ANNIHILATING IDEALS AND TILTING FUNCTORS

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ABSTRACT. We use Kazhdan-Lusztig tensoring to, first, describe annihilating ideals of highest weight modules over an affine Lie algebra in terms of the corresponding VOA and, second, to classify tilting functors, an affine analogue of projective functors known in the case of a simple Lie algebra.

1. INTRODUCTION

This paper grew out of an attempt to carry the classical theory of Harish-Chandra bimodules over to the case of an affine Lie algebra. As we shall explain below, the very definition of a Harish-Chandra bimodule over an affine Lie algebra is not obvious. We do not propose such a definition, but we affinize the notion of *projective functor*, an important tool of the classical theory. In this way we get a semi-simple monoidal category, whose classical counterpart is the subcategory of projective Harish-Chandra bimodules. The Grothendieck ring of this category is shown to be isomorphic to the group algebra of the corresponding affine Weyl group.

It is well-known that when studying representation theory of affine Lie algebras it is important to distinguish between the case when the central charge is positive and the case when it is negative. The results we have just discussed are all valid when the latter is the case. Another part of the classical theory revolving about Duflo's theorem on primitive ideals of the universal enveloping algebra generalizes to affine Lie algebras when the former is the case. The generalization is non-trivial: we establish a bijection between the ideals of the vertex operator algebra attached to a given affine Lie algebra and submodules of a Weyl module with a regular dominant highest weight. This is an analogue of the Dixmier conjecture proved by Joseph [18] which classifies ideals of the universal enveloping algebra in terms of submodules of a Verma module with a regular dominant highest weight.

The main idea employed in this paper is to replace the functor of tensor product with a finite-dimensional module with the functor of Kazhdan-Lusztig tensoring with a Weyl module, or with a module having a filtration by Weyl modules. To make things clearer, we first review a functorial approach to Harish-Chandra bimodules over a simple Lie algebra following the beautiful paper by Bernstein and S.Gelfand [3].

1.1. **Representations of complex groups.** Let $\mathcal{A}$ be a category. Then one can consider the category $Funct(\mathcal{A})$ of functors on $\mathcal{A}$, objects being functors, morphisms being natural transformations of functors. In general, there is no reason to think that $Funct(\mathcal{A})$ is abelian even if $\mathcal{A}$ is so. Here is, however, an important example when $Funct(\mathcal{A})$ contains an abelian full subcategory.

Let $Mod(\mathfrak{g})$ be the category of modules over a simple complex Lie algebra $\mathfrak{g}$ and $Mod(\mathfrak{g}-\mathfrak{g})$ the category of $\mathfrak{g}$-bimodules. "Module" will always mean a space carrying a left action of $\mathfrak{g}$; "bimodule" will always mean a space carrying a left and a right action commuting with each other. Any $H\in Mod(\mathfrak{g}-\mathfrak{g})$ gives rise to the functor

$$\Phi_H : Mod(\mathfrak{g})\to Mod(\mathfrak{g});\ \Phi_H(M)=H\otimes_{\mathfrak{g}} M.$$

It is well-known that

$$Hom_{Mod(\mathfrak{g}-\mathfrak{g})}(H_1,H_2)=Hom_{Funct(Mod(\mathfrak{g}))}(\Phi_{H_1},\Phi_{H_2}).$$

Therefore $Mod(\mathfrak{g}-\mathfrak{g})$ is a full abelian subcategory of $Funct(Mod(\mathfrak{g}))$.

Any $\mathfrak{g}$-bimodule is a $\mathfrak{g}$-module with respect to the diagonal action (that is, the left action minus the right action). A Harish-Chandra bimodule is a finitely generated bimodule such that under the diagonal action it decomposes in a direct sum of finite dimensional $\mathfrak{g}$-modules occurring with finite multiplicities. Consider the category of Harish-Chandra bimodules HCh, and $\mathcal{O}_{\mathfrak{g}}$, the Bernstein-Gelfand-Gelfand category of $\mathfrak{g}$-modules. The condition imposed on the diagonal action ensures that if H is a Harish-Chandra bimodule, then Φ_H preserves $\mathcal{O}_{\mathfrak{g}}$. Therefore the construction we just discussed gives an embedding $HCh\hookrightarrow Funct(\mathcal{O}_{\mathfrak{g}})$ as a full subcategory.

Further, indecomposable projective Harish-Chandra bimodules are exactly those corresponding to direct indecomposable summands of the functor of tensoring by a finite dimensional $\mathfrak{g}$-module V:

$$V\otimes ? : \mathcal{O}_{\mathfrak{g}}\to\mathcal{O}_{\mathfrak{g}},\ M\mapsto V\otimes M. \tag{1}$$

Such functors are naturally called *projective*. To review their classification obtained in [3], we need some facts of representation theory.

$\mathcal{O}_{\mathfrak{g}}$ admits the direct product decomposition with respect to the action of the center of the universal enveloping $U(\mathfrak{g})$:

$$\mathcal{O}_{\mathfrak{g}}=\oplus_\theta\mathcal{O}^\theta,$$

where θ is a central character, $\mathcal{O}^\theta \subset \mathcal{O}_\mathfrak{g}$ is a full subcategory of $\mathfrak{g}$-modules with generalized central character θ. One can, therefore, assume that the functors in question belong to $Funct(\mathcal{O}^{\theta_r}, \mathcal{O}^{\theta_l})$, that is, map from $\mathcal{O}^{\theta_r}$ to $\mathcal{O}^{\theta_l}$. Given a weight λ, denote by M_λ the Verma module with highest weight λ, and P_λ the indecomposable projective module mapping onto the irreducible quotient of M_λ. Let λ_r (λ_l resp.) be a dominant weight such that M_{λ_r} (M_{λ_l} resp.) admits central character θ_r (θ_l resp.) ("Dominant" here means that the corresponding Verma module does not embed in any Verma module different from itself.) ¿From now on it is assumed for simplicity that λ_l, λ_r are regular integral.

Theorem 1.1.1. *There is a bijection between the Weyl group W and the set of isomorphism classes of projective functors in $Funct(\mathcal{O}^{\theta_r}, \mathcal{O}^{\theta_l})$. The bijection is established by assigning to $w \in W$ the functor $\Phi_w \in Funct(\mathcal{O}^{\theta_r}, \mathcal{O}^{\theta_l})$, such that $\Phi_w(M_{\lambda_r}) = P_{w\cdot\lambda_l}$.*

The following result is a key to Theorem 1.1.1.

Theorem 1.1.2. *Any projective functor $\Phi \in Funct(\mathcal{O}^{\theta_r}, \mathcal{O}^{\theta_l})$ is determined up to an isomorphism by $\Phi(M_{\lambda_r})$. The same is true if λ_r is replaced with the antidominant weight admitting θ_r.*

The way to derive Theorem 1.1.1. from Theorem 1.1.2 is to observe that λ_r being dominant, M_{λ_r} is projective and $V \otimes M_{\lambda_r}$ is also. Thus, by Theorem1.1.2, the direct sum decomposition of $V\otimes ?$ is determined by that of $V \otimes M_{\lambda_r}$, the latter being delivered by the theory of projectives in $\mathcal{O}_\mathfrak{g}$.

Remark 1.1.3. *The last sentence of Theorem 1.1.2 suggests an alternative way to prove Theorem 1.1.1, that is, to consider $V \otimes M_{\lambda_r}$ with an antidominant λ_r. Though not a projective, $V \otimes M_{\lambda_r}$ has certain favorable properties summarized by saying that it is a tilting module. It is this approach which generalizes to the affine case.*

Having classified projective functors, it is relatively easy to establish an equivalence of (sub)categories of HCh and $\mathcal{O}_\mathfrak{g}$. Theorem 5.9 of [3] claims that the functor

$$HCh(\theta_l, \theta_r) \to \mathcal{O}^{\theta_l},\ H \mapsto \Phi_H(M_{\lambda_r}), \tag{2}$$

is an equivalence of categories. Here $HCh(\theta_l, \theta_r)$ is a full subcategory of HCh consisting of bimodules admitting right central chatacter θ_r and generalized left central character θ_l.

An important corollary of (2) is the above mentioned description of the 2-sided ideal lattice of $U(\mathfrak{g})_\theta := U(\mathfrak{g})/U(\mathfrak{g})Ker(\theta)$ obtained by

Joseph [18]. Denote by $\Omega(U(\mathfrak{g})_\theta)$ the 2-sided ideal lattice of $U(\mathfrak{g})_\theta$ and by $\Omega(M_\lambda)$ the submodule lattice of M_λ, where λ is the dominant weight related to θ. Then the map

$$\Omega(U(\mathfrak{g})_\theta) \to \Omega(M_\lambda), I \mapsto IM_\lambda \tag{3}$$

is a lattice equivalence. Indeed, $U(\mathfrak{g})_\theta$ is an algebra containing $\mathfrak{g}$, and hence a $\mathfrak{g}$-bimodule; its 2-sided ideals as algebra are its submodules as bimodule. Under the equivalence (2) $U(\mathfrak{g})_\theta$ goes to $M(\lambda)$, because $U(\mathfrak{g})_\theta \otimes_{\mathfrak{g}} M(\lambda) = M(\lambda)$. Thus submodule lattices of $U(\mathfrak{g})_\theta$ and $M(\lambda)$ are equivalent. A little extra work is needed to find the explicit form (3) of this equivalence.

The last result we want to review here belongs to Jantzen (see [16], also [3]) and establishes another equivalence of categories based on the notion of translation functor. Denote by λ the dominant weight lying in the W-orbit of $\lambda_l - \lambda_r$, and by V_λ the simple $\mathfrak{g}$-module with highest weight λ. (Recall that $\lambda_l,\ \lambda_r$ are supposed to be integral.) For any θ denote by $p_\theta : \mathcal{O}_{\mathfrak{g}} \to \mathcal{O}^\theta$ the natural projection. Then the functor

$$T_{\theta_2}^{\theta_1} : \mathcal{O}^{\theta_2} \to \mathcal{O}^{\theta_1},\ T_{\theta_2}^{\theta_1}(M) = p_{\theta_1}(V_\lambda \otimes M) \tag{4}$$

is an equivalence of categories. The functor $T_{\theta_2}^{\theta_1}$ is called translation functor.

We finish our review of the semi-simple case by remarking that many results of [3] are based on, refine and generalize the earlier work, see e.g. [8, 7, 28, 29].

1.2. **An affine analogue.** There are many reasons why it is difficult to give an intelligent definition of a Harish-Chandra bimodule over an affine Lie algebra $\hat{\mathfrak{g}}$. Some of the difficulties become obvious if one considers universal enveloping $U(\hat{\mathfrak{g}})$ as a model example. Under the diagonal action $U(\hat{\mathfrak{g}})$ decomposes in a sum of loop modules. This sum is not direct and multiplicities are infinite. Further, it is easy to see that the composition series of the tensor product of a pair of loop modules always has terms occurring with infinite multiplicities. To avoid difficulties of this kind we adopt a functorial point of view.

