MODULAR INVARIANCE AND UNIQUENESS OF CONFORMAL CHARACTERS

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ABSTRACT. We show that the conformal characters of various rational models of $\mathcal{W}$-algebras can be already uniquely determined if one merely knows the central charge and the conformal dimensions. As a side result we develop several tools for studying representations of $SL(2,\mathbb{Z})$ on spaces of modular functions. These methods, applied here only to certain rational conformal field theories, may be useful for the analysis of many others.

CONTENTS

1. INTRODUCTION

In the last years two-dimensional conformal field theories played a profound role in theoretical physics as well as in mathematics. Starting with the work of A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov [1] in 1984, many new results connecting statistical mechanics and string theory with the theory of topological invariants of 3-manifolds or with number theory were found [2][3]. In mathematical physics the classification of rational conformal field theories (RCFT) became one of the important outstanding problems.

Since one hopes that it is possible to consider all RCFTs as rational models of $\mathcal{W}$-algebras, special vertex operator algebras generalizing in a certain sense Kac-Moody algebras, different methods for the investigation of these algebras and their representations have been developed (for a review see e.g. [4]).

An important tool in the study of rational models of $\mathcal{W}$-algebras are the associated conformal characters. These conformal characters χ_h form a finite set of modular functions satisfying a transformation law

$$\chi_h(A\tau) = \sum_{h'} \rho(A)_{h,h'} \chi_{h'}(\tau).$$

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Here A runs through the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ or through a certain subgroup $G(2)$ (accordingly as the underlying $\mathcal{W}$-algebra is bosonic or fermionic), and ρ is a matrix representation of Γ or $G(2)$, which depends on the rational model under consideration.

It has already been noticed that conformal characters are very distinguished modular functions: First of all, similar to the j-function, their Fourier coefficients are nonnegative integers and they have no poles in the upper half plane. They sometimes admit interesting sum formulas: These formulas, which allow an interpretation as generating functions of the spectrum of certain quasi-particles, can be used to deduce dilogarithm-identities (see e. g. [5,6]). In some cases the conformal characters have simple product expansions. If one has both, sum and product expansions, the resulting identities are what is known in combinatorics as Roger-Ramanujan or, more general, as Andrews-Gordon identities.

In this paper we add one more piece to this theme. We show, for certain rational models, that the central charge and the finite set of conformal dimensions uniquely determine its conformal characters. More precisely, we shall state a few general and simple axioms which are satisfied by the conformal characters of all known rational models of $\mathcal{W}$-algebras. These axioms state essentially not more than the $\mathrm{SL}(2,\mathbb{Z})$-invariance of the space of functions spanned by the conformal characters, the rationality of their Fourier coefficients and an upper bound for the order of their poles. The only data of the underlying rational model occurring in these axioms are the central charge and the conformal dimensions, which give the upper bound for the pole orders and a certain restriction on the $\mathrm{SL}(2,\mathbb{Z})$-invariance. We then prove that, for various sets of central charges and conformal dimensions, there is at most one set of modular functions which satisfies these axioms (cf. the Main Theorem in § 4).

This result has several implications. First, it shows that the simple constraints imposed on modular functions by the indicated axioms are surprisingly restrictive. Apart from giving an aesthetical satisfaction this observation gives further evidence that conformal characters are modular functions of a rather special nature, which may deserve further studies, even independently of the theory of $\mathcal{W}$-algebras.

Secondly, it implies that, in the case of the rational models considered in this article, the conformal characters do a priori not give more information about the underlying rational model than the central charge and the conformal dimensions. This is in perfect accordance with the more general belief that these data already determine completely the rational models of $\mathcal{W}$-algebras which do not contain currents (currents are nonzero elements of dimension 1; see §2). In general one expects that a unique characterization of rational models can be obtained if one takes into account certain additional quantum numbers which can be defined in terms of the Lie algebra spanned by the zero modes of the currents.

Thirdly, our main result has a useful practical consequence for the computation of conformal characters. Apart from several well-understood rational models where one has simple closed formulas for the conformal characters, it is in general difficult to compute them directly. Any attempt to obtain the first few Fourier coefficients by the so-called direct calculations in the $\mathcal{W}$-algebra, the so far only known method in the case where no closed formulas are available, requires considerable computer power. Our result indicates a way to avoid the direct calculations: Once the central

charge and conformal dimensions are determined the computation of the conformal characters can be viewed as a problem which belongs solely to the theory of modular forms, i.e. a problem whose solution affords no further data of the rational model in question. We shall show elsewhere how one can indeed solve this problem in many cases using theta series, and, in particular, how one obtains in this way explicit closed formulas for the conformal characters of certain nontrivial models which could not be computed using known methods [7].

In this paper we restrict our attention to rational models of $\mathcal{W}$-algebras where the associated representation ρ turns out to be irreducible. This restriction is mainly of technical nature: It simplifies the identification of ρ. However, we believe that the Main Theorem holds true in more generality, i.e. that it can be extended to rational models with composite ρ, possibly with a slightly larger set of axioms.

We have organized our article as follows: In §2 we give (axiomatic) definitions of the basic notions concerning $\mathcal{W}$-algebras since there seems to be no satisfactory reference for this. In §3 we give a short overview of those rational models for which we prove our Main Theorem. We do not feel competent to judge the literature cited in this section to be acceptable by a physicist as well as a mathematician in view of its mathematical cleanness, and there might be a dispute whether the existence of various rational models mentioned in §3 is rigorously proved or not. Our policy here is that we simply cite what is asserted in the literature. What is actually needed from this (short) section are solely the Tables 1 and 2. In §4 we state and prove our main result. The sections §4.2 and §4.3, where we develop the necessary tools needed for the proof of the Main Theorem, may be of independent interest for those studying representations ρ arising from conformal characters.

Notation. We use $\mathfrak{H}$ for the complex upper half plane, τ as a variable in $\mathfrak{H}$, $q = e^{2\pi i\tau}$,

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

Γ for the group $\mathrm{SL}(2,\mathbb{Z})$, and

$$\Gamma(n) = \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \mathrm{id} \pmod{n}\}$$

for the principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of level n. We use η for the Dedekind eta function

$$\eta(\tau) = \mathrm{e}^{\pi i\tau/12} \prod_{n\geq 1}(1-q^n).$$

2. Vertex operator algebras, $\mathcal{W}$-algebras and rational models

$\mathcal{W}$-algebras are a special kind of vertex operator algebras. For the reader's convenience we repeat the definition of vertex operator algebras and their representations (see e.g. [8,9]).

Definition (Vertex operator algebra). A vertex operator algebra is a complex $\mathbb{N}$-graded vector space

$$V = \bigoplus_{n \in \mathbb{N}} V_n$$

with $\dim(V_n) < \infty$ for all $n \in \mathbb{N}$ (an element $\phi \in V_n$ is said to be of dimension n), together with a linear map

$$V \to (\mathrm{End}\, V)[[z, z^{-1}]], \qquad \phi \mapsto Y(\phi, z) = \sum_{n \in \mathbb{Z}} \phi_n \, z^{-n-1},$$

(the elements of the image are called vertex operators), and two distinguished elements $1 \in V_0$ (called the vacuum) and $\omega \in V_2$ (called the Virasoro element) satisfying the following axioms:

(1) The map $\phi \mapsto Y(\phi, z)$ is injective.
(2) For all $\phi, \psi \in V$ there exists a n_0 such that $\phi_n \psi = 0$ for all $n \geq n_0$.
(3) For all $\phi, \psi \in V$ and $m, n \in \mathbb{Z}$ one has

$$(\phi_m \psi)_n = \sum_{i \geq 0} (-1)^i \binom{m}{i} (\phi_{m-i} \psi_{n+i} - (-1)^m \psi_{m+n-i} \phi_i).$$

For $m < 0$ this identity has to be read argumentwise: Note that by (2) the sum on the right hand side becomes finite when applied to an element of V.
(4) $Y(1, z) = \mathrm{id}_V$.
(5) Writing $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ one has

$$L_0 \restriction V_n = n \, \mathrm{id}_{V_n},$$

$$Y(L_{-1}\phi, z) = \frac{d}{dz} Y(\phi, z),$$

$$[L_m, L_n] = (m-n) L_{m+n} + \delta_{m+n,0} \, (m^3 - m) \frac{c}{12} \, \mathrm{id}_V,$$

for all $n, m \in \mathbb{Z}$, $\phi \in V$, where c is a complex constant (called the central charge or rank).

