HIGHER ARF FUNCTIONS AND TOPOLOGY OF THE MODULI SPACE OF HIGHER SPIN RIEMANN SURFACES

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ABSTRACT. We describe the space of m-spin structures on a Riemann surface as a finite affine space of $\mathbb{Z}/m\mathbb{Z}$-valued functions on the fundamental group of the surface. We apply this description to prove that any connected component of the space of m-spin structures on compact Riemann surfaces with finite number of punctures and holes is homeomorphic to a quotient of the vector space $\mathbb{R}^d$ by a discrete group action.

1. Introduction

The classical spin structures (theta characteristics) on compact Riemann surfaces play an important role in algebraic geometry since Riemann [Rie53]. Their modern interpretation and classification as complex line bundles such that the tensor square is isomorphic to the cotangent bundle of the surface was given by Atiyah [Ati71] and Mumford [Mum71], a topological interpretation of their results was given by Johnson [Joh80]. They showed a connection between the set of spin bundles on a surface P and the affine space of quadratic (with respect to the index of intersection) forms $H_1(P;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$. Classification of classical spin structures on non-compact Riemann surfaces and the corresponding moduli space were studied in [Nat89, Nat04].

In this paper we study m-spin structures, i.e. complex line bundles such that the m-th tensor power is isomorphic to the cotangent bundle of the surface, on an arbitrary hyperbolic Riemann surfaces with finite fundamental group. We assign (Theorem 4.8) to any m-spin structure on a surface P a unique function on the space of homotopy classes of simple contours on P with values in $\mathbb{Z}/m\mathbb{Z}$. These functions are not quadratic forms. They are described by simple geometric properties.

Definition: We denote by $\pi_1^0(P,p)$ the set of all elements of $\pi_1(P,p)$, which can be represented by simple contours which either do not belong to the kernel of the intersection form or are homologous to a hole or a puncture. A *m-Arf function* is a function $\sigma : \pi_1^0(P,p) \to \mathbb{Z}_m$ satisfying the following conditions

1. $\sigma(bab^{-1}) = \sigma(a)$,
2. $\sigma(a^{-1}) = -\sigma(a)$,
3. $\sigma(ab) = \sigma(a)+\sigma(b)$ if the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a,b\rangle \neq 0$,

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4. $\sigma(ab) = \sigma(a) + \sigma(b) - 1$ if the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b\rangle = 0$ and placed in a neighbourhood of the point p as shown in Figure 4, i.e. in such a way that the oriented contours a, b, and $(ab)^{-1}$ are freely homotopic to pairwise non-intersecting simple contours with orientation opposite to the one induced by the complex structure of the sphere with three holes, which they cut out of P.

Let us outline the construction assigning a m-Arf function to a m-spin structure on $P = \mathbb{H}/\Gamma$. The construction is based on the topological properties of the group $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Aut}(\mathbb{H})$. This group has a unique m-fold covering $G_m \to \mathrm{PSL}(2,\mathbb{R})$. Thereby there is a 1-1-correspondence between m-spin structures on $P = \mathbb{H}/\Gamma$ and lifts into G_m of the group Γ, i.e. subgroups Γ^* of G_m such that the restriction of the covering map $G_m \to \mathrm{PSL}(2,\mathbb{R})$ to Γ^* is an isomorphism $\Gamma^* \to \Gamma$ [Mil75]. We prove that the preimage in G_m of the set of all hyperbolic and parabolic elements of $\mathrm{PSL}(2,\mathbb{R})$ has m connected components, which we identify with elements of the group $\mathbb{Z}_m$. This correspondence induces a map $\sigma : \pi_1^0(P,p) \to \mathbb{Z}_m$. The geometric properties of this map follow from the discreteness criterion [Nat04] for subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

We prove that the the set of all such functions has a structure of an affine space associated with $H^1(P;\mathbb{Z}/m\mathbb{Z})$. We describe the orbits of m-Arf functions under the action of the modular group (the group of homotopy classes of surface autohomeomorphisms). Natural topological invariants of an orbit are the unordered sets of values of the m-Arf functions on the punctures resp. holes. We prove (Theorem 5.2) that the space of m-Arf functions with prescribed genus of the surface and prescribed (unordered) sets of values on punctures resp. holes is connected if m is odd or if m is even and the value of the m-Arf function on one of the punctures or holes is even. Otherwise this space has two connected components distinguished by a topological invariant $\delta \in \{0, 1\}$ (see Definition 5.1).

We use the higher Arf functions to describe the topology of the moduli space of m-spin bundles. Moduli spaces of Riemann surfaces with m-spin structure play an important role in mathematical physics [Wit93, JKV01] and singularity theory [Dol83]. This moduli spaces were studied in [JKV01, CCC04] and other papers. We give a description of the connected components of the moduli space in terms of m-Arf functions and find the number of connected components (Theorem 5.7). For surfaces without holes this number was computed in [Jar00] using methods of algebraic geometry. Moreover we prove (Theorem 5.5) that any connected component is homeomorphic to the space of the form $\mathbb{R}^d/\,\mathrm{Mod}$, where Mod is a discrete group acting on the vector space $\mathbb{R}^d$.

The paper is organized as follows: In section 2 we study the covering groups G_m of the group $\mathrm{PSL}(2,\mathbb{R})$, and in particular the algebraic properties of the preimages in G_m of hyperbolic and parabolic elements of $\mathrm{PSL}(2,\mathbb{R})$. In section 3 we explore the connection between m-spin structures on a Riemann surface $P = \mathbb{H}/\Gamma$ and lifts into the covering G_m of the group Γ. We assign to any lift a function induced by a decomposition of the covering G_m into sheets and choosing a numeration of the sheets and study properties of these functions. In section 4 we define m-Arf functions. We prove that there is a 1-1-correspondence between the set of m-Arf functions on $P=\mathbb{H}/\Gamma$ and the set of functions associated to the lifts of Γ via the numeration of the covering sheets. Hence these two sets are also in 1-1-correspondence with the set of m-spin structures on P. Moreover we show in this section using the explicit description of the Dehn generators of the Teichmüller modular group that the set of all m-Arf functions on a surface P has a structure of an affine space. In the last section we find topological invariants of m-Arf functions and prove that they describe the connected components of the moduli space. Furthermore we show using a version of Theorem of Fricke and Klein that any connected component is homeomorphic to the space of the form $\mathbb{R}^d/\operatorname{Mod}$, where Mod is a discrete group.

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2. The group $\mathrm{PSL}(2,\mathbb{R})$ and its coverings

We consider the universal cover $\tilde{G}=\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the Lie group

$$G=\mathrm{PSL}(2,\mathbb{R})=\mathrm{SL}(2,\mathbb{R})/\{\pm 1\},$$

the group of orientation-preserving isometries of the hyperbolic plane. Here our model of the hyperbolic plane is the upper half-plane $\mathbb{H}=\{z\in\mathbb{C} \mid \operatorname{Im}(z)>0\}$ and the action of an element $[\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)]\in\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$ is by

$$z\mapsto\frac{az+b}{cz+d}.$$

Here we denote by $[A]=[\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)]\in\mathrm{PSL}(2,\mathbb{R})$ the equivalence class of a matrix $A=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{R})$.

Notation. $\mathbb{S}^1=\{z\in\mathbb{C} \mid |z|=1\}\subset\mathbb{C}$, $\mathbb{R}_+=\{x\in\mathbb{R} \mid x>0\}$, $\mathbb{Z}_m=\mathbb{Z}/m\mathbb{Z}$.

2.1. **Description of elements in $G=\mathrm{PSL}(2,\mathbb{R})$.** Elements of $\mathrm{PSL}(2,\mathbb{R})$ can be classified with respect to the fixed point behavior of their action on $\mathbb{H}$. An element is called *hyperbolic* if it has two fixed points which lie on the boundary $\partial\mathbb{H}=\mathbb{R}\cup\{\infty\}$ of $\mathbb{H}$. A hyperbolic element with fixed points in $\mathbb{R}$ is of the form

$$\tau_{\alpha,\beta}(\lambda):=\left[\frac{1}{(\alpha-\beta)\cdot\sqrt{\lambda}}\cdot\begin{pmatrix}\lambda\alpha-\beta & -(\lambda-1)\alpha\beta\\ \lambda-1 & \alpha-\lambda\beta\end{pmatrix}\right],$$

where $\alpha,\beta\in\mathbb{R}$ are the fixed points and $\lambda>0$. A hyperbolic element with one fixed point in ∞ is of the form

$$\tau_{\infty,\beta}(\lambda):=\left[\frac{1}{\sqrt{\lambda}}\cdot\begin{pmatrix}\lambda & -(\lambda-1)\beta\\ 0 & 1\end{pmatrix}\right]$$

or

$$\tau_{\alpha,\infty}(\lambda):=\left[\frac{1}{\sqrt{\lambda}}\cdot\begin{pmatrix}1 & (\lambda-1)\alpha\\ 0 & \lambda\end{pmatrix}\right],$$

where α resp. β is the real fixed points and $\lambda>0$. The parameter $\lambda>0$ is called the *shift parameter*. The *axis* $\ell(g)$ of the element $g=\tau_{\alpha,\beta}(\lambda)$ is the geodesic between the fixed points α and β, oriented from β to α if $\lambda>1$ and from α to β if $\lambda<1$. The element $g=\tau_{\alpha,\beta}(\lambda)$ preserves the geodesic $\ell(g)$ and moves the points on this geodesic in the direction of the orientation. We call a hyperbolic element $\tau_{\alpha,\beta}(\lambda)$ with $\lambda>1$ *positive* if $\alpha<\beta$. The map $\lambda\mapsto\tau_{\alpha,\beta}(\lambda)$ defines a homomorphism $\mathbb{R}_+\to G$ (with respect to the multiplicative structure on $\mathbb{R}_+$). We have

$$(\tau_{\alpha,\beta}(\lambda))^{-1}=\tau_{\alpha,\beta}(\lambda^{-1})=\tau_{\beta,\alpha}(\lambda).$$

An element is called *parabolic* if it has one fixed point which lies on the boundary $\partial\mathbb{H}$. A parabolic element with real fixed point α is of the form

$$\pi_\alpha(\lambda) := \left[\begin{pmatrix} 1-\lambda\alpha & \lambda\alpha^2 \\ -\lambda & 1+\lambda\alpha \end{pmatrix}\right].$$

A parabolic element with fixed point ∞ is of the form

$$\pi_\infty(\lambda) := \left[\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\right].$$

We call a hyperbolic element $\pi_\alpha(\lambda)$ *positive* if $\lambda > 0$. The map $\lambda \mapsto \pi_\alpha(\lambda)$ defines a homomorphism $\mathbb{R} \to G$. (with respect to the additive structure on $\mathbb{R}$). We have

$$(\pi_\alpha(\lambda))^{-1} = \pi_\alpha(-\lambda).$$

An element, which is neither hyperbolic nor parabolic, is called *elliptic*. It has one fixed point that lies in $\mathbb{H}$. Given a base-point $x \in \mathbb{H}$ and a real number φ, let $\rho_x(\varphi) \in G$ denote the rotation through angle φ about the point x. Any elliptic element is of the form $\rho_x(\varphi)$, where x is the fixed point. Thus we obtain a 2π-periodic homomorphism $\rho_x : \mathbb{R} \to G$ (with respect to the additive structure on $\mathbb{R}$). We have

$$\rho_x(\varphi + 2\pi) = \rho_x(\varphi) \quad \text{and} \quad (\rho_x(\varphi))^{-1} = \rho_x(-\varphi).$$

For the fixed point $x = i \in \mathbb{H}$ we have

$$\rho_i(\varphi) = \left[\begin{pmatrix} \cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}\right].$$

For a fixed point $x \in \mathbb{H}\backslash\{i\}$ we obtain $\rho_x(\varphi) = \tau \circ \rho_i(\varphi) \circ \tau^{-1}$, where τ is the hyperbolic element in G such that $\tau(i) = x$.

2.2. **Coverings G_m of G.** As topological space $\mathrm{PSL}(2,\mathbb{R})$ is homeomorphic to the solid torus $\mathbb{S}^1 \times \mathbb{C}$. A homeomorphism $\mathrm{PSL}(2,\mathbb{R}) \to \mathbb{S}^1 \times \mathbb{C}/\{\pm 1\}$ is given by

$$H : \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \mapsto \left(\left(\frac{(a+d)+i(b-c)}{|(a+d)+i(b-c)|}\right)^2, \left[\frac{(a-d)+i(b+c)}{2}\right]\right).$$

The fundamental group of the solid torus G is infinite cyclic. Therefore, for each natural number m there is a unique connected m-fold covering

$$G_m = \tilde{G}/(m \cdot Z(\tilde{G}))$$

of G, where $Z(\tilde{G})$ is the central subgroup of $\tilde{G}$. For $m = 2$ this is the group $G_2 = \mathrm{SL}(2,\mathbb{R})$.

The map $\mu : G \to \mathbb{S}^1$ defined as the composition of the homeomorphism $H : G \to \mathbb{S}^1 \times \mathbb{C}$ and the projection $\mathbb{S}^1 \times \mathbb{C} \to \mathbb{S}^1$ maps an element $[A] = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)] \in G$ to a unit complex number $\mu([A]) = e^{i\psi}$ with

$$\tan\frac{\psi}{2} = \frac{b-c}{a+d}.$$

We shall refer to the number ψ as the argument of the element $[A]$. The map $\mu : G \to \mathbb{S}^1$ lifts to the unique map $\varphi : \tilde{G} \to \mathbb{R}$ of the universal covers such that the following diagram commutes

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\varphi} & \mathbb{R} \\ \downarrow & & \downarrow \\ G & \xrightarrow{\mu} & \mathbb{S}^1 \end{array}$$

Here the map $\mathbb{R} \to \mathbb{S}^1$ is the universal covering map given by $x \mapsto e^{ix}$.

