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PERIODIC AUTOMORPHISMS OF TAKIFF ALGEBRAS, CONTRACTIONS, AND θ-GROUPS

DMITRI I. PANYUSHEV

INTRODUCTION

Let G be a connected reductive algebraic group with Lie algebra $\mathfrak g$. The ground field $\Bbbk$ is algebraically closed and of characteristic zero. Fundamental results in invariant theory of the adjoint representation of G are primarily associated with C. Chevalley and B. Kostant. Especially, one should distinguish the "Chevalley restriction theorem" and seminal article of Kostant [5]. Later, Kostant and Rallis extended these results to the isotropy representation of a symmetric variety [6]. In 1975, E.B. Vinberg came up with the theory of θ-groups. This theory generalises and presents in the most natural form invariant-theoretic results previously known for the adjoint representation and isotropy representations of the symmetric varieties.

Let us remind the main construction and results of Vinberg's article [15]. Let $\theta \in \mathrm{Aut}(\mathfrak g)$ be a periodic (= finite order) automorphism of $\mathfrak g$. The *order* of θ is denoted by $|\theta|$. Fix a primitive root of unity $\zeta = \sqrt[|\theta|]{1}$ and consider the *periodic grading* (or $\mathbb Z_{|\theta|}$-*grading*)

$$\mathfrak g = \bigoplus_{i\in\mathbb Z_{|\theta|}} \mathfrak g_i \ ,$$

where $\mathfrak g_i$ is the ζ^i-eigenspace of θ. In particular, $\mathfrak g_0 = \mathfrak g^\theta$ is the fixed point subalgebra for θ. Let G_0 be the connected subgroup of G with Lie algebra $\mathfrak g_0$. The restriction of the adjoint representation yields the natural homomorphism $G_0 \to GL(\mathfrak g_1)$. The linear groups obtained in this way are called *θ-groups*, and the point is that they have the best possible invariant-theoretic properties:

- $\Bbbk[\mathfrak g_1]^{G_0}$ is a polynomial algebra;
- the quotient morphism $\pi : \mathfrak g_1 \to \mathfrak g_1/\!\!/G_0 = \mathrm{Spec}(\Bbbk[\mathfrak g_1]^{G_0})$ is flat;
- each fibre of π contains finitely many G_0-orbits.

It is a natural problem to extend Vinberg's theory to a more general setting. There can be several possible ways for doing so. Here are at least two of them:

(a) Determine and investigate a wider class of Lie algebras such that their periodic automorphisms lead to representations with similar invariant-theoretic properties.

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(b) Given $(\mathfrak{g}, \theta)$ as above, construct a non-reductive Lie algebra with good invariant-theoretic properties, using $\{\mathfrak{g}_i\}$ as building blocks.

These two ways are not mutually exclusive, but in this article we deal with (a). Our class of Lie algebras consists of Takiff algebras modelled on reductive ones. Although considerable part of the theory can be developed for arbitrary Takiff algebras, substantial applications are related to the reductive case.

For a Lie algebra $\mathfrak{q}$ and $m \in \mathbb{N}$, let $\mathfrak{q}\langle m\rangle$ denote the Takiff algebra modelled on $\mathfrak{q}$. It is an $\mathbb{N}$-graded Lie algebra of dimension $m \dim \mathfrak{q}$, with nonzero components of degrees $0, 1, \dots, m-1$. (See Section 1 for precise definitions.) Our initial observation is that any periodic $\theta \in \mathrm{Aut}(\mathfrak{q})$ gives rise to an automorphism of $\mathfrak{q}\langle m\rangle$ of the same order, denoted $\widehat{\theta}$. The fixed point subalgebra of $\widehat{\theta}$, $\mathfrak{q}\langle m\rangle^{\widehat{\theta}}$, is a "mixture" of eigenspaces of θ, i.e., its component of degree i equals $\mathfrak{q}_i$, $i = 0, 1, \dots, m-1$. Then we consider the representations of $\mathfrak{q}\langle m\rangle^{\widehat{\theta}} = \mathfrak{q}\langle m\rangle_0$ in eigenspaces of $\widehat{\theta}$. If $\mathfrak{q}$ is quadratic (i.e., $\mathfrak{q} \simeq \mathfrak{q}^*$), then the coadjoint representation of $\mathfrak{q}\langle m\rangle_0$ also occurs in this way. Our ultimate goal is to describe several instances in which the algebras of invariants for $(\mathfrak{q}\langle m\rangle_0, \mathrm{ad})$ and $(\mathfrak{q}\langle m\rangle_0, \mathrm{ad}^*)$ are polynomial. (If the explicit formula for $\mathfrak{q}$ is bulky, then we write $\mathsf{Inv}(\mathfrak{q}, \mathrm{ad})$ (resp. $\mathsf{Inv}(\mathfrak{q}, \mathrm{ad}^*)$) in place of $\Bbbk[\mathfrak{q}]^{\mathfrak{q}}$ (resp. $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$).) Another observation is that, for special values of m, $\mathfrak{q}\langle m\rangle_0$ is a contraction of a direct sum of Lie algebras. Namely, $\mathfrak{q}\langle n|\theta|\rangle_0$ is a contraction of $n\mathfrak{q} := \mathfrak{q} \dotplus \ldots \dotplus \mathfrak{q}$ and $\mathfrak{q}\langle n|\theta| + 1\rangle_0$ is a contraction of $n\mathfrak{q} \dotplus \mathfrak{q}_0$. These are examples of the so-called *quasi-graded* contractions, and for such contractions we establish a rather explicit connection between invariants of two Lie algebras. For instance, starting from $\Bbbk[n\mathfrak{q}]^{n\mathfrak{q}}$, we construct an explicit subalgebra of $\mathsf{Inv}(\mathfrak{q}\langle nk\rangle_0, \mathrm{ad})$, denoted $\mathcal{L}_\bullet(\Bbbk[n\mathfrak{q}]^{n\mathfrak{q}})$. The graded algebras $\Bbbk[\mathfrak{q}]^{\mathfrak{q}}$ and $\mathcal{L}_\bullet(\Bbbk[n\mathfrak{q}]^{n\mathfrak{q}})$ have the same Poincaré series. The similar subalgebra for the coadjoint representation is denoted by $\mathcal{L}^\bullet(\mathcal{S}(n\mathfrak{q})^{n\mathfrak{q}})$.

For $\mathfrak{q} = \mathfrak{g}$ reductive, we deal with contractions of reductive algebras and therefore the theory of θ-groups is also at our disposal. Our main result is

Theorem 0.1. *Let θ be a periodic automorphism of $\mathfrak{g}$, $\mathcal{O}^{reg}$ the regular nilpotent G-orbit in $\mathfrak{g}$, and $n \in \mathbb{N}$ arbitrary. Set $k = |\theta|$.*

- (i) *Suppose θ has the property that $\mathfrak{g}_0 \cap \mathcal{O}^{reg} \neq \varnothing$. Then*
  - $\mathcal{L}_\bullet(\Bbbk[n\mathfrak{g}]^{n\mathfrak{g}}) = \mathsf{Inv}(\mathfrak{g}\langle nk\rangle_0, \mathrm{ad})$ *and* $\mathsf{Inv}(\mathfrak{g}\langle nk\rangle_0, \mathrm{ad})$ *is a polynomial algebra of Krull dimension $n{\cdot}\mathrm{rk}\,\mathfrak{g}$.*
  - $\mathcal{L}_\bullet(\Bbbk[n\mathfrak{g} \dotplus \mathfrak{g}_0]^{n\mathfrak{g}\dotplus\mathfrak{g}_0}) = \mathsf{Inv}(\mathfrak{g}\langle nk+1\rangle_0, \mathrm{ad})$ *and* $\mathsf{Inv}(\mathfrak{g}\langle nk+1\rangle_0, \mathrm{ad})$ *is a polynomial algebra of Krull dimension $n{\cdot}\mathrm{rk}\,\mathfrak{g} + \mathrm{rk}\,\mathfrak{g}_0$.*
- (ii) *Suppose θ has the property that $\mathfrak{g}_1$ contains regular semisimple elements of $\mathfrak{g}$ and $\mathcal{O}^{reg}$ meets every irreducible component of the nilpotent cone in $\mathfrak{g}_1$. Then $\mathcal{L}^\bullet(\mathcal{S}(n\mathfrak{g})^{n\mathfrak{g}}) = \mathsf{Inv}(\mathfrak{g}\langle nk\rangle_0, \mathrm{ad}^*)$ and $\mathsf{Inv}(\mathfrak{g}\langle nk\rangle_0, \mathrm{ad}^*)$ is a polynomial algebra of Krull dimension $n{\cdot}\mathrm{rk}\,\mathfrak{g}$.*

The proofs of two parts of this theorem exploit different ideas. Note that $\mathfrak g\langle nk+1\rangle_0$ is quadratic and $\mathfrak g\langle nk\rangle_0^* \simeq \mathfrak g\langle nk\rangle_1$. In particular, in part (ii) we describe the invariants for the $\mathfrak g\langle nk\rangle_0$-module $\mathfrak g\langle nk\rangle_1$.

The paper is organised as follows. Sections 1 contains generalities on quadratic and Takiff algebras and on Lie algebra contractions. In Section 2, we introduce quasi-gradings of Lie algebras and corresponding contractions. We provide a useful construction of quasi-gradings and study the behaviour of invariants. In Section 3, we consider periodic automorphisms of Takiff algebras and their connection with quasi-graded contractions. Section 4 is devoted to the proof of Theorem 0.1(i), and in Section 5 we prove Theorem 0.1(ii). Sections 4 and 5 also contain a number of examples of θ-groups that satisfy the assumptions of Theorem 0.1. In Section 6, we discuss open problems and directions for related investigations.

Notation. The nilpotent radical of a Lie algebra $\mathfrak q$ is denoted by $\mathfrak R_u(\mathfrak q)$. The unipotent radical of an algebraic group Q is $R_u(Q)$. A direct sum of Lie algebras is denoted with $\dot+$.

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1. Preliminaries

1.1. **Quadratic Lie algebras.** A Lie algebra $\mathfrak q$ is called *quadratic*, if there is a $\mathfrak q$-invariant symmetric non-degenerate bilinear form on $\mathfrak q$. Such a form is said to be a *scalar product* on $\mathfrak q$. If B is a scalar product on $\mathfrak q$, we also say that $(\mathfrak q, B)$ is quadratic.

Suppose that $(\mathfrak q, B)$ is quadratic and $\theta \in \mathrm{Aut}(\mathfrak q)$ is of order k. Assume that B is a an eigenvector of θ, i.e., $B(\theta(x),\theta(y)) = \zeta^c B(x,y)$ for all $x, y \in \mathfrak q$ and some $c \in \mathbb Z_k$. Then $B(\mathfrak q_i, \mathfrak q_j) = 0$ unless $i + j = c$ (the equality in $\mathbb Z_k$). It follows that the dual $\mathfrak q_0$-module for $\mathfrak q_i$ is $\mathfrak q_{c-i}$. Thus, $\mathfrak q_0$ is not necessarily quadratic unless $\zeta^c = 1$. However, we have a weaker property that the set of $\mathfrak q_0$-modules $\{\mathfrak q_i\}$ is closed with respect to taking duals.

If $\mathfrak q$ is reductive, then B can always be chosen to be θ-invariant, hence $c = 0$ and $\mathfrak q_i^* \simeq \mathfrak q_{-i}$ for all $i \in \mathbb Z_k$. Actually, $\mathfrak q_0$ is again reductive here.

1.2. **Generalised Takiff algebras** [14]**.** Let Q be a connected algebraic group with Lie algebra $\mathfrak q$. The infinite-dimensional $\Bbbk$-vector space $\mathfrak q_\infty := \mathfrak q \otimes \Bbbk[\mathsf T]$ has a natural structure of a graded Lie algebra such that $[x\otimes \mathsf T^l, y\otimes \mathsf T^k] = [x,y]\otimes \mathsf T^{l+k}$. Then $\mathfrak q_{\geqslant(m)} = \bigoplus_{j\geqslant m} \mathfrak q\otimes \mathsf T^j$ is an ideal of $\mathfrak q_\infty$, and $\mathfrak q_\infty/\mathfrak q_{\geqslant(m)}$ is a *(generalised) Takiff algebra* (modelled on $\mathfrak q$), denoted $\mathfrak q\langle m\rangle$.