Thus we are looking for an interesting subcategory in $Funct(\mathcal{O}_k)$, $\mathcal{O}_k$ being the Bernstein-Gelfand-Gelfand category of $\hat{\mathfrak{g}}$-modules at level k. As an analogue of the functor $V\otimes?$ we choose either

$$V_\lambda^k \dot{\otimes} ? : \mathcal{O}_k \to \mathcal{O}_k,\ B \mapsto V_\lambda^k \dot{\otimes} B,$$

where $\dot{\otimes}$ is the Kazhdan-Lusztig tensoring [21, 22, 23, 12], and V_λ^k is the Weyl module (generalized Verma module in another terminology) induced in a standard way from the finite dimensional $\mathfrak{g}$-module V_λ, or, more generally,

$$A\dot{\otimes}? : \mathcal{O}_k \to \mathcal{O}_k,\ B \mapsto A\dot{\otimes}B,$$

where A has a filtration by Weyl modules. To be more precise, we shall consider the following two cases: (i) $k + h^\vee > 0$, and (ii) $k + h^\vee < 0$, $h^\vee$ being the dual Coxeter number.

The case when $k + h^\vee > 0$. In this case we confine ourselves to the full subcategory $\tilde{\mathcal{O}}_k \subset \mathcal{O}_k$ consisting of modules semi-simple with respect to $\mathfrak{g} \subset \hat{\mathfrak{g}}$.

The Kazhdan-Lusztig tensoring is a subtle thing and many obvious properties of $V\otimes ?$ are hard to carry over to the case of $V_\lambda^k\dot{\otimes}?$. For example, the functor $V_\lambda^k\dot{\otimes}?$ does not seem to be exact in general. There is, however, a case when the analogy is precise – the affine version of translation functor. By [4, 27], there is a direct sum decomposition

$$\tilde{\mathcal{O}}_k = \oplus_{(\lambda,k)\in P_k^+} \tilde{\mathcal{O}}_k{}^\lambda,$$

and thus a projection

$$p_\lambda : \tilde{\mathcal{O}}_k \to \tilde{\mathcal{O}}_k{}^\lambda,$$

where P_k^+ is the set of dominant weights at level $k + h^\vee \in \mathbf{Q}_>$. (This is an analogue of the central character decomposition for $\mathfrak{g}$.) We can therefore define an affine translation functor

$$T_\mu^\lambda : \tilde{\mathcal{O}}_k{}^\mu \to \tilde{\mathcal{O}}_k{}^\lambda,$$

by adjusting definition (4) to the affine case (most notably by replacing $\otimes$ with $\dot{\otimes}$ and the finite dimensional $\mathfrak{g}$-module with an appropriate Weyl module, for details see 4.1). This construction was first proposed in [12] in the case of negative level ($k + h^\vee < 0$) representations, but its meaningfulness in our situation is not quite obvious.

The basic properties of affine translation functors are collected in Proposition 4.3.1. They are summarized by saying that a Weyl module with a dominant highest weight is rigid and the functor of Kazhdan-Lusztig tensoring with such a module is exact. These properties easily imply that $T_\mu^\lambda : \tilde{\mathcal{O}}_k{}^\mu \to \tilde{\mathcal{O}}_k{}^\lambda$ is an equivalence of categories (cf. (4)). This theorem refines results of [4], where a different version of translation functors was defined (in the framework of a general symmetrizable Kac-Moody algebra) by using the standard tensoring with an integrable module.

The study of Kazhdan-Lusztig tensoring is not easy but rewarding. A simple translation of Proposition 4.3.1 in the language of vertex operator algebras (VOA), see 5.1.1, 5.2, gives the following affine analogue of the equivalence (3). Recall that by [15] there is a VOA, $(V_0^k, Y(.,t))$, attached to $\hat{\mathfrak{g}}$. The Fourier components of the fields $Y(v,t), v\in V_0^k$ span a Lie algebra, $U(\hat{\mathfrak{g}})_{loc}$. We prove (Theorem 5.3.1) that the ideal lattice of $U(\hat{\mathfrak{g}})_{loc}$ in the sense of VOA is equivalent to the submodule lattice of the Weyl module V_λ^k with a dominant highest weight $(\lambda,k),\ k+h^\vee\in\mathbf{Q}_>$. Observe that the crucial difference between this statement and (3) is that the associative algebra $U(\mathfrak{g})_\theta$ is replaced with a huge Lie algebra $U(\hat{\mathfrak{g}})_{loc}$. Theorem 5.3.1 generalizes and refines the well-known result that Fourier components of the field $e_\theta(t)^{k+1}$ annihilate all integrable modules at a positive level k; here $e_\theta\in\mathfrak{g}$ is a highest root vector.

The case when $k+h^\vee<0$. In this case we propose the following affine analogue of the notion of projective functor. When $k+h^\vee<0$, the theory of projectives in $\tilde{\mathcal{O}}_k$ and $\mathcal{O}_k$ becomes less convenient than its positive level counterpart. Instead of projectives, we shall use *tilting modules*. These can be defined for both $\tilde{\mathcal{O}}_k$ and $\mathcal{O}_k$ and their formal characters linearly span the Grothendieck rings of each of the categories. To give a couple of examples, remark that a Verma module with an antidominant highest weight is a tilting object of $\mathcal{O}_k$; likewise, a Weyl module with an antidominant highest weight is a tilting object of $\tilde{\mathcal{O}}_k$. (It is worth mentioning that neither $\mathcal{O}_k$ nor $\tilde{\mathcal{O}}_k$ has modules with dominant highest weight.)

Each tilting module is a direct sum of indecomposable ones and the isomorphism classes of the latter are in 1-1 correspondence with highest weights.

We define a *tilting functor* to be a direct summand of the functor $p_{\lambda_l}\circ(W\dot{\otimes}?)\in Funct({}^{fam}\mathcal{O}_k^{\lambda_r},{}^{fam}\,\mathcal{O}_k^{\lambda_l})$, where $W\in\tilde{\mathcal{O}}_k$ is tilting. Here ${}^{fam}\mathcal{O}_k^\lambda$ is a full subcategory of $\mathcal{O}_k^\lambda$ consisting of modules which can be included in a family analytically depending on k. We prove (see Theorem 6.2.1) that isomorphism classes of indecomposable tilting functors are in 1-1 correspondence with elements of affine Weyl group W_k under the asumption that all the highest weights in question are regular. Further, as we mentioned in Remark 1.1.3, the correspondence is established in the same way as the one used in Theorem 1.1.1 except that dominant highest weights are replaced with antidominant ones and projective modules are replaced with tilting ones. It follows that the category of tilting functors is semisimple and is isomorphic to the subcategory of tilting modules in $\mathcal{O}_k^{\lambda_l}$. We remark that Theorem 6.2.1

is derived from Theorem 6.3.1 in the way analogous to the one used to derive Theorem 1.1.1 from Theorem 1.1.2. However, when proving Theorem 6.3.1, we do not have have at our disposal the important tool used by Bernstein and S.Gelfand, namely, the bimodule interpretation of projective functors, in particular the universal enveloping algebra and Kostant's theorem. Instead, we rely on some techniques borrowed from conformal field theory.

Many of the difficulties which one encounters when working with $\dot{\otimes}$ were removed in [22, 23]. For instance, the functor $W\dot{\otimes}?$ is known to be exact and, therefore, induces a homomorphism of the Grothendieck rings. We show that this homomorphism is multiplication by a certain element of the group algebra of W_k. In the case when $\lambda_r = \lambda_l$, the category of tilting functors is naturally a monoidal category, the tensor product being simply the composition of functors, and the previous sentence implies that the Grothendieck ring of this category is isomorphic to the group algebra of W_k, see Theorem 6.2.2.

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2. PRELIMINARIES

2.1. The following is a list of essentials which will be used but will not be explained.

$\mathfrak{g}$ is a simple finite dimensional Lie algebra with a fixed triangular decomposition; in particular with a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$; unless otherwise mentioned the dual $\mathfrak{h}^*$ is understood as the real part $\mathfrak{h}^*_{\mathbf{R}}$;

the action ($\lambda \mapsto w\lambda$) and the shifted (by ρ) action ($\lambda \mapsto w \cdot \lambda$) of the Weyl group W on $\mathfrak{h}^*$ preserving the weight lattice $P \in \mathfrak{h}^*$; denote by $\bar{C}$ the Weyl chamber – a fundamental domain for the shifted action attached to the fixed triangular decomposition; $P^+ = P \cap C$ where $C \subset \bar{C}$ is the interior;

the $\mathcal{O}_{\mathfrak{g}}$ category of $\mathfrak{g}$-modules attached to the triangular decomposition;

a Verma module $M_\lambda \in \mathcal{O}_{\mathfrak{g}},\ \lambda \in \mathfrak{h}$ and a simple finite dimensional module $V_\lambda,\ \lambda \in P^+ \subset P$;

the affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((z)) \oplus \mathbf{C}K$ and the "generalized" Borel subalgebra $\hat{\mathfrak{g}}_\geq = \mathfrak{g} \otimes \mathbf{C}[[z]] \oplus \mathbf{C}K$;

$\mathcal{O}_k$ – the category of $\hat{\mathfrak{g}}$-modules at level k (i.e. $K \mapsto k$), and the full subcategory $\tilde{\mathcal{O}}_k \subset \mathcal{O}_k$ consisting of $\hat{\mathfrak{g}}$-modules semisimple over $\mathfrak{g} \subset \hat{\mathfrak{g}}$;

$M^k_\lambda = Ind^{\hat{\mathfrak{g}}}_{\hat{\mathfrak{g}}_\geq} M_\lambda \in \mathcal{O}_k,\ \lambda \in \mathfrak{h}^*$ is a Verma module; $V^k_\lambda = Ind^{\hat{\mathfrak{g}}}_{\hat{\mathfrak{g}}_\geq} V_\lambda \in \tilde{\mathcal{O}}_k,\ \lambda \in P^+$ is a Weyl module; more generally, if $V \in \mathcal{O}_{\mathfrak{g}}$ is a $\mathfrak{g}$-module, then $V^k \in \mathcal{O}_k$ is a $\hat{\mathfrak{g}}$-module obtained by inducing from V; obviously, V^k_λ is a quotient of M^k_λ; each simple module in $\mathcal{O}_k$ is a quotient of M^k_λ for some $\lambda,\ k$; denote this module by L^k_λ;

if $k \not\in \mathbf{Q}$, then $\tilde{\mathcal{O}}_k$ is semi-simple, each object being a direct sum of Weyl modules; the analogue of this statement for $\mathcal{O}_k$ is the equivalence $\mathcal{O}_k \approx \mathcal{O}_{\mathfrak{g}}$ obtained by Finkelberg [12];

for $k + h^\vee = p/q \in \mathbf{Q}_>$ consider the affine Weyl group $W_k = pQ \propto W$, where Q is a root lattice of $\mathfrak{g}$; there is the usual and the dotted (shifted) action of W_k on $\mathfrak{h}^*$; the fundamental domain for the latter is $\bar{C}_{aff} = \bar{C} \cap \{\lambda : 0 \leq (\lambda + \rho, \theta) \leq p\}$, where θ is the highest root of $\mathfrak{g}$; set $P^+_k = P^+ \cap C_{aff}$ where $C_{aff} \subset \bar{C}_{aff}$ is the interior; at some points it is important to ensure that P^+_k contains at least one non-zero weight; for that, in the case of some exceptional root systems, k should be sufficiently large (see, for example, [12] sect. 2.6), and this will be our assumption throughout the text; call $\lambda \in P^+_k$ (sometimes (λ, k) if λ satisfies this condition) dominant; if $k + h^\vee = p/q \in \mathbf{Q}_<$, one defines W_k and an antidominant weight in a similar way;

by [4, 27], $\mathcal{O}_k = \oplus_{\lambda \in P^+_k} \mathcal{O}^\lambda_k$, where $\mathcal{O}^\lambda_k$ is a full subcategory consisting of modules whose composition series contain only irreducible modules $L^k_{w\cdot\lambda}, w \in W_k$; a similar decomposition is true for $\tilde{\mathcal{O}}_k$.