Definition 2.1. We define $s : \tilde{G} \to \mathbb{Z}$ by

$$s(g) = k \quad \text{if} \quad \varphi(g) \in (-\pi + 2\pi k, \pi + 2\pi k].$$

We define $s_m : G_m \to \mathbb{Z}_m$ on $G_m = \tilde{G}/(m \cdot Z(\tilde{G}))$ by

$$s_m(g \mod (m \cdot Z(\tilde{G}))) = s(g) \mod m$$

for $g \in \tilde{G}$. We recall that $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. All equations involving s_m are to be understood as equations in $\mathbb{Z}_m$, i.e. equations modulo m.

Here is another description of the covering groups G_m of $G = \mathrm{PSL}(2, \mathbb{R})$, which fixes a group structure. Let $\mathrm{Hol}(\mathbb{H}, \mathbb{C}^*)$ be the set of all holomorphic functions $\mathbb{H} \to \mathbb{C}^*$.

Proposition 2.1. *The m-fold covering group G_m of G can be described as*

$$\{(g, \delta) \in G \times \mathrm{Hol}(\mathbb{H}, \mathbb{C}^*) \mid \delta^m(z) = g'(z) \text{ for all } z \in \mathbb{H}\}$$

with multiplication

$$(g_2, \delta_2) \cdot (g_1, \delta_1) = (g_2 \cdot g_1, (\delta_2 \circ g_1) \cdot \delta_1).$$

Proof. Let X be the subspace of $G \times \mathrm{Hol}(\mathbb{H}, \mathbb{C}^*)$ in question. One can check that the space X is connected and that the map $X \to G$ given by $(\gamma, \delta) \mapsto \gamma$ is an m-fold covering of G. Hence the coverings $X \to G$ and $G_m \to G$ are isomorphic. One can check that the operation described above defines a group structure on X and that the covering map $X \to G$ is a homomorphism with respect to this group structure. □

Remark. This description of G_m is inspired by the notion of automorphic differential forms of fractional degree, introduced by J. Milnor in [Mil75]. For a more detailed discussion of this fact see [LV80], section 1.8.

2.3. **Decomposition of G_m into sheets.** Elements of G_m can also be classified with respect to the fixed point behavior of action on $\mathbb{H}$ of their image in $\mathrm{PSL}(2, \mathbb{R})$. We say that an element of G_m is hyperbolic, parabolic, resp. elliptic if its image in G has this property. We also define the absolute value of the trace for an element of G_m as the absolute value of the trace of its image in G.

Any of the homomorphisms $\tau_{\alpha,\beta} : \mathbb{R}_+ \to G$, $\pi_\alpha : \mathbb{R} \to G$, resp. $\rho_x : \mathbb{R} \to G$ lifts to the unique homomorphism $t_{\alpha,\beta} : \mathbb{R}_+ \to \tilde{G}$, $p_\alpha : \mathbb{R} \to \tilde{G}$ resp. $r_x : \mathbb{R} \to \tilde{G}$ into the universal covering group. The elements $t_{\alpha,\beta}(\lambda)$, $p_\alpha(\lambda)$, resp. $r_x(\varphi)$ are hyperbolic, parabolic, resp. elliptic.

For hyperbolic elements we obtain for all $\lambda \in \mathbb{R}_+$ that

$$\varphi(t_{\alpha,\beta}(\lambda)) \in (-\pi, \pi)$$

because of $\mu(\tau_{\alpha,\beta}(1)) = 1$ and $\mu(\tau_{\alpha,\beta}(\lambda)) \neq -1$.

Similarly, for parabolic elements we obtain for all $\lambda \in \mathbb{R}$ that
$$\varphi(p_\alpha(\lambda)) \in (-\pi, \pi)$$
because of $\mu(\pi_\alpha(0)) = 1$ and $\mu(\pi_\alpha(\lambda)) \neq -1$.

Now let us consider elliptic elements. For the trace of the rotation $\rho_x(\varphi)$ it holds
$$|\operatorname{trace}(\rho_x(\varphi))| = |\operatorname{trace}(\rho_i(\varphi))| = |2\cos(\varphi/2)|,$$
hence $\mu(\rho_x(\varphi)) = -1$ if and only if $\varphi \in \pi + 2\pi \cdot \mathbb{Z}$. Because of $\mu(\rho_x(0)) = 1$ this implies for $\varphi \in (-\pi + 2\pi k, \pi + 2\pi k)$ that
$$\varphi(r_x(\varphi)) \in (-\pi + 2\pi k, \pi + 2\pi k).$$
We obtain $s(r_x(\varphi)) = k$ for $\varphi \in (-\pi + 2\pi k, \pi + 2\pi k]$.

Since $\rho_x(2\pi) = \mathrm{id}$, it follows that the lifted element $r_x(2\pi)$ belongs to the central subgroup $Z(\tilde{G})$ of $\tilde{G}$. Note that this element $r_x(2\pi)$ depends continuously on x. But the central subgroup of $\tilde{G}$ is discrete, so this element must remain constant. Let
$$u = r_x(2\pi) \quad \text{for all } x \in \mathbb{H}.$$
The element u is a one of the two generators of the centre of $\tilde{G}$. We would also like to point out that for the lift of an elliptic element $\rho_x(2\pi/p)$ of finite order p it holds
$$(r_x(2\pi/p))^p = r_x(2\pi) = u.$$

Any hyperbolic resp. parabolic resp. elliptic element in $\tilde{G}$ is of the form $t_{\alpha,\beta}(\lambda)\cdot u^k$ resp. $p_\alpha(\lambda) \cdot u^k$ resp. $r_x(\varphi)$. It holds $s(t_{\alpha,\beta}(\lambda) \cdot u^k) = k$, $s(p_\alpha(\lambda) \cdot u^k) = k$, and $s(r_x(\varphi)) = k$ iff $\varphi \in (-\pi + 2\pi k, \pi + 2\pi k]$.

Remark. The subdivision of the solid tube $\tilde{G} \simeq \mathbb{R} \times \mathbb{C}$ is described in more detail in [JN85].

2.4. **Properties of the multiplication of hyperbolic and parabolic elements in G_m.** In this subsection we first study (Lemma 2.2 and 2.3) the behavior of the level functions s_m under inversion and conjugation. The main results of this subsection (Lemma 2.4, 2.6 and 2.7) are statements about the behavior of s_m under multiplication.

In this subsection let us denote by $[\cdot]$ the image of an element in $\tilde{G}$ under the covering map $\tilde{G} \to G_m$.

Lemma 2.2. *We have $s_m(A^{-1}) = -s_m(A)$ for any hyperbolic or parabolic element A in G_m.*

Proof. The hyperbolic resp. parabolic element A is of the form $[t_{\alpha,\beta}(\lambda) \cdot u^k]$ resp. $[p_\alpha(\lambda) \cdot u^k]$. If $A = [t_{\alpha,\beta}(\lambda) \cdot u^k]$ then $A^{-1} = [t_{\alpha,\beta}(\lambda^{-1}) \cdot u^{-k}]$ and $s_m(A^{-1}) = -k = -s_m(A)$. If $A = [p_\alpha(\lambda) \cdot u^k]$ then $A^{-1} = [p_\alpha(-\lambda) \cdot u^{-k}]$ and $s_m(A^{-1}) = -k = -s_m(A)$. □

Lemma 2.3. *For two elements A and B in G_m we have*
$$s_m(B \cdot A \cdot B^{-1}) = s_m(A).$$

Proof. The expression $s_m(BAB^{-1}) - s_m(A)$ is invariant under multiplication of A or B with the central element $[u]$, hence it is sufficient to prove the statement for elements A and B with $s_m(A) = s_m(B) = 0$. The element A in G_m with $s_m(A) = 0$ is of the form $[t_{\alpha,\beta}(\lambda)]$, $[p_\alpha(\lambda)]$ resp. $[r_x(\varphi)]$ with $\varphi \in (-\pi, \pi]$. The element A can be connected to the unit element in G_m via a path $\gamma : I \to G_m$, where I is some closed interval, such that $s_m(\gamma(t)) = 0$ for all $t \in I$. For example, we can take the path $[t_{\alpha,\beta}(t)]$, $t \in [1, \lambda]$, $[p_\alpha(t)]$, $t \in [0, \lambda]$, resp. $[r_x(t)]$, $t \in [0, \varphi]$. The path $\gamma^B : I \to G_m$ given by

$$\gamma^B(t) = B \cdot \gamma(t) \cdot B^{-1}$$

connects the element $B \cdot A \cdot B^{-1}$ with the unit element. It holds for all $t \in I$

$$|\operatorname{trace}(\gamma^B(t))| = |\operatorname{trace}(B \cdot \gamma(t) \cdot B^{-1})| = |\operatorname{trace}(\gamma(t))| \neq 0,$$

and hence $s_m(\gamma^B(t)) = 0$ for all $t \in I$, in particular $s_m(B \cdot A \cdot B^{-1}) = 0$. (Here $|\operatorname{trace} A|$ for an element A in G_m is defined as $|\operatorname{trace} \bar{A}|$, where $\bar{A}$ is the projection of A in G.) □

Lemma 2.4. *For preimages A resp. B in G_m of the elements $\bar{A} = \tau_{\infty,0}(\lambda_1)$ resp. $\bar{B} = \tau_{\alpha,\beta}(\lambda_2)$ for some $\lambda_1, \lambda_2 > 1$ and $\alpha, \beta \in \mathbb{R}\backslash\{0\}$ the difference*

$$s_m(A \cdot B) - s_m(A) - s_m(B)$$

is equal

- $+1$ *if*

$$0 < \frac{\lambda_1 + \lambda_2}{1 + \lambda_1\lambda_2} \cdot \beta < \alpha < \beta,$$

- -1 *if*

$$\beta < \alpha \leqslant \frac{\lambda_1 + \lambda_2}{1 + \lambda_1\lambda_2} \cdot \beta < 0,$$

- 0 *else.*

Proof. The expression $s_m(A \cdot B) - s_m(A) - s_m(B)$ is invariant under multiplication of A or B with the central element $[u]$, hence it is sufficient to prove the statement for elements A and B with $s_m(A) = s_m(B) = 0$, i.e. for $A = [t_{\infty,0}(\lambda_1)]$ and $B = [t_{\alpha,\beta}(\lambda_2)]$. Let us consider the path $\gamma : [0, 1] \to G$ from 1 to

$$\bar{A} \cdot \bar{B} = \tau_{\infty,0}(\lambda_1) \cdot \tau_{\alpha,\beta}(\lambda_2)$$

given by the suitably reparametrised product

$$\gamma(t) := \tau_{\infty,0}(\lambda_1^t) \cdot \tau_{\alpha,\beta}(\lambda_2^t)$$

with $\lambda_j^t = 1 + t(\lambda_j - 1)$ for $j = 1, 2$. Let $\tilde{\gamma} : [0, 1] \to G_m$ be the lift of this path covering with $\tilde{\gamma}(0) = e$. The path γ is homotopic to the path

$$\delta := (\tau_{\infty,0}|_{[1,\lambda_1]}) * (\tau_{\infty,0}(\lambda_1) \cdot \tau_{\alpha,\beta}|_{[1,\lambda_2]}),$$

where $*$ means to go along the first path and then along the second path. It is clear that

$$\tilde{\delta} := ([t_{\infty,0}]|_{[1,\lambda_1]}) * ([t_{\infty,0}(\lambda_1)] \cdot [t_{\alpha,\beta}]|_{[1,\lambda_2]})$$

is the lift of the path δ with the starting point e. The end point of the lifted path $\tilde{\delta}$ is $[t_{\infty,0}(\lambda_1) \cdot t_{\alpha,\beta}(\lambda_2)]$. Since the path γ is homotopic to δ, the lift $\tilde{\gamma}$ has the same end point as $\tilde{\delta}$, hence

$$\tilde{\gamma}(1) = [t_{\infty,0}(\lambda_1) \cdot t_{\alpha,\beta}(\lambda_2)] = A \cdot B.$$

So we have to compute

$$s_m(A\cdot B)-s_m(A)-s_m(B)=s_m(\tilde\gamma(1)).$$

On the other hand it holds

$$\begin{aligned}\gamma(t)&=\tau_{\infty,0}(\lambda_1^t)\cdot\tau_{\alpha,\beta}(\lambda_2^t)\\&=\left[\frac{1}{(\alpha-\beta)\cdot\sqrt{\lambda_1^t\lambda_2^t}}\cdot\begin{pmatrix}\lambda_1^t(\lambda_2^t\alpha-\beta) & -\lambda_1^t(\lambda_2^t-1)\alpha\beta\\ \lambda_2^t-1 & \alpha-\lambda_2^t\beta\end{pmatrix}\right].\end{aligned}$$

Let $\Phi(t)=\varphi(\tilde\gamma(t))$. We obtain

$$\tan\frac{\Phi(t)}{2}=-\frac{(\lambda_1^t\alpha\beta+1)(\lambda_2^t-1)}{(\lambda_1^t\lambda_2^t+1)\alpha-(\lambda_1^t+\lambda_2^t)\beta}.$$

It holds $\Phi(0)=0$. The denominator of this fraction is

$$\begin{aligned}f(t)&:=(\lambda_1^t\lambda_2^t+1)\alpha-(\lambda_1^t+\lambda_2^t)\beta\\&=(\lambda_1^t+\lambda_2^t)(\alpha-\beta)+(\lambda_1^t-1)(\lambda_2^t-1)\alpha.\end{aligned}$$

Since $\lambda_1^t+\lambda_2^t>0$ and $(\lambda_1^t-1)(\lambda_2^t-1)\geqslant 0$, if α and $\alpha-\beta$ are both positive or both negative, i.e. in the cases $\alpha<\beta<0$, $\alpha<0<\beta$, $\beta<0<\alpha$ and $0<\beta<\alpha$, we have $f(t)\neq 0$ for $t\in[0,1]$. This implies $\Phi(1)\in(-\pi,\pi)$ and hence

$$s_m(A\cdot B)-s_m(A)-s_m(B)=s_m(\tilde\gamma(1))=0.$$

In the cases $0<\alpha<\beta$ resp. $\beta<\alpha<0$ we have to look carefully at the argument $\Phi(t)$. The argument $\Phi(t)$ satisfies the equation

$$\tan\frac{\Phi(t)}{2}=\frac{-(\lambda_1^t\alpha\beta+1)(\lambda_2^t-1)}{f(t)}$$

where

$$\begin{aligned}f(t)&=(\lambda_1^t\lambda_2^t+1)\alpha-(\lambda_1^t+\lambda_2^t)\beta\\&=\alpha(\lambda_1-1)(\lambda_2-1)\cdot t^2+(\alpha-\beta)((\lambda_1-1)+(\lambda_2-1))\cdot t+2(\alpha-\beta)\end{aligned}$$

is a quadratic function. Since $\alpha\cdot\beta>0$, $\lambda_1^t>0$ and $\lambda_2^t\geqslant 1$, the sign of $\tan(\Phi(t)/2)$ is opposite to the sign of $f(t)$.

