Max-Planck-Institut für Mathematik Bonn

Solution of a q-difference Noether problem and the quantum Gelfand-Kirillov conjecture for $\mathfrak{gl}_N$

by

Vyacheslav Futorny
Jones T. Hartwig

Max-Planck-Institut für Mathematik
Preprint Series 2011 (71)

Solution of a q-difference Noether problem and the quantum Gelfand-Kirillov conjecture for $\mathfrak{gl}_N$

Vyacheslav Futorny
Jones T. Hartwig

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
University of São Paulo
São Paulo
Brazil

Department of Mathematics
Stanford University
Stanford CA
USA

MPIM 11-71

SOLUTION OF A q-DIFFERENCE NOETHER PROBLEM AND THE QUANTUM GELFAND-KIRILLOV CONJECTURE FOR $\mathfrak{gl}_N$

VYACHESLAV FUTORNY AND JONAS T. HARTWIG

Abstract. It is shown that the q-difference Noether problem for all classical Weyl groups has a positive solution, simultaneously generalizing well known results on multisymmetric functions of Mattuck [Mat] and Miyata [Mi] in the case $q = 1$, and q-deforming the noncommutative Noether problem for the symmetric group [FMO]. It is also shown that the quantum Gelfand-Kirillov conjecture for $\mathfrak{gl}_N$ (for a generic q) follows from the positive solution of the q-difference Noether problem for the Weyl group of type D_n. The proof is based on the theory of Galois rings [FO]. From here we obtain a new proof of the quantum Gelfand-Kirillov conjecture for $\mathfrak{sl}_N$, thus recovering the result of Fauquant-Millet [FM]. Moreover, we provide an explicit description of skew fields of fractions for quantized $\mathfrak{gl}_N$ and $\mathfrak{sl}_N$ generalizing [AD].

1. Introduction

One important tool in the study of different noncommutative domains is a comparison of their skew fields of fractions. One might recall the concepts of birational equivalence in algebraic geometry and of derived equivalence in category theory. This makes the structure problem of division algebras very important. Sometime the situation is especially pleasant: it was shows by Farkas, Schofield, Snider and Stafford [FSSS] that the skew field of fractions of the group algebra of finitely generated torsion free nilpotent group determines the group up to isomorphism. Of course, in general the problem is way more complicated. As it was pointed in [FSSS] very little is known about division algebras which are infinite dimensional over their centers. In particular, it is very difficult to decide when two such algebras are isomorphic.

The classical Gelfand-Kirillov conjecture states that the skew field of fractions (equivalently, quotient division ring) of the universal enveloping algebra of an algebraic Lie algebra over an algebraically closed field of characteristic zero is isomorphic to a Weyl field, that is, a skew field of fractions of the Weyl algebra over a purely transcendental extension of the ground field $\mathbb{k}$. This conjecture was proven by Gelfand and Kirillov [GK] for $\mathfrak{gl}_N$ and $\mathfrak{sl}_N$ and for nilpotent Lie algebras. For solvable Lie algebras the conjecture was proven independently by Borho, Gabriel and Rentschler [BGR], Joseph [Jo] and McConnell [Mc]. Moreover, Alev, Ooms and Van den Bergh [AOV1] proved the conjecture for all Lie algebras of dimension at most eight. However, the same authors found counterexamples to the conjecture for mixed Lie algebras [AOV2]. Also, Premet [P] showed that the conjecture fails for orthogonal Lie algebras and for simple Lie algebras of types E_6, E_7, E_8 and F_4.

An analogue of the Gelfand-Kirillov conjecture was shown for finite W-algebras of type A [FMO].

In this paper we fully solve the quantum Gelfand-Kirillov conjecture for the quantized $\mathfrak{gl}_N$ over $\mathbb{C}$. Let $\mathbb{k}$ be a field, $q \in \mathbb{k}$ be nonzero, $S = (s_{ij})$ be a skew-symmetric $n \times n$ integer matrix. Define the following *quantum polynomial algebra over* $\mathbb{k}$:

$$\mathbb{k}_{q,S}[X_1, \ldots, X_n] := \mathbb{k}\langle X_1, \ldots, X_n \mid X_iX_j = q^{s_{ij}} X_jX_i\rangle. \tag{1.1}$$

A *quantum Weyl field over* $\mathbb{k}$ is the skew field of fractions of an algebra of the form (1.1). We will discuss alternative definitions of quantum Weyl fields in Section 2.3.

We say that a unital associative $\mathbb{C}$-algebra A admitting a skew field of fractions $\operatorname{Frac}(A)$ satisfies the *quantum Gelfand-Kirillov conjecture* if $\operatorname{Frac}(A)$ is isomorphic to a quantum Weyl field over a purely transcendental field extension $\mathbb{k}$ of $\mathbb{C}$ (cf. [BG]).

The quantum Gelfand-Kirillov conjecture for $U_q(\mathfrak{g})$ has been studied for almost 20 years by many authors. Let $\mathfrak{g}$ be any complex finite-dimensional semi-simple Lie algebra, $\mathfrak{n}$ the nilpotent radical of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, and G the simply connected group associated to $\mathfrak{g}$. B. Feigin formulated the quantum Gelfand-Kirillov conjecture at RIMS in 1992 for $U_q(\mathfrak{n})$, which is now known as Feigin's conjecture. For generic values of q, Alev, Dumas [AD], Iohara, Malikov [IM] and Joseph [Jo1] have shown that $\operatorname{Frac} U_q(\mathfrak{n})$ satisfies the quantum Gelfand-Kirillov conjecture, while Caldero [Ca] proved it for $\operatorname{Frac} U_q(\mathfrak{n})$ and $\operatorname{Frac} \mathbb{C}_q[G]$. Panov [Pa] has proved that $U_q(\mathfrak{b})$ (and generalizations) also satisfy the quantum Gelfand-Kirillov conjecture.

That the skew field of fractions of (certain extensions of) $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_3)$ satisfy the quantum Gelfand-Kirillov conjecture was proved in [AD] by explicitly calculating the skew fields. Finally, Fauquant-Millet [FM] proved the quantum Gelfand-Kirillov conjecture for $U_q(\mathfrak{sl}_N)$ by modifying the original proof of Gelfand and Kirillov in the classical case.

We refer the reader to [BG], [G] and references therein for a detailed historical account of the Gelfand-Kirillov conjecture for quantized enveloping algebras.

Our contribution to the quantum Gelfand-Kirillov conjecture consists of explicit calculation of the skew fields for $U_q(\mathfrak{gl}_N)$ and (certain extension of) $U_q(\mathfrak{sl}_N)$ which provides a new proof for the conjecture in these cases. In particular, we recover the results of Alev and Dumas [AD].

Let $\mathcal{O}_q(\mathbb{k}^2)$ denotes the *quantum plane* $\mathbb{k}\langle x, y \mid yx = qxy\rangle$ over a field $\mathbb{k}$, $\bar{q} = (q_1, \ldots, q_n)$ a tuple of nonzero elements of $\mathbb{k}$. Let n be a positive integer and $\mathcal{O}_{\bar{q}}(\mathbb{k}^{2n})$ a quantum affine space:

$$\mathcal{O}_{\bar{q}}(\mathbb{k}^{2n}) := \mathcal{O}_{q_1}(\mathbb{k}^2) \otimes_{\mathbb{k}} \mathcal{O}_{q_2}(\mathbb{k}^2) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} \mathcal{O}_{q_n}(\mathbb{k}^2) \tag{1.2}$$

$$\simeq \mathbb{k}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \mid y_ix_j = q^{\delta_{ij}} x_jy_i,\ [x_i, x_j] = [y_i, y_j] = 0,\ \forall i, j \in [\![1, n]\!]\rangle.$$

When $q_1 = \ldots = q_n = q$ then we simply denote $\mathcal{O}_{\bar{q}}(\mathbb{k}^{2n})$ by $\mathcal{O}_q(\mathbb{k}^{2n})$.

We show

Theorem I. *The quantum Gelfand-Kirillov conjecture holds for $U_q(\mathfrak{gl}_N)$ for $q \in \mathbb{C}$ not a root of unity. Explicitly, there exists a $\mathbb{C}$-algebra isomorphism*

$$\operatorname{Frac}\big(U_q(\mathfrak{gl}_N)\big) \simeq \operatorname{Frac}\Big(\mathcal{O}_q(\mathbb{k}^2)^{\otimes_{\mathbb{k}}(N-1)} \otimes_{\mathbb{k}} \mathcal{O}_{q^2}(\mathbb{k}^2)^{\otimes_{\mathbb{k}}(N-1)(N-2)/2}\Big), \tag{1.3}$$

where $\mathbb{k}$ denotes the field $\mathbb{C}(Z_1, \ldots, Z_N)$ of rational functions in N variables over $\mathbb{C}$.

The proof is based on the reduction of the quantum Gelfand-Kirillov conjecture to the *q-difference Noether problem* for the Weyl group of type D_n.

Let $W_n = W(X_n)$ be the Weyl group of type X_n where $X \in \{A, B, C, D\}$. The group W_n acts naturally on $\mathcal{O}_q(\Bbbk^{2n})$ by $\Bbbk$-algebra automorphisms (see Section 4 for details). Let $\mathcal{F}_{q,n}$ (respectively $\mathcal{F}_{\bar{q},n}$) denote the skew field of fractions of $\mathcal{O}_q(\Bbbk^{2n})$ (respectively $\mathcal{O}_{\bar{q},n}(\Bbbk^{2n})$). The action of W_n on $\mathcal{O}_q(\Bbbk^{2n})$ induces an action of W_n on $\mathcal{F}_{q,n}$. We let

$$\mathcal{F}_{q,n}^{W_n} := \{a \in \mathcal{F}_{q,n} \mid w(a) = a, \, \forall w \in W_n\}$$

denote the subalgebra (skew subfield) of invariants under W_n. Consider the following problem, which we call the *q-difference Noether problem for W_n*:

Problem 1.1. *Do there exist $q_1, \ldots, q_n \in \langle q \rangle := \{q^k \mid k \in \mathbb{Z}\}$ such that*

$$\mathcal{F}_{q,n}^{W_n} \simeq \mathcal{F}_{\bar{q},n}, \tag{1.4}$$

where $\bar{q} = (q_1, \ldots, q_n)$, as $\Bbbk$-algebras?

We answer this question affirmatively and prove our main result:

Theorem II. *The q-difference Noether problem for the group W_n has a positive solution, namely*

$$\mathcal{F}_{q,n}^{W_n} \simeq \mathcal{F}_{\bar{q},n}, \tag{1.5}$$

where

$$\bar{q} = \begin{cases} (q, q, \ldots, q), & \text{if } W_n = W(A_n) = S_n, \\ (q^2, q^2, \ldots, q^2), & \text{if } W_n = W(B_n) = W(C_n), \\ (q, q^2, q^2, \ldots, q^2), & \text{if } W_n = W(D_n). \end{cases}$$

This can be viewed as quantum versions of classical results of Mattuck [Mat] and of Miyata [Mi].

As a corollary we get an isomorphism of $\Bbbk$-algebras

$$\big(\operatorname{Frac}(A_1^q(\Bbbk)^{\otimes_{\Bbbk} n})\big)^{S_n} \simeq \operatorname{Frac}(A_1^q(\Bbbk)^{\otimes_{\Bbbk} n}),$$

where $A_1^q(\Bbbk) := \Bbbk\langle x, y \mid yx - qxy = 1\rangle$ (see Corollary 3.11). This result can be regarded as a q-deformation of the isomorphism $\operatorname{Frac}(A_n(\Bbbk))^{S_n} \simeq \operatorname{Frac}(A_n(\Bbbk))$ proved in [FMO]. Here $A_n(\Bbbk)$ is the n:th Weyl algebra over $\Bbbk$.

Our proof of the quantum Gelfand-Kirillov conjecture relies on the theory of Galois rings [FO]. Using this theory and Gelfand-Tsetlin representations constructed by Mazorchuk and Turowska [MT] we show that $U_q(\mathfrak{gl}_N)$ can be embedded into the $(W_1 \times W_2 \times \cdots \times W_N)$-invariants of a certain skew group ring (Theorem 5.14), where W_m is the Weyl group of type D_m. Using this realization of $U_q(\mathfrak{gl}_N)$ the problem is then reduced to computation of the skew field of the W_m-invariants in the tensor product of m quantum planes. This computation follows from positive solution of the q-difference Noether problem for the Weyl group W_m.

2. Preliminaries

2.1. **Notation.** Unless otherwise stated, the ground field $\Bbbk$ is arbitrary and $q \in \Bbbk$ is only assumed to be nonzero. All rings and algebras will be understood to be associative and unital. By a *skew field* we mean a division ring. The skew field of fractions, provided it exists, of an algebra A will be denoted by $\operatorname{Frac}(A)$. A well-known fact (see for example [D, Sec. 3.2.1]) is that if A is an Ore domain, acted upon by a finite group G with $|G|$ invertible in A, then the invariants $A^G := \{a \in$

$A \mid g(a) = a, \forall g \in G\}$ is also an Ore domain and $\mathrm{Frac}(A^G) = \mathrm{Frac}(A)^G$. We will use the generalized Kronecker delta notation δ_P for a statement P, defined by

$$\delta_P = \begin{cases} 1, & \text{if } P \text{ is true,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

For $a, b \in \mathbb{Z}$ we use the notation $[\![a, b]\!] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. If a group G acts on a ring R by automorphisms, we denote the corresponding skew group ring by $R * G$. We sometimes use the q-commutator notation $[a, b]_q = ab - qba$.

2.2. **The algebra $U_q(\mathfrak{gl}_N)$.** Assume $q^2 \neq 1$. For positive integers N we let $U_N = U_q(\mathfrak{gl}_N)$ denote the unital associative $\Bbbk$-algebra with generators $E_i^{\pm}$, K_j, K_j^{-1}, $i \in [\![1, N-1]\!]$, $j \in [\![1, N]\!]$ and relations [KS, p.163]

$$\begin{aligned}
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad [K_i, K_j] = 0, \quad \forall i, j \in [\![1, N]\!],\\
K_i E_j^{\pm} K_i^{-1} = q^{\pm(\delta_{ij} - \delta_{i,j+1})} E_j^{\pm}, \quad \forall i \in [\![1, N]\!], \forall j \in [\![1, N-1]\!],\\
[E_i^+, E_j^-] = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_{i+1} K_i^{-1}}{q - q^{-1}}, \quad \forall i, j \in [\![1, N-1]\!],\\
[E_i^{\pm}, E_j^{\pm}] = 0, \quad |i - j| > 1,\\
(E_i^{\pm})^2 E_j^{\pm} - (q + q^{-1}) E_i^{\pm} E_j^{\pm} E_i^{\pm} + E_j^{\pm} (E_i^{\pm})^2 = 0, \quad |i - j| = 1.
\end{aligned}$$

2.3. **Quantum Weyl fields.** If n is a positive integer, the Weyl algebra $A_n(\Bbbk)$ is the algebra of differential operators on polynomial ring $\mathcal{O}(\Bbbk^n)$. This algebra is a simple Noetherian domain which allows a skew field of fractions called a *Weyl field*. In this section we recall some well-known results regarding the q-analogue of Weyl fields.

Recall the quantum polynomial algebra (1.1):

$$\Bbbk_{q, \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]}[X_1, X_2] \simeq \mathcal{O}_q(\Bbbk^2). \tag{2.2}$$

Proposition 2.1. *Let n be a positive integer. Let S be a $2n \times 2n$ skew-symmetric integer matrix. Then there exist integers $k_1, \ldots, k_n$ and an algebra isomorphism*

$$\mathrm{Frac}\left(\Bbbk_{q,S}[X_1, \ldots, X_{2n}]\right) \simeq \mathrm{Frac}\left(\mathcal{O}_{q^{k_1}}(\Bbbk^2) \otimes \cdots \otimes \mathcal{O}_{q^{k_n}}(\Bbbk^2)\right). \tag{2.3}$$

Proof. Similar to the proof of [H, Theorem 4.8], but we provide details for convenience. Denote $\Bbbk_q[x, y] = \mathcal{O}_q(\Bbbk^2)$. It is enough to show that the corresponding Laurent analogs, $\Bbbk_{q,S}[X_1^{\pm 1}, \ldots, X_{2n}^{\pm 1}]$ and

$$\Bbbk_{q^{k_1}}[x^{\pm 1}, y^{\pm 1}] \otimes \cdots \otimes \Bbbk_{q^{k_n}}[x^{\pm 1}, y^{\pm 1}]$$

are isomorphic. Consider a change of generators

$$X_i' := X_1^{u_{1i}} \cdots X_{2n}^{u_{2n,i}}, \qquad i = 1, \ldots, 2n,$$

where $U = (u_{ij})$ is an invertible $2n \times 2n$ integer matrix. The new commutation relations are

$$X_i' X_j' = q^{s_{ij}'} X_j' X_i', \qquad i, j = 1, \ldots, 2n, \tag{2.4}$$

where s_{ij}' are the entries of the matrix $S' := U^t S U$. By Theorem IV.1 in [N] there is an invertible $2n \times 2n$ integer matrix U such that $U^t S U$ is block diagonal with skew-symmetric 2×2 blocks on the diagonal. That is,

$$U^t S U = \bigoplus_{i=1}^{n} \begin{bmatrix} 0 & k_i \\ -k_i & 0 \end{bmatrix} \tag{2.5}$$

for some $k_i \in \mathbb{Z}$. Put $x_i = X'_{2i}$ and $y_i = X'_{2i-1}$ for $i = 1, \dots, n$. Then (2.4) and (2.5) imply that $y_i x_i = q^{k_i} x_i y_i$ for all i and $[x_i, x_j] = [x_i, y_j] = [y_i, y_j] = 0$ for all $i \neq j$. Thus there is a $\Bbbk$-algebra isomorphism

$$\Bbbk_{q,S}[X_1^{\pm 1}, \dots, X_{2n}^{\pm 1}] \xrightarrow{\sim} \Bbbk_{q^{k_1}}[x^{\pm 1}, y^{\pm 1}] \otimes \cdots \otimes \Bbbk_{q^{k_n}}[x^{\pm 1}, y^{\pm 1}],$$

determined by

$$x_i \longmapsto 1^{\otimes i-1} \otimes x \otimes 1^{\otimes n-i},$$
$$y_i \longmapsto 1^{\otimes i-1} \otimes y \otimes 1^{\otimes n-i}.$$

□

Let $\bar{q} = (q_1, \dots, q_n) \in (\Bbbk \backslash \{0\})^n$ and $\Lambda = (\lambda_{ij})$ be an $n \times n$ matrix with $\lambda_{ij} \in \Bbbk$, $\lambda_{ij}\lambda_{ji} = \lambda_{ii} = 1$ for all i, j. The *multiparameter quantized Weyl algebra* $A_n^{\bar{q},\Lambda}(\Bbbk)$ was introduced by Maltsiniotis [Mal] (see also [J]). This algebra can be viewed as algebra of q-difference operators on quantum affine space $\mathcal{O}_q(\Bbbk^n)$. It is defined as the associative unital $\Bbbk$-algebra generated by $x_1, \dots, x_n$ and $y_1, \dots, y_n$ with defining relations

$$y_i y_j = \lambda_{ij} y_j y_i, \quad \forall i, j \tag{2.6a}$$
$$x_i x_j = q_i \lambda_{ij} x_j x_i, \quad i < j \tag{2.6b}$$
$$x_i y_j = \lambda_{ji} y_j x_i, \quad i < j \tag{2.6c}$$
$$x_i y_j = q_j \lambda_{ji} y_j x_i, \quad i > j \tag{2.6d}$$
$$x_i y_i - q_i y_i x_i = 1 + \sum_{1 \le k \le i-1} (q_k - 1) y_k x_k. \tag{2.6e}$$

The following proposition is well-known (see for example [BG] and references therein), but we provide a proof containing the explicit isomorphisms which are not always given in the literature.

