(X)$.

It is easy to obtain the following properties:

Lemma 2.10 *For $h\in C^\infty_{b,F}(\mathbb{R}_+\times\mathbb{R}_+, M^\mu_{\mathcal{O}}(X))$ is valid:*

(i) $\kappa_\lambda^{-1}\,\mathrm{op}_M^\gamma(h)\,\kappa_\lambda = \mathrm{op}_M^\gamma(h_\lambda)$ *with* $h_\lambda(t,t',z) = h(\lambda^{-1}t,\lambda^{-1}t',z)$;

(ii) *if we set* $(T^\sigma h)(t,t',z) = h(t,t',z+\sigma)$ *then* $\mathrm{op}_M^\gamma(h)\,t^{-\sigma} = t^{-\sigma}\,\mathrm{op}_M^{\gamma+\sigma}(T^\sigma h)$;

(iii) *for arbitrary* $\gamma,\delta\in\mathbb{R}$ *we have* $\mathrm{op}_M^\gamma(h) = \mathrm{op}_M^\delta(h)$ *on* $C_0^\infty(X^\wedge)$.

Lemma 2.11 *Let X be a smooth compact manifold. Then*

$$M_{\mathcal{O}}^{-\infty}(X;\mathbb{R}^q) = C^\infty(X\times X)\hat{\otimes}_\pi M_{\mathcal{O}}^{-\infty}(\mathbb{R}^q).$$

PROOF: In view of $L^{-\infty}(X;\mathbb{R}^q) = \mathcal{S}(\mathbb{R}^q, C^\infty(X\times X))$ and by a direct comparison of semi-norms we obtain $M_{\mathcal{O}}^{-\infty}(X;\mathbb{R}^q) = C^\infty(X\times X, M_{\mathcal{O}}^{-\infty}(\mathbb{R}^q))$. Then the assertion follows from the nuclearity of $C^\infty(X\times X)$ and the completeness of $M_{\mathcal{O}}^{-\infty}(\mathbb{R}^q)$, see Treves [11]. □

3 A class of parameter-dependent cone operators

3.1 Edge degenerate symbols

Edge-degenerate symbols in the sense of the following definition are motivated by the analysis of pseudodifferential operators on a manifold with edges, cf. [9]. They are operator-valued functions depending on an additional covariable η multiplied by t, the distance variable to the conical singularity. The same kind of degeneracy appears when η is interpreted as a spectral parameter to a Fuchs type operator on a manifold with conical singularities. As a crucial result for the structure of our calculus we also formulate in this section the Mellin quantization for such edge-degenerate symbols.

Definition 3.1 Let us introduce the classes

$$C^\infty_{\deg}(\overline{\mathbb{R}}_+, L^\mu(X;\mathbb{R}^{1+q})) := \left\{p \mid p(t,\tau,\eta) = \tilde{p}(t,t\tau,t\eta) \text{ with } \tilde{p}\in C^\infty(\overline{\mathbb{R}}_+, L^\mu(X;\mathbb{R}^{1+q}))\right\},$$

$$C^\infty_{\deg}(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q)) := \left\{h \mid h(t,z,\eta) = \tilde{h}(t,z,t\eta) \text{ with } \tilde{h}\in C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q))\right\}$$

of edge degenerate symbols of Fourier and holomorphic Mellin type. The operators of *freezing of coefficients at 0* are defined as

$$p\mapsto p_\wedge(t,\tau,\eta) = \tilde{p}(0,t\tau,t\eta), \qquad h\mapsto h_\wedge(t,z,\eta) = \tilde{h}(0,z,t\eta).$$

Note that $C^\infty_{\deg}(\overline{\mathbb{R}}_+, F)\subset C^\infty(\mathbb{R}_+, F)$ for $F = L^\mu(X;\mathbb{R}^{1+q})$ or $F = M^\mu_{\mathcal{O}}(X;\mathbb{R}^q)$. In particular, we may consider the operator-families

$$\mathrm{op}_t(p)(\eta),\, \mathrm{op}_M^\gamma(h)(\eta) : C_0^\infty(X^\wedge)\to C^\infty(X^\wedge),$$

which are elements of $L^{\mu}(X^{\wedge};\mathbb{R}^q)$. Here op_t indicates the pseudodifferential operator based on the Fourier transform with respect to the variable t.

Theorem 3.2 (Mellin quantization) *Let $P \in C^{\infty}_{\deg}(\overline{\mathbb{R}}_+, L^{\mu}(X;\mathbb{R}^{1+q}))$ and let $\phi \in C_0^{\infty}(\mathbb{R}_+)$ be a function such that $\phi \equiv 1$ near to 1. Then there exists an operator-valued symbol $H \in C^{\infty}_{\deg}(\overline{\mathbb{R}}_+, M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^q))$ such that*

$$\mathrm{op}_t(P)(\eta) - \mathrm{op}_M^{\frac{1}{2}}(H)(\eta) = \mathrm{op}_t(Q)(\eta) \in L^{-\infty}(X^{\wedge};\mathbb{R}^q)$$

with $Q(t,t',\tau,\eta) = (1-\phi(t'/t))P(t,\tau,\eta)$ for all $t,t' \in \mathbb{R}_+, \tau \in \mathbb{R}, \eta \in \mathbb{R}^q$.

A proof of this theorem is given in the appendix A.1.

3.2 Operator-valued symbols

Our parameter-dependent operator functions on $X^{\wedge}$ are operator-valued symbols in which the typical rescaling properties of the cone operators together with the degeneracy in η are reflected by a corresponding type of symbol estimates. In this section we summarize some properties of a class, suitable to describe this effect.

In the sequel assume $E_j = \mathcal{K}^{s_j,\gamma_j}(X^{\wedge})^{\varrho_j}$ or $E_j = \mathcal{H}^{s_j,\gamma_j}(X^{\wedge})$ for some reals s_j, γ_j, ϱ_j, and $\{\kappa_\lambda | \lambda > 0\}$ as in Definition 2.5. Moreover, let us fix a smooth function $\eta \mapsto [\eta] : \mathbb{R}^q \to \mathbb{R}_+$ with $[\eta] = |\eta|$ for $|\eta| > c_0$ for some constant $c_0 > 0$.

Definition 3.3 Let $\mu \in \mathbb{R}$. By $S^{\mu}(\mathbb{R}^q; E_0, E_1)$ denote the space of all symbols $a \in C^{\infty}(\mathbb{R}^q, \mathcal{L}(E_0,E_1))$ that satisfy

$$\|\kappa^{-1}_{[\eta]}(D^{\alpha}_{\eta} a(\eta))\kappa_{[\eta]}\|_{\mathcal{L}(E_0,E_1)} \leq c_\alpha\, [\eta]^{\mu-|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^q$, and c_α some constant independent of $\eta \in \mathbb{R}^q$. Further, we set

$$S^{-\infty}(\mathbb{R}^q; E_0, E_1) := \bigcap_{\mu\in\mathbb{R}} S^{\mu}(\mathbb{R}^q; E_0, E_1) = \mathcal{S}(\mathbb{R}^q, \mathcal{L}(E_0,E_1)).$$

If we replace both spaces E_0, E_1 by $\mathbb{C}$ and suppose $\kappa_\lambda \equiv 1$, we obtain the standard scalar-valued Hörmander class $S^{\mu}_{1,0}(\mathbb{R}^q)$ of symbols with constant coefficients.

Of course, such operator-valued symbols can be formulated on $\Omega \times \mathbb{R}^q$ for some open set $\Omega \subset \mathbb{R}^q$ or $\Omega \subset \mathbb{R}^q \times \mathbb{R}^q$. It is also possible to admit arbitrary Banach spaces E_j with strongly continuous group actions, cf. [2], [10].

Remark 3.4 The following properties hold:

i) $S^{\mu_1}(\mathbb{R}^q; E_1, E_2) \cdot S^{\mu_0}(\mathbb{R}^q; E_0, E_1) \subset S^{\mu_0+\mu_1}(\mathbb{R}^q; E_0, E_2)$;

ii) $D^{\alpha}_{\eta} S^{\mu}(\mathbb{R}^q; E_0, E_1) \subset S^{\mu-|\alpha|}(\mathbb{R}^q; E_0, E_1)$;

iii) $S^{\mu}(\mathbb{R}^q; E_1, E_2) \subset S^{\mu}(\mathbb{R}^q; E_0, E_3)$ if $E_0 \hookrightarrow E_1$ and $E_2 \hookrightarrow E_3$.

Example 3.5 A function $f \in C^\infty(\mathbb{R}^q \setminus \{0\}, \mathcal{L}(E_0, E_1))$ is called (twisted) homogeneous of order μ, if

$$f(\lambda\eta) = \lambda^\mu \kappa_\lambda f(\eta) \kappa_\lambda^{-1} \quad \text{for all} \ \ \lambda > 0 \ \ \text{and} \ \ \eta \neq 0.$$

Then, if $\chi(\eta)$ is a excision function at 0 we have $\chi f \in S^\mu(\mathbb{R}^q; E_0, E_1)$.

This concept of homogeneity allows us to introduce classical symbols:

Definition 3.6 A symbol $a \in S^\mu(\mathbb{R}^q; E_0, E_1)$ is called classical, if there is a sequence of functions $a_{(\mu-j)} \in C^\infty(\mathbb{R}^q \setminus \{0\}, \mathcal{L}(E_0, E_1))$ that are homogeneous of degree $\mu - j$, such that for any excision function $\chi(\eta)$ at $\eta = 0$

$$a - \sum_{j=0}^{N-1} \chi a_{(\mu-j)} \in S^{\mu-N}(\mathbb{R}^q; E_0, E_1)$$

for every $N \in \mathbb{N}$. In this case we write $a \in S^\mu_{c\ell}(\mathbb{R}^q; E_0, E_1)$. For $j \in \mathbb{N}_0$ we set

$$\sigma_\wedge^{\mu-j}(a)(y, \eta) := a_{(\mu-j)}(y, \eta).$$

In particular, $\sigma_\wedge^\mu(a)$ plays the role of the homogeneous principal symbol of a.

Example 3.7 Let us consider an element $\tilde{f}(z, \eta) \in L^\mu(X; \Gamma_0 \times \mathbb{R}^q_\eta)$ and set $f(t, z, \eta) = \tilde{f}(z, t\eta)$. Then, if $\omega(t)$ and $\tilde{\omega}(t)$ are arbitrary cut–off functions, and $\nu, \tilde{\nu} \in \mathbb{R}$,

$$a(\eta) := \omega(t[\eta])\, [\eta]^\nu t^{\tilde{\nu}} \mathrm{op}_M^{\frac{1}{2}}(f)(\eta) \tilde{\omega}(t[\eta]) : \mathcal{K}^{s, \frac{n+1}{2}}(X^\wedge)^\varrho \to \mathcal{K}^{s-\mu, \frac{n+1}{2}+\tilde{\nu}}(X^\wedge)^{\varrho'}$$

is a smooth family in η of continuous operators for every fixed $s, \varrho, \varrho' \in \mathbb{R}$, and we have

$$a(\lambda\eta) = \lambda^{\nu-\tilde{\nu}} \kappa_\lambda a(\eta) \kappa_\lambda^{-1}$$

for all $\lambda \geq 1$, $|\eta| \geq c$ for a constant $c > 0$. Then

$$a(\eta) \in S^{\nu-\tilde{\nu}}_{c\ell}(\mathbb{R}^q; \mathcal{K}^{s, \frac{n+1}{2}}(X^\wedge)^\varrho, \mathcal{K}^{s-\mu, \frac{n+1}{2}+\tilde{\nu}}(X^\wedge)^{\varrho'}).$$

Furthermore,

$$\sigma_\wedge^{\nu-\tilde{\nu}}(a)(\eta) = \omega(t|\eta|)\, |\eta|^\nu t^{\tilde{\nu}} \mathrm{op}_M^{\frac{1}{2}}(f)(\eta) \tilde{\omega}(t|\eta|).$$

3.3 Green symbols

Algebra operations between operator families of the above kind lead to additional remainder terms, the so-called Green symbols. The notation is motivated by Green's function in elliptic boundary value problems. For the Laplacian (for instance) this function has, up to a fundamental solution, a boundary symbolic structure of the form of a Green symbol, cf. [10].

Definition 3.8 Let $\gamma, \gamma' \in \mathbb{R}$. The space $R^{\mu}_{G}(\mathbb{R}^q; (\gamma, \gamma'))$ of Green symbols with weight data (γ, γ') consists of all operator-valued functions that satisfy

$$g \in \bigcap_{s,s',\varrho,\varrho' \in \mathbb{R}} S^{\mu}_{c\ell}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge)^{\varrho}, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\varrho'}), \tag{4}$$

$$g^* \in \bigcap_{s,s',\varrho,\varrho' \in \mathbb{R}} S^{\mu}_{c\ell}(\mathbb{R}^q; \mathcal{K}^{s,-\gamma'}(X^\wedge)^{\varrho}, \mathcal{K}^{s',-\gamma}(X^\wedge)^{\varrho'}), \tag{5}$$

where $*$ denotes the pointwise formal adjoint in the sense of Definition 2.4. The subspace $R^{\mu}_{G}(\mathbb{R}^q; (\gamma, \gamma'))_\infty$ consists of all Green symbols, where in (4) we can replace γ' by $\gamma' + \varepsilon$ and in (5) $-\gamma$ by $-\gamma + \varepsilon$, and the intersections are taken also over $\varepsilon > 0$.

The properties of operator-valued symbols, cf. Remark 3.4, carry over in an obvious way to the Green symbols.

Remark 3.9 Due to the mapping property we can conclude that every Green symbol is parameter-dependent smoothing on $X^\wedge$, that is, $g(\eta) \in L^{-\infty}(X^\wedge; \mathbb{R}^q)$.