Let us assume that $0<\alpha<\beta$. The coefficient $\alpha(\lambda_1-1)(\lambda_2-1)$ by t^2 in $f(t)$ is positive, hence the function f is concave. It holds

$$\begin{aligned}f(0)&=2(\alpha-\beta)<0,\\f(1)&=(\lambda_1\lambda_2+1)\alpha-(\lambda_1+\lambda_2)\beta.\end{aligned}$$

There are two cases, $f(1)\leqslant 0$ and $f(1)>0$.

Let us assume that $0<\alpha<\beta$ and $f(1)\leqslant 0$. Then $f(0)<0$, $f(1)\leqslant 0$ and f concave implies $f(t)<0$ for $t\in[0,1)$, hence $\tan(\Phi(t)/2)>0$ for $t\in(0,1)$. This implies $\Phi(t)\in[0,\pi]$ and hence

$$s_m(A\cdot B)-s_m(A)-s_m(B)=s_m(\tilde\gamma(1))=0.$$

Let us assume that $0<\alpha<\beta$ and $f(1)>0$, which is equivalent to

$$0<\frac{\lambda_1+\lambda_2}{1+\lambda_1\lambda_2}\cdot\beta<\alpha<\beta.$$

Then $f(0) < 0$, $f(1) > 0$ and f quadratic implies that there is $t_0 \in (0,1)$ such that $f(t_0) = 0$, $f(t) < 0$ for $t \in [0, t_0)$, and $f(t) > 0$ for $t \in (t_0, 1]$. Hence $\tan(\Phi(t)/2) > 0$ for $t \in (0, t_0)$ and $\tan(\Phi(t)/2) < 0$ for $t \in (t_0, 1]$. This implies $\Phi(1) \in (\pi, 2\pi]$ and hence

$$s_m(A \cdot B) - s_m(A) - s_m(B) = s_m(\tilde{\gamma}(1)) = 1.$$

Let us assume that $\beta < \alpha < 0$. The coefficient $\alpha(\lambda_1 - 1)(\lambda_2 - 1)$ by t^2 in $f(t)$ is negative, hence the function f is convex. It holds

$$f(0) = 2(\alpha - \beta) > 0,$$
$$f(1) = (\lambda_1\lambda_2 + 1)\alpha - (\lambda_1 + \lambda_2)\beta.$$

There are two cases, $f(1) > 0$ and $f(1) \leqslant 0$.

Let us assume that $\beta < \alpha < 0$ and $f(1) > 0$. Then $f(0) > 0$, $f(1) > 0$ and f convex implies $f(t) \neq 0$ for $t \in [0, 1]$, hence $s_m(A \cdot B) - s_m(A) - s_m(B) = 0$ as before.

Let us assume that $\beta < \alpha < 0$ and $f(1) \leqslant 0$, which is equivalent to

$$\beta < \alpha \leqslant \frac{\lambda_1 + \lambda_2}{1 + \lambda_1\lambda_2} \cdot \beta < 0.$$

Then $f(0) > 0$, $f(1) \leqslant 0$ and f quadratic implies that there is $t_0 \in (0, 1]$ such that $f(t_0) = 0$, $f(t) > 0$ for $t \in [0, t_0)$, and $f(t) < 0$ for $t \in (t_0, 1)$. Hence $\tan(\Phi(t)/2) < 0$ for $t \in (0, t_0)$ and $\tan(\Phi(t)/2) > 0$ for $t \in (t_0, 1)$. This implies $\Phi(1) \in [-2\pi, -\pi]$ and hence

$$s_m(A \cdot B) - s_m(A) - s_m(B) = s_m(\tilde{\gamma}(1)) = -1. \quad \square$$

Corollary 2.5. *If the axes of two hyperbolic elements A and B in G_m intersect, or the elements A and B are oriented differently (i.e. A is positive while B is negative or B is positive while A is negative), then*

$$s_m(A \cdot B) = s_m(A) + s_m(B).$$

The results similar to Lemma 2.4 hold also for products of hyperbolic and parabolic elements as well as for products of parabolic elements:

Lemma 2.6. *For preimages A resp. B in G_m of the elements $\bar{A} = \tau_{\infty,0}(\lambda_1)$ resp. $\bar{B} = \pi_\alpha(\lambda_2)$ for some $\lambda_1 > 1$, $\lambda_2 > 0$ and $\alpha > 0$ the difference*

$$s_m(A \cdot B) - s_m(A) - s_m(B)$$

is equal

- *to* 0 *if*

$$\lambda_2\alpha \leqslant \frac{\lambda_1 + 1}{\lambda_1 - 1},$$

- *and to* 1 *else.*

Lemma 2.7. *For preimages A resp. B in G_m of the elements $\bar{A} = \pi_\infty(\lambda_1)$ resp. $\bar{B} = \pi_\alpha(\lambda_2)$ for some $\lambda_1, \lambda_2, \alpha > 0$ the difference*

$$s_m(A \cdot B) - s_m(A) - s_m(B)$$

is equal

- *to* 0 *if* $\lambda_1\lambda_2 \leqslant 2$,
- *and to* 1 *else.*

We omit the proofs of Lemmas 2.6 and 2.7, which are along the lines of the proof of Lemma 2.4.

3. Higher co-spin structures and lifts of Fuchsian groups

3.1. **Higher co-spin structures.** Let $E \to P$ be complex line bundle over a Riemann surface P. Let Γ be a torsionsfree Fuchsian group such that $P = \mathbb{H}/\Gamma$. Let $L \to \mathbb{H}$ be the induced complex line bundle over $\mathbb{H}$. Let $L \simeq \mathbb{H} \times \mathbb{C}$ be a trivialization of the bundle L. With respect to this trivialization the action of Γ on L is given by

$$g \cdot (z,t) = (g(z), \delta(g,z) \cdot t),$$

where $\delta : \Gamma \times \mathbb{H} \to \mathbb{C}^*$ is a map such that the function $\delta_g = \delta|_{\{g\}\times\mathbb{H}}$ is holomorphic for any $g \in \Gamma$ and for any $g_1, g_2 \in \Gamma$ it holds

$$\delta_{g_2 \cdot g_1} = (\delta_{g_2} \circ g_1) \cdot \delta_{g_1}.$$

The map δ is called the *transition map* of the bundle $E \to P$ with respect to the given trivialization.

In particular, if E is the tangent bundle of the surface P, then the transition map can be chosen so that $\delta_g = g'$. Let $E_1 \to P$, $E_2 \to P$ be two complex line bundles over a Riemann surface P, and let δ_1 resp. δ_2 be their transition maps, then $\delta_1 \cdot \delta_2$ is a transition map of the bundle $E_1 \otimes E_2 \to P$. In particular, if δ is the transition map of the bundle $E \to P$, then δ^m is a transition map of the bundle $E^m = E \otimes \cdots \otimes E \to P$ (with respect to the induced trivialization).

A *m-co-spin structure* on a Riemann surface P is a transition map δ of a complex line bundle $E \to P$ which satisfies the condition $\delta_g^m = g'$, i.e. the induced transition map δ^m of the bundle $E^m \to P$ coincides with the transition map of the tangent bundle of P.

Remark. A m-spin structure on a Riemann surface P is a transition map δ of a complex line bundle $E \to P$ which satisfies the condition $\delta_g^m = (g')^{-1}$, i.e. the induced transition map δ^m of the bundle $E^m \to P$ coincides with the transition map of the cotangent bundle of P. There is a one-to-one correspondence between m-spin and m-co-spin structures on a Riemann surface given by taking δ to δ^{-1}. In the following we consider the m-co-spin structures.

Remark. A complex line bundle $E \to P$ is said to be *m-co-spin* if the bundle $E^m \to P$ is isomorphic to the tangent bundle of P. For a compact Riemann surface P there is a 1-1-correspondence between m-co-spin structures on P and m-co-spin bundles over P.

Definition 3.1. A *lift* of the Fuchsian group Γ into G_m is a subgroup Γ^* of G_m such that the restriction of the covering map $G_m \to G$ to Γ^* is an isomorphism between Γ^* and Γ.

Proposition 3.1. *There is a 1-1-correspondence between m-co-spin structures on the Riemann surface $P = \mathbb{H}/\Gamma$ and lifts of Γ into G_m.*

Proof. On the one hand using the description of the covering G_m from Proposition 2.1 we see that there is a 1-1-correspondence between the lifts of Γ into G_m and the families $\{\delta_g\}_{g\in\Gamma}$ of holomorphic functions $\mathbb{H} \to \mathbb{C}^*$ such that for any $g \in \Gamma$

$$\delta_g^m = g'$$

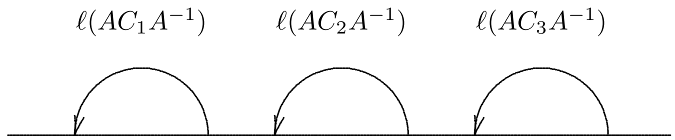

Figure 1: Axes of a sequential set of type $(0,3,0)$

and for any $g_1, g_2 \in \Gamma$

$$\delta_{g_2 \cdot g_1} = (\delta_{g_2} \circ g_1) \cdot \delta_{g_1}.$$

On the other hand as we explained in this section there is a 1-1-correspondence between such families of holomorphic functions and m-co-spin structures on $P = \mathbb{H}/\Gamma$. □

3.2. **Finitely generated Fuchsian groups.**

Definition 3.2. A *sequential set of type* $(0, l_h, l_p)$ with $l_h + l_p = 3$ is a triple of elements (C_1, C_2, C_3) in G, such that the elements C_i, $1 \leqslant i \leqslant l_h$, are positive hyperbolic, the elements C_i, $l_h < i \leqslant 3$, are positive parabolic, it holds $C_1 \cdot C_2 \cdot C_3 = 1$ and for some element $A \in G$ the position of the axes of the elements AC_jA^{-1} is as in Figure 1. (Figure 1 shows the position of the axes for the type $(0,3,0)$, i.e. all elements are hyperbolic. It is clear how the similar picture looks like in presence of parabolic elements.)

Definition 3.3. A *sequential set of type* $(0, l_h, l_p)$ is an n-tuple of elements

$$(C_1, \dots, C_n)$$

with $n = l_h + l_p$ in G such that the elements $C_1, \dots, C_{l_h}$ are hyperbolic, the elements $C_{l_h+1}, \dots, C_n$ are parabolic, and for any $j \in \{1, \dots, n\}$ the triple

$$(C_1 \cdots C_{j-1}, C_j, C_{j+1} \cdots C_n)$$

is a sequential set of type $(0,3,0)$.

Definition 3.4. A *sequential set of type* (g, l_h, l_p) is a $(2g+n)$-tuple of elements

$$(A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_n)$$

with $n = l_h + l_p$ in G such that the elements $A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_{l_h}$ are hyperbolic, the elements $C_{l_h+1}, \dots, C_n$ are parabolic, and the tuple

$$(A_1, B_1A_1^{-1}B_1^{-1}, \dots, A_g, B_gA_g^{-1}B_g^{-1}, C_1, \dots, C_n)$$

is a sequential set of type $(0, 2g + l_h, l_p)$.

Definition 3.5. We call a Riemann surface of genus g with l_h holes and l_p punctures a *Riemann surface of type* (g, l_h, l_p).