Remarks. 1. For $m \geq 0$ property (5) is equivalent to

$$[\psi_m, \phi_n] = \sum_{i \geq 0} \binom{m}{i} (\psi_i \phi)_{m+n-i}.$$

where the left hand side denotes the ordinary commutator of endomorphisms.

2. This commutator identity implies in particular $[L_0, \phi_n] = (L_{-1}\phi)_{n+1} + (L_0 \phi)_n$, hence $[L_0, \phi_n] = (d - n - 1)\phi_n$ for $\phi \in V_d$ (here we used $(L_{-1}\phi)_{n+1} = (-n-1)\phi_n$ from axiom (4)). From this one obtains

$$\phi_n V_m \subseteq V_{m+d-n-1}.$$

Definition (Representation of a vertex operator algebra). A representation of a vertex operator algebra V is a linear map

$$\rho: V \to (\mathrm{End}\, M)[[z, z^{-1}]], \qquad \phi \mapsto Y_M(\phi, z) = \sum_{n \in \mathbb{Z}} \rho(\phi)_n z^{-n-1},$$

where M is a $\mathbb{N}$-graded complex vector space

$$M = \bigoplus_{n \in \mathbb{N}} M_n$$

with $\dim(M_n) < \infty$ for all $n \in \mathbb{N}$, such that the following axioms are satisfied:

(1) For all $\phi \in V_d$ and m, n one has $\rho(\phi)_n M_m \subset M_{m-n-1+d}$.

(2) For all $\phi \in V$ and $v \in M$ there exist a n_0 such that $\rho(\phi)_n v = 0$ for all $n \geq n_0$.

(3) For all $\phi, \psi \in V$ and all $m, n \in \mathbb{Z}$ one has

$$\rho(\phi_m \psi)_n = \sum_{i \geq 0} (-1)^i \binom{m}{i} \left(\rho(\phi)_{m-i}\rho(\psi)_{n+i} - (-1)^m \rho(\psi)_{m+n-i}\rho(\phi)_i\right),$$

where again this identity has to read argumentwise.

(4) $Y_M(1, z) = \mathrm{id}_M$.

(5) Using $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} \rho(L)_n z^{-n-2}$ one has

$$Y_M(L_{-1}\phi, z) = \frac{d}{dz} Y_M(\phi, z),$$

$$[\rho(L)_m, \rho(L)_n] = (m-n)\rho(L)_{m+n} + \delta_{m+n,0}\,(m^3 - m)\frac{c}{12}\,\mathrm{id}_M,$$

for all $n, m \in \mathbb{Z}$, $\phi \in V$, where c is the central charge of V.

The representation ρ is called irreducible if there is no nontrivial subspace of M which is invariant under all $\rho(\phi)_n$.

In the following we shall occasionally use simply the term V-module M instead of representation $\rho: V \to \mathrm{End}(M)[[z, z^{-1}]]$.

Remarks. Note that a vertex operator algebra V is a V-module itself via $\phi \mapsto Y(\phi, z)$ (use remark (2) after the definition of vertex operator algebra for verifying axiom (1) of a representation).

Lemma. *Let $\rho: V \to \mathrm{End}(M)[[z, z^{-1}]]$ be an irreducible representation of the vertex operator algebra V. Then there exists a complex constant h_m such that*

$$\rho(L)_0 \restriction M_n = (h_M + n)\,\mathrm{id}_{M_n}$$

for all $n \in \mathbb{N}$.

Proof. By axiom (1) of a vertex operator algebra representation we have that $\rho(L)_0 M_0 \subseteq M_0$. Hence, since M_0 is finite dimensional, there exists an eigenvector v of $\rho(L)_0$ in M_0. Let h_M be the corresponding eigenvalue. Since ρ is irreducible

the vector space M is generated by the vectors $\rho(\phi)_n v$ ($\phi \subset V_d$, $d \in \mathbb{N}$, $n \in \mathbb{Z}$); for proving this note that the subspace spanned by the latter vectors is invariant under all $\rho(\phi)_n$ as can be deduced from axiom (3)). For $m \in \mathbb{N}$ let M'_m be the subspace generated by all $\rho(\phi)_n v$ with $\phi \in M_d$ and $d - n - 1 = m$. By axiom (1) we have $M'_m \subseteq M_m$, and since M is the sum of all the M'_m we conclude $M'_m = M_m$.

On the other hand, one has $[\rho(L)_0, \rho(\phi)_n] = (d-n-1)\phi_n$ for all n and all $\phi \in V_d$ (similarly as in remark (2) after the definition of vertex operator algebras). From this we obtain $\rho(L) \restriction M'_n = (h_M + n)\,\mathrm{id}_{M'_n}$. This proves the lemma. □

The lemma suggests the following

Definition (Character of a vertex operator algebra module). Let M be an irreducible module of the vertex operator algebra V (with respect to the representation ρ). Then the character χ_M of M is the formal power series defined by

$$\chi_M(q) := \mathrm{tr}_M(q^{\rho(L)_0 - c/24}) := q^{h_M - c/24} \sum_{n \in \mathbb{N}} \dim(M_n) q^n$$

where c is the central charge of V and h_M the conformal dimension of M.

The most important class of vertex operator algebras if given by "rational" vertex operator algebras:

Definition (Rationality of vertex operator algebras). A vertex operator algebra V is called rational if the following axioms are satisfied:

(1) V has only finitely many inequivalent irreducible representations.
(2) Every finitely generated representation of V is equivalent to a direct sum of finitely many irreducible representations.

Here the notions equivalence, finitely generated and direct sum are to be understood in the obvious sense. The importance of the rational algebras becomes clear by the following theorem:

Theorem (Zhu [12]). *Let M_i ($i = 1, \ldots, n$) be a complete set irreducible modules of the rational vertex operator algebra V. Assume, furthermore, that Zhu's finiteness condition is satisfied, i.e.*

$$\dim(V/(V)_{-2}V) < \infty$$

where $(V)_{-2}V \subset V$ is defined by $(V)_{-2}V := \{\phi_{-2}\psi | \phi, \psi \in V\}$. Then the conformal characters χ_{M_i} become holomorphic functions on the upper complex half $\mathfrak{H}$ plane by setting $q = e^{2\pi i \tau}$ with $\tau \in \mathfrak{H}$. Furthermore, the space spanned by the conformal characters χ_{M_i} ($i = 1, \ldots, n$) is invariant under the natural action $(\chi(\tau), A) \mapsto \chi(A\tau)$ of the modular group $\mathrm{SL}(2, \mathbb{Z})$.

We now turn to the definition of $\mathcal{W}$-algebras and rational models of $\mathcal{W}$-algebras. As indicated above we describe these in terms of vertex operator algebras.

Definition ($\mathcal{W}$-algebra). A vertex operator algebra V is called a (bosonic) $\mathcal{W}$-algebra if it satisfies the following additional axioms:

(1) $\dim(V_0) = 1$.
(2) There exist finitely many homogeneous elements $\phi^i \in \ker(L_1)$ ($i = 1, \ldots, n$) which generate V.