Example 3.10 Let $\varphi, \psi \in C^\infty_0(\mathbb{R}_+)$ be arbitrary functions with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$. Then, for $f(t, z, \eta)$ as in Example 3.7, and $\mu, \tilde{\mu} \in \mathbb{R}$,

$$g(\eta) := \varphi(t[\eta])\, [\eta]^{\mu} t^{\tilde{\mu}} \mathrm{op}_M^{\frac{1}{2}}(f)(\eta)\psi(t[\eta])$$

is a Green symbol of order $\mu - \tilde{\mu}$.

Example 3.11 For $P(\eta) \in L^{-\infty}(X^\wedge; \mathbb{R}^q)_0$ and functions $\varphi, \psi \in C^\infty_0(\mathbb{R}_+)$ we have

$$\varphi\, P(\cdot)\psi \in R^{-\infty}_{G}(\mathbb{R}^q; (\gamma, \gamma'))_\infty$$

for every $\gamma, \gamma' \in \mathbb{R}$.

Remark 3.12 Let $\sigma, \tilde{\sigma} \in C^\infty_0(\overline{\mathbb{R}}_+)$ be cut-off functions, and $g \in R^{\mu}_{G}(\mathbb{R}^q; (\gamma, \gamma'))$. Then

(i) $\sigma\, g,\ g\, \sigma \in R^{\mu}_{G}(\mathbb{R}^q; (\gamma, \gamma'))$,

(ii) $(1 - \sigma)g,\ g(1 - \sigma) \in R^{-\infty}_{G}(\mathbb{R}^q; (\gamma, \gamma'))$,

(iii) $\sigma\, g\, \tilde{\sigma} - g \in R^{-\infty}_{G}(\mathbb{R}^q; (\gamma, \gamma'))$,

(iv) $t^k\, g\, t^l \in R^{\mu-k-l}_{G}(\mathbb{R}^q; (\gamma - l, \gamma' + k))$ for $l, k \in \mathbb{R}$.

Example 3.13 Let $b \in \mathcal{S}(\mathbb{R}^2_{(t,t')}, C^\infty(X_x \times X_{x'}))$ be supported in $\mathbb{R}^2_+ \times X^2$, and set

$$(g(\eta)u)(t, x) = [\eta]^{\mu+n+1} \int_0^\infty \int_X b(t[\eta], x, t'[\eta], x')u(t', x')(t')^n \, dt' dx'$$

with $\mu \in \mathbb{R}$. Then $g(\lambda\eta) = \lambda^{\mu}\, \kappa_\lambda g(\eta)\kappa_\lambda^{-1}$ for all sufficiently large $|\eta|$ and $\lambda \geq 1$, i.e. g is twisted homogeneous of degree μ for large $|\eta|$, and we have $g \in R^{\mu}_{G}(\mathbb{R}^q, (\gamma, \gamma'))_\infty$ for all $\gamma, \gamma' \in \mathbb{R}$.

3.4 Complete edge symbols

We now turn to a class of operator-valued symbols that are parameter-dependent families of pseudodifferential operators on the infinite cone $X^\wedge$.

Definition 3.14 For $\gamma, \mu, \nu \in \mathbb{R}$, with $\mu - \nu \in \mathbb{N}_0$, let $R^\nu(\mathbb{R}^q; (\gamma, \gamma - \mu))$ be the space of all

$$a(\eta) = \sigma_1(t)(a_0(\eta) + a_1(\eta))\sigma_0(t) + (1 - \sigma_1)(t)P(\eta)(1 - \sigma_2)(t) + g(\eta)$$

with

$$\begin{aligned} a_0(\eta) &= \omega_1(t[\eta])t^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\omega_0(t[\eta]), \\ a_1(\eta) &= (1 - \omega_1)(t[\eta])t^{-\nu}\mathrm{op}_t(p)(\eta)(1 - \omega_2)(t[\eta]), \end{aligned}$$

where $p \in C^\infty_{\mathrm{deg}}(\overline{\mathbb{R}}_+, L^\nu(X; \mathbb{R}^{1+q}_{\tau,\eta}))$ and $h \in C^\infty_{\mathrm{deg}}(\overline{\mathbb{R}}_+, M^\nu_{\mathcal{O}}(X; \mathbb{R}^q))$ is the Mellin quantization of p. Moreover, $g \in R^\nu_G(\mathbb{R}^q; (\gamma, \gamma - \mu))$ and $P(\eta) \in L^\nu(X^\wedge; \mathbb{R}^q)_0$. Apart from that, σ_j, ω_j, $j = 0, 1, 2$, are cut-off functions satisfying $\omega_2 \prec \omega_1 \prec \omega_0$ and $\sigma_2 \prec \sigma_1 \prec \sigma_0$. We associate to a an additional specific symbolic level, namely the (twisted) homogeneous principal edge symbol

$$\sigma^\nu_\wedge(a)(\eta) := \sigma^\nu_\wedge(a_0)(\eta) + \sigma^\nu_\wedge(a_1)(\eta) + \sigma^\nu_\wedge(g)(\eta), \quad \eta \in \mathbb{R}^q \setminus \{0\},$$

with

$$\begin{aligned} \sigma^\nu_\wedge(a_0)(\eta) &= \omega_1(t|\eta|)t^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_\wedge)\omega_0(t|\eta|), \\ \sigma^\nu_\wedge(a_1)(\eta) &= (1 - \omega_1)(t|\eta|)t^{-\nu}\mathrm{op}_t(p_\wedge)(1 - \omega_2)(t|\eta|), \end{aligned}$$

where $h_\wedge$ and $p_\wedge$ are as in Definition 3.1.

The elements of the space $R^\nu(\mathbb{R}^q; (\gamma, \gamma - \mu))$ are called *complete edge symbols* (without asymptotics, with constant coefficients) as they were originally introduced in [9] (cf. also [2],[10]). In the present form we can prove the following fact:

Proposition 3.15 *For arbitrary $s, \varrho \in \mathbb{R}$ we have*

$$R^\nu(\mathbb{R}^q; (\gamma, \gamma - \mu)) \subset S^\nu(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge)^\varrho).$$

In particular, $\sigma^\nu_\wedge(a) \in C^\infty(\mathbb{R}^q \setminus \{0\}, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge)^\varrho))$ is homogeneous of degree ν, cf. Example 3.5.

For a proof see for example [8].

3.5 New representation of complete edge symbols

This section shows that the concept of holomorphic representations in the above Mellin quantization leads to a very convenient new description of the operator-valued edge symbols, in which the η-dependent cut-off functions are removed from the non-smoothing part. In particular, we get the principal edge symbol in an equivalent simple form.

Definition 3.16 Let $\gamma, \mu, \nu \in \mathbb{R}$, with $\mu - \nu \in \mathbb{N}_0$. The space $\mathcal{R}^\nu(\mathbb{R}^q; (\gamma, \gamma - \mu))$ consists of all operator-families of the form

$$a(\eta) = \sigma_1(t)\, t^{-\nu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(h)(\eta)\, \sigma_0(t) + (1 - \sigma_1)(t) P(\eta)(1 - \sigma_2)(t) + g(\eta), \qquad (\mathcal{R})$$

where $h \in C^\infty_{\deg}(\overline{\mathbb{R}}_+, M^\nu_{\mathcal{O}}(X; \mathbb{R}^q))$, $P(\eta) \in L^\nu(X^\wedge; \mathbb{R}^q)_0$ and $g \in R^\nu_G(\mathbb{R}^q; (\gamma, \gamma - \mu))$. The functions σ_j, $j = 0, 1, 2$, are cut-off functions satisfying $\sigma_2 \prec \sigma_1 \prec \sigma_0$.
As in Definition 3.14 we have the principal edge symbol

$$\sigma^\nu_\wedge(a)(\eta) := t^{-\nu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(h_\wedge)(\eta) + \sigma^\nu_\wedge(g)(\eta), \quad \eta \in \mathbb{R}^q \setminus \{0\}.$$

Recall that $h_\wedge(t, z, \eta) = \tilde{h}(0, z, t\eta)$. In Lemma 4.5 we will prove that every edge symbol is a usual parameter-dependent pseudodifferential operator on $X^\wedge$. In particular, we also have the interior principal symbol $\sigma^\nu_\psi(a)$ in the case of classical operators.

Remark 3.17 The localization with σ_0 allows us to assume, without loss of generality, that in $(\mathcal{R})$ the symbol h is compactly supported in $t \in \overline{\mathbb{R}}_+$.

Theorem 3.18 *The class $R^\nu(\mathbb{R}^q; (\gamma, \gamma - \mu))$ from Definition 3.14 coincides with the class $\mathcal{R}^\nu(\mathbb{R}^q; (\gamma, \gamma - \mu))$. The principal edge symbol $\sigma^\nu_\wedge(a)$ is independent of the representation of the corresponding edge symbol a.*

Proof: Let $a \in \mathcal{R}^\nu(\mathbb{R}^q; (\gamma, \gamma - \mu))$ with the notation as in $(\mathcal{R})$. Setting $a_M(\eta) := t^{-\nu} \mathrm{op}_M^{\gamma - \frac{n}{2}}(h)(\eta)$ we have

$$a(\eta) = \sigma_1(t) a_M(\eta) \sigma_0(t) + (1 - \sigma_1)(t) P(\eta)(1 - \sigma_2)(t) + g(\eta).$$

Let $\omega_2 \prec \omega_1 \prec \omega_0$ be cut-off functions. Then

$$\begin{aligned} a_M(\eta) &= \omega_1(t[\eta]) a_M(\eta) \omega_0(t[\eta]) + (1 - \omega_1)(t[\eta]) a_M(\eta)(1 - \omega_2)(t[\eta]) \\ &\quad + \omega_1(t[\eta]) a_M(\eta)(1 - \omega_0)(t[\eta]) + (1 - \omega_1)(t[\eta]) a_M(\eta) \omega_2(t[\eta]) \\ &\equiv \omega_1(t[\eta]) a_M(\eta) \omega_0(t[\eta]) + (1 - \omega_1)(t[\eta]) a_M(\eta)(1 - \omega_2)(t[\eta]) \pmod{R^\nu_G} \end{aligned} \qquad (6)$$

due to Proposition A.8 and Remark 3.12. Therefore,

$$\begin{aligned} a(\eta) &\equiv \sigma_1(t)(a_0(\eta) + a_1(\eta)) \sigma_0(t) + (1 - \sigma_1)(t) P(\eta)(1 - \sigma_2)(t) \\ &\quad + \sigma_1(1 - \omega_1)(t[\eta]) \{a_M(\eta) - t^{-\nu} \mathrm{op}_t(p)(\eta)\}(1 - \omega_2)(t[\eta]) \sigma_0 \pmod{R^\nu_G}, \end{aligned} \qquad (7)$$

where p and the symbol h of a_M are related via the Mellin quantization, and $a_0(\eta)$, $a_1(\eta)$ are as in Definition 3.14. Now, the latter term in (7) is in R^ν_G due to Proposition A.4. Hence

$$a(\eta) = \sigma_1(t)(a_0(\eta) + a_1(\eta)) \sigma_0(t) + (1 - \sigma_1)(t) P(\eta)(1 - \sigma_2)(t) + \tilde{g}(\eta) \qquad (8)$$

implying that $\mathcal{R}^\nu \subset R^\nu$. The other inclusion follows similarly from the relation

$$\begin{aligned} a_0(\eta) + a_1(\eta) &= \omega_1(t[\eta]) a_M(\eta) \omega_0(t[\eta]) + (1 - \omega_1)(t[\eta]) a_M(\eta)(1 - \omega_2)(t[\eta]) \\ &\quad + (1 - \omega_1)(t[\eta]) \{t^{-\nu} \mathrm{op}_t(p)(\eta) - a_M(\eta)\}(1 - \omega_2)(t[\eta]), \end{aligned}$$

using (6) and applying again Proposition A.4. Finally, by the same calculations for the principal edge symbol (with $t|\eta|$ instead of $t[\eta]$) we obtain as in (8)

$$\sigma^{\nu}_{\wedge,new}(a) = \sigma^{\nu}_{\wedge,old}(a_0) + \sigma^{\nu}_{\wedge,old}(a_1) + \sigma^{\nu}_{\wedge}(\tilde{g}) = \sigma^{\nu}_{\wedge,old}(a)$$

with trivial meaning of the notation. $\square$

4 Elements of the calculus for complete edge symbols

4.1 Calculus for degenerate holomorphic Mellin symbols

Lemma 4.1 *Let $\tilde{h} \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q))$ be independent of $t \in \overline{\mathbb{R}}_+$ for large t. Then*

$$a(s,w) := \big((t,z,\eta) \mapsto \tilde{h}(st, w+z, s\eta) \big) \ \in \ \mathcal{A}(\mathbb{R}_+ \times \mathbb{C}, C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q))),$$

i.e., a is a holomorphic amplitude function with values in $C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q))$.

