OPERADS, MODULI OF SURFACES AND QUANTUM ALGEBRAS

RALPH M. KAUFMANN

ABSTRACT. We review the relationship of moduli spaces of surfaces to operads and deformations of algebras starting by recalling basic facts about operads, examples of them, and their relations to algebras. In particular, we regard the Arc operad as well as several suboperads which can be thought of as cacti operads which play a prominent role in the string topology of Chas and Sullivan. It is recalled that spineless cacti and cacti homotopy equivalent to the little discs and framed little discs operads featured prominently in deformation theory. Furthermore, we analyze operations on operads and use the results to relate the spineless cacti operad to the renormalization Hopf algebra of Connes and Kreimer. Finally, we give a cell decomposition for spineless cacti and show that the cellular chains operate on the Hochschild complex of an associative algebra. This gives a new solution to Deligne's conjecture about the Hochschild complex and furthermore directly relates the Gerstenhaber structures on the loop space and the Hochschild complex.

CONTENTS

INTRODUCTION

In recent years, there has been a dynamic and vibrant interplay between physics and mathematics which is now spreading to biology. For mathematicians the ideas around topological and conformal field theory as well as string theory have found their manifestation in constructions which use the geometry of moduli spaces to describe deformations of algebras. There are two main questions: which geometries govern specific deformations; and given specific geometries, what kind of algebraic

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objects do they represent? The deformations themselves are usually given in terms of an expansion in one or more deformation parameters. The coefficients of these expansions viewed as multilinear functions and their relations are of particular interest. For their study operads provide the right tool being at once an organizing principle and providing a theoretical background and tertium comperationis.

For persons not already inclined towards operads one could say operads are very much like Steenrod operations in the following sense. They both have three aspects: 1) There is an algebraic description. Pertaining to the Steenrod operations this is the fact that they form an algebra. For operads the counterpart is the definition of an operad in terms of its constituents and operations. 2) The nature as universal operations, i.e. regarding the Steenrod algebra as operations on cohomology and for operads considering the classes of algebras over this operad. And 3) The direct realization in a concrete model.

In passing from geometry to algebra one usually uses geometric moduli to produce and describe the structure of deformations. For the physically inclined one could state that one starts out with an interpretation of Feynman rules. These can usually be translated into a graph theoretical picture together with a geometrical description such as a stratification of a moduli space indexed by these graphs. This would be a safe place to start for a critical mathematician. Appealing once more to physics; the idea of the path integral, like a usual integral, also encompasses an additivity property in the domain. This translates to glueing or pasting operations which have been known in mathematics as operads. Combining these two ingredients with the idea of strings one is mathematically led to consider operads of surfaces as we discuss below.

The basic example is that of topological field theory where graphs actually represent surfaces with labelled boundaries and the glueing operation is the glueing along the boundaries - this is discussed below in detail. The relation to string theory is the apparent interpretation that the boundaries are strings and as they move they sweep out the surface. To make this interpretation rigorous, one has to introduce orientations and look at the cobordism category which allows to speak of inputs and outputs. With these inputs and outputs a surface looks like a thickened Feynman graph. These surfaces are naturally the geometric part of the picture; the algebra comes in form of a functor to the category of vector spaces as in Atiyah's [A] version of topological field theory.

Here the geometric fact that any surface can be decomposed into pairs of pants, cylinders and discs is translated into the fact that on the vector space there should be an associative commutative multiplication, a non–degenerate bilinear form and an identity for the multiplication.

Starting with this topological picture one can put more structure on the surfaces which will in turn give more operations on the algebraic side. This leads to deformations or augmentation of the basic structure of an associative commutative algebra. One can think for instance of surfaces together with conformal structures. In the algebro–geometric setting one is led in this case either to consider the open moduli spaces $M_{g,n}$ of n–punctured surfaces of genus zero or their Deligne–Mumford compactification $\bar{M}_{g,n}$. In the latter case, one arrives at Gromov–Witten invariants (see e.g. [M]) or quantum cohomology which is, as the name states, a deformation of the commutative associative algebra. If one looks at the algebras that are governed by $M_{0,n}$ then one arrives at so–called gravity algebras or Lie_∞ in the genus 0 case which are deformations of Lie algebras. The fact that Lie algebras instead of commutative algebras appear has a very nice interpretation in terms of operads, since the operad $H_*(M_{0,n})$ is Koszul dual to $H_*(\bar{M}_{0,n})$ [Ge2, GiK] just as the operads for commutative and Lie algebras are Koszul duals of each other.

An augmentation can be made by adding for instance G–bundles to the picture for a finite group G. In this case one arrives at G–equivariant theories [K1] which are well suited to study so–called "stringy aspects" of global quotients such as symmetric products [K2]. Adding conformal structure one arrives at the G-equivariant setting of GW–invariants [JKK]. For general orbifolds the corresponding invariant structures are contained in [CR].

Staying on the differential–topological or analytic side one can keep the boundaries and add punctures for instance. The natural spaces to study are then Teichmüller spaces and their moduli space quotients. One especially promising aspect of this approach is the natural appearance of graphs in this theory. They arise in the theory of Strebel differentials as well as in Penner's work [P1, P2]. More recently another beautiful view of the topology of movements of strings in terms of operations on loop spaces has been put forth and developed in [CS1, CS2, S]. This or some aspects of the theory also have an interpretation in terms of surfaces decorated by weighted arcs which again form an operad, the arc operad introduced in [KLP]. In the genus zero case this algebra of [CS1] is captured by the cacti operad of Voronov [V] which is a suboperad of the arc operad [KLP, K3]. The algebraic structure that appears is that of a Gerstenhaber Batalin–Vilkovisky algebra up to homotopy (basically this is an odd Poisson algebra whose bracket is the odd commutator of a differential that is not a derivation, but a degree two differential operator; see §2.6 for the definitions). This structure is closely related to the gravity algebra structure [G] as is the space defining the arc operad to the open moduli space [P2].

For the second type of question: what kind of geometry governs deformations, two questions are key examples: deformation quantization and Deligne's conjecture about the Hochschild cochain complex. It has turned out that they are indeed related [Ko, T]. The solving of the deformation quantization [Ko] was indeed done through geometry in terms of integrals over configuration spaces. Deligne's conjecture, on the other hand, is directly a question about the geometry governing deformations. It was realized by Gerstenhaber [G] that deformations of the multiplication of the product in an associative algebra into an associative star product are governed by the Hochschild complex of this algebra. He proved that there is a surprising structure, namely that of an odd Poisson algebra, on the homology level. This structure is actually derived from the cochain level. Since the operad governing Gerstenhaber algebras is the homology of a geometric object, that of little discs, it was conceivable that this correspondence could be lifted to the chain level. This is the content of Deligne's conjecture which we prove below. This conjecture has been proven in various ways [Ko, T, MS, V2, KS] (for a full review see [MSS]) by basically choosing adequate chain models. The virtue of our approach lies in its directness and that it also establishes the surprising and intriguing fact that the Gerstenhaber structure on the loop spaces and the Gerstenhaber structure on the Hochschild cochains have a common natural interpretation in terms of surfaces. The formal similarities of the Gerstenhaber structure on the arc operad or rather of the suboperad of spineless cacti (defined in [K3]) with the Gerstenhaber structure on the Hochschild cochains were first observed by Gerstenhaber himself and have given rise to our new proof of Deligne's conjecture in terms of spineless cacti [K4] which is reviewed below. In fact, spineless cacti are homotopy equivalent to the little discs operad and here is a cell decomposition of the spineless cacti operad indexed by trees which directly give the operation in the Hochschild cochain complex. Furthermore, the spineless cacti can be thought of as surfaces with weighted arcs that satisfy certain natural restrictions and as such can be seen as the substructure of moduli space giving rise to the bracket operations. This fits also well with a path integral description of deformation quantization by [CF]. On the other hand, cacti and spineless cacti [V, K3] are the operads which correspond to the "string topology" of [CS1] and provide the BV and Gerstenhaber structure, respectively. A realization of the Gerstenhaber structure in this setting is given by the Goldman bracket [CS1]. In another direction, we show that the top–dimensional cells of the cell decomposition which provides the solution to Deligne's conjecture is related to the pre-Lie operad and to the renormalization Hopf algebra of Connes and Kreimer [CK]. It is thus tempting to view the $\mathcal{ARC}$ operad which was defined in [KLP] as an underlying "string mechanism" for all of these structures.

The paper is organized as follows:

In the first section, we introduce and discuss several types of trees as they are the language in which operads are most easily spoken about. In the second section, we review the notion of an operad and give many examples of them. These sections also serve to fix notations. In the third section, we review the construction of the $\mathcal{ARC}$ operad of [KLP] which can be seen as a combinatorial version of the operad of moduli spaces of genus g curves with n punctures. In a sense the operad describes hyperbolic field theory. Besides a direct relation discussed in the same volume by Penner, abstractly there is a relation due to the BV nature of the operad which we also review as well as giving a string interpretation for the arc operad. The forth paragraph deals with the cacti operads of [V, K3]. We show how they are naturally sub–operads of the arc operad thus relating the arc operad to the string topology of [CS1]. In §5.3.2, we give a natural cell decomposition for spineless cacti which is the basis of our solution of Deligne's conjecture. Paragraph 6 is an intermezzo about universal operations for operads, if one wishes meta-operads. This analysis is applied to the pre–Lie operad in order to relate it to the Hopf algebra of Connes and Kreimer. Furthermore, we show that the operad of top dimensional cells of our cell decomposition is isomorphic to the pre–Lie operad thus giving a geometric interpretation to this Hopf algebra. In §7 we give a new proof of Deligne's conjecture based on our cell model of the little discs operad which is given by the cellular chain operad of the natural cell decomposition for the spineless cacti operad or better the chain operad of normalized spineless cacti. We do this by providing two natural ways of letting these chains operate on the Hochschild cochains of an associative algebra. The first operation is of the same type as the operation of the chains of the arc operad on themselves self and is related to string topology. The second realization uses the fact that there is a geometric interpretation of the Hochschild cochains as chains of the spineless cacti operad. This view is implemented by a foliage operator and is basically the approach of [KS]. In the last section §8 we speculate on the A_∞ generalization of our results as well as on possible relations of the arc formalism to other "quantum phenomena" and subjects.

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Notation

$\mathbb{S}_n$ the permutation group on n letters. C_n the cyclic group of order n. k is a field.

We will tacitly assume that everything is in the super setting, that is $\mathbb{Z}/2\mathbb{Z}$ graded. For all formulas, unless otherwise indicated, the standard rules of sign (see e.g. [M2]) apply.

1. Trees

There is no paper about operads without trees which appear in many guises. In order to fix notation, we will thus consider them here again.

1.1. General Definitions.

Definition 1.1.1. *A graph is a 1 dim simplicial–complex. We call the 0–dimensional simplices vertices and denote them by $V(\Gamma)$. The 1–dimensional simplices are called edges and denoted by $E(\Gamma)$.*

A tree is an isomorphism class of a connected simply–connected graph.

A rooted tree is a tree with a marked vertex.

We call a rooted tree planted if the root vertex lies on a unique edge. In this case we call the unique edge with a boundary in the root the "root edge".

We usually depict the root by a small square, denote by $root(\tau) \in V(\tau)$ the root vertex and by $e_{root(\tau)} \in E(\tau)$ the root edge.

Notice that an edge of a graph or a tree is given by a set $\{v_1, v_2\}$ of vertices. An ordered edge is an edge $e = \{v_1, v_2\}$ together with an order (v_1, v_2). An edge path is a sequence of ordered edges $e_i = (v_1^i, v_2^i)$ such that $v_2^i = v_1^{i+1}$.

An edge that has v as a vertex is called an adjacent edge to v.

Definition 1.1.2. *Given a tree τ and an edge $e \in E(\tau)$ there is a tree obtained by contracting the edge e which we denote by $con(\tau, e)$.*

More formally let $e = \{v_1, v_2\}$, and consider the equivalence relation $\sim$ on the set of vertices which is given by $\forall w \in V(\tau) : w \sim w$ and $v_1 \sim v_2$. Then $con(\tau, e)$

is the tree whose vertices are $V(\tau)/\sim$ *and whose edges are* $E(\tau)\setminus\{e\}/\sim$ *where* $\sim$ *denotes by abuse of notation the induced equivalence relation* $\{w_1, w_2\} \sim w_1', w_2'$ *if* $w_1 \sim w_1'$ *and* $w_2 \sim w_2'$ *or* $w_2 \sim w_1'$ *and* $w_1 \sim w_2'$*.*

1.1.1. **Labeled trees and trees with tails.**

Definition 1.1.3. *For a finite set* S *an* S *labeling for a tree is a decomposition into disjoint subsets of* $V(\tau) = V_l \amalg V_u$ *and a bijective function from the subset* $Lab : V_l \to S$*. For a rooted tree, we demand that* $root \in V_u$*.*

An n*-labeled tree is a tree labeled by* $\bar{n} := \{1, \dots, n\}$*. For such a tree we call* $v_i := Lab^{-1}(i)$*.*

1.1.2. **Structures on rooted trees.** A rooted tree has a natural orientation, toward the root. For each vertex there is a unique shortest edge path to the root. For a rooted tree τ with root vertex $root \in V(\tau)$ we define the function $N : V(\tau) \setminus \{root\} \to V(\tau)$ by the rule that

$$N(v) = \text{the first vertex on unique path to the root}$$

We call the set $N^{-1}(v)$ the set of incoming edges of v and call the edge $(v, N(v))$ the outgoing edge of v.

Furthermore we define the valence of v to be $|v| := |N^{-1}(v)|$.

CAVEAT: Our $|v|$ is the number of incoming edges, which is the number of adjacent edges minus one.

Definition 1.1.4. *We define the set of leaves* V_{leaf} *of a tree to be the vertices which have valence zero, i.e. the number of incoming edges is zero.*

A tree with tails is a tree together with a set $V_{tail} \subset V_{leaf}$*. The tail edges are the edges which are adjacent to a tail.*

An S *labeled tree with tails is a labeled tree s.t.* $V_l \cap V_{tail} = \emptyset$ *and an* S *labeled tree with* T *labeled tails is an* S *labeled tree with tails together with a* T *labeling of the tails, i.e. a bijection* $V_{tail} \to T$*.*

Remark 1.1.5. *There are forgetful morphisms forgetting the root and in the planted case contracting the root edge. The latter morphism actually induces a bijection between planted rooted and rooted trees. The inverse map is given by adding one additional vertex which is the new root and introducing an edge from the new root to the old root. We call this map* $plant : \mathcal{RT}ree \mapsto \mathcal{PRT}ree$*.*

For a rooted tree there is also a bijection $v2e$ between the set $V(\tau) \setminus \{root\} \to E(\tau)$ which associates to each vertex except the root its unique outgoing edge $v \mapsto (v, N(v))$.

1.2. **More types of trees.**

Definition 1.2.1. *A planar tree is a pair* (τ, p) *of a tree* τ *together with a pinning* p *which is a cyclic order of each of the sets of adjacent edges at each vertex.*

A black and white graph (b/w graph) τ *is a graph together with a function* $\mathrm{color} : V(\tau) \to \{0, 1\}$*.*

We call the set $V(\tau)_w := \mathrm{color}^{-1}(0)$ *the set of white vertices and call the set* $V(\tau)_b := \mathrm{color}^{-1}(1)$ *the set of black vertices.*

A bipartite b/w tree is a b/w tree whose edges only connect vertices of different colors.