Here vectors ϕ^i $(i = 1, \ldots, n)$ are said to generate V if the smallest subspace of V which is invariant under the action of $(\phi^i)_m$ $(i = 1, \ldots, n; m \in \mathbb{Z})$ and contains 1 equals V.

A $\mathcal{W}$-algebra V is said to be of type $\mathcal{W}(d_1, \ldots, d_n)$ if there exists a minimal set of homogeneous generators $\phi^i \in \ker(L_1)$ $(i = 1, \ldots, n)$ whose dimensions equal $d_1, \ldots, d_n$. Here minimal means that no proper subset of the set of the ϕ^i generates V. Note that the d_i occurring here may in general not be unique.

Remarks. 1. Examples of $\mathcal{W}$-algebras can be constructed from the Virasoro and Kac-Moody algebras. They are of type $\mathcal{W}(1, \ldots, 1)$, respectively $\mathcal{W}(2)$ for the Virasoro algebra [9].

2. Note the following for connecting our definition of $\mathcal{W}$-algebras with the corresponding notion used in the physical literature. The right hand side of (5) in the definition of vertex operator algebras is, for $m < 0$, what is usually called the n-th mode $N(\psi, \partial^{-1-m}\phi)_n$ of the normal ordered product of the vertex operators corresponding to ψ and the $(-m-1)$-th derivative of vertex operator corresponding to ϕ (see e.g. [10]) Moreover, the commutator formula in remark (1) after the definition of vertex operator algebras implies the (in the physical literature) well-known formula for the commutator of two homogeneous elements in $\ker(L_1)$ of a $\mathcal{W}$-algebra V (see e.g. [10,11]).

Definition (Rational model). A rational model (or rational model of a $\mathcal{W}$-algebra) is a rational $\mathcal{W}$-algebra V which satisfies Zhu's finiteness condition. The *effective central charge* of a rational model is defined by

$$\tilde{c} = c - 24 \min h_{M_i}$$

where M_i runs through a complete set of inequivalent irreducible representations of V.

Remarks. 1. Examples of rational models are given by certain vertex operator algebras constructed from Kac-Moody algebras [9] or the Virasoro algebra [13] (for more details see also §3).

2. One can show that the effective central charge of a rational model with a minimal generating set of n vectors lies in the range [14]

$$0 \leq \tilde{c} < n.$$

3. Historically the term "rational models" was used in the physical literature [1] for field theories in which the operator product expansion of any two local quantum fields decomposes into finitely many conformal families.

The following theorem justifies the terminology "rational models":

Theorem ([15]). *Assume that the representation of the modular group acting on the space spanned by the conformal characters of a rational model is unitary. Then the central charge and the conformal dimensions of the rational model are rational numbers.*

3. Central charges and conformal dimensions of certain rational models

In this section we review some facts about those rational models which are concerned by the Main Theorem in §4. Note that some of the results summarized in this section are not yet proved on a rigorous mathematical level. However, we shall not be concerned by this since we are only interested in the central charges and sets of conformal dimensions provided by these models. This section serves rather as a motivation than as a background for the considerations in the subsequent sections.

Firstly, we review some known rational models with effective central charge less than 1. The simplest $\mathcal{W}$-algebras are those which can be constructed from the Virasoro algebra (as already mentioned in the foregoing section). The rational models among these are called the Virasoro minimal models (see e.g. [1, 16,13]). They can be parameterized by a set of two coprime integers $p, q \geq 2$. The rational model corresponding to such a set p, q has central charge

$$c = c(p,q) = 1 - 6\frac{(p-q)^2}{pq}$$

and its conformal dimensions are given by:

$$h(p,q,r,s) = \frac{(rp-sq)^2 - (p-q)^2}{4pq} \quad (1 \leq r < q,\ (2,r) = 1,\ s = 1 \leq s < p),$$

where we assume q to be odd.

The Virasoro minimal models are special examples of the larger class of rational models with $\tilde{c} < 1$ which emerges from the ADE-classification of modular invariant partition functions [17,14]. Their central charges and conformal dimensions are given in Table 1: The first column describes the type of modular invariant partition function, the central charge is always $c = c(p,q)$ where p and q are the parameters of the respective row under consideration. Moreover, $c(p,q)$ and $h(p,q,\cdot,\cdot)$ are as defined above. Note that the listed models exist also for p, q, m not necessary prime. The primality restrictions have been added for technical reasons only which will become clear in the next section.

The second list of rational models which we shall consider are special cases of the so-called Casimir $\mathcal{W}$-algebras.

Starting from a Kac-Moody algebra associated to a simple Lie algebra $\mathcal{K}$ one can construct a 1-parameter family $\mathcal{WK}$ of $\mathcal{W}$-algebras, the parameter being the central charge (see e.g. [18]) (Note that this construction is different from the one mentioned in the foregoing section). For all but a finite number of central charges these $\mathcal{W}$-algebras are of type $\mathcal{W}(d_1, \ldots, d_n)$ where n is the rank of $\mathcal{K}$ and the d_i $(i = 1, \ldots, n)$ are the orders of the Casimir operators of $\mathcal{K}$. The remaining ones, called truncated, are of type $\mathcal{W}(d_{i_1}, \ldots, d_{i_r})$ where the d_{i_k} form a proper subfamily of the d_i above. Note that the $\mathcal{W}$-algebras constructed from the Virasoro algebra mentioned in §2 are exactly the Casimir $\mathcal{W}$-algebras associated to $\mathcal{A}_1$. The rational models of Casimir $\mathcal{W}$-algebras (sometimes called minimal models) have been determined, assuming certain conjectures, in [18].

Table 1: Data of certain $\mathcal{W}$-algebras related to the ADE-classification

type	type of $\mathcal{W}$-algebra	$H_{c(p,q)}$ $\quad (I_n := \{1,\dots,n\})$
(A_{q-1}, A_{p-1})	$\mathcal{W}(2)$ $p > q$ prime	$\{h(p,q,r,s) \mid r \in I_{q-1},\ s \in I_{p-1},\ (2,r) = 1\}$
(A_{q-1}, D_{m+1})	$\mathcal{W}(2, \frac{(m-1)(q-2)}{2})$ $p = 2m$ q, m prime	$\{h(p,q,r,s) \mid r \in I_{(q-1)/2},\ s \in I_m,\ (2,s) = 1\}$
(A_{q-1}, E_6)	$\mathcal{W}(2, q-3)$ $p = 12,\ q \geq 5$ q prime	$\{\min(h(p,q,r,1), h(p,q,r,\ 7)) \mid r \in I_{(q-1)/2}\} \cup$ $\{\min(h(p,q,r,5), h(p,q,r,11)) \mid r \in I_{(q-1)/2}\} \cup$ $\{h(p,q,r,4) \mid r \in I_{(q-1)/2}\}$
(A_{q-1}, E_8)	$\mathcal{W}(2, q-5)$ $p = 30,\ q \geq 7$ q prime	$\{\min(h(p,q,r,1), h(p,q,r,11)) \mid r \in I_{(q-1)/2}\} \cup$ $\{\min(h(p,q,r,7), h(p,q,r,13)) \mid r \in I_{(q-1)/2}\}$

In Table 2 we list the central charges c, effective central charge $\tilde{c}$ and the sets of conformal dimensions H_c of 6 rational models with $\tilde{c} > 1$.

The last four are Casimir $\mathcal{W}$-algebras associated to $\mathcal{B}_2, \mathcal{G}_2, \mathcal{E}_7$ and $\mathcal{B}_3$.