Proposition 2.2. *Let n be a positive integer, $\Bbbk$ a field, and $(q_1, \dots, q_n) \in (\Bbbk \setminus \{0, 1\})^n$. Then the skew fields of fractions of the following three algebras are isomorphic:*

(i) *The tensor product of quantum Weyl algebras*

$$A_1^{q_1}(\Bbbk) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A_1^{q_n}(\Bbbk); \tag{2.7}$$

(ii) *The tensor product of quantum planes*

$$\mathcal{O}_{q_1}(\Bbbk^2) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathcal{O}_{q_n}(\Bbbk^2); \tag{2.8}$$

(iii) *The multiparameter quantized Weyl algebra*

$$A_n^{\bar{q},\Lambda}(\Bbbk) \tag{2.9}$$

with parameters $\bar{q} = (q_1, \dots, q_n)$, and $\Lambda = (\lambda_{ij})$, $\lambda_{ij} = 1$ for all $i, j = 1, \dots, n$.

Proof. That (2.7) and (2.8) have isomorphic skew fields of fractions follows from the fact that there is an isomorphism

$$\Bbbk\langle x^{\pm 1}, y \mid yx - qxy = 1\rangle \longrightarrow \Bbbk\langle x^{\pm 1}, y \mid yx = qxy\rangle$$
$$x \longmapsto x$$
$$y \longmapsto (qx - x)^{-1}(y - 1).$$

This is straightforward to check directly. (One can understand this isomorphism as coming from the realization of y in the left hand side as the q-difference operator $f(x)\mapsto \frac{f(qx)-f(x)}{qx-x}$ for $f(x)\in \Bbbk[x,x^{-1}]$ while in the right hand side y can be realized as the q-shift operator $f(x)\mapsto f(qx)$.)

Concerning the multiparameter quantized Weyl algebra, the proof can be derived from [J]. We recall from [J] that the elements $z_i\in A_n^{\bar{q},\Lambda}(\Bbbk)$ defined by

$$z_i := [x_i,y_i] = 1+\sum_{1\le k\le i}(q_k-1)y_kx_k,\quad i=1,\dots,n \tag{2.10}$$

satisfy

$$z_iz_j = z_jz_i,\quad \forall i,j \tag{2.11a}$$

$$z_jy_i = \begin{cases} y_iz_j, & j<i,\\ q_iy_iz_j, & j\ge i.\end{cases} \tag{2.11b}$$

In $\operatorname{Frac}\big(A_n^{\bar{q},\Lambda}(\Bbbk)\big)$, putting

$$z_j' := z_j\cdot z_{j-1}^{-1},\quad \forall j=1,\dots,n, \tag{2.12}$$

where $z_0:=1$, relations (2.11) imply that

$$z_i'z_j' = z_j'z_i' \tag{2.13a}$$

$$z_j'y_i = \begin{cases} y_iz_j', & i\ne j,\\ q_iy_iz_j', & i=j.\end{cases} \tag{2.13b}$$

Since $\lambda_{ij}=1$ for all i,j, (2.6) implies

$$y_iy_j = y_jy_i. \tag{2.13c}$$

Relations (2.13) prove that, there is a $\Bbbk$-algebra homomorphism

$$\begin{aligned}\operatorname{Frac}\big(\mathcal{O}_{q_1}(\Bbbk^2)\otimes_\Bbbk\cdots\otimes_\Bbbk\mathcal{O}_{q_n}(\Bbbk^2)\big) &\longrightarrow \operatorname{Frac}\big(A_n^{\bar{q},\Lambda}(\Bbbk)\big),\\ 1^{\otimes i-1}\otimes x\otimes 1^{\otimes n-i} &\longmapsto y_i,\\ 1^{\otimes i-1}\otimes y\otimes 1^{\otimes n-i} &\longmapsto z_i',\end{aligned}$$

$\bar{q}=(q_1,\dots,q_n)$. It is injective since the domain is a skew field and surjective since in $\operatorname{Frac}\big(A_n^{\bar{q},\Lambda}(\Bbbk)\big)$ we have by (2.10),(2.12)

$$x_i = \frac{y_i^{-1}(z_i-z_{i-1})}{q_i-1} = \frac{y_i^{-1}\Big(\prod_{j=1}^i z_j' - \prod_{j=1}^{i-1} z_j'\Big)}{q_i-1},\quad \forall i=1,\dots,n, \tag{2.14}$$

where $z_0:=1$. □

Remark 2.3. In [AD, Thm 3.5] it is proved that if q_i,λ_{ij} $(i,j=1,\dots,n)$ are powers of some fixed non-root of unity $q\in\Bbbk\setminus\{0\}$, then $\operatorname{Frac}\big(A_n^{\bar{q},\Lambda}(\Bbbk)\big)$ is isomorphic to a quantum Weyl field $\operatorname{Frac}\big(\Bbbk_{q,S}[X_1,\dots,X_{2n}]\big)$ for some $2n\times 2n$ skew-symmetric integer matrix S (see also [P, Sec 5]). Combining this with Proposition 2.1 we get the following result.

Corollary 2.4. *If all parameters q_i,λ_{ij} $(i,j=1,\dots,n)$ are powers of some fixed non-root of unity $q\in\Bbbk\setminus\{0\}$, then there exists a tuple $(k_1,\dots,k_n)\in\mathbb{Z}^n$ such that*

$$\operatorname{Frac}\big(A_n^{\bar{q},\Lambda}(\Bbbk)\big)\simeq \operatorname{Frac}\big(\mathcal{O}_{q^{k_1}}(\Bbbk^2)\otimes_\Bbbk\cdots\otimes_\Bbbk\mathcal{O}_{q^{k_n}}(\Bbbk^2)\big). \tag{2.15}$$

In general, however, the integers k_i occuring in Corollary 2.4 require some work to determine.

3. The q-difference Noether problem for S_n

Let n be a positive integer. Throughout this section, $\Bbbk$ denotes a field of characteristic zero, and q is any nonzero element of $\Bbbk$. Let

$$\Bbbk_q[\bar{x},\bar{y}] = \Bbbk_q[x_1,y_1]\otimes_{\Bbbk}\Bbbk_q[x_2,y_2]\otimes_{\Bbbk}\cdots\otimes_{\Bbbk}\Bbbk_q[x_n,y_n]\simeq \mathcal{O}_q(\Bbbk^{2n}),$$

$\Bbbk_q(\bar{x},\bar{y})$ be the skew field of fractions of $\Bbbk_q[\bar{x},\bar{y}]$ and $\Bbbk_q(\bar{x},\bar{y})^{S_n}$ the subalgebra of S_n invariants.

3.1. **Generators and relations for the skew field of invariants.** In this section we provide a set of generators and relations for the algebra of invariants $\Bbbk_q(\bar{x},\bar{y})^{S_n}$. Let

$$C_n^q := \Bbbk(x_1,\dots,x_n)\langle y_1,\dots,y_n\rangle \tag{3.1}$$

denote the $\Bbbk(x_1,\dots,x_n)$-subring of $\Bbbk_q(\bar{x},\bar{y})$ generated by $\{y_1,\dots,y_n\}$. Note that C_n^q is an S_n-invariant subspace of $\Bbbk_q(\bar{x},\bar{y})$ and that $\mathrm{Frac}(C_n^q)=\Bbbk_q(\bar{x},\bar{y})$. Inspired by [Mat], we observe that the Vandermonde matrix

$$\begin{bmatrix} 1 & x_1 & x_1 & \cdots & x_1^{n-1}\\ 1 & x_2 & x_2 & \cdots & x_2^{n-1}\\ & \vdots & & \ddots & \vdots\\ 1 & x_n & x_n & \cdots & x_n^{n-1}\end{bmatrix} \tag{3.2}$$

is invertible and thus the system of equations

$$t_1 + x_i t_2 + x_i^2 t_3 + \cdots + x_i^{n-1} t_n = y_i, \qquad i=1,\dots,n \tag{3.3}$$

has a unique solution $(t_1,\dots,t_n)\in (C_n^q)^n$. Since the system (3.3) is S_n-invariant,

$$t_i \in (C_n^q)^{S_n}, \qquad \forall i=1,\dots,n. \tag{3.4}$$

The explicit inverse of the matrix (3.2) is well-known and implies the following description of the t_i. If we introduce the generating function $P(X)\in C_n^q[X]$ by

$$P(X)=\sum_{j=1}^{n} t_j X^{j-1}, \tag{3.5}$$

then

$$P(X)=\sum_{j=1}^{n}\left(\prod_{k\in\{1,\dots,n\}\setminus\{j\}}\frac{X-x_k}{x_j-x_k}\right)y_j. \tag{3.6}$$

Explicitly,

$$t_i=\sum_{j=1}^{n}\left(\frac{(-1)^{n-i}e'_{n-i}(x_1,\dots,\widehat{x_j},\dots,x_n)}{\prod_{k\in\{1,\dots,n\}\setminus\{j\}}(x_j-x_k)}\right)y_j \tag{3.7}$$

where e'_i is the degree i elementary symmetric polynomial in $n-1$ variables, $e'_0:=1$, and $\widehat{x_j}$ means that variable should be omitted.

Since the t_i and y_i can be expressed through each other via (3.3) and (3.7) we have

$$C_n^q=\Bbbk(x_1,\dots,x_n)\langle t_1,\dots,t_n\rangle, \tag{3.8}$$

i.e. C_n^q is generated as a $\Bbbk(x_1,\dots,x_n)$-ring by $t_1,\dots,t_n$.

Proposition 3.1. *For any $i, j \in \{1, \ldots, n\}$ we have*

$$[t_i, t_j] = 0. \tag{3.9}$$

The proof of Proposition 3.1 will be given in the Appendix.

We need the following preliminary observation of the commutation relations between t_i and rational functions of $x_1, \ldots, x_n$.

Lemma 3.2. *For any $a \in \Bbbk(x_1, \ldots, x_n)$ and any $i \in \{1, \ldots, n\}$ there are $a_{i1}, \ldots, a_{in} \in \Bbbk(x_1, \ldots, x_n)$ with*

$$t_i a = a_{i1} t_1 + \cdots + a_{in} t_n. \tag{3.10}$$

Proof. From (3.7) we know that

$$t_i = b_{i1} y_1 + \cdots + b_{in} y_n$$

for some $b_{ij} \in \Bbbk(x_1, \ldots, x_n)$. Using the commutation relation $y_j x_k = q^{\delta_{jk}} x_k y_j$ we obtain that

$$t_i a = c_{i1} y_1 + \cdots + c_{in} y_n$$

for some $c_{ij} \in \Bbbk(x_1, \ldots, x_n)$. Now use (3.3) to obtain (3.10) for some a_{ij}. □

Combining (3.8), Proposition 3.1 and Lemma 3.2 we obtain the following result.

Proposition 3.3. *The set*

$$\left\{ t_1^{k_1} \cdots t_n^{k_n} \mid k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0} \right\}$$

spans C_n^q as a left $\Bbbk(x_1, \ldots, x_n)$-module.

We can now prove the following statement about the generators of the invariants of C_n^q.

Proposition 3.4. *The algebra $(C_n^q)^{S_n}$ is generated as a $\Bbbk(x_1, \ldots, x_n)^{S_n}$-ring by $\{t_1, \ldots, t_n\}$.*

Proof. Let $u \in (C_n^q)^{S_n}$. By Proposition 3.3 we have

$$u = \sum_{k \in (\mathbb{Z}_{\geq 0})^n} u_k t_1^{k_1} \cdots t_n^{k_n}$$

for some $u_k \in \Bbbk(x_1, \ldots, x_n)$. Since u and $t_1, \ldots, t_n$ are S_n-fixed we have

$$u = \frac{1}{|S_n|} \sum_{w \in S_n} w(u) = \sum_{k \in (\mathbb{Z}_{\geq 0})^n} \Big(\frac{1}{|S_n|} \sum_{w \in S_n} w(u_k) \Big) t_1^{k_1} \cdots t_n^{k_n}$$

which proves that $u \in \Bbbk(x_1, \ldots, x_n)^{S_n} \langle t_1, \ldots, t_n \rangle$. □

As a corollary we obtain a set of generators for the skew field $\Bbbk_q(\bar{x}, \bar{y})^{S_n}$.

Corollary 3.5. *$\Bbbk_q(\bar{x}, \bar{y})^{S_n}$ is generated as a skew field over $\Bbbk$ by the following set of $2n$ elements:*

$$\{t_1, \ldots, t_n\} \cup \{e_1, \ldots, e_n\}$$

where

$$e_d := \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}, \quad d \in [\![0, n]\!] \tag{3.11}$$

is the degree d elementary symmetric polynomial in $x_1, \ldots, x_n$.

In order to describe precise commutation relations between t_j and e_k, it will be useful to rewrite t_j as follows.

Lemma 3.6. *We have the following formula for* t_j*:*

$$t_j = (-1)^{j-1}\Delta^{-1}\sum_{w\in S_n}\operatorname{sgn}(w)w\big(x_1^{n-2}x_2^{n-3}\cdots x_{n-2}e'_{n-j}y_n\big),\quad \forall j\in [\![1,n]\!],\tag{3.12}$$

where e'_d *denotes the degree* d *elementary symmetric polynomial in the variables* $x_1,\dots,x_{n-1}$ *and* $\Delta=\prod_{1\le i<j\le n}(x_i-x_j)$.

Proof. Let $\Delta'=\prod_{1\le i<j\le n-1}(x_i-x_j)$. Let $\operatorname{Coeff}_{X^j}A(X)$ denote the coefficient of X^j in a polynomial $A(X)$. By (3.5) and (3.6) we have, for any $j\in\{1,\dots,n\}$,

$$\begin{aligned} t_j &= \sum_{i=1}^{n}\Big(\operatorname{Coeff}_{X^{j-1}}\prod_{k\in\{1,\dots,n\}\setminus\{i\}}\frac{X-x_k}{x_i-x_k}\Big)y_i\\ &= \sum_{w\in S_n/S_{n-1}} w\bigg(\frac{(-1)^{n-j}e'_{n-j}}{\prod_{k=1}^{n-1}(x_n-x_k)}y_n\bigg). \end{aligned}$$

Here we mean that w runs through a set of representatives of S_n/S_{n-1}. Since $\Delta/\Delta'=\prod_{k=1}^{n-1}(x_k-x_n)$ and $w(\Delta)=\operatorname{sgn}(w)\Delta$ for all $w\in S_n$, we get

$$t_j = (-1)^{j-1}\Delta^{-1}\sum_{w\in S_n/S_{n-1}}\operatorname{sgn}(w)\cdot w\big(e'_{n-j}\Delta' y_n\big).\tag{3.13}$$

Writing Δ' as a determinant gives $\Delta'=\sum_{w\in S_{n-1}}\operatorname{sgn}(w)w(x_1^{n-2}x_2^{n-3}\cdots x_{n-2})$. Substituting this into (3.13) and using that e'_{n-j} and y_n are fixed by S_{n-1}, gives

$$t_j = (-1)^{j-1}\Delta^{-1}\sum_{\substack{w\in S_n/S_{n-1}\\ w'\in S_{n-1}}}\operatorname{sgn}(ww')ww'(x_1^{n-2}x_2^{n-3}\cdots x_{n-2}e'_{n-j}y_n).\tag{3.14}$$

Since ww' runs through every element of S_n exactly once when w ranges over a set of representatives for S_n/S_{n-1} and w' runs through S_{n-1} we obtain (3.12). □

We now have the following proposition, describing commutation relations between the generators t_j and e_k.

Proposition 3.7. *The following relations hold in* $\Bbbk_q(\bar{x},\bar{y})^{S_n}$*:*

$$[t_i,t_j]=0,\qquad \forall i,j\in[\![1,n]\!],\tag{3.15}$$

$$[e_k,e_l]=0,\qquad \forall k,l\in[\![0,n]\!],\tag{3.16}$$

$$t_je_k - q^{\delta_{j+k>n}}e_kt_j = (q-1)\sum_{i\in\mathbb{Z}\setminus I(n-(j+k))}(-1)^{i+\delta_{i<0}}e_{k+i}t_{j+i},\tag{3.17}$$

for all $j\in[\![1,n]\!]$ *and* $k\in[\![0,n]\!]$*, where* δ_P *is the Kronecker delta* (2.1) *and for all* $k\in\mathbb{Z}$ *we put*

$$I(k) := [\![\min(0,k+1),\max(0,k)]\!] = \begin{cases}[\![0,k]\!], & k\ge 0,\\ [\![k+1,0]\!], & k<0,\end{cases}\tag{3.18}$$

and, by convention, $t_j=0$ *if* $j\notin[\![1,n]\!]$ *and* $e_k=0$ *if* $k\notin[\![0,n]\!]$.

The proof of Proposition 3.7 will be given in Appendix.

3.2. **Simplification of the relations.** In this section we show how to inductively change generators to simplify the relations. The final set of relations are q-commutation relations, which gives a positive solution to the q-difference Noether problem.

We will frequently use the following telescoping sum identities.

Lemma 3.8. *If $\{T_j\}_{j\in\mathbb{Z}}$ is a set of commuting elements of an algebra with at most finitely many nonzero elements, then for all $j,k\in\mathbb{Z}$ the following identities hold:*

$$\sum_{i\in\mathbb{Z}\setminus I(k-j)} (-1)^{\delta_{i<0}} T_{j+i}T_{k-i} = -\delta_{j>k} T_j T_k, \tag{3.19}$$

$$\sum_{i\in\mathbb{Z}\setminus I(-1+k-j)} (-1)^{\delta_{i<0}} T_{j+i}T_{k-i} = \delta_{j<k} T_j T_k, \tag{3.20}$$

where $I(k)$ was defined in (3.18)*.*

Proof. We prove (3.19). The proof of (3.20) is analogous. By shifting the index of T_i we may assume that $j=0$. If $k\geq 0$, then $I(k)=[\![0,k]\!]$ so making the substitution $i\mapsto k-i$ in the left hand side of (3.19) we get the same expression except that $\delta_{i<0}$ has been replaced by $\delta_{k-i<0}$ which equals $1-\delta_{i<0}$ for $i\notin[\![0,k]\!]$. So both sides of (3.19) are zero in this case. If $k<0$, then $I(k)=[\![1+k,0]\!]$. The $i=k$ term in the left hand side of (3.19) equals

$$-T_kT_0. \tag{3.21}$$

Removing this term from the sum gives a sum over the set $\mathbb{Z}\setminus[\![k,0]\!]$ which can be seen to be zero, after substituting $i\mapsto k-i$ as in the previous case. Therefore the left hand side of (3.19) equals (3.21) which in turn is equal to the right hand side of (3.19), since $k<0$. □

The following proposition describes the recursive process for simplifying the relations among the generators.