Proof: *Step 1*: Consider the local situation, i.e. $\tilde{h} \in S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C} \times \mathbb{R}^{n+q})$. Since the following calculations are not affected by the variables of $\mathbb{R}^n \times \mathbb{R}^n$, we omit them in the notation. It is clear that $a \in C^\infty(\mathbb{R}_+, S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{C} \times \mathbb{R}^q))\hat{\otimes}_\pi \mathcal{O}(\mathbb{C})$, where $\mathcal{O}(\mathbb{C})$ is the Fréchet space of entire functions. On $S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{C} \times \mathbb{R}^q)$ we consider the semi-norms

$$p_n(\tilde{h}) = \sup\left\{ |\partial_t^{l'} \partial_z^{k'} \partial_\eta^{\alpha'} \tilde{h}(t, \sigma_0 + i\tau, \eta)| \, \langle \tau,\eta\rangle^{|\alpha'|+k'-\mu} \right\},$$

where the supremum is taken over all $t \le n$, $|\sigma_0| \le n$, $(\tau,\eta) \in \mathbb{R}^{1+q}$, and $l'+k'+|\alpha'| \le n$. We have to show the existence of μ_n, m_n, such that for all $k,l,N \in \mathbb{N}$

$$\sup\left\{ p_n((s\partial_s)^l \partial_w^k a(s, \sigma_1 + i\xi)) g(s)^{-m_n} \langle \xi \rangle^{-\mu_n} \ | \ |\sigma_1| \le N, (s,\xi) \in \mathbb{R}_+ \times \mathbb{R} \right\} < \infty. \tag{9}$$

Here, $g(s) = e^{\langle \log s \rangle}$. If we write $(\eta\partial_\eta)^\alpha = (\eta_1 \partial_{\eta_1})^{\alpha_1} \ldots (\eta_q \partial_{\eta_q})^{\alpha_q}$ we have

$$(s\partial_s)^l a(s,w) = \sum_{m+|\alpha|=l} c_{m\alpha} [(t\partial_t)^m (\eta\partial_\eta)^\alpha \tilde{h}](s\cdot, w+\cdot, s\cdot).$$

Since the map $\tilde{h} \mapsto (t\partial_t)^m (\eta\partial_\eta)^\alpha \tilde{h} : S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{C} \times \mathbb{R}^q) \to S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{C} \times \mathbb{R}^q)$ is continuous and $(t\partial_t)^l (\eta\partial_\eta)^\alpha \tilde{h}$ also satisfies the assumptions on $\tilde{h}$, we can assume in (9) that $l = 0$. The estimate (9) is valid if we can show that

$$\partial_t^{l'} \partial_z^{k+k'} \partial_\eta^\alpha [\tilde{h}(st, \sigma_0 + \sigma_1 + i\tau + i\xi, s\eta)] \le |||\tilde{h}||| \, g(s)^{m_n} \langle \xi \rangle^{\mu_n} \langle \tau,\eta \rangle^{\mu - |\alpha'| - k'} \tag{10}$$

uniformly in $t \le n$, $|\sigma_0| \le n$, $|\sigma_1| \le N$, $l'+k'+|\alpha'| \le n$, $(\tau,\eta) \in \mathbb{R}^{1+q}$, $(s,\xi) \in \mathbb{R}_+ \times \mathbb{R}$, with a semi-norm $|||\cdot|||$ of $S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{C} \times \mathbb{R}^q)$, depending only on k,n, and N. Using the elementary inequality $\langle \tau, s\eta \rangle^\mu \le \max\{s^{-1}, s\}^{|\mu|} \langle \tau,\eta \rangle^\mu$, the left-hand side of (10) is dominated by

$$\begin{aligned}
& s^{l'+|\alpha'|} |(\partial_t^{l'} \partial_z^{k+k'} \partial_\eta^\alpha \tilde{h})(st, \sigma_0 + \sigma_1 + i\tau + i\xi, s\eta)| \\
& \le |||\tilde{h}||| \, s^{l'+|\alpha'|} \langle \tau + \xi, s\eta \rangle^{\mu - k - k' - |\alpha'|} \le c\,|||\tilde{h}||| \, s^{l'+|\alpha'|} \langle \tau, s\eta \rangle^{\mu - k' - |\alpha'|} \langle \xi \rangle^{|\mu - |\alpha'| - k'|} \\
& \le c\,|||\tilde{h}||| \, g(s)^{l'+|\alpha'|+|\mu-k'-|\alpha'||} \langle \xi \rangle^{|\mu-k'-|\alpha'||} \langle \tau,\eta \rangle^{\mu-k'-|\alpha'|} \\
& \le c\,|||\tilde{h}||| \, g(s)^{m_n} \langle \xi \rangle^{\mu_n} \langle \tau,\eta \rangle^{\mu - k' - |\alpha'|}.
\end{aligned}$$

Here we have set $\mu_n = \max_{j=1}^n |\mu - j|$, $m_n = n + \max_{j=1}^n |\mu - j|$. Moreover,

$$|||\tilde{h}||| = \sup\left\{ |(\partial_t^{l'} \partial_z^{k+k'} \partial_\eta^{\alpha'} \tilde{h})(t, \sigma + i\tau, \eta)| \, \langle \tau, \eta \rangle^{k+k'+|\alpha'|-\mu} \right\},$$

with the supremum over all $t \in \mathbb{R}$, $|\sigma| \leq n + N$, $l' + k' + |\alpha'| \leq n$, and $(\tau, \eta) \in \mathbb{R}^{1+q}$.
Step 2: Assume that $\tilde{h} \in C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^{-\infty}(X; \mathbb{R}^q))$. We introduce the following notation: for a Hilbert space E let $S^\mu(\overline{\mathbb{R}}_+ \times \Gamma_\delta \times \mathbb{R}^q; E)$ be the space of all $f \in C^\infty(\overline{\mathbb{R}}_+ \times \Gamma_\delta \times \mathbb{R}^q, E)$ such that

$$\sup\left\{ \left\| \partial_t^l \partial_\tau^k \partial_\eta^\alpha f(t, \delta + i\tau, \eta) \right\|_E \langle \tau, \eta \rangle^{\alpha+k-\mu} \mid (\tau, \xi) \in \mathbb{R}^{1+q}, t \in K, l + k + |\alpha| \leq N \right\} < \infty$$

for all $N \in \mathbb{N}$ and all compact sets $K \subset \overline{\mathbb{R}}_+$. Analogously, we generalize $S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{C} \times \mathbb{R}^q)$ to $S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{C} \times \mathbb{R}^q; E)$. These are Fréchet spaces, and

$$C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^{-\infty}(X; \mathbb{R}^q)) = \operatorname*{pr.lim}_{j \in \mathbb{N}} S^{-j}(\overline{\mathbb{R}}_+ \times \mathbb{C} \times \mathbb{R}^q; H^j(X \times X)).$$

Precisely as in Step 1 we have $a \in \mathcal{A}(\mathbb{R}_+ \times \mathbb{C}, C^\infty(S^{-j}(\overline{\mathbb{R}}_+ \times \mathbb{C} \times \mathbb{R}^q, H^j(X \times X))))$ for each $j \in \mathbb{N}$. Hence $a \in \mathcal{A}(\mathbb{R}_+ \times \mathbb{C}, C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^{-\infty}(X; \mathbb{R}^q)))$ by Lemma 1.4(ii).
Step 3: In the general case, i.e. $\tilde{h} \in C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^{-\infty}(X; \mathbb{R}^q))$, we write

$$\tilde{h}(t, z, \eta) = \sum_{j=1}^N \Phi_j(\theta_j^{-1})_* \operatorname{op}_x(\tilde{h}_j)(t, z, \eta) \, \Psi_j + \tilde{h}_{N+1}(t, z, \eta)$$

with $\tilde{h}_j \in S^\mu(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{C} \times \mathbb{R}^{n+q})$ and $\tilde{h}_{N+1} \in C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^{-\infty}(X; \mathbb{R}^q))$. Here, θ_j, Φ_j, and Ψ_j are as in the proof of Theorem 3.2. Thus, with obvious meaning of notation,

$$a(s, w) = \sum_{j=1}^N \Phi_j(\theta_j^{-1})_* \operatorname{op}_x(a_j(s, w)) \, \Psi_j + a_{N+1}(s, w).$$

In view of Steps 1, 2 and Lemma 1.4(iv), a is an amplitude function as desired. □

Proposition 4.2 *Let $\tilde{h}(t, z, \eta) \in C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^\mu(X; \mathbb{R}^q))$ be independent of t for large t, and $h(t, z, \eta) = \tilde{h}(t, z, t\eta)$. Then*

$$\tilde{h}_R(t', z, \eta) := \iint s^{-i\xi} \tilde{h}(st', z + i\xi, s\eta) \, \frac{ds}{s} đ\xi$$

(convergent in $C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^\mu(X; \mathbb{R}^q))$). Setting $h_R(t', z, \eta) := \tilde{h}_R(t', z, t'\eta)$ we obtain that, for each real γ,

$$\operatorname{op}_M^\gamma(h_R)(\eta) = \operatorname{op}_M^\gamma(h)(\eta).$$

Proof: By Lemma 4.1, the oscillatory integral exists in $C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^\mu(X; \mathbb{R}^q))$.
Step 1: Let $\tilde{h}$ be compactly supported in $t \in \overline{\mathbb{R}}_+$. Then $h(\cdot, \cdot, \eta) \in C_{b,F}^\infty(\mathbb{R}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$. By Remark 2.7

$$h_R(t', z, \eta) = \iint s^{-i\xi} h(st', z + i\xi, \eta) \, \frac{ds}{s} đ\xi$$

exists in $C^\infty_{b,F}(\mathbb{R}_+, L^\mu(X;\Gamma_{\frac{1}{2}-\gamma}))$ and $\mathrm{op}^\gamma_M(h_R)(\eta) = \mathrm{op}^\gamma_M(h)(\eta)$ for each η. Interpreting both oscillatory integrals as such in $L^\mu(X)$ yields $h_R(t', z, \eta) = \tilde{h}_R(t', z, t'\eta)$ in $L^\mu(X)$ for all $t' > 0$, $z \in \mathbb{C}$, and $\eta \in \mathbb{R}^q$. Due to the continuity this is then also true for $t' = 0$.
Step 2: Assume $\tilde{h}(t, z, \eta) = (1-\omega)(t)\tilde{h}_\infty(z, \eta)$ with $\tilde{h}_\infty \in M^\mu_{\mathcal{O}}(X;\mathbb{R}^q)$ and some cut-off function ω. The holomorphy allows us to write

$$\mathrm{op}^\gamma_M(h)(\eta) = \mathrm{op}^\gamma_M(t^{-N}\, T^N h)(\eta)\, t^N.$$

For $N \geq \max\{0, \mu\}$ we have $h^N(t, z, \eta) := t^{-N}\, T^N h(t, z, \eta) \in C^\infty_{b,F}(\mathbb{R}_+, L^\mu(X;\, \Gamma_{\frac{1}{2}-\gamma}))$ for each η (recall that $\tilde{h}$ is supported away from zero). Then, proceeding as in the first step, we obtain that

$$h^N_R(t', z, \eta) = t'^N \iint s^{-i\xi} h^N(st', z + i\xi, \eta)\, \frac{ds}{s} d\!\!{}^-\xi$$

(convergent in $C^\infty_{b,F}(\mathbb{R}_+, L^\mu(X;\Gamma_{\frac{1}{2}-\gamma}))$) satisfies

$$\mathrm{op}^\gamma_M(t^{-N}\, T^N h)(\eta)\, t^N = \mathrm{op}^\gamma_M(h^N_R)(\eta).$$

For fixed $t' > 0$, z, and η,

$$h^N_R(t', z, t'^{-1}\eta) = \iint s^{-i\xi} s^{-N} \tilde{h}(st', z + N + i\xi, s\eta)\, \frac{ds}{s} d\!\!{}^-\xi$$

with convergence in $L^\mu(X)$. But this integral even converges in $C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q))$ and equals $\tilde{h}_R(t', z, \eta)$. Hence $\mathrm{op}^\gamma_M(h_R)(\eta) = \mathrm{op}^\gamma_M(h)(\eta)$.
Finally, a general $\tilde{h}$ can be decomposed into the two parts treated above. □

With similar calculations one can prove the following results:

Proposition 4.3 *Let $h_j \in C^\infty_{\deg}(\overline{\mathbb{R}}_+, M^{\mu_j}_{\mathcal{O}}(X;\mathbb{R}^q))$, $j = 0, 1$, be independent of t for large t. If we define $h \in C^\infty_{\deg}(\overline{\mathbb{R}}_+, M^{\mu_0+\mu_1}_{\mathcal{O}}(X;\mathbb{R}^q))$ by*

$$\tilde{h}(t, z, \eta) := \iint s^{-i\xi} \tilde{h}_0(t, z + i\xi, \eta) \tilde{h}_1(st, z, s\eta)\, \frac{ds}{s} d\!\!{}^-\xi$$

(convergent in $C^\infty(\overline{\mathbb{R}}_+, M^{\mu_0+\mu_1}_{\mathcal{O}}(X;\mathbb{R}^q))$) then for each real γ

$$\mathrm{op}^\gamma_M(h_0)(\eta)\mathrm{op}^\gamma_M(h_1)(\eta) = \mathrm{op}^\gamma_M(h)(\eta).$$

Proposition 4.4 *Let $h \in C^\infty_{\deg}(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q))$ be independent of t for large t. Further, define $h^{(*)} \in C^\infty_{\deg}(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q))$ by*

$$\tilde{h}^{(*)}(t, z, \eta) := \iint s^{-i\xi} \tilde{h}(st, n + 1 - \bar{z} + i\xi, s\eta)^\star\, \frac{ds}{s} d\!\!{}^-\xi$$

(convergent in $C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q))$), where $\star$ is the formal adjoint in $L^\mu(X)$. Then for each real γ

$$\mathrm{op}^\gamma_M(h)(\eta)^* = \mathrm{op}^{-\gamma-n}_M(h^{(*)})(\eta).$$

The operator on the left-hand side is the formal adjoint in the sense of Definition 2.4.

4.2 Some properties of the edge symbols

Lemma 4.5 *Let $a \in \mathcal{R}^\nu(\mathbb{R}^q; (\gamma, \gamma - \mu))$. Then a is an element of $L^\nu(X^\wedge; \mathbb{R}^q)$.*

PROOF: Let a be written as in $(\mathcal{R})$. We see immediately

$$(1-\sigma_1)(t)P(\eta)(1-\sigma_2)(t) + g(\eta) \in L^\nu(X^\wedge; \mathbb{R}^q).$$

Writing now $\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)$ as a pseudodifferential operator with respect to the Fourier transform we get

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta) = \mathrm{op}_t(q)(\eta) \quad \text{with} \quad q \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^\nu(X; \mathbb{R}^{1+q}))$$

so that $\mathrm{op}_t(q)(\eta) \in L^\nu(X^\wedge; \mathbb{R}^q)$. □

Lemma 4.6 *Let $\varphi, \psi \in C_0^\infty(\mathbb{R}_+)$, and $Q(\eta) \in L^\nu(X^\wedge; \mathbb{R}^q)$. Then*

$$\varphi Q(\cdot)\psi \in \mathcal{R}^\nu(\mathbb{R}^q; (\gamma, \gamma-\mu))$$

with vanishing principal edge symbols. Moreover, we may represent $\varphi Q(\eta)\psi$ as in $(\mathcal{R})$ with h and P being compatible, i.e. for some fixed $c' > c > 0$ we have

$$\phi(t^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta) - P(\eta))\tilde{\phi} \in L^{-\infty}(X^\wedge; \mathbb{R}^q) \tag{11}$$

for all $\phi, \tilde{\phi} \in C_0^\infty(\mathbb{R}_+)$ supported in $[c, c']$.