Duality Functors. Given a vector space W, denote by W^d its full dual. If W is a Lie algebra module, then so is W^d.

Given a vector space W carrying a gradation by finite dimensional subspaces, denote by $D(W)$ its restricted dual.

Objects of $\tilde{\mathcal{O}}_k$ are canonically graded. Denote by $D : \tilde{\mathcal{O}}_k \to \tilde{\mathcal{O}}_k$, $M \mapsto D(M)$ the functor such that the $\hat{\mathfrak{g}}$-module structure is defined by precomposing the canonical action on the dual space with an automorphism $\hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$, $g \otimes z^n \mapsto g \otimes (-z)^{-n}$. In a similar manner one defines the duality $D : \mathcal{O}_k \to \mathcal{O}_k$.

The functors $^d, D(.)$ are exact.

There is an involution $^{-}\colon P^+ \to P^+$ so that $V_\lambda^d = V_{\bar\lambda}$.

2.2. **Two lemmas on geometry of weights.**

2.2.1. The following is proved in [17] Lemma 7.7.

Lemma 2.2.1. *Suppose:*
(i) $(\lambda, k), (\mu, k) \in P_k^+$ *are regular;*
(ii)$\bar w \in W$ *satifies* $\bar w(\lambda - \mu) \in P^+$*;*
(iii) ν *is a weight of* $V_{\bar w(\lambda-\mu)}$ *such that* $w_1 \cdot \lambda = w \cdot \mu + \nu$ *for some* $w, w_1 \in W_k$.
Then: $w_1 = w$ *and* $\nu \in W(\lambda - \mu)$.

2.2.2. The Bruhat ordering on W_k determines a partial ordering on $\mathfrak{h}^*$: $\mu <_k \nu$ if and only if there is an (anti)dominant λ so that $\mu = w_1 \cdot \lambda,\ \nu = w_2 \cdot \lambda$ for some $w_1 < w_2 \in W_k$.

Lemma 2.2.2. *Let* λ *be a regular antidominant weight. If* $\lambda + \psi <_k \lambda + \phi$*, then* $|\phi| > |\psi|$*, where* $|.|$ *is the length function coming from the canonical inner product on* $\mathfrak{h}^*$.

Proof repeats word for word the proof of Lemma 1.5 in Appendix 1 of [3] except that instead of hyperplanes one has to consider affine hyperplanes.

3. THE KAZHDAN-LUSZTIG TENSORING

Kazhdan and Lusztig [21, 22, 23] (inspired by Drinfeld [5]) defined a covariant bifunctor

$$\tilde{\mathcal{O}}_k \times \tilde{\mathcal{O}}_k \to \tilde{\mathcal{O}}_k,\ A, B \mapsto A \dot\otimes B. \tag{5}$$

We shall review its definition and main properties.

3.1. **Definition.**

3.1.1. *The set-up.* The notation to be used is as follows:
z is a once and for all fixed coordinate on $\mathbf{CP^1}$;
$L\mathfrak{g}^P, P \in \mathbf{CP^1}$ is the loop algebra attached to P; in other words, $L\mathfrak{g}^P = \mathfrak{g} \otimes \mathbf{C}((z-P)), P \in \mathbf{C}$, and $L\mathfrak{g}^\infty = \mathfrak{g} \otimes \mathbf{C}((z^{-1}))$;
more generally, if $P = \{P_1, ..., P_m\} \subset \mathbf{CP^1}$, then

$$L\mathfrak{g}^P = \oplus_{i=1}^m L\mathfrak{g}^{P_i};$$

$\hat{\mathfrak{g}}^P = L\mathfrak{g}^P \oplus \mathbf{C}K, P \in \mathbf{CP^1}$ is the affine algebra attached to the point P – the canonical central extension of $L\mathfrak{g}^P$; of course, $\hat{\mathfrak{g}}^0 = \hat{\mathfrak{g}}$;

more generally, if $P=\{P_1,...,P_m\}\subset \mathbf{CP^1}$, then $\hat{\mathfrak{g}}^P$ is the direct sum of $\hat{\mathfrak{g}}^{P_i},\ i=1,...,m$ modulo the relation: all canonical central elements K (one in each copy) are equal each other;

$\Gamma=\mathfrak{g}\otimes\mathbf{C}[z,z^{-1},(z-1)^{-1}]$; Γ is obviously a Lie algebra.

The Laurent series expansions at points $\infty,1,0$ produce the Lie algebra homomorphism

$$\epsilon:\Gamma\rightarrow L\mathfrak{g}^{\{\infty,1,0\}}.$$

Lemma 3.1.1. *The map ϵ lifts to a Lie algebra homomorphism*

$$\Gamma\rightarrow\hat{\mathfrak{g}}^{\{\infty,1,0\}}.$$

Proof consists of using the residue theorem, see [22].

By pull-back, any $\hat{\mathfrak{g}}^{\{\infty,1,0\}}$-module is canonically a Γ-module. Further, any $A\in\tilde{\mathcal{O}}_k$ is canonically a $\hat{\mathfrak{g}}^P$-module for any P – by the obvious change of coordinates; refer to this as attaching A to $P\in\mathbf{CP^1}$. Given $A,B,C\in\tilde{\mathcal{O}}_k$, we shall regard $A\otimes B\otimes C$ as a $\hat{\mathfrak{g}}^{\{\infty,1,0\}}$-module meaning that $\hat{\mathfrak{g}}^\infty$ acts on A, $\hat{\mathfrak{g}}^1$ on B, $\hat{\mathfrak{g}}^0$ on C. (There is an obvious ambiguity in this notation.) There arises the space of coinvariants

$$(A\otimes B\otimes C)_\Gamma=(A\otimes B\otimes C)/\Gamma(A\otimes B\otimes C).$$

This construction easily generalizes to the case when instead of three points – $\infty,1,0$ – there are m points, m modules and instead of Γ one considers the Lie algebra of rational functions on $\mathbf{CP^1}$ with m punctures with values in $\mathfrak{g}$. We shall be mostly interested in the case $m=3$ and sometimes in the case $m=2$. If $m=2$, then Γ becomes $\tilde{\mathfrak{g}}=\mathfrak{g}\otimes\mathbf{C}[z,z^{-1}]$.

Lemma 3.1.2. *Suppose $D(B)$ is attached to ∞, A to 0. Then*

$$Hom_{\hat{\mathfrak{g}}}(A,B)=((D(B)\otimes A)_{\tilde{\mathfrak{g}}})^d.$$

Proof can be found in [22]; the reader may also observe that the arguments from 3.2.2 are easily adjusted to this case.

3.1.2. *Definition.* Let $\hat{\Gamma}$ be the central extension of Γ, the cocycle being defined as usual except that one takes the sum of residues at ∞ and 1. Let $\Gamma(0)\subset\hat{\Gamma}$ be the subalgebra consisting of functions vanishing at 0. Obviously, $\Gamma(0)$ can also be regarded as a subalgebra of Γ.

Consider the (full) dual space $(A\otimes B)^d$; it is naturally a $\hat{\Gamma}$-module. $(A\otimes B)^d$ carries the increasing filtration $\{(A\otimes B)^d(N)\}$, where

$$(6)\quad \begin{aligned}&(A\otimes B)^d(N)\\&=\{x\in(A\otimes B)^d:\gamma_1\cdots\gamma_N x=0\text{ if all }\gamma_i\in\Gamma(0),x\in(A\otimes B)\}.\end{aligned}$$

The space $\cup_{N\geq 1}(A\otimes B)^d(N)$ is naturally a $\hat{\mathfrak{g}}$-module. The passage from $(A\otimes B)^d$ to $\cup_{N\geq 1}(A\otimes B)^d(N)$ (or its obvious versions) is often called a functor of smooth vectors.

Define

$$A\dot{\otimes}B = D(\bigcup_{N\geq 1}(A\otimes B)^d(N)). \tag{7}$$

Lemma 3.1.3. *The functor* $\dot{\otimes} : \tilde{\mathcal{O}}_k \times \tilde{\mathcal{O}}_k \to \tilde{\mathcal{O}}_k$ *is right exact in each variable.*

Proof (see *loc. cit.*) The functor $\dot{\otimes}$ is a composition of two dualizations, ^{d} and $D(.)$, and the functor of smooth vectors. It is enough to remark that the first two are exact while the last is only left exact. □

3.2. **Some properties of $\dot{\otimes}$.**

3.2.1. For the future reference we collect some of the properties of $\dot{\otimes}$ in the following

Theorem 3.2.1. *(i)*

$$Hom_{\hat{\mathfrak{g}}}(A\dot{\otimes}B, D(C)) = Hom_{\hat{\mathfrak{g}}}(C, D(A\dot{\otimes}B)) = ((A\otimes B\otimes C)_\Gamma)^d.$$

(ii) If $A, B \in \tilde{\mathcal{O}}_k$ *have a Weyl filtration, then* $A\dot{\otimes}B$ *has also. (Here by Weyl filtration we mean a filtration such that its quotients are Weyl modules.)*

(iii) If $k \notin \mathbf{Q}$, *then* $V_\lambda^k\dot{\otimes}V_\mu^k = (V_\lambda\otimes V_\mu)^k$.

(iv) For any $k \in \mathbf{C}$, $V_\lambda^k\dot{\otimes}V_\mu^k$ *has a Weyl filtration (see(ii)), the multiplicity of* V_ν^k *being equal* $(V_\lambda\otimes V_\mu : V_\nu)$ *(c.f. (iii)).*

(v) There is an isomorphism $A\dot{\otimes}V_0^k \to A$ *for any* $A\in\tilde{\mathcal{O}}_k$.

(vi) There are commutativity and associativity morphisms $A\dot{\otimes}B \approx B\dot{\otimes}A$ *and* $(A\dot{\otimes}B)\dot{\otimes}C \approx A\dot{\otimes}(B\dot{\otimes}C)$ *which endow* $\tilde{\mathcal{O}}_k$ *with the structure of a braided monoidal category.*

3.2.2. *Morphisms and coinvariants.* The description of morphisms in terms of coinvariants (see Theorem 3.2.1(i)) is the hallmark of this theory. Let us briefly explain why (i) holds. There is the obvious isomorphism of vector spaces

$$(A\otimes B\otimes C)^d \to Hom_{\mathbf{C}}(C, (A\otimes B)^d).$$

It induces the map

$$((A\otimes B\otimes C)_\Gamma^d \to Hom_{\hat{\Gamma}}(C, (A\otimes B)^d).$$

By $\hat{\Gamma}$-linearity, it actually gives the map

$$((A\otimes B\otimes C)^d_\Gamma \to Hom_{\hat\Gamma}(C, \bigcup_{N\geq 1}(A\otimes B)^d(N)).$$

It remains to look at (7) and note that $\hat\Gamma$ is dense in $\hat{\mathfrak{g}}$.

