Notes on Quantum Groups and Quantum De Rham Complexes

Yu. I. Manin

Steklov Mat. Institue
42 Vavilova
117966 Moscow

USSR

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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Introduction. Basic examples

0.1. Wess-Zumino de Rham complex of the quantum plane. This paper is devoted to an extension of the approach to quantum groups suggested in [Ma1] and [Ma2]. Namely, we replace the category of associative algebras considered there by a category of differential graded algebras treated as "de Rham complexes of quantum spaces".

This extension was explained in my Kyoto lectures in May 1990, and afterwards elaborated in several courses in Moscow, Utrecht, Cambridge, and at 1991 Bonn's Arbeitstagung. When a preliminary version of these notes was written, I became aware of G. Maltsiniotis' articles [Mal 1], [Mal 2] who has suggested the same generalization. Some of the examples were calculated independently by G. Maltsiniotis and participants of my Moscow seminar E. Demidov, E. Mukhin, D. Zhdanovich: see §3. The general structure results of §§2,3 seem to be new.

It was probably S. L. Woronowicz [Wo] who first developed a non-commutative differential calculus of the type considered here. However, our treatment was principally motivated by the preprint by J. Wess and B. Zumino [WeZu]. To explain its results, I shall start with the basic example $GL_q(2)$.

Let k be a ground field, $q \in k^*$. By definition, the ring of polynomial functions $F = F[GL_q(2)]$ is a Hopf algebra which can be described in the following way. As a k-algebra, it is generated by a, b, c, d and a formal inverse of a central element

$$D = DET_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - q^{-1}bc,$$

where a, b, c, d satisfy the following commutation relations:

$$ab = q^{-1}ba,\ ac = q^{-1}ca,\ cd = q^{-1}dc,\ bd = q^{-1}db,$$
$$bc = cb,\ ad - da = (q^{-1} - q)bc. \tag{0.1}$$

The comultiplication $\Delta : F \to F \otimes F$ is defined by

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{0.2}$$

where the tensor product in the r. h. s. denotes the usual product of matrices in which products like ab are replaced by $a \otimes b$. The counit is given by

$$\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{0.3}$$

Finally, the antipode map $i : F \to F$ is

$$i\begin{pmatrix} a & b \\ c & d \end{pmatrix} = D^{-1}\begin{pmatrix} d & -qb \\ -c/q & a \end{pmatrix}. \tag{0.4}$$

Although it can be checked directly that all these structures are well defined and satisfy the Hopf algebra axioms, the computations are tedious and not very enlightening.

A more conceptual approach consists in introducing two quantum planes $A_q^{2|0}$ and $A_q^{0|2}$, with function rings

$$F[A_q^{2|0}] = k\langle x, y\rangle/(xy - q^{-1}yx), \tag{0.5}$$

$$F[A_q^{0|2}] = k\langle \xi, \eta\rangle/(\xi^2, \eta^2, \xi\eta + q\eta\xi), \tag{0.6}$$

and obtaining (0.1) as solution of the following universal problem (for $q^2 \neq -1$:) the coordinate change

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{0.7}$$

should be compatible with (0.5), (0.6).

In this way, $GL_q(2)$ emerges as a "quantum automorphism group" of a pair of "non-commutative linear spaces", and all its properties can be directly derived from this definition. In particular, (0.2) expresses the composition of two "automorphisms", and D can be calculated from the formula $\xi'\eta' = D\xi\eta$, exactly as in the classical definition of the determinant via a volume form.

In [Ma2] (cf. also [Ma4], Ch. iV) this construction was generalized: it was shown that $F[A_q^{2|0}]$ can be replaced by an arbitrary quadratic algebra (or even arbitrary algebra with a fixed system of generators) considered as a function ring on an abstract quantum linear space (or rather cone).

Until recently, however, it was unclear, why do we need in this construction *two* quantum spaces, and not just one.

A beautiful answer was given by J. Wess and B. Zumino ([WeZu]). Namely, they suggested to consider $F[A_q^{2|0}]$ and $F[A_q^{0|2}]$ as parts of a differential graded algebra $\Omega[A_q^{2|0}]$, *the quantum de Rham complex* of the quantum plane $A_q^{2|0}$.

More precisely, $\Omega[A_q^{2|0}]$ is generated by (x, y, ξ, η) over k and graded by (ξ, η)–degree. To the commutation rules (0.5), (0.6) Wess and Zumino add new cross-commutation relations between (x, y) and (ξ, η) respectively:

$$x\xi = q^{-2}\xi x,\ x\eta = q^{-1}\eta x + (q^{-2} - 1)\xi y,$$
$$y\xi = q^{-1}\xi y,\ y\eta = q^{-2}\eta y. \tag{0.8}$$

The differential d is uniquely defined by the conditions

$$dx = \xi,\ dy = \eta,\ d^2 = 0 \tag{0.9}$$

and the usual (not quantized !) Leibniz formula $d(fg) = df.g + (-1)^{\deg f} f dg$.

Actually, there are *two* sets of good cross-commutation relations between coordinates and differentials: the other one can be obtained from (0.8) by interchanging $x \leftrightarrow y, \xi \leftrightarrow \eta$, and replacing q by q^{-1}.

These sets are the only ones compatible with the action (0.7) of $GL_q(2)$ and satisfying the Wess-Zumino condition

(WZ): *The algebra of the differential forms as a $k-$space must be the tensor product of the function algebra and algebra of differential forms with constant coefficients.*

Compatibility with the action of $GL_q(2)$ formally means that (0.7) defines a graded differential algebra homomorphism

$$\Omega[A_q^{2|0}] \to F[GL_q(2)] \otimes \Omega[A_q^{2|0}], \tag{0.10}$$

(on $F[GL_q(2)]$, the differential vanishes).

Now the next step is almost obvious. In (0.10), the quantum plane is represented by its de Rham complex, whereas the quantum group is represented only by its function ring. We would like to have also $\Omega(GL_q(2))$ instead of $F[GL_q(2)]$ in the r. h. s. of (0.10). In order to construct it, we must redo the theory of [Ma2] from scratch, starting with the differential graded algebras (DGA) as abstract quantum de Rham complexes and constructing their automorphism objects which will then be de Rham complexes of quantum groups, so that e. g. (0.10) becomes replaced by a universal DGA-morphism

$$\Omega[A_q^{2|0}] \to \Omega[GL_q(2)] \otimes \Omega[A_q^{2|0}]. \tag{0.11}$$

Actually, this is a largely formal undertaking, since it involves only work with tensor algebra in more general rigid tensor categories than that of linear spaces. But it is not the end of the story.

The point is that the universal DGA W obtained by this construction (before localization making it Hopf) is not a deformation of the classical de Rham complex of the ring of matrix coefficients. In fact, it has exponential growth order. We are thus facing the same problem of "missing relations" in a new guise. However, it can now be solved without introducing new objects: a calculation shows that W has a unique quotient which is a bialgebra satisfying (WZ). We explain this calculation in some detail in §3, in a slightly generalized setting, for the two-parametric deformation of $GL(2)$. There are two more quotients satisfying (WZ) which are not bialgebras; all in all, we obtain six de Rham complexes of quantum matrices $M_{p,q}(2)$, because the same construction applies to the second de Rham complex of Wess-Zumino.

This approach also enriches our understanding of quantum groups viewed as deformations of the universal enveloping algebras (cf. [Dr]). In fact, classicaly Lie algebras consist of vector fields, which can be defined algebraically as derivations of function algebras: linear maps ∂ satisfying $\partial(fg) = \partial f.g + f\partial g$. This form of Leibniz formula cannot, however, be taken as a definition of a vector field in non-commutative geometry: in the correct definition the second summand is badly twisted (see (2.5) below) because it is a by-product of the usual Leibniz formula for the total differential *and* the cross-commutation relations between functions and differentials. The role of the (WZ)-condition in this context is that it provides the necessary relations. We devote §§3 and 4 to the general study of their structure. A somewhat unexpected outcome is the appearance of quantum matrix semigroups classifying Wess-Zumino type "skew products": see Theorem 1.3.

This chapter of non-commutative differential geometry can be profitably compared with Connes' approach based upon his universal de Rham complex and "cycles"([Co]). Calculating the cohomolgy of our de Rham complexes in the simplest examples, one sees that Connes' definition of a cycles can and should be quantized, by replacing in his condition $\int[\omega, \omega'] = 0$ the usual commutators by twisted ones. However, we have no universal definition for the latters: practically speaking, they are calculated each time anew using the Wess-Zumino type arguments. It is possible that that the correct framework for this type of complexes is the tensor algebra over braided (instead of tensor) categories where a new version of the cyclic cohomology might be defined.

Before turning to the general formalism, we will give the basic formulas of one-dimensional quantum differential geometry which are both amusing and instructive.

0.2. One-dimensional de Rham complex. This is the complex induced by the cross-commutation rules (0.8) on the "axex" of our quantum plane:

$$\Omega = k[x, dx]_v; \ (dx)^2 = 0; \ v dx.x = v^{-1} x.dx \tag{0.12}$$

(we replaced q^{-1} by v^2 in order to conform with notation in Lusztig's papers on representations of quantum groups). For $n \geq 1$, we have

$$d(x^n) = dx.x^{n-1} + x.dx.x^{n-2} + \cdots = dx(1 + v^2 + \cdots + v^{2n-2})x^{n-1} = \\ v^{n-1}[n]_v dx.x^{n-1}, \tag{0.13}$$

where $[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}$ are Gaussian, or quantized, integers. If we put $[0]_v = 0$ and $[-n]_v = -[n]_v$ for positive n, the same formula (0.13) will describe the natural extension of d to $k[x, x^{-1}, dx]_v$. (If $v = \pm 1$, we of course put $v^{n-1}[n]_v = n$).