PROOF: Of course, $P(\eta) := \varphi Q(\eta)\psi \in L^\nu(X^\wedge; \mathbb{R}^q)_0$. For cut-off functions $\sigma_2 \prec \sigma_1 \prec \sigma_0$, the operators $\sigma_1 Q(\eta)(1-\sigma_0)$ and $(1-\sigma_1)Q(\eta)\sigma_2$ are smoothing so that $\sigma_1 P(\eta)(1-\sigma_0)$ and $(1-\sigma_1)P(\eta)\sigma_2$ are both in $R_G^{-\infty}(\mathbb{R}^q; (\gamma, \gamma-\mu))_\infty$ for all $\mu \in \mathbb{R}$ (cf. Example 3.11). Hence

$$P(\eta) \equiv \sigma_1 P(\eta)\sigma_0 + (1-\sigma_1)P(\eta)(1-\sigma_2) \pmod{R_G^{-\infty}}.$$

We now use the Mellin quantization to write $\sigma_1 P(\eta)\sigma_0$ as a sum $\sigma_1 t^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 + g(\eta)$ with suitable h and g. Due to the presence of the functions φ and ψ there is a symbol $q \in C^\infty(\mathbb{R}_+, L^\nu(X; \mathbb{R}^{1+q}))$, compactly supported in $\mathbb{R}_+$, such that $P(\eta) = \mathrm{op}_t(q)(\eta)$ and

$$t^\nu q(t, t^{-1}\tau, t^{-1}\eta) \in C^\infty(\overline{\mathbb{R}}_+, L^\nu(X; \mathbb{R}^{1+q}_{\tau,\eta}))$$

so that $p(t,\tau,\eta) := t^\nu q(t,\tau,\eta) \in C^\infty_{\mathrm{deg}}(\overline{\mathbb{R}}_+, L^\nu(X; \mathbb{R}^{1+q}_{\tau,\eta}))$. This symbol p is also compactly supported in $\mathbb{R}_+$. From Theorem 3.2 there exists an $h \in C^\infty_{\mathrm{deg}}(\overline{\mathbb{R}}_+, M^\nu_{\mathcal{O}}(X; \mathbb{R}^q))$ such that

$$d(\eta) := \mathrm{op}_t(p)(\eta) - \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta) \in L^{-\infty}(X^\wedge; \mathbb{R}^q).$$

Note that h is compactly supported in $\mathbb{R}_+$ as p, and the difference $d(\eta)$ is supported away from $t = 0$. Together with Example 3.11 we obtain

$$\begin{aligned} \sigma_1 P(\eta)\sigma_0 &= \sigma_1 t^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 + \sigma_1 t^{-\nu} d(\eta)\sigma_0 \\ &\equiv \sigma_1 t^{-\nu}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 \pmod{R_G^{-\infty}} \end{aligned}$$

what yields the desired representation

$$P(\eta) = \sigma_1 \, t^{-\nu} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 + (1-\sigma_1)P(\eta)(1-\sigma_2) + g(\eta)$$

with $g \in R_G^{-\infty}(\mathbb{R}^q; (\gamma, \gamma-\mu))_\infty$. The compatibility relation is clearly satisfied for any constants $c' > c > 0$. □

Lemma 4.7 *Every complete edge symbol a can be written as in $(\mathcal{R})$ with compatibility of h and P as in Lemma 4.6.*

Proof: Every $a \in \mathcal{R}^\nu$ is of the form

$$a(\eta) = \sigma_1 \, t^{-\nu} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 + (1-\sigma_1)P(\eta)(1-\sigma_2) + g(\eta).$$

By means of the (inverse) Mellin quantization we find an element $P_1(\eta) \in L^\nu(X^\wedge; \mathbb{R}^q)_0$ being compatible with h. Set

$$a_1(\eta) = \sigma_1 \, t^{-\nu} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0 + (1-\sigma_1)P_1(\eta)(1-\sigma_2) + g(\eta).$$

On the other hand, $a - a_1 = (1-\sigma_1)(P(\eta) - P_1(\eta))(1-\sigma_2)$ can be written with the desired compatibility condition due to Lemma 4.6. Thus we are done since $a = a_1 + (a - a_1)$. □

Corollary 4.8 *Let h and P be compatible in $[c, c']$, and let $\sigma_2 \prec \sigma_1 \prec \sigma_0$ and $\tilde{\sigma}_2 \prec \tilde{\sigma}_1 \prec \tilde{\sigma}_0$ be two sets of cut-off functions supported in $[0, c']$ such that $\sigma_2 = \tilde{\sigma}_2 = 1$ on some open neighbourhood of $[0, c]$. As in $(\mathcal{R})$ let us form a with $\{\sigma_j\}$ as well as $\tilde{a}$ with $\{\tilde{\sigma}_j\}$. Then $a - \tilde{a}$ belongs to $R_G^{-\infty}(\mathbb{R}^q; (\gamma, \gamma-\mu))_\infty$; in other words, the class $\mathcal{R}^\nu(\mathbb{R}^q; (\gamma, \gamma-\mu))$ is independent of the choice of the cut-off functions whenever they respect the compatibility between h and P. The same is true if we simultaneously interchange in $(\mathcal{R})$ $\sigma_1 \leftrightarrow \sigma_0$ and $(1-\sigma_1) \leftrightarrow (1-\sigma_2)$.*

Proof: Choose cut-off functions $\omega_5 \prec \omega_4 \prec \omega_3$ with $\omega_3 \prec \sigma_2$, $\omega_3 \prec \tilde{\sigma}_2$ and such that $\omega_5 = 1$ on $[0, c]$. If we write

$$a = \omega_4 \, a \, \omega_3 + (1-\omega_4)a(1-\omega_5) + \omega_4 \, a(1-\omega_3) + (1-\omega_4)a \, \omega_5$$

then Proposition A.13 yields

$$a \equiv \omega_4 a_M \omega_3 + (1-\omega_4)a(1-\omega_5) \pmod{R_G^{-\infty}} \tag{12}$$

where $a_M(\eta) = \sigma_1 \, t^{-\nu} \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\sigma_0$. In the same manner

$$a \equiv \omega_1 \, a \, \omega_0 + (1-\omega_1)P(1-\omega_2) \pmod{R_G^{-\infty}} \tag{13}$$

for functions $\omega_2 \prec \omega_1 \prec \omega_0$ with $\sigma_0 \prec \omega_2$ and $\tilde{\sigma}_0 \prec \omega_2$. Inserting (13) into (12) we get

$$a \equiv (1-\omega_4)\omega_1 \, a \, \omega_0(1-\omega_5) + \omega_4 a_M \omega_3 + (1-\omega_1)P(1-\omega_2) \pmod{R_G^{-\infty}}.$$

Doing the same for $\tilde{a}$ we obtain

$$a - \tilde{a} \equiv \underbrace{(1-\omega_4)\omega_1}_{=:\varphi_1}(a-\tilde{a})\underbrace{\omega_0(1-\omega_5)}_{=:\varphi_0} \quad (\mathrm{mod}\ R_G^{-\infty}).$$

As in the proof of Lemma 4.6 we have

$$\begin{aligned}\varphi_1(1-\sigma_1)P(1-\sigma_2)\varphi_0 &\equiv \varphi_1(P-\sigma_1 P\sigma_0)\varphi_0 \quad (\mathrm{mod}\ R_G^{-\infty}),\\ \varphi_1(1-\tilde{\sigma}_1)P(1-\tilde{\sigma}_2)\varphi_0 &\equiv \varphi_1(P-\tilde{\sigma}_1 P\tilde{\sigma}_0)\varphi_0 \quad (\mathrm{mod}\ R_G^{-\infty}),\end{aligned}$$

and therefore

$$\varphi_1(a-\tilde{a})\varphi_0 \equiv \varphi_1\sigma_1(a_M - P)\sigma_0\varphi_0 - \varphi_1\tilde{\sigma}_1(a_M-P)\tilde{\sigma}_0\varphi_0 \equiv 0 \quad (\mathrm{mod}\ R_G^{-\infty})$$

due to the compatibility relation (11) and Example 3.11. □

As an immediate consequence of the new representation we get that the class of complete edge symbols is closed with respect to differentiation and pointwise formal adjoint:

Proposition 4.9 *Let $a \in \mathcal{R}^\nu(\mathbb{R}^q;(\gamma,\gamma-\mu))$. Then*

$$D_\eta^\alpha a \in \mathcal{R}^{\nu-|\alpha|}(\mathbb{R}^q;(\gamma,\gamma-\mu)) \ \text{ and } \ a^* \in \mathcal{R}^\nu(\mathbb{R}^q;(-\gamma+\mu,-\gamma)),$$

where a^* is the formal adjoint in the sense of Definition 2.4. Moreover, the principal edge symbols satisfy

$$\sigma_\wedge^{\nu-|\alpha|}(D_\eta^\alpha a)(\eta) = D_\eta^\alpha\sigma_\wedge^\nu(a)(\eta) \ \text{ and } \ \sigma_\wedge^\nu(a^*)(\eta) = \sigma_\wedge^\nu(a)^*(\eta).$$

4.3 The composition theorem

Theorem 4.10 *Let $a_j \in \mathcal{R}^{\nu_j}(\mathbb{R}^q;(\gamma_j,\gamma_j-\mu_j))$, $j=0,1$, with $\gamma_1=\gamma_0-\mu_0$. Then*

$$a_1a_0 \in \mathcal{R}^{\nu_0+\nu_1}(\mathbb{R}^q;(\gamma_0,\gamma_0-\mu_0-\mu_1)).$$

Moreover, the symbols satisfy

$$\sigma_\wedge^{\nu_0+\nu_1}(a_1a_0)(\eta) = \sigma_\wedge^{\nu_1}(a_1)(\eta)\sigma_\wedge^{\nu_0}(a_0)(\eta) \quad \text{for all } \eta\in\mathbb{R}^q\setminus\{0\},$$

and in the case of classical operators $\sigma_\psi^{\nu_0+\nu_1}(a_1a_0) = \sigma_\psi^{\nu_1}(a_1)\sigma_\psi^{\nu_0}(a_0)$.

PROOF: Let us write

$$a_0 = a_{0M} + a_{0P} + g_0 \ \text{ and } \ a_1 = a_{1M}+a_{1P}+g_1$$

with $a_{jM}(\eta) = \sigma_1 t^{-\nu_j}\mathrm{op}_M^{\gamma_j-\frac{n}{2}}(h_j)(\eta)\sigma_0$ and $a_{jP}(\eta) = (1-\sigma_1)P_j(\eta)(1-\sigma_2)$. First of all we consider the term

$$a_1g_0 = a_{1M}g_0 + a_{1P}g_0 + g_1g_0.$$

Clearly, $g_1 g_0 \in R_G^{\nu_0+\nu_1}(\mathbb{R}^q;(\gamma_0,\gamma_0-\mu_0-\mu_1))$ with $\sigma_\wedge^{\nu_0+\nu_1}(g_1g_0)=\sigma_\wedge^{\nu_1}(g_1)\sigma_\wedge^{\nu_0}(g_0)$. Moreover, $a_{1P}g_0 \in R_G^{-\infty}(\mathbb{R}^q;(\gamma_0,\gamma_0-\mu_0-\mu_1))$ due to the presence of $(1-\sigma_1)$ and Remark 3.12. If $\omega_2 \prec \omega_1 \prec \omega_0$ are cut-off functions, Proposition A.8 yields that

$$\begin{aligned} a_{1M}(\eta)g_0(\eta) \equiv\ & \omega_1(t[\eta])a_{1M}(\eta)\omega_0(t[\eta])g_0(\eta) \\ & + (1-\omega_1)(t[\eta])a_{1M}(\eta)(1-\omega_2)(t[\eta])g_0(\eta) \pmod{R_G^{\nu_0+\nu_1}}. \end{aligned} \tag{14}$$

Due to elementary mapping properties of Mellin operators it is easy to see that the first term on the right-hand side belongs to $R_G^{\nu_0+\nu_1}(\mathbb{R}^q;(\gamma_0,\gamma_0-\mu_0-\mu_1))$. By choosing ω_1 appropriately, we may rewrite the second term as

$$\sigma_1\Big\{t^{-N}(1-\omega_1)(t[\eta])\Big\}\Big\{\chi(\eta)t^{-\nu_1}\mathrm{op}_M^{\gamma-\frac{n}{2}}(T^N h_1)(\eta)\Big\}\Big\{t^N(1-\omega_2)(t[\eta])\sigma_0 g_0(\eta)\Big\}$$

for some excision function χ at $\eta=0$ and each $N\in\mathbb{N}$. Choosing N sufficiently large, the mapping properties of the respective three factors yield that the second term on the right-hand side of (14) also belongs to $R_G^{\nu_0+\nu_1}(\mathbb{R}^q;(\gamma_0,\gamma_0-\mu_0-\mu_1))$. By freezing the coefficients of $\tilde h_1$ at $t=0$ it is straightforward to verify that $\sigma_\wedge^{\nu_0+\nu_1}(a_{1M}g_0)=\sigma_\wedge^{\nu_1}(a_{M1})\sigma_\wedge^{\nu_0}(g_0)$. Similarly we obtain $g_1a_{0M}+g_1a_{0P}\in R_G^{\nu_0+\nu_1}(\mathbb{R}^q;(\gamma_0,\gamma_0-\mu_0-\mu_1))$ so that we have

$$a_1\cdot a_0 \equiv a_{1M}\cdot a_{0M}+a_{1P}\cdot a_{0P}+a_{1M}\cdot a_{0P}+a_{1P}\cdot a_{0M} \pmod{R_G^{\nu_0+\nu_1}}$$