Proposition 3.9. *Suppose $T_1,\ldots,T_n$ and $E_0,E_1,\ldots,E_n$ are elements of some skew field $\mathbb{F}$ containing $\Bbbk$ such that*

$$[T_i,T_j]=0, \quad \forall i,j\in[\![1,n]\!], \tag{3.22}$$

$$[E_k,E_l]=0, \quad \forall k,l\in[\![0,n]\!], \tag{3.23}$$

$$T_jE_k - q^{\delta_{j+k>n}}E_kT_j = (q-1)\sum_{i\in\mathbb{Z}\setminus I(n-(j+k))} (-1)^{i+\delta_{i<0}} E_{k+i}T_{j+i}, \tag{3.24}$$

$\forall j\in[\![1,n]\!], k\in[\![0,n]\!]$, where by convention $T_j=0$ for $j\notin[\![1,n]\!]$ and $E_k=0$ for $k\notin[\![0,n]\!]$. Define

$$\widetilde{T}_j = \begin{cases} E_jT_1T_n - (-1)^jE_0T_{n-j}T_1 - (-1)^{n-j}E_nT_{n+1-j}T_n, & j\in[\![1,n-1]\!], \\ 0, & \textit{otherwise,} \end{cases} \tag{3.25}$$

$$\widetilde{E}_k = \begin{cases} T_{k+1}, & k\in[\![0,n-1]\!], \\ 0, & \textit{otherwise.} \end{cases} \tag{3.26}$$

Then

$$[\widetilde{T}_i, \widetilde{T}_j] = 0, \quad \forall i, j \in [\![1, n-1]\!], \tag{3.27}$$

$$[\widetilde{E}_k, \widetilde{E}_l] = 0, \quad \forall k, l \in [\![0, n-1]\!], \tag{3.28}$$

and, where $\circ$ *denotes the opposite multiplication* $a \circ b = ba$,

$$\widetilde{T}_j \circ \widetilde{E}_k - q^{\delta_{j+k>n-1}} \widetilde{E}_k \circ \widetilde{T}_j = (q-1) \sum_{i \in \mathbb{Z} \setminus I(n-1-(j+k))} (-1)^{i+\delta_{i<0}} \widetilde{E}_{k+i} \circ \widetilde{T}_{j+i} \tag{3.29}$$

for all $j \in [\![1, n-1]\!]$, $k \in [\![0, n-1]\!]$. *Moreover, we have the following alternative expression for* $\widetilde{T}_j$*:*

$$q\widetilde{T}_j = T_n T_1 E_j - (-1)^j T_1 T_{n-j} E_0 - (-1)^{n-j} T_n T_{n+1-j} E_n, \quad \forall j \in [\![1, n-1]\!] \tag{3.30}$$

which is equal to the right hand side of (3.25) *calculated in the opposite algebra. Furthermore, the set* $\{E_0, E_n\} \cup \{\widetilde{T}_j\}_{j=1}^{n-1} \cup \{\widetilde{E}_k\}_{k=0}^{n-1}$ *generates the same skew subfield of* $\mathbb{F}$ *as the original generators* $\{T_j\}_{j=1}^{n} \cup \{E_k\}_{k=0}^{n}$.

The proof of Proposition 3.9 will be given in the Appendix.

We can now prove the following theorem which implies Theorem II for the symmetric group S_n.

Theorem 3.10. *Define a set of elements* $e_k^{(i)} \in \mathbb{k}_q(\bar{x}, \bar{y})^{S_n}$ *for* $i \in [\![0, n]\!]$, $k \in [\![0, n-i]\!]$ *recursively by*

$$e_k^{(0)} = e_k, \qquad \forall k \in [\![0, n]\!], \tag{3.31a}$$

$$e_k^{(1)} = t_{k+1}, \qquad \forall k \in [\![0, n-1]\!], \tag{3.31b}$$

$$\begin{aligned} e_k^{(i)} =& e_{k+1}^{(i-2)} e_0^{(i-1)} e_{n-i+1}^{(i-1)} - (-1)^{k+1} e_0^{(i-2)} e_{n-i-k}^{(i-1)} e_0^{(i-1)} \\ &- (-1)^{n-i+1-k} e_{n-i+2}^{(i-2)} e_{n-i+1-k}^{(i-1)} e_{n-i+1}^{(i-1)}, \qquad \forall k \in [\![0, n-i]\!], \forall i \in [\![2, n]\!], \end{aligned} \tag{3.31c}$$

where e_d *and* t_j *were defined in* (3.11) *and* (3.7) *respectively. Let*

$$(X_1, X_2, \ldots, X_n) = (e_n^{(0)}, e_{n-1}^{(1)}, \ldots, e_1^{(n-1)}), \tag{3.32a}$$

$$(Y_1, Y_2, \ldots, Y_n) = (e_0^{(1)}, e_0^{(2)}, \ldots e_0^{(n)}), \tag{3.32b}$$

and put

$$\widehat{X}_1 = X_1, \quad \widehat{X}_i = Y_{i-1}^{(-1)^i} X_i^{(-1)^{i+1}}, \qquad \forall i \in [\![2, n]\!], \tag{3.33a}$$

$$\widehat{Y}_1 = Y_1, \quad \widehat{Y}_2 = Y_1^{-2} Y_2, \quad \widehat{Y}_j = Y_{j-2}^{-1} Y_{j-1}^{-2} Y_j, \qquad \forall j \in [\![3, n]\!], \tag{3.33b}$$

Then there is an isomorphism of $\mathbb{k}$*-algebras*

$$\mathbb{k}_q(\bar{x}, \bar{y}) \xrightarrow{\sim} \mathbb{k}_q(\bar{x}, \bar{y})^{S_n} \tag{3.34}$$

given by

$$x_k \longmapsto \widehat{X}_k, \qquad \forall k \in [\![1, n]\!], \tag{3.35}$$

$$y_k \longmapsto \widehat{Y}_k, \qquad \forall k \in [\![1, n]\!]. \tag{3.36}$$

Proof. First we prove that for each $i \in [\![1, n]\!]$, the elements

$$(E_0, \ldots, E_{n-i+1}) = (e_0^{(i-1)}, \ldots, e_{n-i+1}^{(i-1)}), \tag{3.37a}$$

$$(T_1, \ldots, T_{n-i+1}) = (e_0^{(i)}, \ldots, e_{n-i}^{(i)}), \tag{3.37b}$$

satisfy relations (3.22),(3.23) and (3.24) with n replaced by $n-i+1$, and

$$\mathbb{F}=\mathbb{F}_i:=\begin{cases}\Bbbk_q(\bar{x},\bar{y})^{S_n} & \text{if } i \text{ is odd,}\\ \big(\Bbbk_q(\bar{x},\bar{y})^{S_n}\big)^{\mathrm{op}} & \text{if } i \text{ is even.}\end{cases} \tag{3.38}$$

We prove this by induction on i. For $i=1$ this follows from Proposition 3.7. For $i>1$ we may, by the induction hypothesis, apply Proposition 3.9 with n replaced by $n-i+2$ and

$$(E_0,\ldots,E_{n-i+2})=(e_0^{(i-2)},\ldots,e_{n-i+2}^{(i-2)}), \tag{3.39a}$$

$$(T_1,\ldots,T_{n-i+2})=(e_0^{(i-1)},\ldots,e_{n-i+1}^{(i-1)}), \tag{3.39b}$$

and $\mathbb{F}=\mathbb{F}_{i-1}$. Substituting (3.39) into (3.25), (3.26), we obtain

$$(e_0^{(i-1)},\ldots,e_{n-i+1}^{(i-1)})=(\widetilde{E}_0,\ldots,\widetilde{E}_{n-i+1}), \tag{3.40}$$

and, *in the algebra* $\mathbb{F}_{i-1}$,

$$(e_0^{(i)},\ldots,e_{n-i}^{(i)})=(\widetilde{T}_1,\ldots,\widetilde{T}_{n-i+1}),$$

by the definition of $e_k^{(i)}$. Thus, keeping in mind (3.30), we obtain that in $\Bbbk_q(\bar{x},\bar{y})^{S_n}$,

$$(e_0^{(i)},\ldots,e_{n-i}^{(i)})=q^{\delta_{i-1\in 2\mathbb{Z}}}(\widetilde{T}_1,\ldots,\widetilde{T}_{n-i+1}). \tag{3.41}$$

However, the possible extra factor q does matter; the conclusion from Proposition 3.9 that relations (3.27),(3.28), and (3.29) (with n replaced by $n-i+2$) hold in $\mathbb{F}_{i-1}$ implies that, choosing E_k, T_j as in (3.37), relations (3.22),(3.23),(3.24) (with n replaced by $n-i+1$) hold in the algebra $\mathbb{F}_i$. This proves the induction step.

In particular, by (3.22) and (3.23),

$$[e_j^{(i)},e_0^{(i)}]=0,\quad \forall j\in[\![0,n-i]\!],\forall i\in[\![0,n]\!], \tag{3.42a}$$

$$[e_j^{(i)},e_{n-i}^{(i)}]=0,\quad \forall j\in[\![0,n-i]\!],\forall i\in[\![0,n]\!]. \tag{3.42b}$$

By (8.16) and (8.17) we have, in $\Bbbk(\bar{x},\bar{y})^{S_n}$,

$$[e_j^{(i+1)},e_0^{(i)}]=0,\quad \forall j\in[\![0,n-i-1]\!],\forall i\in[\![0,n-1]\!], \tag{3.43a}$$

$$[e_j^{(i+1)},e_{n-i}^{(i)}]_{q^{(-1)^i}}=0,\quad \forall j\in[\![1,n-i-1]\!],\forall i\in[\![0,n-1]\!]. \tag{3.43b}$$

More generally, the following relations hold:

$$[e_j^{(k)},e_0^{(i)}]=0,\quad \forall j\in[\![0,n-k]\!], 0\le i\le k\le n, \tag{3.44a}$$

$$[e_j^{(k)},e_{n-i}^{(i)}]_{q^{(-1)^i\cdot a_{k-i}}}=0,\quad \forall j\in[\![0,n-k]\!], 0\le i\le k\le n, \tag{3.44b}$$

where $a_k\in\mathbb{Z}$ is defined by the recursion relation

$$a_k=2a_{k-1}+a_{k-2},\ \ a_0=0,\ a_1=1. \tag{3.45}$$

To prove this we use induction on $k-i$. For $k-i=0$ and $k-i=1$, relations (3.44) follow from (3.42) and (3.43) respectively. Assume $k-i>1$. By the induction hypothesis we have, for any j_1,j_2,j_3,

$$[e_{j_1}^{(k-2)}e_{j_2}^{(k-1)}e_{j_3}^{(k-1)},e_0^{(i)}]=0 \tag{3.46}$$

and

$$\begin{aligned} e_{j_1}^{(k-2)}e_{j_2}^{(k-1)}e_{j_3}^{(k-1)}\cdot e_{n-i}^{(i)} &= q^{(-1)^i\cdot(a_{k-2-i}+2a_{k-1-i})}e_{n-i}^{(i)}\cdot e_{j_1}^{(k-2)}e_{j_2}^{(k-1)}e_{j_3}^{(k-1)}\\ &= q^{(-1)^i\cdot a_{k-i}}e_{n-i}^{(i)}\cdot e_{j_1}^{(k-2)}e_{j_2}^{(k-1)}e_{j_3}^{(k-1)} \end{aligned} \tag{3.47}$$

Using (3.46),(3.47) and the definition, (3.31), of $e_j^{(k)}$, we obtain (3.44).

Using X_i, Y_j given in (3.32), relations (3.44) imply that

$$
\begin{aligned}
[Y_k, Y_i] &= 0, \quad \forall k, i \in [\![1, n]\!], && (3.48a)\\
[X_k, X_i]_{q^{(-1)^{i+1}\cdot a_{k-i}}} &= 0, \quad k \geq i, && (3.48b)\\
[Y_k, X_i] &= 0, \quad k < i, && (3.48c)\\
[Y_k, X_i]_{q^{(-1)^{i+1}\cdot a_{k-i+1}}} &= 0, \quad k \geq i. && (3.48d)
\end{aligned}
$$

This means that, putting $(Z_1, \ldots, Z_{2n}) = (X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$, we have

$$Z_i Z_j = q^{s_{ij}} Z_j Z_i, \tag{3.49}$$

where $S = (s_{ij})$ is the $2n \times 2n$ skew-symmetric integer matrix

$$S = \begin{bmatrix}
0 & -1 & -1 & -2 & -2 & -5 & \cdots & -a_{n-1} & -a_n \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 2 & \cdots & a_{n-2} & a_{n-1} \\
2 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
2 & 0 & -1 & 0 & 0 & -1 & \cdots & -a_{n-3} & -a_{n-2} \\
5 & 0 & -2 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1} & 0 & -a_{n-2} & 0 & a_{n-3} & 0 & \cdots & 0 & (-1)^n \\
a_n & 0 & -a_{n-1} & 0 & a_{n-2} & 0 & \cdots & (-1)^{n+1} & 0
\end{bmatrix} \tag{3.50}$$

The matrix S may be brought to normal form as follows. Take

$$U = \begin{bmatrix}
1 & & & & & & & & & & & \\
 & 1 & 1 & -2 & & -1 & & & & & & \\
 & & -1 & & & & & & & & & \\
 & & & 1 & -1 & -2 & & -1 & & & & \\
 & & & & 1 & & & & & & & \\
 & & & & & 1 & 1 & -2 & & -1 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & & \cdots & 1 & (-1)^{n-1} & -2 & & -1 \\
 & & & & & & \cdots & & (-1)^n & & & \\
 & & & & & & \cdots & & & 1 & (-1)^n & -2 \\
 & & & & & & \cdots & & & & (-1)^{n+1} & \\
 & & & & & & \cdots & & & & & 1
\end{bmatrix}$$

where zero entries were omitted. Then $U^t S U$ is block diagonal with n copies of $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ on the diagonal. As in the proof of Proposition 2.1, we see that the matrix U corresponds exactly to the change of variables (3.33). This proves that

$$
\begin{aligned}
\widehat{Y}_i \widehat{X}_j &= q^{\delta_{ij}} \widehat{X}_j \widehat{Y}_i, \quad \forall i, j \in [\![1, n]\!], \\
[\widehat{X}_i, \widehat{X}_j] &= [\widehat{Y}_i, \widehat{Y}_j] = 0, \quad \forall i, j \in [\![1, n]\!].
\end{aligned} \tag{3.51}
$$

Moreover, one can easily check that the set $\{\widehat{X}_1, \ldots, \widehat{X}_n, \widehat{Y}_1, \ldots, \widehat{Y}_n\}$ generate the skew subfield isomorphic to the skew subfield generated by $X_1, \ldots, X_n, Y_1, \ldots, Y_n$. Alternatively, one may prove (3.51) directly by using (3.48) and (3.33).

Now (3.51) implies the existence of a unique $\Bbbk$-algebra homomorphism (3.34) satisfying (3.35),(3.36). Since the domain is a skew field, it is sufficient to show that the homomorphism is surjective. It follows from Proposition 3.9 that the set $\{X_1,\dots,X_n,Y_1,\dots,Y_n\}$ generates $\Bbbk_q(\bar{x},\bar{y})^{S_n}$ as a skew field over $\Bbbk$. Hence

$$\{\widehat{X}_1,\dots,\widehat{X}_n,\widehat{Y}_1,\dots,\widehat{Y}_n\}$$

also generates $\Bbbk_q(\bar{x},\bar{y})^{S_n}$ and thus the homomorphism (3.34) is surjective. This concludes the proof. □

Corollary 3.11. *We have an isomorphism of* $\Bbbk$*-algebras*

$$\big(\operatorname{Frac}(A_1^q(\Bbbk)^{\otimes_{\Bbbk} n})\big)^{S_n} \simeq \operatorname{Frac}(A_1^q(\Bbbk)^{\otimes_{\Bbbk} n}). \tag{3.52}$$

Proof. Follows directly from Theorem 3.10 and Proposition 2.2, noting that the isomorphism in Proposition 2.2 commutes with the S_n-action. □

We will need one more property of the isomorphism (3.34). For $r\in\Bbbk\setminus\{0\}$ we define two automorphisms α_r,β_r of $\Bbbk_q(\bar{x},\bar{y})$ as follows:

$$\alpha_r,\beta_r:\Bbbk_q(\bar{x},\bar{y})\to\Bbbk_q(\bar{x},\bar{y}), \tag{3.53}$$

$$\alpha_r(x_j)=x_j,\quad \alpha_r(y_j)=r\cdot y_j,\quad \forall j\in[\![1,n]\!] \tag{3.54}$$

$$\beta_r(x_j)=x_j,\quad \beta_r(y_j)=r^{\delta_{1j}}\cdot y_j,\quad \forall j\in[\![1,n]\!]. \tag{3.55}$$

Similarly to how one proves the commutation relations

$$[\widehat{X}_1,\widehat{X}_j]=0,\quad \widehat{X}_1\widehat{Y}_j\widehat{X}_1^{-1}=q^{-\delta_{1j}}\widehat{Y}_j,\quad \forall j\in[\![1,n]\!]$$

one can verify the following result.

Lemma 3.12. *The isomorphism* $g:\Bbbk_q(\bar{x},\bar{y})^{S_n}\to\Bbbk_q(\bar{x},\bar{y})$ *constructed in Theorem 3.10 satisfies*

$$g\circ\alpha_r\circ g^{-1}=\beta_r \tag{3.56}$$

for all $r\in\Bbbk\setminus\{0\}$.

4. The q-difference Noether problem for classical Weyl groups

Let $W(B_n)=W(C_n)=S_n\ltimes(\mathbb{Z}/2\mathbb{Z})^n$ be the Weyl group of type B_n (equivalently, of type C_n). The group $W(B_n)$ acts naturally on $\Bbbk_q(\bar{x},\bar{y})\simeq\mathcal{O}_q(\Bbbk^{2n})$ by

$$\zeta(x_i)=x_{\zeta(i)},\quad \zeta(y_i)=y_{\zeta(i)},\quad \forall\zeta\in S_n,\ \forall i\in[\![1,n]\!], \tag{4.1a}$$

$$\alpha(x_i)=(-1)^{\alpha_i}x_i,\quad \alpha(y_i)=(-1)^{\alpha_i}y_i,\quad \forall\alpha\in(\mathbb{Z}/2\mathbb{Z})^n,\ \forall i\in[\![1,n]\!]. \tag{4.1b}$$

Let $\mathcal{E}_n=\{\alpha=(\alpha_1,\dots,\alpha_n)\in(\mathbb{Z}/2\mathbb{Z})^n\mid \alpha_1+\dots+\alpha_n=0\}$ and $W(D_n)=S_n\ltimes\mathcal{E}_n$ be the Weyl group of type D_n.