A labeled b/w tree is a tree with a labeling, s.t. $V_l = V_w$ *and* $V_u = V_b$*.*

For a black and white bipartite tree, we define the set $E_w(\tau)$ to be the edges $(v_1, N(v_1))$ with $N(v_1) \in V_w$ and call the elements white edges and likewise we define E_b with elements called black edges, so that there is a partition $V(\tau) = V_w(\tau) \amalg V_b(\tau)$.

Notation 1.2.2. *By a b/w tree with tails, we understand a b/w tree with tails* and *the assumption that the tails are exactly the black leaves.*

For a rooted planted b/w bipartite tree, we take the condition to be bipartite to mean that both of the vertices of the root edge are black, i.e. root and the vertex $N^{-1}(root)$ is black.

1.2.1. **Structures on planar trees.** A planar tree can be embedded in the plane in such a way that the induced cyclic order from the natural orientation of the plane and the pinning coincide.

The set of all pinnings of a fixed tree is finite and is a principal homogeneous set for the group

$$\mathbb{S}(\tau) := \times_{v \in V(\tau)} \mathbb{S}_{|v|}$$

where each factor acts by permutations on the set of cyclic orders of the edges adjacent to v. This action is given by permuting the edges with the cyclic permutations being trivial $\mathbb{S}_{|v|+1}/C_{|v|+1} \simeq S_{|v|}$.

Given a rooted planar tree there is a linear order at each vertex except for the root which is given by the cyclic order and the outgoing edge as the smallest element. The root vertex has only a cyclic order, though.

On a planar planted tree there is a linear order at all of the vertices, since the root now has only one incoming edge and no outgoing one.

Furthermore, on such a tree there is a path which passes through all the edges exactly twice by starting at the root going along the root edge and at each vertex continuing on the next edge in the cyclic order. We call this path the outside path.

By omitting recurring elements this yields a linear order $\prec^{(\tau,p)}$ starting with the root edge on the set $V(\tau) \amalg E(\tau)$. This order induces an order on the set of vertices $V(\tau)$, on the set of all edges $E(\tau)$, as well as a linear order for all the vertices incident to the vertex $v \prec_v^{(\tau,p)}$ starting with the outgoing edge. We omit the superscript for $\prec^{(\tau,p)}$ if it is clear from the context.

An edge e' is said to be above e if e lies on the edge path of the vertices of e' to the root. With this notion we can define the weight of an edge by $wt(e) := |\{\text{edges above e}\}|$.

Definition 1.2.3. *We say that an n–labeled planar planted tree has a natural labeling if the order of the vertices in the labeling agrees with the order $\prec$, i.e. $v_i \prec v_j$ if $i < j$.*

Remark 1.2.4. *By the remark 1.1.5 for non–planar trees the morphism given by contracting the root edge of a planted tree is a bijection. For planar trees this is not the case. And moreover, the linear order can only be defined for planted rooted trees.*

Therefore, we will not regard rooted planar trees, but only planted planar trees.

1.2.2. **Notation.** N.B. A tree can have several of the attributes mentioned above; for instance, we will look at planar bipartite planted rooted trees. We will now fix several sets of trees in what we hope natural notation

$\mathcal{RT}ree$	The set of all rooted trees
$\mathcal{LL}tree$	The set of leaf labeled planar trees
$\mathcal{PPT}ree$	The set of planted planar trees
$\mathcal{WPT}ree$	The set of planted planar b/w trees with white vertices only (except for the root).
$\mathcal{WPTT}ree$	The set of planted planar b/w trees with tails and white vertices only, except for the root and the tails
$\mathcal{WLPT}ree$	The set of b/w planted planar trees with white leaves only and no black vertices of valence one
$\mathcal{WLPTT}ree$	The set of b/w planted planar trees with tails, white leaves only except for the tails, and no black vertices of valence one
$\mathcal{WLBPT}ree$	The set of b/w planted planar bipartite trees with white leaves only
$\mathcal{WLBPTT}ree$	The set of b/w planted planar bipartite trees with tails and white leaves only except for the tails

1.2.3. **Notation and Conventions.** We use the convention that for a class of trees $\mathcal{CT}ree$ of any of the above $\mathcal{CT}ree(n)$ denotes the n–labeled trees.

Most of the time we wish to look at the free Abelian groups or free vector spaces generated by the sets of trees. We could introduce the notation $Free(\mathcal{CT}ree, \mathbb{Z})$ and $Free(\mathcal{CT}ree, k)$ for the direct sums over the $\mathcal{CT}ree(n)$. But since there is almost no risk of confusion, we will just denote these free objects again by $\mathcal{CT}ree$ to avoid cluttered notation.

We also fix the convention that for a planted b/w tree the root has to be among the black vertices.

1.3. Maps between different types of trees.

1.3.1. **A map from $\mathcal{WLPT}ree \to \mathcal{WLBPT}ree$.** We define a map from $\mathcal{WLPT}ree \to \mathcal{WLBPT}ree$ in the following way: First, we set to zero any tree which has black vertices whose valence is greater than two. Then, we contract all edges which join two black vertices. And lastly, we insert a black vertex into each edge joining two white vertices.

We call this map st_∞, since this map in a certain precise sense forgets the A_∞ structure if an associative structure is present.

1.3.2. **The map from $\mathcal{RT}ree \to \mathcal{PPT}ree$.** There is a map pin from planted trees to planted planar trees.

$$pin(\tau) = \sum_{\text{p} \,\in\, \text{Pinnings}(\tau)} (\tau, p)$$

We extend this map to a map $ppin : \mathcal{RT}ree \to \mathcal{PPT}ree$ by

$$ppin(\tau) := pin(plant(\tau))$$

1.3.3. **A map from $\mathcal{PPT}ree \to \mathcal{WPT}ree$.** We also define a map:

$$bw : \mathcal{PPT}ree \mapsto \mathcal{WPT}ree$$

by coloring all vertices except for the root vertex white.

For the subset $\mathcal{RT}ree[n]$ of trees τ in $\mathcal{RT}ree(n)$ with $|\{V(\tau) \setminus \{root\}\}| = n$ (i.e. all vertices except the root are labeled), we get the induced inclusion for labeled trees.

1.3.4. **A map from $\mathcal{RT}ree \to \mathcal{WLBPT}ree$.**

$$cppin := st_\infty \circ bw \circ ppin : \mathcal{RT}ree \to \mathcal{WLBPT}ree$$

that plants, pins and colors in a bipartite way.

Again, for the subset $\mathcal{RT}ree[n]$ of trees τ in $\mathcal{RT}ree(n)$ with $|\{V(\tau)\setminus\{root\}\}| = n$ (i.e. all vertices except the root are labeled), we get the induced inclusion for labeled trees.

Using the maps $bw \circ ppin$ and $cppin$, we will consider $\mathcal{PPT}ree$ as the subset of $\mathcal{WPT}ree$ of $\mathcal{WLPT}ree$ and as a subset of $\mathcal{WLBPT}ree$.

2. Operads

In this section we briefly review the notion of an operad and give the main examples of operads for the algebras we will be considering.

First let us fix a symmetric monoidal category $(\mathcal{C}, \otimes)$. The usual candidates which we will also use in the following are $\mathcal{S}et, \mathcal{T}op, \mathcal{C}hain, \mathcal{V}ect_k$ — the categories of finite sets with disjoint union, topological spaces with Cartesian products, the category of chain complexes of Abelian groups with tensor product over $\mathbb{Z}$ and the category of chain complexes of vector spaces over a field k together with tensor product.

Operads have been defined in many places (see e.g. [MSS]) and the pure definition seems at first difficult, but the example below 2.2.1 is paradigmatic for the definition, and the uninitiated reader might want to start there. The cognizant reader, however, might want to skip ahead.

2.1. **Operads and induced operads.**

Definition 2.1.1. *An operad $\mathcal{C}$ is a collection of objects $\mathcal{O} := \{O(n) : O(n) \in \mathcal{C}, n \geq 1\}$ together with an $\mathbb{S}_n$ action on $O(n)$ and maps*

$$\circ_i : O(m) \otimes O(n) \to O(m+n-1), i \in \{1, \dots m\} \tag{2.1}$$

which are associative and $\mathbb{S}_n$–equivariant and an element $id \in O(1)$ that satisfies for all $op_n \in O(n), i \in \{1, \dots, n\}$

$$\circ_i(op_n, id) = \circ_1(id, op_n) = op_n$$

i) *Associativity: for $op_k \in O(k), op'_l \in O(l)$ and $op''_m \in O(m)$*

$$(op_k \circ_i op'_l) \circ_j op''_m = \begin{cases} (op_k \circ_j op''_m) \circ_{i+m-1} op'_l) & \text{if } 1 \leq j < i \\ op_k \circ_i (op'_l \circ_{j-i+1} op''_m) & \text{if } i \leq j < i+l \\ (op_k \circ_{i-l+1} op'_l) \circ_j op''_m & \text{if } i+l \leq j \end{cases}$$

ii) *Equivariance: $op_m \in O(m)$ and $op_n \in O(n)$*

$$\sigma_m(op_m) \circ_i \sigma'_n(op_n) = \sigma_m \circ_i \sigma'_n(op_m \circ_{\sigma_m(i)} op_n)$$

where $\sigma_m \circ_i \sigma'_n \in \mathbb{S}_{m+n-1}$ is the permutation that the block or iterated permutation

$$(1, 2, \dots, i-1, (1', \dots, m'), i+1 \dots, n) \mapsto \sigma_n(1, 2, \dots, i-1, \sigma'_m(1', \dots, m'), i+1 \dots, n) \tag{2.2}$$

induces on $(1'', \dots, (m+n-1)'')$

$$j'' = \begin{cases} j & 1 \le j \le i-1 \\ j-i+1' & i \le j \le i+n-1 \\ j-n & i+n \le j \le m+n-1 \end{cases}$$

We call the operads in $\mathcal{S}et, \mathcal{T}op, \mathcal{C}hain, \mathcal{V}ect_k$ combinatorial, topological operads, chain operads and linear operads, respectively. We also call $O(n)$ the n–th component of $\mathcal{O}$.

Definition 2.1.2. *A morphism of two operads $\mathcal{O}, \mathcal{O}'$ in the same monoidal category or an operadic morphism is a collection of morphisms from $O(n) \to O'(n)$ which respect all the structures. A suboperad is an injective morphism of operads. In this case, we call $\mathcal{O}$ a suboperad of $\mathcal{O}'$.*

2.1.1. **Induced operads.** As we mentioned before, the categories we are interested in are $\mathcal{T}op, \mathcal{C}hain, \mathcal{V}ect_k$. Now there is the singular homology functor $H_* : \mathcal{T}op \to \mathcal{V}ect_k$ which given a topological operad yields a linear operad. Using the functor of singular chains $C_* : \mathcal{T}op \to \mathcal{C}hain$ gives a map from topological operads to chain complexes and finally taking homology $H_* : \mathcal{C}hain \to \mathcal{V}ect_k$ of a chain complex yields a functor from chain operads to linear operads.

For the intermediate level, the chain level, there might also be several choices depending on the chosen model.

Any model with a map to the diagonal is fine. Given a CW complex one can define the operads CC_* of cellular chains, and given a triangulated space one has the functor C_*^{Δ} of simplicial chains. In these cases one has to additionally check that the compositions on the chains are indeed (a sum) of chains.

There is also a functor of $\mathcal{F} : \mathcal{S}et \mapsto \mathcal{C}hain$ which associates the free Abelian group to a set, however, usually there are some signs that appear in geometric situations coming from different orientations. In a given case, there might also be a candidate of a possible non–trivial differential.

2.2. **The fundamental examples.** There are two fundamental examples which help to explain the notion of an operad. If one wishes, an operad is the abstraction of the algebraic properties of these examples.

2.2.1. **The Operad of Functions.** Set $Funct(n) := \{\text{functions of } n \text{ variables}\}$. The $\mathbb{S}_n$ action is the action induced by permutation of the variables:

$$\sigma(f)(x_1, \dots, x_n) := f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

and the maps $\circ_i$ are just the substitution maps

$$(f \circ_i g)(z_1, \dots z_{m+n-1}) := f(z_1, \dots .z_{i-1}, g(z_i, \dots, z_{i+n-1}), z_{i+n}, \dots, z_{m+n-1})$$

This map is just the following substitution: say f is a function of the variables x_i and g is a function of the variables y_j then setting $x_i = g(y_1, \dots, y_n)$ yields

$$\begin{aligned} f(x_1, \dots, x_m) \circ_i g(y_1, \dots, y_n) &= f(x_1, \dots .x_{i-1}, g(y_1, \dots, y_n), x_{i+1}, \dots, x_m) \\ &= (f \circ_i g)(z_1, \dots, z_{m+n-1}) \end{aligned}$$

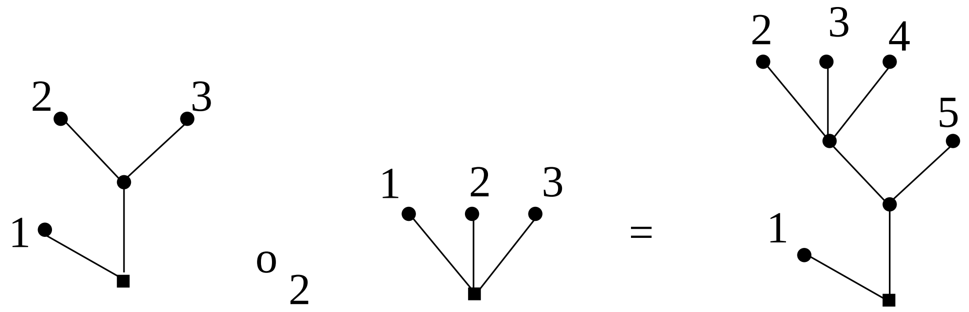


FIGURE 1. Example of grafting two trees

with

$$z_j = \begin{cases} x_j & 1 \le j \le i-1 \\ y_{j-i+1} & i \le j \le i+n-1 \\ x_{j-n} & i+n \le j \le m+n-1 \end{cases} \tag{2.3}$$

The $\mathbb{S}_n$ equivariance means that applying a fixed permutation $\sigma_x \in \mathbb{S}_n$ to the x_j and a permutation $\sigma_y \in \mathbb{S}_m$ to the y_k and then composing there is a unique permutation σ_z of the z_l which has the same effect, i.e.

$$\sigma_x(f) \circ_i \sigma_y(g) = \sigma_z(f \circ_{\sigma_x(i)} g)$$

The permutation which is easily constructed from this definition is also the permutation for the general case.

Finally, the associativity translates to the fact that if one does two substitutions, be they both for the variables x or nested (i.e. for some variables x_i and y_j), the order in which the substitutions are performed does not matter. Notice that this gives three cases.

The associativity of two iterated substitutions is also the associativity condition in the general case.

2.2.2. **Rooted leaf labeled Trees.** Another useful primordial example is that of rooted trees with labeled leaves $\mathcal{LL}tree$.

The n–th component of this operad are rooted trees with n labeled leaves. The $\mathbb{S}_n$ action on on the n–th component is given by permuting the labels.

The operation of grafting defines the composition.