0.3. Cohomology. If v is not a root of unity, $[n]_v \neq 0$ for $n \neq 0$. Assume in addition that all $[n]_v$, $n \neq 0$, are invertible in the ground ring k. Then (0.13) shows that, as in the classical case,

$$H^*(k[x, dx]_v) \cong k; \ H^*(k[x, x^{-1}, dx]_v) \cong k \oplus k dx.x^{-1}. \tag{0.14}$$

Notice that this may happen also in finite characteristics, namely, when v is transcendental over $\mathbf{F}_p$, and when k contains $\mathbf{F}_p[[n]_v^{-1} \mid n \in \mathbf{Z} \setminus \{0\}]$.

For $\omega = dx.f$, $f \in k[x, x^{-1}]$, consider the residue functional

$$\int dx(\sum a_i x^i) = a_{-1}.$$

It satisfies two relations:

$$\int dg = 0; \ \int [dx.x^n, x^m]_{v^m} = 0,$$

where, by definition, $[x,y]_w = wxy - w^{-1}yx$. It is the presence of such twisted commutators in the second formula that distinguishes $(k[x,x^{-1},dx]_v, \int)$ from a usual Connes' cycle.

We shall often meet such commutators later on. Notice that the last relation in (0.12) is $[dx,x]_v = 0$.

Now, let v^2 be a primitive root of unity of degree $l > 1$. Then

$$[1]_v \neq 0, \ldots, [l-1]_v \neq 0;\ [l]_v = 0; [n+lm]_v = v^{lm}[n]_v. \tag{0.15}$$

Therefore, assuming again that non-vanishing $[n]_v$ are invertible, we have

$$H^*(k[x,x^{-1},dx]_v) \cong \oplus_{m\in\mathbf{Z}}(kx^{ml} \oplus kdx.x^{ml-1}). \tag{0.16}$$

In the classical case, such is the structure of the de Rham cohomology in characteristic l. In the quantum context, k may have any characteristic, including zero.

This is probably the simplest example of a general phenomenon: quantizarion at special parameter values tends to reproduce some effects that we are accustomed to see as characteristic-dependent, in all characteristics. Here is one more example: characteristic-independent Frobenius.

0.4. Gauss binomial coefficients. Assume that

$$[x,y]_v = 0 \tag{0.17}$$

for certain elements x, y of an associative k-algebra. Then the following binomial formula is valid:

$$(x+y)^n = \sum_{j=0}^{n} v^{j(n-j} \begin{bmatrix} n \\ j \end{bmatrix}_v x^{n-j}y^j, \tag{0.18}$$

where for $0 \le j \le n$ we put

$$\begin{bmatrix} n \\ j \end{bmatrix}_v = \frac{[n]!_v}{[j]!_v[n-j]_v}\ [n]!_v = [1]_v \ldots [n]_v,\ [0]!_v = 1. \tag{0.19}$$

0.4.1. Proof of (0.18). Using (0.19), we can directly check that

$$\begin{bmatrix} n+1 \\ j \end{bmatrix}_v = v^j \begin{bmatrix} n \\ j \end{bmatrix}_v + v^{-n-1} \begin{bmatrix} n \\ j-1 \end{bmatrix}_v.$$

Multiplying this by t^j and summing over $j = 0, \ldots, n+1$, we get

$$F_{n=1}(t) = F_n(vt)(1+v^{-n}t),$$

where

$$F_n(t) := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_v t^j.$$

Hence by induction

$$F_n(t) = \prod_{k=0}^{n-1}(1 + v^{2k-n+1}t).$$

Replacing t by $v^{n-1}t$, we get

$$\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_v v^{j(n-1)}t^j = \prod_{k=0}^{n-1}(1 + v^{2k}t).$$

Now, if $[x, y]_v = 0$, by formally inverting x we obtain

$$(x+y)^n = x^n \prod_{i=0}^{n-1}(1 + v^{2i}x^{-1}y) = x^n \sum_{j=0}^{n-1}\begin{bmatrix} n \\ j \end{bmatrix}_v v^{j(n-1)}(x^{-1}y)^j,$$

which leads to (0.18) because $(x^{-1}y)^j = v^{-j(j-1)}x^{-j}y^j$.

Assume now that v^2 is a primitive root of unity of degree $l > 1$. Then from (0.15) and (0.18) we obtain a simple Frobenius type formula valid for $[x, y]_v = 0$:

$$(x+y)^l = x^l + y^l. \tag{0.20}$$

Notice also that if l is odd, we have

$$(xy)^l = v^{l(l-1)}x^l y^l = x^l y^l = y^l x^l.$$

On a deeper level, these formulas lead to:

a). Existence of "unramified coverings" of classical simple groups in the category of quantum groups, icluding lifts of Frobenius morphisms to characteristic zero. This can be vaguely interpreted as existence of hidden fundamental groups of the classical algebraic groups, which become visible only in non-commutative geometry.

b). A parallelism between the representation theory of quantum groups in characteristic zero at roots of unity , and that of classical groups in finite characteristic, investigated by Lusztig.

0.5. Frobenius map for $GL_q(2)$. In this subsection, we return to the notation of 0.1. Let q be a primitive root of unity of odd degree l, a, b, c, d the matrix elements of $GL_q(2)$. Then we have ([PW]):

i). a^l, b^l, c^l, d^l are central in $F[GL_q(2)]$.

ii). $\Delta\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} \otimes \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$.

iii). $\det\begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix} = \left[DET_q\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]^l$.

In the language of non-commutative geometry, this can be rewritten as an exact sequence of quantum groups

$$1 \to H_q \to GL_q(2) \to GL_1(2) = GL(2) \to 1, \tag{0.22}$$

where the third arrow is the Frobenius morphism Φ_l. The kernel H_q is a finite-dimensional Hopf algebra defined by relations (0.1) to which are added

$$a^l = d^l = 1,\ b^l = c^l = 0.$$

It is flat over $\mathbf{Z}[q, q^{-1}]$ which is a natural definition ring for the whole setting. If l is an odd prime, (0.22) can be reduced modulo its prime divisor in this ring, and Φ_l reduces to a usual Frobenius morphism of $GL(2)$.

We sketch here a proof of i) and ii). Obviously, b^l and c^l commute with a, b, c, d, and a^l commutes with b, c. It takes slightly more efforts to check that $a^l d = da^l$. Since the determinant D is central (for all q), we have

$$a^l d = a^{l-1}(D + q^{-1}bc) = Da^{l-1} + q^{1-2l}bca^{l-1} =$$

$$(D + qbc)a^{l-1} = da^l.$$

One can treat d^l similarly.

In order to check ii), that is, to prove that, say, $\Delta(a^l) = \Delta(a)^l$, one applies (0.20) to the matrix elements of $\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This is possible, because $\Delta(a) = a \otimes a + b \otimes c$, and

$$(a \otimes a)(b \otimes c) = q^{-2}(b \otimes c)(a \otimes a),$$

where q^{-2} is still a primitive root of degree l.

After this digression, we return to one-dimensional differential geometry.

0.6. Differential operators. Define $\partial_v :\ A = k[x, x^{-1}] \to A$ by $df = dx.\partial_v f$. When $[n]_v$ for all $n \neq 0$ are invertible, the ring of differential operators can be defined as the ring geenrated by ∂_v and multiplications. Otherwise "quantized divided powers" $\partial_v^i/[i]!_v$ should be introduced as primary objects.

Let σ be the automorphism of A over k with $\sigma(x) = v^2 x$. Then

$$f dx = dx.\sigma(f)$$

so that

$$\partial_v(fg) = \partial_v f.g + \sigma(f)\partial_v g. \tag{0.23}$$

In particular, $\partial_v \circ x - v^2 x \circ \partial_v = 1$, or

$$[\partial_v, x]_{v^{-1}} = v^{-1}\mathrm{id} \tag{0.24}$$

as operators on A. This is a quantized version of the Heisenberg commutation relation.

From (0.23) one easily deduces that if $[i]_v$ are invertible for $i \neq 0$, then the ring of differential operators is a free $A-$module freely generated by $1, \partial_v, \partial_v^2, \dots$.

0.7. Vector fields. The space of vector fields $A\partial_v$ is spanned by $L_n = vx^{n+1}\partial_v$. The usual commutation relations now deform to

$$[L_n, L_m]_{v^{n-m}} = [m-n]_v L_{n+m} \tag{0.25}$$

(apply both parts to x^r).

Notice that the parameter v^{n-m} of quantized commutator in (0.25) varies with n, m. A quantized version of the Virasoro algebra with zero central charge may be defined as an associative algebra generated by abstract symbols L_n subject to (0.25). However, if we want to define on it a comultiplication, we must take into account one more complication. Namely, the usual coalgebra structure upon $U(g)$, where g is a Lie algebra of vector fields, $\Delta(X) = X \otimes 1 + 1 \otimes X$, is just a translation of the Leibniz rule $X(fg) = Xf.g + fXg$. But in our case it is replaced by (0.23):

$$L_n(fg) = L_n f.g + \sigma(f).L_n g,\ \sigma(fg) = \sigma(f)\sigma(g).$$

This means that to obtain a closed formula for Δ, we must add σ to $\{L_i\}$. But then one can delete $\{L_0\}$ which can be expressed via $K_0 = \sqrt{\sigma}$:

$$K_0(x^r) = v^r x^r,\ \Delta(K_0) = K_0 \otimes K_0,$$

$$L_0(x^r) = v^r[r]_v x^r = \frac{K_0^2 - 1}{v - v^{-1}} x^r.$$

Summarizing, we have (assuming v and $v - v^{-1}$ invertible) the following commutation relations and comultiplication rules:

$$\Delta(L_i) = L_i \otimes 1 + K_0^2 \otimes L_i;\ \Delta(K_0) = K_0 \otimes K_0. \tag{0.26}$$

$$[L_n, L_m]_{v^{n-m}} = \begin{cases} [m-n]_v L_{n+m}, & \text{for } n+m \neq 0 \\ -[2n]_v \frac{K_0^2-1}{v-v^{-1}}, & \text{for } n+m = 0. \end{cases} \tag{0.27}$$

$$K_0 L_n K_0^{-1} = v^n L_n. \tag{0.28}$$

One must also check that relations are compatible with comultiplication. See §2 below for a more general discussion of this problem.