Choosing a cut-off function $\tilde\sigma \prec \sigma_1$ we can write

$$a_{1M}(\eta)a_{0P}(\eta)=\{\tilde\sigma a_{1M}(\eta)(1-\sigma_1)\}P_0(\eta)(1-\sigma_2)+(1-\tilde\sigma)a_{1M}(\eta)a_{0P}(\eta).$$

In view of Proposition A.13 the first term on the right-hand side is in $R_G^{-\infty}(\mathbb{R}^q;(\gamma_0,\gamma_0-\mu_0-\mu_1))_\infty$. The second term belongs to $\mathcal{R}^{\nu_0+\nu_1}(\mathbb{R}^q;(\gamma_0,\gamma_0-\mu_0-\mu_1))$ in view of Lemma 4.6. The product $a_{1P}(\eta)a_{0M}(\eta)$ is treated analogously. In other words, there is a $P(\eta)\in L^{\nu_0+\nu_1}(X^\wedge;\mathbb{R}^q)_0$ such that

$$a_{1P}\cdot a_{0P}+a_{1M}\cdot a_{0P}+a_{1P}\cdot a_{0M}\equiv(1-\sigma_1)P(\eta)(1-\sigma_2) \pmod{R_G^{-\infty}}.$$

As a direct consequence we have that this term does not contribute to the principal edge symbol of the composition. Now, with $h_0'(t,z,\eta):=\sigma_1(t)h_0(t,z,\eta)$ and applying Lemma 2.10 we obtain

$$\begin{aligned} a_{1M}(\eta)a_{0M}(\eta) &= \sigma_1\, t^{-\nu_0-\nu_1}\mathrm{op}_M^{\gamma_1+\nu_0-\frac{n}{2}}(T^{\nu_0}h_1)(\eta)\,\mathrm{op}_M^{\gamma_1+\nu_0-\frac{n}{2}}(h_0')(\eta)\sigma_0 \\ &= \sigma_1\, t^{-\nu_0-\nu_1}\mathrm{op}_M^{\gamma_0-\frac{n}{2}}(h)(\eta)\sigma_0 \quad \text{on } C_0^\infty(X^\wedge), \end{aligned}$$

where $h\in C^\infty_{\deg}(\overline{\mathbb{R}}_+,M_{\mathcal{O}}^{\nu_0+\nu_1}(X;\mathbb{R}^q))$ due to Proposition 4.3. More precisely, $h(t,z,\eta)=\tilde h(t,z,t\eta)$ with

$$\tilde h(t,z,\eta)=\iint s^{-i\xi}\,\tilde h_1(t,z+\nu_0+i\xi,\eta)\sigma_1(st)\tilde h_0(st,z,s\eta)\frac{ds}{s}đ\xi.$$

Thus the relation for the principal edge symbols is clearly satisfied as well as the relation for the interior symbols. □

A Further results from the cone theory

A.1 Proof of the global Mellin quantization

Theorem A.1 *Let $V \subset \mathbb{R}^n$ be open. Let $\tilde{p} \in S^\mu(\overline{\mathbb{R}}_+ \times V \times \mathbb{R}^{1+n+q})$, $\mu \in \mathbb{R}$, and set $p(t,x,\tau,\xi,\eta) := \tilde{p}(t,x,t\tau,\xi,t\eta)$. Let further $\phi \in C_0^\infty(\mathbb{R}_+)$ be a function such that $\phi \equiv 1$ near to 1. Then there exists an $\tilde{h} \in S^\mu(\overline{\mathbb{R}}_+ \times V \times \mathbb{C} \times \mathbb{R}^{n+q})$ such that*

$$\mathrm{op}_{t,x}(\phi(t'/t)p)(\eta) = \mathrm{op}_M^{\frac{1}{2}}(\mathrm{op}_x(h))(\eta)$$

for any $\eta \in \mathbb{R}^q$, where $h(t,x,z,\xi,\eta) := \tilde{h}(t,x,z,\xi,t\eta)$.

A proof of this theorem was given in [4, Theorem 2.3]. In fact, we have explicitly

$$\tilde{h}(t,x,z,\xi,\eta) = v_z(t)\mathrm{op}_M^{\frac{1}{2}}(\phi(t'/t)\tilde{g})(x,\xi,\eta)v_{-z}(t) \tag{15}$$

with $\tilde{g}(t,t',x,i\tau,\xi,\eta) := M(t,t')t'\tilde{p}(t,x,-M(t,t')t\tau,\xi,\eta) \in S^\mu(\mathbb{R}_+^2 \times V \times \Gamma_0 \times \mathbb{R}^{n+q})$. Here $M(t,t') := \frac{\log t - \log t'}{t-t'} > 0$ for $t,t' \in \mathbb{R}_+$, and $v_z(t) := t^z \in C^\infty(\mathbb{R}_+, C^\infty(X))$.

Proof of Theorem 3.2

We begin with the trivial identity $\mathrm{op}_t(P) = \mathrm{op}_t(\phi(t'/t)P) + \mathrm{op}_t(Q)$. Since

$$\mathrm{op}_t(Q)(\eta) = \mathrm{op}_t(Q_N)(\eta) \quad \text{for all } \; N \in \mathbb{N}$$

with $Q_N(t,t',\tau,\eta) = (1-\phi(t'/t))(t'/t-1)^{-N}(D_\tau^N\tilde{P})(t,t\tau,t\eta) \in C^\infty(\mathbb{R}_+^2, L_{c\ell}^{\mu-N}(X;\mathbb{R}^{1+q}))$, then $\mathrm{op}_t(Q)(\eta) \in L^{-\infty}(X^\wedge;\mathbb{R}^q)$. Next, let $\{U_1,\dots,U_N\}$ be an open covering of X by coordinate neighborhoods with corresponding charts $\theta_j : U_j \to V_j \subset \mathbb{R}^n$. Further let $\{\varphi_1,\dots,\varphi_N\}$ be a subordinate partition of unity, and $\{\psi_1,\dots,\psi_N\}$ be a system of functions satisfying $\psi_j \in C_0^\infty(U_j)$ and $\varphi_j\psi_j = \varphi_j$ for all $j = 1,\dots,N$. Then the non-degenerate part corresponding to P can be written

$$\tilde{P}(t,\tau,\eta) = \sum_{j=1}^N \Phi_j\,\tilde{P}_j(t,\tau,\eta)\Psi_j + \tilde{P}_\infty(t,\tau,\eta), \quad \text{and so}$$

$$\mathrm{op}_t(\phi(t'/t)P)(\eta) = \sum_{j=1}^N \Phi_j\,\mathrm{op}_t[\phi(t'/t)(\theta_j^{-1})_*\,\mathrm{op}_x(p_j)](\eta)\Psi_j + \mathrm{op}_t(\phi(t'/t)P_\infty)(\eta), \tag{16}$$

where $p_j(t,x,\tau,\xi,\eta) := \tilde{p}_j(t,x,t\tau,\xi,t\eta)$ with local symbols $\tilde{p}_j \in S^\mu(\overline{\mathbb{R}}_+ \times V_j \times \mathbb{R}^{1+n+q}_{\tau,\xi,\eta})$, $j = 1,\dots,N$; $\tilde{P}_\infty \in C^\infty(\overline{\mathbb{R}}_+, L^{-\infty}(X;\mathbb{R}^{1+q}_{\tau,\eta}))$, and Φ_j, Ψ_j are the operators of multiplication by the corresponding functions φ_j, ψ_j. From Theorem A.1 there are Mellin symbols $\tilde{h}_j \in S^\mu(\overline{\mathbb{R}}_+ \times V \times \mathbb{C} \times \mathbb{R}^{n+q})$, $j = 1,\dots,N$, such that

$$\mathrm{op}_t[\phi(t'/t)(\theta_j^{-1})_*\,\mathrm{op}_x(p_j)](\eta) = \mathrm{op}_M^{\frac{1}{2}}(H_j)(\eta) \quad \text{with } \; H_j(t,z,\eta) = (\theta_j^{-1})_*\,\mathrm{op}_x(h_j)(t,z,\eta)$$

for any $\eta \in \mathbb{R}^q$. To handle the remainder term in (16) let us set

$$\tilde{G}_\infty(t,t',i\tau,\eta) := \phi(t'/t)M(t,t')t'\tilde{P}_\infty(t,-M(t,t')t\tau,\eta) \text{ and}$$
$$\tilde{H}_\infty(t,z,\eta) := v_z(t)\mathrm{op}_M^{\frac{1}{2}}(\tilde{G}_\infty)(\eta)v_{-z}(t).$$

We have $\mathrm{op}_t(\phi(t'/t)P_\infty)(\eta) = \mathrm{op}_M^{\frac{1}{2}}(H_\infty)(\eta)$ with $H_\infty(t,z,\eta) := \tilde{H}_\infty(t,z,t\eta)$, cf. (15). Moreover, we claim that $\tilde{H}_\infty$ belongs to $C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{-\infty}(X;\mathbb{R}^q))$. To prove this we first observe that $\tilde{G}_\infty \in C^\infty(\mathbb{R}_+^2, L^{-\infty}(X;\Gamma_0 \times \mathbb{R}^q))$. Further,

$$L^{-\infty}(X;\Gamma_0 \times \mathbb{R}^q) = C^\infty(X \times X)\hat{\otimes}_\pi \mathcal{S}(\Gamma_0 \times \mathbb{R}^q) \text{ and}$$
$$M_{\mathcal{O}}^{-\infty}(X;\mathbb{R}^q) = C^\infty(X \times X)\hat{\otimes}_\pi M_{\mathcal{O}}^{-\infty}(\mathbb{R}^q).$$

For that reason it is sufficient to show that $\tilde{H}_\infty \in C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{-\infty}(\mathbb{R}^q))$ whenever $\tilde{G}_\infty \in C^\infty(\mathbb{R}_+^2, \mathcal{S}(\Gamma_0 \times \mathbb{R}^q))$, that is, we have to show holomorphy in z, and the boundedness of the semi-norms

$$|\tilde{H}_\infty|_{m,N} := \sup\left\{ |\partial_t^k \partial_{\varrho,\eta}^\alpha \tilde{H}_\infty(t,\beta+i\varrho,\eta)| \langle \varrho,\eta\rangle^N \right\},$$

where the supremum is taken over all $(\varrho,\eta) \in \mathbb{R}^{1+q}, |\alpha| \le N, |\beta| \le N, k \le m, t \le m$. Now for $z = \beta + i\varrho \in \mathbb{C}$, using the change $t' \to tr$, $\tau \to \tau + \varrho$, we have

$$\tilde{H}_\infty(t,z,\eta) = \iint \left(\frac{t}{t'}\right)^{-i\tau+z} \tilde{G}_\infty(t,t',i\tau,\eta)\frac{dt'}{t'}đ\tau$$
$$= \iint r^{i\tau-\beta}\chi(r)\tilde{P}_\infty(t,-M(r,1)(\tau+\varrho),\eta)dr\,đ\tau$$

with $\chi(r) = \phi(r)M(r,1) \in C_0^\infty(\mathbb{R}_+)$. Hence for $\alpha \in \mathbb{N}_0^{1+q}$ and $k \in \mathbb{N}_0$

$$\partial_t^k \partial_{\varrho,\eta}^\alpha \tilde{H}_\infty(t,\beta+i\varrho,\eta) = \iint r^{i\tau-\beta}\chi_1(r)(\partial_t^k\partial_{\varrho,\eta}^\alpha \tilde{P}_\infty)(t,-M(r,1)(\tau+\varrho),\eta)dr\,đ\tau,$$

and so, for any $l > 0$ we obtain

$$|\partial_t^k \partial_{\varrho,\eta}^\alpha \tilde{H}_\infty(t,\beta+i\varrho,\eta)| \le \tilde{c}_{k,\alpha} \iint r^{-\beta}\chi_2(r)\langle \tau+\varrho,\eta\rangle^{-l} dr\,d\tau,$$

where $\chi_{1,2} \in C_0^\infty(\mathbb{R}_+)$. In particular, for $l \ge N+2$ we have $\langle \tau+\varrho,\eta\rangle^{-l} \le \langle\tau\rangle^{-2}\langle\varrho,\eta\rangle^{-N}$. Then $|\partial_t^k \partial_{\varrho,\eta}^\alpha \tilde{H}_\infty(t,\beta+i\varrho,\eta)| \le c_{k,\alpha}(\beta)\langle\varrho,\eta\rangle^{-N}$ with $c_{k,\alpha}(\beta)$ depending continuously on β. Thus $|\tilde{H}_\infty|_{m,N} < \infty$ for every $m, N \in \mathbb{N}_0$. Finally, writing $\tilde{H}_\infty$ as above but without the change $\tau \to \tau + \varrho$, and looking now at the semi-norms of $\tilde{H}_\infty(t,z,\cdot)$ in $\mathcal{S}(\mathbb{R}^q)$, we can analogously verify that for every $t \in \overline{\mathbb{R}}_+$ the function $z \mapsto \tilde{H}_\infty(t,z,\cdot)$ is holomorphic. Summing up, the function

$$H(t,z,\eta) = \sum_{j=1}^N \Phi_j\, H_j(t,z,\eta)\Psi_j + H_\infty(t,z,\eta),$$

belongs to $C_{\deg}^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^\mu(X;\mathbb{R}^q))$ and the relation $\mathrm{op}_t(\phi(t'/t)P)(\eta) = \mathrm{op}_M^{\frac{1}{2}}(H)(\eta)$ is valid for every $\eta \in \mathbb{R}^q$. $\square$

A.2 Green remainders induced by the Mellin quantization

For $\mu, m \in \mathbb{R}$ we denote by $S^{\mu,m}(\overline{\mathbb{R}}_+^2 \times \mathbb{R}^{2n} \times \mathbb{R}^{1+n} \times \mathbb{R}^q)$ the Fréchet space of all functions $p \in C^\infty(\overline{\mathbb{R}}_+^2 \times \mathbb{R}^{2n} \times \mathbb{R}^{1+n} \times \mathbb{R}^q)$ with

$$\sup\left\{\left|\partial_t^\beta\, \partial_{t',x,x'}^\gamma\, \partial_{\tau,\xi,\eta}^\alpha\, p(t,t',x,x',\tau,\xi,\eta)\right| \langle t\rangle^{|\beta|-m}\, \langle(\tau,\xi,\eta)\rangle^{|\alpha|-\mu}\right\} < \infty$$

for all multi-indices α, β, γ; the supremum is taken over all involved variables.