In Theorem 3.10 we solved the q-difference Noether problem for the Weyl group of type A_n. In this section we will show that the other cases (B_n,C_n,D_n) can be reduced to that case. First note that by replacing y_i by x_iy_i in $\Bbbk_q(\bar{x},\bar{y})$ we can, and will, assume that $(\mathbb{Z}/2\mathbb{Z})^n$ fixes y_i for all i, so that (4.1b) is replaced by

$$\alpha(x_i)=(-1)^{\alpha_i}x_i,\quad \alpha(y_i)=y_i,\quad \forall\alpha\in(\mathbb{Z}/2\mathbb{Z})^n,\ \forall i\in[\![1,n]\!]. \tag{4.2}$$

We start with the case B_n, which is the easiest.

Theorem 4.1. *The q-difference Noether problem for the Weyl group of type* B_n *as a positive solution. More precisely, there exist* $\Bbbk$*-algebra isomorphisms*

$$\Bbbk_q(\bar{x},\bar{y})^{W(B_n)}\simeq\Bbbk_{q^2}(\bar{x},\bar{y})^{S_n}\simeq\Bbbk_{q^2}(\bar{x},\bar{y}). \tag{4.3}$$

Proof. Using that

$$\{x_1^{k_1}\cdots x_n^{k_n}\cdot y_1^{k_{n+1}}\cdots y_n^{k_{2n}} \mid k\in\mathbb{Z}^{2n}\}$$

is a $\Bbbk$-basis for $\Bbbk_q[\bar{x},\bar{y}]$ it is easy to see that there is an isomorphism of $\Bbbk$-algebras

$$\Bbbk_{q^2}[\bar{x},\bar{y}] \xrightarrow{\sim} \Bbbk_q[\bar{x},\bar{y}]^{(\mathbb{Z}/2\mathbb{Z})^n}$$

given by

$$\begin{aligned} x_i &\longmapsto x_i^2,\\ y_i &\longmapsto y_i. \end{aligned}$$

Taking skew field of fractions on both sides, followed by taking S_n-invariants we obtain that

$$\Bbbk_{q^2}(\bar{x},\bar{y})^{S_n} \simeq \left(\Bbbk_q(\bar{x},\bar{y})^{(\mathbb{Z}/2\mathbb{Z})^n}\right)^{S_n} = \Bbbk_q(\bar{x},\bar{y})^{W(B_n)},$$

which together with (3.34) proves (4.3). □

For the remaining type D_n case, we need the following lemma.

Lemma 4.2. *The algebra $\Bbbk_q(\bar{x},\bar{y})^{W(D_n)}$ is free as a left $\Bbbk_q(\bar{x},\bar{y})^{W(B_n)}$-module with basis $\{1, x_1x_2\cdots x_n\}$.*

Proof. We must prove that

$$\Bbbk_q(\bar{x},\bar{y})^{W(B_n)} \oplus \Bbbk_q(\bar{x},\bar{y})^{W(B_n)}\cdot x_1x_2\cdots x_n = \Bbbk_q(\bar{x},\bar{y})^{W(D_n)} \tag{4.4}$$

Let $\gamma\in W(B_n)$ be a representative for the nontrivial element in $W(B_n)/W(D_n)\simeq \mathbb{Z}/2\mathbb{Z}$. For example we may take $\gamma=(1,0,\dots,0)\in(\mathbb{Z}/2\mathbb{Z})^n\subseteq W(B_n)$. Then γ acts as an order two $\Bbbk$-algebra automorphism of $\Bbbk_q(\bar{x},\bar{y})^{W(D_n)}$. By polarization, we get a decomposition of $\Bbbk_q(\bar{x},\bar{y})^{W(D_n)}$ into ±1 eigenspaces. The $+1$ eigenspace of γ is obviously equal to $\Bbbk_q(\bar{x},\bar{y})^{W(B_n)}$. Since $x_1x_2\cdots x_n$ belongs to the -1 eigenspace and is invertible, it is easy to see that the -1 eigenspace of γ equals

$$\Bbbk_q(\bar{x},\bar{y})^{W(B_n)}\cdot x_1x_2\cdots x_n.$$

This proves (4.4). □

We are now ready to prove the following.

Theorem 4.3. *The q-difference Noether problem for the Weyl group $W_n=W(D_n)$ of type D_n has a positive solution. Explicitly, there exists a $\Bbbk$-algebra isomorphism*

$$\Bbbk_q(\bar{x},\bar{y})^{W_n} \simeq \operatorname{Frac}\left(\Bbbk_q[x,y]\otimes_{\Bbbk}\Bbbk_{q^2}[x,y]^{\otimes_{\Bbbk}(n-1)}\right). \tag{4.5}$$

Proof. The isomorphism $g=g_2\circ g_1$ where $g_1:\Bbbk_q(\bar{x},\bar{y})^{W(B_n)}\xrightarrow{\sim}\Bbbk_{q^2}(\bar{x},\bar{y})^{S_n}$ and $g_2:\Bbbk_{q^2}(\bar{x},\bar{y})^{S_n}\xrightarrow{\sim}\Bbbk_{q^2}(\bar{x},\bar{y})$, obtained in the proof of Theorem 4.1, satisfies $g(x_1^2x_2^2\cdots x_n^2)=x_1$. We also have a $\Bbbk$-algebra monomorphism

$$\begin{aligned} k:\Bbbk_{q^2}(\bar{x},\bar{y}) &\hookrightarrow \operatorname{Frac}\left(\Bbbk_q[x_1,y_1]\otimes_{\Bbbk}\Bbbk_{q^2}[x_2,y_2]\otimes_{\Bbbk}\cdots\otimes_{\Bbbk}\Bbbk_{q^2}[x_n,y_n]\right),\\ x_1 &\mapsto x_1^2,\\ x_i &\mapsto x_i, \quad \forall i\in [\![2,n]\!],\\ y_i &\mapsto y_i, \quad \forall i\in [\![1,n]\!]. \end{aligned}$$

Similarly to Lemma 4.2 we have a direct sum decomposition

$$\operatorname{Frac}\left(\Bbbk_q[x_1,y_1]\otimes_{\Bbbk}\Bbbk_{q^2}[x_2,y_2]\otimes_{\Bbbk}\cdots\otimes_{\Bbbk}\Bbbk_{q^2}[x_n,y_n]\right) = \operatorname{im}k\oplus(\operatorname{im}k)\cdot x_1. \tag{4.6}$$

Using Lemma 4.2, we now define

$$f:\Bbbk_q(\bar{x},\bar{y})^{W(D_n)}\longrightarrow \operatorname{Frac}\big(\Bbbk_q[x_1,y_1]\otimes_{\Bbbk}\Bbbk_{q^2}[x_2,y_2]\otimes_{\Bbbk}\cdots\otimes_{\Bbbk}\Bbbk_{q^2}[x_n,y_n]\big) \tag{4.7}$$

by

$$f(a+b\cdot x_1x_2\cdots x_n)=(k\circ g)(a)+(k\circ g)(b)\cdot x_1,\quad \forall a,b\in\Bbbk_q(\bar{x},\bar{y})^{W(B_n)}. \tag{4.8}$$

By (4.6), f is a surjective map. Furthermore, the restriction of f to $\Bbbk_q(\bar{x},\bar{y})^{W(B_n)}$ is a homomorphism and $x_1^2=f\big((x_1x_2\cdots x_n)^2\big)$. Thus, to prove that f is a homomorphism it is thus enough to show that

$$(k\circ g)\big(x_1x_2\cdots x_n\cdot a\cdot(x_1x_2\cdots x_n)^{-1}\big)=x_1\cdot(k\circ g)(a)\cdot x_1^{-1},\quad \forall a\in\Bbbk_q(\bar{x},\bar{y})^{W(B_n)}. \tag{4.9}$$

Recall the automorphisms α_r,β_r from Lemma 3.12. We have

$$x_1\cdot(k\circ g)(a)\cdot x_1^{-1}=(k\circ\beta_{q^{-1}}\circ g_2\circ g_1)(a). \tag{4.10}$$

By Lemma 3.12, $\beta_{q^{-1}}=g_2\circ\alpha_{q^{-1}}\circ g_2^{-1}$. So (4.10) equals

$$(k\circ g_2\circ\alpha_{q^{-1}}\circ g_1)(a)=(k\circ g)\big(x_1x_2\cdots x_n\cdot a\cdot(x_1x_2\cdots x_n)^{-1}\big)$$

which proves (4.9). This proves that f is a surjective $\Bbbk$-algebra homomorphism. It is injective since its domain is a skew-field. □

Theorem 4.1 and Theorem 4.3 complete the proof of Theorem II.

Remark 4.4. We note that a positive solution to the q-difference Noether problem for classical Weyl groups in the case $q=1$ can be deduced from [Mi, Remark 3].

5. Reduction via Galois rings

For the rest of the paper we specialize to $\Bbbk=\mathbb{C}$ as ground field, and assume that $q\in\mathbb{C}\setminus\{0\}$ is not a root of unity.

We use the theory of Galois rings [FO] to reduce the quantum Gelfand-Kirillov conjecture to the q-difference Noether problem.

5.1. Galois rings.

In this subsection, Γ denotes an integral domain, K the field of fractions of Γ, $K\subseteq L$ a finite Galois extension with Galois group $G=\operatorname{Gal}(L/K)$, and $\mathcal{M}$ a monoid acting on L by automorphisms. We will assume that $\mathcal{M}$ is K-separating, that is $m_1|_K=m_2|_K$ implies $m_1=m_2$ for $m_1,m_2\in\mathcal{M}$. The group G acts naturally on $\mathcal{M}$ by conjugations and thus on the skew monoid ring $L*\mathcal{M}$ by automorphisms. We denote the G-invariants in $L*\mathcal{M}$ by $(L*\mathcal{M})^G$.

If $u=\sum_{m\in\mathcal{M}}a_m m\in L*\mathcal{M}$, we put $\operatorname{Supp}(u)=\{m\in\mathcal{M}\mid a_m\neq 0\}$. For $\varphi\in\mathcal{M}$, let $\operatorname{Stab}_G(\varphi)$ be the stabilizer subgroup of G at φ and $T_\varphi\subseteq G$ be a set of representatives for $G/\operatorname{Stab}_G(\varphi)$ (the set of orbits of the action of $\operatorname{Stab}_G(\varphi)$ on G by conjugations). For $a\in L$, put

$$[a\varphi]:=\sum_{g\in T_\varphi}a^g\varphi^g. \tag{5.1}$$

Then $[a\varphi]\in(L*\mathcal{M})^G$, [FO, Lemma 2.1].

Definition 5.1 ([FO], Definition 3)**.** A finitely generated Γ-subring $U\subseteq(L*\mathcal{M})^G$ is called a *Galois* Γ*-ring* if $UK=KU=(L*\mathcal{M})^G$.

Proposition 5.2 ([FO], Proposition 4.1)**.** *Suppose U is a Γ-subring of $(L*\mathcal{M})^G$ generated by $u_1,\dots,u_k\in U$. If $\cup_{i=1}^k\operatorname{Supp}(u_i)$ generate $\mathcal{M}$ as a monoid, then U is a Galois Γ-ring in $(L*\mathcal{M})^G$.*

Proof. Since the proof of [FO, Proposition 4.1] is rather sketchy we provide the details for convenience. Consider a K-subbimodule $V = Ku_1K + \cdots + Ku_kK$ in $(L * \mathcal{M})^G$. It follows from the proof of [FO, Lemma 4.1] that for any i and any $m \in \operatorname{Supp}(u_i)$ there exists $a \in L$ such that $[am] \in Ku_iK$. Thus the bimodule V contains the elements $[a_1\varphi_1], \ldots, [a_t\varphi_t]$, where $\varphi_1^g, \ldots, \varphi_t^g$, $g \in G$, generate $\mathcal{M}$. Now consider a subalgebra $U' \subset U$ generated over Γ by $[a_i\varphi_i]$, $i = 1, \ldots, t$. Since

$$\operatorname{Supp}([am]\Gamma[a'm']) = \operatorname{Supp}[am]\operatorname{Supp}[a'm'],$$

then given $\varphi \in \mathcal{M}$ one can find $a \in L$ such that $[a\varphi] \in U'$. Moreover, $a \in L^{\operatorname{Stab}_G(\varphi)}$. Now we use the fact that $K\varphi(\Gamma) = \varphi(K)$ and hence

$$K(\Gamma[a\varphi]\Gamma) = [K\Gamma\varphi(\Gamma)a\varphi] = [K\varphi(K)a\varphi].$$

Thus $K(\Gamma[a\varphi]\Gamma) = [L^{\operatorname{Stab}_G(\varphi)}\varphi]$ and $KU \simeq (L * \mathcal{M})^G$. Similarly, $UK \simeq (L * \mathcal{M})^G$. We conclude that U is a Galois Γ-ring in $(L * \mathcal{M})^G$.

□

5.2. **The center of $U_q(\mathfrak{gl}_N)$.** It is known that the center Z_N of $U_N = U_q(\mathfrak{gl}_N)$ is generated by the quantum Casimir operators constructed by Bracken, Gould and Zhang [BGZ] and by the element $(K_1 \ldots K_N)^{-1}$ [Li]. Here we recall some facts that will be used in later sections.

Let U_N^0, (respectively $U_N^{\pm}$) be the subalgebra of U_N generated by $K_i, K_i^{-1}, i \in [\![1, N]\!]$ (respectively $E_j^{\pm}, j \in [\![1, N-1]\!]$). By the quantum PBW theorem we have $U_N = U_N^+U_N^0U_N^-$. Thus each $a \in U_N$ can be uniquely decomposed as $a = a^{(0)} + a'$, where $a^{(0)} \in U_N^0$ and $a' \in \sum_j E_j^+U_N + U_NE_j^-$. The *quantum Harish-Chandra homomorphism* $h_N : Z_N \to U_N^0$ is defined by $h_N(z) = z^{(0)}$.

Put $\widetilde{K}_i = q^{-i}K_i$. We may regard U_N^0 as a Laurent polynomial algebra in the variables $\widetilde{K}_i$. Let $W_N = S_N \ltimes \mathcal{E}_N$, the Weyl group of type D_N, act on U_N^0 by permutations and sign changes of $\widetilde{K}_i$, $i \in [\![1, N]\!]$. The following lemma give a description of the center of U_N.

Lemma 5.3. *We have $\mathbb{C}$-algebra isomorphisms*

$$Z_N \overset{h_N}{\simeq} (U_N^0)^{W_N} \simeq \mathbb{C}[z_1, \ldots, z_{N-1}][z_N^{\pm 1}]. \tag{5.2}$$

Proof. Let $(U_N^0)_{\mathrm{ev}}$ denote the subalgebra of U_N^0 generated by $K_i^{\pm 2}$, $i \in [\![1, N]\!]$. By [Li, Lemma 2.1], h_N is injective and its image is generated by $\big((U_N^0)_{\mathrm{ev}}\big)^{S_N}$ and the element I_N^{-1}, where $I_N := K_1K_2 \cdots K_N$. Note that $\mathcal{E}_N$ fixes $K_i^{\pm 2}$ for all $i \in [\![1, N]\!]$ and also fixes I_N^{-1} since there are only an even number of sign changes. Thus the image of h_N is contained in $(U_N^0)^{W_N}$. For the converse inclusion, one can check that the order two $\mathbb{C}$-algebra automorphism of U_N^0 given by $K_j \mapsto (-1)^{\delta_{1j}}K_j$ for $j \in [\![1, N]\!]$ preserves the subalgebra $(U_N^0)^{W_N}$. The $+1$ eigenspace of $(U_N^0)^{W_N}$ coincides with $\big((U_N^0)_{\mathrm{ev}}\big)^{S_N}$. The element I_N belongs to the -1 eigenspace of $(U_N^0)^{W_N}$. Multiplying any element of the -1 eigenspace by I_N we get an element of the $+1$ eigenspace. Since I_N is invertible, it follows that the -1 eigenspace of $(U_N^0)^{W_N}$ is equal to $I_N^{-1} \cdot (U_N^0)_{\mathrm{ev}}$. This proves that the image of h_N equals $(U_N^0)^{W_N}$.

For the second map in (5.2) we define

$$\begin{aligned} f : \mathbb{C}[z_1,\ldots,z_{N-1}][z_N^{\pm 1}] &\longrightarrow (U_N^0)^{W_N}, \\ z_d &\longmapsto e_d(\widetilde{K}_1^2,\ldots,\widetilde{K}_N^2), \quad \forall d\in [\![1,N-1]\!], \\ z_N &\longmapsto \widetilde{K}_1\widetilde{K}_2\cdots\widetilde{K}_N, \end{aligned}$$

where e_d is the elementary symmetric polynomial in N variables of degree d. Since I_N is invertible and U_N^0 is commutative, f is a well-defined $\mathbb{C}$-algebra homomorphism. By the previous paragraph, any element of $(U_N^0)^{W_N}$ can be written as a sum of elements of the form $I_N^{-k}\cdot u$, where $k\in\mathbb{Z}_{\geq 0}$ and u is a symmetric polynomial in $\widetilde{K}_i^2$, $i\in[\![1,N]\!]$. By Newton's theorem and that $f(z_N^2)=e_N(\widetilde{K}_1^2,\ldots,\widetilde{K}_N^2)$, we conclude that u, hence $I_N^{-k}\cdot u$ lies in the image of f. This proves that f is surjective. To prove that f is injective, it is enough to prove that $f(z_1),\ldots,f(z_N)$ are algebraically independent over $\mathbb{C}$. By applying the involution $K_j\mapsto(-1)^{\delta_{1j}}K_j$ from the previous paragraph, it is enough to prove that $f(z_1),\ldots,f(z_{N-1}),f(z_N)^2$ are algebraically independent, which follows from Newton's theorem. □

Remark 5.4. We note that [Li, Eq. (2.5)] can be regarded as a q-deformation of a formula of Zhelobenko [Zh].

5.3. **Gelfand-Tsetlin modules over $U_q(\mathfrak{gl}_N)$.** Gelfand-Tsetlin bases for finite-dimensional irreducible representations of $U_q(\mathfrak{gl}_N)$ were obtained in [UTS]. Similarly to the classical $U(\mathfrak{gl}_N)$-case, the bases consist of finite sets of *tableaux*, i.e. double-indexed families $(\lambda_{mi})_{1\leq i\leq m\leq N}$ of integers, satisfying certain conditions. The action of the generators $E_i^{\pm}$ and K_j on these tableaux are given by q-analogues of the classical Gelfand-Tsetlin formulas.

Mazorchuk and Turowska [MT] used these formulas to define a family of $U_q(\mathfrak{gl}_N)$-modules (in fact they used the algebra obtained from $U_{q^2}(\mathfrak{gl}_N)$ by adjoining $K_j^{\pm 1/2}$, but the results are the same), the so called *generic Gelfand-Tsetlin modules*, which are always infinite-dimensional and not necessarily simple. The bases are now parametrized by tableaux with complex entries $\lambda=(\lambda_{mi})_{1\leq i\leq m\leq N}\in\mathbb{C}^{N(N+1)/2}$. The only restriction on the tableaux is that they should be *admissible*. By definition, a tableau λ is admissible if $q^{2(k+\lambda_{mi}-\lambda_{mj})}\neq 1$ for all $k\in\mathbb{Z}$ and all $1\leq i,j\leq m\leq N$.