3.2.3. *Using the spaces of coinvariants.* A lot about the functor $\dot\otimes$ easily follows from Theorem 3.2.1(i). As an example, let us derive (v). By (i),

$$Hom_{\hat{\mathfrak{g}}}(A\dot\otimes V_0^k, B) = ((A\otimes V_0^k\otimes D(B))_\Gamma)^d \text{ for any } B\in\tilde{\mathcal{O}}_k.$$

As $V_0^k = Ind_{\hat{\mathfrak{g}}_\geq}^{\hat{\mathfrak{g}}}\mathbf{C}$, the Frobenius reciprocity gives

$$(A\otimes V_0^k\otimes B)_\Gamma = (A\otimes D(B))_{\tilde{\mathfrak{g}}},$$

the latter space being $Hom_{\hat{\mathfrak{g}}}(A,B)$ by Lemma 3.1.2. We see that the spaces of morphisms of the modules A and $A\dot\otimes V_0^k$ are equal, hence so are the modules.

Replacing in this argument $\mathbf{C}$ with a suitable finite dimensional $\mathfrak{g}$-module and repeating it three times one gets

$$Hom_{\hat{\mathfrak{g}}}(V_\lambda^k\dot\otimes V_\mu^k, D(V_\nu^k)) = Hom_{\mathfrak{g}}(V_\lambda\otimes V_\mu, V_{\bar\nu}). \tag{8}$$

As for generic k $D(V_\nu^k)\approx V_\nu^k$ (see 2.1), (8) along with Theorem 3.2.1(i) implies Theorem 3.2.1(iii).

4. Affine translation functors

4.1. **Definition.** For any $(\lambda,k)\in P_k^+$ denote by $\tilde{\mathcal{O}}_k^{\lambda}$ the full subcategory of $\tilde{\mathcal{O}}_k$ consisting of modules whose composition factors all have highest weights lying in the orbit $W_k\cdot(\lambda,k)$. There arises the projection

$$p_\lambda : \tilde{\mathcal{O}}_k \to \tilde{\mathcal{O}}_k^{\lambda}.$$

This all has been reviewed in 2.1.

Given $(\lambda,k),(\mu,k)\in P_k^+$, pick $\bar w\in W$ so that $\bar w(\lambda-\mu)\in P^+$. It is easy to see that then $(\bar w(\lambda-\mu),k)\in P_k^+$.

Define the translation functor

$$T_\mu^\lambda:\ \tilde{\mathcal{O}}_k^{\mu}\to\tilde{\mathcal{O}}_k^{\lambda} A\mapsto p_\lambda(V^k_{\bar w(\lambda-\mu)}\dot\otimes A). \tag{9}$$

This functor was first introduced by Finkelberg [12] who, however, considered it only for $k<0$.

As an immediate corollary of the definition, one has

$$T_\lambda^\mu = p_\mu \circ ((V^d_{\bar{w}(\lambda-\mu)})^k \dot{\otimes} ?) \tag{10}$$

4.2. **Rigidity of Weyl modules with dominant highest weight.**

Lemma 4.2.1. *If $(\lambda, k), (\mu, k)$ are regular (i.e. off the affine walls) and $w \in W_k$ satisfies $w \cdot \mu \in P^+$, then*

$$T_\mu^\lambda(V^k_{w\cdot\mu}) = V^k_{w\cdot\lambda}.$$

Proof. By Theorem 3.2.1 (iv), $T_\mu^\lambda(V^k_{w\cdot\mu})$ has a filtration with quotients isomorphic to $V^k_{w_1\cdot\lambda}$, $w_1 \in W_k$ such that $w_1 \cdot \lambda = w \cdot \mu + \nu$, ν being a weight of $V_{\bar{w}(\lambda-\mu)}$. By Lemma 2.2.1, $w_1 = w$. This implies that this filtration has only one term, $V^k_{w\cdot\lambda}$. $\square$

Corollary 4.2.2. *If $(\lambda, k) \in P_k^+$ is regular, then V_0^k is a direct summand of $V_\lambda^k \dot{\otimes} V_{\bar{\lambda}}^k$.*

Proof. Of course $(0, k)$ is dominant regular and $p_0 A$ is a direct summand of A. It remains to observe that $T^0_{\bar{\lambda}} V^k_{\bar{\lambda}} = p_0(V^k_\lambda \dot{\otimes} V^k_{\bar{\lambda}})$ and use Lemma 4.2.1 to get $T^0_{\bar{\lambda}} V^k_{\bar{\lambda}} = V_0^k$. $\square$

We get the maps

$$i_\lambda : V_0^k \to V_\lambda^k \dot{\otimes} V_{\bar{\lambda}}^k, \ e_\lambda : V_{\bar{\lambda}}^k \dot{\otimes} V_\lambda^k \to V_0^k.$$

Observing that the maps between $\dot{\otimes}$-products of Weyl modules are uniquely determined by the induced maps of the corresponding finite dimensional $\mathfrak{g}$-modules (Theorem 3.2.1 and (8)), we see that we can normalize i_λ, e_λ so that the compositions

$$V_\lambda^k = V_0^k \dot{\otimes} V_\lambda^k \stackrel{i_\lambda \dot{\otimes} id}{\to} V_\lambda^k \dot{\otimes} V_{\bar{\lambda}}^k \dot{\otimes} V_\lambda^k \stackrel{id \dot{\otimes} e_\lambda}{\to} V_\lambda^k \tag{11}$$

$$V_{\bar{\lambda}}^k = V_{\bar{\lambda}}^k \dot{\otimes} V_0^k \stackrel{id \dot{\otimes} i_\lambda}{\to} V_{\bar{\lambda}}^k \dot{\otimes} V_\lambda^k \dot{\otimes} V_{\bar{\lambda}}^k \stackrel{e_\lambda \dot{\otimes} id}{\to} V_{\bar{\lambda}}^k,$$

are equal to the identity. By definition (see e.g. [23] III, Appendix) we have

Corollary 4.2.3. *If $(\lambda, k) \in P_k^+$, then V_λ^k and $V_{\bar{\lambda}}^k$ are rigid.*

Consider the functor $V_\lambda^k \dot{\otimes} ? : \tilde{\mathcal{O}}_k \to \tilde{\mathcal{O}}_k, M \mapsto V_\lambda^k \dot{\otimes} M$.

Corollary 4.2.4. *(i)If $(\lambda, k) \in P_k^+$, then the functors $V_\lambda^k \dot{\otimes} ?$ and $V_{\bar{\lambda}}^k \dot{\otimes} ?$ are adjoint, i.e. there is a functor ismorphism*

$$Hom_{\hat{\mathfrak{g}}}(V_\lambda^k \dot{\otimes} A, B) = Hom_{\hat{\mathfrak{g}}}(A, V_{\bar{\lambda}}^k \dot{\otimes} B).$$

(ii) If $(\lambda, k) \in P_k^+$, then the functors $V_\lambda^k \dot{\otimes} ?$ and $V_{\bar{\lambda}}^k \dot{\otimes} ?$ are exact, i.e. send exact short sequences to exact ones.

Proof is standard; for the reader's convenience we reproduce the one from [23] III, Appendix. To prove (i), consider two composition maps

$$\phi : Hom_{\hat{\mathfrak{g}}}(V_\lambda^k \dot{\otimes} A, B) \to Hom_{\hat{\mathfrak{g}}}(V_{\bar{\lambda}}^k \dot{\otimes} V_\lambda^k \dot{\otimes} A, V_{\bar{\lambda}}^k \dot{\otimes} B) \stackrel{i_{\bar{\lambda}}}{\to} Hom_{\hat{\mathfrak{g}}}(A, V_{\bar{\lambda}}^k \dot{\otimes} B),$$

$$\psi : Hom_{\hat{\mathfrak{g}}}(A, V_{\bar{\lambda}}^k \dot{\otimes} B) \to Hom_{\hat{\mathfrak{g}}}(V_\lambda^k \dot{\otimes} A, V_\lambda^k \dot{\otimes} V_{\bar{\lambda}}^k \dot{\otimes} B) \stackrel{e_{\bar{\lambda}}}{\to} Hom_{\hat{\mathfrak{g}}}(V_\lambda^k \dot{\otimes} A, B).$$

By (11), the compositions $\phi \circ \psi$ and $\psi \circ \phi$ are equal to the identity.

(ii) is an easy consequence of (i): we have to prove that $B_1 \to B_2$ is a monomorphism implies that $V_\lambda^k \dot{\otimes} B_1 \to V_\lambda^k \dot{\otimes} B_2$ is also, or, equivalently, that for any $A \in \tilde{\mathcal{O}}_k$ the induced map

$$Hom_{\hat{\mathfrak{g}}}(A, V_\lambda^k \dot{\otimes} B_1) \to Hom_{\hat{\mathfrak{g}}}(A, V_\lambda^k \dot{\otimes} B_2)$$

is also a monomorphism. By (i), it is equivalent to proving that

$$Hom_{\hat{\mathfrak{g}}}(V_{\bar{\lambda}}^k \dot{\otimes} A, B_1) \to Hom_{\hat{\mathfrak{g}}}(V_{\bar{\lambda}}^k \dot{\otimes} A, B_2)$$

is a monomorphism, but this is an obvious corollary of injectivity of the map $B_1 \to B_2$. $\square$

4.3. **Properties of affine translation functors.** Recall that there is the notion of formal character chA for any $A \in \tilde{\mathcal{O}}_k{}^\lambda$, see e.g. [4]. There arises an abelian group of characters, each of the following sets being a topological basis in it:

$\{chV_{w\cdot\lambda}^k,\ w \in W_k\}$, $\{chL_{w\cdot\lambda}^k,\ w \in W_k\}$. Of course the symbols $chV_{w\cdot\lambda}^k$, $chL_{w\cdot\lambda}^k$ should be ignored unless $w \cdot \lambda \in P^+$. Observe that

$$chA = \sum_{w \geq w_o} \bar{n}_w chL_{w\cdot\mu}^k \Leftrightarrow chA = \sum_{w \geq w_o} n_w chV_{w\cdot\mu}^k \tag{12}$$

Proposition 4.3.1. *Let $(\lambda, k), (\mu, k)$ be regular dominant.*

(i) T_μ^λ is exact;

(ii) $T_\mu^\lambda, T_\lambda^\mu$ are adjoint to each other;

(iii) If $chA = \sum_{w\in W_k} n_w chV_{w\cdot\mu}^k$, then $chT_\mu^\lambda A = \sum_{w\in W_k} n_w chV_{w\cdot\lambda}^k$.

(iv) $T_\mu^\lambda(L_{w\cdot\mu}^k) = L_{w\cdot\lambda}^k$;

(v) More generally, $T_\mu^\lambda(.)$ establishes an equivalence of the submodule lattices of $V_{w\cdot\mu}^k$ and $V_{w\cdot\lambda}^k$.

Proof. (i) T_μ^λ is exact as a composition of the exact functors p_λ and $(V_{\bar{w}(\lambda-\mu)}^d)^k \dot{\otimes} ?$, see Corollary 4.2.4 (ii).