The first two $\mathcal{W}$-algebras are "tensor products" of the rational $\mathcal{W}$-algebra with $c = -22/5$ constructed from the Virasoro algebra and the rational $\mathcal{W}$-algebras with $c = 14/5$ or $c = 26/5$ constructed from the Kac-Moody algebras associated to $\mathcal{G}_2$ or $\mathcal{F}_4$, respectively. We denote them by $\mathcal{W}_{\mathcal{G}_2}(2, 1^{14})$ and $\mathcal{W}_{\mathcal{F}_4}(2, 1^{14})$, respectively. Here the construction of the $\mathcal{W}$-algebras in question is the one mentioned in §2.

Table 2: Data of the six rational models

$\mathcal{W}$-algebra	c	$\tilde{c}$	H_c
$\mathcal{W}_{G_2}(2, 1^{14})$	$-\frac{8}{5}$	$\frac{16}{5}$	$\frac{1}{5}\{0, -1, 1, 2\}$
$\mathcal{W}_{F_4}(2, 1^{26})$	$\frac{4}{5}$	$\frac{28}{5}$	$\frac{1}{5}\{0, -1, 2, 3\}$
$\mathcal{W}(2,4)$	$-\frac{444}{11}$	$\frac{12}{11}$	$-\frac{1}{11}\{0, 9, 10, 12, 14, 15, 16, 17, 18, 19\}$
$\mathcal{W}(2,6)$	$-\frac{1420}{17}$	$\frac{20}{17}$	$-\frac{1}{17}\{0, 27, 30, 37, 39, 46, 48, 49, 50,$ $52, 53, 55, 57, 58, 59, 60\}$
$\mathcal{W}(2,8)$	$-\frac{3164}{23}$	$\frac{28}{23}$	$-\frac{1}{23}\{0, 54, 67, 81, 91, 94, 98, 103, 111,$ $112, 116, 118, 119, 120, 122, 124,$ $125, 129, 130, 131, 132, 133\}$
$\mathcal{W}(2,4,6)$	$-\frac{13}{15}$	$\frac{17}{15}$	$\frac{1}{180}\{0, -15, -8, -3, 12, 37, 57, 60, 100,$ $117, 120, 132, 145, 252, 285, 405\}$

We give some comments on these 6 rational models. Using [16] and [19] the central charges, conformal characters and dimensions of the two composite rational models can be computed. For the rational models of type $\mathcal{W}(2,d)$ lists of the associated conformal dimension can be found in [14]. The conformal dimensions of the last rational model of type $\mathcal{W}(2,4,6)$ have been calculated in [20].

As it will turn out in the next section the first five rational models in Table 2 exhibit some interesting analogy: The representations of Γ afforded by their conformal characters belong to one and the same series ρ_l (cf. §4.4 for details). So one could ask whether there exist more rational models whose representations of Γ belongs to the series ρ_l. A more detailed investigation of the fusion algebras associated to such potentially existing models showed that this is not the case [21] (cf. also the speculation in [14]).

4. Uniqueness of conformal characters of certain rational models

4.1 Statement of the main theorem.

Main theorem. *Let c be any of the central charges of Table 1 or 2, let H_c denote the set of corresponding conformal dimensions, and let H be a subset of H_c containing 0. Assume that there exist nonzero functions $\xi_{c,h}$ ($h \in H$), holomorphic on the upper half plane, which satisfy the following conditions:*

(1) *The functions $\xi_{c,h}$ are modular functions for some congruence subgroup of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$.*
(2) *The space of functions spanned by the $\xi_{c,h}$ ($h \in H$) is invariant under Γ with respect to the action $(A,\xi) \mapsto \xi(A\tau)$.*
(3) *For each $h \in H$ one has $\xi_{c,h} = \mathcal{O}(q^{-\tilde{c}/24})$ as $\mathrm{Im}(\tau)$ tends to infinity, where $\tilde{c} = c - 24 \min H$.*
(4) *For each $h \in H$ the function $q^{-(h-\frac{c}{24})}\xi_{c,h}$ is periodic with period 1.*
(5) *The Fourier coefficients of the $\xi_{c,h}$ are rational numbers.*

Then $H = H_c$, and, for each $h \in H$, the function $\xi_{c,h}$ is unique up to multiplication by a scalar.

Remarks. 1. Note that the theorem only ensures the uniqueness of the functions $\xi_{c,h}$ but not their existence. However, they do indeed exist. For Table 1 the existence of the corresponding functions is a well-known fact [17, 14]: explicit formulas for them can be given in terms of the Riemann-Jacobi theta series

$$\sum_{\substack{x \in \mathbf{Z} \\ x \equiv \lambda \bmod 2k}} \exp(2\pi i \tau x^2/4k).$$

The existence of the functions $\xi_{c,h}$ related to Table 2 will be proved elsewhere [7].

2. Note that the conformal characters χ_M of a rational model with H as set of conformal dimensions satisfy the properties listed under (2) – (5) by the very definition of rational models and Zhu's theorem if we set $\xi_{c,h} = \chi_M$ (h = conformal dimension of M). Property (1) is not part of this definition, and it is not clear whether it is implied by the axioms for rational models. However, there is evidence that it always holds true (cf. the discussion below).

3. If we assume for a rational model corresponding to a row in Table 1 or Table 2 that its conformal characters satisfy (1) we can conclude from our theorem that the corresponding set H_c is exactly the set of its conformal dimensions and that the properly normalized functions $\xi_{c,h}$ ($h \in H_c$) are its conformal characters.

4. For the proof of the theorem for the first 5 models of Table 2 the assumption $0 \in H$ is not needed, and it can possibly be dropped in all cases. However, we did

not pursue this any further: From the physical point of view the assumption $0 \in H$ is natural since $h = 0$ corresponds to the vacuum representation of the underlying $\mathcal{W}$-algebra, i.e. the representation given by the $\mathcal{W}$-algebra itself.

For the first two cases of Table 2 the requirement that the $\xi_{c,h}$ are modular functions on some congruence subgroup is not necessary. Here we have the

Supplement to the main theorem. *For $c = -\frac{8}{5}$ and $c = \frac{4}{5}$ and with H_c as in Table 2 the equality $H = H_c$ and the uniqueness of the $\xi_{c,h}$ ($h \in H$) are already implied by properties (2) to (5).*

For the other cases we do not know whether the statement about the uniqueness of H and the $\xi_{c,h}$ remains true if one takes also into account non-modular functions or non-congruence subgroups.

However, as already mentioned, it seems to be reasonable to expect that the conformal characters associated to rational models satisfy (1). Evidence for this is given by the following:

There is no example of a conformal character of any rational model which is not a modular function on a congruence subgroup.

As mentioned above the functions $\xi_{c,h}$, whose uniqueness is ensured by the Main theorem, exist. As it turns out they can be normalized so that their Fourier coefficients are always nonnegative integers (for the case of Table 2 cf. [7]). This gives further evidence that they are identical with the conformal characters of the corresponding $\mathcal{W}$-algebra models whence the latter therefore satisfy (1).

According to the Main theorem, for each H_c of Table 1 and 2 the Γ-module spanned by the $\xi_{c,h}$ is uniquely determined. In particular the S-matrix (i.e. the matrix representing the action of S with respect to the basis given by the $\xi_{c,h}$ with the normalization indicated in the preceding remark) is unique. Closed formulas for the S-matrices corresponding to the first four rows of Table 2 can be found in [7]. They can be compared with the S-matrix of the corresponding $\mathcal{W}(2,4)$ rational model with $c = -\frac{444}{11}$ as numerically computed in [22] using so-called direct calculations in the $\mathcal{W}$-algebra. Both S-matrices coincide within the range of the numerical precision.

All rational models listed in Table 2 are minimal models of Casimir $\mathcal{W}$-algebras for which formulas for the corresponding conformal characters have been obtained in [18] under the assumption of a certain conjecture. Once more, the conformal characters so obtained are modular functions on congruence subgroups [7].