Write $Q\langle m\rangle$ for the corresponding connected algebraic group. Clearly, $\mathfrak q\langle m\rangle$ is $\mathbb N$-graded and $\dim\mathfrak q\langle m\rangle = m\dim\mathfrak q$. An alternate notation for $\mathfrak q\langle m\rangle$ used below is

$$\mathfrak q\ltimes\mathfrak q\ltimes\ldots\ltimes\mathfrak q \qquad (m \text{ factors}),$$

where the consecutive factors from the left to right comes from the subspaces $\mathfrak q\otimes T^i$, $i=0,1,\ldots,m-1$. In particular, $\mathfrak q\langle 2\rangle=\mathfrak q\ltimes\mathfrak q$ is the usual semi-direct product. The image of $\mathfrak q\otimes T^i$ in $\mathfrak q\ltimes\mathfrak q\ltimes\ldots\ltimes\mathfrak q$ is sometimes denoted by $\mathfrak q[i]$. We may (and will) represent the elements of $\mathfrak q\langle m\rangle$ as vectors:

$$\vec{x}=(x_0,x_1,\ldots,x_{m-1}), \text{ where } x_i\in\mathfrak q[i].$$

Recall that $\mathcal S(\mathfrak q)$ is the symmetric algebra of $\mathfrak q$ over $\Bbbk$ and $\Bbbk[\mathfrak q]=\mathcal S(\mathfrak q^*)$. For reductive $\mathfrak g$, $\mathfrak g[0]$ is a Levi subalgebra of $\mathfrak g\langle m\rangle$ and $\mathfrak R_u(\mathfrak g\langle m\rangle)=0\ltimes\mathfrak g\ltimes\cdots\ltimes\mathfrak g$. It is shown in [14] that $\Bbbk[\mathfrak g\langle m\rangle]^{\mathfrak g\langle m\rangle}=\Bbbk[\mathfrak g\langle m\rangle]^{G\langle m\rangle}$ is polynomial. More precisely, there is an explicit procedure for constructing elements of $\Bbbk[\mathfrak g\langle m\rangle]^{\mathfrak g\langle m\rangle}$ from those of $\Bbbk[\mathfrak g]^{\mathfrak g}$, which enables us to prove the polynomiality. The semisimple case with $m=2$ was considered by Takiff in 1971. In [11, Sect. 11], we extend results of [14] to more general Lie algebras and prove that the algebra $\Bbbk[\mathfrak g\langle m\rangle]^{R_u(G\langle m\rangle)}$ is also polynomial.

1.3. **Contractions.** We only consider Lie algebra contractions of the following form. Let $\mathsf c_t:\mathfrak q\to\mathfrak q$, $t\in\Bbbk^\times$, be a polynomial linear action of $\Bbbk^\times$ on $\mathfrak q$. That is, $\mathsf c_1=\mathrm{id}$, $\mathsf c_{t'}\mathsf c_{t''}=\mathsf c_{t't''}$, and the matrix entries of $\mathsf c_t$ are polynomials in t. Define the new Lie algebra structure on the vector space $\mathfrak q$ by

$$[x,y]_{(t)}:=\mathsf c_t^{-1}[\mathsf c_t(x),\mathsf c_t(y)]. \tag{1.1}$$

The corresponding Lie algebra is denoted by $\mathfrak q_{(t)}$. Then $\mathfrak q_{(1)}=\mathfrak q$ and all these algebras are isomorphic. If $\lim_{t\to 0}[x,y]_{(t)}$ exists for all $x,y\in\mathfrak q$, then we obtain a new Lie algebra, say $\mathfrak s$, which is a contraction of $\mathfrak q$. To express this fact, we write $\lim_{t\to 0}\mathfrak q_{(t)}=\mathfrak s$ or merely $\mathfrak q\rightsquigarrow\mathfrak s$. We identify $\mathfrak q$ and $\mathfrak s$ as vector spaces.

There is a relation between invariants of $\mathfrak q$ and $\mathfrak s$. Let $\Bbbk[\mathfrak q]_m\simeq\Bbbk[\mathfrak q_{(t)}]_m\simeq\Bbbk[\mathfrak s]_m$ be the space of polynomials of degree m and $\Bbbk[\mathfrak q_{(t)}]_m^{\mathfrak q_{(t)}}$ the subspace of invariants of the adjoint representation of $\mathfrak q_{(t)}$. Then $\dim\Bbbk[\mathfrak q_{(t)}]_m^{\mathfrak q_{(t)}}$ does not depend on $t\in\Bbbk^\times$. Therefore $\lim_{t\to 0}\Bbbk[\mathfrak q_{(t)}]_m^{\mathfrak q_{(t)}}$ exists in the appropriate Grassmannian and is clearly contained in $\Bbbk[\mathfrak s]_m^{\mathfrak s}$. Gathering together components of all degrees, we obtain the subalgebra

$$\lim_{t\to 0}(\Bbbk[\mathfrak q_{(t)}]^{\mathfrak q_{(t)}})\subset\Bbbk[\mathfrak s]^{\mathfrak s}.$$

It can be described more explicitly, as follows. For $F\in\Bbbk[\mathfrak q]_m^{\mathfrak q}$, set $F_{(t)}(x)=F(\mathsf c_t(x))$ and expand $F_{(t)}=\sum_{j=a}^b F_j{\cdot}t^j$ with $F_a\ne 0$ and $F_b\ne 0$. We say that F_a (resp. F_b) is the *initial* (resp. *highest*) component for F and write $F_\bullet$ (resp. $F^\bullet$) for it. Set $\mathcal L_\bullet(\Bbbk[\mathfrak q]_m^{\mathfrak q})=\{F_\bullet\mid F\in\Bbbk[\mathfrak q]_m^{\mathfrak q}\}$.

Proposition 1.1. *Given $F \in \Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m$, we have*

(i) $F_{(t)} \in \Bbbk[\mathfrak{q}_{(t)}]^{\mathfrak{q}_{(t)}}_m$ *for any* $t \in \Bbbk^\times$;
(ii) $F_\bullet \in (\lim_{t\to 0} \Bbbk[\mathfrak{q}_{(t)}]^{\mathfrak{q}_{(t)}}_m) \subset \Bbbk[\mathfrak{s}]^{\mathfrak{s}}_m$;
(iii) $\lim_{t\to 0} \Bbbk[\mathfrak{q}_{(t)}]^{\mathfrak{q}_{(t)}}_m = \mathcal{L}_\bullet(\Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m)$.

Proof. (i) Consider the representation of $\mathfrak{q}$ in $\Bbbk[\mathfrak{q}]$. Let $e * (?)$ denote the operator corresponding to $e \in \mathfrak{q}$. Then $F \in \Bbbk[\mathfrak{q}]^{\mathfrak{q}}$ if and only if $e * F = 0$ for any $e \in \mathfrak{q}$. The corresponding operator for $e \in \mathfrak{q}_{(t)}$ is denoted by $e *_{(t)} (?)$. Note that $\mathsf{c}_t : \mathfrak{q} \to \mathfrak{q}$ can be regarded as isomorphism of Lie algebras $\mathfrak{q}$ and $\mathfrak{q}_{(t)}$. Therefore $(e *_{(t)} F_{(t)})(x) = (\mathsf{c}_t(e) * F)(\mathsf{c}_t(x))$, which yields the assertion.

(ii) Consider $F_{(t)}$ as a curve in the projectivisation of $\Bbbk[\mathfrak{q}]_m$. Then $\lim_{t\to 0} F_{(t)} = F_\bullet$.

(iii) The inclusion "$\supset$" is already proved. Hence it suffices to show that $\dim \Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m = \dim \mathcal{L}_\bullet(\Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m)$. Consider the finite ascending filtration $\mathcal{F}_\bullet$ of $\Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m$:

$$\Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m = \mathcal{F}_0 \supset \cdots \supset \mathcal{F}_j \supset \dots,$$

where

$$\mathcal{F}_j := \{F \in \Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m \mid F_{(t)} = \sum_{j \geqslant i} F_j t^j\}.$$

Let $\mathcal{F}_N$ be the last nonzero term of $\mathcal{F}_\bullet$. Set $\mathrm{Gr}(\Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m) = \bigoplus_{i=0}^N \mathcal{F}_i/\mathcal{F}_{i+1}$. Then $\dim \Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m = \dim \mathrm{Gr}(\Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m)$, and for each i there is a natural linear mapping $\varphi_i : \mathcal{F}_i/\mathcal{F}_{i+1} \to \mathcal{L}_\bullet(\Bbbk[\mathfrak{q}]^{\mathfrak{q}}_m)$, $F + \mathcal{F}_{i+1} \mapsto F_\bullet$. Clearly, each φ_i is injective. Furthermore, $\varphi_i(\mathcal{F}_i/\mathcal{F}_{i+1})$ consists of polynomials of weight $(-i)$ with respect to the induced action of $\Bbbk^\times$ in $\Bbbk[\mathfrak{q}]_m$. Hence, the subspaces $\varphi_i(\mathcal{F}_i/\mathcal{F}_{i+1})$ are linearly independent. □

Similar results hold for coadjoint representations and algebras of invariants $\mathcal{S}(\mathfrak{q}_{(t)})^{\mathfrak{q}_{(t)}}$. The only notable difference is that for $F \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ one have to take the highest component $F^\bullet$.

2. Quasi-graded contractions and invariants

A Lie algebra $(\mathfrak{q}, [\ ,\ ])$ is said to be *quasi-graded* if there is a vector space decomposition $\mathfrak{q} = \bigoplus_{i=0}^{k-1} \mathfrak{q}_i$ such that $[\mathfrak{q}_i, \mathfrak{q}_j] \subset \mathfrak{q}_{i+j}$ whenever $i + j \leqslant k-1$. There are no conditions on $[\mathfrak{q}_i, \mathfrak{q}_j]$ if $i + j \geqslant k$. The family of subspaces $\Gamma = \{\mathfrak{q}_i\}_{i=0}^{k-1}$ is said to form a *quasi-graded structure (of order k)* on $\mathfrak{q}$. Define the new Lie algebra structure, $[\ ,\ ]_\Gamma$, on the vector space $\mathfrak{q}$ as follows: for $x_i \in \mathfrak{q}_i$, we set $[x_i, x_j]_\Gamma := \begin{cases} [x_i, x_j], & \text{if } i+j \leqslant k-1; \\ 0, & \text{if } i+j \geqslant k. \end{cases}$. The resulting $\mathbb{N}$-graded Lie algebra is denoted by $\mathfrak{C}_\Gamma(\mathfrak{q})$ or $\mathfrak{q}_0 \ltimes \mathfrak{q}_1 \ltimes \ldots \ltimes \mathfrak{q}_{k-1}$. It is a contraction of $\mathfrak{q}$

in the sense of Subsection 1.3. Indeed, for $t \in \Bbbk^\times$, define linear operators $\mathsf{c}_t : \mathfrak q \to \mathfrak q$ by $\mathsf{c}_t|_{\mathfrak q_i} = t^i{\cdot}\mathrm{id}$ and define $\mathfrak q_{(t)}$ as above. Then

$$[x_i, x_j]_{(t)} = \begin{cases} [x_i, x_j], & \text{if } i{+}j \leqslant k{-}1; \\ t{\cdot}(\text{a polynomial expression in } t), & \text{if } i{+}j \geqslant k. \end{cases}$$

It follows that $\lim_{t\to 0} \mathfrak q_{(t)} = \mathfrak C_\Gamma(\mathfrak q)$. The passage $\mathfrak q \rightsquigarrow \mathfrak C_\Gamma(\mathfrak q)$ as well as $\mathfrak C_\Gamma(\mathfrak q)$ itself is called the Γ-*contraction* or a *quasi-graded contraction* of $\mathfrak q$. We identify $\mathfrak q$ and $\mathfrak C_\Gamma(\mathfrak q)$ as graded vector spaces.

Example 2.1. Here we provide important examples of quasi-graded structures.

1) Let $\theta \in \mathrm{Aut}(\mathfrak q)$ be of order k. Then the corresponding $\mathbb Z_k$-grading of $\mathfrak q$ is a quasi-grading. The quasi-graded contraction associated with θ is said to be *cyclic* or a $\mathbb Z_k$-*contraction*. It is denoted by $\mathfrak C_\theta(\mathfrak q)$.

2) Let $\mathfrak h$ be a subalgebra of $\mathfrak q$. Suppose that there is an $\mathrm{ad}(\mathfrak h)$-stable subspace $\mathfrak m \subset \mathfrak q$ such that $\mathfrak q = \mathfrak h \oplus \mathfrak m$. Then $\Gamma = \{\mathfrak h, \mathfrak m\}$ is a quasi-graded structure of order 2 on $\mathfrak q$. The passage $\mathfrak q \rightsquigarrow \mathfrak h \ltimes \mathfrak m$ is called the $\mathfrak h$-*isotropy contraction* of $\mathfrak q$. Here $\mathfrak h \ltimes \mathfrak m$ is the *semi-direct product* of (the Lie algebra) $\mathfrak h$ and (the $\mathfrak h$-module) $\mathfrak m$.

Remark. Any $\mathbb Z_2$-contraction is an isotropy contraction; $\mathbb Z_2$-contractions of reductive Lie algebras have already been considered in [11, § 9] and [12].

There is a natural construction of certain quasi-gradings from periodic gradings.