(ii) By Corollary 4.2.4 (i), one has for any $A \in \tilde{\mathcal{O}}_k{}^\mu$, $B \in \tilde{\mathcal{O}}_k{}^\lambda$

(13)
$$\begin{aligned} Hom_{\hat{\mathfrak{g}}}(T_\mu^\lambda A, B) &= Hom_{\hat{\mathfrak{g}}}(p_\lambda(V^k_{\bar{w}(\lambda-\mu)}\dot{\otimes}A), B) = Hom_{\hat{\mathfrak{g}}}(V^k_{\bar{w}(\lambda-\mu)}\dot{\otimes}A, B) \\ &= Hom_{\hat{\mathfrak{g}}}(A, (V^d_{\bar{w}(\lambda-\mu)})^k\dot{\otimes}B) = Hom_{\hat{\mathfrak{g}}}(A, p_\mu(V^d_{\bar{w}(\lambda-\mu)})^k\dot{\otimes}B) \\ &= Hom_{\hat{\mathfrak{g}}}(A, T_\lambda^\mu B). \end{aligned}$$

(iii) follows at once from (i) (if one uses the local composition series, see e.g. [4]).

(iv) Let $T_\mu^\lambda(L^k_{w_0\cdot\mu})$ be reducible. There arises an exact sequence with non-zero N
$$0 \to N \to T_\mu^\lambda(L^k_{w_0\cdot\mu}) \to L^k_{w_0\cdot\lambda} \to 0.$$

Applying T_λ^μ to it one gets
$$0 \to T_\lambda^\mu(N) \to T_\lambda^\mu(T_\mu^\lambda(L^k_{w_0\cdot\mu})) \to T_\lambda^\mu(L^k_{w_0\cdot\lambda}) \to 0.$$

By (iii) and (12), $ch(T_\lambda^\mu(T_\mu^\lambda(L^k_{w_0\cdot\mu}))) = chL^k_{w_0\cdot\mu}$ and $chT_\lambda^\mu(N) \neq 0$; therefore $chT_\lambda^\mu(L^k_{w_0\cdot\lambda}) < chL^k_{w_0\cdot\mu}$. Contradiction.

(v) Here proof is an obvious version of that of (iv). By (ii) it is enough to show that if $A \subset B \subset V^k_{w\cdot\mu}$, then $T_\mu^\lambda(A) \subset T_\mu^\lambda(B) \subset V^k_{w\lambda}$. Using (12) and passing to quotients, if necessary, the problem is reduced to the case when B is a highest weight module. In this case the arguments of (ii) go through practically unchanged. $\square$

4.4.

Theorem 4.4.1. *The functor $T_\mu^\lambda : \tilde{\mathcal{O}}_k{}^\mu \to \tilde{\mathcal{O}}_k{}^\lambda$ is an equivalence of categories.*

Proof. It is enough show that $T_\mu^\lambda \circ T_\lambda^\mu : \tilde{\mathcal{O}}_k{}^\lambda \to \tilde{\mathcal{O}}_k{}^\lambda$ is equivalent to the identity. In other words, we want to show that $id : A \to A,\ A \in \tilde{\mathcal{O}}_k{}^\lambda$ is transformed into an isomorphism in $Hom_{\hat{\mathfrak{g}}}(T_\mu^\lambda \circ T_\lambda^\mu(A), A)$. We already know this when A is simple, see Corollary 4.3.1 (ii). Using the local composition series one proves it for an arbitrary A.

An alternative way to prove the theorem is to observe that by Corollary 4.2.2 the action of $T_\mu^\lambda \circ T_\lambda^\mu$ is equivalent to that of $V_0^k\dot{\otimes}?$, the latter being equivalent to id by Theorem 3.2.1 (v). $\square$

4.5. **Generalizing from $\tilde{\mathcal{O}}_k$ to $\mathcal{O}_k$.** Our two key results – Proposition 4.3.1 and Theorem 4.4.1 – can be carried over to the category $\mathcal{O}_k$. Let us briefly explain it. We will be using subcategories $\mathcal{O}_k^\lambda \subset \mathcal{O}_k$ (see 2.1) only when $k + h^\vee \in \mathbf{Q}_>$ and λ is integral, although the last condition can be easily relaxed.

It is non-trivial (if at all meaningful) to carry the Kazhdan-Lusztig tensoring over to the entire $\mathcal{O}_k$. It is however straightforward to extend it to the functor

$$\dot{\otimes} : \tilde{\mathcal{O}}_k \times \mathcal{O}_k \to \mathcal{O}_k,$$

as proposed by Finkelberg [12]. One basic property of this operation absolutely analogous (along with the proof) to Theorem 3.2.1 (iv) is as follows.

Lemma 4.5.1. *If $A \in \tilde{\mathcal{O}}_k$ has a filtration by Weyl modules and $B \in \mathcal{O}_k$ has a filtration by Verma modules, then $A\dot{\otimes}B$ also has a filtration by Verma modules. Further the multipliciites are the same as in the finite dimensional case; for instance*

$$(V^k_\lambda \dot{\otimes} M^k_\mu : M^k_\nu) = (V_\lambda \otimes M_\mu : M_\nu). \tag{14}$$

Given this one can easily inspect our exposition of affine translation functors and observe that quite a lot carries over to the setting of $\mathcal{O}_k$ word for word except that at the appropriate places Weyl modules are to be changed for the corresponding Verma modules. Here are some examples:

(i) definition of $T^\lambda_\mu : \mathcal{O}^\mu_k \to \mathcal{O}^\lambda_k$;

(ii) the Verma filtration of $V^k_\lambda \dot{\otimes} M^k_{w\cdot\mu}, w \in W_k$ and Lemma 2.2.1 imply that $T^\lambda_\mu(M^k_{w\cdot\mu}) = M^k_{w\cdot\lambda}$ if $(\mu, k), (\lambda, k)$ are regular (c.f. Lemma 4.2.1); observe that we can now drop the condition that $w \cdot \mu \in P^+$;

(iii) therefore Proposition 4.3.1 holds with the indicated changes.

We get

Theorem 4.5.2. *The functor $T^\lambda_\mu : \mathcal{O}^\mu_k \to \mathcal{O}^\lambda_k$ is an equivalence of categories if λ, μ are integral and both belong to the same Weyl chamber.*

5. Annihilating ideals of highest weight modules

5.1. **Fields and coinvariants.** The usual tensor functor $\otimes : M, N \mapsto M \otimes N$ has the following fundamental (and trivial) property: there is a natural map

$$\begin{aligned} &N \to Hom_{\mathbf{C}}(M, M \otimes N) \\ &\qquad\qquad n \mapsto n(.) \text{ such that } n(m) = m \otimes n. \end{aligned} \tag{15}$$

Here we shall explain the $\dot{\otimes}$-analogue of this map

5.1.1. By Theorem 3.2.1 (v), $A\dot{\otimes}V_0^k \approx A$ for any $A \in \tilde{\mathcal{O}}_k$. Therefore by Theorem 3.2.1 (i), there is a natural isomorphism

$$((A \otimes V_0^k \otimes D(A))_\Gamma)^d \approx Hom_{\hat{\mathfrak{g}}}(A, B),$$

for any $B \in \tilde{\mathcal{O}}_k$.

Recall that the space $((A \otimes V_0^k \otimes D(A))_\Gamma)^d$ was defined by means of Γ, the latter being defined by choosing three points, $\infty, 1, 0$, see the end of 3.2.3. The choice of points was, of course, rather arbitrary. Keeping $\infty, 0$ fixed and A, $D(B)$ attached to $\infty, 0$ resp., we shall allow the third point to vary. We get then the family of Lie algebras $\Gamma_t, t \in \mathbf{C}^*$ and the family of the one-dimensional spaces (c.f. 3.1.1)

$$< A, V_0^k, D(B) >_t := ((A \times V_0^k \times D(B))_{\Gamma_t})^d,\ t \in \mathbf{C}^*.$$

These naturally arrange in a trivial line bundle over $\mathbf{C}^*$, the fiber being isomorphic to

$$< A, V_0^k, D(B) >_t = (A \otimes D(B))_{\tilde{\mathfrak{g}}} = Hom_{\hat{\mathfrak{g}}}(A, B),$$

by the arguments using Frobenius reciprocity as in 3.2.3. Pick a section of this bundle by choosing $\phi \in Hom_{\hat{\mathfrak{g}}}(A, B)$.

Hence we get a trilinear functional (depending on $t \in \mathbf{C}^*$)

$$\Phi_t^\phi \in < A, V_0^k, D(B) >_t \subset (A \otimes V_0^k \otimes D(B))^d.$$

Reinterprete it as the linear map:

$$\Phi_t^\phi(.) : V_0^k \to (A \otimes D(B))^d, \tag{16}$$

or, equivalently,

$$\tilde{\Phi}_t^\phi(.) : V_0^k \to Hom_{\mathbf{C}}(A, D(B)^d),\ t \in \mathbf{C}^*. \tag{17}$$

The latter map is an analogue of $N \to Hom_{\mathbf{C}}(M, M \otimes N)$ mentioned above. To analyze its properties observe that there is an obvious embedding $B \to (D(B))^d$. It does not, of course, allow us to interprete $\tilde{\Phi}_t^\phi(v),\ v \in V_0^k$ as an element of $Hom_{\mathbf{C}}(A, B)$ depending on t. But, as the following lemma shows, Fourier coefficients of $\tilde{\Phi}_t^\phi(v),\ v \in V_0^k$ are actually elements of $Hom_{\mathbf{C}}(A, B)$. To formulate this lemma observe that there is a natural gradation on A and B consistent with that of $\tilde{\mathfrak{g}}$; e.g. $A = \oplus_{n \geq 0} A[n],\ dim A[n] < \infty$.

Lemma 5.1.1. *Let B be either A or a quotient of A, $id : A \to B$ be the natural projection. Then:*

(i) $\Phi_t^{id}(vac)(x, y) = y(x)$, where vac is understood as the generator of V_0^k;

(ii) more generally, if $v \in V_0^k[n]$, $x \in A[m]$, $y \in D(B)[l]$, *then*

$$\Phi_t^{id}(v)(x,y) \in \mathbf{C} \cdot t^{-l+m-n}.$$

Proof. Given $g \in \mathfrak{g}$, denote by $g_n \in \hat{\mathfrak{g}}^P$ the element $g \otimes (z-P)^n$ or $g \otimes z^{-n}$ if $P = \infty$. (It should be clear from the context which P is meant.) Thus $g_n x = (g \otimes z^{-n})x$ if $x \in A$, the A being attached to ∞; similarly, $g_n x = (g \otimes z^n)x$ if $x \in D(B)$, the $D(B)$ being attached to 0.