Lemma A.2 *Let $\tilde{p}(t,t',x,x',\tau,\xi,\eta) \in S^{\mu,m}(\overline{\mathbb{R}}_+^2 \times \mathbb{R}^{2n} \times \mathbb{R}^{1+n} \times \mathbb{R}^q)$ be compactly supported in $(x,x') \in \mathbb{R}^{2n}$. Let $\chi \in C^\infty(\mathbb{R}^q)$ be an excision function at 0, and $\phi \in C_0^\infty(\mathbb{R}_+)$ with $\phi \equiv 1$ near to 1. Further, set*

$$\begin{aligned} q(t,t',x,x',\tau,\xi,\eta) &:= (1-\phi(t'/t))\tilde{p}(t,t',x,x',t\tau,\xi,t\eta) \quad \text{and} \\ Q(\eta) &:= \chi(\eta)(1-\omega)(t[\eta])\mathrm{op}_{t,x}(q)(\eta)(1-\omega_1)(t[\eta]). \end{aligned}$$

Then for each $s, \delta, \tilde{\mu}, \tilde{m} \in \mathbb{R}$

$$Q \in S^0(\mathbb{R}^q; H^{s,\delta}(\mathbb{R}^{1+n}), H^{s-\tilde{\mu},\delta-\tilde{m}}(\mathbb{R}^{1+n})).$$

If $\tilde{p}$ is additionally compactly supported in $t \in \overline{\mathbb{R}}_+$, we can omit χ in the definition of Q. The spaces $H^{s,\delta}$ are the usual weighted Sobolev spaces.

Proof: The variables (x, x', ξ) are irrelevant in the following calculations, so without loss of generality we assume $n = 0$. First observe that integration by parts yields $\mathrm{op}_t(q)(\eta) = \mathrm{op}_t(q_N)(\eta)$, where

$$q_N(t,t',\tau,\eta) = (1-\phi(t'/t))(t'/t-1)^{-N}(D_\tau^N\tilde{p})(t,t',t\tau,t\eta)$$

for every $N \in \mathbb{N}$. We define

$$q_{N,1}(t,t',\tau,\eta) := \chi(\eta)(1-\omega)(t[\eta])(1-\omega_1)(t'[\eta])q_N(t,t',\tau,\eta).$$

Clearly $Q(\eta) = \mathrm{op}_t(q_{N,1})(\eta)$. We shall prove that $q_{N,1} \in C^\infty(\mathbb{R}^q; S^{\mu_N,m_N}(\mathbb{R}^2_{t,t'} \times \mathbb{R}_\tau))$ with $\mu_N = \mu - N$, $m_N = m + \mu - N$. To verify the corresponding symbol estimates let us set $f_N(t,t',\eta) := (1-\omega)(t[\eta])(1-\omega_1)(t'[\eta])(1-\phi(t'/t))(t'/t-1)^{-N}$ and study for every $k, k', l \in \mathbb{R}$ the derivatives

$$\begin{aligned} I_{k,k',l}(q_{N,1}) &:= \left|\partial_t^k \partial_{t'}^{k'} \partial_\tau^l\, q_{N,1}(t,t',\tau)\right| \\ &= \left|\chi(\eta)\partial_t^k\partial_{t'}^{k'}\, t^l f_N(t,t',\eta)(\partial_\tau^{N+l}\tilde{p})(t,t',t\tau,t\eta)\right|. \end{aligned}$$

Due to the Leibniz formula $I_{k,k',l}$ leads to a finite sum of terms like

$$\left|(\partial_t^{k_1} t^l)(\partial_t^{k_2}\partial_{t'}^{k_2'} f_N(t,t',\eta))(\partial_t^{k_3}\tilde{p}_{N,l}(t,t',t\tau,t\eta))\right| =: |I_1 I_2 I_3|$$

with $\tilde{p}_{N,l} = \partial_{t'}^{k_3'}\partial_\tau^{N+l}\tilde{p}$, $\sum k_j = k$, $\sum k_j' = k'$, and where $t, t' \in [c_0, \infty)$ and $|\eta| \geq c_1$ for some constants $c_0, c_1 > 0$ depending on ω, ω_1 and χ.

Obviously, $|I_1| \leq C_1 \langle t \rangle^{l-k_1}$. To estimate I_2 we apply again the Leibniz formula and the chain rule. If some $\partial_t(1-\omega)$ appears then I_2 has compact support in t. For the remaining terms we have to consider expressions like

$$\left| t^{-k_2-l'}(\partial_r^L \phi)(t'/t)(t'/t-1)^{L-N-l'} \right| \leq c \langle t \rangle^{-k_2} \left| (\partial_r^L \phi)(t'/t)(t'/t-1)^{L-N-l'} \right|$$

with $c>0$, $l' \leq k_2'$ and $L \leq k_2 + l'$. Take $\varepsilon, \tilde{\varepsilon} > 0$ such that $\phi(t'/t) = 1$ for $|t'/t-1| \leq \varepsilon$ and $\phi(t'/t) = 0$ for $|t'/t-1| \geq \tilde{\varepsilon}$, then $(\partial_r^L \phi)(t'/t)|t'/t-1|^{L-N-l'}$ is uniformly bounded for each L, N, l'. We conclude that $|I_2| \leq C_2 \langle t \rangle^{-k_2}$ for all $t \in \mathbb{R}_+$. Further, we have

$$I_3 = \sum_{|\gamma|=k_3} a_\gamma(\eta) \tau^{\gamma_2} (\partial_{t,\tau,\eta}^\gamma \tilde{p}_{N,l})(t, t', t\tau, t\eta),$$

where $a_\gamma \in S^{k_3}(\mathbb{R}^q)$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}_0^{1+1+q}$. Moreover, the absolute value of every term in the sum is bounded by

$$c \langle \tau \rangle^{\gamma_2} \langle t \rangle^{m-\gamma_1} \langle t\tau, t\eta \rangle^{\mu-N-l-\gamma_2-|\gamma_3|} \leq c' \langle t \rangle^{m+\mu-N-l-k_3} \langle \tau \rangle^{\mu-N-l}; \tag{17}$$

here we use the inequalities

$$\langle t\tau, t\eta \rangle^{-L} \leq c \langle t\tau, t \rangle^{-L} \leq c' \langle t \rangle^{-L} \langle \tau \rangle^{-L} \quad \text{and} \quad \langle t\tau, t\eta \rangle^{L} \leq c'' \langle t \rangle^{L} \langle \tau \rangle^{L} \langle \eta \rangle^{L}$$

for every $L \geq 0$, which are true for $t \geq c_0$ and $|\eta| \geq c_1$.
Thus $|I_3| \leq C_3 \langle t \rangle^{m+\mu-N-l-k_3} \langle \tau \rangle^{\mu-N-l}$. Combining all these estimates we obtain

$$I_{k,k',l}(q_{N,1}) \leq C_{k,k',l}(\eta) \langle t \rangle^{m+\mu-N-k} \langle \tau \rangle^{\mu-N-l} \tag{18}$$

for every $k, k', l \in \mathbb{R}$, $\eta \in \mathbb{R}^q$. In the same way we estimate $I_{k,k',l}(\partial_\eta^\alpha q_{N,1})$ and get expressions like (18) with another $C_{k,k',l}^\alpha(\eta)$ of the same kind. Hence the assertion about $q_{N,1}$ holds.
Let us finally define

$$q_{N,2}(t, t', \tau, \eta) := \chi(\eta) f_N(t[\eta]^{-1}, t'[\eta]^{-1}, \eta)(D_\tau^N \tilde{p})(t[\eta]^{-1}, t'[\eta]^{-1}, t\tau, t[\eta]^{-1}\eta).$$

Clearly $q_{N,2} \in C^\infty(\mathbb{R}^q; S^{\mu_N, m_N}(\mathbb{R}^2_{t,t'} \times \mathbb{R}_\tau))$ too, and it holds $\kappa_{[\eta]}^{-1} Q(\eta) \kappa_{[\eta]} = \mathrm{op}_t(q_{N,2})(\eta)$. In particular, $q_{N,2}(\eta) \in S^{\tilde{\mu}, \tilde{m}}(\mathbb{R}^2_{t,t'} \times \mathbb{R}_\tau)$ for all N such that $\mu_N \leq \tilde{\mu}$ and $m_N \leq \tilde{m}$. Moreover, due to the Calderón-Vaillancourt Theorem, the operator norm of $\mathrm{op}_t(q_{N,2})(\eta)$ in $\mathcal{L}(H^{s,\delta}, H^{s-\tilde{\mu}, \delta-\tilde{m}})$ can be majorized by a finite number of expressions like

$$c \sup_{t,t',\tau} \{ I_{k,k',l}(q_{N,2}) \langle t \rangle^{k-\tilde{m}} \langle \tau \rangle^{l-\tilde{\mu}} \},$$

where t, t' run over $[c_0, \infty)$, $|\eta| \geq c_1$. The constant $c > 0$ as well as the supremum above depend continuously on $\tilde{p}$. Similar to (18) we have here

$$I_{k,k',l}(q_{N,2}) \leq \tilde{C}_{k,k',l}(\eta) \left\langle t[\eta]^{-1} \right\rangle^{m} \langle t \rangle^{\mu-N-k} \langle \tau \rangle^{\mu_N-l} \leq \tilde{C}_{k,k',l}(\eta) \langle t \rangle^{|m|+\mu-N-k} \langle \tau \rangle^{\mu_N-l}$$

with $\tilde{C}_{k,k',l}(\eta)$ bounded in $\eta \in \mathbb{R}^q$ since η appears in $q_{N,2}$ together with $[\eta]^{-1}$. Now, for N large enough the supremum above exists and is uniformly bounded in $\eta \in \mathbb{R}^q$. Therefore

$$\left\| \kappa_{[\eta]}^{-1} Q(\eta) \kappa_{[\eta]} \right\|_{\mathcal{L}(H^{s,\delta}, H^{s-\tilde{\mu},\delta-\tilde{m}})} \leq C \quad \text{for every } s, \delta, \tilde{\mu}, \tilde{m} \in \mathbb{R}.$$

The calculations for $\partial_\eta^\alpha Q(\eta)$ are similar. Note that ∂_η^α generates a factor $t^{|\alpha|}$ ($\tilde{p}$ depends on $t\eta$), and $\kappa_{[\eta]}^{-1} t^{|\alpha|} \kappa_{[\eta]} = t^{|\alpha|}[\eta]^{-|\alpha|}$. □

Remark A.3 If we interchange the variables $t \leftrightarrow t'$ and $x \leftrightarrow x'$ in the symbol of Q, then Lemma A.2 still holds. This will be used for adjoint operators.

Proposition A.4 *Let $h \in C^\infty_{\deg}(\overline{\mathbb{R}}_+, M^\nu_{\mathcal{O}}(X;\mathbb{R}^q))$ and $p \in C^\infty_{\deg}(\overline{\mathbb{R}}_+, L^\nu(X;\mathbb{R}^{1+q}))$ be related via the Mellin quantization (Theorem 3.2). Then*

$$g(\eta) = \sigma(t)(1-\omega)(t[\eta])\{\mathrm{op}_t(p)(\eta) - \mathrm{op}_M^{\frac{1}{2}}(h)(\eta)\}(1-\omega_1)(t[\eta])\sigma_0(t)$$

is an element of $R^0_G(\mathbb{R}^q;(\gamma,\gamma))_\infty$.