Lemma 2.2. *Suppose $\theta \in \mathrm{Aut}(\mathfrak q)$ is of order k. Let $\mathfrak q_0$ be the corresponding fixed point subalgebra of $\mathfrak q$. Then $\mathfrak q_0 \dotplus \mathfrak q$ has a natural quasi-graded structure of order $k+1$. The corresponding quasi-graded contraction is isomorphic to $\mathfrak q_0 \ltimes \mathfrak q_1 \ltimes \ldots \ltimes \mathfrak q_{k-1} \ltimes \mathfrak q_0$.*

Proof. Consider the family of subspaces $\Gamma = \{\mathfrak s_j\}_{j=0}^k$ such that $\mathfrak s_0 = \{(x, x) \mid x \in \mathfrak q_0\} \subset \mathfrak q_0 \dotplus \mathfrak q_0 \subset \mathfrak q_0 \dotplus \mathfrak q$, $\mathfrak s_i = \mathfrak q_i \subset \mathfrak q$ for $i = 1, \ldots, k-1$, and $\mathfrak s_k = \mathfrak q_0 \subset \mathfrak q$. Then $\mathfrak q_0 \dotplus \mathfrak q = \bigoplus_{i=0}^k \mathfrak s_i$ and

$$[\mathfrak s_i, \mathfrak s_j] \subset \begin{cases} \mathfrak s_{i+j}, & \text{if } i+j \leqslant k, \\ \mathfrak s_{i+j-k}, & \text{if } i+j \geqslant k+1. \end{cases} \tag{2.1}$$

Therefore Γ is a quasi-grading. Obviously, $\mathfrak C_\Gamma(\mathfrak q_0 \dotplus \mathfrak q) \simeq \mathfrak q_0 \ltimes \mathfrak q_1 \ltimes \ldots \ltimes \mathfrak q_{k-1} \ltimes \mathfrak q_0$. □

Remark. The quasi-graded contraction in the lemma is not cyclic.

Below, we explicitly describe a connection between invariants of $\mathfrak q$ and $\mathfrak C_\Gamma(\mathfrak q)$. In the special case of $\mathbb Z_2$-contractions of reductive Lie algebras, such a connection is explained in [12, Proposition 3.1]. Discussing "invariants of $\mathfrak q$", we always mean invariants of the adjoint and coadjoint representations, i.e., the algebras $\Bbbk[\mathfrak q]^{\mathfrak q}$ and $\mathcal S(\mathfrak q)^{\mathfrak q}$. If an explicit formula for $\mathfrak q$ appears to be bulky, we also use notation $\mathsf{Inv}(\mathfrak q, \mathrm{ad})$ and $\mathsf{Inv}(\mathfrak q, \mathrm{ad}^*)$.

Any quasi-grading $\Gamma = \{\mathfrak{q}_i\}_{i=0}^{k-1}$ determines a polygrading of $\mathcal{S}(\mathfrak{q})$. Using the vector space decomposition $\mathfrak{q} = \bigoplus_{j=0}^{k-1} \mathfrak{q}_j$, any $F \in \mathcal{S}(\mathfrak{q})$ can be written as $F = \sum F_{i_0,i_1,\dots,i_{k-1}}$, where $F_{i_0,i_1,\dots,i_{k-1}} \in \bigotimes_{j=0}^{k-1} \mathcal{S}^{i_j}(\mathfrak{q}_j)$. Define a specialisation of this polygrading by

$$\deg_\Gamma(F_{i_0,i_1,\dots,i_{k-1}}) = i_1 + 2i_2 + \dots + (k-1)i_{k-1}.$$

We say that $\deg_\Gamma$ is the Γ-*degree* in $\mathcal{S}(\mathfrak{q})$. Note that the usual degree in $\mathcal{S}(\mathfrak{q})$ is defined by $\deg(F_{i_0,i_1,\dots,i_{k-1}}) = i_0 + i_1 + \dots + i_{k-1}$. If we refer to homogeneous polynomials, then the usual degree is meant, unless otherwise stated. If F is homogeneous, then $\deg(F) = \deg(\mathrm{gr}^\bullet F) = \deg(\mathrm{gr}_\bullet F)$. Using the dual decomposition of $\mathfrak{q}^*$, one similarly defines the Γ-degree for polynomials in $\Bbbk[\mathfrak{q}]$.

Definition 1. For $F \in \mathcal{S}(\mathfrak{q})$, let $\mathrm{gr}^\bullet F$ (resp. $\mathrm{gr}_\bullet F$) denote the component of F of the *maximal* (resp. *minimal*) Γ-degree. The same notation applies to $F \in \Bbbk[\mathfrak{q}]$.

As $\mathfrak{q}$ and $\mathfrak{C}_\Gamma(\mathfrak{q})$ are identified as vector spaces, each $F \in \mathcal{S}(\mathfrak{q})$ (resp. $F \in \Bbbk[\mathfrak{q}]$) can also be regarded as element of $\mathcal{S}(\mathfrak{C}_\Gamma(\mathfrak{q}))$ (resp. $\Bbbk[\mathfrak{C}_\Gamma(\mathfrak{q})]$).

Theorem 2.3. *Let $\Gamma = \{\mathfrak{q}_i\}_{i=0}^{k-1}$ be an arbitrary quasi-graded structure on $\mathfrak{q}$.*

(i) *If $F \in \mathcal{S}(\mathfrak{q})^\mathfrak{q}$, then $\mathrm{gr}^\bullet F \in \mathsf{Inv}(\mathfrak{C}_\Gamma(\mathfrak{q}), \mathrm{ad}^*)$.*
(ii) *If $F \in \Bbbk[\mathfrak{q}]^\mathfrak{q}$, then $\mathrm{gr}_\bullet F \in \mathsf{Inv}(\mathfrak{C}_\Gamma(\mathfrak{q}), \mathrm{ad})$.*

Proof. (i) Let $\{\ ,\ \}$ (resp. $\{\ ,\ \}_\Gamma$) denote the Poisson bracket in $\mathcal{S}(\mathfrak{q})$ (resp. $\mathcal{S}(\mathfrak{C}_\Gamma(\mathfrak{q}))$). As is well-known, $F \in \mathcal{S}(\mathfrak{q})^\mathfrak{q}$ if and only if $\{x, F\} = 0$ for any $x \in \mathfrak{q}$.

If $x_i \in \mathfrak{q}_i$, then $\deg_\Gamma(x_i) = i$. Therefore

$$\deg_\Gamma\{x_i, x_j\} \begin{cases} = i+j, & \text{if } i+j < k, \\ < i+j, & \text{if } i+j \geqslant k. \end{cases}$$

If $\deg_\Gamma(F_{i_0,i_1,\dots,i_{k-1}}) = m$, then $\deg_\Gamma(\{x_j, F_{i_0,i_1,\dots,i_{k-1}}\}_\Gamma) = m + j$. Furthermore, comparing the commutators in $\mathfrak{q}$ and $\mathfrak{C}_\Gamma(\mathfrak{q}) = \mathfrak{q}_0 \ltimes \dots \ltimes \mathfrak{q}_{k-1}$ shows that

$$\{x_j, F_{i_0,i_1,\dots,i_{k-1}}\} = \{x_j, F_{i_0,i_1,\dots,i_{k-1}}\}_\Gamma + (\text{terms of } \Gamma\text{-degree} < j + m).$$

It follows that $\{x_j, \mathrm{gr}^\bullet F\}_\Gamma$ is the component of the maximal possible Γ-degree in $\{x_j, F\}$. As $\{x_j, F\} = 0$, we also must have $\{x_j, \mathrm{gr}^\bullet F\}_\Gamma = 0$.

(ii) This follows from Proposition 1.1, since $\mathrm{gr}_\bullet F$ is the initial component of $F_{(t)}$ with respect to t. □

Remark. Proposition 1.1 can be adapted for obtaining another proof of part (i). In the context of quasi-graded contractions, we prefer to use notation $\mathrm{gr}_\bullet F$ in place of $F_\bullet$ (and likewise for $\mathrm{gr}^\bullet F$).

Recall that $\mathcal{L}_\bullet(\Bbbk[\mathfrak q]^{\mathfrak q}) := \{\mathrm{gr}_\bullet F \mid F \in \Bbbk[\mathfrak q]^{\mathfrak q}\}$ is a subalgebra of $\Bbbk[\mathfrak C_\Gamma(\mathfrak q)]^{\mathfrak C_\Gamma(\mathfrak q)}$, and it follows from Proposition 1.1(iii) that the graded algebras $\Bbbk[\mathfrak q]^{\mathfrak q}$ and $\mathcal{L}_\bullet(\Bbbk[\mathfrak q]^{\mathfrak q})$ have equal Poincaré series. Similarly, $\mathcal{L}^\bullet(\mathcal S(\mathfrak q)^{\mathfrak q})$ is a subalgebra of $\mathcal S(\mathfrak C_\Gamma(\mathfrak q))^{\mathfrak C_\Gamma(\mathfrak q)}$.

3. Periodic automorphisms of Takiff algebras and related contractions

Throughout this section, θ is a periodic automorphism of $\mathfrak q$ and $\zeta = \sqrt[|\theta|]{1}$; usually $|\theta| = k$.

Every periodic $\theta \in \mathrm{Aut}(\mathfrak q)$ can be extended to an automorphism of $\mathfrak q\langle m\rangle$ of the <u>same</u> order. There are at least two ways for doing so:

$$\widehat{\theta}|_{\mathfrak q[i]} = \theta;$$

$$\widehat{\theta}|_{\mathfrak q[i]} = \zeta^{-i}\theta. \tag{3.1}$$

In both cases, it is easily seen that $\widehat{\theta} \in \mathrm{Aut}(\mathfrak q\langle m\rangle)$. Notice that no relation between m and $|\theta|$ is required! In the first case, the fixed point subalgebra is again a Takiff algebra. Therefore we only work with the second case, which is certainly more interesting. That is, from now on, $\widehat{\theta}$ is defined by Eq. (3.1). Then the ζ^i-eigenspace of $\widehat{\theta}$ is

$$\mathfrak q\langle m\rangle_i = \mathfrak q_i \ltimes \mathfrak q_{i+1} \ltimes \dots \ltimes \mathfrak q_{i+m-1},$$

where $\mathfrak q_i$ is the ζ^i-eigenspace of θ and all subscripts are regarded as elements of $\mathbb Z_{|\theta|}$. In particular, the fixed point subalgebra for $\widehat{\theta}$ is

$$\mathfrak q\langle m\rangle^{\widehat{\theta}} = \mathfrak q\langle m\rangle_0 = \mathfrak q_0 \ltimes \mathfrak q_1 \ltimes \dots \ltimes \mathfrak q_{m-1}.$$

If $(\mathfrak q, B)$ is quadratic, then $\mathfrak q\langle m\rangle$ is also a quadratic Lie algebra. For, we can extend B to $\mathfrak q\langle m\rangle$ by letting

$$\hat B(\vec x, \vec y) = \sum_{i=0}^{m-1} B(x_i, y_{m-1-i})\ . \tag{3.2}$$

Lemma 3.1. *Suppose $(\mathfrak q, B)$ is quadratic and B is a θ-eigenvector with eigenvaue ζ^c. Then $\hat B$ is a $\widehat{\theta}$-eigenvector with eigenvalue ζ^{c+1-m}.*

Proof. This follows from Eq. (3.1) and (3.2). □

Thus, even if a scalar product on $\mathfrak q$ is θ-invariant, then extending it to a Takiff algebra $\mathfrak q\langle m\rangle$ we may obtain the scalar product with a non-trivial $\widehat{\theta}$-eigenvalue. Combining Lemma 3.1 and Subsection 1.1, we obtain

Corollary 3.2. *If $(\mathfrak q, B)$ is quadratic and B is a θ-eigenvector with eigenvaue ζ^c, then the dual $\mathfrak q\langle m\rangle_0$-module for $\mathfrak q\langle m\rangle_i$ is $\mathfrak q\langle m\rangle_{c+1-m-i}$. In particular, if $c = 0$, then both $(\mathfrak q_0, B|_{\mathfrak q_0})$ and $(\mathfrak q\langle nk + 1\rangle_0, \hat B)$, $n \in \mathbb N$, are also quadratic.*

We are going to study the algebra of invariants for the adjoint and coadjoint representations of algebras of the form $\mathfrak q\langle m\rangle_0$. Since $\mathfrak q\langle m\rangle_0^* \simeq \mathfrak q\langle m\rangle_{c+1-m}$, the coadjoint representation of $\mathfrak q\langle m\rangle_0$ can also be realised as $\widehat{\theta}$-representation.