0.8. Twisted $U(sl(2))$. For $v = 1$, $\{L_n, L_0, L_{-n}\}$ generate a Lie subalgebra isomorphic to $sl(2)$. In our case, we take $\{L_n, K_0, L_{-n}\}$. In order to get the commutation relations in a more symmetric form, put (for a fixed $n > 0$):

$$e = -v^{-n/2}[2n]_v^{-1} L_n,\ f = K_0^{-1} L_{-n}, k = K_0$$

(in this subsection, k has this meaning, and not that of a ground ring). Then we have

$$[e, f]_{v^{3n/2}} = \frac{k - k^{-1}}{v - v^{-1}}; \tag{0.29}$$

$$kek^{-1} = v^n e;\ kfk^{-1} = v^{-n} f; \tag{0.30}$$

$$\Delta(e) = e \otimes 1 + k^2 \otimes e; \Delta(f) = f \otimes k^{-1} + k \otimes f. \tag{0.31}$$

This should be compared with the usual $U_{v^2}(sl(2))$:

$$[E, F] = \frac{K - K^{-1}}{v - v^{-1}}; \tag{0.32}$$

$$KEK^{-1} = v^2 E;\ KFK^{-1} = v^{-2} F; \tag{0.33}$$

$$\Delta(E) = E \otimes 1 + K \otimes E;\ \Delta(F) = F \otimes K^{-1} + 1 \otimes F;\ \Delta(K) = K \otimes K. \tag{0.34}$$

The principal difference is between (0.29) and (0.32): the commutator in (0.32) is twisted. We can also obtain a usual commutator in our setting, but then the r. h. s. of (0.32) becomes spoiled: put

$$\hat{e} = -[2n]_v^{-1} K_0^{-2} L_n,\ \hat{f} = K_0^{-2} L_{-n},\ \hat{k} = K_0^2;$$

then

$$[\hat{e}, \hat{f}] = \frac{\hat{k}^{-2} - 1}{v - v^{-1}}; \tag{0.35}$$

$$\hat{k}\hat{e}\hat{k}^{-1} = v^{2n}\hat{e};\ \hat{k}\hat{f}\hat{k}^{-1} = v^{-2n}\hat{f}; \tag{0.36}$$

$$\Delta(\hat{e}) = \hat{e} \otimes \hat{k}^{-1} + 1 \otimes \hat{e};\ \Delta(\hat{f}) = \hat{f} \otimes \hat{k}^{-1} + 1 \otimes \hat{f}. \tag{0.37}$$

0.9. The standard $U_{v^2}(g_A)$. For reader's convenience, we reproduce here the Drinfeld-Jimbo presentation of $U_{v^2}(g_A)$, where g_A is the Lie algebra corresponding to a symmetrizable generalized Cartan matrix A.

Recall that $A = (a_{ij})$, $1 \le i, j \le n$, $a_{ii} = 2$, $a_{ij} \le 0$ for $i \ne j$; $d_i a_{ij} = d_j a_{ji}$, $d_i \in \mathbf{Z}$; $\gcd(d_i) = 1$.

We need n triples $(E_i, F_i, K_i^{\pm 1}, i = 1, \dots, n$. They satisfy analogues of (0.32)-(0.34), and more generally

$$K_i K_j = K_j K_i; \tag{0.38}$$

$$K_i E_j K_i^{-1} = v^{d_i a_{ij}} E_j; K_i F_j K_i^{-1} = v^{-d_i a_{ij}}; \tag{0.39}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}. \tag{0.40}$$

But the most essential novelty is the deformed Serre's relations:

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{v^{d_i}} E_i^r E_j E_i^s = 0,\ i \ne j; \tag{0.41}$$

and similarly for $\{F_i\}$.

We cannot, of course, "explain" these relations using only one-dimensional de Rham complex. A very intriguing new approach to them was recently developed by A. A. Beilinson et al. in Duke MJ.

It is based upon the observation that, for $q = v^2$ a prime power, $v^{n-1}[n]_v$ coincides with the number of $\mathbf{F}_q$–points of $\mathbf{P}^{n-1}$. A sophisticated version of this simple remark gives a geometric interpretation of (0.41) in terms of the geometry of flag manifolds over finite fields. This suggests a new kind of relations between the finite characteristic geometry and non-commutative geometry.

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§1. Skew products

1.0. Notation. We denote by k a $\mathbf{Z}_2$-graded supercommutative ring. All k-modules are $\mathbf{Z}_2$-graded; $\tilde{a}$ denotes the $\mathbf{Z}_2$-degree, or parity, of a. All morphisms are homogeneous; we consider also odd morphisms of $k-$modules. All $k-$modules are assumed to be free; we consider mostly direct free submodules. All $k-$algebras are (super-)central. The tensor product of two k-algebras is endowed with multiplication $(a\otimes b)(a'\otimes b') = (-1)^{\tilde{a}'\tilde{b}}aa'\otimes bb'$.

1.1. Skew products. Let $A, B \subset C$ br two subalgebras of an associative k-algebra C. We shall say that they are skew-commuting if the multiplication maps induce injections of k-modules $m:\ A\otimes B \to C,\quad m:\ B\otimes A \to C$ with the same image.

This image is then a subalgebra of C denoted $AB = BA$. If it coincides with C, we say that C is a skew product of A and B. Obviously, $A\otimes B$ is a skew product. Given A, B, we want to classify their skew products. Intuitively, this amounts to classifying "good" cross-commutation relations between elements of the type $a\otimes 1$ and $1\otimes b$ in $A\otimes B$ defining new multiplication on $A\otimes B$.

1.2. Basic construction. Assume that a skew product $C = AB = BA$ is given. It defines (and is defined by) the k-linear isomorphism $r = r_C:\ B\otimes A \to A\otimes B$ induced by multiplication in C. Consider for every $b\in B$ two operators $\tau_b:\ A \to A\otimes B$, $\sigma_b:\ A \to B\otimes A$ defined by

$$\tau_b(a) = r(b\otimes a);\qquad \sigma_b(a) = (-1)^{\tilde{a}\tilde{b}}r^{-1}(a\otimes b).$$

Clearly, τ_b and σ_b are left $k-$linear in a, of parity $\tilde{b}$. Their dependence of b is even left $k-$linear.

We want to consider τ_b (resp. σ_b) as elements of $k-$algebras $\mathbf{End}_{k-mod}(A)\otimes B$ (resp. $B\otimes \mathbf{End}_{k-mod}(A)^{op}$) where **End** is the algebra of both even and odd endomorphisms.

We will work everything out in coordinates. Choose a $\mathbf{Z}_2-$graded free $k-$basis $\{b_i\}, i\in I$, of B. For variable $a\in A$, denote by $\tau_{b,j}(a)$ coefficients in the following equivalent expressions:

$$\tau_b(a) = r(b\otimes a) = \sum_j (-1)^{\tilde{b}_j\tilde{a}}\tau_{b,j}(a)\otimes b_j \qquad (\text{in } A\otimes B),$$

$$ba = \sum_j (-1)^{\tilde{b}_j\tilde{a}}\tau_{b,j}(a)b_j \qquad (\text{in } C) \tag{1.1}$$

and write (1.1) symbolically as

$$\tau_b = \sum_j \tau_{b,j}\otimes b_j. \tag{1.2}$$

Similarly, put

$$\sigma_b = \sum_i b_i \otimes \sigma_{b,i} \tag{1.3}$$

meaning that for all $a \in A$,

$$(-1)^{\tilde{a}\tilde{b}} \sigma_b(a) = r^{-1}(a \otimes b) = \sum_i (-1)^{\tilde{a}\tilde{b}} b_i \otimes \sigma_{b,i}(a),$$

$$ab = \sum_i (-1)^{\tilde{a}\tilde{b}} b_i \sigma_{b,i}(a). \tag{1.4}$$

The sums (1.2), (1.3) are generally infinite. However, they are finite, if B is finite dimensional, or if B is graded by finite dimensional submodules B^n, $AB^n = B^n A$, and $\{b_i\}$ is homogeneous with respect to this grading. In the sequel, we shall assume one of these conditions.

1.2.1. LEMMA. *$\tau_{b,j}$ and $\sigma_{b,j}$ are k–mod morphisms $A \to A$ of parity $\tilde{b} + \tilde{b}_j$.*

PROOF: For $c \in k$, we have from (1.1):

$$\tau_b(ca) = r(b \otimes ca) = (-1)^{\tilde{b}\tilde{c}} c r(b \otimes a) = (-1)^{\tilde{b}\tilde{c}} c \sum_j (-1)^{\tilde{b}_j \tilde{a}} \tau_{b,j}(a) \otimes b_j.$$

On the other hand,

$$\tau_b(ca) = \sum_j (-1)^{\tilde{b}_j(\tilde{c}+\tilde{a})} \tau_{b,j}(ca) \otimes b_j,$$

so that $\tau_{b,j}(ca) = (-1)^{(\tilde{b}+\tilde{b}_j)\tilde{c}} \tau_{b,j}(a)$.

Similarly, from (1.4),

$$\sigma_b(ca) = (-1)^{\tilde{c}+\tilde{a}\tilde{b}} r^{-1}(ca \otimes b) = c(-1)^{(\tilde{a}+\tilde{c})\tilde{b}} r^{-1}(a \otimes b) =$$

$$c \sum_i (-1)^{\tilde{c}\tilde{b}} b_i \otimes \sigma_{b,i}(a) = c(-1)^{\tilde{c}\tilde{b}} \sigma_b(a),$$

and from (1.3)

$$\sigma_b(ca) = \sum_i b_i \otimes \sigma_{b,i}(ca),$$

so that

$$\sigma_{b,i}(ca) = (-1)^{\tilde{c}(\tilde{b}+\tilde{b}_i)} c \sigma_{b,i}(a).$$

1.2.2. LEMMA. *The maps*

$$\tau: \ B \to \mathbf{End}_{k-mod}(A) \otimes B \ : \ b \mapsto \tau_b, \tag{1.5}$$

$$\sigma: \ B \to B \otimes \mathbf{End}_{k-mod}(A)^{op} \ : \ b \mapsto \sigma_b \tag{1.6}$$

are $\mathbf{Z}_2$*−graded ring homomorphisms, such that for* $c \in k$, $\tau_c = 1 \otimes c = c \otimes 1 = \sigma_c$.