To graft two rooted leaf labeled trees τ and τ' fix the vertex v_i of τ' and fix the equivalence relation $\sim$ on $V := V(\tau) \amalg V(\tau')$ where $\forall w \in V : w \sim w$ and $root(\tau) \sim v_i$.

Then the grafting $\tau \circ_i \tau'$ of two rooted trees is the tree with vertices $V/\sim$ and edges $(E(\tau) \amalg E(\tau'))/\sim$ where $\sim$ is the induced equivalence relation as defined as in 1.1.2. The root of the tree is the image of the root of τ and the labeling is analogous to the example 2.2.1 in equation (2.3).

An example of grafting is as depicted in figure 1.

The $\mathbb{S}_n$ equivariance is as above and the associativity states, that the resulting tree does not depend on how it was grown.

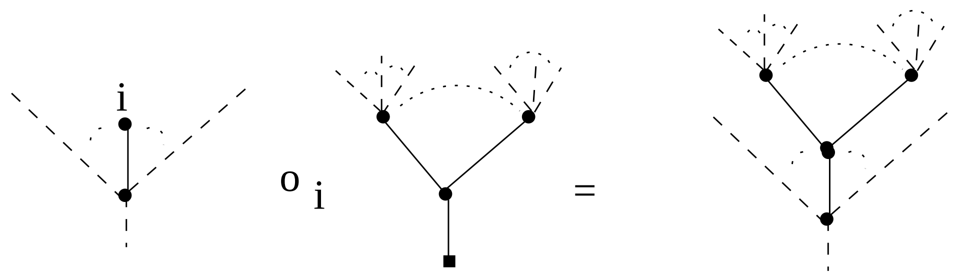


FIGURE 2. Grafting two planted planar trees

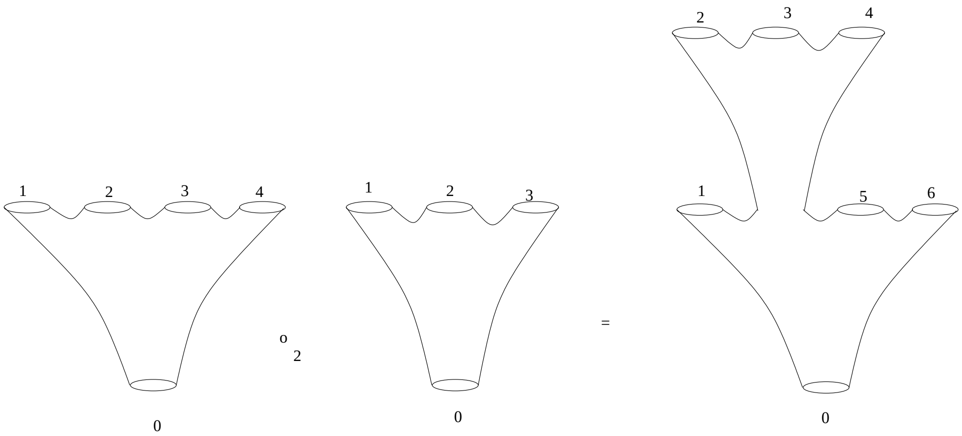


FIGURE 3. Example of grafting two surfaces

Remark 2.2.1. *The grafting procedure still makes sense for planar planted trees with labeled leaves. However, we fix by convention that $\tau \circ_i \tau'$ denotes the planted planar tree in which after the grafting the image of the root edge of τ' is contracted.*

An example of his procedure is as depicted in figure 2.

2.2.3. **Bordered surfaces, corollas.** We can also look at bordered topological surfaces with or without punctures and genus. Here the space $TFT(n)$ is the space of homeomorphism classes of bordered surfaces $(\Sigma, \partial\Sigma)$ whose boundary is homeomorphic to the disjoint union of $n+1$ circles which are labeled by $0, \dots, n$, $\partial\Sigma \sim_{homeo} \amalg_{i=0,\dots,n} S^1$.

The operad structure comes from glueing surfaces. I.e. $\Sigma \circ_i \Sigma'$ is the surface obtained by glueing the boundary 0 of Σ' to the boundary i of Σ as depicted in Figure 3.

The $\mathbb{S}_n$ action being as usual by permuting the labels, and the labeling after the grafting as in equation 2.3.

Another way to encode this is by representing each surface by a tree which has a root vertex and n labeled leaves. Such a tree is called a (labeled) corolla. The operad structure comes from glueing corollas τ, τ' and then contracting the image of the edge $(root(\tau), v_i)$, the $\mathbb{S}_n$ action being again permutation the labels.

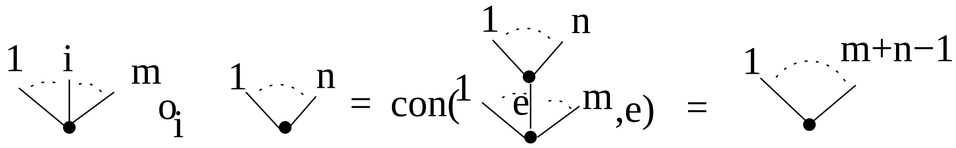


FIGURE 4. Example of grafting two corollas

2.3. **Insertion operads.** Lately, another type of operad of trees has appeared [KS, CL] where the glueings are not restricted to the leaves and there is a grafting procedure into inner vertices as well. The origin of these operads is discussed in section 6.

We will need the following variant of this operad structure on $\mathcal{BWBPT}ree$ where $\mathcal{BWBPT}ree(n)$ are n–labeled rooted planar planted bipartite trees. Recall that this means that the white vertices are labeled.

The description in words is as follows: We cut off the branches above v_i graft on τ' and then graft the cut off branches onto the planar tree taking care that the order of the branches given by the linear order at v_i is respected by the grafting procedure. I.e. that the branches also appear in the same order as previously on the grafted tree.

Definition 2.3.1. *Given $\tau \in \mathcal{BWBPT}ree(m)$ and $\tau' \in \mathcal{BWBPT}ree(n)$, we define the tree*

$$\tau \circ_i \tau' := \sum_{(gr:N^{-1}(v_i)\to V_w(\tau'),p(gr))}{}' (\tau_{gr}, p(gr)) \in \mathcal{BWBPT}ree(m+n-1)$$

where

- *gr is a bijection,*
- *τ_{gr} is the tree whose vertices are $V_{\tau\circ_i\tau'} := (V_\tau \setminus \{v_i\}) \amalg (V_{t'}) \setminus root(\tau'))$ and the following edges: the edges that do not hit v_i are kept, the outgoing edge of v_i is connected to $root_{\tau_i}$ giving rise to an edge $(root(\tau'), n(v_i))$, the image of the root edge of τ' is contracted and finally the incoming edges of v_i are connected according to gr, i.e. give rise to edges $(v, gr(v))$ for $v \in N^{-1}(v_i)$.*
- *$p(gr)$ is a pinning of τ_{gr} and*
- *the $\prime$ on $\sum'$ indicates that the sum is over all compatible pairs of a bijection gr and a pinning $p(gr)$ that preserve the linear order. Here compatibility is the following: Let $\prec^{p(gr)}$ be the linear order induced by the pinning $p(gr)$. We call gr and $p(gr)$ compatible if the linear order on the edges τ' as well as for the edges of τ that are kept is respected and $e_l = (v_l, v_i) \prec e_k = (v_k, v_i)$ in τ implies that $(v_l, gr(v_l)) \prec^{p(gr)} (v_k, gr(v_k))$ in the linear order $\prec^{p(gr)}$ of τ_{gr}.*

An example of such an insertion is given in figure 5.

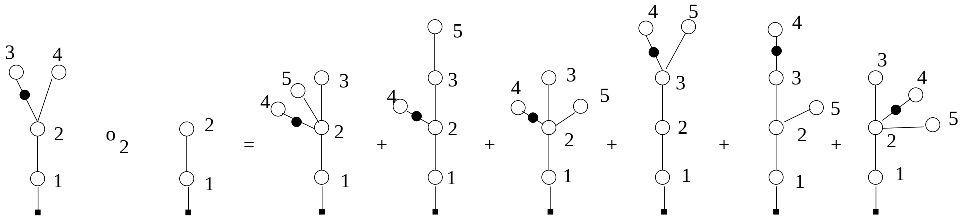


FIGURE 5. Example of the insertion of a planted planar tree

2.3.1. **Signs.** As discussed later on, if the trees have a geometrical interpretation, in terms of cells of a complex, then it is necessary to introduce signs

$$\tau \circ_i \tau' := \pm \sum_{(gr:N^{-1}(v_i)\to V_w(\tau'),p(gr))}{}' (\tau_{gr}, p(gr))$$

into the concatenations, which are dictated by the orientations of the cells.

Remark 2.3.2. *With the above compositions $\mathcal{WLBPT}ree$ is a suboperad of $\mathcal{BWBPT}ree$.*

2.3.2. **Other tree insertion operads and compatibilities.** There are two tree insertion operads structures already present in the literature on $\mathcal{RT}ree$ [CL] and, historically the first, on $\mathcal{WLPT}ree$ [KS].

For $\mathcal{RT}ree$ one omits the condition on the order and for $\mathcal{WLPT}ree$ one also allows to glue to black vertices.

The signs for the first gluing are all plus [CL] and in the second gluing are dictated by a chain interpretation [KS].

Remark 2.3.3. *The operad structure defined above restricted to $\mathcal{WLBPT}ree$ is compatible with those of $\mathcal{RT}ree$ and $\mathcal{WLPT}ree$ under the maps cppin and st_∞. In 5.4, we define signs for our tree operad by giving a cell interpretation. The result is that there is an orientation for top–dimensional cells which makes cppin into an operadic map and an orientation of cells which makes st_∞ into an operadic map, see §6.6 and 7.4*

Topological versions of this type of insertion glueing are given in [V, KLP, K3] as explained in §3 and §4.

2.4. **Variations of operads.** There are some variations of the structure of operads which we will need.

Definition 2.4.1. *The same data as in 2.1.1 without the identity is called a non–unital operad. Omitting the $\mathbb{S}_n$ action and the $\mathbb{S}_n$ equivariance yields the structure of a non–Σ operad.*

It is clear that any given operad can be indexed by arbitrary sets instead.

We will also need the following weakening of the data:

Definition 2.4.2. *A quasi–operad is an operad where the associativity need not hold.*

Remark 2.4.3. *If a quasi–operad in the topological category is homotopy associative then its homology has the structure of an operad. In certain cases, like the* *ones we will consider, the structure of operad already exists on the level of a chain model.*

Definition 2.4.4. *Lastly in a partial operad the concatenations need not all be defined on the whole components of the operads, but when they are defined the axioms hold.*

Definition 2.4.5. *A cyclic operad is an operad together with an action of $\mathbb{S}_{n+1}$ on each $\mathcal{O}(n)$ where $\mathbb{S}_{n+1}$ acts on the set $\{0, 1, \dots, n\}$ extending the action of $\mathbb{S}_n$ on $\{1, \dots, n\}$ and additionally satisfying the following compatibility. Let * denote the action of the cycle $(012\dots n) \in \mathbb{S}_{n+1}$ then for $a \in O(n)$*

$$(a \circ_n b)^* = (-1)^{|a||b|} b^* \circ_1 a^*$$

where we denoted the possible $\mathbb{Z}/2\mathbb{Z}$ degree of an element a by $|a|$.

The additional axiom means that the root is not distinguished any more.

Examples are the $TFT(n)$ and the operad of rooted trees with grafting on leaves and also the operads $\bar{M}_{g,n}$, $M_{g,n}$ and $\mathcal{ARC}$ (see below).

2.5. **Algebras over operads.** If we take the example 2.2.1 to be linear functions of a fixed vector space V onto itself, we obtain the linear operad $\mathcal{H}om_V$ with $Hom_V(n) := Hom_{k-lin}(V^{\otimes n}, V)$

Definition 2.5.1. *An algebra V over a linear operad $\mathcal{O}$ is a vector space V over k together with an operadic map $\mathcal{O} \to \mathcal{H}om_V$.*

The idea of the definition is that each element of $O(n)$ defines a n–ary multilinear operation on V. The ur–operation one has in mind is an algebra multiplication $V^{\otimes 2} \to V$.

Definition 2.5.2. *An algebra over a cyclic operad $\mathcal{O}$ is vector space V together with a symmetric non–degenerate bilinear form $\langle\, ,\, \rangle$ and a map of operads $\phi : \mathcal{O} \to \mathcal{H}om_V$ s.t.*

$$\langle v_0, \phi(op_n)(v_1, \dots, v_n)\rangle = \langle v_n, \phi(op_n^*)(v_0, v_1, \dots, v_{n-1})\rangle$$

for $op_n \in O(n)$.

2.6. **Operads classifying algebras.** More examples of operads come from operads classifying algebras. I.e. operads s.t. each algebra over them is of a certain type, like associative commutative etc. and vice–versa each algebra of the given type is indeed an algebra over this operad.

2.6.1. **The operad for commutative algebras.** The operad $\mathcal{COM}$ of associative, commutative algebras over a field k is given by

$$Com(n) := k \text{ as the trivial } \mathbb{S}_n \text{ module}$$

With glueing maps given by the identification $k \otimes_k k \simeq k$.

It is easily seen that this operad coincides with the operad TFT or corollas if one takes their k–linear span.

For an algebra over this operad the operation for $1 \in k = Com(2)$ defines a map $V^{\otimes 2} \to V : v_1 \otimes v_2 \mapsto v_1 \cdot v_2$ which has to be commutative since it is $\mathbb{S}_2$ invariant. Furthermore, it is associative from glueing two two–corollas in the two possible different ways to obtain the three–corolla.

The element $1 \in k = Com(n)$ represented by the n–corolla is necessarily the map $V^{\otimes n} \mapsto V : v_1 \otimes \cdots \otimes v_n \mapsto v_1 \cdot \cdots \cdot v_n$ which is uniquely defined since the algebra is associative and independent of the order due to the $\mathbb{S}_n$ invariance.

2.6.2. **The operad for associative algebras.** The operad $\mathcal{ASSOC}$ of associative algebras over a field k is given by

$$Assoc(n) := \text{The regular representation of } \mathbb{S}_n$$

with glueing maps given by composition of permutations as in the axiom of $\mathbb{S}_n$ equivariance of 2.1.1.

It is easily seen that this operad coincides with planar corollas.

An algebra over this operad has two multiplications $\cdot$ and $\cdot^{op}$. The multiplication $\cdot$ is associative and the planar n–corollas corresponding to the basis element e_σ of the regular representation of $\mathbb{S}_n$ represent the maps $v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \cdot \cdots \cdot v_{\sigma(n)}$

This operad is the same as the natural one for planar planted corollas.