In the rest of §4 we prove our main theorem. To this end we will develop some general tools dealing with modular representations, i.e. with representations of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ on spaces of modular functions or forms. These methods are introduced in the next two subsections. In §4.4 we conclude with the proof of the main theorem.

4.2 A dimension formula for spaces of vector valued modular forms.

In this section we state dimension formulas for spaces of vector valued modular forms on $\mathrm{SL}(2,\mathbb{Z})$. These formulas are one of the main tools in the proof of the main theorem. It is quite natural in the context of conformal characters, or more generally in the context of modular representations, to ask for such formulas: The vector χ whose entries are the conformal characters of a rational model, multiplied

by a suitable power of η, is exactly what we shall call a vector valued modular form, and is as such an element of a finite dimensional space. (The latter holds true at least in the case where the characters are invariant under a subgroup of finite index in Γ; see the assumptions in the theorem below).

Multiplying χ by an odd power of η yields a vector valued modular form of half-integral weight. However, because of the ambiguity of the squareroot of $c\tau + d$ (c, d being the lower entries of a matrix in Γ) we now do not deal with a vector valued modular form on $\mathrm{SL}(2,\mathbb{Z})$ but rather on a certain double cover $D\Gamma = \mathrm{DSL}(2,\mathbb{Z})$ of this group.

We now make these notions precise.

The double cover $D\Gamma$ is defined as follows: the group elements are the pairs (A, w), where A is a matrix in Γ and w is a holomorphic function on $\mathfrak{H}$ satisfying $w^2(\tau) = c\tau + d$ with c, d the lower row of A. The multiplication of two such pairs is defined by

$$(A, w(\tau)) \cdot (A', w'(\tau)) = (AA', w(A'\tau) \cdot w'(\tau)).$$

For any $k \in \mathbb{Z}$ we have an action of $D\Gamma$ on functions f on $\mathfrak{H}$ given by

$$(f|_k(A, w))(\tau) = f(\tau)\, w(\tau)^{-2k}.$$

Note that for integral k this action factors to an action of Γ, which is nothing else than the usual "$|_k$"-action of Γ given by $(f|_k A)(\tau) = f(\tau)(c\tau + d)^{-k}$.

For a subgroup Δ of Γ we will denote by $D\Delta \subset D\Gamma$ the preimage of Δ with respect to the natural projection $D\Gamma \to \Gamma$ mapping elements to their first component.

Special subgroups of $D\Gamma$ which we have to consider below are the groups

$$\Gamma(4m)^\sharp = \{(A, j(A,\tau)) | A \in \Gamma(4m)\}.$$

Here, for $A \in \Gamma(4m)$, we use

$$j(A,\tau) = \vartheta(A\tau)/\vartheta(\tau)$$

where $\vartheta(\tau) = \sum_{n\in\mathbf{Z}} q^{n^2}$. It is well-known that indeed $j(A,\tau) = \epsilon(A)\sqrt{c\tau + d}$ where c, d are the lower row of A and $\epsilon(A) = \pm 1$. Explicit formulas for $\epsilon(A)$ can be found in the literature, e.g. [23].

We can now define the notion of a vector valued modular form on Γ or $D\Gamma$.

Definition. For any representation $\rho: D\Gamma \to \mathrm{GL}(n,\mathbb{C})$ and any number $k \in \frac{1}{2}\mathbb{Z}$ denote by $M_k(\rho)$ the space of all holomorphic maps $F: \mathfrak{H} \to \mathbb{C}^n$ which satisfy $F|_k\alpha = \rho(\alpha)F$ for all $\alpha \in D\Gamma$, and which are bounded in any region $\mathrm{Im}(\tau) \geq r > 0$. Denote by $S_k(\rho)$ the subspace of all forms $F(\tau)$ in $M_k(\rho)$ which tend to 0 as $\mathrm{Im}(\tau)$ tends to infinity.

If ρ is a representation of Γ and k is integral we use $M_k(\rho)$ for $M_k(\rho \circ \pi)$, where π is the projection of $D\Gamma$ onto the first component. Clearly, in this case the transformation law for the functions F of $M_k(\rho)$ is equivalent to $F|_k A = \rho(A)F$ for all $A \in \Gamma$. In general, if k is integral, the group $D\Gamma$ may be replaced by Γ in all of the following considerations.

Finally, for a subgroup Δ of $D\Gamma$ or Γ we use $M_k(\Delta)$ for the space of modular forms of weight k on Δ in the usual sense. In the case $\Delta \subset \Gamma$ the weight k has of course to be integral. The reader may not mix the two kinds of spaces $M_k(\rho)$ and $M_k(\Delta)$; it will always be clear from the context whether ρ and Δ refer to a representation or a group.

Clearly, if the image of ρ is finite, i.e. if the kernel of ρ is of finite index in $D\Gamma$ then the components of an F in $M_k(\rho)$ are modular forms of weight k on this kernel. In particular, the space $M_k(\rho)$ is then finite dimensional. Formulas for the dimension of these spaces can be obtained as follows: Let V be the complex vector space of row vectors of length $n = \dim \rho$, equipped with the $D\Gamma$-right action $(z, \alpha) \mapsto \rho(\alpha)^t z$, where $()^t$ means transposition. The space $M_k(\rho)$ can then be identified with the space $\mathrm{Hom}_{D\Gamma}(V, M_k(\Delta))$ of $D\Gamma$-homomorphisms from V to $M_k(\Delta)$, where $\Delta = \ker \rho$, via the correspondence

$$M_k(\rho) \ni F \mapsto \text{ the map which associates } z \in V \text{ to } z^t \cdot F \in M_k(\Delta).$$

By orthogonality of group characters the dimension of $\mathrm{Hom}_{D\Gamma}(V, M_k(\Delta))$ can be expressed in terms of the traces of the endomorphisms defined by the action of elements of $D\Gamma$ on $M_k(\Delta)$. These traces in turn can be explicitly computed by using the Eichler-Selberg trace formula. In this way one can derive the following theorem (cf. [23, pp. 100] for a complete proof):

Theorem (Dimension formula [23]). *Let $\rho : \mathrm{DSL}(2, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{C})$ be a representation with finite image and such that $\rho((\epsilon^2 \,\mathrm{id}, \epsilon)) = \epsilon^{-2k}\,\mathrm{id}$ for all fourth roots of unity ϵ. and let $k \in \frac{1}{2}\mathbb{Z}$. Then the dimension of $M_k(\rho)$ is given by the following formula*

$$\begin{aligned}\dim M_k(\rho) - \dim S_{2-k}(\overline{\rho}) = &\frac{k-1}{12} \cdot n + \frac{1}{4}\,\mathrm{Re}\left(\mathrm{e}^{\pi i k/2}\,\mathrm{tr}\,\rho((S, \sqrt{\tau}))\right) \\ &+ \frac{2}{3\sqrt{3}}\,\mathrm{Re}\left(\mathrm{e}^{\pi i(2k+1)/6}\,\mathrm{tr}\,\rho((ST, \sqrt{\tau+1}))\right) \\ &+ \frac{1}{2}a(\rho) - \sum_{j=1}^{n} \mathbb{B}_1(\lambda_j).\end{aligned}$$

Here the λ_j $(1 \le j \le n)$ are complex numbers such that $\mathrm{e}^{2\pi i \lambda_j}$ runs through the eigenvalues of $\rho(T)$, we use $a(\rho)$ for the number of j such that $\mathrm{e}^{2\pi i \lambda_j} = 1$, and we use $\mathbb{B}_1(x) = x' - 1/2$ if $x \in x' + \mathbb{Z}$ with $0 < x' < 1$, and $\mathbb{B}_1(x) = 0$ for x integral. Moreover, for $\tau \in \mathfrak{H}$, we use $\sqrt{\tau}$ and $\sqrt{\tau+1}$ for those square roots which have positive real parts.