PROOF: We first remark that these maps are invariantly defined. In the definition of C^{op} for a $\mathbf{Z}_2$−graded ring C, we of course introduce signs: C^{op} has the same additive group as C, and multiplication $c_1 * c_2 = (-1)^{\tilde{c}_1\tilde{c}_2} c_2 c_1$.

Now, identifying via multiplication $B \otimes A$ and $A \otimes B$ with their images in C, we have in view of (1.1):

$$\tau_{b'b}(a) = r(b'b \otimes a) = b'ba = \sum_l (-1)^{\tilde{b}_l \tilde{a}} b' \tau_{b,l}(a) b_l =$$

$$\sum_l (-1)^{\tilde{b}_l \tilde{a}} \sum_j (-1)^{\tilde{b}_j \widetilde{\tau_{b,l}(a)}} \tau_{b',j}(\tau_{b,l}(a)) \otimes b_j b_l.$$

On the other hand, by (1.2):

$$(\tau_{b'}\tau_b)(a) = \left(\sum_j \tau_{b',j} \otimes b_j \right) \left(\sum_l \tau_{b,l} \otimes b_l \right) (a) =$$

$$\sum_{j,l} (-1)^{\tilde{b}_j \tilde{t}au_{b,l}} (\tau_{b',j}\tau_{b,l} \otimes b_j b_l)(a) =$$

$$\sum_{j,l} (-1)^{\tilde{b}_j(\tilde{b}+\tilde{b}_l)+\tilde{a}(\tilde{b}_j+\tilde{b}_l)} \tau_{b',j}\tau_{b,l}(a) \otimes b_j b_l.$$

It remains to compare signs:

$$\tilde{b}_l \tilde{a} + \tilde{b}_j(\tilde{a} + \tilde{b} + \tilde{b}_l) = \tilde{b}_j(\tilde{b} + \tilde{b}_l) + \tilde{a}(\tilde{b}_j + \tilde{b}_l).$$

Similarly, by (1.4),

$$\sigma_{b'b}(a) = (-1)^{\tilde{a}(\tilde{b}+\tilde{b}')} r^{-1}(a \otimes b'b) = (-1)^{\tilde{a}(\tilde{b}+\tilde{b}')}(ab')b =$$

$$(-1)^{\tilde{a}(\tilde{b}+\tilde{b}')}(-1)^{\tilde{a}\tilde{b}'} \sum_i b_i(\sigma_{b',i}(a)b) =$$

$$(-1)^{\tilde{a}\tilde{b}} \sum_i \sum_j (-1)^{(\tilde{b}'+\tilde{b}_i+\tilde{a})\tilde{b}} b_i b_j \otimes \sigma_{b,j}(\sigma_{b',i}(a)) =$$

$$\sum_{i,j} (-1)^{(\tilde{b}'+\tilde{b}_i)\tilde{b}} b_i b_j \otimes \sigma_{b,j}\sigma_{b',i}(a).$$

Finally, by (1.3), taking into account the reverse multiplication in $\mathbf{End}(A)^{op}$:

$$(\sigma_{b'}\sigma_b)(a) = \left(\sum_i b_i \otimes \sigma_{b',i}\right)\left(\sum_j b_j \otimes \sigma_{b,j}\right)(a) =$$

$$\sum_{i,j}(-1)^{\tilde{b}_j(\tilde{b}'+\tilde{b}_i)} b_i b_j \otimes (-1)^{(\tilde{b}'+\tilde{b}_i)(\tilde{b}+\tilde{b}_j)} \sigma_{b,j}\sigma_{b',i}(a).$$

1.2.3. Structural action. Consider now the two universal algebra morphisms, which are automatically universal bialgebra coactions, linear (and homogeneous) upon $\{b_i\}$:

$$\delta_l : \ B \to E_l(B) \otimes B : \quad \delta_l(b_i) = \sum_j z_{ij} \otimes b_j,$$

$$\delta_r : \ B \to B \otimes E_r(B) : \quad \delta_r(b_i) = \sum_j b_j \otimes \hat{z}_{ji}.$$

Here $z_{ij} \in E_l(B)$ (resp. $\hat{z}_{ji} \in E_r(B)$) are algebra generators arranged in a format of a multiplicative matrix. In particular, the parity of z_{ij}, $\hat{z}_{ji}$ is $\tilde{b}_i + \tilde{b}_j$.

There exists a natural map

$$E_r(B) \to E_l(B) : \qquad \hat{z}_{ji} \mapsto (-1)^{(\tilde{b}_j+\tilde{b}_i)\tilde{b}_j} z_{ij}$$

("supertransposition") which is an isomorphism of bialgebras

$$st : \qquad (E_r(B), \Delta_r) \to (E_l(B), \Delta_l^{op}),$$

where Δ_r, Δ_l are the respective comultiplications.

Using (1.5), (1.6), and universality, one sees that instead of τ, σ one can give the two algebra homomorphisms

$$\lambda : \qquad E_l(B) \to \mathbf{End}_{k-mod}(A);$$

$$\rho : \qquad E_r(B)^{op} \to \mathbf{End}_{k-mod}(A).$$

This means that, starting with a skew product of A and B, we have constructed actions of $E_l(B)$ and $E_r(B)^{op}$ upon the left k-module A. We shall call them *structural actions.*

The comultiplication map $\Delta_r : \quad E_r(B) \to E_r(B) \otimes E_r(B)$ is also a comultiplicatuon map for $E_r(B)^{op}$. Using it, we can define the action ϕ of $E_r(B)^{op}$ upon $A \otimes A$: for $e \in E_r(B)^{op}$, $c \in A \otimes A$, put

$$\phi(e)(c) = (\rho \otimes \rho)(\Delta(e))(c).$$

1.2.4. LEMMA. *The multiplication morphism $m : \ A \otimes A \to A$ is a morphism of $E_r(B)^{op}$-modules.*

PROOF: For a basis $\{b_j\}$ of B, put as above

$$\delta_r(b_j) = \sum_i b_i \otimes \hat{z}_{ij}, \qquad \hat{z}_{ij} \in E_r(B)^{op}, \tag{1.7}$$

and

$$\sigma_{ij} = \sigma_{b_j,i}.$$

Comparing (1.7) and (1.3) one sees that $\hat{z}_{ij}$ acts upon A by the operator $\rho(\hat{z}_{ij}) = \sigma_{ij}$. To prove the Lemma, we must check that for every $z \in E(B)^{op}$, $c \in A \otimes A$, we have

$$m[\Delta_r(z)(c)] = z(m(c)) \tag{1.8}$$

(we should have written here $\rho(z)$ instead of z, and $(\rho \otimes \rho)(\Delta(z)$ instead of $\Delta(z)$).

First, it suffices to verify (1.8) for generators $z = \hat{z}_{ij}$. In fact, both sides are linear in z, and if (1.8) holds for z_1, z_2 and all c, it holds also for $z_1 z_2$ and all c:

$$m[\Delta(z_1 z_2)(c)] = m[\Delta(z_1)(\Delta(z_2)(c))] =$$

$$z_1(m[\Delta(z_2)(c)]) = z_1 z_2 m(c).$$

Second, it suffices to check (1.8) for decomposable $c = f \otimes g$, because both sides are biadditive in c. Now,

$$\Delta(\hat{z}_{ij}) = \sum_k \hat{z}_{ik} \otimes \hat{z}_{kj}.$$

Therefore, (1.8) reduces to the equality of the following two expressions:

$$m[\Delta(\hat{z}_{ij})(f \otimes g)] = m\left[\left(\sum_k \hat{z}_{ik} \otimes \hat{z}_{kj}\right)(f \otimes g)\right] =$$

$$\sum_k (-1)^{(\bar{b}_k + \bar{b}_j)\bar{f}} \sigma_{ik}(f) \sigma_{kj}(g),$$

$$\hat{z}_{ij}(m(f \otimes g)) = \sigma_{ij}(fg).$$

To this end we apply (1.4) in turn to the cases $b = b_j$, and $a = g, f, fg$:

$$(fg)(b_j) = f(gb_j) = \sum_k (-1)^{\bar{g}\bar{b}_j} (fb_k)\sigma_{kj}(g) =$$

$$\sum_k (-1)^{\bar{g}\bar{b}_j} \sum_i (-1)^{\bar{f}\bar{b}_k} b_i \sigma_{ik}(f) \sigma_{kj}(g);$$

$$(fg)b_j = \sum_i (-1)^{(\bar{f}+\bar{g})\bar{b}_j} b_i \sigma_{ij}(fg).$$

Remark. For operators τ, one can prove a similar identity (but notice different orders of f, g in two sides):

$$\tau_{ik}[m(f \otimes g)] = m[\Delta(\tau_{ik})(g \otimes f)].$$

1.2.5. LEMMA. *Put*

$$\sigma_{ij} = \sigma_{b_j,i}, \quad \tau_{kl} = \tau_{b_k,l}, \quad \hat{\tau}_{kl} = (-1)^{\tilde{b}_k(\tilde{b}_k+\tilde{b}_l)}\tau_{lk}.$$

Then we have

$$\sum_j \sigma_{kj}\hat{\tau}_{ji} = \delta_{ki} id_A, \tag{1.9}$$

$$\sum_j \hat{\tau}_{kj}\sigma_{ji} = \delta_{ki} id_A. \tag{1.10}$$

PROOF: Rewrite $b_i a$ by first applying (1.1) and then (1.4):

$$b_i a = \sum_j (-1)^{\tilde{b}_j \tilde{a}} \tau_{ij}(a) b_j = \sum_j (-1)^{\tilde{b}_j \tilde{a}} \sum_k (-1)^{\tilde{b}_j(\tilde{a}+\tilde{b}_i+\tilde{b}_j)} b_k \sigma_{kj}(\tau_{ij}(a)).$$

This gives (1.9).