2.6.3. **The operad for Gerstenhaber algebras.**

Definition 2.6.1. *A Gerstenhaber algebra A is a graded vector space together with two operations, a graded commutative and associative product $\cdot$ of degree 0, a bracket $\{\,\bullet\,\}$ of degree 1 (which is a Lie graded bracket on ΣA, the suspension of A sometimes also called an odd Lie–bracket) such that the bracket is odd Poisson for the multiplication $\cdot$.*

More precisely: if we denote the degree of $x \in A$ by $|x|$ and by $|sx| = |x| + 1$ (the degree of x in ΣA), then the following equations hold:

$$\begin{aligned}
(x \cdot y) &= (-1)^{|x||y|} y \cdot x \\
x \cdot (y \cdot z) &= (x \cdot y) \cdot z \\
\{x \bullet y\} &= -(-1)^{|sx||sy|}\{y \bullet x\} \\
\{x \bullet \{y \bullet z\}\} &= \{\{x \bullet y\} \bullet z\} + (-1)^{|sx||sy|}\{y \bullet \{x \bullet z\}\} \\
\{x \bullet y \cdot z\} &= \{x \bullet y\} \cdot z + (-1)^{|sx||y|} y \cdot \{x \bullet z\}.
\end{aligned}$$

The second last equation is sometimes called odd Jacobi and the last equation is the odd Poisson property. The idea is that $\bullet$ is of degree 1. We will actually see later on that $\bullet$ can be interpreted geometrically as a one dimensional simplex.

Definition 2.6.2. *The little discs operad D_2 is the operad given by $D_2(n) :=$ $\{$configurations of n discs labeled from 1 to n which are embedded into the unit disc of $\mathbb{R}^2\}$ together with the glueing described below and the $\mathbb{S}_n$ action by permuting the labels.*

Let us call the unit disc the outer disc. The glueing $\circ_i$ is defined by scaling the second configuration by a homothety so that the diameter of the outer disc coincides with the diameter of the i–th disc of the first configuration and then glueing in the scaled second configuration into the i-th disc of the first configuration. This is done by identifying the scaled outer disc of the second configuration and the i–th disc of the first configuration and erasing the identified boundary.

The following proposition is well known [C1, C2].

Proposition 2.6.3 (Cohen)**.** *Any Gerstenhaber algebra is an algebra over the little discs operad and vice versa.*

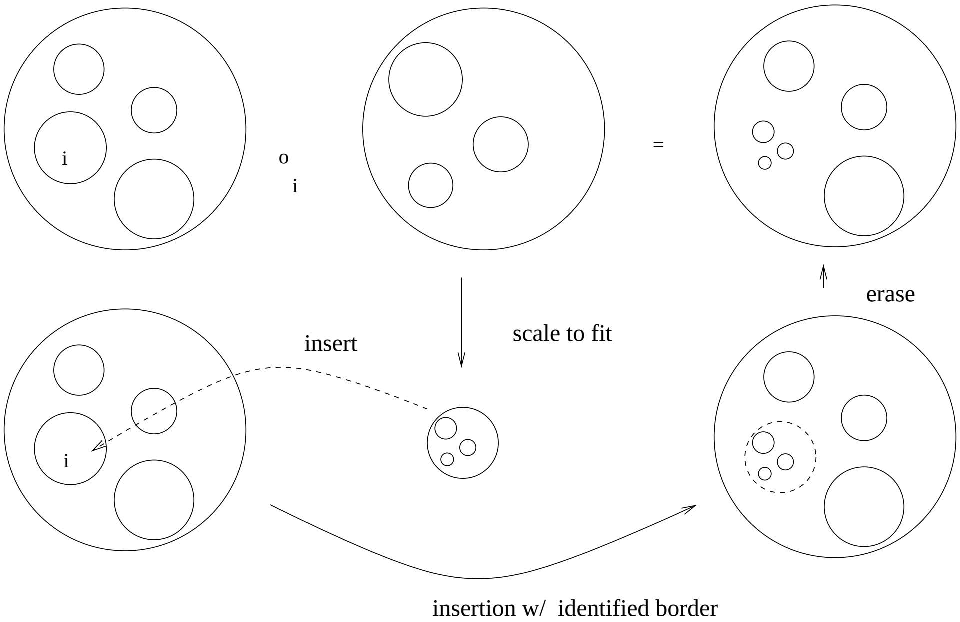


FIGURE 6. The composition maps for little discs

2.6.4. **The operad for Batalin–Vilkovisky (BV) algebras.**

Definition 2.6.4. *A Batalin–Vilkovisky (BV) algebra is an associative super–commutative algebra A together with an operator Δ of degree 1 that satisfies*

$$\begin{aligned} \Delta^2 &= 0 \\ \Delta(abc) &= \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{|sa||b|}b\Delta(ac) - \Delta(a)bc \\ &\quad -(-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c) \end{aligned}$$

Proposition 2.6.5. *For any BV–algebra (A, Δ) define*

$$\{a \bullet b\} := (-1)^{|a|}\Delta(ab) - (-1)^{|a|}(\Delta(a))b - a\Delta(b) \tag{2.4}$$

Then $(A, \{\bullet\})$ is a Gerstenhaber algebra.

We call a triple $(A, \{\bullet\}, \Delta)$ a GBV–algebra if (A, Δ) is a BV algebra and $\{\bullet\} : A \otimes A \to A$ satisfies the equation (2.4). By the Proposition above $(A, \{\bullet\})$ is a Gerstenhaber algebra.

Definition 2.6.6. *The framed little discs operad fD_2 is the operad given by $fD_2(n) :=$ {configurations of n discs labeled from 1 to n which are embedded into the unit disc of $\mathbb{R}^2$ together with an orientation (i.e. an angle $\theta \in [0, 2\pi]$) of each of the n discs} together with the permutation action of $\mathbb{S}_n$ on the labels and the glueing $\circ_i$ which is given by first rotating the second configuration by the angle θ_i, then scaling the configuration and finally inserting it.*

Proposition 2.6.7 (Getzler)**.** *Any Gerstenhaber algebra is an algebra over the framed little discs operad and vice versa.*

2.6.5. **The pre–Lie operad.**

Definition 2.6.8 (Gerstenhaber)**.** *A pre–Lie algebra is a graded vector space V together with a bilinear operation $*$ that satisfies*

$$(x * y) * z - x * (y * z) = (-1)^{|y||x|}((x * z) * y - x * (z * y))$$

Proposition 2.6.9. *(Gerstenhaber) Define*

$$\{a \bullet b\} := a * b - (-1)^{(|a|+1)(|b|+1)} b * a \tag{2.5}$$

Then $(V, \{\ \bullet\ \})$ is an odd Lie algebra.

In the case of no signs the operad which defines pre–Lie algebras is actually isomorphic to an insertion operad of trees. It is the operad structure on $\mathcal{RT}ree$ discussed in 2.3.

Proposition 2.6.10 (Livernet–Chapoton)**.** *Any pre–Lie algebra is an algebra over the operad of rooted trees $\mathcal{P}reLie$ and vice versa.*

To keep the signs is more tricky, but see 7.5.

2.7. **Operads of Moduli Spaces of curves.** The Deligne–Mumford compactifications $\bar{M}_{g,n}$ of the moduli spaces of curves of genus g with n marked points form an operad. The glueing is essentially glueing of two curves at two chosen marked points. If one allows self– glueing one obtains a so–called modular operad [GeK]. The theory of Gromov–Witten invariants yields the statement that the cohomology of a smooth variety is an algebra over the modular operad $H_*(\bar{M}_{g,n})$.

The spaces $\bar{M}_{0,n}$ form a suboperad (not modular) and the cohomology of a smooth variety as an algebra over $H_*(\bar{M}_{0,n})$ is usually called quantum cohomology.

One can also consider the open moduli spaces $M_{g,n}$ and these also form a modular operad [GeK]. The algebras over the suboperad $H_*(M_{0,n})$ are called gravity algebras. The relationship between $H_*(M_{0,n})$ and $H_*(\bar{M}_{0,n})$ is that they are Koszul dual to each other as quadratic operads [Ge2].

3. The Arc operad

3.1. **The operad structure.**

3.1.1. **The space.** In [KLP] we constructed an operad based on bordered surfaces with arcs projectively weighted by non–negative real numbers. This operad whose definition we will briefly recall is an extension of the bordered surface operad.

We fix a surface of genus g with r punctures and $n+1$ boundary components that are labelled from 0 to n and call it $F = F^s_{g,n+1}$. We also fix a window which is a closed proper subset $W_i \subsetneq \partial_i F$, for each boundary component $\partial_i F$.

Definition 3.1.1. *An essential arc in F is an embedded path a in F whose endpoints lie among the windows, where we demand that a is not isotopic rel endpoints to a path lying in ∂F. Two arcs are said to be parallel if there is an isotopy between them which fixes each $\partial_i - W_i$ pointwise (and fixes each W_i setwise) for $i = 1, 2, \dots, r$. An arc family in F is the isotopy class of an unordered collection of disjointly embedded essential arcs in F, no two of which are parallel. Thus, there is a well-defined action of the pure mapping class group on arc families. Where the pure mapping class group $PMC = PMC(F)$ is the group of isotopy classes of all orientation-preserving homeomorphisms of F which fix each $\partial_i - W_i$ pointwise (and fix each W_i setwise), for each $i = 1, 2, \dots, r$.*

The arc families have a natural partial order which is given by inclusion. This allows to build a simplicial cell complex whose k skeleton is composed of simplices indexed by arc families with k+1 arcs. These are attached to the k-1 skeleton by the face maps given by deleting one arc from the collection.

Definition 3.1.2. *We define $Arc'(F)$ to be the complex obtained in the above manner and $Arc(F)$ to be the topological space obtained as quotient of $Arc'(F)$ by the action of PMC. We also define $DArc(F) := Arc'(F) \times \mathbb{R}_{>0}$.*

We would like to remark that the points of $Arc(F)$ can and should be thought of as a mapping class group orbit of a projectively weighted (by positive real numbers) arc family on F and points of $DArc(F)$ as mapping class group orbits of weighted (by positive real numbers) arc families.

Any point of $Arc(F)$ lies in the interior of some cell of minimal dimension or on a vertex of the complex. If the vertex lies inside a cell, then assigning barycentric coordinates. We can now look at the point as being given by positive real co–ordinates w_j assigned to the arcs which make up the family; we call the w_j weights. If we drop the condition that the sum of the coordinates is one, then we obtain $DArc(F)$ by picking the coordinate on $\mathbb{R}_{>0}$ to be given by the sum and the coordinates on $Arc(F)$ to be given by the normalization. Finally, viewing $Arc(F)$ as the quotient of $DArc(F)$ by the action of $\mathbb{R}_{>0}$ scaling all coordinates at the same time, we obtain projective weights. If it lies on a vertex the point corresponds to a single arc which we can think of having any non–zero weight.

We will represent such a point by choosing a representative family and representative weights.

The underlying topological space for the n–th component our operad is an open subset of $\bigcup_{g,s} Arc(F^s_{g,n+1})$.

Definition 3.1.3. *An arc family is said to be exhaustive if for each boundary component ∂_i, for $i = 1, 2, \ldots, r$, there is at least one component arc in α with its endpoints in the window W_i. Likewise, a PMC-orbit of arc families is said to be exhaustive if some (that is, any) representing arc family is so. Define the topological spaces*

$$Arc^s_g(n) = \{[\alpha] \in Arc(F^s_{g,n+1}) : \alpha \text{ is exhaustive}\} \tag{3.1}$$

3.1.2. **The operad structure alias the glueing maps.** The definition of the glueing maps is best and most naturally done in the setting of partial measured foliations. The details can be found in [KLP]. The basic idea however is the following: First we glue two surfaces in the standard operadic fashion 2.2.2, i.e. boundary 0 of the second surface to the boundary of the first surface. Secondly, we have to give a procedure, how to glue together the bands. To this end we would like to think of the weighted arcs as bands of width given by the weights; this can be done by thickening the arcs into a partially measured foliation – which we view as a collection of bands. Now we arrange the bands in the window, such that in a neighborhood of the window the bands looks is depicted in figure 7 IV. In this way, the bands or the partially measured foliation can equivalently be thought of as a partition on an interval as depicted in figure 8 I.

Now if the sum of the weights of the arcs hitting the two boundary components of the two surfaces that are to be glued happen to coincide, we can "splice" the bands according to the largest common refinement of the two partitions; this is depicted in figure 9 I.

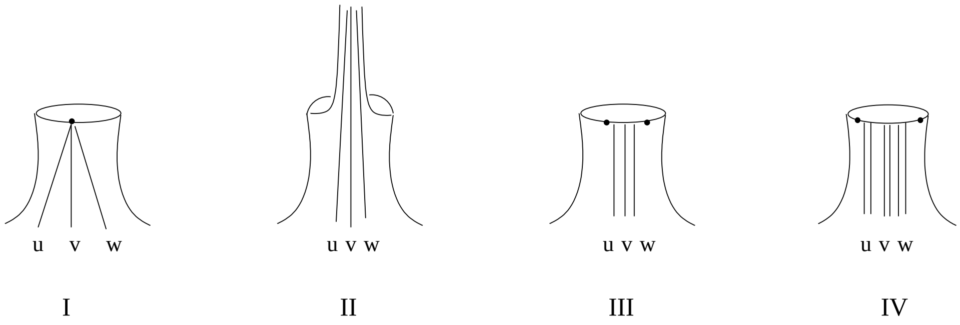


FIGURE 7. I. Arcs running to a point on the boundary; II. Arcs running to a point at infinity; III. Arcs in a window; IV. Bands in a window

3.1.3. **Several models for arcs.** To elucidate the role of the windows, we would like to briefly recall from [KLP] several geometric models for the common underlying combinatorics of arc families.

For the first such model, let us choose a distinguished point $d_i \in \partial_i$, for $i = 1, 2, \dots, r$, and consider the space of all complete finite-area metrics on F of constant Gauss curvature -1 (so-called "hyperbolic metrics") so that each $\partial_i^\times = \partial_i - \{d_i\}$ is totally geodesic (so-called "quasi hyperbolic metrics") on F. To explain this, consider a hyperbolic metric with geodesic boundary on a once-punctured annulus A and the simple geodesic arc a in it asymptotic in both directions to the puncture; the induced metric on a component of $A - a$ gives a model for the quasi hyperbolic structure on $F^\times = F - \{d_i\}_1^r$ near $\partial_i^\times$. The first geometric model for an arc family α in F is as a set of disjointly embedded geodesics in $F^\times$, each component of which is asymptotic in both directions to some distinguished point d_i; see part II of figure 7.

In the homotopy class of each ∂_i, there is a unique geodesic $\partial_i^* \subset F^\times$. Excising from $F - \cup\{\partial_i^\times\}_1^r$ any component which contains a point of $\partial^\times$, we obtain a hyperbolic structure on the surface $F^* \subseteq F^\times$ with geodesic boundary (where in the special case of an annulus, F^* collapses to a circle). Taking $\alpha \cap F^*$, we find a collection of geodesic arcs connecting boundary components (where in the special case of the annulus, we find two points in the circle).

This is our second geometric model for arc families. We may furthermore choose a distinguished point $p_i \in \partial_i^*$ and a regular neighborhood U_i of p_i in ∂_i^*, for $i = 1, 2, \dots, r$. Provided $p_i \notin \alpha$, we may take U_i sufficiently small that $U_i \cap \alpha = \emptyset$, so the arc $V_i = \partial_i^* - U_i$ forms a natural "window" containing $\alpha \cap \partial_i^*$. There is then an ambient isotopy of F^* which shrinks each window V_i down to a small arc $W_i \subseteq \partial_i^*$, under which α is transported to a family of (non-geodesic) arcs with endpoints in the windows W_i. In case p_i does lie in α, then let us simply move p_i a small amount in the direction of the natural orientation (as a boundary component of F^*) along ∂_i^* and perform the same construction; see part III of figure 7.