The following theorem gives their construction. For $x\in\mathbb{C}$ we put

$$[x]_q := \frac{q^x-q^{-x}}{q-q^{-1}}.$$

Theorem 5.5 ([MT], Theorem 2)**.** *To each admissible tableau λ there exists a $U_q(\mathfrak{gl}_N)$-module $V(\lambda)$ with basis $B(\lambda)=\big\{[\lambda+\gamma]\mid\gamma\in\mathbb{Z}^{N(N-1)/2}\big\}$ and action given by*

$$\begin{aligned} E_m^{\pm}[\mu] &= \sum_{i=1}^{m} a_{mi}^{\pm}(\mu)[\mu\pm\delta^{mi}], \qquad m=1,\ldots,N-1, \\ K_m[\mu] &= q^{\sum_{i=1}^m \mu_{mi}-\sum_{i=1}^{m-1}\mu_{m-1,i}}[\mu], \qquad m=1,\ldots,N, \end{aligned} \tag{5.3}$$

for any $\mu\in B(\lambda)$, where δ^{mi} is the Kronecker tableau given by $(\delta^{mi})_{kj}=\delta_{mk}\delta_{ij}$ and

$$a_{mi}^{\pm}(\mu) := \mp\frac{\prod_{j=1}^{m\pm 1}[\widetilde{\mu}_{m\pm 1,j}-\widetilde{\mu}_{mi}]_q}{\prod_{j\in\{1,\ldots,m\}\setminus\{i\}}[\widetilde{\mu}_{mj}-\widetilde{\mu}_{mi}]_q}, \tag{5.4}$$

where $\widetilde{\mu}_{mi}:=\mu_{mi}-i$ for all $1\leq i\leq m\leq N$.

Note that the denominator in (5.4) is always nonzero since λ is admissible. The following result will also be used.

Theorem 5.6 ([MT], in Proof of Theorem 4)**.** *The intersection of all annihilators of the $U_q(\mathfrak{gl}_N)$-modules $V(\mu)$ as μ ranges over all admissible tableaux, is zero.*

For $1 \leq m \leq N$, put $U_m = U_q(\mathfrak{gl}_m)$. Denote by $Z_m = Z(U_m)$ the center of the algebra U_m. Let Γ be the *Gelfand-Tsetlin subalgebra* of U_N generated by $Z_1, \dots, Z_N$.

A finitely generated $U_q(\mathfrak{gl}_N)$-module M is called a *Gelfand-Tsetlin module* if

$$M = \bigoplus_{\mathfrak{m} \in \operatorname{Specm} \Gamma} M(\mathfrak{m}), \tag{5.5}$$

where $M(\mathfrak{m}) = \big\{x \in M \mid \mathfrak{m}^k x = 0 \text{ for some } k \geq 0\big\}$ and $\operatorname{Specm}\Gamma$ denotes the set of maximal ideals in Γ. The following result shows that the terminology is sensible.

Lemma 5.7. *For any admissible tableau λ, the generic Gelfand-Tsetlin module $V(\lambda)$ is a Gelfand-Tsetlin module. Moreover, Γ acts diagonally in the basis $B(\lambda)$ of a generic Gelfand-Tsetlin module $V(\lambda)$.*

Proof. By [MT, Thm. 2], $V(\lambda)$ has finite length and is therefore finitely generated. That Γ acts diagonally in the basis $B(\lambda)$ follows from [MT, Proof of Theorem 2]. In particular $V(\lambda)$ has a decomposition of the form (5.5) and thus is a Gelfand-Tsetlin module. □

5.4. **Realization of $U_q(\mathfrak{gl}_N)$ as a Galois Γ-ring.** Let $U_N = U_q(\mathfrak{gl}_N)$ and $\mathcal{M} = \mathbb{Z}^{N(N-1)/2}$ with $\mathbb{Z}$-basis $\{\delta^{mi}\}_{1 \leq i \leq m \leq N-1}$. Let Γ be the Gelfand-Tsetlin subalgebra of $U_q(\mathfrak{gl}_n)$. Let $\Lambda = \mathbb{C}[X_{mi}^{\pm 1} \mid 1 \leq i \leq m \leq N]$ be a Laurent polynomial algebra in $N(N+1)/2$ variables. The group $\mathcal{M}$ acts on Λ by $\delta^{mi} X_{kj} = q^{-\delta_{mk}\delta_{ij}} X_{kj}$ for all $1 \leq i \leq m \leq N-1$ and $1 \leq j \leq k \leq N$. Let L be the field of fractions of Λ. Let $S \subseteq \Lambda$ be the multiplicative subset generated by $\{q^{2l} X_{mj}^2 - q^{2k} X_{mi}^2 \mid k, l \in \mathbb{Z}, 1 \leq i, j \leq m, i \neq j\}$, and let Λ_S be the localization. Then S is $\mathcal{M}$-invariant, thus $\mathcal{M}$ acts also on Λ_S. The skew monoid ring $\Lambda_S * \mathcal{M}$ acts on any generic Gelfand-Tsetlin module $V(\lambda)$ as follows:

$$\begin{aligned} &\rho_\lambda : \Lambda_S * \mathcal{M} \to \operatorname{End}\big(V(\lambda)\big), \\ &\rho_\lambda(\delta^{mi})[\mu] = [\mu + \delta^{mi}], \qquad \forall 1 \leq i \leq m \leq N-1, \\ &\rho_\lambda(X_{mi})[\mu] = q^{\widetilde{\mu}_{mi}}[\mu], \qquad \forall 1 \leq i \leq m \leq N, \end{aligned} \tag{5.6}$$

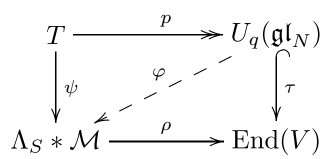
for all $[\mu] \in B(\lambda)$. Note that action of s^{-1} for $s \in S$ is well-defined since λ is admissible.

Lemma 5.8. *If $a \in \Lambda_S * \mathcal{M}$ acts diagonally in the basis $B(\lambda)$ of a generic Gelfand-Tsetlin module $V(\lambda)$ for some admissible tableaux λ, then $a \in \Lambda_S$.*

Proof. Follows from the fact that the set $\{m[\lambda]\}_{m \in \mathcal{M}}$ is linearly independent over $\mathbb{C}$. □

Proposition 5.9. *There exists an injective algebra homomorphism $\varphi : U_N \to \Lambda_S * \mathcal{M}$ determined by*

$$\varphi(E_m^{\pm}) = \sum_{i=1}^{N} (\pm\delta^{mi}) A_{mi}^{\pm}, \qquad \varphi(K_m) = A_m^0 e \tag{5.7}$$



(5.11)

Figure 1. A commutative diagram.

where $\delta^{mi} \in \mathcal{M}$ *are the tableaux units,* $e \in \mathcal{M}$ *is the neutral element, and* $A^{\pm}_{mi}$*,* $A^0_m \in \Lambda_S$ *are given by*

$$A^{\pm}_{mi} = \mp(q-q^{-1})^{-1\mp 1} \frac{\prod_{j=1}^{m\pm 1}\big(X_{m\pm 1,j}X_{mi}^{-1} - X_{m\pm 1,j}^{-1}X_{mi}\big)}{\prod_{j\in\{1,\dots,m\}\setminus\{i\}}\big(X_{mj}X_{mi}^{-1} - X_{mj}^{-1}X_{mi}\big)}, \tag{5.8}$$

$$A^0_m = q^m \prod_{i=1}^{m} X_{mi} \prod_{i=1}^{m-1} X_{m-1,i}^{-1}. \tag{5.9}$$

Proof. Let T be the free associative unital $\mathbb{C}$-algebra generated by $\{E_i^{\pm}, K_j^{\pm}, \mid i = 1,\dots,N-1; j = 1,\dots,N\}$. Let $p : T \to U_q(\mathfrak{gl}_n)$ denote the canonical projection $E_i^{\pm} \mapsto E_i^{\pm}, K_j^{\pm} \mapsto K_j^{\pm 1}$. Let $\psi : T \to \Lambda_S * \mathcal{M}$ be given by

$$\psi(E_m^{\pm}) = \sum_{i=1}^{N} (\pm\delta^{mi})A^{\pm}_{mi}, \qquad \psi(K_m^{\pm}) = (A^0_m)^{\pm 1}e. \tag{5.10}$$

Let λ be an admissible tableaux, $V(\lambda)$ the corresponding generic Gelfand-Tsetlin module over $U_q(\mathfrak{gl}_N)$, and $\tau_\lambda : U_q(\mathfrak{gl}_N) \to \operatorname{End}\big(V(\lambda)\big)$ the associated representation. Recall the representation ρ_λ from (5.6). Note that algebra homomorphisms $\rho_\lambda \circ \psi$ and $\tau_\lambda \circ p$ coincide on the generators of T, hence they coincide on all of T. Let V be the direct product of all $V(\lambda)$ as λ runs through the set of all admissible tableaux. Thus V is the set of families $(v_\lambda)_\lambda$ indexed by admissible tableaux λ and where $v_\lambda \in V(\lambda)$ are arbitrary, not necessarily only finitely many nonzero. Let $\tau : U_q(\mathfrak{gl}_N) \to \operatorname{End}(V)$ and $\rho : \Lambda_S * \mathcal{M} \to \operatorname{End}(V)$ be the respective product representations. The two key points now are that $\rho \circ \psi = \tau \circ p$ (since they are component-wise equal) and that, by Theorem 5.6, τ is injective. These facts and a quick diagram-chasing in Figure 1 imply that $\ker(\psi) \subseteq \ker(p)$. Thus, since p is surjective, we get an induced map $\varphi : U_q(\mathfrak{gl}_N) \to \Lambda_S * \mathcal{M}$ defined by $\varphi(a) = \psi(p^{-1}(a))$, which is the required map. Furthermore, φ is injective. Indeed, assume that $\varphi(a) = 0$. Thus $\rho \circ \varphi(a) = 0$. By the commutativity of (5.11), we get $\rho \circ \varphi(a) = \tau(a)$. Since τ is injective, this implies that $a = 0$. □

Let W_N be the Weyl group of type D_N, $W_N = S_N \ltimes \mathcal{E}_N$. Let $G = \prod_{m=1}^{N} W_m$. Then G acts on Λ by

$$g(X_{mi}) = (-1)^{\alpha_{mi}} X_{m\zeta_m(i)}, \quad 1 \le i \le m \le n, \tag{5.12a}$$

for $g = (\zeta_1\alpha_1, \cdots \zeta_N\alpha_N) \in G$ where $\zeta_m \in S_m$, $\alpha_m = (\alpha_{m1},\dots,\alpha_{mm}) \in \mathcal{E}_m$. Note also that S is a G-invariant set, thus G acts also on Λ_S. Viewing $\mathcal{M}$ as a subset of $\operatorname{End}(\Lambda_S)$, G acts naturally on $\mathcal{M}$ by conjugations. Explicitly,

$$g(\delta^{mi}) = \delta^{m\zeta_m(i)}, \quad 1 \le i \le m \le n-1, \tag{5.12b}$$

for $g = (\zeta_1\alpha_1, \cdots \zeta_N\alpha_N) \in G$. Note that the subgroups $\mathcal{E}_m$ act trivially on $\mathcal{M}$ for any $m = 1, \dots, n$. Hence G acts on the skew group ring $\Lambda_S * \mathcal{M}$ by $\mathbb{C}$-algebra automorphisms.

Proposition 5.10. $\operatorname{im}\varphi \subseteq (\Lambda_S * \mathcal{M})^G$.

Proof. By definition of φ, this is equivalent to showing that $\operatorname{im}\psi \subseteq (\Lambda_S * \mathcal{M})^G$ for ψ defined above. Since $(\Lambda_S * \mathcal{M})^G$ is an algebra, it is enough to show that $\psi(a) \in (\Lambda_S * \mathcal{M})^G$ for all a in a generating set of T. We claim that $\psi(E_m^{\pm}) = [\delta^{m1} A_{m1}^{\pm}]$ with notation as in (5.1). Indeed, $G/\operatorname{Stab}_G(\delta^{m1}) \simeq \mathbb{Z}/m\mathbb{Z}$ with a set of representatives in G given by $\{(1), (12)_m, (13)_m, \dots (1m)_m\}$, where $(ij)_m \in G$ is the element with the transposition (ij) placed in the m:th factor of G and identity elements in the other $N-1$ places. It is easy to check that $(A_{m1}^{\pm})^{(1i)_m} = A_{mi}^{\pm}$ from which the claim follows. By [FO, Lemma 2.1] it follows that $\psi(E_m^{\pm}) \in (\Lambda_S * \mathcal{M})^G$. It is visible from (5.9) that the copy of S_k in G acts trivially on $\psi(K_m)$ for any $k, m = 1, \dots, N$. Likewise, any $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{E}_k$ fixes $\psi(K_m)$ since $(-1)^{\alpha_1 + \cdots + \alpha_k} = 1$. □

For $m \in [\![1, n]\!]$, let $\Lambda_m = \mathbb{C}[X_{m1}^{\pm 1}, \dots, X_{mm}^{\pm 1}] \subseteq \Lambda$ and let $\xi_m : \Lambda_m \to U_m^0$ be the isomorphism given by $\xi_m(X_{mi}) = \widetilde{K}_i = q^{-i} K_i$ for all i. Note that ξ_m commutes with the action of W_m, when the W_m-action on U_m^0 is defined as in Section 5.2.

The following result shows that the restriction of φ to Z_m can be identified with the quantum Harish-Chandra homomorphism.

Proposition 5.11. $\varphi\big|_{Z_m} = \xi_m^{-1} \circ h_m$

Proof. Let M be a type 1 finite-dimensional irreducible representation of U_N. As is well-known, it has a Gelfand-Tsetlin basis, see e.g. [KS]. This means that the action of U_N on M is given by the exact same formulas as the generic Gelfand-Tsetlin modules, except that the action of $E_i^{\pm}$ on a basis vector is zero if the result lies outside the support. Thus, when $z \in Z_m$ acts on a basis vector $[\mu]$ of M, the resulting expression will be the same as if $[\mu]$ were a basis vector of a generic Gelfand-Tsetlin module. That is, they are given by the same Laurent polynomial in $q^{\mu_{mi}}$. From the generic case, we know that this Laurent polynomial is $\varphi(z)$ evaluated by substituting X_{mi} by $q^{\tilde{\mu}_{mi}}$. From the finite-dimensional case we get the polynomial $h_m(z) \in \mathbb{C}[K_1^{\pm}, \dots, K_m^{\pm}]$ evaluated by substituting K_i by $q^{\mu_{mi}}$, $i \in [\![1, m]\!]$. This proves the claim. □

Proposition 5.12. *Let* $K := \operatorname{Frac}\big(\varphi(\Gamma)\big)$. *Then* $K = L^G$.

Proof. It follows from Proposition 5.11 and Lemma 5.3 that $\varphi(\Gamma) = \Lambda^G$. Thus $K = L^G$. □

Proposition 5.13. (a) $\mathcal{M}$ *is* K*-separating;*
(b) $K \subseteq L$ *is a finite Galois extension with Galois group* G.

Proof. (a) That $\mathcal{M}$ is K-separating is easily seen by acting with $\mathcal{M}$ on $X_{m1}^2 + \cdots + X_{mm}^2 \in \Lambda^G \subseteq K$ for $m \in [\![1, N-1]\!]$ and using that q is not a root of unity.

(b) Proposition 5.12 gives $K = L^G$. The field extension $K \subseteq L$ is normal since L is the splitting field of the following polynomial in $K[x]$:

$$p(x) = \prod_{m=1}^{N} (x^2 - X_{m1}^2) \cdots (x^2 - X_{mm}^2)(x - X_{m1} \cdots X_{mm}).$$

Thus, since $\operatorname{char} K = 0$, $K \subseteq L$ is a Galois extension. □

We are now ready to prove that $U_q(\mathfrak{gl}_N)$ can be realized as a Galois Γ-ring.

Theorem 5.14. *The image of φ is a Galois $\varphi(\Gamma)$-ring in $(L * \mathcal{M})^G$.*

Proof. Since we have proved that we have the required setup of Section 5.1, then the claim follows from Proposition 5.2 by taking u_i to be the images under φ of the generators $E_i^{\pm}, K_j$ of U_N. □

6. Proof of the quantum Gelfand-Kirillov conjecture

In this section we prove Theorem I by showing that the quantum Gelfand-Kirillov conjecture follows from a positive solution to the q-difference Noether problem.

By Theorem 5.14 we have

$$\begin{aligned}\operatorname{Frac}(U_N) &\simeq \operatorname{Frac}\big((L*\mathcal{M})^G\big) \simeq \big(\operatorname{Frac}(\Lambda * \mathcal{M})\big)^G \\ &\simeq \operatorname{Frac}\Big(\bigotimes_{m=1}^{N-1}\big(\operatorname{Frac}(\Lambda_m * \mathbb{Z}^m)\big)^{W_m} \otimes (\operatorname{Frac}\Lambda_N)^{W_N}\Big),\end{aligned} \tag{6.1}$$

where W_m is the Weyl group of type D_m, $\Lambda_m = \mathbb{C}[X_{m1}^{\pm 1}, \dots, X_{mm}^{\pm 1}]$ and $\otimes = \otimes_{\mathbb{C}}$.

Lemma 6.1. *There is an algebra isomorphism*

$$\iota : \mathbb{C}_q(\bar{x}, \bar{y}) \xrightarrow{\sim} \operatorname{Frac}(\Lambda_m * \mathbb{Z}^m)$$

where $\bar{x} = (x_1, \dots, x_m)$ and $\bar{y} = (y_1, \dots, y_m)$, uniquely defined by

$$x_i \mapsto X_{mi}^{-1}, \quad y_i \mapsto X_{mi}^{-1}\delta^{mi}, \quad \forall i \in [\![1, n]\!].$$

Moreover, this isomorphism commutes with the W_m-action defined on both sides.