Similarly, rewrite ab_i by first applying (1.4) and afterwards (1.1):

$$ab_i = \sum_j (-1)^{\tilde{a}\tilde{b}_i} b_j \sigma_{ji}(a) =$$

$$\sum_j (-1)^{\tilde{a}\tilde{b}_i} \sum_k (-1)^{\tilde{b}_k(\tilde{a}+\tilde{b}_i+\tilde{b}_j)} \tau_{jk}\sigma_{ji}(a) b_k.$$

This gives (1.10).

We can now state the main result of this section.

1.3. THEOREM. *The construction described above establishes a bijection between the following two sets:*
(SP) Skew products of A and B (compatible with the gradation of B).
(EA) Actions of $E_r(B)^{op}$ upon A, defined by an invertible operator-valued matrix (σ_{kj}), such that $m : A \otimes A \to A$ is a morphism of $E_r(B)^{op}-$modules. Here $(E_r(B), \Delta_r)$ is the universal coalgebra coacting upon B (compatibly with gradation).

PROOF: We have already constructed the map (SP)→(EA). It is injective because the knowledge of an $E_r(B)^{op}-$action allows one to reconstruct σ and then the crosscommutation relations in $B \otimes A$ corresponding to the given skew product structure.

It remains to show that this map is surjective. This means that, starting with an action, we must define an associative multiplication, say, on $B \otimes A$, and to check that it defines a skew product. Obviously, the multiplication must be defined by the formula

$$(b \otimes a)(b' \otimes a') = (m_B \otimes m_A)(b \otimes \sigma_{b'}(a) \otimes a'),$$

where σ is reconstructed from the action. The rest of the proof consists of checks that have essentially be made already in the course of proving our Lemmas 1.2.1 - 1.2.4. We will only show associativity in some detail.

It suffices to establish that the product of three elements of the type $b_j \otimes a$ does not depend on the bracket configuration. We will omit the tensor product sign. Take the three elements $b_j a'$, $b_l a''$, $b_n a'''$. We can assume that $b_j = 1$, $a''' = 1$, because they will cancel in the identity. So finally we want

$$(a' b_l a'') b_n = a' (b_l a'' b_n).$$

First calculate the l.h.s. expression:

$$(a' b_l a'') b_n = \sum_k (-1)^{\tilde{a}' \tilde{b}_l} b_k \sigma_{kl}(a') a'' b_n =$$

$$\sum_k (-1)^{\tilde{a}' \tilde{b}_l} b_k \sum_m (-1)^{\tilde{b}_n (\tilde{a}'' + \tilde{b}_k + \tilde{b}_l + \tilde{a}')} b_m \sigma_{mn} [\sigma_{kl}(a') a'']$$

and take into account that

$$\sigma_{mn}[\sigma_{kl}(a') a''] = \sum_i (-1)^{(\tilde{b}_i + \tilde{b}_n)(\tilde{a}' + \tilde{b}_k + \tilde{b}_l)} (\sigma_{mi} \sigma_{kl})(a') \sigma_{in}(a'').$$

Second, calculate the r.h.s. expression:

$$a'(b_l a'' b_n) = a' b_l \sum_i (-1)^{\tilde{a}'' \tilde{b}_n} b_i \sigma_{in}(a'') =$$

$$\sum_k (-1)^{\tilde{a}' \tilde{b}_l} b_k \sigma_{kl}(a') \sum_i (-1)^{\tilde{a}'' \tilde{b}_n} b_i \sigma_{in}(a'').$$

It remains to pass b_i to the left beyond $\sigma_{kl}(a')$. We will obtain a sum of members of the type $b_m(\sigma_{mi}\sigma_{kl})(a')\sigma_{in}(a'')$ with some signs. A direct verification shows that signs will be the same as for the l.h.s.

1.4. Remarks. The following questions deserve a further study.

a). In the context of de Rham complexes of the type considered here, A and B are usually both graded so that the same skew product can be obtained dually from the action of $E_r(A)^{op}$ on B. A particularly symmetric situation arises if the same coalgebra acts and coacts upon A and B simultaneously. (Notice however that the differential breaks the symmetry).

b). If the action of $E_r(B)^{op}$ factorizes through the Hopf envelope, the matrix (σ_{ij}) is automatically invertible (apply the antipode map). The case when this Hopf algebra is commutative, i.e. consists of functions on a group scheme acting on B is especially interesting.

c). The quantization parameters in our examples can be viewed as describing the structural action. Hence intuitively introduction of $E_r(B)^{op}$ can be imagined as "a quantization of Planck's constant".

§2. Quantum de Rham complexes and differential operators

2.0. Notation. In this section, we study differential $(\mathbf{Z}, \mathbf{Z}_2)$–graded k–algebras Ω. We call the $\mathbf{Z}_2$–degree parity, and the $\mathbf{Z}$–degree dimension, denote by Ω^i the k–module of elements of dimension i, and assume that $\Omega^0 = k$, $\Omega^{<0} = 0$. We denote sometimes Ω^0 by A ("functions").

Differential d is of degree (1,1), and is left k–linear in the sense of superalgebra. The Leibniz formula $d(ab) = (da)b + (-1)^{\tilde{a}} adb$ involves parity rather than dimension.

We assume also given a bigraded k–submodule $B = \oplus_{i=0}^{\infty} B^i$ ("differential forms with constant coeficients") where all B^i are free of finite rank. The strongest condition we impose on B is that it is a quadratic subalgebra of Ω consisting of closed elements, such that Ω is a skew product of A and B. Weaker conditions sufficient for establishing some of the algebraic properties of Ω are stated at appropriate places below, and marked C1 etc.

2.1. Differentials and vector fields. Here we assume the following.

C1. *Multiplication in Ω induces linear isomorphisms $B^1 \otimes A \to \Omega^1 \leftarrow A \otimes B^1$.*

Let $\{b_i\}$ be a homogeneous basis of B^1. Then, as in §1, we can define operators σ_{ij}, τ_{ij} : $A \to A$ by

$$fb_j = \sum_i (-1)^{\tilde{f}\tilde{b}_j} b_i \sigma_{ij}(f); \tag{2.1}$$

$$b_i f = \sum_j (-1)^{\tilde{f}\tilde{b}_j} \tau_{ij}(f) b_j. \tag{2.2}$$

where $f \in A$.

Moreover, we can define a family of "quantum vector fields" $\partial_i : A \to A$ dual to $\{b_i\}$ by

$$df = \sum_{i=1}^{n} b_i \partial_i f, \ f \in A. \tag{2.3}$$

Clearly, ∂_i is left k–linear of parity $b_i + 1$. From the Leibniz formula for d,

$$d(\omega\nu) = d\omega.\nu + (-1)^{\tilde{\omega}} \omega d\nu \tag{2.4}$$

and (2.1), (2.3) one obtains the twisted Leibniz formulas for ∂_i: if $f, g \in A$,

$$\partial_i(fg) = \partial_i f.g + \sum_j (-1)^{\tilde{f}\tilde{\partial}_j} \sigma_{ij}(f) \partial_j(g). \tag{2.5}$$

We supplement (2.5) by the respective "Leibniz formulas" for σ_{ij} and τ_{ij}:

$$\sigma_{ij}(fg) = \sum_k (-1)^{(\tilde{b}_j + \tilde{b}_k)\tilde{f}} \sigma_{ik}(f) \sigma_{kj}(g); \tag{2.6}$$

$$\tau_{ij}(fg) = \sum_k (-1)^{(\tilde{b}_j+\tilde{b}_k)\tilde{g}} \tau_{ij}(f)\tau_{jk}(g). \tag{2.7}$$

They folow from the identities

$$(fg)b_j = f(gb_j),\ b_j(fg) = (b_jf)g.$$

2.2. Basic problem. In the setting of 2.0, 2.1, we can construct an algebra U of operators on A generated by ∂_i, σ_{ij} and possibly also τ_{ij} and some left or right multiplications by elements of A. It is natural to consider such algebras as a replacement in non-commutative geometry of Lie algebras generated by vector fields, and generally of rings of differential operators. We want to understand the following problems.

a). What are the relations between $\sigma_{ij}, \tau_{ij}, \partial_i$? (The relations between these operators and multiplications are given by Leibniz formulas).

b). Is it possible to define the structure of a coalgebra upon U in such a way that its comultiplication Δ on the generators would be given by the formulas compatible with (2.5), (2.6):

$$\Delta(\partial_i) = \partial_i \otimes 1 + \sum_j \sigma_{ij} \otimes \partial_j. \tag{2.8}$$

$$\Delta(\sigma_{ij}) = \sum_k \sigma_{ik} \otimes \sigma_{kj}. \tag{2.9}$$

In other words, we want $m: A \otimes A \to A$ to be a morphism of $U-$modules, the structure of an $U-$module upon $A \otimes A$ being defined via Δ.

The results of §1 give a partial answer to these questions. Namely, if Ω is a skew product of A and B, we have described the "universal part" of the relations between σ_{ij}. More precisely, we know the relations between generators of $E_r(B)^{op}$, and they are universal in the sense that they do not depend neither on A nor on the particular choice of the skew product structure AB. And (2.9) actually defines a comultiplication compatible with the Leibniz formulas (Lemma 1.2.4).

We proceed now to describe relations between ∂_i themselves, and cross-commutation relations between ∂_i, and σ_{ij}, τ_{ij}, again trying to understand only the universal part, and mostly assuming quadraticity. In 2.6 we will address U directly.