This leads to our final geometric model of arc families, namely, the model we used to define our spaces: arcs in a bounded surface with endpoints in windows. This third model is in the spirit of train tracks and measured foliations (cf. [PH]) as we shall see and is most convenient for describing the operadic structure.

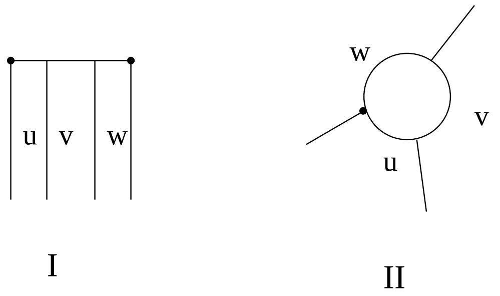


FIGURE 8. I. Bands ending on an interval; II. Bands ending on a circle

In this picture there is also then a unique orientation-preserving mapping [KLP]

$$c_i^{(\alpha)} : \partial_i(\alpha) \to S^1 \tag{3.2}$$

which maps the (class of the) first point of $\partial_i(\alpha)$ in the orientation of the window W_i to $0 \in S^1$, contracts the parts of the boundary outside the window and not hit by the bands, and maps the boundary points which are part of the bands to S^1 according to the normalized partial measure on the bands, which in our rudimentary discussion is just given by the weight and the normalizations means that the total weight is one. If we do not normalize, the map will take us to a circle of radius given by the total weight.

3.1.4. **Pictorial representations of arc families.** As explained before, there are several ways in which to imagine weighted arc families near the boundary. They are illustrated in figure 7. It is also convenient to view arcs near a boundary component as coalesced into a single wide band by collapsing to a point each interval in the window complementary to the bands; this *interval model* is illustrated in figure 8, part I. It is also sometimes convenient to further take the image under the maps 3.2 to produce the *circle model* as is depicted in figure 8, part II.

3.1.5. **Glueing weighted arc families.** Given two weighted arc families (α') in $F^s_{g,m+1}$ and (β') in $F^t_{h,n+1}$ so that $\mu_i(\partial_i(\alpha')) = \mu_0(\partial_0(\beta'))$, for some $1 \leq i \leq m$, we shall next make choices to define a weighted arc family in $F^{s+t}_{g+h,m+n}$ as follows.

First of all, let $\partial_0, \partial_1, \ldots, \partial_m$ denote the boundary components of $F^s_{g,m+1}$, let $\partial'_0, \partial'_1, \ldots, \partial'_n$ denote the boundary components of $F^t_{h,n+1}$, and fix some index $1 \leq i \leq m$. Each boundary component inherits an orientation in the standard manner from the orientations of the surfaces, and we may choose any orientation-preserving homeomorphisms $\xi : \partial_i \to S^1$ and $\eta : \partial'_0 \to S^1$ each of which maps the initial point of the respective window to the base-point $0 \in S^1$. Glueing together ∂_i and ∂'_0 by identifying $x \in S^1$ with $y \in S^1$ if $\xi(x) = \eta(y)$ produces a space X homeomorphic to $F^{s+t}_{g+h,m+n}$, where the two curves ∂_i and ∂'_0 are thus identified to a single separating curve in X. There is no natural choice of homeomorphism of X with $F^{s+t}_{g+h,m+n}$, but there are canonical inclusions $j : F^s_{g,m+1} \to X$ and $k : F^t_{h,n+1} \to X$.

We enumerate the boundary components of X in the order

$$\partial_0, \partial_1, \ldots, \partial_{i-1}, \partial'_1, \partial'_2, \ldots \partial'_n, \partial_{i+1}, \partial_{i+2}, \ldots \partial_m$$

and re-index letting ∂_j, for $j = 0, 1, \ldots, m+n-1$, denote the boundary components of X in this order. Likewise, first enumerate the punctures of $F^s_{g,m+1}$ in order and then those of $F^t_{h,n+1}$ to determine an enumeration of those of X, if any. Let us choose an orientation-preserving homeomorphism $H : X \to F^{s+t}_{g+h,m+n}$ which preserves the labeling of the boundary components as well as those of the punctures, if any.

In order to define the required weighted arc family, consider the partial measured foliations $\mathcal{G}$ in $F^s_{g,m+1}$ and $\mathcal{H}$ in $F^t_{h,n+1}$ corresponding respectively to (α') and (β'). By our assumption that $\mu_i(\partial_i(\alpha')) = \mu_0(\partial_0(\beta'))$, we may produce a corresponding partial measured foliation $\mathcal{F}$ in X by identifying the points $x \in \partial_i(\alpha')$ and $y \in \partial_0(\beta')$ if $c^{(\alpha)}_i(x) = c^{(\beta)}_0(y)$. The resulting partial measured foliation $\mathcal{F}$ may have simple closed curve leaves which we must simply discard to produce yet another partial measured foliation $\mathcal{F}'$ in X. The leaves of $\mathcal{F}'$ thus run between boundary components of X and therefore, as in the previous section, decompose into a collection of bands B_i of some widths w_i, for $i = 1, 2, \dots I$. Choose a leaf of $\mathcal{F}'$ in each such band B_i and associate to it the weight w_i given by the width of B_i to determine a weighted arc family (δ') in X which is evidently exhaustive. Let $(\gamma') = H(\delta')$ denote the image in $F^{s+t}_{g+h,m+n}$ under H of this weighted arc family.

It is a fact that this is well defined also on PMC orbits, see [KLP].

Definition 3.1.4. *Given $[\alpha] \in Arc^s_g(m)$ and $[\beta] \in Arc^t_h(n)$ and an index $1 \le i \le m$, let us choose respective deprojectivizations (α') and (β') and write the weights*

$$w(\alpha') = (u_0, u_1, \dots, u_m),$$
$$w(\beta') = (v_0, v_1, \dots, v_n).$$

Define

$$\rho_0 = \sum_{\{b\in\beta:\partial b\cap\partial_0\neq\emptyset\}} v_i,$$
$$\rho_i = \sum_{\{a\in\alpha:\partial a\cap\partial_i\neq\emptyset\}} u_i,$$

where in each sum the weights are taken with multiplicity, e.g., if a has both endpoints at ∂_0, then there are two corresponding terms in ρ_0.

Since both arc families are exhaustive, $\rho_i \neq 0 \neq \rho_0$, and we may re-scale

$$\rho_0 w(\alpha') = (\rho_0 u_0, \rho_0 u_1, \dots, \rho_0 u_m),$$
$$\rho_i w(\beta') = (\rho_i v_0, \rho_i v_1, \dots, \rho_i v_n),$$

so that the 0^{th} entry of $\rho_i w(\beta')$ agrees with the i^{th} entry of $\rho_0 w(\alpha')$.

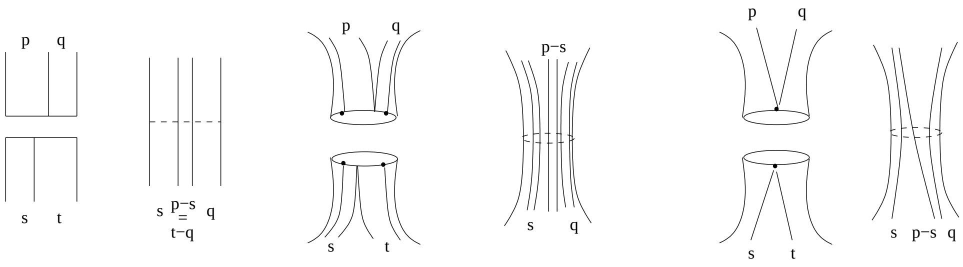
Thus, we may apply the composition of 3.1.5 to the re-scaled arc families to produce a corresponding weighted arc family (γ') in $F^{s+t}_{g+h,m+n}$, whose projective class is denoted $[\gamma] \in Arc^{s+t}_{g+h}(m+n-1)$. We let

$$[\alpha] \circ_i [\beta] = [\gamma],$$

in order to define the composition

$$\circ_i : Arc^s_g(m) \times Arc^t_h(n) \to Arc^{s+t}_{g+h}(m+n-1), \text{ for any } i = 1, 2, \dots, m.$$

3.1.6. **A pictorial representation of the glueing.** A graphical representation of the glueing can be found in figure 9, where we present the glueing in three of the different models.



I II III

FIGURE 9. The glueing: I. in the interval picture, II. in the windows with bands picture and III. in the arcs running to a marked point version.

Definition 3.1.5. *For each $n \geq 0$, let $\mathcal{ARC}_{cp}(n) = Arc_0^0(n)$ (where the "cp" stands for compact planar), and furthermore, define $\mathcal{ARC}(n)$ of the union over all $Arc_g^s(n)$, which we give the direct limit topology as $g, s \to \infty$.*

Theorem 3.1.6. *The compositions $\circ_i$ of Definition 3.1.4 imbue the collection of spaces $\mathcal{ARC}(n)$ with the structure of a topological operad under the natural $\mathbb{S}_n$–action on labels on the boundary components. The operad has a unit $1 \in \mathcal{ARC}(1)$ given by the class of an arc in the cylinder meeting both boundary components, and the operad is cyclic for the natural $\mathbb{S}_{n+1}$–action permuting the labels of the boundary components. Furthermore the spaces $\mathcal{ARC}_{cp}$ form a cyclic suboperad.*

3.1.7. **The deprojectivized spaces $\mathcal{D}\mathcal{A}rc$.** For the following it is convenient to introduce deprojectivized arc families. This amounts to adding a factor $\mathbb{R}_{>0}$ for the overall scale.

Let $\mathcal{D}\mathcal{A}rc(n) = \mathcal{ARC}(n) \times \mathbb{R}_{>0}$ be the space of weighted arc families; it is clear that $\mathcal{D}\mathcal{A}rc(n)$ is homotopy equivalent to $\mathcal{ARC}(n)$.

As the definition 3.1.4 of gluing was obtained by lifting to weighted arc families and then projecting back, we can promote the compositions to the level of the spaces $\mathcal{D}\mathcal{A}rc(n)$. This endows the spaces $\mathcal{D}\mathcal{A}rc(n)$ with a structure of a cyclic operad as well. Moreover, by construction the two operadic structures are compatible. This type of composition can be compared to the composition of loops, where such a rescaling is also inherent. In our case, however, the scaling is performed on both sides which renders the operad cyclic.

In this context, the total weight at a given boundary component given by the sum of the individual weights w_t of incident arcs makes sense, and thus the map 3.2 can be naturally viewed as map to a circle of radius $\sum_t w_t$.

3.2. **Notation.** We denote the operad on the collection of spaces $\mathcal{ARC}(n)$ by $\mathcal{ARC}$ and the operad on the collection of spaces $\mathcal{D}\mathcal{A}rc(n)$ by $\mathcal{D}\mathcal{A}rc$. By an "Arc algebra", we mean an algebra over the homology operad of $\mathcal{ARC}$. Likewise, $\mathcal{ARC}_{cp}$ and $\mathcal{D}\mathcal{A}rc_{cp}$ are comprised of the spaces $\mathcal{ARC}_{cp}(n)$ and $\mathcal{D}\mathcal{A}rc_{cp}(n)$ respectively, and an "Arc$_{cp}$ algebra" is an algebra over the homology of $\mathcal{ARC}_{cp}$

3.2.1. **Sub–operads, Props.** There are several natural suboperads for the arc operad, given by imposing certain conditions on the arcs.

For example, one may specify a symmetric $(n+1)$-by-$(n+1)$ matrix $A^{(n)}$ as well as an $(n+1)$-vector $R^{(n)}$ of zeroes and ones over $\mathbb{Z}/2\mathbb{Z}$ and consider the subspace of $Arc(F^s_{g,n+1})$ where arcs are allowed to run between boundary components i and j if and only if $A^{(n)}_{ij} \neq 0$ and are required to meet the boundary component k if and only if $R^{(n)}_k \neq 0$. For instance, the case of interest in this paper corresponds to $A^{(n)}$ the matrix whose entries are all one, and $R^{(n)}$ the vector whose entries are also all one.

For a class of examples of props (the generalization of operads with arbitrary inputs and outputs, for a definition see e.g. [MSS]), consider a partition of $\{0, 1, \dots n\} = I^{(n)} \sqcup O^{(n)}$, into "inputs" and "outputs", where $A^{(n)}_{ij} = 1$ if and only if $\{i, j\} \cap I^{(n)}$ and $\{i, j\} \cap O^{(n)}$ are each singletons, and $R^{(n)}$ is the vector whose entries are all one.

Definition 3.2.1. *The trees suboperad is defined for arc families in surfaces with $g = s = 0$ in the notation explained above by the allowed incidence matrix $A^{(n)}$, whose non–zero entries are $a_{0i} = 1 = a_{i0}$, for $i = 1, \dots, n$, and required incidence relations $R^{(n)}$, whose entries are all equal to one.*

This is a suboperad of $\mathcal{ARC}_{cp}$, and a representation of it as a collection of labeled trees can be found in [K3].

Dropping the requirement that $g = s = 0$, we obtain a suboperad of $\mathcal{ARC}$ called the rooted graphs *or* Chinese trees *suboperad.*

3.2.2. **Linearity Condition.** We say that an element of the (cyclic Chinese) trees suboperad satisfies the Linearity Condition if the linear orders match, i.e., the bands hitting each boundary component in their linear order are a subchain of all the bands in their linear order derived from the 0–th boundary.

It is easy to check that this condition is stable under composition.

We call the suboperad of elements satisfying the Linearity Condition of the (cyclic Chinese) trees operad the *(cyclic Chinese) linear trees operad*.

The following proposition form [KLP] clarifies the role of this condition.

Proposition 3.2.2. *The suboperad generated by (cyclic Chinese) linear trees and $\mathcal{ARC}_{cp}(1)$ inside $\mathcal{ARC}$ coincides with (cyclic Chinese) trees.*

3.3. **String interpretation.** One way that one can view the $\mathcal{ARC}$ operad is as tracing n closed strings as they move, split and recombine to form one loop. If one traces fixed base points of the strings and keeps track of the length of the strings that the closed strings may break into, one arrives exactly at the arc picture. Here one arc of weight w or better band of width w depicts the movement of a piece of string of length w. Starting with n strings that may move, break and recombine to become one string, one obtains the trees suboperad.

If the strings have in this way swept over a genus zero surface (with no punctures) this information is enough to recover the surface, since one only needs to know the number of boundary components. If there is non–trivial topology, however, this information may not be enough, but one can consider all graphs of the traces of the base point on a given fixed topological surface. This is what the arc operad does, cf. Figure 10.

If one wishes to see n strings move and recombine into m strings one arrives at the notion of the props discussed above. This corresponds to having a kind of Morse function for the surface which gives the state of the strings at a given time. In this

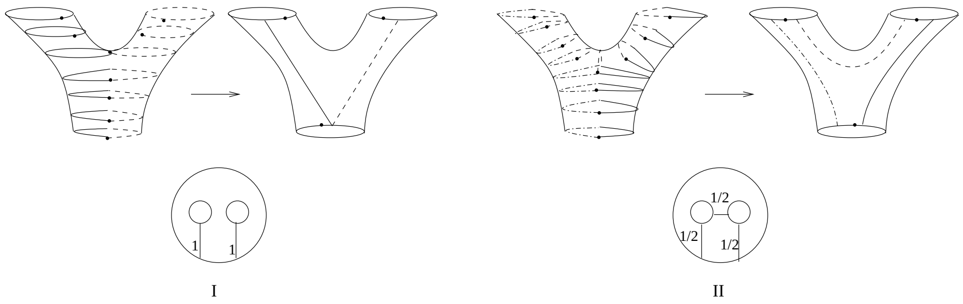


FIGURE 10. Movements of strings combining I and breaking, annihilating and combining II

view, the singular level sets are the important ingredients and a careful analysis of this picture leads to the $\mathcal{C}acti$ and the consideration of [CS1].