Proof. We have $[X_{mi}, X_{mj}] = 0 = [\delta^{mi}, \delta^{mj}]$ for any $i, j \in [\![1, m]\!]$. By the definition of the action of $\mathcal{M}$ on Λ we have the commutation relation $\delta^{mi} X_{mj} = q^{-\delta_{ij}} X_{mj} \delta^{mi}$, hence $X_{mj}^{-1}\delta^{mj} X_{mi}^{-1} = q^{\delta_{ij}} X_{mi}^{-1} X_{mj}^{-1} \delta^{mj}$ for all $i, j \in [\![1, m]\!]$. Since $y_j x_i = q^{\delta_{ij}} x_i y_j$, this proves that the map ι is well-defined, and is clearly bijective. That it intertwines the W_m-actions is clear by the definitions, (4.1) and (5.12), of the respective W_m-actions. □

Hence Lemma 6.1 reduces the quantum Gelfand-Kirillov conjecture for $\mathfrak{gl}_N$ to the q-difference Noether problem for W_N. By Theorem 4.3 the right hand side of (6.1) is isomorphic to

$$\operatorname{Frac}\Big(\bigotimes_{m=1}^{N-1} \operatorname{Frac}\big(\mathbb{C}_q[x,y] \otimes_{\mathbb{C}} \mathbb{C}_{q^2}[x,y]^{\otimes_{\mathbb{C}}(m-1)}\big) \otimes_{\mathbb{C}} \operatorname{Frac}(\Lambda_N)^{W_N}\Big). \tag{6.2}$$

Since W_N is the Weyl group of type D_N, it is in particular a complex reflection group. Thus, by the Chevalley-Shephard-Todd theorem, $\mathbb{C}[X_{N1}, \dots, X_{NN}]^{W_N}$ is a polynomial algebra in N variables. Hence $\operatorname{Frac}(\Lambda_N)^{W_N}$ is isomorphic to a field $\Bbbk = \mathbb{C}(Z_1, \dots, Z_N)$ of rational functions in N variables over $\mathbb{C}$. Thus

$$\begin{aligned}&\operatorname{Frac}\Big(\bigotimes_{m=1}^{N-1} \operatorname{Frac}\big(\mathbb{C}_q[x,y] \otimes_{\mathbb{C}} \mathbb{C}_{q^2}[x,y]^{\otimes_{\mathbb{C}}(m-1)}\big) \otimes \operatorname{Frac}(\Lambda_N)^{W_N}\Big) \\ &\qquad \simeq \operatorname{Frac}\big(\Bbbk_q[x,y]^{\otimes_{\Bbbk}(N-1)} \otimes_{\Bbbk} \Bbbk_{q^2}[x,y]^{\otimes_{\Bbbk}(N-1)(N-2)/2}\big)\end{aligned} \tag{6.3}$$

where $\Bbbk = \mathbb{C}(Z_1, \dots, Z_N)$. The proof of Theorem I is completed.

7. The quantum Gelfand-Kirillov conjecture for $U_q^{\mathrm{ext}}(\mathfrak{sl}_N)$

Let $U_q(\mathfrak{sl}_N)$ be the quantized enveloping algebra of $\mathfrak{sl}_N$ [KS]. The *extented quantum group* $U_q^{\mathrm{ext}}(\mathfrak{sl}_N)$ can be defined as the quotient of $U_q(\mathfrak{gl}_N)$ by the ideal $\langle K_1K_2\cdots K_N-1\rangle$ (see [KS, Sec. 8.5.3]). Denoting the images of $E_i^{\pm}$ and K_j by $E_i^{\pm}$ and $\widehat{K}_j$ respectively, there is an embedding

$$U_q(\mathfrak{sl}_N) \longrightarrow U_q^{\mathrm{ext}}(\mathfrak{sl}_N) \tag{7.1}$$

given by the usual embedding $U_q(\mathfrak{sl}_N) \to U_q(\mathfrak{gl}_N)$ followed by the canonical projection. That is,

$$\begin{aligned} E_i^{\pm} &\longmapsto E_i^{\pm}, \\ K_i &\longmapsto \widehat{K}_i\widehat{K}_{i+1}^{-1}, \end{aligned}$$

for $i \in [\![1, N-1]\!]$. Moreover, as is observed in [KS, Sec. 8.5.3], $U_q^{\mathrm{ext}}(\mathfrak{sl}_N)$ is isomorphic to the algebra obtained from $U_q(\mathfrak{sl}_N)$ by adjoining the N:th roots

$$(K_1K_2^2\cdots K_{N-1}^{N-1})^{\pm 1/N}. \tag{7.2}$$

The isomorphism maps E_i to E_i and $\widehat{K}_i$ to K_i for $i \in [\![1, N-1]\!]$ and maps $\widehat{K}_N$ to the element (7.2).

The following result shows that the quantum Gelfand-Kirillov conjecture holds for $U_q^{\mathrm{ext}}(\mathfrak{sl}_N)$.

Theorem 7.1. *There exists a* $\mathbb{C}$*-algebra isomorphism*

$$\operatorname{Frac}\left(U_q^{\mathrm{ext}}(\mathfrak{sl}_N)\right) \simeq \operatorname{Frac}\left(\Bbbk_q[x,y]^{\otimes_{\Bbbk}(N-1)} \otimes_{\Bbbk} \Bbbk_{q^2}[x,y]^{\otimes_{\Bbbk}(N-1)(N-2)/2}\right) \tag{7.3}$$

where $\Bbbk = \mathbb{C}(Z_1, \ldots, Z_{N-1})$.

Proof. The element $K_1K_2\cdots K_N$ is a central element of U_N and, by Proposition 5.11,

$$\varphi(K_1K_2\cdots K_N) = q^{N(N+1)/2}X_{N1}X_{N2}\cdots X_{NN} \in (\Lambda_S * \mathcal{M})^G.$$

Therefore, the result follows by the isomorphisms in Section 6, by using that $q^{N(N+1)/2}X_{N1}X_{N2}\cdots X_{NN}$ can be taken as one of the algebraically independent generators of $\mathbb{C}[X_{N1}, \ldots, X_{NN}]^{W_N}$ and thus that

$$\Lambda_N^{W_N}/\langle q^{N(N+1)/2}X_{N1}X_{N2}\cdots X_{NN}-1\rangle \simeq \mathbb{C}[Z_1, \mathbb{Z}_2, \ldots, Z_{N-1}].$$

□

7.1. **Alev and Dumas' result for $\mathfrak{sl}_3$.**

Recall the multiparameter quantized Weyl algebras $A_n^{\bar{q},\Lambda}(\Bbbk)$ from Section 2.3. In [AD, Sec. 4.4], the authors define a certain algebra, denoted $U_q^{AD}(\mathfrak{sl}_3)$, and prove in [AD, Thm. 4.6] that

$$\operatorname{Frac}\left(U_q^{AD}(\mathfrak{sl}_3)\right) \simeq \operatorname{Frac}\left(A_3^{\bar{q},\Lambda}\left(\mathbb{C}(Z_1, Z_2)\right)\right), \tag{7.4}$$

where $\bar{q} = (q, q, q^4)$ and $\Lambda = (\lambda_{ij})$ with $\lambda_{ij} = 1$ for all i, j and $\mathbb{C}(Z_1, \mathbb{Z}_2)$ is the field of rational functions in two variables. Following [KS], let $\breve{U}_{q^2}(\mathfrak{sl}_3)$ denote the algebra with generators $K_1^{\pm 1}, K_2^{\pm 1}$, $E_1^{\pm}, E_2^{\pm}$ and relations

$$\begin{aligned} K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad [K_i, K_j] = 0, \quad \forall i,j \in \{1,2\}, \\ K_iE_j^{\pm}K_i^{-1} = q^{\pm a_{ij}}E_j^{\pm}, \quad \forall i,j \in \{1,2\}, \\ [E_i^+, E_j^-] = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^2 - q^{-2}}, \quad \forall i,j \in \{1,2\}, \\ [E_i^{\pm}, E_j^{\pm}] = 0, \quad |i-j| > 1, \\ (E_i^{\pm})^2E_j^{\pm} - (q^2+q^{-2})E_i^{\pm}E_j^{\pm}E_i^{\pm} + E_j^{\pm}(E_i^{\pm})^2 = 0, \quad |i-j| = 1. \end{aligned}$$

where $(a_{ij}) = \left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]$ is the Cartan matrix of $\mathfrak{sl}_3$. Alev and Dumas' algebra $U_q^{AD}(\mathfrak{sl}_3)$ is obtained from $\breve{U}_{q^2}(\mathfrak{sl}_3)$ by adjoining $(K_1^2K_2)^{\pm 1/3}$. By viewing $U_{q^2}^{\mathrm{ext}}(\mathfrak{sl}_3)$ as an extension of $U_{q^2}(\mathfrak{sl}_3)$, we observe that there is a homomorphism

$$\begin{aligned} U_{q^2}^{\mathrm{ext}}(\mathfrak{sl}_3) &\longrightarrow U_q^{AD}(\mathfrak{sl}_3) \\ E_i^{\pm} &\longmapsto E_i^{\pm}, \quad i \in \{1,2\}, \\ K_i &\longmapsto K_i^2, \quad i \in \{1,2\}, \\ (K_1K_2^2)^{1/3} &\longmapsto (K_1^2K_2)^{1/3} \cdot K_2, \end{aligned}$$

Therefore we may equivalently view $U_q^{AD}(\mathfrak{sl}_3)$ as being obtained from $U_{q^2}^{\mathrm{ext}}(\mathfrak{sl}_3)$ by adjoining $K_1^{1/2}$ and $K_2^{1/2}$.

So let us define $U_q^{AD}(\mathfrak{sl}_N)$ for general N as the algebra obtained from $U_{q^2}^{\mathrm{ext}}(\mathfrak{sl}_N)$ by adjoining $K_j^{1/2}$ for $j \in [\![1, N-1]\!]$. By Proposition 5.11,

$$\varphi(K_1K_2\cdots K_m) = q^{m(m+1)/2}X_{m1}X_{m2}\cdots X_{mm}$$

for any $m \in [\![1, N]\!]$. Furthermore, the isomorphism in Theorem 3.10 maps $\widehat{X}_1 = e_n = x_1x_2\cdots x_n$ to $x_1 \in \Bbbk_q(\bar{x}, \bar{y})$. Following through the isomorphisms, this means that for $m \in [\![1, N-1]\!]$, $K_1\cdots K_m$ is mapped under the map

$$\mathrm{Frac}\left(U_{q^2}^{\mathrm{ext}}(\mathfrak{sl}_N)\right) \xrightarrow{\sim} \mathrm{Frac}\left(\Bbbk_{q^2}[x,y]^{\otimes_{\Bbbk}(N-1)} \otimes_{\Bbbk} \Bbbk_{q^4}[x,y]^{\otimes_{\Bbbk}(N-1)(N-2)/2}\right) \tag{7.5}$$

to some nonzero $\Bbbk$-multiple of the element

$$x_m = 1^{\otimes(m-1)} \otimes x \otimes 1^{\otimes(N-1)(N-2)/2-m}.$$

Therefore, adjoining the square roots $K_j^{1/2}$ for $j \in [\![1, N-1]\!]$, or equivalently $(K_1K_2\cdots K_j)^{1/2}$ for $j \in [\![1, N-1]\!]$, to $U_{q^2}^{\mathrm{ext}}(\mathfrak{sl}_N)$, corresponds to adjoining the square roots $x_m^{1/2}$ for $m = 1, \ldots, N-1$. This shows that

$$\mathrm{Frac}\left(U_q^{AD}(\mathfrak{sl}_N)\right) \simeq \mathrm{Frac}\left(\Bbbk_q[x,y]^{\otimes_{\Bbbk}(N-1)} \otimes_{\Bbbk} \Bbbk_{q^4}[x,y]^{\otimes_{\Bbbk}(N-1)(N-2)/2}\right) \tag{7.6}$$

In particular, for $N = 3$ we recover (7.4), bearing in mind Proposition 2.2.

8. Appendix

8.1. **Proof of Proposition 3.1.** The statement is equivalent to proving that

$$[P(X), P(Y)] = 0 \tag{8.1}$$

in $C_n^q[X,Y]$. Put

$$Q_j(X) = \left(\prod_{k\in\{1,\dots,n\}\setminus\{j\}} \frac{X - x_k}{x_j - x_k} \right) y_j. \tag{8.2}$$

so that $P(X) = \sum_{i=1}^n Q_i(X)$. Observe that

$$w\big(Q_j(X)\big) = Q_{w(j)}(X), \qquad w \in S_n. \tag{8.3}$$

Thus, to prove (8.1), it is enough to show the following two identities:

$$[Q_1(X), Q_1(Y)] = 0, \tag{8.4}$$

$$[Q_1(X), Q_2(Y)] + [Q_2(X), Q_1(Y)] = 0. \tag{8.5}$$

Since $y_1 x_i = q^{\delta_{i1}} x_i y_1$ we have

$$\begin{aligned} Q_1(X)Q_1(Y) &= \prod_{k=2}^n \frac{X - x_k}{x_1 - x_k} y_1 \prod_{k=2}^n \frac{Y - x_k}{x_1 - x_k} y_1 = \\ &= \prod_{2\le k\le n} \frac{(X - x_k)(Y - x_k)}{(x_1 - x_k)(qx_1 - x_k)} y_1^2 \end{aligned}$$

which is symmetric in X, Y. This proves (8.4).

Next we prove (8.5). Let

$$R_j(X) = \prod_{k=3}^n \frac{X - x_k}{x_j - x_k}, \qquad j = 1, 2.$$

Then

$$Q_1(X) = \frac{X - x_2}{x_1 - x_2} R_1(X) y_1, \qquad Q_2(X) = \frac{X - x_1}{x_2 - x_1} R_2(X) y_2,$$
$$[y_1, R_2(X)] = [y_2, R_1(X)] = 0,$$

and

$$R_1(X)R_2(Y) = R_1(Y)R_2(X) = R_2(X)R_1(Y) = R_2(Y)R_1(X).$$

We have

$$\begin{aligned} &[Q_1(X), Q_2(Y)] + [Q_2(X), Q_1(Y)] = Q_1(X)Q_2(Y) - Q_1(Y)Q_2(X) \\ &\quad + Q_2(X)Q_1(Y) - Q_2(Y)Q_1(X) \\ &= \frac{X - x_2}{x_1 - x_2} \cdot \frac{Y - qx_1}{x_2 - qx_1} R_1(X)R_2(Y) y_1 y_2 - \frac{Y - x_2}{x_1 - x_2} \cdot \frac{X - qx_1}{x_2 - qx_1} R_1(Y)R_2(X) y_1 y_2 \\ &\quad + \frac{X - x_1}{x_2 - x_1} \cdot \frac{Y - qx_2}{x_1 - qx_2} R_2(X)R_1(Y) y_1 y_2 - \frac{Y - x_1}{x_2 - x_1} \cdot \frac{X - qx_2}{x_1 - qx_2} R_2(Y)R_1(X) y_1 y_2 \\ &= \Bigg(\frac{(X - x_2)(Y - qx_1) - (Y - x_2)(X - qx_1)}{(x_1 - x_2)(x_2 - qx_1)} \\ &\qquad + \frac{(X - x_1)(Y - qx_2) - (Y - x_1)(X - qx_2)}{(x_2 - x_1)(x_1 - qx_2)} \Bigg) R_1(X)R_2(Y) y_1 y_2 \\ &= \left(\frac{(x_2^2 - q^2 x_1^2)X - (x_2 - qx_1)Y}{(x_1 - x_2)(x_2 - qx_1)} + \frac{(x_1 - qx_2)X - (x_1 - qx_2)Y}{(x_2 - x_1)(x_1 - qx_2)} \right) R_1(X)R_2(Y) y_1 y_2 \\ &= 0 \end{aligned}$$

This shows (8.5) and completes the proof that $[t_i, t_j] = 0$ for all i, j.

8.2. **Proof of Proposition 3.7.** The relation (3.15) holds by Proposition 3.1, while (3.16) holds by the definition, (3.11), of e_d. Relation (3.17) is trivial for $k=0$.

Using (3.12) and that $w(e_k)=e_k$ for any $w\in S_n$ we have, for any $j,k\in\{1,\dots,n\}$,

$$(-1)^{j-1}\Delta\cdot t_je_k=\sum_{w\in S_n}\operatorname{sgn}(w)w\Big(x_1^{n-2}x_2^{n-3}\cdots x_{n-2}e'_{n-j}e_k(x_1,\dots,x_{n-1},qx_n)y_n\Big)$$

Substituting $y_n=t_1+x_nt_2+\dots+x_n^{n-1}t_n$ and using that $w(t_i)=t_i$ for all $w\in S_n$ we get

$$(-1)^{j-1}\Delta\cdot t_je_k=$$
$$\sum_{i=1}^n\sum_{w\in S_n}\operatorname{sgn}(w)w\Big(x_1^{n-2}x_2^{n-3}\cdots x_{n-2}\cdot x_n^{i-1}e'_{n-j}e_k(x_1,\dots,x_{n-1},qx_n)\Big)t_i.$$

Write e'_{n-j} as a sum of monomials $x_{i_1}\cdots x_{i_{n-j}}$ and $1\le i_1<\dots<i_{n-j}\le n-1$. We claim that the only way to get a nonzero contribution is when $i_r=r$ for all r. Indeed, suppose $i_r>r$ for some r chosen minimal. Then the product

$$x_1^{n-2}x_2^{n-3}\cdots x_{n-2}\cdot x_{i_1}\cdots x_{i_{n-j}}\cdot x_n^{i-1}e_k(x_1,\dots,x_{n-1},qx_n)$$

will be fixed by the transposition $(i_r-1\;\;i_r)$. Therefore, after anti-symmetrization, the term will cancel out. In other words, the substitution $w\mapsto w\cdot(i_r-1\;\;i_r)$ in the sum

$$\sum_{w\in S_n}\operatorname{sgn}(w)w\Big(x_1^{n-2}x_2^{n-3}\cdots x_{n-2}\cdot x_{i_1}\cdots x_{i_{n-j}}\cdot x_n^{i-1}e_k(x_1,\dots,x_{n-1},qx_n)\Big)t_i$$

gives the same expression with opposite sign, proving it is zero. Thus, noting also that

$$e_k(x_1,\dots,x_{n-1},qx_n)=e'_k+qx_ne'_{k-1},$$

we have

$$\begin{aligned}&(-1)^{j-1}\Delta\cdot t_je_k\\&=\sum_{i=1}^n\sum_{w\in S_n}\operatorname{sgn}(w)w\Big(x_1^{n-1}\cdots x_{n-j}^j\cdot x_{n-j+1}^{j-2}\cdots x_{n-2}x_n^{i-1}(e'_k+qx_ne'_{k-1})\Big)t_i.\end{aligned}\tag{8.6}$$

The term $i=j$: Write $e'_k=\sum_{1\le i_1<\dots<i_k\le n-1}x_{i_1}\cdots x_{i_k}$. Consider

$$x_1^{n-1}\cdots x_{n-j}^jx_{n-j+1}^{j-2}\cdots x_{n-2}x_n^{j-1}\cdot x_{i_1}\cdots x_{i_k}$$

An expression like this containing factors $(x_rx_{r'})^s$ $(r\neq r')$ will become zero after anti-symmetrization. If $n-j\ge k$ there is a unique way to get a nonzero result, namely to choose $(i_1,\dots,i_k)=(1,2,\dots,k)$. If $n-j<k$ there is no way to get nonzero result. Thus

$$\begin{aligned}&\sum_{w\in S_n}\operatorname{sgn}(w)w\Big(x_1^{n-1}\cdots x_{n-j}^j\cdot x_{n-j+1}^{j-2}\cdots x_{n-2}x_n^{j-1}e'_k\Big)\\&=\begin{cases}a(n,n-1,\dots,n-k+1,n-k-1,\dots,j,j-2,\dots,1,0,j-1), & j+k\le n\\ 0, & j+k>n\end{cases}\end{aligned}$$

where $a(i_1,\dots,i_n) := \sum_{w\in S_n}\operatorname{sgn}(w)w(x_1^{i_1}\cdots x_n^{i_n})$. Use that $w(a(i_1,\dots,i_n)) = \operatorname{sgn}(w)a(i_1,\dots,i_n)$ with

$$w = (n-j+1 \quad n-j+2 \quad \cdots \quad n),$$

which is a cycle of length j, to get

$$\begin{aligned}&a(n,n-1,\dots,n-k+1,n-k-1,\dots,j,j-2,\dots,1,0,j-1)\\&\qquad\qquad = (-1)^{j-1}a(n,n-1,\dots,n-k+1,n-k-1,\dots,0).\end{aligned}$$