2.3. Quadratic relations between vector fields. In the setting of 2.1, denote by D a free $k-$module given together with an *odd* non-degenerate pairing $\langle | \rangle : B^1 \otimes_k D \to k$. The sign rules are:

$$\langle \alpha b|\partial\rangle = \alpha\langle b|\partial\rangle = (-1)^{\tilde{\alpha}(\tilde{b}+1)}\langle b|\alpha\partial\rangle; \alpha \in k, b \in B^1, \partial \in D \tag{2.10}$$

(mnemonically, $|$ is odd). Parity of $\langle b|\partial\rangle$ is $\tilde{b} + \tilde{\partial} + 1$.

Next, introduce an even pairing

$$(\ ,\ ):\ (B^1)^{\otimes 2} \otimes_k D^{\otimes 2} \to k$$

by the formula

$$(b \otimes b', \partial \otimes \partial') = (-1)^{(\tilde{\partial}+\tilde{b}')(\tilde{\partial}'+1)} \langle b|\partial'\rangle \langle b'|\partial\rangle. \tag{2.11}$$

It is well defined. In fact, using (2.10) we check first that the r.h.s actually depends only on $b \otimes b'$ and $\partial \otimes \partial'$ that is, gives the same result when evaluated upon $b\alpha \otimes b'$ and $b \otimes \alpha b'$ etc. Moving α from b to b' at the r.h.s. we get an extra sign:

$$[(-1)^{\tilde{\alpha}(\tilde{\partial}'+1)}]^2 = 1.$$

Similarly, replacing $\partial\alpha \otimes \partial'$ by $\partial \otimes \alpha\partial'$:

$$(-1)^{\tilde{\alpha}(\tilde{\partial}'+1)+\tilde{\alpha}(\tilde{\partial}+\tilde{b}')}(-1)^{\tilde{\alpha}(\tilde{\partial}+1+\tilde{b}'+\tilde{\partial}')} = 1.$$

Notice that (2.11) is $k-$linear in $b \otimes b'$ but only semilinear in $\partial \otimes \partial'$:

$$(b \otimes b'\alpha, \partial \otimes \partial') = (-1)^{\tilde{\alpha}}(b \otimes b', \alpha\partial \otimes \partial'). \tag{2.12}$$

We can make it bilinear by considering instead ${}^tD^{\otimes 2}$ which is $D^{\otimes 2}$ as an additive group, but has a twisted left $k-$module structure:

$$\alpha\partial \otimes \partial' \text{ (in } {}^tD^{\otimes 2}) = (-1)^{\alpha}\alpha\partial \otimes \partial' \text{ (in } D^{\otimes 2}).$$

Clearly, ${}^tD^{\otimes 2}$ and $D^{\otimes 2}$ have the same lattices of submodules and the same sets of bases of submodules.

Let now $\{b_i\}$, $\{D_j\}$ be dual bases of B^1, $D : \langle b_i|D_j\rangle = \delta_{ij}$. Then, from (2.11),

$$(b_i \otimes b_j), D_k \otimes D_l) = (-1)^{\tilde{b}_i}\delta_{il}\delta_{jk}. \tag{2.13}$$

Denote by $R \subset (B^1)^{\otimes 2}$ the $k-$submodule of quadratic relations, that is, the kernel of the multiplication map $(B^1)^{\otimes 2} \to \Omega$. Assume that R is a free direct submodule, and define $R_D \subset D^{\otimes 2}$ as orthogonal complement to R w.r.t. (2.11). Denote by B^2 the image of $(B^1)^{\otimes 2}$ in Ω and impose the next condition:

C2. *Multiplication in Ω induces a linear injection $B^2 \otimes A \to \Omega$.*

Then we have:

2.3.1. PROPOSITION. *Choose dual bases $\langle b_i|D_j\rangle = \delta_{ij}$ as above. Assume that $db_i = 0$. Define $\partial_j : A \to A$ by (2.3). Then every element of R_D vanishes (as an operator on A) when evaluated on $\{\partial_i\}$ instead of $\{D_i\}$.*

Comment. Assume that B is a quadratic graded superalgebra generated by B^1. Denote by B^τ the "odd dual opposite" quadratic superalgebra which is defined as $T(D)/(R_D)$ where $T(D)$ is the tensor algebra of D. Our Prop. 2.3.1 says that the operator algebra generated by $\{\partial_j\}$ is a quotient of B^τ.

On the other hand, $E_r(B)^{op}$ in this case can be described as $(B^! \bullet B)^{op}$ where $B^!$ is the even dual quadratic superalgebra, and the "black product" $\bullet$ is defined via tensor multiplication of relation modules (see [Ma1]-[Ma3]). For the sake of completeness, and in order to fix all signs in the presence of odd variables and constants, we shall reproduce below in 2.4 a coordinate description.

The simplest classical case is $A = k[x_1, ..., x_r]$, $B = \Lambda[dx_1, ..., dx_r]$. Proposition 2.3.1 generalizes the fact that $\partial/\partial x_i$ pairwise commute whereas dx_i anticommute.

PROOF: From (2.3) we see that for every $f \in A$

$$0 = d^2 f = \sum_{ij} (-1)^{i} b_i b_j \partial_j \partial_i f. \tag{2.14}$$

Choose a free $\mathbf{Z}_2$-graded k-graded right k-basis $r_\alpha \in (B^1)^{\otimes 2}$ of R, and complement it by elements $s_\beta \in (B^1)^{\otimes 2}$ such that $\{r_\alpha, s_\beta\}$ form a right k-basis of $(B^1)^{\otimes 2}$; $\alpha = 1, ..., m; \beta = m+1, ..., n^2$.

Denote by $\{S_\alpha, R_\beta\}$ the dual left basis of ${}^t D^{\otimes 2}$ such that

$$\begin{gathered}(r_\alpha, R_\beta) = (S_\alpha, s_\beta) = 0;\\ (r_{\alpha_1}, S_{\alpha_2}) = \delta_{\alpha_1,\alpha_2};\ (s_{\beta_1}, R_{\beta_2}) = \delta_{\beta_1,\beta_2}.\end{gathered} \tag{2.15}$$

In particular, R_β form a basis of $R_D \subset D^{\otimes 2}$. Then we have in $(B^1)^{\otimes 2} \otimes {}^t D^{\otimes 2}$:

$$\sum_{ij} (-1)^{\tilde{b}_i} b_i \otimes b_j \otimes D_j \otimes D_i = \sum_\alpha r_\alpha \otimes S_\alpha + \sum_\beta s_\beta \otimes R_\beta \tag{2.16}$$

because of (2.13) and (2.15). Denote by $S_\alpha(\partial)$ (resp. $R_\beta(\partial)$) the operators obtained by replacing D_i's by ∂_i's. Similarly, denote by $r_\alpha(b)$ (resp. $s_\beta(b)$) the image of r_α (resp. s_β) in B^2. Then $r_\alpha(b) = 0$, and $s_\beta(b)$ are right linearly independent over A. From (2.14) and (2.16) we see that

$$\sum_\beta s_\beta(b) R_\beta(\partial) f = 0$$

for all f, so that $R_\beta(\partial) = 0$.

2.4. Quadratic relations between τ_{ij}, and σ_{ij}. Keeping notation of 2.3, put

$$r_\alpha = \sum_{ij} c_\alpha^{ij} b_i \otimes b_j,\ c_\alpha^{ij} \in k. \tag{2.17}$$

Denote by B^{1*} the right even dual B^1, given together with an even non-degenerate scalar product with the obvious sign conventions. Extend it to an even scalar product $(B^1)^{\otimes 2} \otimes (B^{1*})^{\otimes 2} \to k$ by the rule

$$(b_1 \otimes b_2, b^1 \otimes b^2) = (-1)^{\tilde{b}^1 \tilde{b}_2} (b_1, b^1)(b_2, b^2).$$

Denote by $B^!$ the quadratic algebra $T(B^{1*})/(R^c)$ where R^c is the orthogonal complement to R.

Let $\{b_i\}$, $\{b^j\}$ be dual bases in B^1, B^{1*}. As above, choose a free basis $\{r_\alpha, s_\beta\}$ of $B^1 \otimes B^1$, and let $\{s^\alpha, r^\beta\}$ be the dual basis of $B^{1*} \otimes B^{1*}$. Then $\{r^\beta\}$ is a basis of R^c. Put

$$r^\beta = \sum_{kl} b^k \otimes b^l c_{kl}^\beta,\ c_{kl}^\beta \in k. \tag{2.18}$$

We have then for every k, l

$$b_k \otimes b_l = \sum_\beta (-1)^{\tilde{b}_k \tilde{b}_l} c_{kl}^\beta s_\beta \quad \mod R. \tag{2.19}$$

To see it, take the scalar product of both sides with all r^β. Put now

$$E = T(B^1 \otimes B^{1*})/(S_{(23)}(R \otimes R^c)).$$

This algebra has generators $b_i \otimes b^j = z_{ij}$ satisfying relations $S_{(23)}(r_\alpha \otimes r^\beta)$, or explicitely

$$\sum_{ijkl} c_\alpha^{ij} (-1)^{\tilde{b}_j \tilde{b}_k} z_{ik} z_{jl} c_{kl}^\beta = 0. \tag{2.20}$$

Put $B = T(B^1)/(R)$. Then the map $\delta : \ B \to E \otimes B$ transforming $\{b_i\}$ by (z_{ij}) is the universal left coaction.

2.4.1. PROPOSITION. *$\{\tau_{ik}\}$ satisfy (2.20), whereas $\{\sigma_{ik}\}$ satisfy the opposite super-transposed relations.*

PROOF: We have, using (2.19):

$$0 = r_\alpha(b) f = \sum_{ij} c_\alpha^{ij} b_i b_j f = \sum_{ij} c_\alpha^{ij} b_i \sum_l (-1)^{\tilde{f}\tilde{b}_l} \tau_{jl}(f) b_l =$$

$$\sum_\alpha^{ij} \sum_l (-1)^{\tilde{f}\tilde{b}_l} \sum_k (-1)^{(\tilde{f}+\tilde{\tau}_{jl})\tilde{b}_k} \tau_{ik}\tau_{jl}(f) b_k b_l =$$

$$\sum_{ijkl\beta} c_\alpha^{ij} (-1)^{\tilde{f}(\tilde{b}_k + \tilde{b}_l) + \tilde{b}_j \tilde{b}_k} \tau_{ik}\tau_{jl}(f) c_{kl}^\beta s_\beta(b).$$

It remains to simplify the sign using

$$\tilde{c}_{kl}^\beta = \tilde{r}^\beta + \tilde{b}_k + \tilde{b}_l$$

and to take into account that s_β are right $A-$independent. One can treat σ similarly.