Let us point out two important cases:

$$\text{If } m=k, \text{ then } \mathfrak q\langle k\rangle_0 = \mathfrak q_0 \ltimes \mathfrak q_1 \ltimes \ldots \ltimes \mathfrak q_{k-1},$$
$$\text{If } m=k+1, \text{ then } \mathfrak q\langle k+1\rangle_0 = \mathfrak q_0 \ltimes \mathfrak q_1 \ltimes \ldots \ltimes \mathfrak q_{k-1} \ltimes \mathfrak q_0.$$

In the first case, each $\mathfrak q_i$ occurs exactly once. Furthermore, if $c=0$, then $\mathfrak q\langle k\rangle_0^* \simeq \mathfrak q\langle k\rangle_1$ and $\mathfrak g\langle k+1\rangle_0$ is quadratic. The utility of these (and some other related) cases is that the fixed point subalgebra for $\widehat{\theta}$ appears to be a quasi-graded contraction.

Proposition 3.3. (i) *The passage $\mathfrak q \rightsquigarrow \mathfrak q\langle k\rangle_0$ is the $\mathbb Z_k$-contraction associated with θ;* (ii) *The passage $\mathfrak q_0 \dotplus \mathfrak q \rightsquigarrow \mathfrak q\langle k+1\rangle^{\widehat{\theta}} = \mathfrak q\langle k+1\rangle_0$ is a quasi-graded contraction of order $k+1$.*

Proof. (i) is obvious; (ii) follows from Lemma 2.2. □

Recall that the cyclic contraction of $\mathfrak q$ associated with θ is denoted by $\mathfrak C_\theta(\mathfrak q)$. Therefore Proposition 3.3(i) can be expressed as the equality $\mathfrak C_\theta(\mathfrak q) \simeq \mathfrak q\langle|\theta|\rangle^{\widehat{\theta}}$. To generalise previous observations, we need some preparations.

Lemma 3.4. *Let θ be a periodic automorphism of $\mathfrak q$. For any $n\in\mathbb N$, there is a periodic automorphism $\widetilde{\theta}$ of $n\mathfrak q := \mathfrak q \dotplus \ldots \dotplus \mathfrak q$ (n summands) such that $|\widetilde{\theta}| = n|\theta|$. Furthermore, for any $j\in\mathbb Z_{n|\theta|}$, we have $(n\mathfrak q)_j \simeq \mathfrak q_{\bar j}$, where $\bar j$ is the image of j in $\mathbb Z_{|\theta|}$.*

Proof. Let $k=|\theta|$ and $\zeta = \sqrt[k]{1}$. Using the $\mathbb Z_k$-grading $\mathfrak q = \bigoplus_{i\in\mathbb Z_k}\mathfrak q_i$, we consider $n\mathfrak q$ as the direct sum of spaces $n(\mathfrak q_i) = \mathfrak q_i \dotplus \ldots \dotplus \mathfrak q_i$. All these spaces are to be $\widetilde{\theta}$-stable, and we define $\widetilde{\theta}$ for each i separately. For $(a_1,\ldots,a_n)\in n(\mathfrak q_i)$, set

$$\widetilde{\theta}(a_1,\ldots,a_n) = (a_2,\ldots,a_n,\zeta^i a_1).$$

Obviously, $\widetilde{\theta}\in \mathrm{Aut}(n\mathfrak q)$ and $\widetilde{\theta}^n|_{n(\mathfrak q_i)} = \zeta^i{\cdot}\mathrm{id}$. Hence $|\widetilde{\theta}| = nk$. To describe the eigenspaces of $\widetilde{\theta}$, we choose a primitive root $\mu = \sqrt[nk]{1}$ such that $\mu^n=\zeta$. Write $(n\mathfrak q)_j$ for the μ^j-eigenspace of $\widetilde{\theta}$. We claim that

$$(n\mathfrak q)_j = \{x, \mu^j x, \ldots, \mu^{(n-1)j}x \mid x\in\mathfrak q_{\bar j}\}.$$

Indeed, let $\mathfrak r_j$ denote the right-hand side. It is easily seen that $\mathfrak r_j \subset (n\mathfrak q)_j$, $\mathfrak r_j \simeq \mathfrak q_{\bar j}$, and $\mathfrak r_j \subset n(\mathfrak q_{\bar j})$. Moreover, for any $j\in\{0,1,\ldots,k-1\}$, the sum $\sum_{l=0}^{n-1}\mathfrak r_{j+lk}$ is direct (use the Vandermonde determinant!). Whence $n\mathfrak q = \bigoplus_{j=0}^{nk-1}\mathfrak r_j$, and we are done. □

Our general result on contractions associated with Takiff algebras is

Theorem 3.5. *Given a periodic $\theta\in\mathrm{Aut}(\mathfrak q)$ and $n\in\mathbb N$, consider $\widetilde{\theta}\in\mathrm{Aut}(n\mathfrak q)$ as above. Then*

(i) *For* $\widehat{\theta} \in \mathrm{Aut}(\mathfrak{q}\langle n|\theta|\rangle$, *the fixed point subalgebra* $\mathfrak{q}\langle n|\theta|\rangle^{\widehat{\theta}}$ *is the cyclic contraction of* $n\mathfrak{q}$ *associated with* $\widetilde{\theta}$, *i.e.,* $\mathfrak{q}\langle n|\theta|\rangle^{\widehat{\theta}} \simeq \mathfrak{C}_{\widetilde{\theta}}(n\mathfrak{q})$;

(ii) *For* $\widehat{\theta} \in \mathrm{Aut}(\mathfrak{q}\langle n|\theta|+1\rangle$, *the fixed point subalgebra* $\mathfrak{q}\langle n|\theta|+1\rangle^{\widehat{\theta}}$ *is a quasi-graded contraction (of order* $nk+1$*) of* $\mathfrak{q}_0 \dotplus n\mathfrak{q}$.

Proof. This is an immediate consequence of our previous results. First, we consider the $\mathbb{Z}_{nk}$-grading of $n\mathfrak{q}$ constructed in Lemma 3.4, which yields (i). For (ii), we endow $\mathfrak{q}_0 \dotplus n\mathfrak{q}$ with the quasi-graded structure of order $nk+1$ using Lemmas 2.2 and 3.4. We also need the fact that $(n\mathfrak{q})_j \simeq \mathfrak{q}_{\bar{j}}$ for any $j = 0, 1, \dots, nk-1$. □

Corollary 3.6. *Let* $\widetilde{\theta}$ *be the cyclic permutation of the summands in* $n\mathfrak{q}$. *Then* $\mathfrak{q}\langle n\rangle \simeq \mathfrak{C}_{\widetilde{\theta}}(n\mathfrak{q})$.

Proof. This is the particular case of Theorem 3.5(i) with $\theta = \mathrm{id}$ and hence $\widehat{\theta} = \mathrm{id}$. □

Note that Proposition 3.3 corresponds to the case $n = 1$ in Theorem 3.5.

Remark 3.7. The $\mathbb{Z}_{nk}$-contraction of $n\mathfrak{q}$ is the $\mathbb{N}$-graded algebra $(n\mathfrak{q})_0 \ltimes (n\mathfrak{q})_1 \ltimes \dots \ltimes (n\mathfrak{q})_{nk-1}$, but using the isomorphisms $(n\mathfrak{q})_j \simeq \mathfrak{q}_{\bar{j}}$ we can write it as $\mathfrak{q}\langle nk\rangle^{\widehat{\theta}} = \mathfrak{q}_0 \ltimes \mathfrak{q}_1 \ltimes \dots \ltimes \mathfrak{q}_{k-1}$, where each $\mathfrak{q}_i$ occurs n times. Yet, one should not forget that different summands $\mathfrak{q}_i$ in the last expression arise from different subspaces of $n\mathfrak{q}$. For future use, we record the fact that the subalgebra $\mathfrak{q}_0$ in $\mathfrak{q}\langle nk\rangle^{\widehat{\theta}}$ or $\mathfrak{q}\langle nk+1\rangle^{\widehat{\theta}}$ (the very first summand) corresponds under the isomorphisms of Lemma 3.4 and Theorem 3.5 to the diagonally embedded subalgebra $\mathfrak{q}_0$ in $n(\mathfrak{q}_0) \subset n\mathfrak{q}$ or in $(n+1)\mathfrak{q}_0 \subset \mathfrak{q}_0 \dotplus n\mathfrak{q}$.

Combining Theorems 2.3 and 3.5, we obtain

Theorem 3.8. *Let* $\theta \in \mathrm{Aut}(\mathfrak{q})$ *be periodic and* $n \in \mathbb{N}$. *Consider the graded structure of* $n\mathfrak{q}$ *and quasi-graded structure of* $\mathfrak{q}_0 \dotplus n\mathfrak{q}$ *determined by* θ.

(i) *If* $F \in \mathsf{Inv}(n\mathfrak{q}, \mathrm{ad}^*)$, *then* $\mathrm{gr}^\bullet F \in \mathsf{Inv}(\mathfrak{q}\langle n|\theta|\rangle_0, \mathrm{ad}^*)$;

(ii) *If* $F \in \mathsf{Inv}(n\mathfrak{q}, \mathrm{ad})$, *then* $\mathrm{gr}_\bullet F \in \mathsf{Inv}(\mathfrak{q}\langle n|\theta|\rangle_0, \mathrm{ad})$.

(iii) *If* $F \in \mathsf{Inv}(\mathfrak{q}_0 \dotplus n\mathfrak{q}, \mathrm{ad}^*)$, *then* $\mathrm{gr}^\bullet F \in \mathsf{Inv}(\mathfrak{q}\langle n|\theta|+1\rangle_0, \mathrm{ad}^*)$;

(iv) *If* $F \in \mathsf{Inv}(\mathfrak{q}_0 \dotplus n\mathfrak{q}, \mathrm{ad})$, *then* $\mathrm{gr}_\bullet F \in \mathsf{Inv}(\mathfrak{q}\langle n|\theta|+1\rangle_0, \mathrm{ad})$.

Let us point out possible applications of these procedures. For definiteness, consider part (ii). Taking each (homogeneous) $F \in \Bbbk[n\mathfrak{q}]^{n\mathfrak{q}} \simeq \mathsf{Inv}(\mathfrak{q}, \mathrm{ad})^{\otimes n}$ to $\mathrm{gr}_\bullet F$, we obtain the subalgebra of $\mathsf{Inv}(\mathfrak{q}\langle n|\theta|\rangle_0, \mathrm{ad})$, which is denoted by $\mathcal{L}_\bullet(\Bbbk[n\mathfrak{q}]^{n\mathfrak{q}})$. We know that $\Bbbk[n\mathfrak{q}]^{n\mathfrak{q}}$ and $\mathcal{L}_\bullet(\Bbbk[n\mathfrak{q}]^{n\mathfrak{q}})$ have the same Poincaré series with respect to the usual degree. If one knows somehow that $\Bbbk[n\mathfrak{q}]^{n\mathfrak{q}}$ and $\mathsf{Inv}(\mathfrak{q}\langle n|\theta|\rangle_0, \mathrm{ad})$ also have equal Poincaré series, then we obtain the equality $\mathcal{L}_\bullet(\Bbbk[n\mathfrak{q}]^{n\mathfrak{q}}) = \mathsf{Inv}(\mathfrak{q}\langle n|\theta|\rangle_0, \mathrm{ad})$. This sometimes allows us to prove that good properties of $\mathsf{Inv}(\mathfrak{q}, \mathrm{ad})$ are carried to $\mathsf{Inv}(\mathfrak{q}\langle nk\rangle_0, \mathrm{ad})$ over. Instances of such a phenomenon are discussed in the following sections.

4. Good cases for adjoint representations

From now on, our initial object is a reductive Lie algebra $\mathfrak g$ (in place of an arbitrary $\mathfrak q$). As above, it is assumed that $\theta\in\mathrm{Aut}(\mathfrak g)$ and $|\theta|=k$. If an algebra of invariants appears to be graded polynomial, then the elements of any set of algebraically independent homogeneous generators will be referred to as *basic invariants*. E.g., one can consider basic invariants in $\mathcal S(\mathfrak g)^{\mathfrak g}=\mathcal S(\mathfrak g)^G$ or $\Bbbk[\mathfrak g_1]^{G_0}$.

Let us start with some remarks concerning the reductive case. We assume that the scalar product on $\mathfrak g$ is θ-invariant. Hence $c=0$ in Corollary 3.2 and $\mathfrak g\langle nk+1\rangle_0$ is quadratic for any $n\in\mathbb N$. Recall that $\mathfrak g[0]\simeq\mathfrak g$ is a Levi subalgebra of $\mathfrak g\langle m\rangle$ and $\mathfrak R_u(\mathfrak g\langle m\rangle)=0\ltimes\mathfrak g\ltimes\ldots\ltimes\mathfrak g$. Similarly, $\mathfrak g_0\subset\mathfrak g[0]$ is a Levi subalgebra of $\mathfrak g\langle m\rangle_0$ and $\mathfrak R_u(\mathfrak g\langle m\rangle_0)=0\ltimes\mathfrak g_1\ltimes\ldots\ltimes\mathfrak g_{m-1}$.