(i) can be proved by an obvious induction on the degree of x and y using the following formula (which follows from the definition of $((A \times V_0^k \times D(B))_\Gamma)^d$ and the Laurent expansions of z^{-n} at ∞ and 0):

$$\Phi_t^{id}(vac)(g_n x, y) = -\Phi_t^{id}(vac)(x, g_{-n} y).$$

To prove (ii) observe, first, that (i) is a particular case of (ii) when $v = vac$. One then proceeds by induction on n using the formula (which again follows from the definition of $((A \times V_0^k \times D(B))_\Gamma)^d$ and the Laurent expansions of $(z-t)^{-n}$ at ∞ and 0):

$$\begin{aligned} (18) \quad & (-1)^{n-1}(n-1)!\Phi_t^{id}(g_{-n}v)(x,y) \\ & = (\frac{d}{dt})^{n-1}\{\sum_{i=1}^{\infty} t^{i-1}\Phi_t^{id}(v)(g_i x, y) - \sum_{i=0}^{\infty} t^{-i-1}\Phi_t^{id}(v)(x, g_i y)\}. \end{aligned}$$

□

Observe that the spaces A, B being graded, the space $Hom_{\mathbf{C}}(A, D(B))$ is also. Lemma 5.1.1 means that although the map $\tilde{\Phi}_t^{id}(.)$ from (17) cannot be interpreted as an element of $Hom_{\mathbf{C}}(A, B)$, its Fourier components can because they are homogeneous. To compare with [14] introduce the following notation: for any $v \in V_0^k[n]$ set

$$Y(v,t) = \sum_{i \in \mathbf{Z}} v_i t^{-i-n}, \tag{19}$$

where

$$v_i := \oint \tilde{\Phi}_t^{id}(v) t^{i+n-1}\, dt : A[l] \to B[l+i], \tag{20}$$

for all $l \geq 0$, and call the generating functions $Y(v,t)$ *fields.* For example, it easily follows from the formulae above that

$$x(t) := Y(x_{-1}vac, t) = \sum_{i \in \mathbf{Z}} x_i t^{-i-1}, \tag{21}$$

producing the famous *current* $x(t)$. Another fact easily reconstructed from the formulae above (especially from the proof of Lemma 5.1.1) is that

$$(-1)^{n-1}(n-1)!Y(x_{-n}v,t) =: x(t)^{(n-1)}Y(v,t) :, \tag{22}$$

where we set

$$: x(t)^{(n-1)}Y(v,z) : = (x(z)^{(n-1)})_{-}Y(v,t) + Y(v,t)(x(z)^{(n-1)})_{+},$$

$(x(z)^{(n-1)})_{\pm}$ being defined as usual (see e.g. [15]). It follows that all fields are infinite combinations of elements of $\hat{\mathfrak{g}}$.

The expressions $Y(v,t)$ are not only formal generating functions. In this notation Lemma 5.1.1 can be rewritten as follows.

Corollary 5.1.2. *Under the assumptions of Lemma 5.1.1,*

$$\Phi_t^{id}(v)(x,y) = y(Y(v,t)x).$$

5.1.2. *Generalization – vertex operators.* The considerations of 5.1.1 are easily generalized as follows. (We shall skip the proofs as they essentially repeat those in 5.1.1.)

Replace V_0^k with V_λ^k and pick $A, B \in \tilde{\mathcal{O}}_k$ so that the space $< A, V_\lambda^k, D(B) >_t \neq 0$. For any $\phi \in < A, V_\lambda^k, D(B) >_t$ we get a map

$$\begin{aligned} Y(.,t):\ & V_\lambda^k \to Hom_{\mathbf{C}}(A, B((t,t^{-1}))) \\ & V_\lambda^k \ni v \mapsto Y(v,t) = \sum_{i\in\mathbf{Z}} v_i t^{-i-\tilde{v}},\ v_i \in Hom_{\mathbf{C}}(A,B). \end{aligned} \tag{23}$$

$Y(v,t),\ v \in V_\lambda^k$ is a generating function having all properties its counterpart from 5.1.1 with one notable exception. Consider the "upper floor" of V_λ^k: $V_\lambda \subset V_\lambda^k$. The Fourier components of the fields $Y(v,t),\ v \in V_\lambda,\ \lambda \neq 0$ generate a $\hat{\mathfrak{g}}$-submodule of $Hom_{\mathbf{C}}(A,B)$ isomorphic to the loop module $L(V_\lambda) = V_\lambda \otimes \mathbf{C}[z,z^{-1}]$. Strange as it may seem to be, if $\lambda = 0$, then instead of $\mathbf{C}[z,z^{-1}]$ this construction gives simply $\mathbf{C}$ – this was explained above.

The embedding $L(V_\lambda) \subset Hom_{\mathbf{C}}(A,B)$ is called *a vertex operator.* It is easy to see that all vertex operators are obtained via the described construction.

5.2. **vertex operator algebras.** We now recall that a vertex operator algebra (VOA) is defined to be a graded vector space $\bigcup_{i\in\mathbf{Z}} V[i],\ dimV_i < \infty$ along with a map

$$Y(.,t) : V \to End(V)((t,t^{-1})),$$

satisfying certain axioms among which we mention *associativity* and *commutativity* axioms, see e.g. [14, 15]. Similarly one defines the notion of a module (submodule) over a VOA. A VOA is a module over itself; call an ideal of a VOA a submodule of a VOA as a module over itself. Observe that it follows from the associativity axiom that the Fourier components of fields $Y(v,t)$, $v\in V$ close in a Lie algebra, $Lie(V)$. In this way, an ideal of a VOA V produces an ideal of $Lie(V)$ in the Lie algebra sense. Not any ideal of $Lie(V)$ can be obtained in this way. We shall call to such an ideal *vertex ideal of* $Lie(V)$.

It follows from [15] (see also [9]) that the constructions of 5.1.1 give: $(V_0^k, Y(.,t))$ is a vertex operator algebra and each $A\in\tilde{\mathcal{O}}_k$ is a module over it. $Lie(V_0^k)$ is habitually denoted $U(\hat{\mathfrak{g}})_{loc}$ and called *a local completion* of $U(\hat{\mathfrak{g}})$, even though it is not an associative algebra! A moment's thought shows that the vertex ideal lattice of $U(\hat{\mathfrak{g}})_{loc}$is isomorphic with the submodule lattice of V_0^k considered as a $\hat{\mathfrak{g}}$-module.

5.3. Here we prove the following theorem – one of the main results of this paper.

Theorem 5.3.1. *Let $k\in\mathbf{Q}_>$, $(\lambda,k),(0,k)\in P_k^+$ be regular. Denote by $\Omega(V_\lambda^k)$ the submodule lattice of V_λ^k, and by $\Omega(U(\hat{\mathfrak{g}})_{loc})$ the vertex ideal lattice of $U(\hat{\mathfrak{g}})_{loc}$ at the level k. There is a lattice equivalence*

$$\omega:\Omega(U(\hat{\mathfrak{g}})_{loc})\rightarrow\Omega(V_\lambda^k),\quad \Omega(U(\hat{\mathfrak{g}})_{loc})\ni I\mapsto IV_\lambda^k. \tag{24}$$

Proof.

First of all, by definition 5.2 ω is equivalently reinterpreted as a map of the submodule lattices of the $\hat{\mathfrak{g}}$-modules: $\omega:\ \Omega(V_0^k)\rightarrow\Omega(V_\lambda^k)$. In what follows we shall make use of this reinterpretation.

Consider the translation functor: T_0^λ. If $N\subset V_0^k$ is a submodule, then on the one hand we have

$$T_0^\lambda(V_0^k)=V_\lambda^k\dot{\otimes}V_0^k(=V_\lambda^k),$$

and therefore

$$T_0^\lambda(V_0^k/N)=V_\lambda^k\dot{\otimes}(V_0^k/N).$$

By Theorem 3.2.1 and Corollary 5.1.2,

$$\begin{aligned}Hom_{\hat{\mathfrak{g}}}(T_0^\lambda(V_0^k/N),?) &=< V_0^k/N, V_\lambda^k, D(?)>_t\\ &= Hom_{\hat{\mathfrak{g}}}(V_\lambda^k/\omega(N),?).\end{aligned} \tag{25}$$

On the other hand, by Proposition 4.3.1 (i)

$$T_0^\lambda(V_0^k/N)=V_\lambda/T_0^\lambda(N).$$

We conclude immediately that $\omega(N) = T_0^\lambda(N)$. It remains to recollect that T_0^λ is an isomorphism of the submodule lattices by Proposition 4.3.1 (v). $\square$

An application of this result to annihilating ideals of *admissible representations* is as follows. Recall that if $k + h^\vee \in \mathbf{Q}_>$, $(\lambda, k) \in P_k^+$ is regular, then L_λ^k is called admissible [20]. L_λ^k is an irreducible quotient of V_λ^k be a submodule N_λ^k generated by one singular vector, see also [20]. By Theorem 5.3.1, $\omega(N_0^k) = N_\lambda^k$. We get

Corollary 5.3.2. *The annihilating ideal of an admissible representation equals $Lie(N_0^k)$; in particular, it is generated (in the sense of VOA) by one singular vector of V_0^k.*

Remarks. (i) Theorem 5.3.1 reduces the problem of classifying annihilating ideals to the easier problem of classifying submodules of V_0^k. What can be said about the latter? It has been known for a while that in the simplest case of $\hat{sl}_2$ V_0^k contains a unique proper non-trivial submodule. In general, multiplicities in composition series are described by Kazhdan-Lusztig polynomials. In the recent work [24] Lusztig exhibits an example when infinitely many multiplicities are non-zero and thus there are infinitely many different submodules. This result makes one believe that this is " usually" the case.

(ii) In the case $\mathfrak{g} = \mathfrak{s}l_2$, Corollary 5.3.2 follows from the more general results of [10], see also [11].

(iii) If the Feigin-Frenkel conjecture on the singular support of L_0^k (theorem in the sl_2-case, see [11]) is correct, then Corollary 5.3.2 implies its validity for any admissible representation from $\tilde{\mathcal{O}}_k$ and thus would give a new example of rational conformal field theory.

(iv) Another way to think of Corollary 5.3.2 is that L_0^k is a VOA and L_λ^k is a module over it; in the sl_2-case, this point of view is adopted in [1, 6].

6. TILTING FUNCTORS – THE NEGATIVE LEVEL CASE

¿From now on $k + h^\vee$ is a negative rational number. The structure of $\mathcal{O}_k$ can be described as follows. Consider an antidominant weight (λ, k) with integral λ. Denote by $\mathcal{O}_k^\lambda$ the full subcategory consisting of modules containing only irreducibles with highest weights lying in the W_k-orbit of an antidominant weight (λ, k). By [4, 27], there is the decomposition $\mathcal{O}_k = \oplus_\lambda \mathcal{O}_k^\lambda$ (c.f. 2).

It follows from [19] that any module from $\mathcal{O}_k^\lambda$ has a finite length. There arises the Grothendieck ring, each of the following sets $\{ch(M_{w\cdot\lambda}^k),\ w \in W_k\}$, $\{ch(L_{w\cdot\lambda}^k),\ w \in W_k\}$ being a basis of it.

All this carries over to the case of $\tilde{\mathcal{O}}_k$ by replacing M_λ^k with V_λ^k and making sure that in the latter case λ is dominant integral.

6.1. **Tilting Modules. Definition** A module W from $\mathcal{O}_k$ ($\tilde{\mathcal{O}}_k$ resp.) is called *tilting* if both W and $D(W)$ possess a filtration by Verma (Weyl resp.) modules.

For instance, a Verma module with an antidominant highest weight is a tilting module in $\mathcal{O}_k$, for it is irreducible and, therefore, isomorphic to its dual. Likewise, a Weyl module with an antidominant highest weight is a tilting module in $\tilde{\mathcal{O}}_k$.