Using that the Schur function

$$s_\lambda = a(\lambda_1+n-1,\lambda_2+n-2,\dots,\lambda_n)/a(n-1,n-2,\dots,0),$$

defined for a partition $\lambda=(\lambda_1,\dots,\lambda_n)$, $\lambda_1\geq\cdots\geq\lambda_n\geq 0$, satisfies $s_{1^k0^{n-k}} = e_k$ and that $\Delta = a(n-1,n-2,\dots,0)$ we get that

$$\begin{aligned}&\sum_{w\in S_n}\operatorname{sgn}(w)w\Big(x_1^{n-1}\cdots x_{n-j}^{j}\cdot x_{n-j+1}^{j-2}\cdots x_{n-2}x_n^{j-1}e'_k\Big)\\&\qquad\qquad = \begin{cases}(-1)^{j-1}\Delta\cdot e_k, & j+k\leq n,\\ 0, & j+k>n.\end{cases}\end{aligned} \tag{8.7}$$

Similarly, if we look at the term containing $qx_ne'_{k-1}$, there is at most one tuple $(i_1,\dots,i_{k-1})$, $1\leq i_1<\cdots<i_{k-1}\leq n-1$ such that the antisymmetrization of

$$qx_1^{n-1}\cdots x_{n-j}^{j}x_{n-j+1}^{j-2}\cdots x_{n-2}x_n^{j}x_{i_1}\cdots x_{i_{k-1}}$$

is nonzero, namely $(i_1,\dots,i_{k-1}) = (1,\dots,k-1)$ and this time, due to the presence of x_n^j, it gives nonzero result if and only if $k-1\geq n-j$ i.e. $j+k>n$. Thus

$$\begin{aligned}&\sum_{w\in S_n}\operatorname{sgn}(w)w\Big(x_1^{n-1}\cdots x_{n-j}^{j}\cdot x_{n-j+1}^{j-2}\cdots x_{n-2}\cdot qx_n^{j}e'_{k-1}\Big)\\&= \begin{cases}0, & j+k\leq n\\ qa(n,n-1,\dots,j+1,j-1,\dots,n-k+1,n-k-1,\dots,1,0,j), & j+k>n\end{cases}\end{aligned}$$

To get a descending sequence inside the parenthesis we apply the cyclic permutation which places j between $j-1$ and $j+1$. This cycle has length j, giving a factor $(-1)^{j-1}$. As before, this gives

$$\begin{aligned}&\sum_{w\in S_n}\operatorname{sgn}(w)w\Big(x_1^{n-1}\cdots x_{n-j}^{j}\cdot x_{n-j+1}^{j-2}\cdots x_{n-2}\cdot qx_n^{j}e'_{k-1}\Big)\\&\qquad\qquad = \begin{cases}0, & j+k\leq n\\ (-1)^{j-1}q\Delta\cdot e_k, & j+k>n,\end{cases}\end{aligned} \tag{8.8}$$

Combining (8.7) and (8.8) yields

$$\begin{aligned}&(-1)^{j-1}\Delta\cdot(t_je_k - q^{\delta_{j+k>n}}e_kt_j)\\&= \sum_{i\in\{1,\dots,n\}\setminus\{j\}}\sum_{w\in S_n}\operatorname{sgn}(w)w\Big(x_1^{n-1}\cdots x_{n-j}^{j}\cdot x_{n-j+1}^{j-2}\cdots x_{n-2}x_n^{i-1}(e'_k+qx_ne'_{k-1})\Big)t_i.\end{aligned} \tag{8.9}$$

The terms where $i > j$: We first look at the e'_k term in (8.9). That $i > j$ means the exponent $i-1$ of x_n occurs in one of the exponents in $x_1^{n-1}x_2^{n-2}\cdots x_{n-j}^j$, namely in $x_{n-(i-1)}^{i-1}$. Therefore $x_{i_1}\cdots x_{i_k}$ must contain $x_1x_2\cdots x_{n-(i-1)}$. In particular $k \geq n-(i-1)$. The remaining factors must be $x_{n-j+1}x_{n-j+2}\cdots$ and they cannot continue beyond x_{n-1} meaning that $k-(n-i+1)+(n-j)\leq n-1$. Thus the following inequalities are necessary conditions in order to avoid having two variables with the same exponent:

$$k \geq n-i+1, \quad \text{and} \quad k+i-j-1 \leq n-1,$$

i.e.

$$n-k+1 \leq i \leq n-k+j.$$

If these inequalities hold there is a unique tuple

$$(i_1,\ldots,i_k) = (1,2,\ldots,n-i+1,n-j+1,n-j+2,\ldots,k+i-j-1)$$

with $1 \leq i_1 < \cdots < i_k \leq n-1$ such that

$$\sum_{w\in S_n} \operatorname{sgn}(w) w\Big(x_1^{n-1}\cdots x_{n-j}^j \cdot x_{n-j+1}^{j-2}\cdots x_{n-2}^2 x_n^{i-1}\cdot x_{i_1}\cdots x_{i_k}\Big).$$

is nonzero. With this choice we get

$$\begin{aligned}
&\sum_{w\in S_n} \operatorname{sgn}(w) w\Big(x_1^{n-1}\cdots x_{n-j}^j \cdot x_{n-j+1}^{j-2}\cdots x_{n-2} x_n^{i-1}\cdot x_{i_1}\cdots x_{i_k}\Big)\\
&\quad = a(n,\ldots,i,i-2,\ldots,n-(k+i-j)+1,n-(k+i-j)-1,\ldots,0,i-1)\\
&\quad = (-1)^i a(n,n-1,\ldots,n-(k+i-j)+1,n-(k+i-j)-1,\ldots,0)\\
&\quad = (-1)^i \Delta\cdot e_{k+i-j} \qquad (8.10)
\end{aligned}$$

where we applied the cyclic permutation $(n-i+2 \;\; n-i+3 \;\; \cdots \;\; n-1 \;\; n)$ of length $i-1$ in the second equality.

The argument for the term containing $qx_1e'_{k-1}$ is analogous, but gives an extra minus sign. Together with (8.10) one obtains that for $i > j$ we have

$$\begin{aligned}
&\sum_{w\in S_n} \operatorname{sgn}(w) w\Big(x_1^{n-1}\cdots x_{n-j}^j \cdot x_{n-j+1}^{j-2}\cdots x_{n-2} x_n^{i-1}(e'_k + qx_ne'_{k-1})\Big)t_i\\
&\qquad = \begin{cases} (-1)^{i+1}(q-1)\Delta\cdot e_{k+i-j}t_i, & n-k+1\leq i\leq n-k+j,\\ 0, & \text{otherwise.}\end{cases} \qquad (8.11)
\end{aligned}$$

The terms where $i < j$: We look at the e'_k term in (8.9). Necessary conditions for nonzero contribution are $k \geq j-i$ and $k-(j-i)\leq n-j$, i.e.

$$j-k \leq i \leq n-k. \qquad (8.12)$$

After a similar computation as the $i > j$ case we obtain

$$\begin{aligned}
&\sum_{w\in S_n} \operatorname{sgn}(w) w\Big(x_1^{n-1}\cdots x_{n-j}^j \cdot x_{n-j+1}^{j-2}\cdots x_{n-2} x_n^{i-1}(e'_k + qx_ne'_{k-1})\Big)t_i\\
&\qquad = \begin{cases} (-1)^{i}(q-1)\Delta\cdot e_{k+i-j}t_i, & j-k\leq i\leq n-k\\ 0, & \text{otherwise.}\end{cases} \qquad (8.13)
\end{aligned}$$

Combining (8.13), (8.11) and (8.9) we obtain

$$t_j e_k - q^{\delta_{j+k>n}} e_k t_j = (q-1) \sum_{\substack{i>j \\ n-k+1 \le i \le n-k+j}} (-1)^{j-1+i+1} e_{k+i-j} t_i + (q-1) \sum_{\substack{i<j \\ j-k \le i \le n-k}} (-1)^{j-1+i} e_{k+i-j} t_i \quad (8.14)$$

Making the change of summation variables $i \mapsto i+j$ we get

$$t_j e_k - q^{\delta_{j+k>n}} e_k t_j = (q-1) \sum_{\substack{i>0 \\ n-(j+k)+1 \le i \le n-k}} (-1)^{i+\delta_{i<0}} e_{k+i} t_{j+i} + (q-1) \sum_{\substack{i<0 \\ -k \le i \le n-(j+k)}} (-1)^{i+\delta_{i<0}} e_{k+i} t_{j+i}. \quad (8.15)$$

In the first sum, the condition $i \le n-k$ is redundant since, by the notational convention, $e_{k+i} = 0$ for $i > n-k$. Similarly, $-k \le i$ is superfluous in the second sum. Thus we obtain (3.17).

8.3. **Proof of Proposition 3.9.** First note that (3.24) implies that

$$[T_j, E_0] = 0, \quad \forall j \in [\![1, n]\!], \quad (8.16)$$

$$[T_j, E_n]_q = 0, \quad \forall j \in [\![1, n]\!]. \quad (8.17)$$

We now prove (3.29). Let $j \in [\![1, n-1]\!]$ and $k \in [\![0, n-1]\!]$. Then the left hand side of (3.29) equals

$$[\widetilde{E}_k, \widetilde{T}_j]_{q^{\delta_{j+k>n-1}}} = [T_{k+1}, E_j T_1 T_n - (-1)^j E_0 T_{n-j} T_1 - (-1)^{n-j} E_n T_{n+1-j} T_n]_{q^{\delta_{j+1+k>n}}} \quad (8.18)$$

$$= (q-1) \sum_{i \in \mathbb{Z} \setminus I(n-1-j-k)} (-1)^{i+\delta_{i<0}} E_{j+i} T_{k+1+i} T_1 T_n \quad (8.19)$$

$$- (1 - q^{\delta_{j+k+1>n}})(-1)^j E_0 T_{k+1} T_{n-j} T_1 \quad (8.20)$$

$$- (q - q^{\delta_{j+k+1>n}})(-1)^{n-j} E_n T_{k+1} T_{n+1-j} T_n. \quad (8.21)$$

By (3.19) with (j,k) replaced by $(k+1, n-j)$, the term (8.20) equals

$$-(q-1) \sum_{i \in \mathbb{Z} \setminus I(n-1-j-k)} (-1)^{j+\delta_{i<0}} E_0 T_{k+1+i} T_{n-j-i} T_1. \quad (8.22)$$

Similarly, applying (3.20) with (j,k) replaced by $(k+1, n+1-j)$ shows that (8.21) is equal to

$$-(q-1) \sum_{i \in \mathbb{Z} \setminus I(n-1-j-k)} (-1)^{n-j+\delta_{i<0}} E_n T_{k+1+i} T_{n+1-j-i} T_n. \quad (8.23)$$

Adding together (8.22), (8.23) and (8.19) gives the right hand side of (3.29). This proves (3.29).

In particular, taking $k = 0$ and $k = n-1$ in (3.29) we get

$$T_1 \widetilde{T}_j - \widetilde{T}_j T_1 = 0, \quad (8.24)$$

$$T_n \widetilde{T}_j - q \widetilde{T}_j T_n = 0, \quad (8.25)$$

for all $j \in [\![1, n-1]\!]$. Using these identities, together with $[T_j, E_0] = 0$ and $[T_j, E_n]_q = 0$ which follow from (3.24), one can check that

$$q\widetilde{T}_j = T_1 T_n \widetilde{T}_j (T_1 T_n)^{-1} = T_n T_1 E_j - (-1)^j T_1 T_{n-j} E_0 - (-1)^{n-j} T_n T_{n+1-j} E_n,$$

proving (3.30).

That (3.28) holds is trivial from the assumption (3.22).

We now prove (3.27). Let $j,k \in [\![1, n-1]\!]$. We will bring $\widetilde{T}_j \widetilde{T}_k$ to the normal form where all the E's are to the left of all the T's and prove that the resulting expression is symmetric in j,k. We may assume $j \neq k$. Using (8.24), (8.25) and (3.29), we have

$$\begin{aligned}
\widetilde{T}_j\widetilde{T}_k =&(E_j T_1 T_n - (-1)^j E_0 T_{n-j} T_1 - (-1)^{n-j} E_n T_{n+1-j} T_n)\widetilde{T}_k \\
=&qE_j\widetilde{T}_k T_1 T_n - (-1)^j E_0 T_{n-j}\widetilde{T}_k T_1 - (-1)^{n-j} qE_n T_{n+1-j}\widetilde{T}_k T_n \\
=&qE_j\widetilde{T}_k T_1 T_n \\
&- (-1)^j E_0 \Big(q^{\delta_{-j+k>0}} \widetilde{T}_k T_{n-j} + (q-1) \sum_{i\in\mathbb{Z}\setminus I(j-k)} (-1)^{i+\delta_{i<0}} \widetilde{T}_{k+i} T_{n-j+i} \Big) T_1 \\
&- (-1)^{n-j} qE_n \Big(q^{\delta_{1-j+k>0}} \widetilde{T}_k T_{n+1-j} + \\
&\quad (q-1) \sum_{i\in\mathbb{Z}\setminus I(-1+j-k)} (-1)^{i+\delta_{i<0}} \widetilde{T}_{k+i} T_{n+1-j+i} \Big) T_n \\
=&qE_j E_k T_1^2 T_n^2 - (-1)^k qE_j E_0 T_{n-k} T_1^2 T_n - (-1)^{n-k} qE_j E_n T_{n+1-k} T_1 T_n^2 \\
&- (-1)^j q^{\delta_{k>j}} E_0 E_k T_{n-j} T_1^2 T_n + (-1)^{j+k} q^{\delta_{k>j}} E_0^2 T_1^2 T_{n-k} T_{n-j} + \\
&\quad + (-1)^{n-k+j} q^{\delta_{k>j}} E_0 E_n T_{n-j} T_{n+1-k} T_1 T_n \\
&- (-1)^j (q-1) \sum_{i\in\mathbb{Z}\setminus I(j-k)} (-1)^{i+\delta_{i<0}} E_0 \Big(E_{k+i} T_1 T_n \\
&\quad - (-1)^{k+i} E_0 T_{n-k-i} T_1 - (-1)^{n-k-i} E_n T_{n+1-k-i} T_n \Big) T_{n-j+i} T_1 \\
&- (-1)^{n-j} q^{1+\delta_{k\geq j}} E_n E_k T_1 T_n T_{n+1-j} T_n + (-1)^{n-j+k} q^{1+\delta_{k\geq j}} E_n E_0 T_{n-k} T_1 T_{n+1-j} T_n \\
&\quad + (-1)^{2n-j-k} q^{1+\delta_{k\geq j}} E_n^2 T_{n+1-k} T_n^2 T_{n+1-j} \\
&- (-1)^{n-j} q(q-1) \sum_{i\in\mathbb{Z}\setminus I(-1+j-k)} (-1)^{i+\delta_{i<0}} E_n \Big(E_{k+i} T_1 T_n \\
&\quad - (-1)^{k+i} E_0 T_{n-k-i} T_1 - (-1)^{n-k-i} E_n T_{n+1-k-i} T_n \Big) T_{n+1-j+i} T_n.
\end{aligned} \tag{8.26}$$

We prove that all parts of this expression are symmetric in j,k. The first term, containing $E_j E_k$, is trivially symmetric.

The terms containing $E_0^2 T_1^2$. There are two terms in (8.26) containing $E_0^2 T_1^2$:

$$(-1)^{j+k} q^{\delta_{k>j}} E_0^2 T_{n-k} T_{n-j} T_1^2 + (-1)^{j+k} (q-1) \sum_{i\in\mathbb{Z}\setminus I(j-k)} (-1)^{\delta_{i<0}} E_0^2 T_1^2 T_{n-k-i} T_{n-j+i}. \tag{8.27}$$

Applying (3.19) with (j,k) replaced by $(n-j, n-k)$ we get that (8.27) equals

$$(-1)^{j+k} E_0^2 T_{n-j} T_{n-k} T_1^2$$

which is symmetric in j, k.

The terms containing $E_n^2 T_n^2$.

$$\begin{aligned}(-1)^{2n-j-k} q^{1+\delta_{k\geq j}} E_n^2 T_n^2 T_{n+1-k} T_{n+1-j} \\ + (-1)^{2n-j-k} q(q-1) \sum_{i\in\mathbb{Z}\setminus I(-1+j-k)} (-1)^{\delta_{i<0}} E_n^2 T_n^2 T_{n+1-k-i} T_{n+1-j+i}\end{aligned} \tag{8.28}$$

Here we can apply (3.20) with (j, k) replaced by $(n+1-j, n+1-k)$ to see that (8.28) equals

$$(-1)^{j+k} q^2 E_n^2 T_n^2 T_{n+1-j} T_{n+1-k}$$

which is symmetric in j, k.

The terms containing $E_0 T_1^2 T_n$.

$$\begin{aligned}- E_0 \Big((-1)^k q E_j T_{n-k} + (-1)^j q^{\delta_{k>j}} E_0 E_k T_{n-j} \\ + (-1)^j (q-1) \sum_{i\in\mathbb{Z}\setminus I(j-k)} (-1)^{i+\delta_{i<0}} E_0 E_{k+i} T_{n-j+i} \Big) T_1^2 T_n\end{aligned} \tag{8.29}$$

The parenthesis equals

$$\begin{aligned}(-1)^k E_j T_{n-k} + (-1)^j E_k T_{n-j} + \\ (-1)^k (q-1) E_j T_{n-k} + (-1)^j (q^{\delta_{k>j}} - 1) E_k T_{n-j} + \\ (-1)^j (q-1) \sum_{i\in\mathbb{Z}\setminus I(j-k)} (-1)^{i+\delta_{i<0}} E_{k+i} T_{n-j+i}.\end{aligned} \tag{8.30}$$

If $j \geq k$, we can include $(-1)^k (q-1) E_j T_{n-k}$ as the term $i = j-k$ in the sum. If $j < k$, the term $(-1)^k (q-1) E_j T_{n-k}$ cancels the term $i = j-k$ in the sum, and $(-1)^j (q^{\delta_{k>j}} - 1) E_k T_{n-j}$ may be included in the sum as $i = 0$. Thus (8.30) can be written

$$\begin{aligned}(-1)^k E_j T_{n-k} + (-1)^j E_k T_{n-j} \\ + (-1)^j (q-1) \sum_{i\in \begin{cases}\mathbb{Z}\setminus [\![0, j-k-1]\!], & j\geq k \\ \mathbb{Z}\setminus [\![j-k, -1]\!], & j<k\end{cases}} (-1)^{i+\delta_{i<0}} E_{k+i} T_{n-j+i}.\end{aligned} \tag{8.31}$$

Making the change of variables $i \mapsto i + j - k$ in this sum gives the same expression but with j and k interchanged. Thus it is symmetric in j and k.