Let $\mathfrak N$ be the nilpotent cone of $\mathfrak g$ and $\mathcal O^{reg}\subset\mathfrak N$ the regular nilpotent G-orbit.

Theorem 4.1. *Suppose θ has the property that $\mathfrak g_0\cap\mathcal O^{reg}\neq\varnothing$ and $n\in\mathbb N$ is arbitrary. Then*

(i) $\mathcal L_\bullet(\Bbbk[n\mathfrak g]^{n\mathfrak g})=\mathsf{Inv}(\mathfrak g\langle nk\rangle_0,\mathrm{ad})$ *and* $\mathsf{Inv}(\mathfrak g\langle nk\rangle_0,\mathrm{ad})$ *is a polynomial algebra of Krull dimension $n{\cdot}\mathrm{rk}\,\mathfrak g$.*

(ii) $\mathcal L_\bullet(\Bbbk[n\mathfrak g\dotplus\mathfrak g_0]^{n\mathfrak g\dotplus\mathfrak g_0})=\mathsf{Inv}(\mathfrak g\langle nk{+}1\rangle_0,\mathrm{ad})$ *and* $\mathsf{Inv}(\mathfrak g\langle nk{+}1\rangle_0,\mathrm{ad})$ *is a polynomial algebra of Krull dimension $n{\cdot}\mathrm{rk}\,\mathfrak g+\mathrm{rk}\,\mathfrak g_0$.*

Proof. (i) Consider the chain of Lie algebra contractions:

$$n\mathfrak g\underset{(1)}{\rightsquigarrow}\mathfrak g\langle nk\rangle_0=\mathfrak g_0\ltimes\mathfrak g_1\ltimes\ldots\ltimes\mathfrak g_{k-1}\underset{(2)}{\rightsquigarrow}\mathfrak g_0\ltimes(\underbrace{\mathfrak g_1\oplus\ldots\oplus\mathfrak g_{k-1}}_{nk-1})=:\mathfrak h\ ,$$

where "$\underset{(1)}{\rightsquigarrow}$" is constructed in Theorem 3.5(i) and "$\underset{(2)}{\rightsquigarrow}$" stands for the $\mathfrak g_0$-isotropy contraction of $\mathfrak g\langle nk\rangle_0$. In other words, $\mathfrak R_u(\mathfrak g\langle nk\rangle_0)$ is proclaimed to be abelian in $\mathfrak h$. The direct passage $n\mathfrak g\rightsquigarrow\mathfrak h$ can also be regarded as the $\mathfrak g_0$-isotropy contraction of $n\mathfrak g$, where $\mathfrak g_0$ is the diagonally embedded subalgebra $\mathfrak g_0\subset n(\mathfrak g_0)\subset n\mathfrak g$ (cf. Remark 3.7). Consider the corresponding transformations of algebras of invariants

$$\Bbbk[n\mathfrak g]^{n\mathfrak g}\rightsquigarrow\mathcal L^{(1)}_\bullet(\Bbbk[n\mathfrak g]^{n\mathfrak g})\hookrightarrow\mathsf{Inv}(\mathfrak g\langle nk\rangle_0,\mathrm{ad})\rightsquigarrow\mathcal L^{(2)}_\bullet(\mathsf{Inv}(\mathfrak g\langle nk\rangle_0,\mathrm{ad}))\hookrightarrow\Bbbk[\mathfrak h]^{\mathfrak h}. \tag{4.1}$$

Here the functor $\mathcal L^{(i)}_\bullet$ corresponds to the contraction $\underset{(i)}{\rightsquigarrow}$ and each arrow "$\rightsquigarrow$" preserves the Poincaré series. By [11, Theorem 6.2], the algebra $\Bbbk[\mathfrak h]^{\mathfrak h}$ is polynomial. That is, both extreme algebras are polynomial. Furthermore, the degrees of the basic invariants for both are the same [11, Theorem 9.5(2)]. (The crucial property is that the Levi subalgebra $\mathfrak g_0\subset\mathfrak h$ arises from the diagonally embedded $\mathfrak g_0\subset n(\mathfrak g_0)\subset n\mathfrak g$, and therefore a regular nilpotent element of $\mathfrak g_0$ is still regular in $n\mathfrak g$.)

The equality for the degrees implies that both embeddings in Eq. (4.1) are actually isomorphisms, and all the algebras involved have one and the same Poincaré series. Let $F_1,\ldots,F_l$ be the basic invariants in $\Bbbk[\mathfrak h]^{\mathfrak h}$. Then there are homogeneous $\tilde F_i\in\Bbbk[n\mathfrak g]^{n\mathfrak g}$ such that $\mathrm{gr}^{(2)}_{\bullet}(\mathrm{gr}^{(1)}_{\bullet}(\tilde F_i)) = F_i$ for all i. It is then easily seen that $\{\tilde F_i\}$ and $\{\mathrm{gr}^{(1)}_{\bullet}\tilde F_i\}$ must be algebraically independent. Since $\deg F_i = \deg(\mathrm{gr}^{(1)}_{\bullet}\tilde F_i)$, $\mathsf{Inv}(\mathfrak g\langle nk\rangle_0, \mathrm{ad})$ is freely generated by the $\{\mathrm{gr}^{(1)}_{\bullet}\tilde F_i\}$'s.

(ii) We begin with the chain of Lie algebra contractions:

$$\mathfrak g_0 \dotplus n\mathfrak g \underset{(1)}{\rightsquigarrow} \mathfrak g\langle nk+1\rangle_0 = \mathfrak g_0 \ltimes \mathfrak g_1 \ltimes \ldots \ltimes \mathfrak g_{k-1} \ltimes \mathfrak g_0 \underset{(2)}{\rightsquigarrow} \mathfrak g_0 \ltimes (\underbrace{\mathfrak g_1 \oplus \ldots \oplus \mathfrak g_{k-1} \oplus \mathfrak g_0}_{nk}) .$$

The rest is essentially the same as in part (i). □

The list of θ satisfying the assumption $\mathfrak g_0 \cap \mathcal O^{reg} \ne \varnothing$ is not long. For simple $\mathfrak g$, all possible pairs $\mathfrak g \supset \mathfrak g_0$ are pointed out below:

$$\begin{aligned} &|\theta| = 2: \quad \mathfrak{sl}_{2n} \supset \mathfrak{sp}_{2n},\ \mathfrak{sl}_{2n+1} \supset \mathfrak{so}_{2n+1},\ \mathfrak{so}_{2n} \supset \mathfrak{so}_{2n-1},\ \mathbf{E}_6 \supset \mathbf{F}_4; \\ &|\theta| = 3: \quad \mathbf{D}_4 \supset \mathbf{G}_2. \end{aligned} \tag{4.2}$$

For semisimple Lie algebras, the only new possibility is the cyclic permutation of summands in $n\mathfrak g$, which leads to $\mathfrak g\langle n\rangle$, cf. Corollary 3.6.

Example 4.2. We give realisations of algebras $\mathfrak g\langle nk\rangle_0$ and $\mathfrak g\langle nk+1\rangle_0$, associated with the list in (4.2), as centralisers of nilpotent elements. If θ is an involution, then there are only two eigenspaces, $\mathfrak g_0$ and $\mathfrak g_1$, and we will use the more suggestive notation $\mathfrak L_m(\mathfrak g_0, \mathfrak g_1)$ in place of $\mathfrak g\langle m\rangle_0$. For $\mathfrak g$ simple, $\mathfrak L_m(\mathfrak g_0, \mathfrak g_1)$ is quadratic if and only if m is odd. Irreducible $\mathfrak g_0$-modules occurring in $\mathfrak g_1$ are depicted by their highest weights. Namely, $\mathsf R(\lambda)$ is a simple module with highest weight λ. The i-th fundamental weight of a simple Lie algebra is denoted by ϖ_i, with the numbering from [16]. We write $1\!\!1$ for the trivial 1-dimensional module. The symbol $\mathfrak z_n$ stands for the n-dimensional centre. Items 1^o–4^o below provide realisations of the algebras $\mathfrak L_m(\mathfrak g_0, \mathfrak g_1)$ associated with the *outer* involutions of $\mathfrak g = \mathfrak{sl}_N$ or $\mathfrak{gl}_N$.

- 1^o. Let $e \in \tilde{\mathfrak g} = \mathfrak{sp}_{2nm}$ be a nilpotent element with partition $(\underbrace{2m, 2m, \ldots, 2m}_{n}) =: ((2m)^n)$. Then $\tilde{\mathfrak g}^e \simeq \mathfrak L_{2m}(\mathfrak{so}_n, \mathsf R(2\varpi_1) \oplus 1\!\!1) \simeq \mathfrak L_{2m}(\mathfrak{so}_n, \mathsf R(2\varpi_1)) \dotplus \mathfrak z_m$. Here $\mathfrak{so}_n \oplus (\mathsf R(2\varpi_1) \oplus 1\!\!1) = \mathfrak{gl}_n$ and $\tilde{\mathfrak g}^e$ is a contraction of $m(\mathfrak{gl}_n)$.
- 2^o. Let $e \in \tilde{\mathfrak g} = \mathfrak{so}_{n(2m+1)}$ be a nilpotent element with partition $((2m+1)^n)$. Then $\tilde{\mathfrak g}^e \simeq \mathfrak L_{2m+1}(\mathfrak{so}_n, \mathsf R(2\varpi_1) \oplus 1\!\!1) \simeq \mathfrak L_{2m+1}(\mathfrak{so}_n, \mathsf R(2\varpi_1)) \dotplus \mathfrak z_m$. Here $m(\mathfrak{gl}_n) \dotplus \mathfrak{so}_n \rightsquigarrow \tilde{\mathfrak g}^e$.
- 3^o. Let $e \in \tilde{\mathfrak g} = \mathfrak{so}_{4nm}$ be a nilpotent element with partition $((2m)^{2n})$. Then $\tilde{\mathfrak g}^e \simeq \mathfrak L_{2m}(\mathfrak{sp}_{2n}, \mathsf R(\varpi_2) \oplus 1\!\!1) \simeq \mathfrak L_{2m}(\mathfrak{sp}_{2n}, \mathsf R(\varpi_2)) \dotplus \mathfrak z_m$. Here $\mathfrak{sp}_{2n} \oplus (\mathsf R(\varpi_2) \oplus 1\!\!1) = \mathfrak{gl}_{2n}$ and $\tilde{\mathfrak g}^e$ is a contraction of $m(\mathfrak{gl}_{2n})$.
- 4^o. Let $e \in \tilde{\mathfrak g} = \mathfrak{sp}_{2n(2m+1)}$ be a nilpotent element with partition $((2m+1)^{2n})$. Then $\tilde{\mathfrak g}^e \simeq \mathfrak L_{2m+1}(\mathfrak{sp}_{2n}, \mathsf R(\varpi_2) \oplus 1\!\!1) \simeq \mathfrak L_{2m+1}(\mathfrak{sp}_{2n}, \mathsf R(\varpi_2)) \dotplus \mathfrak z_m$. Here $m(\mathfrak{gl}_{2n}) \dotplus \mathfrak{sp}_{2n} \rightsquigarrow \tilde{\mathfrak g}^e$.
- 5^o. Let $e \in \tilde{\mathfrak g} = \mathfrak{so}_{2n+2}$ be a nilpotent element with partition $(3, 1^{2n-1})$. Then $\tilde{\mathfrak g}^e \simeq (\mathfrak{so}_{2n-1} \ltimes \mathsf R(\varpi_1)) \dotplus \Bbbk e$, where the first summand is a contraction of $\mathfrak{so}_{2n}$.

6^o. Let $e \in \tilde{\mathfrak{g}} = \mathbf{E}_7$ be a nilpotent element with weighted Dynkin diagram $\left(\begin{array}{ccccccccccc} 2&-&0&-&0&-&0&-&0&-&0 \\ &&&&&&|&&&& \\ &&&&&&0&&&& \end{array}\right)$. Then $\tilde{\mathfrak{g}}^e = (\mathbf{F}_4 \ltimes \mathsf{R}(\varpi_1)) \dotplus \Bbbk e$.

7^o. For $n = 6, 7, 8$, let $e \in \tilde{\mathfrak{g}} = \mathbf{E}_n$ be nilpotent elements with weighted Dynkin diagrams $\left(\begin{array}{ccccccccc} 2&-&0&-&0&-&0&-&2 \\ &&&&|&&&& \\ &&&&0&&&& \end{array}\right)$, $\left(\begin{array}{ccccccccccc} 2&-&2&-&0&-&0&-&0&-&2 \\ &&&&&&|&&&& \\ &&&&&&0&&&& \end{array}\right)$, $\left(\begin{array}{ccccccccccccc} 2&-&2&-&2&-&0&-&0&-&0&-&2 \\ &&&&&&&&|&&&& \\ &&&&&&&&0&&&& \end{array}\right)$, respectively. Then $\tilde{\mathfrak{g}}^e \simeq (\mathbf{G}_2 \ltimes \mathsf{R}(\varpi_1) \ltimes \mathsf{R}(\varpi_1)) \dotplus \mathfrak{z}_{n-4}$, where the first summand is a $\mathbb{Z}_3$-contraction of $\mathbf{D}_4$.