Remark. For $k \ge 2$ the theorem gives an explicit formula for $\dim M_k(\rho)$ since in this case $\dim(S_{2-k}(\rho)) = 0$ (the components of a vector valued modular form are ordinary modular forms on $\ker \rho$, and there exist no nonzero modular forms of negative weight and no cusp forms of weight 0).

For $k = 1/2,\ 3/2$ and $\ker(\rho) \supset \Gamma(4m)^{\natural}$ it is still possible to give an explicit formula for $M_k(\rho)$ [23]. However, we do not need those dimension formulas in full generality but need only the following consequence of them:

Supplement to the dimension formula [23]. *Let $\rho : \mathrm{DSL}(2,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{C})$ be an irreducible representation with $\Gamma(4m)^{\sharp} \subset \ker(\rho)$ for some integer m. Then one has $\dim(M_{1/2}(\rho)) = 0, 1$. Furthermore, if $\dim(M_{1/2}(\rho)) = 1$ then the eigenvalues of $\rho(T)$ are of the form $e^{2\pi i \frac{l^2}{4m}}$ with integers l.*

Remark. A complete list of all those representations ρ for which $\dim(M_k(\rho)) = 1$ can be found in [23].

A proof of this supplement can be found in [23]. It uses a theorem of Serre-Stark describing explicitly the modular forms of weight $1/2$ on congruence subgroups.

4.3 Three basic lemmas on representations of $\mathrm{SL}(2,\mathbb{Z})$.

In this section we will prove some lemmas which are useful for identifying a given representation ρ of Γ if one has certain informations about ρ, which can e.g. easily computed from the central charge and the conformal dimensions of a rational model.

Assume that the conformal characters of a rational model are modular functions on some a priori unknown congruence subgroup. Then the first step for determining the representation ρ, given by the action of Γ on the the conformal characters, consists in finding a positive integer N such that ρ factors through $\Gamma(N)$. The next theorem tells us that the optimal choice of N is given by the order of $\rho(T)$.

Theorem (Factorization criterion). *Let $\rho\colon \Gamma \longrightarrow \mathrm{GL}(n,\mathbb{C})$ be a representation, and let $N > 0$ be an integer. Assume that $\rho(T^N) = 1$, and, if $N > 5$, that the kernel of ρ is a congruence subgroup. Then ρ factors through a representation of $\Gamma/\Gamma(N)$.*

Proof. The kernel Γ' of ρ contains the normal hull in Γ of the subgroup generated by T^N. Call this normal hull $\Delta(N)$. By a result of [24] (but actually going back to Fricke-Klein) one has $\Delta(N) = \Gamma(N)$ for $N \leq 5$. If $N > 5$ then by assumption we have $\Gamma' \supset \Gamma(N')$ for some integer N'. Thus Γ' contains $\Delta(N)\Gamma(NN')$, which, once more by [24], equals $\Gamma(N)$.

By the last theorem the determination of the representation ρ associated to a rational model with modular functions as conformal characters is reduced to the investigation of the finite list of irreducible representations of $\Gamma/\Gamma(N) \approx \mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ with some easily computable N. The following theorem, or rather its subsequent corollary, allows to reduce this list dramatically.

Theorem (K-Rationality of modular representations). *Let k and $N > 0$ be integers, let $K = \mathbb{Q}(\mathrm{e}^{2\pi i/N})$. Then the K-vector space $M_k^K(\Gamma(N))$ of all modular forms on $\Gamma(N)$ of weight k whose Fourier developments with respect to $\mathrm{e}^{2\pi i\tau/N}$ have coefficients in K is invariant under the action $(f, A) \mapsto f|_k A$ of Γ.*

Proof. Let $j(\tau)$ denote the usual j-function, which has Fourier coefficients in $\mathbb{Z}$ and satisfies $j(A\tau) = j(\tau)$ for all $A \in \Gamma$. Assume that k is even. Then the map $f \mapsto f/j'^{k/2}$ defines an injection of the K-vector space $M_k^K(\Gamma(N))$ into the field of all modular functions on $\Gamma(N)$ whose Fourier expansions have coefficients in K. It clearly suffices to show that the latter field is invariant under Γ. A proof for this can be found in [25, p. 140, Prop. 6.9 (1), equ. (6.1.3)]. The case k odd can

be reduced to the case k even by considering the squares of the modular forms in $M_k^K(\Gamma(N))$.

Corollary. *Let $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ be a representation whose kernel contains $\Gamma(N)$ for some positive integer N, and let $K = \mathbb{Q}(e^{2\pi i/N})$. If, for some integer k, there exists a nonzero element in $M_k(\rho)$ whose Fourier development has Fourier coefficients in K^n, then $\rho(\Gamma) \subset \mathrm{GL}(n, K)$.*

I. f one assumes that a vector valued modular form is related to the conformal characters of a rational model which are modular functions of some congruence subgroup then obviously all the Fourier coefficients are rational so that the corollary applies.

Proof. If $F \in M_k(\rho)$ has Fourier coefficients in K^n then $F|_k A$, by the preceding theorem, has Fourier coefficients in K^n too. Here A is any element in Γ. From $F|A = \rho(A)F$ we deduce that $\rho(A)$ has entries in K.

4.4 Proof of the main theorem. We will now prove our main theorem stated in §4.1. Pick one of the central charges c in Table 1 or Table 2. Assume that for some $H \subset H_c$ containing 0 there exist functions $\xi_{c,h}$ $(h \in H)$ which satisfy the properties (1) to (5) of the main theorem. Let ξ denote the vector whose components are the functions $\xi_{c,h}$ ordered with increasing h. Note that the h-values are pairwise different modulo 1. By (3) the $\xi_{c,h}$ are thus linearly independent. Hence, we have a well-defined $|H|$-dimensional representation ρ of the modular group if we set $\xi(A\tau) = \rho(A)\xi(\tau)$ for $A \in \Gamma$. Finally, recall that the Dedekind eta function η is a modular form of weight $1/2$ for $D\Gamma$, more precisely, that there exists a one-dimensional representation θ of $D\Gamma$ on the group of 24-th roots of unity such that $\eta \in M_{\frac{1}{2}}(\theta)$.

For any half integer $k \in \frac{1}{2}\mathbb{Z}$ such that

$$k \geq \tilde{c}/2$$

we have $F := \eta^{2k}\xi \in M_k(\rho \otimes \theta^{2k})$, as is immediate from property (3) and the assumption that the $\xi_{c,h}$ are holomorphic in the upper half plane. Let k be the smallest possible half integer satisfying this inequality. The actual value is given in Table 3 below.

We shall show that by property (1) to (5) the representation ρ is uniquely determined (up to equivalence) and, in fact, is conjugate to the representation in the last column of Table 3, respectively (notations will be explained below). In particular, ρ has dimension equal to the cardinality of H_c, and hence we conclude $H = H_c$. The h-values are pairwise incongruent modulo 1, i.e. $\rho(T)$ has pairwise different eigenvalues. Since $\rho(T)$ is a diagonal matrix the representation ρ is thus unique up to conjugacy by diagonal matrices.

Finally, the kernel of ρ is a congruence subgroup by property (1). In particular, $\rho \otimes \theta^{2k}$ has a finite image. Thus we can apply the dimension formulas stated in §4.2. It will turn out that $M_k(\rho \otimes \theta^{2k})$ is one-dimensional. Thus, if there actually exist functions $\xi_{c,h}$ satisfying (1) to (5) then $M_k(\rho \otimes \theta^{2k}) = \mathbb{C} \cdot \xi\eta^{2k}$. Since ρ is unique up to conjugacy by diagonal matrices we conclude that ξ is unique up to multiplication by such matrices, and this proves the theorem. We now give the details.