The terms containing $E_n T_1 T_n^2$.

$$\begin{aligned}- q E_n \Big((-1)^{n-k} E_j T_{n+1-k} + (-1)^{n-j} q^{\delta_{k\geq j}} E_k T_{n+1-j} \\ + (-1)^{n-j} (q-1) \sum_{i\in\mathbb{Z}\setminus I(-1+j-k)} (-1)^{i+\delta_{i<0}} E_{k+i} T_{n+1-j+i} \Big) T_1 T_2^2\end{aligned} \tag{8.32}$$

Similarly to the previous case, the expression inside the parenthesis can be written as

$$
\begin{aligned}
&(-1)^{n-k} q E_j T_{n+1-k} + (-1)^{n-j} q E_k T_{n+1-j} \\
&+ (-1)^{n-j}(q-1) \sum_{i \in \begin{cases} \mathbb{Z} \setminus [\![1, j-k]\!], & j > k \\ \mathbb{Z} \setminus [\![j-k+1, 0]\!], & j \le k \end{cases}} (-1)^{i+\delta_{i\le 0}} E_{k+i} T_{n+1-j+i}.
\end{aligned}
\tag{8.33}
$$

Substituting $i \mapsto i-k+j$ one checks this is symmetric in j and k.

The terms containing $E_0 E_n T_1 T_n$. Finally, there are four terms in (8.26) containing $E_0 E_n T_1 T_n$:

$$
\begin{aligned}
E_0 E_n \Big(&(-1)^{n-k+j} q^{\delta_{k>j}} T_{n+1-k} T_{n-j} + (-1)^{n-j+k} q^{1+\delta_{k\ge j}} T_{n-k} T_{n+1-j} \\
&+ (-1)^{n-k+j}(q-1) \sum_{i \in \mathbb{Z} \setminus I(j-k)} (-1)^{\delta_{i<0}} T_{n+1-k-i} T_{n-j+i} \\
&+ (-1)^{n-j+k} q(q-1) \sum_{i \in \mathbb{Z} \setminus I(-1+j-k)} (-1)^{\delta_{i<0}} T_{n-k-i} T_{n+1-j+i} \Big) T_1 T_n
\end{aligned}
\tag{8.34}
$$

Applying (3.19) with (j,k) replaced by $(n-j+1, n-k)$ and (3.20) with (j,k) replaced by $(n-j, n-k+1)$ we obtain that the parenthesis in (8.34) equals

$$(-1)^{n+j-k} q T_{n+1-k} T_{n-j} + (-1)^{n+k-j} q T_{n+1-j} T_{n-k}$$

which is symmetric in j and k. This completes the proof that (8.26) is symmetric in j and k. Thus (3.27) holds.

The last statement about generators follows from the fact that (3.25) and (3.26) can be used to express E_j for $j \in [\![1, n-1]\!]$ and T_k for $k \in [\![1, n]\!]$, in terms of the new generators $\{E_0, E_n\} \cup \{\widetilde{T}_j\}_{j=1}^{n-1} \cup \{\widetilde{E}_k\}_{k=0}^{n-1}$.

8.4. **Example: The case $n = 2$.** If $n = 2$ then (3.3) becomes

$$y_1 = t_1 + x_1 t_2, \qquad y_2 = t_1 + x_2 t_2$$

and from this, or using (3.12), we get

$$
\begin{aligned}
t_1 &= (x_1 - x_2)^{-1}(x_1 y_2 - x_2 y_1), \\
t_2 &= -(x_1 - x_2)^{-1}(y_2 - y_1).
\end{aligned}
$$

By definition (3.11), we have

$$e_0 = 1, \quad e_1 = x_1 + x_2, \quad e_2 = x_1 x_2.$$

By Corollary 3.5, $\Bbbk_q(\bar{x}, \bar{y})^{S_2}$ is generated as a skew field over $\Bbbk$ by e_1, e_2, t_1, t_2. By Proposition 3.7 we have the following relations:

$$
\begin{aligned}
t_1 t_2 &= t_2 t_1, \\
e_1 e_2 &= e_2 e_1, \\
t_1 e_2 &= q e_2 t_1, \\
t_2 e_2 &= q e_2 t_2, \\
t_1 e_1 &= e_1 t_1 + (1-q) e_2 t_2, \\
t_2 e_1 &= q e_1 t_2 + (q-1) t_1.
\end{aligned}
$$

Using the notation in (3.32) and (3.31) we have

$$\begin{aligned} X_1 &= e_2^{(0)} = e_2, \\ X_2 &= e_1^{(1)} = t_2, \\ Y_1 &= e_0^{(1)} = t_1, \\ Y_2 &= e_0^{(2)} = e_1^{(0)} e_0^{(1)} e_1^{(1)} + e_0^{(0)} e_0^{(1)} e_0^{(1)} + e_2^{(0)} e_1^{(1)} e_1^{(1)} = \\ &= e_1 t_1 t_2 + e_0 t_1^2 + e_2 t_2^2. \end{aligned}$$

By (3.48) or direct computations,

$$\begin{aligned} [Y_1, Y_2] &= 0, & [X_2, X_1]_q &= 0, \\ [Y_1, X_2] &= 0, & [Y_2, X_1]_{q^2} &= 0, \\ [Y_1, X_1]_q &= 0, & [Y_2, X_2]_{q^{-1}} &= 0. \end{aligned}$$

Thus, $(Z_1, Z_2, Z_3, Z_4) = (X_1, Y_1, X_2, Y_2)$ satisfy $Z_i Z_j = q^{s_{ij}} Z_j Z_i$ with

$$(s_{ij}) = \begin{bmatrix} 0 & -1 & -1 & -2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & -1 & 0 \end{bmatrix}.$$

Using the definition (3.33),

$$\widehat{X}_1 = X_1, \quad \widehat{X}_2 = Y_1 X_2^{-1}, \quad \widehat{Y}_1 = Y_1, \quad \widehat{Y}_2 = Y_1^{-2} Y_2.$$

By Theorem 3.10, $\widehat{X}_1, \widehat{X}_2, \widehat{Y}_1, \widehat{Y}_2$ generate $\Bbbk(\bar{x}, \bar{y})^{S_2}$ as a skew field and the following relations hold:

$$\begin{aligned} [\widehat{X}_1, \widehat{X}_2] &= 0, \qquad [\widehat{Y}_1, \widehat{Y}_2] = 0, \\ \widehat{Y}_i \widehat{X}_j &= q^{\delta_{ij}} \widehat{X}_j \widehat{Y}_i, \qquad \forall i, j \in \{1, 2\}. \end{aligned}$$

This shows that $\Bbbk_q(x_1, x_2, y_1, y_2)^{S_2} \simeq \Bbbk_q(x_1, x_2, y_1, y_2)$.

8.5. **Example: The case $n = 3$.** The elementary symmetric polynomials e_d are

$$\begin{aligned} e_0 &= 1, \\ e_1 &= x_1 + x_2 + x_3, \\ e_2 &= x_1 x_2 + x_2 x_3 + x_3 x_1, \\ e_3 &= x_1 x_2 x_3. \end{aligned}$$

By (3.12) we have

$$\begin{aligned} t_1 &= \Delta^{-1} \cdot \big((x_2^2 x_3 - x_3^2 x_2) y_1 + (x_3^2 x_1 - x_1^2 x_3) y_2 + (x_1^2 x_2 - x_2^2 x_1) y_3 \big), \\ t_2 &= \Delta^{-1} \cdot \big((x_2^2 - x_3^2) y_1 + (x_3^2 - x_1^2) y_2 + (x_1^2 - x_2^2) y_3 \big), \\ t_3 &= \Delta^{-1} \cdot \big((x_2 - x_3) y_1 + (x_3 - x_1) y_2 + (x_1 - x_2) y_3 \big), \end{aligned}$$

where

$$\Delta = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

By Corollary 3.5, $\Bbbk_q(\bar{x},\bar{y})^{S_3}$ is generated as a skew field over $\Bbbk$ by e_1,e_2,e_3,t_1,t_2,t_3 and by Proposition 3.7 or direct computations, we have the following relations:

$$\begin{aligned}
[t_i,t_j] &= 0, \quad \forall i,j\in\{1,2,3\},\\
[e_i,e_j] &= 0, \quad \forall i,j\in\{1,2,3\},\\
[t_i,e_3]_q &= 0, \quad \forall i\in\{1,2,3\},\\
[t_1,e_1] &= (q-1)e_3t_3,\\
[t_2,e_1] &= (q-1)(t_1-e_2t_3),\\
[t_3,e_1]_q &= (q-1)t_2,\\
[t_1,e_2] &= (1-q)e_3t_2,\\
[t_2,e_2]_q &= (1-q)(e_3t_3-e_1t_1),\\
[t_3,e_2]_q &= (1-q)t_1.
\end{aligned}$$

By (3.32) and (3.31),

$$\begin{aligned}
X_1 &= e_3^{(0)} = e_3 = x_1x_2x_3,\\
X_2 &= e_2^{(1)} = t_3,\\
X_3 &= e_1^{(2)} = e_2^{(0)}e_0^{(1)}e_2^{(1)} - e_0^{(0)}e_0^{(1)}e_0^{(1)} + e_3^{(0)}e_1^{(1)}e_2^{(1)} =\\
&= e_2t_1t_3 - e_0t_1^2 + e_3t_2t_3,\\
Y_1 &= e_0^{(1)} = t_1,\\
Y_2 &= e_0^{(2)} = e_1^{(0)}e_0^{(1)}e_2^{(1)} + e_0^{(0)}e_1^{(1)}e_0^{(1)} - e_3^{(0)}e_2^{(1)}e_2^{(1)} =\\
&= e_1t_1t_3 + e_0t_2t_1 - e_3t_3^2,\\
Y_3 &= e_0^{(3)} = e_1^{(1)}e_0^{(2)}e_1^{(2)} + e_0^{(1)}e_0^{(2)}e_0^{(2)} + e_2^{(1)}e_1^{(2)}e_1^{(2)} =\\
&= t_2Y_2X_3 + t_1Y_2^2 + t_3X_3^2.
\end{aligned}$$

By (3.48),

$$\begin{aligned}
&[Y_k,Y_i]=0, \quad \forall k,i\in\{1,2,3\},\\
&[X_2,X_1]_q=0, \quad [X_3,X_1]_{q^2}=0, \quad [X_3,X_2]_{q^{-1}}=0,\\
&[Y_1,X_2]=0, \quad [Y_1,X_3]=0, \quad [Y_2,X_3]=0,\\
&[Y_1,X_1]_q=0, \quad [Y_2,X_1]_{q^2}=0, \quad [Y_3,X_1]_{q^5}=0,\\
&[Y_2,X_2]_{q^{-1}}=0, \quad [Y_3,X_2]_{q^{-2}}=0, \quad [Y_3,X_3]_q=0.
\end{aligned}$$

Thus, if we let $(Z_1,Z_2,\ldots,Z_6)=(X_1,Y_1,X_2,Y_2,X_3,Y_3)$, then $Z_iZ_j=q^{s_{ij}}Z_jZ_i$ with

$$(s_{ij}) = \begin{bmatrix} 0 & -1 & -1 & -2 & -2 & -5\\ 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 1 & 2\\ 2 & 0 & -1 & 0 & 0 & 0\\ 2 & 0 & -1 & 0 & 0 & -1\\ 5 & 0 & -2 & 0 & 1 & 0 \end{bmatrix}. \tag{8.35}$$

By performing simultaneous elementary row and column transformations, this matrix can be brought to the skew normal form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}. \tag{8.36}$$

As in (3.33), changing generators to

$$\widehat{X}_1 = X_1, \quad \widehat{X}_2 = Y_1 X_2^{-1}, \quad \widehat{X}_3 = Y_2^{-1} X_3,$$
$$\widehat{Y}_1 = Y_1, \quad \widehat{Y}_2 = Y_1^{-2} Y_2, \quad \widehat{Y}_3 = Y_1^{-1} Y_2^{-2} Y_3.$$

one can also verify directly that

$$[\widehat{X}_i, \widehat{X}_j] = [\widehat{Y}_i, \widehat{Y}_j] = 0, \quad \forall i, j \in \{1, 2, 3\},$$
$$\widehat{Y}_i \widehat{X}_j = q^{\delta_{ij}} \widehat{X}_j \widehat{Y}_i, \quad \forall i, j \in \{1, 2, 3\},$$

which means that there is an isomorphism of skew fields

$$\mathbb{k}_q(\bar{x}, \bar{y}) \xrightarrow{\sim} \mathbb{k}_q(\bar{x}, \bar{y})^{S_3}$$
$$x_i \longmapsto \widehat{X}_i, \qquad \forall i \in \{1, 2, 3\},$$
$$y_i \longmapsto \widehat{Y}_i, \qquad \forall i \in \{1, 2, 3\}.$$

9. Acknowledgment

The first author is grateful to the Max Planck Institute for Mathematics in Bonn for support and hospitality during his visit. The first author is supported in part by the CNPq grant (301743/2007-0) and by the Fapesp grant (2010/50347-9).

The authors are grateful to Michel Van den Bergh, Fedor Malikov, Eugene Mukhin and Alan Weinstein for encouraging discussions.

References

[AD] Alev J., Dumas F., *Sur le corps des fractiones de certaines algebres quantiques*, J. Algebra 170 (1994), 229-265.

[AOV1] Alev J., Ooms A., Van den Bergh M., *The Gelfand-Kirillov conjecture for Lie algebras of dimension at most eight*, J. Algebra 227 (2000), 549-581. Corrigendum, J. Algebra 230 (2000), 749.

[AOV2] Alev J., Ooms A., Van den Bergh M., *A class of counterexamples to the Gelfand-Kirillov conjecture*, Trans. Amer. Math. Soc. 348 (1996), 1709-1716.

[BGR] Borho W., Gabriel P., Rentschler R., *Primideale in Einhullenden auflosbarer Lie-Algebren*, Lecture Notes in Math. vol 357, Springer, Berlin and New York, 1973.

[BGZ] Bracken A., Gould M., Zhang R., *Quantum group invariants and link polynomials*, Comm. Math. Phys., 137 (1991), 13-27.

[BG] Brown K.A., Goodearl K.R., *Lectures on algebraic quantum groups*, Advance course in Math. CRM Barcelona, vol 2., Birkhauser Verlag, Basel, 2002

[Ca] Caldero P., *On the Gelfand-Kirillov conjecture for quantum algebras* , Proc. AMS, 128 (1999), 943-951.

[C] Cauchon G., *Effacement des dérivations et spectres premiers des algèbres quantiques*, J.Algebra, 260 (2003), 476-518.

[D] Dumas F., *An introduction to noncommutative polynomial invariants*, Lecture notes, "Homological methods and representations of non-commutative algebras", Mar del Plata, Argentina March 6-17, 2006.

[FSSS] Farkas D., Schofield A., Snider R., Stafford J., *The isomorphism question for division rings of group rings*, Proc. AMS, 85 (1982), 327-330.

[FM] Fauquant-Millet F., *Quantification de la localisation de de Dixmier de* $U(sl_{n+1}(\mathbb{C}))$, J. Algebra 218 (1999), 93-116.

[FMO] Futorny V., Molev A., Ovsienko S., *The Gelfand-Kirillov Conjecture and Gelfand-Tsetlin modules for finite* W*-algebras*, Advances in Mathematics, 223 (2010), 773-796.

[FO] Futorny V., Ovsienko S., *Galois orders in skew monoid rings*, J.Algebra, 324 (2010), 598-630.

[G] Goodearl K. R., *Prime spectra of quantized coordinate rings*, in "Interactions between ring theory and representations of algebras (Murcia)", 205-237, Lecture Notes in Pure and Appl. Math., 210, Dekker, New York, 2000.

[GK] Gelfand I.M. et. Kirillov A.A., Sur les corps liés cor aux algèbres enveloppantes des algèbres de Lie, Publ. Inst. Hautes Sci., 31, (1966), 5-19.

[H] Hartwig J., *Locally finite simple weight modules over twisted generalized Weyl algebras*, J. Algebra 303 (2006), 42-76.

[IM] Iohara K., Malikov F., *Rings of skew polynomials and Gelfand-Kirillov conjecture for quantum groups*, Comm. Math. Phys. 164 (1994), 217-237.

[Jo] Joseph A., *Proof of the Gelfand-Kirillov conjecture for solvable Lie algebras*, Proc. Amer. Math. Soc. 45 (1974), 1-10.

[Jo1] Joseph A., *Sur une conjecture de Feigin*, C. R. Acad. Sci. Paris Ser.I Math. 320 (1995), no. 12, 1441-1444.

[J] Jordan D., *A simple localization of the quantized Weyl algebra*, J. Algebra 174 (1995), 267-281.

[KS] Klimyk A., Schmudgen K., *Quantum groups and their representations*, Springer-Verlag, Berlin Heidelberg, 1997.

[Li] Li J., *The quantum Casimir operators of* $U_q(\mathfrak{gl}_n)$ *and their eigenvalues*, J. Physics A: Mathematical and Theoretical, 43 (2010), 345202 (9pp).

[Mal] Maltsiniotis G., *Calcul differentiel quantique*, Groupe de travail, Universite Paris VII, 1992.

[Mat] Mattuck A., *The field of multisymmetric functions*, Proc. Amer. Math. Soc. 19 (1968), 764-765.

[MT] Mazorchuk V., Turowska L., *On Gelfand-Zetlin modules over* $U_q(\mathfrak{gl}_n)$, Czechoslovak J. Physics, 50 (2000), 139-144.

[Mc] McConnel J.C., *Representations of solvable Lie algebras and the Gelfand-Kirillov conjecture*, Proc. London Math. Soc., 29 (1974), 453-484.

[Mi] Miyata T., *Invariants of certain groups. 1*, Nagoya Math. Journal, 41 (1971), 69-73.

[N] Newman M., *Integral matrices* , Academic Press, New York and London, 1972.

[Pa] Panov A., *Fields of fractions of quantum solvable algebras* , J. Algebra, 236 (2001), 110-121.

[P] Premet A., *Modular Lie algebras and the Gelfand-Kirillov conjecture*, Inventiones Mathematicae 181 (2010), 395-420.

[UTS] Ueno K., Takebayashi T., Shibukawa Y., *Gelfand-Tsetlin basis for* $U_q(gl(N+1))$, Letters Math. Phys., 18 (1989), 215-221.

[Zh] Zhelobenko D., *Compact Lie groups and their representations*, Translations of Mathematical Monographs, AMS, 1973.

Department of Mathematics, University of São Paulo, São Paulo, Brazil and Max Planck Institute for Mathematics, Bonn, Germany
E-mail address: futorny@ime.usp.br

Department of Mathematics, Stanford University, Stanford, CA, USA
E-mail address: jonas.hartwig@gmail.com