Theorem 4.1 applies to items 3^o–7^o, and to 1^o–2^o if n is odd. The centralisers in 2^o and 4^o are quadratic.

Remark 4.3. If $|\theta| = 2$ and $n = 1$, then the algebra $\mathsf{Inv}(\mathfrak{g}_0 \ltimes \mathfrak{g}_1, \mathrm{ad})$ is always polynomial and the basic invariants can explicitly be described [11, Theorem 6.2]. However, the equality $\mathsf{Inv}(\mathfrak{g}_0 \ltimes \mathfrak{g}_1, \mathrm{ad}) = \mathrm{gr}_\bullet(\Bbbk[\mathfrak{g}]^{\mathfrak{g}})$ holds if and only if $\mathfrak{g}_0 \cap \mathcal{O}^{reg} \neq \varnothing$.

5. Good cases for coadjoint representations

In this section, we prove a counterpart of Theorem 4.1 for invariants of coadjoint representations of $\mathfrak{g}\langle nk\rangle_0$. To this end, we need some preparations.

The *index* of a Lie algebra $\mathfrak{q}$, $\mathrm{ind}\,\mathfrak{q}$, is the minimal dimension of stabilisers of elements of $\mathfrak{q}^*$ with respect to the coadjoint representation. Let $\mathfrak{q}^*_{reg}$ be the set of regular elements of $\mathfrak{q}^*$, i.e., those with minimal dimension of the stabiliser. We say that $\mathfrak{q}$ has the *codim–2* (resp. *codim–3*) property, if $\mathrm{codim}\,(\mathfrak{q}^* \setminus \mathfrak{q}^*_{reg}) \geqslant 2$ (resp. $\geqslant 3$). Set $b(\mathfrak{q}) = (\dim \mathfrak{q} + \mathrm{ind}\,\mathfrak{q})/2$. We will need the following result, which is explicitly stated in [12, Thm. 1.2] and based on an earlier work of Odesskii-Rubtsov [7].

Theorem 5.1. *Suppose $\mathfrak{q}$ has the codim–2 property and $\mathrm{trdeg}\,\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathrm{ind}\,\mathfrak{q}$. Set $l = \mathrm{ind}\,\mathfrak{q}$. Let $f_1, \dots, f_l \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ be arbitrary homogeneous algebraically independent polynomials. Then*

- (i) $\sum_{i=1}^{l} \deg f_i \geqslant b(\mathfrak{q})$;
- (ii) *If $\sum_{i=1}^{l} \deg f_i = b(\mathfrak{q})$, then $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is freely generated by $f_1, \dots, f_l$ and $\xi \in \mathfrak{q}^*_{reg}$ if and only if $(df_1)_\xi, \dots, (df_l)_\xi$ are linearly independent.*

In order to apply this result to algebras $\mathfrak{g}\langle nk\rangle_0$, we must have the *codim–2* property and know the index of $\mathfrak{g}\langle nk\rangle_0$.

Given a periodic $\theta \in \mathrm{Aut}(\mathfrak{g})$, we have the flat quotient morphism $\pi : \mathfrak{g}_1 \to \mathfrak{g}_1/\!\!/G_0$ (see Introduction). The fibre $\mathfrak{N}_1 := \pi^{-1}(\pi(0))$ consists of all nilpotent elements in $\mathfrak{g}_1$. It can be reducible; moreover, if $|\theta| \geqslant 3$, then some components can be reduced, while some other not.

Let us say that θ (or the corresponding $\mathbb{Z}_k$-grading) is

- $\mathcal{S}$-*regular*, if $\mathfrak{g}_1$ contains regular semisimple elements of $\mathfrak{g}$;

- $\mathcal{N}$-*regular*, if $\mathfrak{g}_1$ contains regular nilpotent elements of $\mathfrak{g}$ (i.e., $\mathfrak{g}_1 \cap \mathcal{O}^{reg} \neq \varnothing$);
- *very* $\mathcal{N}$-*regular*, if $\mathcal{O}^{reg}$ meets each irreducible component of $\mathfrak{N}_1$.

Some structure results on $\mathcal{S}$- and $\mathcal{N}$-regular gradings are obtained in [10].

Lemma 5.2. *Suppose θ is both $\mathcal{S}$-regular and very $\mathcal{N}$-regular. Then* $\operatorname{codim}(\mathfrak{g}_1 \setminus (\mathfrak{g}_1 \cap \mathfrak{g}_{reg})) \geqslant 2$.

Proof. Let $\pi^{-1}(\beta) = \mathcal{F}_\beta$ be an arbitrary fibre of π and $\mathcal{O}_\beta$ the dense G_0-orbit in an irreducible component of $\mathcal{F}_\beta$. Since $\dim \mathcal{F}_\beta = \dim \mathfrak{N}_1$, the *associated cone* of $\mathcal{O}_\beta$ (see [2, § 3]) is an irreducible component of $\mathfrak{N}_1$. Then the assumption on $\mathfrak{N}_1$ implies that $\mathcal{O}_\beta \subset \mathfrak{g}_{reg}$. In other words, if $\mathcal{O} \subset \mathfrak{g}_1$ is a G_0-orbit of maximal dimension, then $\mathcal{O} \subset \mathfrak{g}_{reg}$. The assumption on regular semisimple elements shows that a generic fibre of π is a (closed) G_0-orbit, i.e., the whole such fibre belongs to $\mathfrak{g}_1 \cap \mathfrak{g}_{reg}$. The union of all other fibres is a proper subvariety of $\mathfrak{g}_1$ (actually, it is a divisor). However, the open G_0-orbits in all other fibres belong to $\mathfrak{g}_{reg}$, too. Hence the complement is of codimension at least 2, as required. □

Proposition 5.3. (i) *Suppose that $\mathfrak{g}_1 \cap \mathfrak{g}_{reg} \neq \varnothing$. Then* $\operatorname{ind} \mathfrak{g}\langle nk\rangle_0 = n{\cdot}\operatorname{rk}\mathfrak{g}$.
(ii) *If θ is both $\mathcal{S}$-regular and very $\mathcal{N}$-regular, then $\mathfrak{g}\langle nk\rangle_0$ has the codim–2 property.*

Proof. (i) Set $\mathfrak{q} = \mathfrak{g}\langle nk\rangle_0$. Then $\mathfrak{q}^* = \mathfrak{g}\langle nk\rangle_1$. Recall that

$$\mathfrak{g}\langle nk\rangle_0 = \mathfrak{g}_0 \ltimes \mathfrak{g}_1 \ltimes \ldots \ltimes \mathfrak{g}_{k-1}$$
$$\mathfrak{g}\langle nk\rangle_1 = \mathfrak{g}_1 \ltimes \mathfrak{g}_2 \ltimes \ldots \ltimes \mathfrak{g}_0 \qquad (nk \text{ factors in both cases})$$

Take any $x \in \mathfrak{g}_1 \cap \mathfrak{g}_{reg}$. Write $\mathfrak{g}_i^x$ (resp. $\mathfrak{g}^x$) for the centraliser of x in $\mathfrak{g}_i$ (resp. $\mathfrak{g}$). Then $\mathfrak{g}^x = \oplus_{i=0}^{k-1}\mathfrak{g}_i^x$ and $\dim \mathfrak{g}^x = \operatorname{rk}\mathfrak{g}$. Consider $\boldsymbol{\xi} = (x, 0, \ldots, 0)$ as element of $\mathfrak{q}^*$. Then the stabiliser of $\boldsymbol{\xi}$ in $\mathfrak{q}$ is $\mathfrak{g}_0^x \ltimes \mathfrak{g}_1^x \ltimes \cdots \ltimes \mathfrak{g}_{k-1}^x$, i.e., we get $\mathfrak{g}_i^x$ inside every component $\mathfrak{g}_i$ occurring in $\mathfrak{q}$. Since each $\mathfrak{g}_i$ occurs n times, $\dim \mathfrak{q}^{\boldsymbol{\xi}} = n{\cdot}\operatorname{rk}\mathfrak{g}$. Therefore $\operatorname{ind}\mathfrak{q} \leqslant n{\cdot}\operatorname{rk}\mathfrak{g}$. On the other hand, $\mathfrak{q}$ is a contraction of $n\mathfrak{g}$ (Theorem 3.5(i)). Hence $\operatorname{ind}\mathfrak{q} \geqslant \operatorname{ind}(n\mathfrak{g}) = n{\cdot}\operatorname{rk}\mathfrak{g}$. Notice that we also proved that if $x \in \mathfrak{g}_1 \cap \mathfrak{g}_{reg}$, then $\boldsymbol{\xi} = (x, 0, \ldots, 0) \in \mathfrak{q}^*_{reg}$.

(ii) Set $\mathfrak{g}_1^{reg} = \mathfrak{g}_1 \cap \mathfrak{g}_{reg}$. By Lemma 5.2, we have $\operatorname{codim}(\mathfrak{g}_1 \setminus \mathfrak{g}_1^{reg}) \geqslant 2$. Now, let $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_{nk}) \in \mathfrak{q}^*$, where $\xi_i \in \mathfrak{g}_{\bar{i}}$. We claim that if $\xi_1 \in \mathfrak{g}_1^{reg}$, then $\boldsymbol{\xi} \in \mathfrak{q}^*_{reg}$. This yields the desired *codim–2* property. Hence it suffices to prove the claim. Consider $\boldsymbol{\xi}(t) = (\xi_1, t\xi_2, \ldots, t^{nk-1}\xi_{nk})$, where $t \in \Bbbk$. It is easily seen that if $(x_0, x_1, \ldots, x_{nk-1}) \in \mathfrak{q}^{\boldsymbol{\xi}(t)}$ for $t \neq 0$, then $(t^{nk-1}x_0, t^{nk-2}x_1, \ldots, x_{nk-1}) \in \mathfrak{q}^{\boldsymbol{\xi}}$. It follows that, for $t \neq 0$, $\dim \mathfrak{q}^{\boldsymbol{\xi}(t)}$ does not depend on t. Because $\lim_{t\to 0}\boldsymbol{\xi}(t) = (\xi_1, 0, \ldots, 0) \in \mathfrak{q}^*_{reg}$, we conclude that all elements $\boldsymbol{\xi}(t)$ are regular. That is, $(\xi_1, \xi_2, \ldots, \xi_{nk}) \in \mathfrak{q}^*_{reg}$ whenever $\xi_1 \in \mathfrak{g}_1^{reg}$. □

Example 5.4. It can happen that θ is both $\mathcal{S}$- and $\mathcal{N}$-regular, but not very $\mathcal{N}$-regular. This may lead to the absence of the *codim–2* property for $\mathfrak{g}\langle|\theta|\rangle_0$. Consider $\mathfrak{g} = \mathfrak{sp}_4$ and an automorphism of order 4 such that $\mathfrak{g}_0$ is a Cartan subalgebra, say $\mathfrak{t}$. If α, β are simple roots with respect to $\mathfrak{t}$ (α is short), so that $2\alpha + \beta$ is the highest root, then $\mathfrak{g}_1$ is the sum of root spaces corresponding to α, β, and $-2\alpha-\beta$. Therefore $\mathfrak{g}_1$ contains regular semisimple and regular nilpotent elements of $\mathfrak{g}$. However, a direct verification shows that the corresponding $\mathbb{Z}_4$-contraction does not have the *codim–2* property. In this case, $\mathfrak{N}_1$ has three irreducible components and $\mathcal{O}^{reg}$ meets only two of them.

Theorem 5.5. *Suppose θ is both $\mathcal{S}$-regular and very $\mathcal{N}$-regular and $n \in \mathbb{N}$. Then $\mathsf{Inv}(\mathfrak{g}\langle nk\rangle_0, \mathrm{ad}^*)$ is a polynomial algebra of Krull dimension $n{\cdot}\mathrm{rk}\,\mathfrak{g}$. Moreover, $\mathcal{L}^\bullet(\mathcal{S}(n\mathfrak{g})^{n\mathfrak{g}}) = \mathsf{Inv}(\mathfrak{g}\langle nk\rangle_0, \mathrm{ad}^*)$.*

Proof. First, assume that $n = 1$. In view of Theorem 5.1 and Proposition 5.3, it suffices to show that there are basic invariants $F^{(1)}, \dots, F^{(\mathrm{rk}\,\mathfrak{g})} \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $\{\mathrm{gr}^\bullet F^{(i)}\}$ remain algebraically independent.