Determination of the representation ρ. We first determine the equivalence class of the representation ρ.

Denote by d the lowest common denominator of the rational numbers $h - c/24$ ($h \in H_c$), and for any integer k' let $l(k')$ be the lowest common denominator of the numbers $h - c/24 + k'/12$ ($h \in H_c$), i.e. let

$$l(k') = 12d/\gcd(12, k'd).$$

Clearly, the order of $(\rho \otimes \theta^{2k'})(T)$ divides $l(k')$. Let k' the smallest nonnegative integer such that $l = l(k')$ is minimal, and set $\widetilde{\rho} = \rho \otimes \theta^{2k'}$. The values of k' and l are given in Table 3.

Note that k' integral implies that $\widetilde{\rho}$ can be regarded as a representation of Γ (rather than $\mathrm{DSL}(2,\mathbb{Z})$). By property (1) its kernel is a congruence subgroup. Thus we can apply the factorization criterion of §4.3 to conclude that this kernel contains $\Gamma(l)$. Note that here the assumption (1), namely that the $\xi_{c,h}$, are invariant under a congruence subgroup, is crucial if $l > 5$. For $l \leq 5$, this assumption is not necessary, which explains the supplement to the main theorem.

Table 3: Representations of Γ and weights related to certain rational models

$\mathcal{W}$-algebra	c	k	k'	l	$\widetilde{\rho} = \rho \otimes \theta^{2k'}$
$\mathcal{W}(2)$	$1 - 6\frac{(p-q)^2}{pq}$	$\frac{1}{2}$	2	$8pq$	$\sigma_p^q \otimes \sigma_q^p \otimes D_8^{pq}$
$\mathcal{W}(2, \frac{(m-1)(q-2)}{2})$	$1 - 3\frac{(2m-q)^2}{mq}$	$\frac{1}{2}$	$\frac{1-3mq}{2} \bmod 12$	mq	$\sigma_q^{2m} \otimes \tau_m^{2q}$
$\mathcal{W}(2, q-3)$	$1 - \frac{(12-q)^2}{2q}$	$\frac{1}{2}$	$-1 - q \bmod 3$	$16q$	$\sigma_q^3 \otimes D_{16}^q$
$\mathcal{W}(2, q-5)$	$1 - \frac{(30-q)^2}{5q}$	$\frac{1}{2}$	$\frac{1-5q}{2} \bmod 12$	$5q$	$\sigma_q^{30} \otimes \sigma_5^q$
$\mathcal{W}_{G_2}(2, 1^{14})$	$-\frac{8}{5}$	2	4	5	ρ_5
$\mathcal{W}_{F_4}(2, 1^{26})$	$\frac{4}{5}$	3	10	5	ρ_5
$\mathcal{W}(2,4)$	$-\frac{444}{11}$	1	6	11	ρ_{11}
$\mathcal{W}(2,6)$	$-\frac{1420}{17}$	1	2	17	ρ_{17}
$\mathcal{W}(2,8)$	$-\frac{3164}{23}$	1	10	23	ρ_{23}
$\mathcal{W}(2,4,6)$	$-\frac{13}{15}$	1	1	360	$\sigma_5^1 \otimes D_8^0 \otimes R_2(1,-)$

Recall that in Table 3 the integers p, q and m are primes with $q \neq p, m$.

We shall say that a representation of Γ has level N if its kernel contains $\Gamma(N)$. Since any representation of level N factors to a representation of

$$\Gamma/\Gamma(N) \approx \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z}),$$

it has a unique decomposition as sum of irreducible level N representations. Furthermore, there are only finitely many irreducible level N representation, and each such representation π has a unique product decomposition

$$\pi = \prod_{p^\lambda \| l} \pi_{p^\lambda}$$

with irreducible level p^λ representations π_{p^λ}. Finally, $\pi_{p^\lambda}(T)$ has order dividing p^λ, i.e. its eigenvalues are p^λ-th roots of unity. Since any N-th root of unity ζ has a unique decomposition as product of the p^λ-th roots of unity $\zeta^{\frac{N}{p^\lambda}x_p}$ with $\frac{N}{p^\lambda}x_p \equiv 1 \bmod p^\lambda$, we conclude:

Lemma. *Let ζ_j $(1 \le j \le n = \dim \pi)$ be the eigenvalues of $\pi(T)$. Then, for each $p^\lambda \| N$, the eigenvalues $\neq 1$ of $\pi_{p^\lambda}(T)$ (counting multiplicities) are exactly those among the numbers $\zeta_j^{\frac{N}{p^\lambda}x_p}$ $(1 \le j \le n)$ which are not equal to 1.*

The representation ρ in line 1 to 4 of Table 3. First, we consider the rational models corresponding to the first 4 rows of Table 3. By assumption $h = 0$ is in H, i.e. $\mu = \exp(2\pi i(-c/24 + k'/12))$ is an eigenvalue of $\widetilde{\rho}(T)$. Let π be that irreducible level l representation in the sum decomposition of $\widetilde{\rho}$ such that $\pi(T)$ has the eigenvalue μ. Since π is irreducible it has a decomposition as product of irreducible representations π_{p^λ} as above. Since μ is a primitive l-th root of unity the lemma implies that the π_{p^λ} are nontrivial.

The minimal dimension of a nontrivial irreducible level p^λ representation is 2, 3 or $(p-1)/2$ accordingly if p^λ equals 8, 16 or is an odd prime [26, p. 521ff]. Hence we have the inequalities

$$\dim \pi \ge \begin{cases} (p-1)(q-1)/2 & \text{for row 1} \\ (m-1)(q-1)/4 & \text{for row 2} \\ 3(q-1)/2 & \text{for row 3} \\ q-1 & \text{for row 4} \end{cases}.$$

For row 1, 3 and 4 the right hand side equals the cardinality of H_c respectively. In these cases we thus conclude that $\widetilde{\rho} = \pi$ is irreducible, that it is equal to a product of nontrivial level p^λ representations with minimal dimensions, and, in particular, that $H = H_c$.

For row 2 the right hand side is smaller than the cardinality of H_c. However, here we can sharpen the above inequality: First we note that the level p representations of dimension $(p-1)/2$ have parity $(-1)^{(p+1)/2}$, whence the product of the corresponding level m and q representations has parity $(-1)^{(mq-1)/2}$. On the other hand any irreducible subrepresentation has the same parity $\widetilde{\rho}$, i.e. the parity $(-1)^{k'} = (-1)^{(mq+1)/2}$. Hence π cannot equal a product of two nontrivial level m and q representations of minimal dimension. The dimension of the second smallest nontrivial irreducible level p representations is $(p+1)/2$. Under each of these representations T affords eigenvalue 1. Since T under $\widetilde{\rho}$ affords no m-th root of unity as eigenvalue we conclude that π cannot be equal to a product of a $(q+1)/2$ dimensional level q and a $(m-1)/2$ dimensional level m representation. Thus,

$$\dim \pi \ge (m+1)(q-1)/4.$$

The right hand side equals $|H_c|$, and we conclude as above that $H = H_c$, that ρ is irreducible, and that $\widetilde{\rho}$ equals a product of an irreducible $(q-1)/2$ dimensional level q and an irreducible $(m+1)/2$ dimensional level m representation.

To identify ρ it thus remains to examine the nontrivial level p^λ representations with small dimensions (cf. [26, p. 521ff]).