Definition 3.6. We define the product ab of two contours a and b in $\pi_1(P, p)$ as the contour given by the path of b followed by the path of a. A *standard basis* of a fundamental group $\pi_1(P, p)$ of a surface P of type (g, l_h, l_p) is a set of generators

$$\{a_i, b_i \ (i = 1, \dots, g), c_i \ (i = g+1, \dots, n)\}$$

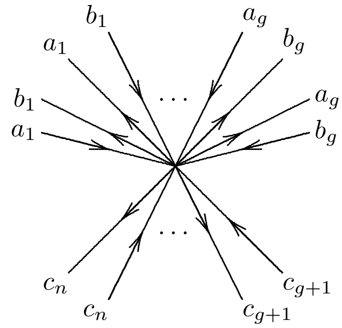

Figure 2: Standard basis

with a single defining relation

$$\prod_{i=1}^{g}[a_i,b_i]\prod_{i=g+1}^{n}c_i=1$$

and represented by a set of simple contours

$$\{\tilde{a}_i,\tilde{b}_i(i=1,\dots,g),\tilde{c}_i(i=g+1,\dots,n)\}$$

with the following properties:

1) the contour $\tilde{c}_i$ encloses a hole in P for $i=g+1,\dots,g+l_h$ and a puncture for $i=g+l_h+1,\dots,n$,
2) $\tilde{a}_i\cap\tilde{b}_j=\tilde{a}_i\cap\tilde{c}_j=\tilde{b}_i\cap\tilde{c}_j=\tilde{c}_i\cap\tilde{c}_j=\{p\}$.
3) in a neighbourhood of the point p, the contours are placed as is shown in Figure 2.

The relation between sequential sets and Fuchsian groups is exploited in [Nat04] (Chapter 1, Theorem 1.1, Lemma 2.1, Theorem 2.1). We recall here the results:

Theorem 3.2. *A sequential set V of type (g,l_h,l_p) generates a Fuchsian group Γ such that the surface $P=\mathbb{H}/\Gamma$ is of type (g,l_h,l_p). The isomorphism $\Phi:\Gamma\to\pi_1(P,p)$, induced by the natural projection $\Psi:\mathbb{H}\to P$, maps the sequential set V to a standard basis of $\pi_1(P,p)$.*

Theorem 3.3. *Let Γ be a Fuchsian group such that the surface $P=\mathbb{H}/\Gamma$ is of type (g,l_h,l_p). let $p\in P$. Let $\Psi:\mathbb{H}\to P$ be the natural projection. Choose $q\in\Psi^{-1}(p)$ and let $\Phi:\Gamma\to\pi_1(P,p)$ be the induced isomorphism. Let*

$$v=\{a_i,b_i\ (i=1,\dots,g),c_i\ (i=g+1,\dots,n)\}$$

be a standard basis of $\pi_1(P,p)$. In this case,

$$\begin{aligned}V=\Phi^{-1}(v)&=\{\Phi^{-1}(a_i),\Phi^{-1}(b_i)(i=1,\dots,g),\Phi^{-1}(c_i)(i=g+1,\dots,n)\}\\&=\{A_i,B_i\ (i=1,\dots,g),C_i\ (i=g+1,\dots,n)\}\end{aligned}$$

is a sequential set of type (g,l_h,l_p).

Now we recall the classification of free Fuchsian groups of rank 2 (see [Nat04], Chapter 1, Lemma 3.2, 3.3):

Lemma 3.4. *The set $(C_1=\tau_{\infty,0}(\lambda_1),C_2=\tau_{\alpha,\beta}(\lambda_2),C_3)$ with $\lambda_1,\lambda_2>1$ is a sequential set of type $(0,3,0)$ or $(0,2,1)$ if and only if*

$$C_3=(C_1\cdot C_2)^{-1}$$

and

$$0<\left(\frac{\sqrt{\lambda_1}+\sqrt{\lambda_2}}{1+\sqrt{\lambda_1\lambda_2}}\right)^2\beta\leqslant\alpha<\beta<\infty.$$

Then the set (C_1,C_2,C_3) *is of type* $(0,2,1)$*, i.e. the element* C_3 *is parabolic, if and only if there is an equality in the last inequality.*

Lemma 3.5. *The set* $(C_1=\tau_{\infty,0}(\lambda_1),C_2=\pi_\alpha(\lambda_2),C_3)$ *with* $\lambda_1>1$*,* $\lambda_2>0$ *is a sequential set of type* $(0,2,1)$ *or* $(0,1,2)$ *if and only if*

$$C_3=(C_1\cdot C_2)^{-1}$$

and

$$\lambda_2\alpha\geqslant\frac{\sqrt{\lambda_1}+1}{\sqrt{\lambda_1}-1}.$$

Then the set (C_1,C_2,C_3) *is of type* $(0,1,2)$*, i.e. the element* C_3 *is parabolic, if and only if the last inequality is an equality.*

3.3. **Lifting sets of generators of Fuchsian groups.** In this subsection let us denote by $[\cdot]$ the image of an element in $\tilde{G}$ under the covering map $\tilde{G}\to G_m$.

Lemma 3.6. *Let* Γ *be a Fuchsian group of type* (g,l_h,l_p) *generated by the sequential set*

$$\begin{aligned}\bar{V}&=\{\bar{A}_i,\bar{B}_i\ (i=1,\dots,g),\bar{C}_i\ (i=g+1,\dots,n)\}\\&=\{\bar{D}_j\ (j=1,\dots,n+g)\}.\end{aligned}$$

Let

$$\begin{aligned}V&=\{A_i,B_i\ (i=1,\dots,g),C_i\ (i=g+1,\dots,n)\}\\&=\{D_j\ (j=1,\dots,n+g)\}\end{aligned}$$

be a set of the lifts of the elements of the sequential set $\bar{V}$ *into* G_m*, i.e. the image of* D_j *in* G *is* $\bar{D}_j$*. Then the subgroup* Γ^* *of* G_m *generated by* V *is a lift of* Γ *into* G_m *if and only if*

$$\prod_{i=1}^{g}[A_i,B_i]\cdot\prod_{i=g+1}^{n}C_i=e.$$

Proof. For any choice of the set of lifts V the restriction of the covering map $G_m\to G$ to the group Γ^* generated by V is a homomorphism with image Γ. There is only one relation

$$\prod_{i=1}^{g}[\bar{A}_i,\bar{B}_i]\cdot\prod_{i=g+1}^{n}\bar{C}_i=e$$

in Γ, hence the equality

$$\prod_{i=1}^{g}[A_i,B_i]\cdot\prod_{i=g+1}^{n}C_i=e$$

ensures injectivity of this homomorphism. □

Lemma 3.7. *Let* (C_1,C_2,C_3) *be a triple of elements in* G_m *such that there images* $(\bar{C}_1,\bar{C}_2,\bar{C}_3)$ *in* G *form a sequential set of type* $(0,l_h,l_p)$ *with* $l_h+l_p=3$*. Then we have* $C_1\cdot C_2\cdot C_3=e$ *if and only if*

$$s_m(C_1)+s_m(C_2)+s_m(C_3)=-1.$$

Moreover it holds

$$s_m(C_1 \cdot C_2) = s_m(C_1) + s_m(C_2) + 1.$$

Proof. We first prove $s_m(C_1 \cdot C_2) = s_m(C_1) + s_m(C_2) + 1$ separately for sequential sets of types $(0,3,0)$ and $(0,2,1)$ using Lemma 2.4, for sequential sets of types $(0,1,2)$ and $(0,2,1)$ using Lemma 2.6, and for sequential sets of type $(0,0,3)$ using Lemma 2.7.

- We first assume that the elements C_1 and C_2 are hyperbolic. Up to conjugation we can assume that $\bar{C}_1 = \tau_{\infty,0}(\lambda_1)$ and $\bar{C}_2 = \tau_{\alpha,\beta}(\lambda_2)$ for some $\lambda_1, \lambda_2 > 1$ and $\alpha, \beta \in \mathbb{R}\backslash\{0\}$. Lemma 3.4 implies that $\alpha, \beta > 0$ and
$$1 > \frac{\alpha}{\beta} > \left(\frac{\sqrt{\lambda_1} + \sqrt{\lambda_2}}{1 + \sqrt{\lambda_1 \lambda_2}}\right)^2 = \frac{\lambda_1 + \lambda_2 + 2\sqrt{\lambda_1 \lambda_2}}{1 + \lambda_1 \lambda_2 + 2\sqrt{\lambda_1 \lambda_2}} > \frac{\lambda_1 + \lambda_2}{1 + \lambda_1 \lambda_2}.$$
According to Lemma 2.4 the inequalities $\alpha \cdot \beta > 0$ and
$$\frac{\lambda_1 + \lambda_2}{1 + \lambda_1 \lambda_2} \cdot \beta < \alpha < \beta$$
imply
$$s_m(C_1 \cdot C_2) = s_m(C_1) + s_m(C_2) + 1.$$
- We now assume that the element C_1 is hyperbolic and the element C_2 is parabolic. Up to conjugation we can assume that $\bar{C}_1 = \tau_{\infty,0}(\lambda_1)$ and $\bar{C}_2 = \pi_\alpha(\lambda_2)$ for some $\lambda_1 > 1$, $\lambda_2 > 0$, and $\alpha > 0$. Lemma 3.5 implies that
$$\lambda_2 \alpha \geqslant \frac{\sqrt{\lambda_1} + 1}{\sqrt{\lambda_1} - 1} = \frac{(\sqrt{\lambda_1} + 1)^2}{(\sqrt{\lambda_1} + 1) \cdot (\sqrt{\lambda_1} - 1)} = \frac{\lambda_1 + 1 + 2\sqrt{\lambda_1}}{\lambda_1 - 1} > \frac{\lambda_1 + 1}{\lambda_1 - 1}.$$
According to Lemma 2.6 the inequalities $\alpha > 0$ and
$$\lambda_2 \alpha > \frac{\lambda_1 + 1}{\lambda_1 - 1}$$
imply
$$s_m(C_1 \cdot C_2) = s_m(C_1) + s_m(C_2) + 1.$$
- We now assume that the elements C_1 and C_2 are parabolic. Up to conjugation we can assume that
$$\bar{C}_1 = \pi_\infty(1) = \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right] \quad \text{and} \quad \bar{C}_2 = \pi_1(4) = \left[\begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix}\right].$$
Then according to Lemma 2.7
$$s_m(C_1 \cdot C_2) = s_m(C_1) + s_m(C_2) + 1.$$
- In all three cases we proved that $s_m(C_1 \cdot C_2) = s_m(C_1) + s_m(C_2) + 1$. For the inverse element this implies
$$s_m((C_1 \cdot C_2)^{-1}) = -s_m(C_1 \cdot C_2) = -(s_m(C_1) + s_m(C_2) + 1).$$
The image of the element $C_1 \cdot C_2 \cdot C_3$ is $\bar{C}_1 \cdot \bar{C}_2 \cdot \bar{C}_3 = 1$, hence $C_1 \cdot C_2 \cdot C_3 = [u]^l$ for some $l \in \mathbb{Z}$. This implies
$$s_m(C_3) = s_m((C_1 \cdot C_2)^{-1} \cdot u^l) = l - (s_m(C_1) + s_m(C_2) + 1),$$
hence we obtain that $C_1 \cdot C_2 \cdot C_3 = e$ if and only if $l = 0$, i.e. if and only if
$$s_m(C_1) + s_m(C_2) + s_m(C_3) = -1. \quad \square$$

Lemma 3.8. *Let (A_1, B_1, C_1) be a triple of elements in G_m such that there images $(\bar{A}_1, \bar{B}_1, \bar{C}_1)$ in G form a sequential set of type $(1, 1, 0)$. Then we have $[A_1, B_1]\cdot C_1 = e$ if and only if*

$$s_m(C_1) = -1.$$

Moreover, it holds

$$s_m(A_1 \cdot B_1) = s_m(A_1) + s_m(B_1).$$

Proof. Lemma 3.5 implies that the axes of the hyperbolic elements A_1 and B_1 intersect. According to corollary 2.5 this implies

$$s_m(A_1 \cdot B_1) = s_m(A_1) + s_m(B_1).$$

The triple $(\bar{A}_1, \bar{B}_1, \bar{C}_1)$ is a sequential set of type $(1, 1, 0)$. By the definition of sequential sets the triple $(\bar{A}_1, \bar{B}_1\bar{A}_1^{-1}\bar{B}_1^{-1}, \bar{C}_1)$ is a sequential set of type $(0, 3, 0)$. According to Lemma 3.7 we have

$$[A_1, B_1] \cdot C_1 = A_1 \cdot (B_1A_1^{-1}B_1^{-1}) \cdot C_1 = e$$

if and only if

$$s_m(A_1) + s_m(B_1A_1^{-1}B_1^{-1}) + s_m(C_1) = -1.$$

Since by Lemma 2.3

$$s_m(B_1A_1^{-1}B_1^{-1}) = s_m(A_1^{-1}) = -s_m(A_1),$$

the last condition is equivalent to

$$s_m(C_1) = -1. \quad \square$$

Lemma 3.9. *Let*

$$V = \{A_i, B_i\ (i = 1, \dots, g), C_i\ (i = g + 1, \dots, n)\}$$

be a tuple of elements in G_m such that the image

$$\bar{V} = \{\bar{A}_i, \bar{B}_i\ (i = 1, \dots, g), \bar{C}_i\ (i = g + 1, \dots, n)\}$$

in G form a sequential set. Then we have

$$\prod_{i=1}^{g}[A_i, B_i] \cdot \prod_{i=g+1}^{n} C_i = e$$

if and only if

$$\sum_{i=g+1}^{n} s_m(C_i) = (2 - 2g) - (n - g).$$

(In the case $n = g$ this means $2 - 2g = 0 \mod m$.)