Proof: ¿From Theorem 3.2 we know that for some suitable $\phi \in C_0^\infty(\mathbb{R}_+)$

$$\mathrm{op}_t(p)(\eta) - \mathrm{op}_M^{\frac{1}{2}}(h)(\eta) = \mathrm{op}_t(Q)(\eta) \quad \text{for all } \eta \in \mathbb{R}^q;$$

with $Q(t,t',\tau,\eta) = (1-\phi(t'/t))p(t,\tau,\eta)$. Therefore, to obtain the assertion we only need to analyze locally operators of the form

$$G_k(\eta) = (1-\omega)(t[\eta])\mathrm{op}_{t,x}(r_k)(\eta)(1-\omega_1)(t[\eta]), \quad k=0,1,$$

where $r_0(t,t',x,x',\tau,\xi,\eta) = (1-\phi(t'/t))\tilde{q}(t,t',x,x',t\tau,\xi,t\eta)$ and $r_1(t,t',x,x',\tau,\xi,\eta) = (1-\phi(t/t'))\tilde{q}(t',t,x',x,t'\tau,\xi,t'\eta)$ with some $\tilde{q} \in S^\nu(\overline{\mathbb{R}}^2_+ \times \mathbb{R}^{2n} \times \mathbb{R}^{1+n} \times \mathbb{R}^q)$ being compactly supported in $\overline{\mathbb{R}}^2_+ \times \mathbb{R}^{2n}$. Lemma A.2 then give us $G_k \in S^0(\mathbb{R}^q; H^{s,\delta}, H^{s-\tilde{\nu},\delta-\tilde{m}})$ for every $s,\delta,\tilde{\nu},\tilde{m} \in \mathbb{R}$. Since G_k is supported away from $t=0$, it induces a continuous operator between cone Sobolev spaces on $X^\wedge$. Moreover, if $\chi \in C^\infty(\mathbb{R}^q)$ is an excision function at 0 then $(1-\chi)G_k \in S^{-\infty}(\mathbb{R}^q; H^{s,\delta}, H^{s-\tilde{\nu},\delta-\tilde{m}})$. For that reason, the function g above is a Green symbol if we can prove that χG_k is classical in the sense of Definition 3.6. To this end we assume for simplicity $\dim X = 0$, put $\mathbf{t} = (t,t')$, and apply the Taylor expansion

$$\tilde{q}(\mathbf{t},\tau,\eta) = \sum_{|\alpha|<N} \mathbf{t}^\alpha \tilde{q}_\alpha(\tau,\eta) + \sum_{|\alpha|=N} \mathbf{t}^\alpha \tilde{q}_\alpha(\mathbf{t},\tau,\eta)$$

with $\tilde{q}_\alpha \in S^\nu(\mathbb{R}_\tau \times \mathbb{R}^q_\eta)$ for $|\alpha| < N$ and $\tilde{q}_\alpha \in S^{\nu,0}(\overline{\mathbb{R}}^2_+ \times \mathbb{R}_\tau \times \mathbb{R}^q_\eta)$ for $|\alpha| = N$. Further, set $G := G_0$ (for G_1 the procedure is the same). Then

$$\chi(\eta)G(\eta) = \sum_{j=0}^{N-1} G_{(j)}(\eta) + G_{(N)}(\eta),$$

where $G_{(j)}(\eta) = \chi(\eta)(1-\omega)(t[\eta])\sum_{|\alpha|=j}\mathrm{op}_{t,x}(r_\alpha)(\eta)(1-\omega_1)(t[\eta])$ for every $0 \leq j \leq N$, and $r_\alpha = (1-\phi(t'/t))\mathbf{t}^\alpha \tilde{q}_\alpha(\mathbf{t},t\tau,t\eta)$. In view of Lemma A.2 each $G_{(j)}$ belongs to $S^{-j}(\mathbb{R}^q; H^{s,\delta}, H^{s-\tilde{\nu},\delta-\tilde{m}})$. Moreover, for each $0 \leq j < N$ the function $G_{(j)}$ is homogeneous of order $-j$ for large $|\eta|$. Therefore χG is classical and the proof is done. □

A.3 Green symbols generated by holomorphic Mellin symbols

Lemma A.5 *Let $s, \gamma \in \mathbb{R}$ be given and choose a cut-off function ω. Then*

(i) *for each $L \in \mathbb{R}$ and all $\gamma' \in \mathbb{R}$*

$$(1-\omega)(t[\eta])t^{-L} \in S^L_{c\ell}(\mathbb{R}^q; \mathcal{H}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma'}(X^\wedge)^\varrho)$$

with an appropriate $\varrho = \varrho(s, \gamma, L)$;

(ii) *for every $\gamma', \varrho \in \mathbb{R}$ there exists an $L = L(s, \gamma, \gamma', \varrho) \geq 0$ such that*

$$(1-\omega)(t[\eta])t^{-L} \in S^L_{c\ell}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{H}^{s,\gamma'}(X^\wedge)).$$

Lemma A.6 *Let $s, \gamma, \varrho, \gamma', \varrho' \in \mathbb{R}$. Then*

(i) $\tilde{\omega}(t[\eta])t^L \in S^{-L}_{c\ell}(\mathbb{R}^q; \mathcal{H}^{s,\gamma'}(X^\wedge), \mathcal{K}^{s,\gamma'+L}(X^\wedge)^{\varrho'})$ *for any $L \in \mathbb{R}$;*

(ii) $\tilde{\omega}(t[\eta])t^L \in S^{-L}_{c\ell}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{H}^{s,\gamma}(X^\wedge))$ *for $L \geq 0$.*

In fact, all symbols of the previous two lemmas are twisted homogeneous for large $|\eta|$ of degree L and $-L$, respectively.

Lemma A.7 *For $\omega_2 \prec \omega_1$ and $N \in \mathbb{N}$ let*

$$f(t, t', \eta) = \omega_2(t[\eta])(\log t/t')^{-N}(1-\omega_1)(t'[\eta]).$$

for $t, t' \in \mathbb{R}_+$ and $\eta \in \mathbb{R}^q$. Then the following is true:

(i) $f(\lambda^{-1}t, \lambda^{-1}t', \lambda\eta) = f(t, t', \eta)$ *for all $\lambda \geq 1$, $t, t' > 0$, and all $|\eta| \geq$ const;*

(ii) *For each $k, k' \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^q$,*

$$\sup_{t,t',\eta} \left\{ |(t\partial_t)^k (t'\partial_{t'})^{k'} \partial_\eta^\alpha f(t, t', \eta)| [\eta]^{|\alpha|} \right\} < \infty.$$

In particular, $(t, t') \mapsto [\eta]^{|\alpha|}\partial_\eta^\alpha f(t, t', \eta)$ is bounded in $\eta \in \mathbb{R}^q$ with respect to the topology in $C^\infty_{b,F}(\mathbb{R}_+ \times \mathbb{R}_+)$.

Proof: (i) is obvious. By induction, $(t\partial_t)^k (t'\partial_{t'})^{k'} \partial_\eta^\alpha f(t, t', \eta)$ is a linear combination of terms

$$a(\eta)[(t\partial_t)^l \omega_2](t[\eta])(\log t/t')^{-(N+M)}[(t'\partial_{t'})^{l'}(1-\omega_1)](t'[\eta])$$

with $l \leq |\alpha| + k$, $l' \leq |\alpha| + k'$, $M \leq k + k'$, and $a \in S^{-|\alpha|}(\mathbb{R}^q)$. Since $\omega_2 \prec \omega_1$, there is an $\varepsilon > 0$ such that $\omega_2(t)(1-\omega)(t') = 0$ whenever $|1 - t/t'| < \varepsilon$. Then also

$$[(t\partial_t)^l \omega_2](t[\eta])[(t'\partial_{t'})^{l'}(1-\omega)](t'[\eta]) = 0 \quad \text{for all } \ |1 - t/t'| < \varepsilon \ \text{ and } \ \eta \in \mathbb{R}^q.$$

Thus (ii) is valid since $\sup\{|\log t/t'|^{-1} \mid |1 - t/t'| \geq \varepsilon\} < \infty$. □

Proposition A.8 *Let $\tilde{h}(t,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q))$ be independent of t for large t, and $h(t,z,\eta) = \tilde{h}(t,z,t\eta)$. Moreover, let $\omega_2 \prec \omega_1$. Then both*

$$g_0(\eta) = \omega_2(t[\eta])\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\,(1-\omega_1)(t[\eta]) \quad \text{and}$$
$$g_1(\eta) = (1-\omega_1)(t[\eta])\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\,\omega_2(t[\eta])$$

are elements of $R^0_G(\mathbb{R}^q;(\gamma,\gamma))_\infty$.

Proof: First we will show that

$$g_0 \in S^0_{c\ell}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\varrho'}) \tag{19}$$

for arbitrary $s, s', \gamma', \varrho, \varrho'$. Integration by parts shows that

$$g_0(\eta) = \mathrm{op}_M^{\gamma-\frac{n}{2}}(f(t,t',\eta)\partial_z^N h(t,z,\eta))$$

for each $N \in \mathbb{N}$. Here, f is the function from Lemma A.7. Now choose ω, $\tilde{\omega}$ such that $\omega_2 \prec \tilde{\omega}$ and $\omega \prec \omega_1$. Moreover choose $L \geq 0$ so large that Lemma A.5(ii) is satisfied. Due to the holomorphy of h we get

$$g_0(\eta) = \tilde{\omega}(t[\eta])t^L \mathrm{op}_M^{\gamma'-\frac{n}{2}}(f(t,t',\eta)T^{-L}\partial_z^N h(t,z,\eta))(1-\omega)(t[\eta])t^{-L}$$

(as operators on $C_0^\infty(X^\wedge)$). In view of Lemma A.5(ii) and Lemma A.6(i) the relation (19) follows if we can show that

$$a(\eta) := \mathrm{op}_M^{\gamma'-\frac{n}{2}}(f(t,t',\eta)T^{-L}\partial_z^N h(t,z,\eta)) \in S^0_{c\ell}(\mathbb{R}^q; \mathcal{H}^{s,\gamma'}(X^\wedge), \mathcal{H}^{s-\mu+N,\gamma'}(X^\wedge))$$

for then we can choose N so large that $s-\mu+N \geq s'$. In order to simplify the notation we replace $T^L\partial_z^N h$ by h, and therefore assume that $\tilde{h} \in C^\infty(\overline{\mathbb{R}}_+, M^{\mu-N}_{\mathcal{O}}(X;\mathbb{R}^q))$. A Taylor expansion yields

$$\tilde{h}(t,z,\eta) = \sum_{j=0}^{J-1} t^j \tilde{h}_j(z,\eta) + t^J \tilde{h}_J(t,z,\eta)$$

with $\tilde{h}_j \in M^{\mu-N}_{\mathcal{O}}(X;\mathbb{R}^q)$ and $\tilde{h}_J \in C^\infty(\overline{\mathbb{R}}_+, M^{\mu-N}_{\mathcal{O}}(X;\mathbb{R}^q))$. Let h_j and h_J be the corresponding degenerate symbols. Hence we get

$$a(\eta) = \sum_{j=0}^{J} t^j \mathrm{op}_M^{\gamma'-\frac{n}{2}}(f(t,t',\eta)h_j(t,z,\eta)) =: \sum_{j=0}^{J} t^j a_j(\eta)$$

with obvious meaning of notation. By Lemma A.7(ii), in particular, $t^j f(t,t',\eta)h_j(t,z,\eta)$ is an element of $C^\infty(\mathbb{R}^q_\eta, C^\infty_{b,F}(\mathbb{R}_+\times\mathbb{R}_+, M^{\mu-N}_{\mathcal{O}}(X)))$ for each j. Therefore

$$t^j a_j(\eta) \in C^\infty(\mathbb{R}^q_\eta, \mathcal{L}(\mathcal{H}^{s,\gamma'}(X^\wedge), \mathcal{H}^{s-\mu+N,\gamma'}(X^\wedge))).$$

Lemma A.7(i) provides that, for $0 \leq j < J$, each $t^j a_j(\eta)$ is homogeneous of degree $-j$ for large $|\eta|$. Thus it remains to verify that

$$t^J a_J(\eta) = a(\eta) - \sum_{j=0}^{J-1} t^j a_j(\eta) \in S^{-J}(\mathbb{R}^q_\eta; \mathcal{H}^{s,\gamma'}(X^\wedge), \mathcal{H}^{s-\mu+N,\gamma'}(X^\wedge)).$$

For $\alpha \in \mathbb{N}_0^q$ we obtain

$$\partial_\eta^\alpha [f(t,t',\eta) h_J(t,z,\eta)] = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial_\eta^\beta f(t,t',\eta) t^{|\alpha-\beta|} (\partial_\eta^{\alpha-\beta} \tilde{h}_J)(t,z,t\eta),$$

where the differentiation is with respect to the topology in $C_{b,F}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, M_{\mathcal{O}}^{\mu-N}(X))$. This yields

$$\kappa_{[\eta]}^{-1}\, \partial_\eta^\alpha (t^J a_J(\eta))\, \kappa_{[\eta]} [\eta]^{|\alpha|+J} = \mathrm{op}_M^{\gamma'-\frac{n}{2}} (v_\alpha(t,t',\eta))$$

with

$$v_\alpha(t,t',\eta) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial_\eta^\beta f)(\frac{t}{[\eta]}, \frac{t'}{[\eta]}, \eta) [\eta]^{|\beta|} t^{J+|\alpha-\beta|} (\partial_\eta^{\alpha-\beta} \tilde{h}_J)(\frac{t}{[\eta]}, z, t\frac{\eta}{[\eta]}).$$

Now, v_α is a bounded function of η with values in $C_{b,F}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, M_{\mathcal{O}}^{\mu-N}(X))$ in view of Lemma A.7(ii) and the compact t-support of v_α. Thus

$$\sup_{\eta \in \mathbb{R}^q} \left\| \mathrm{op}_M^{\gamma'-\frac{n}{2}} (v_\alpha)(\eta) \right\|_{\mathcal{L}(\mathcal{H}^{s,\gamma'}, \mathcal{H}^{s-\mu+N,\gamma'})} < \infty,$$

and (19) holds. To treat g_1 we use Proposition 4.2 and rewrite $\mathrm{op}_M^{\gamma-\frac{n}{2}}(h) = \mathrm{op}_M^{\gamma-\frac{n}{2}}(h_R)$ with a right-symbol $h_R \in C_{\mathrm{deg}}^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^\mu(X;\mathbb{R}^q))$. Then we proceed analogously as above, i.e., we choose L in such a way that $\frac{n+1}{2} - \gamma + L \ge \varrho'$, and write

$$g_1(\eta) = (1-\omega)(t[\eta]) t^{-L} \mathrm{op}_M^{\gamma-\frac{n}{2}} ((-1)^N f(t',t,\eta) T^L \partial_z^N h_R(t',z,\eta)) \tilde{\omega}(t[\eta]) t^L.$$

With Lemmas A.5(i), A.6(ii), A.7, and a Taylor expansion of $\tilde{h}_R$ in t' at $t'=0$ one can verify that g_1 also satisfies (19). To consider the formal adjoints g_j^* it suffices to observe that, after rewriting g_1 with a right-symbol, g_j^* looks like g_{1-j} $(j=0,1)$. □

Definition A.9 Let E be a Fréchet space. We define $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+, E)$ as the subspace of all functions $k \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, E)$ that satisfy

$$\sup \left\{ p(\partial_t^l \partial_{t'}^{l'} k(t,t'))\, \langle t \rangle^N \, \langle t' \rangle^N \mid t,t' \ge \varepsilon \right\} < \infty$$

for each $\varepsilon > 0$, $l, l', N \in \mathbb{N}_0$ and each semi-norm p of E. This expressions provide a semi-norm system that induces a Fréchet topology on $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+, E)$.