Finally, I state relations between τ_{ij} and ∂_k. Unfortunately, I was unable to interpret them conceptually.

2.5.1. PROPOSITION. *For i, β fixed, $f \in A$, we have*

$$\sum_{jkl} (-1)^{\tilde{f}\tilde{b}_j + (\tilde{\partial}_k + \tilde{b}_i + \tilde{f})\tilde{b}_l} \tau_{kl}(\partial_k(\tau_{ij}(f))) c_{lj}^\beta =$$

$$\sum_{jkl} (-1)^{\tilde{b}_i + \tilde{b}_k(\tilde{f} + \tilde{\partial}_j) + (\tilde{b}_j + \tilde{\partial}_j + \tilde{f})\tilde{b}_l} \tau_{il}(\tau_{jk}(\partial_j(f))) c_{lk}^\beta.$$

There are similar relations for σ. To prove them, one should differentiate (2.1) and (2.2).

2.6. Bialgebra structures on operator rings. Returning now to the setting of 2.2, let us deiscuss the comultiplication problem in relation to Leibniz formula. This is well-known, and we include it for the sake of completeness.

Generally, let A be a k–algebra, and $U \subset \mathbf{End}_k(A)$ a k–module of left linear operators upon A. Assume that A admits an augmentation map $\epsilon : \ A \to k$, and for $u \in U, \ f \in A$ put

$$\langle u, f \rangle = \epsilon(u(f)). \tag{2.21}$$

2.6.1. PROPOSITION. *Assume that for every $u \in U$, there exist $u_i, v_i \in U, \ i = 1, ..., n$ such that for all $f, g \in A$*

$$u(fg) = \sum_i (-1)^{\tilde{f}\tilde{v}_i} u_i(f) v_i(g). \tag{2.22}$$

Assume also that (2.21) has trivial left kernel. Then U admits a unique comultiplication map $\Delta : \ U \to U \otimes U$ with the property

$$\langle \Delta(u), f \otimes g \rangle = \langle u, fg \rangle \tag{2.23}$$

for all $f, g \in A$. It is coassociative, and admits a counit $\epsilon_U : \ u \mapsto \epsilon(u1)$. If, in addition, U is an algebra, then (U, Δ, ϵ_U) is a bialgebra.

SKETCH OF PROOF: Existence follows from (2.22): put $\Delta(u) = \sum u_i \otimes v_i$. Uniqueness follows from non-degeneracy. We leave the rest to the reader.

2.6.2. Example. For $A = k[x, x^{-1}, dx]_v$ (see sec. 0.2), $U = \oplus_{i \neq 0} kL_i$ (see sec. 0.6), $\langle u, f \rangle$ has trivial left kernel precisely when v is not a root of unity.

§3. Examples continued: de Rham complexes of quantum spaces and $GL_{Q,\lambda}(n)$

3.1. Two-parametric family of (2,2)-quantum matrices. Here we assume k pure even. Choose two invertible even elements $p, q \in k$. The ring of polynomial functions E on the non-commutative quantum space $M_{p,q}(2)$ is generated by even variables a, b, c, d with the following commutation relations:

$$ab = p^{-1}ba;\ ac = q^{-1}ca;\ ad = da + (q^{-1} - p)cb;$$
$$bc = pq^{-1}cb;\ bd = q^{-1}db;\ cd = p^{-1}dc. \tag{3.1}$$

This ring is a bialgebra with the standard comultiplication and counit. For $pq \neq -1$, it can be defined as universally coacting algebra on the function algebras of two quantum planes:

$$A_q:\ k\langle x, y\rangle/(xy - q^{-1}yx);\ x, y \text{ even}; \tag{3.2}$$

$$B_p:\ k\langle \xi, \eta\rangle/(\xi^2, \eta^2, \xi\eta + p\eta\xi);\ \xi, \eta \text{ odd}. \tag{3.3}$$

Adding the relation $ad - p^{-1}bc = 1$, we get the coordinate ring of $SL_{p,q}(2)$.

Order the generators by $d < b < c < a$. The Diamond Lemma is applicable, and shows that monomials $d^kc^lb^ma^n$ form a k–basis of E. (Of course, a similar statement is true and can be easily checked also for A_q, B_p.)

A slight generalization of the Wess-Zumino construction gives the following result.

3.2. PROPOSITION. *There are precisely two skew products Ω of A_q and B_p, admitting a differential with $dx = \xi$, $dy = \eta$, such that all these structures are compatible with the coaction of E (with zero differential).*

SKETCH OF PROOF: We actually prove slightly more: the skew product property should be postulated only on forms of degree two and three in (x, y, ξ, η); the full strength then follows from the Diamond Lemma.

Comparing dimensions, we see that we must find a system of four cross-commutation relations expressing $x\xi, x\eta, y\xi, y\eta$ linearly via $\xi x, \xi y, \eta x, \eta y$. (Actually, one relation is known: $d(xy - q^{-1}yx) = 0$). We have sixteen indeterminate coefficients which are successively constrained by the rest of conditions. It is convenient to use them in the following order.

i). Apply the differential to the cross-commutation relations. The result should be a consequence of (3.3). This eliminates about half of coefficients.

ii). Use compatibility with the coaction of E. This leaves one parameter free. Of course, this can also be explained conceptually, by analyzing the decomposition of the space

$$(kx \oplus ky) \otimes (k\xi \oplus k\eta) \oplus (k\xi \oplus k\eta) \otimes (kx \oplus ky)$$

with respect to the coaction of E. The result is:

$$x\xi = [(p + q^{-1})r - 1]\xi x;\ y\eta = [(p + q^{-1})r - 1]\eta y;$$

$$x\eta = (rq^{-1} - 1)\xi y + pq^{-1}r\eta x; \;\; y\xi = r\xi y + (pr-1)\eta x \tag{3.4}$$

where r is the last undetermined coefficient.

iii). Constraint r by checking linear relations between cubic monomials. In practice, it suffices to resolve the "overlap ambiguity" $x\eta\xi$, that is, to reduce this monomial first as $x(\eta\xi)$ and second as $(x\eta)\xi$. In both cases, the result is proportional to $\xi\eta x$, but the coefficients coincide iff $r^2 - (p^{-1}+q)r + qp^{-1} = 0$, that is, $r = p^{-1}$ or $r = q$. The net result is:

Variant 1: $\Omega^{(1)}_{p,q}$

$$\begin{gathered} x\xi = (pq)^{-1}\xi x; \;\; y\eta = \eta y; \\ x\eta = (p^{-1}q^{-1} - 1)\xi y + q^{-1}\eta x; y\xi = p^{-1}\xi y. \end{gathered} \tag{3.5}$$

Variant 2: $\Omega^{(2)}_{p,q}$

$$\begin{gathered} x\xi = pq\xi x; \;\; y\eta = pq\eta y; \\ x\eta = p\eta x; \;\; y\xi = q\xi y + (pq-1)\eta x. \end{gathered} \tag{3.6}$$

Notice that (3.5) and (3.6) are connected by the isomorphism interchanging x and y, ξ and η, p and p^{-1}, which is compatible with the isomorphism interchanging a and d, b and c.

iv). Finally, one checks that the combined relations (3.2), (3.3) and either (3.5), or (3.6), satisfy all the conditions of the Diamond Lemma, that is, all ambiguities of cubic monomials are resolvable (with respect to any of the orders $\xi < \eta < x < y$ or $x < y < \xi < \eta$.

We now turn to the calculation of the de Rham complex of $M_{p,q}(2)$. We start with determining the universal differential quadratic algebra $W^{(i)}_{p,q}$ left coacting upon $\Omega^{(i)}_{p,q}$. Denote $\alpha, \beta, \gamma, \delta$ the differentials of a, b, c, d respectively. We assume $pq \neq -1$.

3.3. PROPOSITION. *$W^{(1)}_{p,q}$ is defined by the relations (3.1) and the following ones:*
a). Relations between differentials of matrix coefficients:

$$\begin{gathered} \alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 0; \\ \beta\alpha = -q^{-1}\alpha\beta; \;\; \gamma\alpha = -p^{-1}\alpha\gamma; \\ \delta\gamma = -q^{-1}\gamma\delta; \;\; \delta\beta = -p^{-1}\beta\delta; \end{gathered} \tag{3.7}$$

$$pq\delta\alpha + p\gamma\beta = -\alpha\delta - q\beta\gamma. \tag{3.8}$$

b). Cross-commutation relations:

$$\begin{gathered} a\alpha = (pq)^{-1}\alpha a; \\ a\beta = (p^{-1}q^{-1} - 1)\alpha b + p^{-1}\beta a; \\ a\gamma = q^{-1}\gamma a + (p^{-1}q^{-1} - 1)\alpha c; \end{gathered}$$

$$b\alpha = q^{-1}\alpha b;$$

$$b\beta = (pq)^{-1}\beta b;$$

$$b\delta = q^{-1}\delta b + (p^{-1}q{-1} - 1)\beta d;$$

$$c\alpha = p^{-1}\alpha c;$$

$$c\gamma = (pq)^{-1}\gamma c;$$

$$c\delta = p^{-1}\delta c + (p^{-1}q^{-1} - 1)\gamma d;$$

$$d\beta = p^{-1}\beta d;$$

$$d\gamma = q^{-1}\gamma d;$$

$$d\delta = (pq)^{-1}\delta d; \tag{3.9}$$

$$a\delta + pc\beta = (p^{-1}q^{-1} - 1)\alpha d + (q^{-1} - p)\gamma b + p^{-1}\beta c + \delta a;$$

$$pqd\alpha + qb\gamma = \alpha d + p\gamma b;$$

$$pqd\alpha + pc\beta = \alpha d + q\beta c; \tag{3.10}$$

$W^{(2)}_{p,q}$ can be obtained from here by the involution described above.