On the other hand, from the point of view of closed string field theory [Z] there is actually no big distinction of incoming and outgoing circles in the sense of our prop definitions, as strings may also annihilate.

Thus even in genus zero, we do not need to restrict to the trees in the arc operad, but see all of $\mathcal{ARC}_{cp}$ and of course in general all of $\mathcal{ARC}$. The depiction of a pair of pants according to closed string field theory is not a figure eight as before, but a theta shape, as indicated in figure 10 and figure 24. This type of shape has recently also played a role in the geometry of the so–called stringor bundles [ST].

3.4. **Relation to Moduli spaces.** In this section, we would like to very briefly digress on the relation of the spaces $Arc(F)$ and the $\mathcal{ARC}$ operad to moduli spaces, due to Penner. All the details are contained in the papers [P2, P3].

Assume that $r \neq 0$ (and allow both cases $s = 0$ or $s > 0$). Enumerate the (smooth) boundary components of F as ∂_i, where $i = 1, \dots, r$ and set $\partial = \cup\{\partial_i\}_1^r$. Let $Hyp(F)$ be the space of all hyperbolic metrics with geodesic boundary on F. Define the moduli space to be

$$M = M(F) = [Hyp(F)(\prod_1^r \partial_i)]/\sim$$

where $\sim$ is the equivalence relation generated by push-forward of metric under orientation- preserving diffeomorphism

$$f_*(\Gamma, (\xi_i)_1^r) = (f_*(\Gamma), f(\xi_i)_1^r)$$

where $\Gamma \mapsto f_*(\Gamma)$ is the usual push-forward of metric on $Hyp(F)$.

Now let $M(F)/\mathbb{R}_{>0}$ denote the quotient of the moduli space of the bordered surface F by the action of $\mathbb{R}_{>0}$ by homothety on the tuple of hyperbolic lengths of the geodesic boundary components. Denote by $Arc_\#(g,r)^s$ the space of quasi-filling arc families, which means that complementary regions are either a polygon or a once punctured polygon.

Notice that for $g = s = r = 0$ $Arc_\#(0,0)^0 = Arc_{cp}$ and in general $Arc_\#(g,r)^s \subset Arc^s_{g,r}$.

Theorem 3.4.1. [P2] *For any bordered surface $F \neq F^0_{0,2}$, $Arc_\#(F)$ is proper homotopy equivalent to $M(F)/\mathbb{R}_{>0}$.*

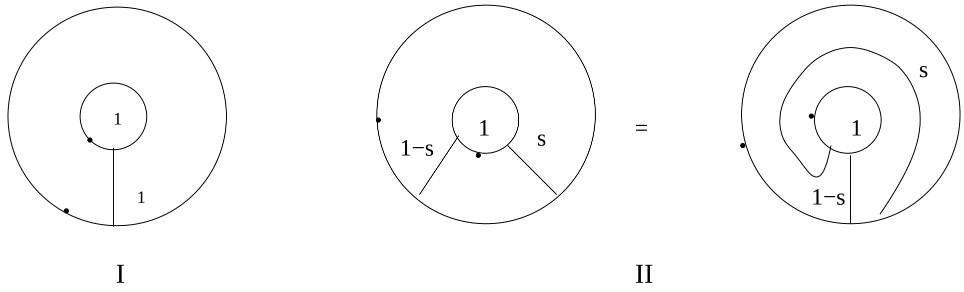


FIGURE 11. I. The identity and II. the arc family δ yielding the BV operator

Furthermore, each of $M(F)/\mathbb{R}_{>0}$ and $Arc_{\#}(F)$ admit natural $(S^1)^r$–actions (moving distinguished points in the boundaries for the former and "twisting" arc families around the boundary components for the latter); the proper homotopy equivalence in the Theorem above is in fact a map of $(S^1)^r$–spaces.

It is astonishing that in the case of genus zero with no punctures, there is another relationship on the algebraic level. It was shown by Getzler [Ge] that the homology operad of the spaces $M_{0,n}$ yields the notion of a gravity or L_∞ algebra. The corresponding cohomology classes are also present in our $\mathcal{ARC}$ picture through the BV structure discussed below, since this structure allows to define the higher brackets [Ge] as well. Actually there are good candidates for arc families leading to these brackets which we will discuss elsewhere [KLP2].

In a sense, what the $\mathcal{ARC}$ operad does is to replace the conformal field theory given by the operads $H_*(M_{g,n})$ by its hyperbolic counterpart. The benefit, as always when dealing with hyperbolic aspects of moduli theory, the formulas become discrete and all spaces tend to have a PL–structure which makes everything very manageable.

The BV structure below legitimizes this point of view, since it is what one would expect from the non–compactified moduli. These are in turn related to the operad constituted by the Deligne–Mumford compactified spaces by Koszul duality for quadratic operads.

It is also interesting that the cell decomposition of trees we present in §5.3.2 is also related to Strebel differentials via the indexing set of trees as described in [KS].

3.5. **Arc families and their induced operations.** The points in $\mathcal{ARC}_{cp}(1)$ are parameterized by the circle, which is identified with $[0,1]$, where 0 is identified to 1. To describe a parameterized family of weighted arcs, we shall specify weights that depend upon the parameter $s \in [0,1]$. Thus, by taking $s \in [0,1]$ figure 11 describes a cycle $\delta \in C_1(1)$ that spans $H_1(\mathcal{ARC}_{cp}(1))$.

As stated above, there is an operation associated to the family δ. For instance, if F_1 is any arc family $F_1 : k_1 \to \mathcal{ARC}_{cp}$, δF_1 is the family parameterized by $I \times k_1 \to \mathcal{ARC}_{cp}$ with the map given by the picture by inserting F_1 into the position 1. By definition,

$$\Delta = -\delta \in C_1(1).$$

In $C_*(2)$ we have the basic families depicted in figure 12 which in turn yield operations on C_*.

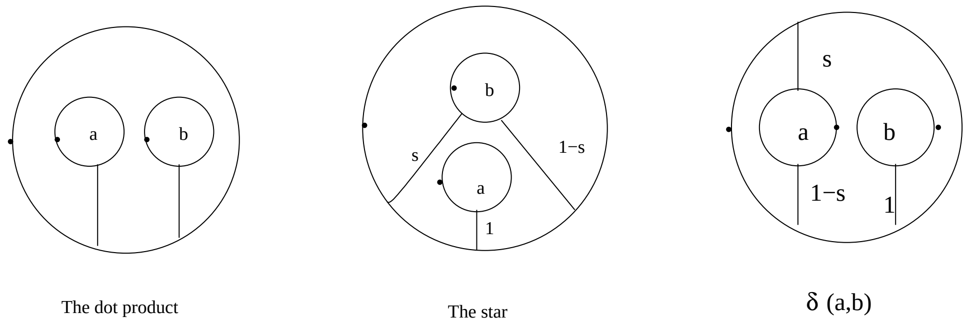


FIGURE 12. The binary operations

To fix the signs, we fix the parameterizations we will use for the glued families as follows: say the families F_1, F_2 are parameterized by $F_1 : k_1 \to \mathcal{ARC}_{cp}$ and $F_2 : k_2 \to \mathcal{ARC}_{cp}$ and $I = [0,1]$. Then $F_1 \cdot F_2$ is the family parameterized by $k_1 \times k_2 \to \mathcal{ARC}_{cp}$ as defined by figure 12 (i.e., the arc family F_1 inserted in boundary a and the arc family F_2 inserted in boundary b).

Interchanging labels 1 and 2 and using $*$ as a chain homotopy as in figure 13 yields the commutativity of $\cdot$ up to chain homotopy

$$d(F_1 * F_2) = (-1)^{|F_1||F_2|} F_2 \cdot F_1 - F_1 \cdot F_2 \tag{3.3}$$

Notice that the product $\cdot$ is associative up to chain homotopy.

Likewise $F_1 * F_2$ is defined to be the operation given by the second family of figure 12 with $s \in I = [0,1]$ parameterized over $k_1 \times I \times k_2 \to \mathcal{ARC}_{cp}$.

By interchanging the labels, we can produce a cycle $\{F_1, F_2\}$ as shown in figure 13 where now the whole family is parameterized by $k_1 \times I \times k_2 \to \mathcal{ARC}_{cp}$.

$$\{F_1, F_2\} := F_1 * F_2 - (-1)^{(|F_1|+1)(|F_2|+1)} F_2 * F_1.$$

Remark 3.5.1. *We have defined the following elements in C_*:*
δ and $\Delta = -\delta$ in $C_1(1)$;
$\cdot$ in $C_0(2)$, which is commutative and associative up to a boundary.
$*$ and $\{-,-\}$ in $C_1(2)$ with $d(*) = \tau \cdot - \cdot$ and $\{-,-\} = * - \tau *$.
Note that $\delta, \cdot$ and $\{-,-\}$ are cycles, whereas $*$ is not.

3.6. **The BV operator.** The operation corresponding to the arc family δ is easily seen to square to zero in homology. It is therefore a differential and a natural candidate for a derivation or a higher order differential operator. It is easily checked that it is *not a derivation*, but it *is a BV operator.*

Proposition 3.6.1. *The operator Δ satisfies the relation of a BV operator up to chain homotopy.*

$$\begin{aligned} \Delta^2 &\sim 0 \\ \Delta(abc) &\sim \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{|sa||b|}b\Delta(ac) - \Delta(a)bc \\ &\quad -(-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c) \end{aligned} \tag{3.4}$$

Thus, any Arc algebra and any Arc_{cp} algebra is a BV algebra.

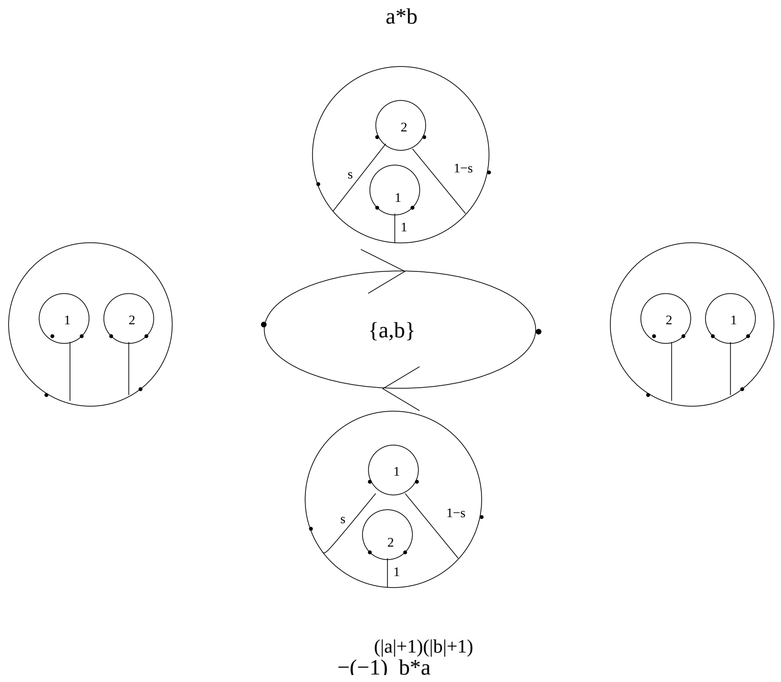


FIGURE 13. The definition of the Gerstenhaber bracket

Lemma 3.6.2.

$$\delta(a,b,c) \sim (-1)^{(|a|+1)|b|} b\delta(a,c) + \delta(a,b)c - \delta(a)bc \tag{3.5}$$

Proof. The proof is contained in figure 14. Let $a : k_a \to \mathcal{ARC}_{cp}$, $b : k_b \to \mathcal{ARC}_{cp}$ and $c : k_c \to \mathcal{ARC}_{cp}$, be arc families then the two parameter family filling the square is parameterized over $I \times I \times k_a \times k_b \times k_c$. This family gives us the desired chain homotopy. □

Given arc families $a : k_a \to \mathcal{ARC}_{cp}$, $b : k_b \to \mathcal{ARC}_{cp}$ and $c : k_c \to \mathcal{ARC}_{cp}$, we define the two parameter family defined by the figure 15 where the families in the rectangles are the depicted two parameter families parameterized over $I \times I \times k_a \times k_b \times k_c$ and the triangle is not filled, but rather its boundary is the operation $\delta(abc)$.

From the diagram we get the chain homotopy consisting of three, and respectively twelve, terms.

Remark 3.6.3. *The fact that the chain operads of $\mathcal{ARC}$ and as we show below $\mathcal{C}act(i)$ or $\mathcal{C}act^1(i)$ all possess the structure of a G(BV) algebra up to homotopy means that for any algebra V over them the algebra as well as $\mathcal{H}om_V$ have the structure of G(BV). If one is in the situation that one can lift the algebra to the chain level, then the G(BV) will exist on the chain level up to homotopy.*

Remark 3.6.4. *We would like to point out that the symbol $\bullet$ in the standard super notation of odd Lie–brackets $\{a \bullet b\}$, which is assigned to have an intrinsic degree of 1, corresponds geometrically in our situation to the one–dimensional interval I.*

3.7. **The associator.** It is instructive to do the calculation in the arc family picture with the operadic notation. For the glueing $* \circ_1 *$ we obtain the elements in $C_2(2)$

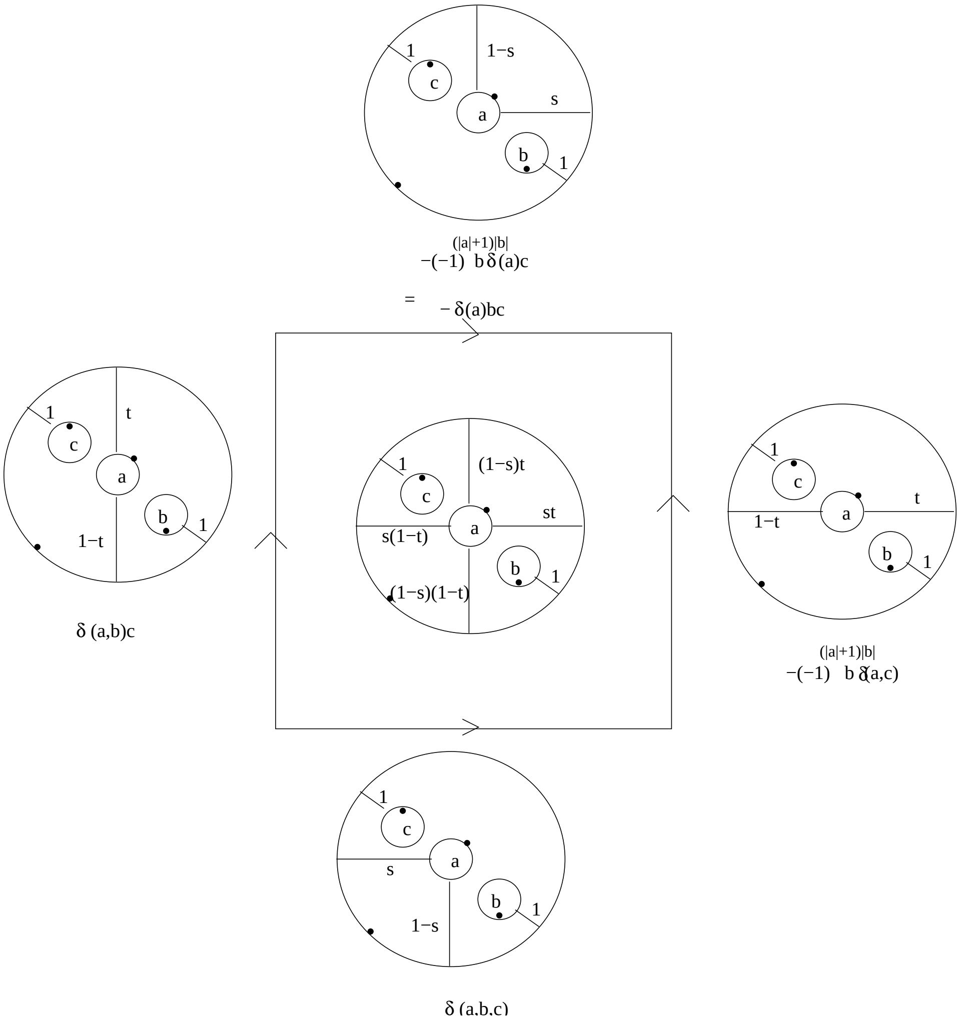


FIGURE 14. The basic chain homotopy responsible for BV

presented in figure 16 to which we apply the homotopy of changing the weight on the boundary 3 from 2 to 1 while keeping everything else fixed. We call this normalization.