Proof. We discuss the case $g = 0$ first, and then we reduce the general case to the case $g = 0$.

- Let $g = 0$. We prove that the statement is true for lifts of sequential sets of type $(0, l_h, l_p)$ by induction on $l_h + l_p$. The case $l_h + l_p = 3$ is covered by Lemma 3.7. Assume that the statement is true for $l_h + l_p \leqslant n - 1$ and consider the case $l_h + l_p = n$. By the definition of sequential sets the set $(\bar{C}_1 \cdot \bar{C}_2, \bar{C}_3, \dots, \bar{C}_n)$ is a sequential set. Hence by our assumption $(C_1 \cdot C_2) \cdot C_3 \cdots C_n = e$ if and only if

$$s_m(C_1 \cdot C_2) + s_m(C_3) + \dots + s_m(C_n) = 2 - (n - 1) = (2 - n) + 1.$$

Moreover, by the definition of sequential sets the set $(\bar{C}_1, \bar{C}_2, \bar{C}_3 \cdots \bar{C}_n)$ is a sequential set too, hence by Lemma 3.7 we have

$$s_m(C_1 \cdot C_2) = s_m(C_1) + s_m(C_2) + 1.$$

The last two equations imply that

$$C_1 \cdots C_n = e$$

if and only if

$$s_m(C_1) + \cdots + s_m(C_n) = 2 - n.$$

- We now consider the general case. By the definition of sequential sets the set

$$(\bar{A}_1, \bar{B}_1\bar{A}_1^{-1}\bar{B}_1^{-1}, \ldots, \bar{A}_g, \bar{B}_g\bar{A}_g^{-1}\bar{B}_g^{-1}, \bar{C}_1, \ldots, \bar{C}_n)$$

is a sequential set of type $(0, 2g + l_h, l_p)$, hence

$$\prod_{i=1}^{g} [A_i, B_i] \cdot \prod_{i=g+1}^{n} C_i = \prod_{i=1}^{g} (A_i \cdot B_i A_i^{-1} B_i^{-1}) \cdot \prod_{i=g+1}^{n} C_i = e$$

if and only if

$$\sum_{i=1}^{g} (s_m(A_i) + s_m(B_i A_i^{-1} B_i^{-1})) + \sum_{i=g+1}^{n} s_m(C_i) = 2 - (n+g) = (2-2g) - (n-g).$$

From Lemma 2.3 we obtain that

$$s_m(B_i A_i^{-1} B_i^{-1}) = s_m(A_i^{-1}) = -s_m(A_i),$$

and hence

$$s_m(A_i) + s_m(B_i A_i^{-1} B_i^{-1}) = 0. \quad \square$$

4. Higher Arf functions

Let Γ be a Fuchsian group and $P = \mathbb{H}/\Gamma$ of type (g, l_h, l_p) with $l_h + l_p = n - g$. Let $p \in P$. Let $\Psi : \mathbb{H} \to P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \to \pi_1(P, p)$ be the induced isomorphism.

4.1. **Definition of higher Arf functions.**

Let Γ^* be a lift of Γ in G_m.

Definition 4.1. Let us consider a function $\hat{\sigma}_{\Gamma^*} : \pi_1(P, p) \to \mathbb{Z}_m$ such that the following diagram commutes

$$\begin{array}{ccc} \Gamma & \xrightarrow{\cong} & \Gamma^* \\ \Phi \big\downarrow & & \big\downarrow s_m|_{\Gamma^*} \\ \pi_1(P,p) & \xrightarrow{\hat{\sigma}_{\Gamma^*}} & \mathbb{Z}_m \end{array}$$

As for the function s_m, all equations involving $\hat{\sigma}_{\Gamma^*}$ are to be understood as equations in $\mathbb{Z}_m$.

Lemma 4.1. *Let α, β, and γ be simple contours in P intersecting pairwise in exactly one point p. Let a, b, and c be the corresponding elements of $\pi_1(P, p)$. We assume that a, b, and c satisfy the relations $a, b, c \neq 1$ and $abc = 1$. Let $\langle \cdot, \cdot \rangle$ be the intersection form on $\pi_1(P, p)$. Then for $\hat{\sigma} := \hat{\sigma}_{\Gamma^*}$*

1. *$\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b)$ if the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b \rangle \neq 0$,*

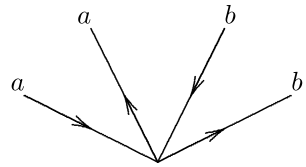

Figure 3: $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) + 1$

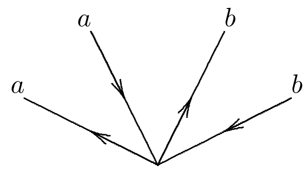

Figure 4: $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) - 1$

2. $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) + 1$ *if the elements* a *and* b *can be represented by a pair of simple contours in* P *intersecting in exactly one point* p *with* $\langle a, b\rangle = 0$ *and placed in a neighbourhood of the point* p *as shown in Figure 3.*
3. $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) - 1$ *if the elements* a *and* b *can be represented by a pair of simple contours in* P *intersecting in exactly one point* p *with* $\langle a, b\rangle = 0$ *and placed in a neighbourhood of the point* p *as shown in Figure 4.*
4. *for any standard basis*

$$v = \{a_i, b_i \ (i = 1, \dots, g), c_i \ (i = g+1, \dots, n)\}$$

of $\pi_1(P, p)$ *it holds*

$$\sum_{i=g+1}^{n} \hat{\sigma}(c_i) = (2 - 2g) - (n - g).$$

Proof. According to Theorem 3.3 either the set

$$V := \{\Phi^{-1}(a), \Phi^{-1}(b), \Phi^{-1}(c)\}$$

or the set

$$V^{-1} := \{\Phi^{-1}(a^{-1}), \Phi^{-1}(b^{-1}), \Phi^{-1}(c^{-1})\}$$

is sequential. This sequential set can be of type $(0, k, m)$ (with $k + m = 3$) or of type $(1, 1, 0)$. If V is a sequential set of type $(1, 1, 0)$, then according to Lemma 3.8 we obtain

$$\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b).$$

If V is a sequential set of type $(0, k, m)$, then according to Lemma 3.7 we obtain

$$\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) + 1.$$

If V^{-1} is a sequential set of type $(0, k, m)$, then according to Lemma 3.7 we obtain

$$\hat{\sigma}(b^{-1}a^{-1}) = \hat{\sigma}(a^{-1}) + \hat{\sigma}(b^{-1}) + 1$$

and hence

$$\begin{aligned}
-\hat{\sigma}(ab) &= \hat{\sigma}((ab)^{-1}) = \hat{\sigma}(b^{-1}a^{-1}) \\
&= \hat{\sigma}(a^{-1}) + \hat{\sigma}(b^{-1}) + 1 \\
&= -\hat{\sigma}(a) - \hat{\sigma}(b) + 1 \\
&= -(\hat{\sigma}(a) + \hat{\sigma}(b) - 1).
\end{aligned}$$

To prove the forth property of $\hat{\sigma}$ we consider the sequential set that corresponds to the standard basis v and apply Lemma 3.9. □

We now formalize the properties of the function $\hat{\sigma}$ in the following definition:

Definition 4.2. We denote by $\pi_1^0(P,p)$ the set of all elements of $\pi_1(P,p)$, which can be represented by simple contours which either do not belong to the kernel of the intersection form or are homologous to a hole or a puncture. A *m-Arf function* is a function $\sigma : \pi_1^0(P,p) \to \mathbb{Z}_m$ satisfying the following conditions

1. $\sigma(bab^{-1}) = \sigma(a)$,
2. $\sigma(a^{-1}) = -\sigma(a)$,
3. $\sigma(ab) = \sigma(a)+\sigma(b)$ if the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b\rangle \neq 0$,
4. $\sigma(ab) = \sigma(a) + \sigma(b) - 1$ if the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b\rangle = 0$ and placed in a neighbourhood of the point p as shown in Figure 4.

As for the function $\hat{\sigma}_{\Gamma^*}$, all equations involving σ are to be understood as equations in $\mathbb{Z}_m$.

Remark. One can prove that in the case $m = 2$ there is a 1-1-correspondence between the 2-Arf functions in the sense of Definition 4.2 and Arf functions in the sense of [Nat04], Chapter 1, Section 7. Namely, a function $\sigma : \pi_1^0(P,p) \to \mathbb{Z}_2$ is a 2-Arf function if and only if $\omega = 1 - \sigma$ is an Arf function in the sense of [Nat04].

The following property of m-Arf functions follows immediately from Properties 4 and 2 in Definition 4.2:

Proposition 4.2. *It holds $\sigma(ab) = \sigma(a) + \sigma(b) + 1$, if the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b\rangle = 0$ and placed in a neighbourhood of the point p as shown in Figure 3.*

Proposition 4.3. *For any standard basis*

$$v = \{a_i, b_i \ (i = 1, \ldots, g), c_i \ (i = g+1, \ldots, n)\}$$

of $\pi_1(P,p)$ it holds

$$\sum_{i=g+1}^{n} \sigma(c_i) = (2 - 2g) - (n - g).$$

Proof. We discuss the case $g = 0$ first, and then we reduce the general case to the case $g = 0$.

- Let $g = 0$. We prove that the statement is true for lifts of sequential sets of type $(0, l_h, l_p)$ by induction on $l_h + l_p$. In the case $l_h + l_p = 3$ Proposition 4.2 implies

$$\sigma(c_1c_2) = \sigma(c_1) + \sigma(c_2) + 1,$$

while Property 2 implies

$$\sigma(c_1c_2) = \sigma(c_3^{-1}) = -\sigma(c_3).$$

Combining these two equations we obtain

$$\sigma(c_1) + \sigma(c_2) + \sigma(c_3) = -1.$$

Assume that the statement is true for $l_h + l_p \leqslant n - 1$ and consider the case $l_h + l_p = n$. By our assumption

$$\sigma(c_1 \cdot c_2) + \sigma(c_3) + \cdots + \sigma(c_n) = 2 - (n-1) = (2-n) + 1.$$

Moreover, by Proposition 4.2 it holds

$$\sigma(c_1c_2) = \sigma(c_1) + \sigma(c_2) + 1,$$

The last two equations imply that

$$\sum_{i=1}^{n} \sigma(c_i) = 2 - n.$$

- We now consider the general case. The set

$$(a_1, b_1a_1^{-1}b_1^{-1}, \dots, a_g, b_ga_g^{-1}b_g^{-1}, c_1, \dots, c_n)$$

is a standard basis of a surface of type $(0, 2g + l_h, l_p)$, hence

$$\sum_{i=1}^{g} (\sigma(a_i) + \sigma(b_ia_i^{-1}b_i^{-1})) + \sum_{i=g+1}^{n} \sigma(c_i) = 2 - (n+g) = (2-2g) - (n-g).$$

From Properties 1 and 2 of m-Arf functions we obtain that

$$\sigma(b_ia_i^{-1}b_i^{-1}) = \sigma(a_i^{-1}) = -\sigma(a_i),$$

and hence

$$\sigma(a_i) + \sigma(b_ia_i^{-1}b_i^{-1}) = 0. \quad \square$$

Definition 4.3. Let $\hat{\sigma}_{\Gamma^*} : \pi_1(P,p) \to \mathbb{Z}_m$ be the function associated to a lift Γ^* as in definition 4.1, then the function $\sigma_{\Gamma^*} := \hat{\sigma}_{\Gamma^*}|_{\pi_1^0(P,p)}$ is an m-Arf function according to Lemma 4.1, 2.2, and 2.3. We call the function σ_{Γ^*} the *m-Arf function associated to the lift* Γ^*.

4.2. **Higher Arf functions and Dehn twists.** We recall the definition of the Dehn twists ([Deh39], [Nat04] Chapter 1, Lemma 7.4).

Definition 4.4. Under Dehn twists we understand some transformations from a standard basis

$$v = \{a_i, b_i \ (i = 1, \dots, g), c_i \ (i = g+1, \dots, n)\}$$

of $\pi_1(P,p)$ to another standard basis

$$v' = \{a_i', b_i' \ (i = 1, \dots, g), c_i' \ (i = g+1, \dots, n)\}.$$

Any of these transformations induces a homotopy class of autohomeomorphisms of the surface P which map holes to holes and punctures to punctures. By Theorem of

Dehn [Deh39] the Dehn twists generate the whole group of such homotopy classes of autohomeomorphisms of P. There are Dehn twists of the following types:

1. $a_1' = a_1 b_1$.
2. $a_1' = (a_1 a_2) a_1 (a_1 a_2)^{-1}$,
   $b_1' = (a_1 a_2) a_1^{-1} b_1 a_2^{-1} (a_1 a_2)^{-1}$,
   $a_2' = a_1 a_2 a_1^{-1}$,
   $b_2' = b_2 a_2^{-1} a_1^{-1}$.
3. $a_g' = (b_g^{-1} c_1) b_g^{-1} (b_g^{-1} c_1)^{-1}$,
   $b_g' = (b_g^{-1} c_1 b_g) c_1^{-1} b_g a_g b_g^{-1} (b_g^{-1} c_1 b_g)^{-1}$,
   $c_1' = b_g^{-1} c_1 b_g$.
4. $a_k' = a_{k+1}$,
   $b_k' = b_{k+1}$,
   $a_{k+1}' = (c_{k+1}^{-1} c_k) a_k (c_{k+1}^{-1} c_k)^{-1}$,
   $b_{k+1}' = (c_{k+1}^{-1} c_k) b_k (c_{k+1}^{-1} c_k)^{-1}$.
5. $c_k' = c_{k+1}$,
   $c_{k+1}' = c_{k+1}^{-1} c_k c_{k+1}$.