Lemma A.10 *Let $\tilde{h} \in C_{b,F}^\infty(\mathbb{R}_+ \times \mathbb{R}_+, M_{\mathcal{O}}^{-\infty}(X;\mathbb{R}^q))$, and*

$$h_0(t,t',z,\eta) = \tilde{h}(t,t',z,t\eta), \qquad h_1(t,t',z,\eta) = \tilde{h}(t,t',z,t'\eta).$$

Since $M_{\mathcal{O}}^{-\infty}(X;\mathbb{R}^q) = M_{\mathcal{O}}^{-\infty}(\mathbb{R}^q) \hat{\otimes}_\pi C^\infty(X \times X)$, we may associate to each h_j a kernel

$$k_j(t,t',x,x',\eta) = \int \left(\frac{t}{t'}\right)^{-i\tau} h_j(t,t',x,x',i\tau,\eta)\, d\tau.$$

Then $k_j \in C^\infty(\mathbb{R}^q, C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times X \times X))$ and $k_j \in C^\infty(\mathbb{R}^q \setminus \{0\}, \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+, C^\infty(X \times X)))$. The corresponding mappings $\tilde{h} \mapsto k_j$ are continuous.

PROOF: By a standard tensor-product argument we can assume that the base X is a point. The part concerning the smoothness of k_j is clear. It remains to verify that, for each given $\varepsilon > 0$, $N \in \mathbb{N}_0$, and each compact set $K \subset \mathbb{R}^q \setminus \{0\}$,

$$\sup\left\{|\partial_t^l \partial_{t'}^{l'} \partial_\eta^\alpha k_j(t,t',\eta)| \, \langle t\rangle^N \langle t'\rangle^N \mid t,t' \geq \varepsilon,\ \eta \in K,\ l,l',|\alpha| \leq N\right\} < \infty.$$

Consider the case $j = 0$. Since $h_0(t,t',z,\eta)$ is holomorphic in z and decreases as a Schwartz function on each Γ_β uniformly for β in compact intervals, the Cauchy formula implies that

$$k_0(t,t',\eta) = t^N t'^{-N} \int \left(\frac{t}{t'}\right)^{-i\tau} h_0(t,t',i\tau - N,\eta)\, đ\tau.$$

Noting that

$$(t\partial_t)^l[f(t,t\eta)] = \sum_{|\beta|+k=l} c_{\beta k}[(t\partial_t)^k(\eta\partial_\eta)^\beta f](t,t\eta),$$

we see that $(t\partial_t)^l(t'\partial_{t'})^{l'}\partial_\eta^\alpha k_0(t,t',\eta)$ is a linear combination of terms

$$t^{N+|\alpha|}t'^{-N}\int\left(\frac{t}{t'}\right)^{-i\tau}[(t\partial_t)^n(t'\partial_{t'})^{n'}(\eta\partial_\eta)^\beta\partial_\eta^\alpha\tilde{h}](t,t',i\tau-N,t\eta)\tau^m\, đ\tau.$$

with $n+|\beta| \leq l$, $n' \leq l'$, and $m \leq l+l'$. For each $M > 0$ there is an appropriate semi-norm $|||\cdot|||$ of $C^\infty_{b,F}(\mathbb{R}_+\times\mathbb{R}_+, M_{\mathcal{O}}^{-\infty}(\mathbb{R}^q))$ (independent of $\tilde{h}$) such that the integrand is bounded from above by $c_K |||\tilde{h}|||\,\langle\tau\rangle^{-2}\langle t\rangle^{-M}$ uniformly in $\eta \in K$ (recall that $\tilde{h}$ is a Schwartz function in (τ,η) and $\langle t\eta\rangle^{-M} \leq c_K\langle t\rangle^{-M}$ for $\eta \in K$). Choosing $M = 3N$ we then obtain that

$$|(t\partial_t)^l(t'\partial_{t'})^{l'}\partial_\eta^\alpha k_0(t,t',\eta)| \leq c_{\varepsilon,K}|||\tilde{h}|||\,\langle t\rangle^{-N}\langle t'\rangle^{-N}$$

uniformly for $t,t' \geq \varepsilon$ and $\eta \in K$. This is then also true if we replace $(t\partial_t)^l(t'\partial_{t'})^{l'}$ by $\partial_t^l\partial_{t'}^{l'}$ since

$$\partial_s^k = s^{-k}\sum_{j=1}^k c_j(s\partial_s)^j$$

for certain constants c_j, and $t^{-l}t'^{-l'}$ is uniformly bounded for $t,t' \geq \varepsilon$. The case $j = 1$ can be treated in the same manner. □

Proposition A.11 *Let $\tilde{h} \in C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{-\infty}(X;\mathbb{R}^q))$, and set*

$$h_0(t,z,\eta) = \tilde{h}(t,z,t\eta), \qquad h_1(t',z,\eta) = \tilde{h}(t',z,t'\eta).$$

If $\chi(\eta)$ is an excision function at $\eta = 0$ then

$$g_j(\eta) = \chi(\eta)(1-\omega)(t[\eta])\operatorname{op}_M^{\gamma-\frac{n}{2}}(h_j)(\eta)\,(1-\omega_1)(t[\eta]) \ \in\ R_G^0(\mathbb{R}^q;(\gamma,\gamma))_\infty.$$

Moreover, if $\sigma,\sigma_0 \in C_0^\infty(\overline{\mathbb{R}}_+)$ are cut-off functions then

$$\tilde{g}_j(\eta) = \sigma(t)\,(1-\omega)(t[\eta])\operatorname{op}_M^{\gamma-\frac{n}{2}}(h_j)(\eta)\,(1-\omega_1)(t[\eta])\,\sigma_0(t) \ \in\ R_G^0(\mathbb{R}^q;(\gamma,\gamma))_\infty.$$

PROOF: At first we will show that, for arbitrary $s, s', \gamma', \varrho, \varrho' \in \mathbb{R}$,

$$g_0 \in S^0_{c\ell}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\varrho'}). \tag{20}$$

A Taylor expansion yields

$$\tilde{h}(t,z,\eta) = \sum_{j=0}^{N-1} t^j \tilde{f}_j(z,\eta) \;+\; t^N \tilde{f}_N(t,z,\eta)$$

with $\tilde{f}_j \in M^{-\infty}_{\mathcal{O}}(X;\mathbb{R}^q)$ for $0 \le j < N$, and $\tilde{f}_N \in C^\infty_{b,F}(\mathbb{R}_+, M^{-\infty}_{\mathcal{O}}(X;\mathbb{R}^q))$. Let f_j be the degenerate symbols corresponding to $\tilde{f}_j$. Due to the holomorphy of the involved symbols,

$$g_0(\eta) = \sum_{j=0}^{N} \chi(\eta)(1-\omega)(t[\eta])\, t^j\, K_j(\eta)\,(1-\omega_1)(t[\eta]) =: \sum_{j=0}^{N} g_{(j)}(\eta)$$

with integral operators $K_j(\eta)$ that have kernels

$$k_j(t,t',\eta) = t'^{-1} \int \left(\frac{t}{t'}\right)^{-i\tau} f_j(t,i\tau,\eta)\, đ\tau \tag{21}$$

with respect to the metric $dt'dx$ (for convenience we suppress the x,x'-variables from the notation). Because of Lemma A.10 and the presence of the excision function χ, each $g_{(j)}$ is a family of integral operators with kernel in $C^\infty(\mathbb{R}^q_\eta, \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+, C^\infty(X\times X)))$, whose kernel is supported away from $(\{t=0\}\times X)\times(\{t'=0\}\times X)$ locally uniformly in η. This implies that

$$g_0 \in C^\infty(\mathbb{R}^q; \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\varrho'})).$$

For $0 \le j < N$ each $g_{(j)}$ is homogeneous of degree $-j$ for large $|\eta|$, since $\tilde{f}_j$ is independent of t. Hence (20) is true if we can show that $g_{(N)} \in S^{-N}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\varrho'})$, i.e. we have to verify that

$$\left\| \kappa^{-1}_{[\eta]} \{\partial^\alpha_\eta g_{(N)}(\eta)\} \kappa_{[\eta]} \right\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\varrho'})} \le c[\eta]^{-N-|\alpha|} \tag{22}$$

with some constant $c > 0$ independent of $\eta \in \mathbb{R}^q$. When calculating $\partial^\alpha_\eta g_{(N)}$ we can omit all terms where χ is differentiated, since the resulting terms are operator-families that are compactly supported in η and thus satisfy (22). The remaining terms of interest are of the form

$$\chi(\eta) t^{N+n_1+|\beta|} t'^{n_2} a_{n_3}(\eta) \psi_{n_1}(t[\eta]) \tilde{\psi}_{n_2}(t'[\eta]) K_\beta(\eta),$$

where $n_1 + n_2 + n_3 + |\beta| = |\alpha|$, $\psi_{n_1} = \partial^{n_1}_t (1-\omega)$, $\tilde{\psi}_{n_2} = \partial^{n_2}_{t'}(1-\omega_1)$, and $a_{n_3} \in S^{-n_3}(\mathbb{R}^q)$. Furthermore, $K_\beta(\eta)$ is the integral operator with kernel as in (21), where $f_N(t,i\tau,\eta)$ is replaced by $(\partial^\beta_\eta \tilde{f}_N)(t,i\tau,t\eta)$. Conjugation with $\kappa_{[\eta]}$ yields operator-families $[\eta]^{-(N+n_1+|\beta|+n_2)} a_{n_3}(\eta) \tilde{K}(\eta)$ with integral operators $\tilde{K}(\eta)$ that have kernels

$$\tilde{k}(t,t',\eta) = t^{N+n_1+|\beta|} t'^{n_2-1} \psi_{n_1}(t) \tilde{\psi}_{n_2}(t') a_{n_3}(\eta) \phi(\eta) \int \left(\frac{t}{t'}\right)^{-i\tau} (\partial^\beta_\eta \tilde{f}_N)(\frac{t}{[\eta]}, i\tau, t\frac{\eta}{[\eta]})\, đ\tau.$$

As in Lemma A.10 one can show that this is a bounded function of $\eta \in \mathbb{R}^q$ with values in $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+, C^\infty(X \times X))$, obviously being supported away from $(\{t = 0\} \times X) \times (\{t' = 0\} \times X)$ uniformly in $\eta \in \mathbb{R}^q$. Hence $\tilde{K}(\eta)$ is a bounded family of operators in $\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge)^\varrho, \mathcal{K}^{s',\gamma'}(X^\wedge)^{\varrho'})$. This, together with the fact that $[\eta]^{-(N+n_1+|\beta|+n_2)}|a_{n_3}(\eta)| \leq c[\eta]^{-(N+|\alpha|)}$, implies (22). Analogously, g_1 fulfills (20). Finally, g_j^* is of the form g_{1-j} for $j = 0, 1$. To handle $\tilde{g}_j$, write

$$\tilde{g}_j(\eta) = (1-\chi)(\eta)\,\tilde{g}_j(\eta) + \sigma(t)\,g_j(\eta)\,\sigma_0(t).$$

The first term on the right-hand side has a kernel in $C_0^\infty(\mathbb{R}^q, C_0^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times X \times X))$. In particular, $(1-\chi)(\eta)\tilde{g}_j(\eta) \in R_G^{-\infty}(\mathbb{R}^q; (\gamma,\gamma))_\infty$. In view of Remark 3.12, the second term is an element of $R_G^0(\mathbb{R}^q; (\gamma,\gamma))_\infty$. □

Remark A.12 Let $h \in C_{\deg}^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^\mu(X;\mathbb{R}^q))$, and let $\varphi, \psi \in C_0^\infty(\overline{\mathbb{R}}_+)$ be functions with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$. Then there is an element $h_\infty \in C_{\deg}^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^{-\infty}(X;\mathbb{R}^q))$ such that

$$\varphi \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\psi = \varphi \operatorname{op}_M^{\gamma-\frac{n}{2}}(h_\infty)(\eta)\psi.$$

Proposition A.13 *Let $h \in C_{\deg}^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^\mu(X;\mathbb{R}^q))$, and let $\sigma, \tilde{\sigma}, \sigma_1, \sigma_2 \in C_0^\infty(\overline{\mathbb{R}}_+)$ be cut-off functions with $\sigma_2 \prec \sigma_1$. Further set $a_M(\eta) = \sigma \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\,\tilde{\sigma}$. Then*

$$\sigma_2\, a_M(\eta)(1-\sigma_1) \quad \textit{and} \quad (1-\sigma_1)a_M(\eta)\sigma_2$$

are elements of $R_G^{-\infty}(\mathbb{R}^q; (\gamma,\gamma))_\infty$.

Proof: Choose cut-off functions $\omega_2 \prec \omega_1$ such that $\omega_2 \prec \sigma_2$ and $\omega_1 \prec \sigma_1$. Then

$$\sigma_2(t)a_M(\eta)(1-\sigma_1)(t) = \sigma_2(t)(g_1(\eta) + g_2(\eta))(1-\sigma_1)(t) =: g(\eta)$$

with $g_1(\eta) = \omega_2(t[\eta])a_M(\eta)(1-\omega_1)(t[\eta])$ and $g_2(\eta) = (1-\omega_2)(t[\eta])a_M(\eta)(1-\omega_1)(t[\eta])$. Proposition A.8 yields $g_1 \in R_G^0(\mathbb{R}^q; (\gamma,\gamma))_\infty$. Using Lemma A.12 we may assume that $h \in C_{\deg}^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^{-\infty}(X;\mathbb{R}^q))$ so that $g_2 \in R_G^0(\mathbb{R}^q; (\gamma,\gamma))_\infty$ because of Proposition A.11. Finally, $g \in R_G^{-\infty}(\mathbb{R}^q; (\gamma,\gamma))_\infty$ in view of Remark 3.12. The family $(1-\sigma_1)a_M(\eta)\sigma_2$ can be treated in the same way. □

Remark A.14 For $h \in C_{\deg}^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^{-\infty}(X;\mathbb{R}^q))$, and cut-off functions σ and $\tilde{\sigma}$ we have

$$\sigma \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(\eta)\,\tilde{\sigma} \in R_G^0(\mathbb{R}^q; (\gamma,\gamma)) \quad \text{for every } \gamma \in \mathbb{R}.$$

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