Unraveling the definitions for the normalized version yields figure 17, where in the different cases the glueing of the bands is shown in figure 18.

The glueing $* \circ_2 *$ in arc families is simpler and yields the glueing depicted in figure 19 to which we apply a normalizing homotopy — by changing the weights on the bands emanating from boundary 1 from the pair $(2s, 2(1-s))$ to $(s, 1-s)$ using pointwise the homotopy $(\frac{1+t}{2}2s, \frac{1+t}{2}(1-s))$ for $t \in [0,1]$:

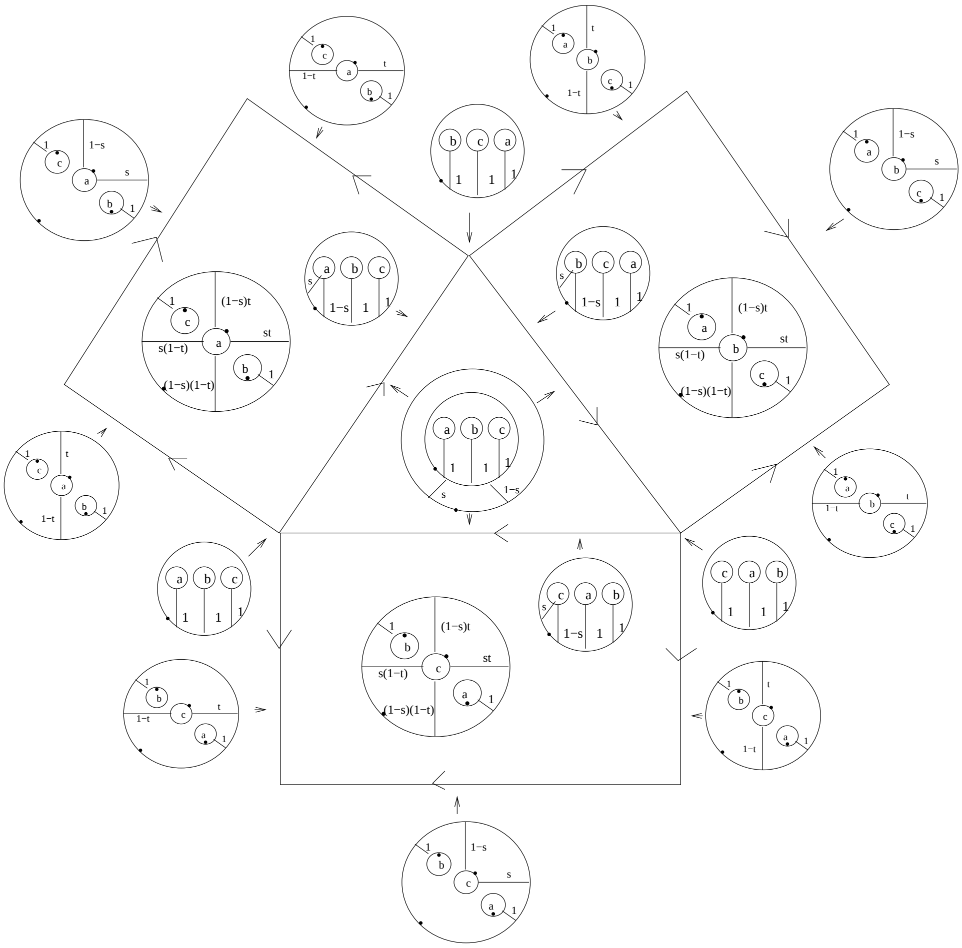


FIGURE 15. The homotopy BV equation

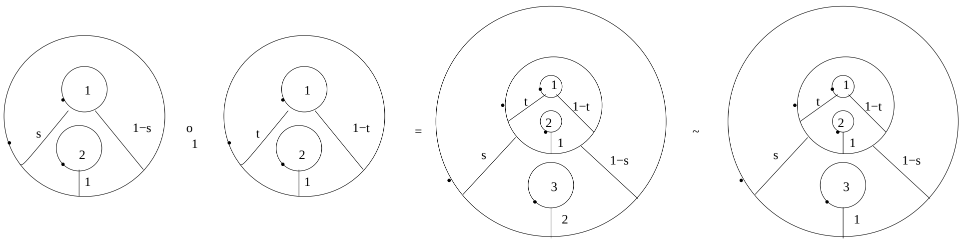


FIGURE 16. The first iterated glueing of $*$

Combining figures 17 and 19 while keeping in mind the parameterizations we can read off the pre–Lie relation:

$$F_1 * (F_2 * F_3) - (F_1 * F_2) * F_3 \sim (-1)^{(|F_1|+1)(|F_2|+1)}(F_2 * (F_1 * F_3) - (F_2 * F_1) * F_3) \tag{3.6}$$

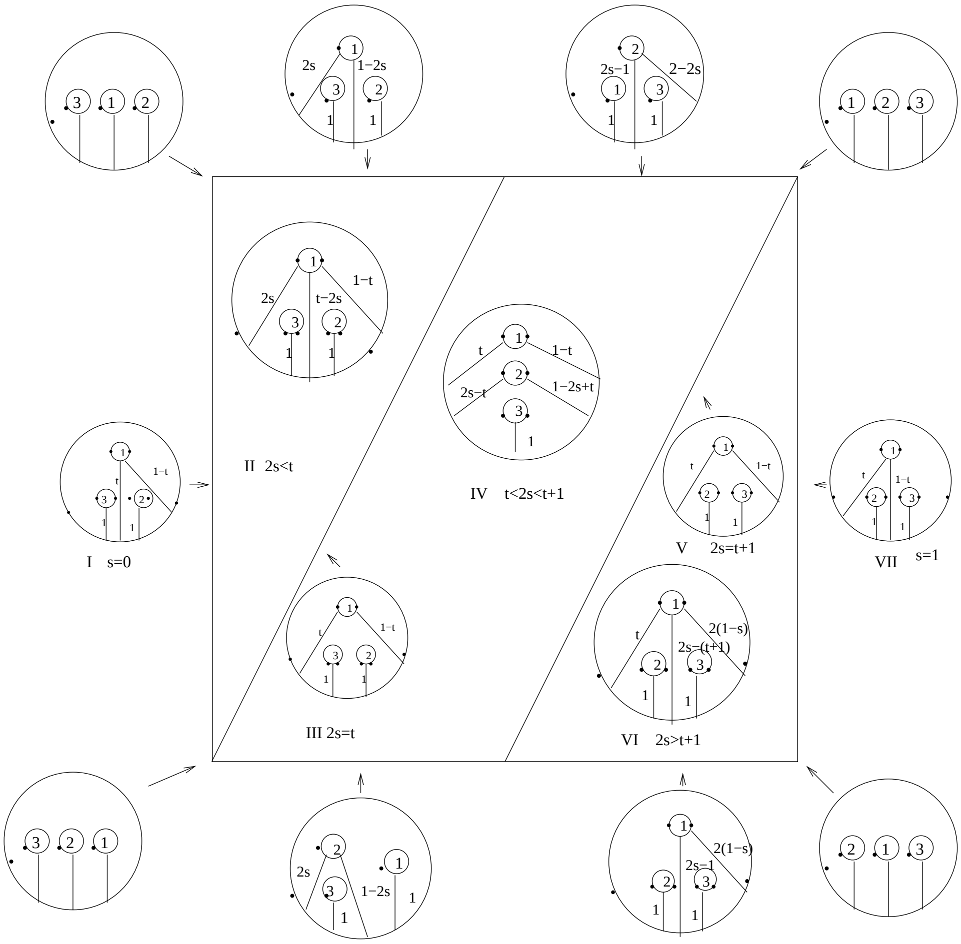


FIGURE 17. The glued family after normalization

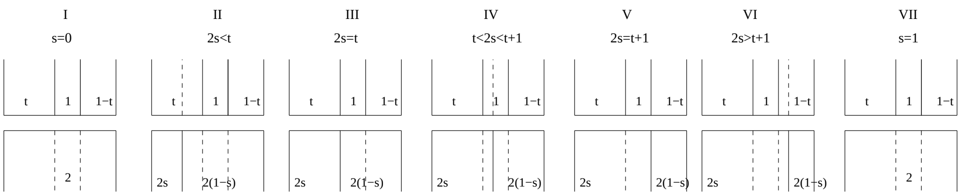


FIGURE 18. The different cases of glueing the bands

which shows that the associator is symmetric in the first two variables and thus following Gerstenhaber [G] we obtain:

Corollary 3.7.1. $\{F_1, F_2\}$ *satisfies the odd Jacobi identity.*

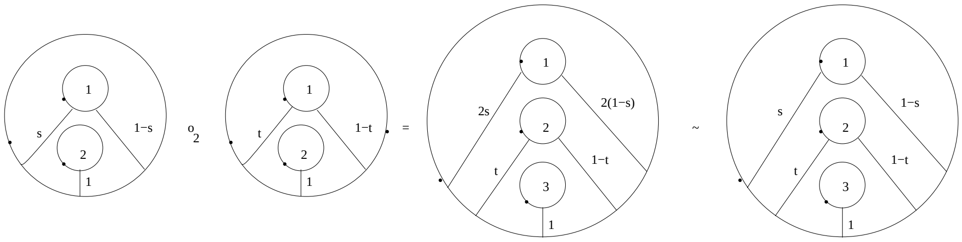


FIGURE 19. The other iteration of $*$

4. Species of Cacti and their relations to other operads

4.1. **Configurations of loops and their graphs.** There are several species of cacti, which are defined in [K3], to which we refer the reader for details. We briefly recall the main definitions here. In words, the cacti operad which was introduced by Voronov [V] has as its n–th component defined in connected, planar tree-like configurations of parameterized loops (of possibly different circumference), together with a marked point on the configuration. The spineless cacti defined in [K3] are the suboperad where the zero of the parameterization corresponds to the lowest intersection point. There are also normalized versions of these configurations, also introduced in [K3], where the circumference of each loop is fixed to be one.

To give a more precise definition we need the following definitions.

We denote the standard circle of radius r by $S^1_r := \{(x,y) \subset \mathbb{R}^2 : x^2 + y^2 = r^2\}$.

Definition 4.1.1. *A configuration of n parameterized loops is a collection $(l_1, \dots, l_n)$ of n orientation preserving continuous injections —called loops— $l_i : S^1_{r_i} \to \mathbb{R}^2$ considered up to isotopy. Where the isotopy is required to fix the incidence conditions, that is if $h^i_t : S^1_{r_i} \times I$ are the isotopies and $h^i_0(p) = l_i(p) = l_j(q) = h^j_0(q)$ then for all t: $h^i_t(p) = h^j_t(q)$ and vice–versa if $h^i_0(p) = l_i(p) \neq l_j(q) = h^j_0(q)$ then for all t: $h^i_t(p) \neq h^j_t(q)$. A pointed configuration is a configuration together with a marked component l_i and marked point on this component $* \in S^1_{r_i}$.*

Definition 4.1.2. *For a configuration of n parameterized loops, with only finitely many intersection points, we can define a bipartite b/w graph as follows: There is one white vertex for each loop and one black vertex for each intersection point. We join a white vertex and a black vertex by an edge, if the intersection point corresponding to the black vertex lies on the loop corresponding to the white vertex. We call this black and white graph the graph of the configuration. We also endow each vertex with the cyclic order coming from the orientation in the plane.*

For a pointed configuration, we include one more black vertex for the marked point and a second black vertex, the root and draw edges from the vertex for the marked point to each white vertex of components the marked point lies on. We also endow the vertex for the marked point by the linear order given by the cyclic order induced from the plane and the choice of the smallest element being the root edge followed by the edge for the marked component.

4.2. **Cacti and spineless Cacti.**

Definition 4.2.1. *The cacti operad which was introduced by Voronov* [V] *has as its n–th component pointed configurations of n parameterized loops whose image is* *connected and whose graph is a tree. The space $\mathcal{C}act(n)$ is endowed with the action of $\mathbb{S}_n$ by permuting the labels.*

Definition 4.2.2. *Notice that the tree (graph) of a cactus is actually a white–leaved planar planted tree which thus has a linear order on all of the vertices. We choose to plant the tree to reflect the linear order at the root.*

We will allow ourselves to talk about the image of a cactus in $\mathbb{R}^2$ by picking a representative $(l_1, \dots, l_n)$ and considering $\bigcup_i l(S^1_{r_i})$ keeping in mind that this is only defined up to isotopy.

The loops, or rather the inside of these loops, are sometimes called lobes — again the above remark applies.

Definition 4.2.3. *Given a cactus $(l_1, \dots, l_n)$ whose loops have radii r_i there is a surjective orientation preserving map from $S^1_{R=\sum r_i} \mapsto \bigcup_i l(S^1_{r_i})$, whose only multiple points are the intersection points of the loop. This map is defined as follows. Start at the marked point (the global zero) and go around the marked loop counterclockwise; if a double point is hit continue on the next loop to the right (i.e. the next in the cyclic order) and continue in this manner until one returns to the marked point. We will call this map the "outside circle" and sometimes refer to the marked point as the "global zero", since it is the image of $0 \in S^1_R$.*

4.2.1. **Glueing for Cacti.** We define the following operations

$$\circ_i : Cacti(n) \times Cacti(m) \to Cacti(n+m-1) \tag{4.1}$$

by the following procedure: given two cacti without spines we reparameterize the outside circle of the second cactus to have length r_i which is the length of the i-th circle of the first cactus. Then glue in the second cactus by identifying the outside circle of the second cactus with the i–th circle of the first cactus.