Here $c_i = [a_i, b_i]$ for $i = 1, \dots, g$, in 4 we consider $k \in \{1, \dots, g\}$, in 5 we consider $k \in \{g+1, \dots, n\}$. If a_i', b_i' resp. c_i' is not described explicitly, this means $a_i' = a_i$, $b_i' = b_i$ resp. $c_i' = c_i$.

Now we compute the values of σ on the standard basis v' from the values of σ on the standard basis v.

Lemma 4.4. *Let $\sigma : \pi_1^0(P, p) \to \mathbb{Z}_m$ be an m-Arf function and D a Dehn twist of the types described above, which maps the standard basis*

$$v = \{a_i, b_i \ (i = 1, \dots, g), c_i \ (i = g+1, \dots, n)\}$$

into the standard basis

$$v' = D(v) = \{a_i', b_i' \ (i = 1, \dots, g), c_i' \ (i = g+1, \dots, n)\}.$$

Let

$$\{\alpha_i, \beta_i \ (i = 1, \dots, g), \gamma_i \ (i = g+1, \dots, n)\}$$

resp.

$$\{\alpha_i', \beta_i' \ (i = 1, \dots, g), \gamma_i' \ (i = g+1, \dots, n)\}$$

be the values of σ on the elements of v resp. v'. Then for the Dehn twists of types 1–5 we obtain

1. $\alpha_1' = \alpha_1 + \beta_1$.
2. $\beta_1' = \beta_1 - \alpha_1 - \alpha_2 - 1$,
   $\beta_2' = \beta_2 - \alpha_2 - \alpha_1 - 1$.
3. $\alpha_g' = -\beta_g$,
   $\beta_g' = \alpha_g - \gamma_1 - 1$.
4. $\alpha_k' = \alpha_{k+1}$,
   $\beta_k' = \beta_{k+1}$,
   $\alpha_{k+1}' = \alpha_k$,
   $\beta_{k+1}' = \beta_k$.
5. $\gamma_k' = \gamma_{k+1}$,
   $\gamma_{k+1}' = \gamma_k$.

Proof. We assume that the Dehn twist D belongs to one of the types described in the definition above. In the first case according to Property 3 of m-Arf functions we obtain

$$\sigma(a_1') = \sigma(a_1 b_1) = \sigma(a_1) + \sigma(b_1).$$

In the second case according to Property 1 we obtain

$$\begin{aligned}
\sigma(a_1') &= \sigma((a_1 a_2) a_1 (a_1 a_2)^{-1}) = \sigma(a_1),\\
\sigma(b_1') &= \sigma((a_1 a_2) a_1^{-1} b_1 a_2^{-1} (a_1 a_2)^{-1}) = \sigma(a_1^{-1} b_1 a_2^{-1}),\\
\sigma(a_2') &= \sigma(a_1 a_2 a_1^{-1}) = \sigma(a_2),\\
\sigma(b_2') &= \sigma(b_2 a_2^{-1} a_1^{-1})
\end{aligned}$$

According to Properties 3 and 2 we obtain

$$\begin{aligned}
\sigma(a_1^{-1} b_1) &= \sigma(a_1^{-1}) + \sigma(b_1) = -\sigma(a_1) + \sigma(b_1),\\
\sigma(b_2 a_2^{-1}) &= \sigma(b_2) + \sigma(a_2^{-1}) = \sigma(b_2) - \sigma(a_1).
\end{aligned}$$

In the following computations we illustrate the position of the contours on the surface with figures showing the position of the axes of the corresponding elements in Γ. Let

$$V = \{A_i, B_i \ (i = 1, \dots, g), C_i \ (i = g+1, \dots, n)\}$$

be the sequential set corresponding to the standard basis v.

The mutual position of the axes of the elements $A_1^{-1} B_1$ and A_2^{-1} is as in Figure 5.

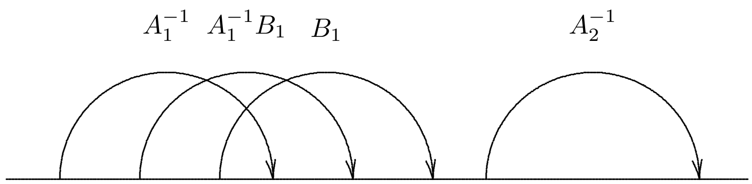

Figure 5: Axes of $A_1^{-1}B_1$ and A_2^{-1}

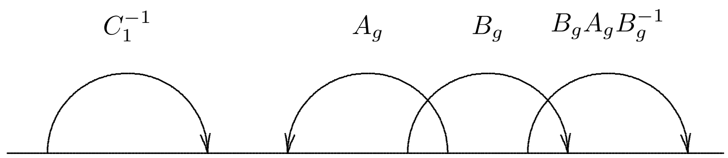

Figure 6: Axes of C_1^{-1} and $B_gA_gB_g^{-1}$

According to Properties 4 and 2 we obtain

$$\begin{aligned}\sigma(b_1') &= \sigma((a_1^{-1}b_1)a_2^{-1})\\ &= \sigma(a_1^{-1}b_1) + \sigma(a_2^{-1}) - 1\\ &= (\sigma(b_1) - \sigma(a_1)) - \sigma(a_2) - 1\\ &= \sigma(b_1) - \sigma(a_1) - \sigma(a_2) - 1.\end{aligned}$$

Similarly

$$\sigma(b_2') = \sigma(b_2) - \sigma(a_2) - \sigma(a_1) - 1.$$

In the third case we obtain according to Properties 2 and 1

$$\begin{aligned}&\sigma(a_g') = \sigma((b_g^{-1}c_1)b_g^{-1}(b_g^{-1}c_1)^{-1}) = \sigma(b_g^{-1}) = -\sigma(b_g),\\ &\sigma(b_g') = \sigma((b_g^{-1}c_1b_g)c_1^{-1}b_ga_gb_g^{-1}(b_g^{-1}c_1b_g)^{-1}) = \sigma(c_1^{-1}b_ga_gb_g^{-1}),\\ &\sigma(c_1') = \sigma(b_g^{-1}c_1b_g) = \sigma(c_1).\end{aligned}$$

The mutual position of the axes of the elements C_1^{-1} and $B_gA_gB_g^{-1}$ is as in Figure 6. According to Properties 4 and 1 we obtain

$$\begin{aligned}\sigma(b_g') &= \sigma(c_1^{-1}\cdot(b_ga_gb_g^{-1}))\\ &= \sigma(c_1^{-1}) + \sigma(b_ga_gb_g^{-1}) - 1\\ &= \sigma(c_1^{-1}) + \sigma(a_g) - 1.\end{aligned}$$

In the forth and fifth case computations are easy, we only use Property 1 of m-Arf functions. $\square$

4.3. **Correspondence between higher Arf functions and co-spin structures.**

Lemma 4.5. *The difference $\sigma_1 - \sigma_2 : \pi_1^0(P,p) \to \mathbb{Z}_m$ of two Arf functions σ_1 and σ_2 induces a linear function $\ell : H_1(P;\mathbb{Z}_m) \to \mathbb{Z}_m$. The set $\mathrm{Arf}^{P,m}$ of all m-Arf functions on $\pi_1^0(P,p)$ has a structure of an affine space, i.e. the set $\{\sigma - \sigma_0 \bigm| \sigma \in \mathrm{Arf}^{P,m}\}$ is a free module of rang $n+g-\chi$ over $\mathbb{Z}_m$ for any $\sigma_0 \in \mathrm{Arf}^{P,m}$, where $\chi = 1$ if $n > g$ and $\chi = 0$ else.*

Proof. Let

$$\begin{aligned} v &= \{a_i, b_i \ (i = 1,\dots,g), c_i \ (i = g+1,\dots,n)\} \\ &= \{d_j \ (j = 1,\dots,n+g)\} \end{aligned}$$

be a standard basis of $\pi_1(P,p)$. For an element $a \in \pi_1(P,p)$ let us denote by $[a]$ the image of a in $H_1(P;\mathbb{Z}_m)$. Let

$$\begin{aligned} [v] &= \{[a_i], [b_i] \ (i = 1,\dots,g), [c_i] \ (i = g+1,\dots,n)\} \\ &= \{[d_j] \ (j = 1,\dots,n+g)\} \end{aligned}$$

be the induced basis of $H_1(P;\mathbb{Z}_m)$. Let us define $\delta : \pi_1^0(P,p) \to \mathbb{Z}_m$ by $\delta = \sigma_1 - \sigma_2$. Let us define a function $\ell : H_1(P;\mathbb{Z}_m) \to \mathbb{Z}_m$ on $[v]$ by

$$\ell[d_j] = \delta(d_j)$$

for all $j = 1,\dots,n+g$ and then extend this function linear on the whole $H_1(P;\mathbb{Z}_m)$. (Here we write $\ell[c]$ instead of $\ell([c])$ to simplify the notation.)

Claim: We claim that $\ell[a] = \delta(a)$ for any element $a \in \pi_1^0(P,p)$.

Observation: We first observe that if a standard basis

$$\begin{aligned} v' &= \{a'_i, b'_i \ (i = 1,\dots,g), c'_i \ (i = g+1,\dots,n)\} \\ &= \{d'_j \ (j = 1,\dots,n+g)\} \end{aligned}$$

is the image of the standard basis v under a Dehn twist, then

$$\ell[d'_j] = \delta(d'_j)$$

for all $j = 1,\dots,n+g$. Indeed, for the Dehn twists of type 1, 2, and 3 we obtain by Lemma 4.4 and by linearity of ℓ

1. $\delta(a'_1) = \delta(a_1) + \delta(b_1)$,
   $\ell[a'_1] = \ell[a_1 b_1] = \ell[a_1] + \ell[b_1]$,
2. $\delta(b'_1) = \delta(b_1) - \delta(a_1) - \delta(a_2)$,
   $\ell[b'_1] = \ell[(a_1 a_2) a_1^{-1} b_1 a_2^{-1} (a_1 a_2)^{-1}] = \ell[b_1] - \ell[a_1] - \ell[a_2]$,
   $\delta(b'_2) = \delta(b_2) - \delta(a_2) - \delta(a_1)$,
   $\ell[b'_2] = \ell[b_2 a_2^{-1} a_1^{-1}] = \ell[b_2] - \ell[a_1] - \ell[a_2]$,
3. $\delta(a'_g) = -\delta(b_g)$,
   $\ell[a'_g] = \ell[(b_g^{-1} c_1) b_g^{-1} (b_g^{-1} c_1)^{-1}] = -\ell[b_g]$,
   $\delta(b'_g) = \delta(a_g) - \delta(c_1)$,
   $\ell[b'_g] = \ell[(b_g^{-1} c_1 b_g) c_1^{-1} b_g a_g b_g^{-1} (b_g^{-1} c_1 b_g)^{-1}] = \ell[a_g] - \ell[c_1]$.

Here if $\delta(d'_j)$ and $\ell[d'_j]$ are not described explicitly, this means $\delta(d'_j) = \delta(d_j)$ and $\ell[d'_j] = \ell[d_j]$.

Let a be any contour not belonging to the kernel of the intersection form. Then a is an element a''_1 in some standard basis v'' of $\pi_1(P,p)$. By Theorem of Dehn [Deh39] the standard basis v'' of $\pi_1(P,p)$ can be obtained from v as an image under a sequence of Dehn twists. Then the observation implies that $\delta(a) = \ell[a]$. Similarly any simple contour homologous to a hole or a puncture is an element c_{g+1} in some standard basis of $\pi_1(P,p)$, hence $\delta(c) = \ell[c]$. So we have proved our claim. Hence the function $\delta = \sigma_1 - \sigma_2$ induces a linear function on $H_1(P;\mathbb{Z}_m)$.

On the contrary, if σ_0 is an m-Arf function on P and ℓ is a linear function on $H_1(P;\mathbb{Z}_m)$, then the function $\sigma : \pi_1^0(P,p) \to \mathbb{Z}_m$ defined by

$$\sigma(a) = \sigma_0(a) + \ell[a]$$

is an m-Arf function on P. Using the properties of the m-Arf function σ_0 and linearity of the function ℓ we can prove that the function σ also has the properties of m-Arf functions. For example, we show Property 4: If $\langle c_1, c_2\rangle = 0$ and the elements c_1 and c_2 can be represented by a pair of simple contours in P intersecting in exactly one point p and placed in a neighbourhood of the point p as is shown in Figure 4, then

$$\sigma(c_1c_2) = \sigma(c_1) + \sigma(c_2) - 1.$$

Linearity of the function ℓ and Property 4 for σ_0 imply

$$\begin{aligned}\sigma(c_1c_2) &= \sigma_0(c_1c_2) + \ell[c_1c_2]\\ &= (\sigma_0(c_1) + \sigma_0(c_2) - 1) + (\ell[c_1] + \ell[c_2])\\ &= \sigma(c_1) + \sigma(c_2) - 1.\end{aligned}$$

The proofs of the other properties are similar. □

Corollary 4.6. *An m-Arf function is uniquely determined by its values on the elements of some standard basis of $\pi_1(P,p)$.*

Corollary 4.7. *For a surface of type (g, l_h, l_p) there are $m^{2g+l_h+l_p-\chi}$ different m-Arf functions on $\pi_1^0(P,p)$, where $\chi = 1$ in the case $l_h + l_p > 0$ and $\chi = 0$ in the case $l_h + l_p = 0$ and $2g - 2 = 0 \mod m$. In the case $l_h + l_p = 0$ and $2g - 2 \neq 0 \mod m$ there are no m-Arf functions on $\pi_1^0(P,p)$.*

Theorem 4.8. *There is a 1-1-correspondence between*

1) m-co-spin structures on $P = \mathbb{H}/\Gamma$.
2) lifts of Γ into G_m,
3) m-Arf functions $\sigma : \pi_1^0(P,p) \to \mathbb{Z}_m$.