Take the basic invariants in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that each $F^{(i)}$ is an eigenvector of θ. Let $\deg F^{(i)} = d_i$. We want to better understand the structure of $\mathrm{gr}^\bullet F^{(i)}$. Take $e \in \mathcal{O}^{reg} \cap \mathfrak{g}_1$. By [5], the differentials $\{dF^{(i)}_e\}$ are linearly independent. In particular, $dF^{(i)}_e \neq 0$ for each i. Consider the polygrading of $\mathcal{S}(\mathfrak{g})$ corresponding to the decomposition $\mathfrak{g} = \bigoplus_{i=0}^{k-1} \mathfrak{g}_i$. We also regard $\mathfrak{g}_1$ as the first factor of $\mathfrak{g}\langle k\rangle_1 \simeq \mathfrak{g}\langle k\rangle_0^*$. Since $\mathfrak{g}_{k-1}^* \simeq \mathfrak{g}_1$ and $\mathfrak{g}_{k-1}$ is the last factor of $\mathfrak{g}\langle k\rangle_0$, the condition $dF^{(i)}_e \neq 0$ implies that $F^{(i)}$ has a nonzero summand of the form

$$(\clubsuit) \qquad F^{(i)}_{(0,\dots,0,d_i)} \quad \text{or} \quad F^{(i)}_{(\dots,1,\dots,d_i-1)}.$$

We have $k-1$ possibilities for the position of '1' in the second expression, hence totally k possibilities in ($\clubsuit$). In fact, there is the following precise assertion:

Lemma 5.6. *(i) A homogeneous θ-eigenvector $F \in \mathcal{S}(\mathfrak{g})$ can have at most one nonzero summand of the form ($\clubsuit$); (ii) If this is the case, then $\mathrm{gr}^\bullet F$ contains that summand.*

Proof. (i) This follows from the fact that, for the nonzero summnads $F_{(i_0,\dots,i_{k-1})}$, $i_0 + \ldots + i_{k-1} = \deg F$ and all the sums $\sum_{j=1}^{k-1} j i_j$ have one and the same residue $\pmod k$, which is determined by the θ-eigenvalue of F. The k possibilities in ($\clubsuit$) just correspond to k possible eigenvalues.

(ii) If $\deg F = d$ and F has the summand $F_{(0,\dots,0,d)}$, then the latter is clearly $\mathrm{gr}^\bullet F$. If F has the summand $F_{(\dots,1,\dots,d-1)}$ (with '1' in position s, $0 \leqslant s \leqslant k-2$), then the summands $F_{(i_0,\dots,i_{k-1})}$ occurring in $\mathrm{gr}^\bullet F$ should satisfy the relations

$$\begin{cases} \sum_{j=0}^{k-1} i_j = d, \\ \sum_{j=1}^{k-1} j i_j \equiv s + (d-1)(k-1) \pmod k, \\ \sum_{j=1}^{k-1} j i_j \ \text{ is maximal posiible.} \end{cases}$$

It is not hard to prove that the maximal value of the last sum is $s + (d-1)(k-1)$. Hence one of the solutions is $(\dots, \underset{s}{1}, \dots, d-1)$. $\square$

It follows from Lemma 5.6(ii) that $dF^{(i)}_e = d(\mathrm{gr}^\bullet F^{(i)})_e$. Hence $\{\mathrm{gr}^\bullet F^{(i)}\}$ remain algebraically independent. This completes the proof of theorem for $n=1$.

For arbitrary $n\in\mathbb{N}$, we consider $\widetilde\theta\in\mathrm{Aut}(n\mathfrak g)$ and the eigenspaces $(n\mathfrak g)_j$, which are described in Lemma 3.4. Here $(n\mathfrak g)_1=\{(x,\mu x,\dots,\mu^{n-1}x)\mid x\in\mathfrak g_1\}$, where $\mu=\sqrt[nk]{1}$, and we work with $\boldsymbol{e}=(e,\mu e,\dots,\mu^{n-1}e)$, which is a regular nilpotent element of $n\mathfrak g$. We regard $(n\mathfrak g)_1$ as the first factor in $\mathfrak g\langle nk\rangle_1$ and choose basic invariants in $\mathcal S(n\mathfrak g)^{n\mathfrak g}$ that are $\widetilde\theta$-eigenvectors. The rest is the same. □

The property of being "very $\mathcal N$-regular" is difficult to check directly. There are, however, useful sufficient conditions.

For $|\theta|=2$, all irreducible component of $\mathfrak N_1$ are conjugate with respect to the action of certain (non-connected) group containing G_0, see [6, Theorem 6]. Therefore $\mathcal N$-regularity coincides with very $\mathcal N$-regularity. Furthermore, an involution is $\mathcal S$-regular if and only if it is $\mathcal N$-regular [1]. Thus, it suffices to assume that $\mathfrak g_1\cap\mathcal O^{reg}\ne\varnothing$. Finally, an involution has the last property if and only if the corresponding Satake diagram has no black nodes.

To state another sufficient condition, we recall that, for $\mathfrak g$ simple, the set of basic invariants in $\Bbbk[\mathfrak g]^G$ contains a unique polynomial of maximal degree. This degree equals the Coxeter number of $\mathfrak g$, denoted $\mathsf c(\mathfrak g)$. Let $F_{\mathsf c(\mathfrak g)}$ be such a basic invariant. It is known that $d(F_{\mathsf c(\mathfrak g)})_e=0$ for any $e\in\mathfrak N\setminus\mathcal O^{reg}$ (see a description of the ideal of $\mathfrak N\setminus\mathcal O^{reg}$ in [3, 4.7–4.9]).

Proposition 5.7. *Let $\mathfrak g$ be simple. Suppose θ is $\mathcal N$-regular and $F_{\mathsf c(\mathfrak g)}|_{\mathfrak g_1}\not\equiv 0$. If either* (i) *$\mathfrak g_0$ is semisimple or* (ii) *$G_0\subset SL(\mathfrak g_1)$ and $(G_0:\mathfrak g_1)$ is locally free, then θ is very $\mathcal N$-regular.*

Proof. Since $\mathfrak g_1\cap\mathcal O^{reg}\ne\varnothing$, the restriction homomorphism $\Bbbk[\mathfrak g]^G\to\Bbbk[\mathfrak g_1]^{G_0}$ is onto, and $F_{\mathsf c(\mathfrak g)}|_{\mathfrak g_1}$ is a basic invariant in $\Bbbk[\mathfrak g_1]^{G_0}$ [10, Theorem 3.5]. Let $\mathcal O$ be a dense G_0-orbit in an irreducible component of $\mathfrak N_1$. In other words, $\mathcal O$ is a nilpotent G_0-orbit of maximal dimension. Then in both cases, the differentials of basic invariants in $\Bbbk[\mathfrak g_1]^{G_0}$ are linearly independent at any $v\in\mathcal O$. For (i) (resp. (ii)), we refer to [9, Cor. 5(i)] (resp. [8, Cor. 1]). In particular, we have $d(F_{\mathsf c(\mathfrak g)})_v\ne 0$. Hence $v\in\mathcal O^{reg}$. □

Remark. In Example 5.4, the condition that $G_0\subset SL(\mathfrak g_1)$ is not satisfied.

Here are some other related results:

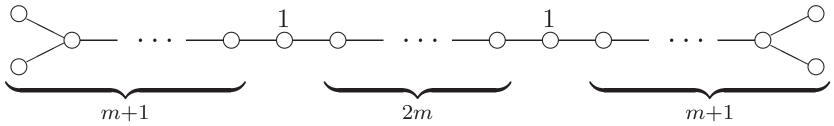
Proposition 5.8.
(i) *If θ is $\mathcal S$-regular and the G_0-action on $\mathfrak g_1$ is locally free, then θ is $\mathcal N$-regular* [10, Thm. 4.2(iii)];
(ii) *If $\theta\in\mathrm{Int}(\mathfrak g)$ and θ is $\mathcal N$-regular, then $F_{\mathsf c(\mathfrak g)}|_{\mathfrak g_1}\not\equiv 0$ if and only if $|\theta|$ divides* $\mathsf c(\mathfrak g)$ [10, Cor. 3.6].

In the rest of the section, we provide examples of $\mathcal S$-regular and very $\mathcal N$-regular automorphisms and thereby examples where Theorem 5.5 applies.

Example 5.9. We give some serial examples of $\mathcal{S}$-regular and very $\mathcal{N}$-regular periodic automorphisms related to classical algebras.

1) The Lie algebra $\mathfrak{gl}_{nk}$ has an automorphism of order k such that $\mathfrak{g}_0 = k\mathfrak{gl}_n$. In more invariant terms, let V_i be a n-dimensional space, $i = 1, \dots, k$, and $\mathfrak{g} = \mathfrak{gl}(V_1 \oplus \cdots \oplus V_k)$. Define $A \in GL(V_1 \oplus \cdots \oplus V_k)$ by $A|_{V_i} = \zeta^i{\cdot}\mathrm{id}$. Let θ be the inner automorphism of $\mathfrak{g}$ determined by A. Then $\mathfrak{g}_0 = \mathfrak{gl}(V_1) \dotplus \ldots \dotplus \mathfrak{gl}(V_k)$ and $\mathfrak{g}_1 = \bigoplus_{i=1}^k \mathrm{Hom}(V_i, V_{i+1})$, where $V_{k+1} = V_1$. Here $\dim \mathfrak{g}_1 = kn^2$. The generic stabiliser in $\mathfrak{g}_0$ is $\mathfrak{t}_n$ (the diagonally embedded Cartan subalgebra). Using the matrix realisation, one easily verifies that θ is both $\mathcal{S}$- and $\mathcal{N}$-regular. A more careful argument shows that $\mathfrak{N}_1$ has k irreducible components, and each contains regular nilpotent elements of $\mathfrak{g}$. Therefore θ is very $\mathcal{N}$-regular.

2) The algebra $\mathbf{D}_{4m+3}$ has an inner automorphism θ of order 4 such that $G_0 = \mathbf{D}_{m+1} \times \mathbf{D}_{m+1} \times \mathbf{A}_{2m} \times T_1$. The corresponding Kac's diagram is



We refer to [15, § 8, Prop. 17] for a complete account on Kac's diagrams of periodic automorphisms. (Partial explanations are also given in [10, Example 4.5], where we have drawn black nodes in place of nodes with labels '1'.) Here $\dim \mathfrak{g}_0 = (2m+1)(4m+3)$ and $\dim \mathfrak{g}_1 = (2m+1)(4m+4)$. Let us prove that θ is $\mathcal{S}$-regular and very $\mathcal{N}$-regular.

Since $\mathsf{c}(\mathbf{D}_{4m+3}) = 8m+4$, $|\theta|$ divides it. The representation of G_0 in $\mathfrak{g}_1$ can be read off the Kac's diagram. Here $\mathfrak{g}_1$ is the sum of two simple G_0-modules of the *same* dimension, and T_1 acts with opposite weights on them. Therefore $G_0 \subset SL(\mathfrak{g}_1)$. One also readily verifies that the action $(G_0 : \mathfrak{g}_1)$ is stable and locally free. Therefore $\dim \mathfrak{c} = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 = 2m+1$. For stable θ-groups, the dimension of a generic semisimple G-orbit meeting $\mathfrak{g}_1$ can be computed as $|\theta|(\dim \mathfrak{g}_1 - \dim \mathfrak{c})$ (cf. [10, Prop. 2.1(i)]), which is equal in this case to $\dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g}$. Hence, θ is $\mathcal{S}$-regular. Combining Proposition 5.7(ii) and 5.8, we then conclude that θ is very $\mathcal{N}$-regular.

Example 5.10. The following table contains some sporadic examples, mostly for exceptional Lie algebras. Here $\mathfrak{g}_0$ is always semisimple and $|\theta|$ divides $\mathsf{c}(\mathfrak{g})$; $\mathrm{ind}\,(\theta)$ denotes the order of θ in $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$. In the last two columns, the dimension of Cartan subspaces and the generic stabiliser for the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ are displayed (this information is borrowed from [15, § 9]). The G_0-action on $\mathfrak{g}_1$ is locally free if and only if this generic stabiliser is trivial.

In most cases, the proof is similar to that given in the previous example. The only exception is the second item for $\mathbf{E}_7$, where θ is not an involution and $(G_0 : \mathfrak{g}_1)$ is not locally free. However, in this case $\mathfrak{N}_1$ appears to be irreducible [4, § 4], hence $\mathcal{N}$-regularity is sufficient.