Proposition 6.1.1. *(i) Any tilting module from $\tilde{\mathcal{O}}_k$ is a direct sum of indecomposable tilting modules. For any (μ, k), μ being dominant integral, there is a uniquely determined indecomposable tilting module $\tilde{W}_{\mu,k}$ such that*

$$ch(\tilde{W}_{\mu,k}) = ch(V_\mu^k) + \sum_{\nu <_k \mu} c_\nu ch(V_\nu^k).$$

(For definition of $<_k$ see 2.2.2.) The map $\mu \mapsto \tilde{W}_{\mu,k}$ establishes a bijection between the set of weights satisfying the above mentioned condition and the set of isomorphism classes of indecomposable tilting modules in $\tilde{\mathcal{O}}_k$.

(ii) Likewise, any tilting module from $\mathcal{O}_k$ is a direct sum of indecomposable tilting modules. For any (μ, k), there is a uniquely determined indecomposable tilting module $W_{\mu,k}$ such that

$$ch(W_{\mu,k}) = ch(M_\mu^k) + \sum_{\nu <_k \mu} c_\nu ch(M_\nu^k).$$

The map $(\mu, k) \mapsto W_{\mu,k}$ establishes a bijection between the set of weights and the set of isomorphism classes of indecomposable tilting modules in $\mathcal{O}_k$.

Proof (i) is proved in [23] using general results of Ringel [26]; (ii) can be proved in the same way. A lucid exposition suited for the $\mathcal{O}$-category case can also be found in [2]. $\square$

Corollary 6.1.2. *The set of characters of tilting modules in $\mathcal{O}_k$ ($\tilde{\mathcal{O}}_k$ resp.) is a basis of the Grothendieck ring of $\mathcal{O}_k$ ($\tilde{\mathcal{O}}_k$ resp.)*

Proof – obvious.

Kazhdan and Lusztig prove that the category $\tilde{\mathcal{O}}_k$ is rigid. Some of the consequences of this fact are collected in the following

Lemma 6.1.3. *(i) For any $A \in \tilde{\mathcal{O}}_k$, the functor $A\dot{\otimes}? : \mathcal{O}_k \to \mathcal{O}_k$ is exact.*

(ii) For any $A \in \tilde{\mathcal{O}}_k$, $B \in \mathcal{O}_k$, one has: $D(A\dot{\otimes}B) = D(A)\dot{\otimes}D(B)$.

(iii) If $A \in \tilde{\mathcal{O}}_k$, $B \in \mathcal{O}_k$ are both tilting, then $A\dot{\otimes}B$ is also.

Proof (i) is proved in the same way Corollary 4.2.4 was proved.

(ii) is a general fact about monoidal categories, see [23] III, Proposition A.1.

(iii) is an immediate consequence of (ii) and Lemma 4.5.1 which is valid along with its proof for all values of central charge. $\square$

6.2. **Tilting Functors.**

6.2.1. Modules depending on a parameter. Let t be an indeterminate and $R(t)$ the ring of rational functions having poles only on a positive real ray. Along with $\mathcal{O}_k$, introduce the category $\mathcal{O}_t$ defined in the same way except that $\mathbf{C}$ as a ground field is replaced with $R(t)$, that is, $\hat{\mathfrak{g}}$ is regarded as an $R(t)$-algebra and objects of $\mathcal{O}_t$ are required to be free $R(t)$-modules. There is the specialization functor

$$sp_k : \mathcal{O}_t \to \mathcal{O}_k,\ M \mapsto M(k) \stackrel{\text{def}}{=} M/(t-k)M.$$

Let $\mathcal{A}_t \subset \mathcal{O}_t$ be the full subcategory of modules V such that $V(k)$ has a Weyl filtration. Define

$$^{fam}\mathcal{O}_k \stackrel{\text{def}}{=} sp_k(\mathcal{A}_t).$$

In other words, $^{fam}\mathcal{O}_k$ consists of modules having a Verma filtration and allowing inclusion in a family analytically depending on k.

One defines in a similar way $^{fam}\mathcal{O}_k^\lambda \subset \mathcal{O}_k^\lambda$ and the Weyl versions $^{fam}\tilde{\mathcal{O}}_k$, $^{fam}\tilde{\mathcal{O}}_k{}^\lambda \subset^{fam} \tilde{\mathcal{O}}_k$.

Examples

(i) $V_\lambda^k \in^{fam} \tilde{\mathcal{O}}_k$, $M_\lambda^k \in^{fam} \mathcal{O}_k$.

(ii) $V_\lambda^k\dot{\otimes}V_\mu^k \in^{fam} \tilde{\mathcal{O}}_k$, $V_\mu^k\dot{\otimes}M_\lambda^k \in^{fam} \mathcal{O}_k$.

(iii) Direct summands of any of the modules from (ii) belong to the corresponding categories.

6.2.2. Main Results. ¿From now on we shall have two fixed regular antidominant weights (λ_r, k) and (λ_l, k) so that λ_r, λ_l are integral.

Definition Tilting functor is a direct summand of the functor

$$p_{\lambda_l} \circ (W\dot{\otimes}?) :^{fam} \mathcal{O}_k^{\lambda_r} \to^{fam} \mathcal{O}_k^{\lambda_l},$$

where $W \in \tilde{\mathcal{O}}_k$ is tilting.

The following is one of our main results.

Theorem 6.2.1. *(i) Each tilting functor is a direct sum of indecomposable ones.*

(ii) There is a 1-1 correspondence between tilting functors and elements of W_k. The functor $\Phi_w \in Funct(^{fam}\mathcal{O}_k^{\lambda_r}, ^{fam}\mathcal{O}_k^{\lambda_l})$ attached to $w \in W_k$ is uniquely determined by the condition that

$$\Phi_w(M^k_{\lambda_r}) = W_{w\cdot\lambda_l,k}.$$

We shall prove this theorem in 6.3. One consequence of this theorem is that the category of tilting functors, as a subcategory of $Funct(^{fam}\mathcal{O}_k^{\lambda_r}, ^{fam}\mathcal{O}_k^{\lambda_l})$, is semi-simple and isomorphic to the category of tilting modules in $\mathcal{O}_k^{\lambda_l}$. Denote this category by $Tilt(\lambda_l, \lambda_r)$.

Suppose now that we are given three antidominant highest weights, $(\lambda_1, k), (\lambda_2, k), (\lambda_3, k)$, and two tilting functors: $\Phi \in Funct(^{fam}\mathcal{O}_k^{\lambda_1}, ^{fam}\mathcal{O}_k^{\lambda_2})$ and $\Psi \in Funct(^{fam}\mathcal{O}_k^{\lambda_2}, ^{fam}\mathcal{O}_k^{\lambda_3})$. Then $\Psi \circ \Phi$ is also a tilting functor. This follows from Lemma 6.1.3(iii) and the associativity morphism: $W_2\dot{\otimes}(W_1\dot{\otimes}B) = (W_2\dot{\otimes}W_1)\dot{\otimes}B$. If in addition $\lambda_l = \lambda_r = \lambda$, then the category $Tilt(\lambda) \stackrel{def}{=} Tilt(\lambda_r, \lambda_l)$ is closed under $\circ$ and the pair $(Tilt(\lambda), \circ)$ becomes a semi-simple monoidal category. The following is another main result concerning tilting functors.

Theorem 6.2.2. *The Grothendieck ring of $(Tilt(\lambda), \circ)$ is isomorphic to the group algebra of W_k.*

Proof of this theorem is to be found in the next section.

6.3. **Further theorems on tilting functors and proofs.**

6.3.1. Analogously to the semi-simple case (see Theorems 1.1.1 and 1.1.2), the key to the proof of Theorem 6.2.1 is the following result.

Theorem 6.3.1. *Let $\Phi, \Psi \in Funct(^{fam}\mathcal{O}_k^{\lambda_r}, ^{fam}\mathcal{O}_k^{\lambda_l})$ be tilting functors. The natural map*

$$Mor(\Phi, \Psi) \to Hom_{\hat{\mathfrak{g}}}(\Phi(M^k_{\lambda_r}), \Psi(M^k_{\lambda_r}))$$

is a vector space isomorphism.

Proof It is obviously enough to consider the case when $\Phi = A_1\dot{\otimes}?$, $\Psi = A_2\dot{\otimes}?$, A_1, A_2 being tilting modules. As $\tilde{\mathcal{O}}_k$ is rigid, we have

$$Hom_{\hat{\mathfrak{g}}}(A_1\dot{\otimes}?, A_2\dot{\otimes}?) = Hom_{\hat{\mathfrak{g}}}(D(A_2)\dot{\otimes}A_1\dot{\otimes}?, ?).$$

Therefore it is enough to prove that the natural map

$$Mor(A\dot{\otimes}?, id) \to Hom_{\hat{\mathfrak{g}}}(A\dot{\otimes}M^k_{\lambda_r}, M^k_{\lambda_r}) \tag{26}$$

is an isomorphism for any tilting A.

Injectivity of (26) Let $\phi \in Mor(A\dot{\otimes}?, id)$ be such that its value $\phi(id)$ on $id \in End(M^k_{\lambda_r})$ is zero. Consider the exact sequence

$$0 \to M^k_{\lambda_r} \to M^k_{u\cdot\lambda_r} \to M^k_{u\cdot\lambda_r}/M^k_{\lambda_r} \to 0.$$

By Lemma 6.1.3 (i), the sequence

$$0 \to A\dot{\otimes}M^k_{\lambda_r} \to A\dot{\otimes}M^k_{u\cdot\lambda_r} \to A\dot{\otimes}(M^k_{u\cdot\lambda_r}/M^k_{\lambda_r}) \to 0$$

is also exact. Therefore,

$$\phi(id) \in Hom_{\hat{\mathfrak{g}}}(A\dot{\otimes}M^k_{u\cdot\lambda_r}, M^k_{u\cdot\lambda_r})$$

factors through to the map

$$A\dot{\otimes}(M^k_{u\cdot\lambda_r}/M^k_{\lambda_r}) \to M^k_{u\cdot\lambda_r}.$$

To show that this map can only be zero we make use of the notion of singular support.

Recall that under certain technical assumptions (which are satisfied in the $\mathcal{O}$-category case,see e.g. [11] for the necessary definitions) one defines the singular support of a Lie algebra module to be the zero set of the annihilating ideal of the corresponding graded object; thus singular support is a conical subset of the dual to the Lie algebra in question. For example, the singular support of a Verma module is all functionals vanishing on the Borel subalgebra. If a module has a finite filtration, then its singular support is the union of the singular supports of the successive quotients. Thus any module with a Verma filtration, $A\dot{\otimes}M^k_{u\cdot\lambda_r}$ for example, has the same support as a Verma module.

A quotient of a Verma module by a proper submodule, however, has a smaller singular support; it is obtained by imposing additional equations, those coming from the symbols of the submodule. By the same token, the singular support of $A\dot{\otimes}(M^k_{u\cdot\lambda_r}/M^k_{\lambda_r})$ is strictly less than that of a Verma module. Therefore, the latter module may not contain a Verma module as a subquotient and the map

$$A\dot{\otimes}(M^k_{u\cdot\lambda_r}/M^k_{\lambda_r}) \to M^k_{u\cdot\lambda_r}$$

can only be zero. Thus $\phi(id)$ is zero on any Verma module and, by exactness, is also on any module from $\mathcal{O}_k$. Injectivity has been proven.