Proof. According to Proposition 3.1 there is a 1-1-correspondence between m-co-spin structures on P and the lifts of Γ into G_m. In Definition 4.3 we attached to any lift Γ^* of Γ into G_m an m-Arf function σ_{Γ^*}. On the other hand we can attach to any m-Arf function σ a subset of G_m

$$\Gamma^*_\sigma := \{g \in G_m \mid \pi(g) \in \Gamma,\ s_m(g) = \sigma(\Phi(\pi(g)))\},$$

where $\pi : G_m \to G$ is the covering map. It remains to prove that this subset of G_m is actually a lift of Γ. Let

$$\begin{aligned} v &= \{a_i, b_i \ (i = 1, \dots, g), c_i \ (i = g+1, \dots, n)\} \\ &= \{d_j \ (j = 1, \dots, n+g)\} \end{aligned}$$

be a standard basis of $\pi_1(P, p)$ and let

$$\begin{aligned} \bar{V} &= \{\bar{A}_i, \bar{B}_i \ (i = 1, \dots, g), \bar{C}_i \ (i = g+1, \dots, n)\} \\ &= \{\bar{D}_j \ (j = 1, \dots, n+g)\} \\ &= \{\Phi^{-1}(d_j) \ (j = 1, \dots, n+g)\} \end{aligned}$$

be the corresponding sequential set. Let

$$\begin{aligned} V &= \{A_i, B_i \ (i = 1, \dots, g), C_i \ (i = g+1, \dots, n)\} \\ &= \{D_j \ (j = 1, \dots, n+g)\} \end{aligned}$$

be a lift of the sequential set $\bar{V}$, i.e. $\pi(D_j) = \bar{D}_j$, such that $s_m(D_j) = \sigma(d_j)$. Then we obtain

$$\sum_{i=g+1}^{n} s_m(C_i) = \sum_{i=g+1}^{n} \sigma(c_i) = (2 - 2g) - (n - g),$$

hence by Lemma 3.9 it holds

$$\prod_{i=1}^{g} [A_i, B_i] \cdot \prod_{i=g+1}^{n} C_i = e.$$

This implies according to Lemma 3.6 that the subgroup Γ^* of G_m generated by V is a lift of Γ into G_m. Let us compare the corresponding Arf function σ_{Γ^*} with the Arf function σ. It holds for all j

$$\sigma_{\Gamma^*}(d_j) = s_m(D_j) = \sigma(d_j),$$

i.e. the Arf functions σ_{Γ^*} and σ coincide on the standard basis v. Thus by Corollary 4.6 the Arf functions σ_{Γ^*} and σ coincide on the whole $\pi_1^0(P, p)$. From the definition of σ_{Γ^*} and Γ^*_σ we see that this implies that $\Gamma^* = \Gamma^*_\sigma$, hence Γ^*_σ is indeed a lift of Γ into G_m. It is clear from the definitions that the mappings $\Gamma^* \mapsto \sigma_{\Gamma^*}$ and $\sigma \mapsto \Gamma^*_\sigma$ are inverse to each other. □

Remark. Let us define an ∞-*Arf function* as a function $\sigma : \pi_1^0(P, p) \to \mathbb{Z}$ satisfying the four properties in Definition of m-Arf functions (Definition 4.2). Then slightly modifying the previous discussion of m-Arf functions we obtain similar results for ∞-Arf functions:

1) The set $\mathrm{Arf}^{P,\infty}$ of all ∞-Arf functions on $\pi_1^0(P, p)$ has a structure of an affine space associated to $H^1(P; \mathbb{Z})$.
2) For a surface $P = \mathbb{H}/\Gamma$ there is a 1-1-correspondence between ∞-Arf functions $\sigma : \pi_1^0(P, p) \to \mathbb{Z}$ and lifts of the group Γ into the universal cover $\tilde{G}$ of $\mathrm{PSL}(2, \mathbb{R})$.

Corollary 4.9. *For a surface P of type (g, l_h, l_p) there are $m^{2g+l_h+l_p-\chi}$ different m-co-spin structures on P, where $\chi = 1$ in the case $l_h + l_p > 0$ and $\chi = 0$ in the case $l_h + l_p = 0$ and $2g - 2 = 0 \mod m$. In the case $l_h + l_p = 0$ and $2g - 2 \neq 0 \mod m$ there are no m-co-spin structures on P. The set of all m-co-spin structures on P has a structure of an affine space.*

5. Moduli spaces of higher co-spin structures

5.1. **Topological classification of higher Arf functions.** In Theorem 4.8 we put m-co-spin structures on P in 1-1-correspondence with m-Arf functions $\sigma : \pi_1^0(P,p) \to \mathbb{Z}_m$. This correspondence allows us to reduce the problem of finding the number of connected components of the moduli space of higher spinor Riemann surfaces to the problem of finding the number of orbits of the action of the group of autohomeomorphisms on the set of m-Arf functions. We describe the orbit of an m-Arf function under the action of the modular group of the corresponding space of Riemann surfaces.

Let P be a Riemann surface of type (g, l_h, l_p) with $l_h + l_p = n - g$. Let $p \in P$.

Definition 5.1. Let $\sigma : \pi_1^0(P,p) \to \mathbb{Z}_m$ be an m-Arf function. For even m we define the *Arf invariant* $\delta = \delta(P,\sigma)$ as $\delta = 0$ if there is a standard basis

$$\{a_i, b_i \ (i = 1, \dots, g), c_i \ (i = g+1, \dots, n)\}$$

of the fundamental group $\pi_1(P,p)$ such that

$$\sum_{i=1}^{g} (1 - \sigma(a_i))(1 - \sigma(b_i)) = 0 \mod 2$$

and as $\delta = 1$ else. For odd m we set $\delta = 0$.

Definition 5.2. Let $\sigma : \pi_1^0(P,p) \to \mathbb{Z}_m$ be an m-Arf function and

$$v = \{a_i, b_i \ (i = 1, \dots, g), c_i \ (i = g+1, \dots, n)\}$$

a standard basis of the fundamental group $\pi_1(P,p)$. In the standard basis v the elements $c_{g+1}, \dots, c_{g+l_h}$ correspond to the holes and the elements $c_{g+l_h+1}, \dots, c_n$ correspond to the punctures. The set of holes

$$c_{g+1}, \dots, c_{g+l_h}$$

is divided into m sets

$$D_j = \{c_i \mid \sigma(c_i) = j, \ g < i \leqslant g + l_h\}.$$

We denote by $n_j^h = n_j^h(P,\sigma)$, $j \in \mathbb{Z}_m$, the cardinality of the set D_j. Similarly, the punctures

$$c_{g+l_h+1}, \dots, c_n$$

are divided into m sets of the form

$$E_j = \{c_i \mid \sigma(c_i) = j, \ g + l_h < i \leqslant n\}.$$

Let $n_j^p = n_j^p(P,\sigma)$, $j \in \mathbb{Z}_m$, be the cardinality of the set E_j. It holds

$$n_0^h + \dots + n_{m-1}^h = l_h \quad \text{and} \quad n_0^p + \dots + n_{m-1}^p = l_p.$$

We set $n_j = n_j^h + n_j^p$. By the *type of the m-Arf function* (P,σ) we mean the tuple

$$(g, \delta, n_0^h, \dots, n_{m-1}^h, n_0^p, \dots, n_{m-1}^p),$$

where δ is the Arf invariant of σ defined above.

Lemma 5.1. *Let $\sigma : \pi_1^0(P,p) \to \mathbb{Z}_m$ be an m-Arf function, then there is a standard basis*

$$v = \{a_i, b_i\ (i = 1, \dots, g), c_i\ (i = g+1, \dots, n)\}$$

of $\pi_1(P,p)$ such that

$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, \xi, 1, \dots, 1)$$

with $\xi \in \{0,1\}$ and

$$\sigma(c_{g+1}) \leqslant \dots \leqslant \sigma(c_{g+l_h}), \quad \sigma(c_{g+l_h+1}) \leqslant \dots \leqslant \sigma(c_n).$$

If m is odd or there is a contour around a hole or a puncture such that the value of σ on this contour is even, then the basis can be chosen in such a way that $\xi = 1$.

Proof.

a) Let us fix some standard basis

$$v_0 = \{a_i, b_i\ (i = 1, \dots, g), c_i\ (i = g+1, \dots, n)\}$$

and consider the sequence of values of the Arf function σ on this basis v_0

$$\begin{aligned}&(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_{g+1}, \dots, \gamma_n)\\ &= (\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g), \sigma(c_{g+1}), \dots, \sigma(c_n)).\end{aligned}$$

Any other standard basis v is an image of the basis v_0 under an autohomeomorphisms of the surface, i.e. under a sequence of Dehn twist. Hence according to Lemma 4.4 the sequence of values of σ on the basis v is the image of the corresponding sequence with respect to the basis v_0 under the group generated by the transformations, which change the first $2g$ components $(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ as follows

1a. $(\alpha_1, \beta_1, \dots, \alpha_i, \beta_i, \dots, \alpha_g, \beta_g)$
$\mapsto (\alpha_1, \beta_1, \dots, \alpha_i \pm \beta_i, \beta_i, \dots, \alpha_g, \beta_g)$,

1b. $(\alpha_1, \beta_1, \dots, \alpha_i, \beta_i, \dots, \alpha_g, \beta_g)$
$\mapsto (\alpha_1, \beta_1, \dots, \alpha_i, \beta_i \pm \alpha_i, \dots, \alpha_g, \beta_g)$,

2. $(\alpha_1, \beta_1, \dots, \alpha_i, \beta_i, \dots, \alpha_j, \beta_j, \dots, \alpha_g, \beta_g)$
$\mapsto (\alpha_1, \beta_1, \dots, \alpha_i, \beta_i - \alpha_j - 1, \dots, \alpha_j, \beta_j - \alpha_i - 1, \dots, \alpha_g, \beta_g)$,

3. $(\alpha_1, \beta_1, \dots, \alpha_i, \beta_i, \dots, \alpha_g, \beta_g)$
$\mapsto (\alpha_1, \beta_1, \dots, -\beta_i, \alpha_i - \gamma_j - 1, \dots, \alpha_g)$,

4. $(\alpha_1, \beta_1, \dots, \alpha_i, \beta_i, \dots, \alpha_j, \beta_j, \dots, \alpha_g, \beta_g)$
$\mapsto (\alpha_1, \beta_1, \dots, \alpha_j, \beta_j, \dots, \alpha_i, \beta_i, \dots, \alpha_g, \beta_g)$,

5a. $(\alpha_1, \beta_1, \dots, \alpha_i, \beta_i, \dots, \alpha_g, \beta_g)$
$\mapsto (\alpha_1, \beta_1, \dots, -\alpha_i, -\beta_i, \dots, \alpha_g, \beta_g)$,

5b. $(\alpha_1, \beta_1, \dots, \alpha_i, \beta_i, \dots, \alpha_g, \beta_g)$
$\mapsto (\alpha_1, \beta_1, \dots, -\beta_i, \alpha_i, \dots, \alpha_g, \beta_g)$

and change $(\gamma_{g+1}, \dots, \gamma_n)$ by all possible permutations of $(\gamma_{g+1}, \dots, \gamma_{g+l_h})$ and $(\gamma_{g+l_h+1}, \dots, \gamma_n)$. The inequalities between the values of σ on the elements c_i are easy to satisfy, because the transformation group contains all possible permutations of $(\gamma_{g+1},\dots,\gamma_{g+l_h})$ and $(\gamma_{g+l_h+1},\dots,\gamma_n)$. Our aim is to show that, if m is odd or one of the numbers $\gamma_{g+1},\dots,\gamma_n$ is even, any tuple $(\alpha_1,\beta_1,\dots,\alpha_g,\beta_g)$ can be transformed into the $2g$-tuple

$$(0,1,1,\dots,1),$$

while otherwise any such tuple can be transformed into one of the tuples

$$(0,0,1,\dots,1) \quad \text{or} \quad (0,1,1,\dots,1).$$

b) We claim that the group of transformations described in (a) contains the transformation of the form

$$(\dots,\alpha_i,\beta_i,\dots,\alpha_j,\beta_j,\dots) \mapsto (\dots,\alpha_i,\beta_i-2,\dots,\alpha_j,\beta_j,\dots):$$