$\mathfrak g$	$\lvert\theta\rvert$	ind (θ)	$\mathfrak g_0$	$\mathfrak g_1$	dim $\mathfrak c$	$\mathfrak s$
$\mathbf{E}_6$	2	1	$\mathbf{A}_5\times\mathbf{A}_1$	$\mathsf{R}(\varpi_3)\otimes\mathsf{R}(\varpi_1)$	4	$\mathfrak t_2$
	3	1	$\mathbf{A}_2\times\mathbf{A}_2\times\mathbf{A}_2$	$\mathsf{R}(\varpi_1)\otimes\mathsf{R}(\varpi_1)\otimes\mathsf{R}(\varpi_1)$	3	$\{0\}$
	2	2	$\mathbf{C}_4$	$\mathsf{R}(\varpi_4)$	6	$\{0\}$
	4	2	$\mathbf{A}_3\times\mathbf{A}_1$	$\mathsf{R}(2\varpi_1)\otimes\mathsf{R}(\varpi_1)$	2	$\{0\}$
$\mathbf{E}_7$	2	1	$\mathbf{A}_7$	$\mathsf{R}(\varpi_4)$	7	$\{0\}$
	3	1	$\mathbf{A}_5\times\mathbf{A}_2$	$\mathsf{R}(\varpi_2)\otimes\mathsf{R}(\varpi_1)$	3	$\mathfrak t_1$
$\mathbf{E}_8$	2	1	$\mathbf{D}_8$	$\mathsf{R}(\varpi_7)$	8	$\{0\}$
	3	1	$\mathbf{A}_8$	$\mathsf{R}(\varpi_3)$	4	$\{0\}$
	5	1	$\mathbf{A}_4\times\mathbf{A}_4$	$\mathsf{R}(\varpi_2)\otimes\mathsf{R}(\varpi_1)$	2	$\{0\}$
$\mathbf{F}_4$	2	1	$\mathbf{C}_3\times\mathbf{A}_1$	$\mathsf{R}(\varpi_3)\otimes\mathsf{R}(\varpi_1)$	4	$\{0\}$
	3	1	$\mathbf{A}_2\times\mathbf{A}_2$	$\mathsf{R}(2\varpi_1)\otimes\mathsf{R}(\varpi_1)$	2	$\{0\}$
$\mathbf{G}_2$	2	1	$\mathbf{A}_1\times\mathbf{A}_1$	$\mathsf{R}(3\varpi_1)\otimes\mathsf{R}(\varpi_1)$	2	$\{0\}$
$\mathbf{D}_4$	3	3	$\mathbf{A}_2$	$\mathsf{R}(3\varpi_1)$	2	$\{0\}$

6. Some remarks and open problems

6.1. **Invariants of the nilpotent radical.** In [11], we explore several instances of representations of non-reductive Lie algebras $\mathfrak q$ such that, for certain nilpotent ideal $\mathfrak r\lhd\mathfrak q$, the invariants of $\mathfrak r$ are polynomial. This includes the adjoint and coadjoint representations of $\mathfrak q$, i.e., the algebras $\Bbbk[\mathfrak q]^{\mathfrak r}$ or $\mathcal S(\mathfrak q)^{\mathfrak r}$. Often, $\mathfrak r=\mathfrak R_u(\mathfrak q)$. This fact was used as a step toward proving the polynomiality of algebras of $\mathfrak q$-invariants. (See Theorems 6.2, 7.1, and 11.1 in loc. cit.)

Similar approach applies to our results in Sections 4 and 5. For instance, in order to describe $\mathsf{Inv}(\mathfrak g\langle k\rangle_0,\mathrm{ad})$, one can first consider the algebra of $\mathfrak R_u(\mathfrak g\langle k\rangle_0)$-invariants in $\Bbbk[\mathfrak g\langle k\rangle_0]$, say $\mathcal A$. Under the assumption of Theorem 4.1, it can be shown that $\mathcal A$ is polynomial, of Krull dimension $\dim\mathfrak g_0+(\mathrm{rk}\,\mathfrak g-\mathrm{rk}\,\mathfrak g_0)$. Furthermore, the induced representation of $\mathfrak g_0\simeq\mathfrak g\langle k\rangle_0/\mathfrak R_u(\mathfrak g\langle k\rangle_0)$ in $\mathrm{Spec}(\mathcal A)$ is isomorphic to the adjoint representation of $\mathfrak g_0$ plus the trivial representation of dimension $\mathrm{rk}\,\mathfrak g-\mathrm{rk}\,\mathfrak g_0$. In this way, one obtains another proof of polynomiality of $\mathsf{Inv}(\mathfrak g\langle k\rangle_0,\mathrm{ad})$. The reason for success in this and other similar cases is that one can explicitly construct a natural set of elements of $\mathcal A$ (presumable basic invariants). Then using Igusa's lemma (see e.g. [11, Lemma 6.1]), one proves that these elements are algebraically independent and generate the algebra $\mathcal A$. The details will appear elsewhere.

6.2. **Index of fixed point subalgebras.** Let us summarise what is known about the index of algebras $\mathfrak g\langle m\rangle_0$. Recall that $\lvert\theta\rvert=k$.

Since $\mathfrak{g}\langle nk+1\rangle_0$ is quadratic, the description of $\mathsf{Inv}(\mathfrak{g}\langle nk+1\rangle_0, \mathrm{ad})$ in Theorem 4.1(ii) shows that $\mathrm{ind}(\mathfrak{g}\langle nk+1\rangle_0) = n{\cdot}\mathrm{rk}\,\mathfrak{g} + \mathrm{rk}\,\mathfrak{g}_0$. That is, index does not change under the contraction $n\mathfrak{g} \dotplus \mathfrak{g}_0 \rightsquigarrow \mathfrak{g}\langle nk+1\rangle_0$ whenever $\mathfrak{g}_0 \cap \mathcal{O}^{reg} \neq \varnothing$. (Actually, this equality can be proved if $\mathfrak{g}_0 \cap \mathfrak{g}_{reg} \neq \varnothing$.) Similarly, if $\mathfrak{g}_1 \cap \mathfrak{g}_{reg} \neq \varnothing$, then $\mathrm{ind}(\mathfrak{g}\langle nk\rangle_0) = n{\cdot}\mathrm{rk}\,\mathfrak{g}$ (Proposition 5.3).

If $|\theta| = 2$, then we always have $\mathrm{ind}(\mathfrak{g}_0 \ltimes \mathfrak{g}_1) = \mathrm{ind}\,\mathfrak{g}$ [11, Corollary 9.4]. The reason is that here G/G_0 is a spherical homogeneous space. From this, it is not hard to deduce that $\mathrm{ind}(\mathfrak{L}_{2n}(\mathfrak{g}_0, \mathfrak{g}_1)) = n{\cdot}\mathrm{rk}\,\mathfrak{g}$ and $\mathrm{ind}(\mathfrak{L}_{2n+1}(\mathfrak{g}_0, \mathfrak{g}_1)) = n{\cdot}\mathrm{rk}\,\mathfrak{g} + \mathrm{rk}\,\mathfrak{g}_0$. (See notation introduced in Example 4.2.) However, for $|\theta| \geqslant 3$, the general answer is not known.

Problem 6.1. *Compute* $\mathrm{ind}\,(\mathfrak{g}\langle nk\rangle_0)$ *and* $\mathrm{ind}\,(\mathfrak{g}\langle nk+1\rangle_0)$ *for an arbitrary* $\theta \in \mathrm{Aut}(\mathfrak{g})$ *with* $|\theta| = k \geqslant 3$*. Is it true that* $\mathrm{ind}(\mathfrak{g}\langle nk+1\rangle_0) = n{\cdot}\mathrm{rk}\,\mathfrak{g} + \mathrm{rk}\,\mathfrak{g}_0$ *and* $\mathrm{ind}(\mathfrak{g}\langle nk\rangle_0) = n{\cdot}\mathrm{rk}\,\mathfrak{g}$*?*

(The existence of contractions $n\mathfrak{g} \dotplus \mathfrak{g}_0 \rightsquigarrow \mathfrak{g}\langle nk+1\rangle_0$ and $n\mathfrak{g} \rightsquigarrow \mathfrak{g}\langle nk\rangle_0$ shows that in both cases the inequality "$\geqslant$" holds.) For instance, consider the outer automorphism of $\mathbf{D}_4$ of order 3 whose fixed point subalgebra is $\mathbf{G}_2$. The corresponding $\mathbb{Z}_3$-contraction is $\mathfrak{q} = \mathbf{G}_2 \ltimes \mathsf{R}(\varpi_1) \ltimes \mathsf{R}(\varpi_1)$. It occurs in Example 4.2(7^o). Here $\mathfrak{g}_1 \simeq \mathsf{R}(\varpi_1)$ does not contain regular elements of $\mathbf{D}_4$. However, it is not hard to verify that $\mathrm{ind}\,\mathfrak{q} = 4$.

6.3. **On Poisson-commutative subalgebras.** A subalgebra $\mathcal{A}$ of $\mathcal{S}(\mathfrak{q})$ is said to be *Poisson-commutative* if $\{f, g\} = 0$ for all $f, g \in \mathcal{A}$. There is a procedure (the so-called *argument shift method*, see e.g. [13]) for constructing "large" Poisson-commutative subalgebras of $\mathcal{S}(\mathfrak{q})$, which begins with $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ and a $\xi \in \mathfrak{q}^*_{reg}$. The resulting subalgebra is denoted by $\mathcal{F}_\xi(\mathcal{S}(\mathfrak{q})^{\mathfrak{q}})$. It is proved in [13] that if (1) $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is polynomial, (2) the sum of degrees of the basic invariants of $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ equals $b(\mathfrak{q})$, and (3) $\mathfrak{q}$ has the *codim–3* property, then $\mathcal{F}_\xi(\mathcal{S}(\mathfrak{q})^{\mathfrak{q}})$ is a maximal Poisson-commutative subalgebra for any $\xi \in \mathfrak{q}^*_{reg}$. Furthermore, $\mathcal{F}_\xi(\mathcal{S}(\mathfrak{q})^{\mathfrak{q}})$ is a polynomial algebra of Krull dimension $b(\mathfrak{q})$.

Our goal is to realise when that result applies to algebras $\mathfrak{g}\langle m\rangle_0$. First of all, Lie algebras $\mathfrak{q}$ occurring in Theorems 4.1(ii) and 5.5 satisfy properties (1) and (2). We also proved that $\mathfrak{g}\langle nk\rangle_0$ has the *codim–2* property (Proposition 5.3(ii)). However, the *codim–3* property does not always holds for $\mathfrak{g}\langle nk\rangle_0$. But for algebras $\mathfrak{g}\langle nk+1\rangle_0$ the situation is better.

Proposition 6.2. *Suppose* θ *has the property that* $\mathfrak{g}_0 \cap \mathcal{O}^{reg} \neq \varnothing$*. Then* $\mathfrak{g}\langle nk+1\rangle_0$ *has the codim–3 property.*

Proof. The proofs of Lemma 5.2 and Prop. 5.3 can be adapted to this situation. Recall that

$$\mathfrak{g}\langle nk+1\rangle_0 \simeq \mathfrak{g}\langle nk+1\rangle_0^* = \mathfrak{g}_0 \ltimes \mathfrak{g}_1 \ltimes \ldots \ltimes \mathfrak{g}_{k-1} \ltimes \mathfrak{g}_0 \quad (nk+1 \text{ factors}).$$

As in Lemma 5.2, we prove that if $x \in \mathfrak{g}_0$ is regular in $\mathfrak{g}_0$, then it is regular in $\mathfrak{g}$. (In doing so, we use the assumption $\mathcal{O}^{reg} \cap \mathfrak{g}_0 \neq \varnothing$ and the fact that the nilpotent cone in $\mathfrak{g}_0$ is irreducible.) Then, as in Prop. 5.3, we prove that $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_{nk}) \in (\mathfrak{g}\langle nk+1\rangle_0^*)_{reg}$ whenever $\xi_0 \in (\mathfrak{g}_0)_{reg}$. Since $\mathrm{codim}\,(\mathfrak{g}_0 \setminus (\mathfrak{g}_0)_{reg}) = 3$, we are done. $\square$

6.4. **Flatness.** Although we have found a number of periodic automorphisms of Takiff algebras such that $\mathsf{Inv}(\mathfrak{g}\langle m\rangle_0, \mathrm{ad})$ or $\mathsf{Inv}(\mathfrak{g}\langle m\rangle_0, \mathrm{ad}^*)$ is polynomial, we do not have substantial results on the flatness of respective quotient morphisms. Actually, I believe that the quotient morphisms are flat in the context of Theorems 4.1 and 5.5. Partial affirmative results for $|\theta| = 2$ are contained in [11, Theorem 9.13] (the adjoint representation of $\mathfrak{g}_0 \ltimes \mathfrak{g}_1$) and [12, Sect. 5] (the coadjoint representation of $\mathfrak{g}_0 \ltimes \mathfrak{g}_1$) .

For the centraliser $\tilde{\mathfrak{g}}^e$ from Example 4.2(7^o) and its adjoint representation, we can also prove, using ad hoc methods, that the quotient morphism is flat.

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Independent University of Moscow, Bol'shoi Vlasevskii per. 11, Moscow 119002, Russia
Institute for Information Transmission Problems, B. Karetnyi per. 19, Moscow 101447

E-mail address: panyush@mccme.ru