Let $p^\lambda = p$ be an odd prime. There exist exactly two irreducible level p representations with dimension $(p-1)/2$. The image of T under these representations has exactly the eigenvalues $\exp(2\pi i \varepsilon x^2/p)$ $(1 \le x \le (p-1)/2)$ where for one of them ε is a quadratic residue modulo p, and a quadratic non-residue for the other one [26]. Call these representations accordingly σ_p^ε. Similarly there exist exactly 2 irreducible level p representations with dimension $(p+1)/2$, denoted by τ_p^ε (with ε being a quadratic residue or non-residue modulo p). The eigenvalues of $\tau_p^\varepsilon(T)$ are $\exp(2\pi i \varepsilon x^2/p)$ $(0 \le x \le (p-1)/2)$.

Let $p^\lambda = 8$. There exist exactly 4 irreducible two dimensional level 8 representations which we denote by D_8^x (x being an integer modulo 4). The eigenvalues of the image of T under the representation D_8^x are $\exp(2\pi i(1+2x)/8)$ and $\exp(2\pi i(7+2x)/8)$.

Let $p^\lambda = 16$. There are 16 irreducible three dimensional level 16 representations. These can be distinguished by their eigenvalues of the image of T. In particular, there are four of these representations, denoted by D_{16}^x (x mod 4), where the image of T has the eigenvalues $\exp(2\pi i(2x+3)/8), \exp(2\pi i(3x-6)/16), \exp(2\pi i(3x+2)/16)$.

Summarizing we find $\widetilde{\rho} = \sigma_p^{n_p} \otimes \sigma_q^{n_q} \otimes D_8^{n_8}$, $= \sigma_q^{n_q} \otimes \tau_m^{n_m}$, $= \sigma_q^{n_p} \otimes D_{16}^{n_16}$ or $= \sigma_q^{n_q} \otimes \sigma_5^{n_5}$, respectively, with suitable numbers $n_p, \ldots$. The latter can be easily determined using the Lemma and the description of H_c in Table 1. The resulting values are given in Table 3.

The representation ρ in line 5 to 9 of Table 3. We now consider the rational models corresponding to row 5 to 9 of Table 3. Here the level of $\widetilde{\rho}$ is a prime l, the dimension of ρ is $\le l-1$, and the eigenvalues of $\rho(T)$ are pairwise different primitive l-th root of unity.

We show that $\widetilde{\rho}$ is irreducible with dimension $l-1$. Assume that $\widetilde{\rho}$ is reducible or has dimension $< (l-1)$. The only irreducible level l representations with dimension $< (l-1)$ for which the image of T does not afford eigenvalue 1 are σ_l^ε. Thus there only two possibilities: (a) $\widetilde{\rho} = \sigma_l^\varepsilon$ or (b) $\widetilde{\rho} = \sigma_l^\varepsilon \oplus \sigma_l^{\varepsilon'}$. For $l = 5, 17$ the representations σ_l^ε have parity -1, whereas $\widetilde{\rho}$ has parity $+1$, a contradiction. For $l = 11, 23$ we note that $\xi\eta^2$ is an element of $M_1(\widetilde{\rho} \otimes \theta^{2-2k'})$. We shall show in moment that the dimension of $M_1(\sigma_l^\varepsilon \otimes \theta^{2-2k'})$ is 0, which gives the desired contradiction (to recognize the contradiction in case (b) note that the "functor" $\rho \mapsto M_k(\rho)$ respects direct sums).

Since the dimension formula gives explicit dimensions only for $k \ne 1$ we cannot apply it directly for calculating the dimension of $M = M_1(\sigma_l^\varepsilon \otimes \theta^{2-2k'})$. For $l = 11$ we note that $\eta^2 M$ is a subspace of $M_2(\sigma_l^\varepsilon \otimes \theta^{4-2k'})$. To the latter we can apply the dimension formula, and find (using $\operatorname{tr} \sigma_l^\varepsilon(S) = 0$, $\operatorname{tr} \sigma_l^\varepsilon(ST) = -1$) that its dimension is 0. For $l = 23$ and $\varepsilon = 1$ we consider $M_{3/2}(\sigma_l^1 \otimes \theta^{3-2k'})$ which contains ηM. We find that its dimension equals

$$\dim S_{1/2}(\sigma_l^{-1} \otimes \theta^{-(3-2k')}) \le \dim M_{1/2}(\sigma_l^{-1} \otimes \theta^{-(3-2k')}),$$

which equals 0 by the supplement in §4.2 (for applying the supplement note that $\sigma_l^{-1} \otimes \theta^{-(3-2k')}$ has a kernel containing $\Gamma(23 \cdot 24)^\sharp$ and represents T with eigenvalues

$\exp(2\pi i(-24x^2 + 17 \cdot 23)/23 \cdot 24)$). Finally, by the dimension formula we find

$$\dim M_1(\sigma_l^{-1} \otimes \theta^{2-2k'}) = \dim S_1(\sigma_l^1 \otimes \theta^{-(2-2k')}),$$

and the right hand side equals 0 since $\dim S_{3/2}(\sigma_l^1 \otimes \theta^{-(1-2k')}) = 0$ by the supplement.

Thus, $\widetilde{\rho}$ is irreducible of dimension $l-1$, which implies in particular $H = H_c$. There exist exactly $(l-1)/2$ irreducible level l representations of dimension $l-1$ [27, p. 228]. We now use property (5) of the main theorem, which implies that the Fourier coefficients of $\xi \cdot \eta^{2k'}$ are rational. Hence, by the corollary in §4.3 we find that $\widetilde{\rho}$ takes values in $\mathrm{GL}(l-1, K)$ with K being the field of l-th roots of unity. There is exactly one irreducible level l representations of dimension $l-1$ whose character takes values in K [27, p. 228]; denote it by ρ_l. Then $\widetilde{\rho} = \rho_l$.

The representation ρ in line 10 of Table 3. Finally, we consider the last rational model of Table 3. Here $\widetilde{\rho}$ has level $360 = 8 \cdot 5 \cdot 9$. The eigenvalue of $\widetilde{\rho}(T)$ corresponding to $h = 0$ is a primitive 360-th root of unity. Hence by the lemma there exists an irreducible subrepresentation π of $\widetilde{\rho}$ which factors as a product of nontrivial irreducible representations of level 8, 5 and 9, respectively. The minimal dimension of an irreducible nontrivial level 8, 5 or 9 representation is 2, 2 and 4, respectively [26, p. 521]. Thus $\dim \pi \geq 16 = |H_c|$, and hence $H = H_c$ and $\widetilde{\rho} = \pi$. The eigenvalues of $\widetilde{\rho}(T)$ can be read off from Table 2. Using the lemma and the representations D_8^x and σ_5^ξ introduced above, we find

$$\widetilde{\rho} = D_8^0 \otimes \sigma_5^1 \otimes R$$

for an irreducible level 9 representation R with dimension 4, which represents T with eigenvalues $\exp(2\pi i x^2)$ $(1 \leq x \leq 4)$, and is odd. Looking up [26] we find that there is exactly one such representation, following [26] we denote it by $R_2(1,-)$.

Computation of dimensions. It remains to show $d = \dim M_k(\widetilde{\rho} \otimes \theta^{2k-2k'}) \leq 1$. For the first 4 rows of Table 3 this follows from the supplement in §4.2 and the irreducibility of ρ (in fact it can be shown that $d = 1$ [23]). For row 5 and 6 we find $d = 1$ by the dimension formula and using $\operatorname{tr} \rho_l(S) = 0$, $\operatorname{tr} \rho_l(ST) = 1$ (valid for arbitrary primes l). For the remaining cases (where $k = 1$) we multiply $M_1(\widetilde{\rho} \otimes \theta^{2-2k'})$ by η for obtaining $d' = \dim M_{3/2}(\widetilde{\rho} \otimes \theta^{3-2k'})$ as upper bound. Again, using the dimension formula and its supplement we find $d' = 1$.

This concludes the proof of the main theorem. □

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