Let us assume $(i,j)=(1,2)$ in order to simplify the notation. We apply transformations 2, 5a, again 2 and again 5a and obtain

$$\begin{aligned}
&(\alpha_1,\beta_1,\alpha_2,\beta_2,\dots)\\
&\mapsto (\alpha_1,\beta_1-\alpha_2-1,\alpha_2,\beta_2-\alpha_1-1,\dots)\\
&\mapsto (\alpha_1,\beta_1-\alpha_2-1,\alpha_2,\beta_2-\alpha_1-1,\dots)\\
&\mapsto (\alpha_1,\beta_1-\alpha_2-1,-\alpha_2,-(\beta_2-\alpha_1-1),\dots)\\
&\mapsto (\alpha_1,\beta_1-\alpha_2-1-(-\alpha_2)-1,-\alpha_2,-\beta_2+\alpha_1+1-\alpha_1-1,\dots)\\
&= (\alpha_1,\beta_1-2,-\alpha_2,-\beta_2,\dots)\\
&\mapsto (\alpha_1,\beta_1-2,\alpha_2,\beta_2,\dots).
\end{aligned}$$

c) Furthermore, we claim that the group of transformations contains a transformation of the form

$$(\alpha_1,\beta_1,\alpha_2,\beta_2,\dots)\mapsto(0,\xi,1,1,\dots), \quad \text{where} \quad \xi\in\{0,1\}:$$

With the help of the transformation described in (b) we can transform

$$(\alpha_1,\beta_1,\alpha_2,\beta_2,\dots)\mapsto(\alpha_1',\beta_1',\alpha_2',\beta_2',\dots), \quad \text{where} \quad \alpha_1',\beta_1',\alpha_2',\beta_2'\in\{0,1\}.$$

Applying the inverse of transformation 2 we obtain

$$(0,0,0,0,\dots)\mapsto(0,1,0,1,\dots).$$

Applying transformation 1a resp. 1b we obtain

$$(0,1,\dots)\mapsto(1,1,\dots) \quad \text{and} \quad (1,0,\dots)\mapsto(1,1,\dots)$$

d) By successive application of the transformation described in (c) we obtain that any $2g$-tuple can be transformed to a tuple of the form $(0,\xi,1,\dots,1)$.

e) If $m=2r+1$ is odd, then we use the transformation from (b) to map

$$\begin{aligned}
(0,1,1,\dots,1) &\mapsto (0,1-2\cdot(r+1),1,\dots,1)\\
&= (0,-m,1,\dots,1)=(0,0,1,\dots,1),
\end{aligned}$$

hence the $2g$-tuple $(0,0,1,\dots,1)$ can be transformed to the tuple $(0,1,1,\dots,1)$.

f) If $\gamma_i = 2r$ is even for some i, then we use the transformation 3 and the transformation from (b) to map

$$(0,0,1,\dots,1) \mapsto (0,0-2r-1,1,\dots,1) \mapsto (0,1,1,\dots,1),$$

hence also in this case the $2g$-tuple $(0,0,1,\dots,1)$ can be transformed to the tuple $(0,1,1,\dots,1)$. □

Theorem 5.2. *A tuple* $t = (g,\delta,n^h_0,\dots,n^h_{m-1},n^p_0,\dots,n^p_{m-1})$*, where*

$$n^h_0 + \dots + n^h_{m-1} = l_h \quad \text{and} \quad n^p_0 + \dots + n^p_{m-1} = l_p,$$

is the type of a hyperbolic m-Arf function on a Riemann surface of type (g,l_h,l_p) if and only if

1) $\sum\limits_{j\in\mathbb{Z}_m} j\cdot n_j = (2-2g)-(n-g)$*, where* $n_j = n^h_j + n^p_j$*,*

2) if m is odd or $n_j \neq 0$ for some even $j \in \mathbb{Z}$, then $\delta = 0$.

Proof. Let us first assume that the tuple t is the type of a hyperbolic m-Arf function on a Riemann surface of type (g,l_h,l_p) and that σ is such a function. Then Proposition 4.3 implies that

$$(2-2g)-(n-g) = \sum_{i=g+1}^{n} \sigma(c_i) = \sum_{j\in\mathbb{Z}_m} j\cdot n_j.$$

If m is odd or $n_j \neq 0$ for some even $j \in \mathbb{Z}$, then according to Lemma 5.1 there is a standard basis

$$v = \{a_i, b_i \ (i=1,\dots,g), c_i \ (i=g+1,\dots,n)\}$$

of $\pi_1(P,p)$ such that

$$(\sigma(a_1),\sigma(b_1),\dots,\sigma(a_g),\sigma(b_g)) = (0,1,1,\dots,1),$$

hence $\delta(P,\sigma) = 0$ by definition. Now let us assume that $t = (g,\delta,n^h_j,n^p_j)$ is a tuple satisfying the conditions (1) and (2). Let us fix some standard basis

$$v_0 = \{a_i, b_i \ (i=1,\dots,g), c_i \ (i=g+1,\dots,n)\}$$

The condition (1) together with Proposition 4.3 and Corollary 4.6 imply that there exist Arf functions σ^0 and σ^1 such that $n^h_j(\sigma^\delta) = n^h_j$, $n^p_j(\sigma^\delta) = n^p_j$ for $j \in \mathbb{Z}_m$ and

$$(\sigma^\delta(a_1),\sigma^\delta(b_1),\dots,\sigma^\delta(a_g),\sigma^\delta(b_g)) = (0,1-\delta,1,\dots,1).$$

It holds $\delta(\sigma^0) = 0$ by definition. It remains to prove, in the case that m is even and all γ_i are odd, that $\delta(\sigma^1) = 1$. To this end we observe, using the explicit description of the action of Dehn twists on m-Arf functions in Lemma 4.4, that the parity of the sum

$$\sum_{i=1}^{g}(1-\sigma(a_i))(1-\sigma(b_i))$$

is preserved under the Dehn twists and hence is equal 1 modulo 2 for any standard basis. □

5.2. **Connected components of the moduli space.** We recall the results in [Nat04] on the moduli space of Riemann surfaces.

Let $\Gamma_{g,n}$ be the group generated by the elements

$$v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_n\}$$

with a single defining relation

$$\prod_{i=1}^{g} [a_i, b_i] \prod_{i=g+1}^{n} c_i = 1.$$

We denote by T_{g,l_h,l_p} the set of monomorphisms $\psi : \Gamma_{g,n} \to \mathrm{Aut}(\mathbb{H})$ such that

$$\psi(v) = \{a_i^\psi, b_i^\psi \ (i = 1, \dots, g), c_i^\psi \ (i = g+1, \dots, n)\}$$

is a sequential set of type (g, l_h, l_p). Here $l_h + l_p = n - g$ and we assume that $6g + 3l_h + 2l_p > 6$.

The group $\mathrm{Aut}(\mathbb{H})$ acts on $\tilde{T}_{g,l_h,l_p}$ by conjugation. We set

$$T_{g,l_h,l_p} = \tilde{T}_{g,l_h,l_p} / \mathrm{Aut}(\mathbb{H}).$$

We parametrise the space $\tilde{T}_{g,l_h,l_p}$ by the fixed points and shift parameters of the elements of the sequential sets $\psi(v)$. We use here the following result similar to Theorem of Fricke and Klein [FK65].

Theorem 5.3. ([Nat04] (Chapter 1, Theorem 4.1))

The space T_{g,l_h,l_p} is diffeomorphic to an open domain in $\mathbb{R}^{6g+3l_h+2l_p-6}$ which is homeomorphic to $\mathbb{R}^{6g+3l_h+2l_p-6}$.

For an element $\psi : \Gamma_{g,n} \to \mathrm{Aut}(\mathbb{H})$ of $\tilde{T}_{g,l_h,l_p}$ we write

$$\widetilde{\mathrm{Mod}}^{\psi} = \widetilde{\mathrm{Mod}}^{\psi}_{g,l_h,l_p} = \{\alpha \in \mathrm{Aut}(\Gamma_{g,n}) \mid \psi \circ \alpha \in \tilde{T}_{g,l_h,l_p}\}.$$

One can show that $\widetilde{\mathrm{Mod}}^{\psi}$ does not depend on ψ, hence we write $\widetilde{\mathrm{Mod}}$ instead of $\widetilde{\mathrm{Mod}}^{\psi}$. Let $I\widetilde{\mathrm{Mod}}$ be the subgroup of all inner automorphisms of $\Gamma_{g,n}$ and let

$$\mathrm{Mod}_{g,l_h,l_p} = \mathrm{Mod} = \widetilde{\mathrm{Mod}} / I\widetilde{\mathrm{Mod}}.$$

We now recall the description of the moduli space of Riemann surfaces

Theorem 5.4. ([Nat04], Chapter 1, Section 5)

The groups $\mathrm{Mod} = \mathrm{Mod}_{g,l_h,l_p}$ and the group of homotopy classes of orientation preserving autohomeomorphisms of the surface of type (g, l_h, l_p) are naturally isomorphic. The group Mod_{g,l_h,l_p} acts naturally on T_{g,l_h,l_p} by diffeomorphisms. This action is discrete. The quotient set $T_{g,l_h,l_p} / \mathrm{Mod}_{g,l_h,l_p}$ can be identified naturally with the moduli space M_{g,l_h,l_p} of Riemann surfaces of type (g, l_h, l_p).

We denote by $S(g, \delta, n_j^h, n_j^p)$ the set of all m-co-spin structures on all Riemann surfaces of type (g, l_h, l_p) such that the associated m-Arf function is of type $t = (g, \delta, n_j^h, n_j^p)$.

Theorem 5.5. *Let $t = (g, \delta, n_0^h, \dots, n_{m-1}^h, n_0^p, \dots, n_{m-1}^p)$ be a tuple satisfying the hypotheses of Theorem 5.2. The space $S(g, \delta, n_j^h, n_j^p)$ is homeomorphic to $T_t / \operatorname{Mod}_t$, where*

$$T_t \cong \mathbb{R}^{6g+3l_h+2l_p-6}$$

and Mod_t acts on T_t as a subgroup of finite index in the group $\operatorname{Mod}_{g,l_h,l_p}$.

Proof. Let us consider an element ψ of the space T_{g,l_h,l_p}. By definition ψ is an homomorphism $\psi : \Gamma_{g,n} \to \operatorname{Aut}(\mathbb{H})$. To the homomorphism ψ we attach a Riemann surface $P_\psi = \mathbb{H}/\psi(\Gamma_{g,n})$, a standard basis

$$v_\psi = \{a_i^\psi, b_i^\psi \ (i = 1, \dots, g), c_i^\psi \ (i = g+1, \dots, n)\}$$

of $\pi_1(P_\psi, p)$, and an m-Arf function σ_ψ on this surface given by

$$\begin{aligned}
(\sigma_\psi(a_1^\psi), \sigma_\psi(b_1^\psi), \sigma_\psi(a_2^\psi), \sigma_\psi(b_2^\psi), \dots, \sigma_\psi(a_g^\psi), \sigma_\psi(b_g^\psi)) \\
= (0, 1-\delta, 1, \dots, 1), \\
(\sigma_\psi(c_{g+1}^\psi), \dots, \sigma_\psi(c_{g+l_h}^\psi)) = (\underbrace{0, \dots, 0}_{n_0^h}, \underbrace{1, \dots, 1}_{n_1^h}, \dots, \underbrace{m-1, \dots, m-1}_{n_{m-1}^h}), \\
(\sigma_\psi(c_{g+l_h}^\psi), \dots, \sigma_\psi(c_n^\psi)) = (\underbrace{0, \dots, 0}_{n_0^p}, \underbrace{1, \dots, 1}_{n_1^p}, \dots, \underbrace{m-1, \dots, m-1}_{n_{m-1}^p}).
\end{aligned}$$

By Theorem 4.8, the m-Arf function σ_ψ on the surface P_ψ corresponds to a m-co-spin structure δ_ψ on P_ψ. The correspondence $\psi \mapsto \delta_\psi$ defines a map

$$T_{g,l_h,l_p} \to S(g, \delta, n_j^h, n_j^p).$$

According to Theorem 5.2 this map is surjective. For any point in $S(g, \delta, n_j^h, n_j^p)$ its preimage in T_{g,l_h,l_p} consists of an orbit of the subgroup Mod_t of $\operatorname{Aut}(P_\psi) = \operatorname{Mod}_{g,l_h,l_p}$ that preserves the m-Arf function σ_ψ. Thus

$$S(g, \delta, n_j^h, n_j^p) = T_{g,l_h,l_p} / \operatorname{Mod}_t.$$

□

Corollary 5.6. *There is a 1-1-correspondence between the connected components of the space of m-co-spin structures on the hyperbolic surfaces of type (g, l_h, l_p) and the topological types of m-Arf functions described in Theorem 5.2. Any connected component is homeomorphic to*

$$\mathbb{R}^{6g+3l_h+2l_p-6} / \operatorname{Mod}_t,$$

where Mod_t acts discrete on $\mathbb{R}^{6g+3l_h+2l_p-6}$.

Summarizing the results of Theorems 5.2 and 5.5 we obtain the following

Theorem 5.7. *The connected components of the space of m-co-spin structures on the hyperbolic surfaces of type (g, l_h, l_p) are those sets $S(g, \delta, n_j^h, n_j^p)$, which are not empty. Here $S(g, \delta, n_j^h, n_j^p)$ is the set of all m-co-spin structures on all hyperbolic surfaces of type (g, l_h, l_p) such that the associated m-Arf function is of type $t = (g, \delta, n_j^h, n_j^p)$ (see Definition 5.2). The set $S(g, \delta, n_j^h, n_j^p)$ is not empty if and only* *if the type* $t = (g, \delta, n_j^h, n_j^p)$ *satisfies the conditions of Theorem 5.2. If the set* $S(g, \delta, n_j^h, n_j^p)$ *is not empty, it is homeomorphic to*

$$\mathbb{R}^{6g+3l_h+2l_p-6}/\operatorname{Mod}_t,$$

where Mod_t *acts discrete on* $\mathbb{R}^{6g+3l_h+2l_p-6}$.

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