Comment. The proof consists of direct calculations which we will omit. The answer is not quite satisfactory because we want the de Rham complex of $M_{p,q}(2)$ to be of the same size as that of the commutative polynomial ring of four variables. For this to hold in degree two, we miss one quadratic relation between differentials and one cross-comutation relation. They are supplied by the following result, again stressing the role of skew products.

3.4. THEOREM. *For $pq \neq 1$, $W^{(i)}_{p,q}$ has precisely three quadratic differential quotients for which quadratic and cubic components are freely spanned by the lexicographically ordered monomials in $\alpha < \beta < \gamma < \delta < d < c < b < a$. Every such quotient is a skew product of $k[a,b,c,d]$ and $k[\alpha,\beta,\gamma,\delta]$ (modulo relations), and their Hilbert series are the same as in the commutative case. For $pq = 1$, these quotients coincide.*

For $W^{(1)}_{p,q}$, the missing relations are as follows:

Variant 1:

$$d\alpha = \alpha d;\ \ \delta\alpha = -\alpha\delta. \tag{3.11}$$

Variant 2:

$$c\beta = p^2\beta c;\ \ \gamma\beta = -p^2\beta\gamma. \tag{3.12}$$

Variant 3:

$$b\gamma = q^2\gamma b; \beta\gamma = -q^2\gamma\beta. \tag{3.13}$$

SKETCH OF PROOF: we start with writing a missing cross-commutation relation, say, for $b\gamma$ as a linear combination of the lexicographically ordered monomials, and then check the resolution of overlaps in cubic monomials. Differentiating the answer, we get a missing relation between the differentials. Finally, we check the conditions of the Diamond Lemma for cubic overlaps.

3.5. Remarks. The long calculations needed to check the Theorem 3.4 were made by D. Zhdanovich and the author. The Proposition 3.3 was independently proved by G. Maltsiniotis [Mal 1], who has also discivered the missing relations (3.11), but not (3.12), (3.13). The reason was that Maltsiniotis was only looking for the DGA-quotients of $W_{p,q}^{(1)}$ compatible with the coalgebra structure, and thus leading to the de Rham complex of $GL_{p,q}(2)$. Only (3.11) satisfy this condition. (The existence of $W_{p,q}^{(2)}$ also is not mentioned by Maltsiniotis).

D. Zhdanovich remarked that (3.7), (3.8), and $(3.11)_{\alpha,\delta}$ determine the algebra of the differential forms with constant coefficients which is naturally isomorphic to $M_{q,p}(2)^{!}$ (even quadratic dual). This does not hold for the other two quotients.

The following remark is also due to D. Zhdanovich. Relations (3.12) and (3.13) can be obtained by imitating the Wess-Zumino approach. More precisely, let us *start* with an algebra of functions determined by (3.1), and its differentials defined by (3.7), (3.8), and either $(3.12)_{\gamma,\beta}$ or $(3.13)_{\gamma,\beta}$. Let us then construct the universal bialgebra Fcoacting upon these algebras (by the same (4,4)-matrix on coordinates and their differentials). Then the cross-commutation relations (3.9), (3.10), and $(3.12)_{c,\beta}$ (resp. $(3.14)_{b,\gamma}$) are uniquely defined by the Wess-Zumino type conditions.

Question. Is F a flat deformation of the polynomial algebra of four variables?

3.6. n-dimensional Wess-Zumino de Rham complex. The following generalization of the DGA (3.5) was considered by several people, among them G. Maltsiniotis, E. Demidov, E. Mukhin, after the discovery by M. Artin, J. Tate, A. Sudbery et. al. of the correct pair of n-dimensional quantum spaces leading to the construction of many-parametric quantum group $GL_{P,Q}(n)$ by the universal coaction method. This algebra is generated by $x_i,\ i = 1, \ldots, n;\ \xi_i = dx_i$, subject to the following commutation relations:

$$x_i x_j = q_{ji}^{-1} x_j x_i,\ i < j; \tag{3.14}$$

$$\xi_i^2 = \xi_i\xi_j + \lambda_j q_{ji}^{-1}\xi_j\xi_i = 0,\ i < j; \tag{3.15}$$

$$\xi_i x_i = \lambda_i x_i \xi_i; \tag{3.16}$$

$$|xi_i x_j = \lambda_j q_{ji}^{-1} x_j \xi_i;\ i < j; \tag{3.17}$$

$$x_i\xi_j = q_{ji}^{-1}\xi_j x_i - (1 - \lambda_j^{-1})\xi_i x_j;\ i < j. \tag{3.18}$$

Here $q_{ij} (i < j), \lambda_j$ are invertible quantization parameters. One can check without serious difficulties that this DGA $\Omega_{Q,\lambda}$ is well defined and is a flat deformation of the de

Rham complex of the polynomial ring in n variables. In fact, the lexicographically ordered monomials in x, ξ form its free $k-$basis.

G.Maltsionitis has applied the universal coaction construction in the DGA-category to $\Omega_{Q,\lambda}$ and established the following result ([Mal2]):

3.7. THEOREM. *The universal coacting DGA-algebra for $\Omega_{Q,\lambda}$ has a quotient which is a flat deformation of the standard de Rham complex of the polynomials in matrix entries iff $\lambda_j = \lambda$ does not depend on j. This quotient is then unique, and is a skew product of two subalgebras, generated by matrix entries, and their differentials, respectively.*

For a list of relations, see [Mal2].

3.8. Question. Do there exist other quotients, with the standard Hilbert function, generalizing (3.12) and (3.13) to general n? Up to know, problems of this type have been treated by a direct application of the Diamond Lemma. G. Maltsionitis says it took him eghty pages of calculations to prove Theorem 3.7. About forty pages were spent to the discovery of (3.12) and (3.13). Clearly, a more intelligent approach is highly desirable.

3.9. Exercise. Calculate operator algebras discussed in §2 in the examplea above. In particular, calculate the algebras generated by σ_{ij} and $x_i\partial_j$ as possible analogs of $U(gl(n))$.

§4. Cohomology of elementary extensions

4.1. Elementary extensions. Let $U \subset V$ be an embedding of differential algebras, $t \in V$ an even element, dt its differential. We shall say that V is *an elementary extension of* V by (t, dt), if the following conditions are satisfied:

i). $\{t^i; dt.t^j\}$ for $i, j \geq 0$ form a free basis of V as a right U–module.

ii). $(dt)^2 = 0$; $tdt = v^2 dt.t$ for an invertible element $v \in k$.

We can then slightly generalize the calculations of 0.2, 0.3 and prove the following result.

4.2. THEOREM. *Assume that all non-vanishing $[i]_v$ are invertible in k.*

a). If v is not a root of unity, then the embeding $U \to V$ defines the isomorphism of the cohomology spaces $H^*(U) \to H^*(V)$.

b). If v is a primitive root of unity of degree l, then $H^*(V)$ as a right $H^*(V)$–module is freely generated by the cohomology classes of cocycles t^{jl}, $j \geq 0$, and $dt.t^{jl-1}$, $j \geq 1$.

PROOF: Write a generic element of V as

$$s = f + \sum_{i \geq 1} t^i g_i + \sum_{j \geq 0} dt.t^j h_j; \quad f, g_i, h_j \in U. \tag{4.1}$$

Then, by (0.13)

$$ds = df + \sum_{i \geq 1} t^i dg_i + \sum_{j \geq 0} dt.t^j (v^j [j+1]_v g_{j+1} - dh_j). \tag{4.2}$$

Hence s is a cycle iff f, g_i $(i \geq 1)$ are cycles, and

$$v^j [j+1]_v g_{j+1} = dh_j. \tag{4.3}$$

Now, if all $[j+1]_v$ for $j \geq 0$ are non-vanishing and hence invertible, then the last summands of (4.1) add up to a boundary, because

$$t^{j+1} g_{j+1} + dt.t^j h_j = v^{-j}[j+1]_v^{-1} t^{j+1} dh_j + dt.t^j h_j =$$

$$v^{-j}[j+1]_v^{-1} d(t^{j+1} h_j).$$

Hence the cohomology class of s coincides with that of f, a cycle in U, and from (4.2)one easily sees that $H^*(U) \to H^*(V)$ is also injective.

If $[j+1]_v = 0$ precisely for $j+1 \equiv 0 \mod l$, the same argument shows that s is equivalent to

$$s' = f + \sum_{j \geq 1} t^{lj} g_{lj} + \sum_{j \geq 1} dt.t^{lj-1} h_{lj-1},$$

where f, g_{lj}, h_{lj-1} are cycles in U.

Then (4.2) shows that these cycles are defined uniquely modulo boundaries in U.

4.3. Application to the n−dimensional Wess-Zumino-de Rham complex. Let Ω_n be the DGA described in 3.6. One easily checks that the natural embedding $\Omega_{n-1} \subset \Omega_n$ (with compatible sets of quantization constants) makes Ω_n a left elementary extension of Ω_{n-1}, for which $v_n^2 = \lambda_n^{-1}$, which allows us to calculate $H^*(\Omega_n)$. From the Theorem 3.7 one infers that Ω_n is "quantum homogeneous" exactly when $\lambda_1 = \ldots \lambda_n = \lambda$, so that the cohomology of Ω_n is then determined by this unique parameter.

Actually, Ω_n is a skew product of Ω_{n-1} and $k[x_n, dx_n]_{v_n}$. To check this, it suffices to prove that the standard ordered monomials in x_i, ξ_i can be expressed as linear combinations of monomials ordered by decreasing indices. To put ξ_n to the leftmost place, one should use (3.18) successively.

4.4. Question. Can one apply a similar reasoning to the de Rham complexes of matrix bialgebras described in §3?

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