Surjectivity of (26) What remains to be done is to show that any element of $Hom_{\hat{\mathfrak{g}}}(A\dot{\otimes}M^k_{\lambda_r}, M^k_{\lambda_r})$ naturally determines an element of $Hom_{\hat{\mathfrak{g}}}(A\dot{\otimes}B, B)$ for any $B \in^{fam} \mathcal{O}^{\lambda_r}_k$ and any tilting $A \in \tilde{\mathcal{O}}_k$. We begin with calculating the space $Hom_{\hat{\mathfrak{g}}}(A\dot{\otimes}M^k_{\lambda_r}, M^k_{\lambda_r})$. Recall that A has a Weyl filtration with quotients $V^k_{\mu_i}$, $i = 1, ..., n$.

Lemma 6.3.2.

$$Hom_{\hat{\mathfrak{g}}}(A\dot{\otimes}M^k_{\lambda_r}, M^k_{\lambda_r}) = \oplus_{i=1}^n V_{\mu_i}[0],$$

where $V_{\mu_i}[0]$ stands for the zero weight subspace of the $\mathfrak{g}$-module V_{μ_i}.

Proof Let, first, $A = V^k_\mu$. Then by Theorem 3.2.1 we have

$$\begin{aligned} (27) \quad Hom_{\hat{\mathfrak{g}}}(V^k_\mu\dot{\otimes}M^k_{\lambda_r}, M^k_{\lambda_r}) &= [(V^k_\mu \otimes M^k_{\lambda_r} \otimes D(M^k_{\lambda_r}))_\Gamma]^d \\ &= [(V^k_\mu \otimes M^k_{\lambda_r} \otimes M^k_{\lambda_r})_\Gamma]^d = V_\mu[0], \end{aligned}$$

where the second equality follows from the isomorphism $M^k_{\lambda_r} = D(M^k_{\lambda_r},$ while the third one follows from the Frobenius reciprocity (c.f. the derivation of (8) in 3.2.3.

To treat the case when A has a Weyl filtration with more than one term, observe that $A\dot{\otimes}M^k_{\lambda_r}$ has a Verma filtration. As in the proof of Proposition 20.1 (6) in [23] III sect.20, one shows that

$$Ext^1(B, D(M^k_{\lambda_r})) = 0$$

for any B carrying a Verma filtration. The long cohomology sequence implies then that the space

$$Hom_{\hat{\mathfrak{g}}}(B, D(M^k_{\lambda_r}))$$

behaves as if B were a direct sum of Verma modules. Lemma is proved. $\square$

Essential for the proof of Lemma 6.3.2 were the following properties of the modules in question: $A\dot{\otimes}M^k_{\lambda_r}$ is a free $\hat{\mathfrak{n}_-}$-module (this is equivalent to carrying a Verma filtration), and $D(M^k_{\lambda_r})$ is a co-induced module, that is, dual to an induced one. Therefore, the same arguments give the following more general result.

Lemma 6.3.3. *Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and $\hat{\mathfrak{g}} = \hat{\mathfrak{n}_-} \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}_+}$ be corresponding triangular decompositions. Further, let $C \in \mathcal{O}_k$ be freely generated by $\hat{\mathfrak{n}_-}$ from a finite dimensional $\mathfrak{h}$-space V and*

$$B = Ind^{\hat{\mathfrak{g}}}_{\hat{\mathfrak{h}}\oplus\hat{\mathfrak{n}_+}} W,\ dim W < \infty.$$

Then

$$Hom_{\hat{\mathfrak{g}}}(C, D(B)) = Hom_{\hat{\mathfrak{h}\oplus\mathfrak{n}_+}}(C, D(W)) = Hom_{\mathfrak{h}}(V, D(W)).$$

What does not allow us to extend Lemma6.3.2 to all $B \in \mathcal{O}^k$ is that in general $B \neq D(B)$. Let, however, k be generic, that is, irrational. Then $\tilde{\mathcal{O}}_k$ is semi-simple and $A = \oplus_i V^k_{\mu_i}$; $\mathcal{O}_k$ is equivalent to $\mathcal{O}_{\mathfrak{g}}$, the $\mathcal{O}$-category of $\mathfrak{g}$-modules; under this equivalence A goes to$\oplus_i V_{\mu_i}$ and $\dot{\otimes}$ is transformed $\otimes$, see [12]. Therefore in this case, the surjectivity of (26) becomes the corresponding statement of the "semi-simple theory" – Theorem 3.5 of [3].

Now the strategy becomes obvious: include A and B in a family of modules depending on $k \in \mathbf{C}$ and prove that all morphisms existing generically admit continuation to our particular value of k; it is at this point we use the fact that $B \in^{fam} \mathcal{O}_k$, $A \in^{fam} \tilde{\mathcal{O}}_k$. It suffices to prove the following lemma.

Lemma 6.3.4. *Let* $\mathbf{A} \in \tilde{\mathcal{O}}_t$, $\mathbf{B} \in \mathcal{O}_t$ *be such that* $\mathbf{A}(k) = A$, $\mathbf{B}(k) = B$. *Then*

(i) $Hom_{\mathcal{O}_t}(\mathbf{A}\dot{\otimes}\mathbf{B}, \mathbf{B})$ *is a free* $R(t)$*-module;*

(ii) the natural map

$$Hom_{\mathcal{O}_t}(\mathbf{A}\dot{\otimes}\mathbf{B}, \mathbf{B})(k) \to Hom_{\mathcal{O}_k}(A\dot{\otimes}B, B)$$

is an embedding.

Proof follows the lines of the proof of Lemma 22.8 in [23].

(i) If $\mathbf{B} = D(\mathbf{B}')$ for some induced representation $\mathbf{B}'$, then (i) follows from Lemma 6.3.3. In general, there is an induced representation $\mathbf{B}'$ such that $\mathbf{B}'$ maps onto $D(\mathbf{B})$ and therefore $\mathbf{B}$ embeds in $D(\mathbf{B}')$. Hence $Hom_{\mathcal{O}_t}(\mathbf{A}\dot{\otimes}\mathbf{B}, \mathbf{B})$ is a submodule of a free $R(t)$-module and thus is also free.

(ii) Let $h \in Hom_{\mathcal{O}_t}(\mathbf{A}\dot{\otimes}\mathbf{B}, \mathbf{B})(k)$ be such that the image of $h(k)$ in $Hom_{\mathcal{O}_k}(A\dot{\otimes}B, B)$ is zero. It means that h evaluated on any element of $\mathbf{A}\dot{\otimes}\mathbf{B}$ belongs to $(t-k)\mathbf{B}$. Therefore $h \in (t-k)Hom_{\mathcal{O}_t}(\mathbf{A}\dot{\otimes}\mathbf{B}, \mathbf{B})$ and hence $h(k) = 0$. □

Theorem 6.3.1 has been proved. □

6.3.2. *Proof of Theorem 6.2.1.* By Theorem 6.3.1, tilting functors Φ_1, Φ_2 are isomorphic if and only if $\Phi_1(M^k_{\lambda_r}) \approx \Phi_2(M^k_{\lambda_r})$ and direct summands of any $W\dot{\otimes}?$ are in 1-1 correspondence with direct summands of $W\dot{\otimes}M^k_{\lambda_r}$. By Lemma 6.1.3, $W\dot{\otimes}M^k_{\lambda_r}$ is tilting. Proposition 6.1.1 implies that for any indecomposable tilting functor Φ there is $w \in W_k$ so that

$$\Phi(M_{\lambda_r}) = W_{w\cdot\lambda_l,k}.$$

It remains to prove that for any $w \in W_k$ there is a Φ so that $\Phi(M_{\lambda_r}) = W_{w\cdot\lambda_l,k}$. Pick a dominant integral $\mu \in \mathfrak{h}^*$ such that $w\cdot\lambda_l - \lambda_r$ is an extremal weight of V_μ. Consider tilting module $\tilde{W}_{\mu,k} \in \tilde{\mathcal{O}}_k$. It follows from Lemma 4.5.1 that the composition series of $\tilde{W}_{\mu,k}\dot{\otimes}M^k_{\lambda_r}$ contains $M^k_{w\cdot\lambda_l}$ and by Lemma 2.2.2 $w\cdot\lambda_l$ is maximal among ν such that M^k_ν appears in the composition series of $\tilde{W}_{\mu,k}\dot{\otimes}M^k_{\lambda_r}$. Proposition 6.1.1 immediately gives then that $\tilde{W}_{\mu,k}\dot{\otimes}M^k_{\lambda_r}$ contains a unique direct summand isomorphic to $W_{w\cdot\lambda_l,k}$. $\square$

6.3.3. Being exact by Lemma 6.1.3, each tilting functor induces a homomorphism of the Grothendieck rings. On the other hand, the Grothendieck rings carry a natural action of W_k.

Theorem 6.3.5. *The homomorphism of the Grothendieck rings induced by a tilting functor is W_k-linear.*

Proof For any exact functor Φ denote by Φ^k the induced homomorphism of Grothendieck rings (or K-rings, hence the notation). If $W \in \tilde{\mathcal{O}}_k$, then $(W\dot{\otimes}?)^K$ is W_k-linear as it equals the operator of multiplication by the formal character of the corresponding finite dimensional $\mathfrak{g}$-module, see Lemma 4.5.1. Therefore it suffices to prove that given a tilting functor Φ, Φ^K equals a linear combination of $(W\dot{\otimes}?)^K$ with tilting $W \in \tilde{\mathcal{O}}_k$.

Let Φ be such that

$$\Phi(M^k_{\lambda_r}) = W_{w\cdot\lambda_l,k},\ w \in W_k.$$

Pick a dominant integral $\mu \in \mathfrak{h}^*$ such that $w\cdot\lambda_l - \lambda_r$ is an extremal weight of V_μ. Consider $\tilde{W}_{\mu,k}\dot{\otimes}?$. As we saw when proving Theorem6.2.1, its direct sum decomposotion is of the form

$$\tilde{W}_{\mu,k}\dot{\otimes}? = \Phi_w \oplus \oplus_{v<w} c_v\Phi_v.$$

Hence

$$(\tilde{W}_{\mu,k}\dot{\otimes}?)^K = (\Phi_w)^K + \sum_{v<w} c_v(\Phi_v)^K.$$

The obvious induction on the length of w allows us to "solve" the latter equality for $(\Phi_w)^K$ in terms of $(\tilde{W}_{v\cdot\mu,k}\dot{\otimes}?)^K$, $v \in W_k$. $\square$

6.3.4. *Proof of Theorem 6.2.2.* Given tilting functor $\Phi_w,\ w \in W_k$ (see Theorem 6.2.1), set $q_w = ch(\Phi_w(M^k_\lambda))$ and regard it as an element of the group algebra of W_k. By Theorem 6.3.5,

$$ch(\Phi_w(M^k_{u\cdot\lambda})) = uq_w,\ u \in W_k.$$

As the set $\{ch(M^k_{u\cdot\lambda}),\ u \in W_k\}$ is a basis of the Grothendieck ring of $\mathcal{O}^\lambda_k$, we get that the action of Φ_w is multilication by q_w. To complete the proof it is enough to remark that by Corollary 6.1.2, the set $\{q_w,\ w \in W_k\}$ is a basis of the group algebra of W_k. $\square$

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