Definition 4.2.4. *The spineless variety of cacti is obtained by postulating that the local zeros defined by the parameterizations of the loops coincide with the first intersection point of the perimeter with a loop (sometimes called a "lobe") of the cactus. Here the first intersection point is the point given by the black vertex which lies on the outgoing edge of the white vertex representing the parameterized loop under consideration. This suboperad inherits the permutation action of $\mathbb{S}_n$ on the labels.*

Remark 4.2.5. *The restriction of the glueing for Cacti*

$$\circ_i : Cact(n) \times Cact(m) \to Cact(n+m-1) \tag{4.2}$$

makes $\mathcal{C}act$ into an operad which is a suboperad of $\mathcal{C}acti$.

Definition 4.2.6. *The spaces $\mathcal{C}act^1(n)$ are the subspaces of $\mathcal{C}act(n)$ with the restriction that all the radii of the lobes are fixed to one.*

4.2.2. **Scaling of a cactus, projective cacti.** Cacti and spineless cacti both come with a universal scaling operation of $\mathbb{R}_{>0}$ which simultaneously scales all radii by the same factor $\lambda \in \mathbb{R}_{>0}$. This action is a free action and the glueing descends to the quotient by this action. We sometimes call these operads projective cacti or spineless projective cacti.

4.2.3. **Left, right and symmetric cacti operads.** For the glueing above one has three basic possibilities to scale in order to make the size of the outer loop of the cactus that is to be inserted match the size of the lobe into which the insertion should be made.

(1) Scale down the cactus which is to be inserted. This is the original version – we call it the right scaling version.
(2) Scale up the cactus into which will be inserted. We call it the left scaling version.
(3) Scale both cacti. The one which is to be inserted by the size of the lobe into which it will be inserted and the cactus into which the insertion is going to be taking place by the size of the outer loop of the cactus which will be inserted. We call this it the symmetric scaling version.

All of these versions are of course homotopy equivalent and in the quotient operad of $\mathcal{C}acti$ by overall scalings, the projective cacti $\mathcal{C}acti/\mathbb{R}_{>0}$ they all descend to the same glueing.

The advantages of the different versions are that version (1) is the original one and inspired by the rescaling of loops, i.e. the size of the outer loop of the first cactus is constant. Version (2) has the advantage that cacti whose lobes have integer sizes are a suboperad. We will use this later on. And version (3), the symmetric version, is the one we also used in $\mathcal{DA}rc$ and as shown below, in this version there is an embedding of the cacti operad into the cyclic operad $\mathcal{DA}rc$.

4.3. **Cacti as a suboperad of $\mathcal{DA}rc$.**

4.3.1. **Framing of a cactus.** We will give a map of cacti into $\mathcal{ARC}$ called a framing. First notice that a cactus can be decomposed by the initial point and the intersection points and the local zeros into a sequence of arcs following the natural orientation given by the data. These arcs are labeled by their lengths as parts of $l_i(S^1_{r_i})$. To frame a given cactus, draw a pointed circle around it and run an arc from each arc of the cactus to the outside circle respecting the linear order given by the outside loop, i.e. starting with the initial arc of the cactus as the first arc emanating from the outside circle in its orientation. Label each such arc by the parameter associated to the arc of the cactus.

We can think of attaching wide bands to the arc of the cactus. The widths of the bands are just the lengths of the arcs to which they are attached. Using these bands we identify the outside circle with the circumference of the cactus. Notice that this "outside" circle appears in the glueing formalism for cacti.

The marked points on the inside boundaries correspond to the local zeros of the inside circles viz. lobes of the cactus.

Two examples of this procedure are provided in figure 20.

Remark 4.3.1. *If one frames a spineless cactus, then the image is in the linear trees.*

4.4. **The Loop of an arc family.** Given a surface with arcs we can forget some of the structure and in this way either produce a collection of loops or one loop which is given by using the arcs as an equivalence relation.

4.4.1. **The boundary circles.** Given an exhaustive weighted arc family (α) in the surface F, we can consider the measure-preserving maps

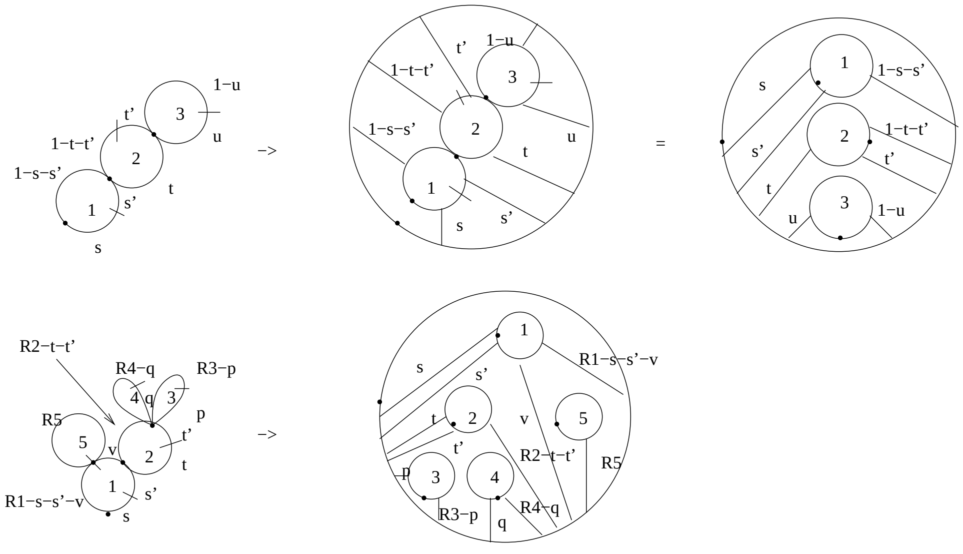


FIGURE 20. Framings of cacti

$$\tilde{c}_i^{(\alpha)} : \partial_i(\alpha) \to S^1_{m_i} \tag{4.3}$$

where S^1_r is a circle of radius r and $m_i = \mu^i(\partial_i(\alpha))$ is the total weight of the arc family at the i–th boundary. Combining these maps, we obtain

$$\tilde{c} : \partial(\alpha) \to \coprod_i S^1_{m_i}. \tag{4.4}$$

Choosing a measure on ∂F as in §1 to identify $\partial(\alpha)$ with ∂F, we finally obtain a map

$$circ : \partial F \to \coprod_i S^1_{m_i} \tag{4.5}$$

Notice that the image of the initial points of the bands give well-defined base–points $0 \in S^1_{m_i}$ for each i.

4.5. **The equivalence relations induced by arcs.** On the set $\partial(\alpha)$ there is a natural reflexive and symmetric relation given by $p \sim_{fol} q$ if p and q are on the same leaf of the partial measured foliation.

Definition 4.5.1. *Let $\sim$ be the equivalence relation on $\coprod_i S^1_{m_i}$ generated by $\sim_{fol}$. In other words $p \sim q$ if there are leaves l_j, for $j = 1, \dots m$, so that $p \in \tilde{c}(\partial(l_1)), q \in \tilde{c}(\partial(l_n))$ and $\tilde{c}(\partial(l_j)) \cap \tilde{c}(\partial(l_{j+1})) \neq \emptyset$.*

Remark 4.5.2. *It is clear that neither the image of circ —which will denote by $circ((\alpha))$— as a collection of parameterized circles nor the relation $\sim$ depends upon the choice of measure on ∂F.*

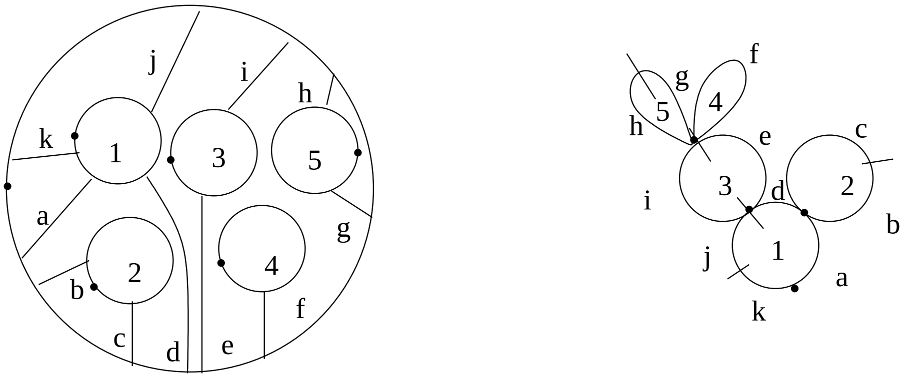


FIGURE 21. An arc family whose loop is a cactus

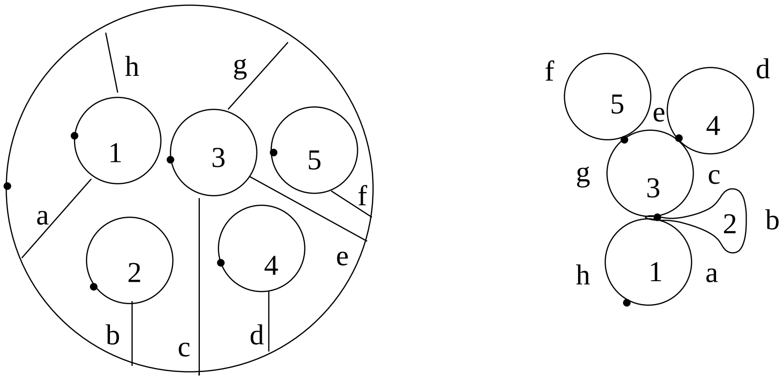


FIGURE 22. An arc family whose loop is a cactus without spines

Definition 4.5.3. *Given a deprojectivized arc family* $(\alpha) \in \mathcal{DA}rc$, *we define* $Loop((\alpha)) = circ((\alpha))/\sim$ *and denote the projection map* $\pi : circ((\alpha)) \to Loop((\alpha))$.

Furthermore, we define two maps taking values in the monoidal category of pointed spaces:

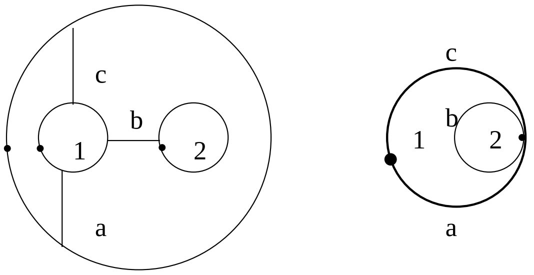
$$int((\alpha)) = \coprod_{i=1}^{n} (\pi(\tilde{c}_i^{(\alpha)}(\partial_i(\alpha)), \pi(*_i)) \tag{4.6}$$

$$ext((\alpha)) = (\pi(\tilde{c}_0^{(\alpha)}(\partial_0(\alpha)), \pi(*_0)) \tag{4.7}$$

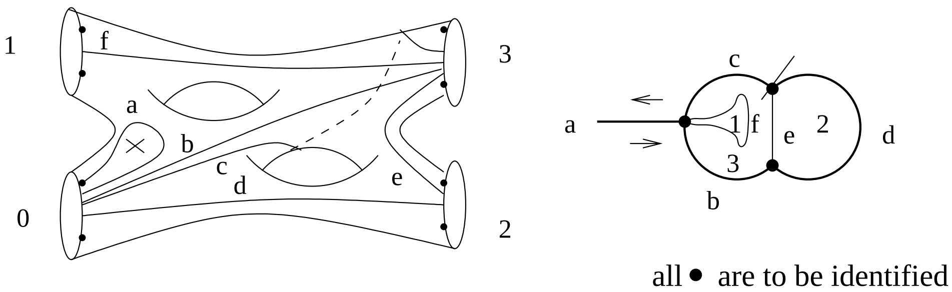
and call them the internal and external loops of (α) *in* $Loop((\alpha))$. *We denote the space with induced topology given by the collection of images* $Loop((\alpha))$ *of all* $(\alpha) \in \mathcal{DA}rc(n)$ *by* $Loop(n)$.

Notice that there are $n+1$ marked points on $Loop((\alpha))$ for $(\alpha) \in \mathcal{DA}rc(n)$.

Examples of loops of an arc family are depicted in figures 21–23. In figure 23 I the image of the boundary 1 runs along the outside circle and then around the inside circle. The same holds for the boundary 3 in figure 23 II. In both 23 I and II, the outside circle and its base–point are in bold.



I



II

FIGURE 23. Loops of arc families not yielding cacti: I. genus 0 case; II. genus 2 with one puncture.

Remark 4.5.4. *There are two types of intersection points for pairs of loops. The first are those coming from the interiors of the bands; these points are double points and occur along entire intervals. The second type of multiple point arises from the boundaries of the bands via the transitive closure; they can have any multiplicity, but are isolated.*

4.6. **From loops to arcs.** If the underlying surface of an arc family satisfies $g = s = 0$, then its Loop together with the parameterizations uniquely determines the arc family. In other words, the map $frame$ is a section of *Loop*.

Definition 4.6.1. *A configuration of circles is the image of a surjection* $p : \coprod_i S^1_{m_i} \to L$ *of metric spaces such that each point of* L *lies in the image of at least two components and the intersections of the images of more than two components are isolated. Let* $Config(n)$ *be the space of all such configurations of* $n+1$ *circles with the natural topology. We call a configuration of circles* planar*, if* L *can be embedded in the plane with the natural orientation for all images* $S^1_i : i \neq 0$ *coinciding with the induced orientation and the opposite orientation for* S^1_0*. We call the space of planar configurations of* $n+1$ *circles* $Config_p(n)$*.*

Proposition 4.6.2. *The map* $Loop : \mathcal{D}\mathcal{A}rc(n) \to Config_p(n)$ *is surjective.*

Definition-Proposition 4.6.3. *The deprojectivized arc families such that* $\pi|_{\tilde{c}(\partial_0)}(\alpha)) = Loop((\alpha))$ *constitute a suboperad of* $\mathcal{ARC}$*. We call this suboperad* $\mathcal{L}oop$*.*

Proposition 4.6.4. *If $(\alpha) \in \mathcal{L}oop$ then $Loop((\alpha))$ is a cactus. Furthermore, the operad $\mathcal{L}oop$ is identical to the operad of Chinese trees.*

Using the symmetric scaling version of cacti (see 4.2.3), we obtain:

Theorem 4.6.5. *The framing of a cactus is a section of Loop and is thus an embedding. This embedding identifies (normalized and/or spineless) cacti as (normalized and/or linear) trees.*

$$\begin{tikzcd}
\textit{(spineless) cacti} \arrow[r, shift right, "frame"'] \arrow[d, "\pi"'] & \mathcal{D}\mathcal{A}rc \arrow[l, shift right, "Loop"'] \arrow[d, "\pi"] \\
\textit{(spineless) cacti}/\mathbb{R}_{>0} \arrow[r, shift left, "\pi frame"] & \mathcal{ARC} \arrow[l, shift left, "\pi Loop"]
\end{tikzcd}$$

where $\pi Loop$ is defined by choosing any lift.

By inspection of the diagrams for the Gerstenhaber relations we see that up to homotopy all are defined in the image of $\mathcal{C}act$ and the BV structure can be defined in the image of $\mathcal{C}acti$.

Thus we have:

Corollary 4.6.6. *The chains of $\mathcal{C}act$ carry the structure of a Gerstenhaber algebra up to homotopy. The chains of $\mathcal{C}acti$ carry the structure of a GBV algebra up to homotopy.*

Remark 4.6.7. *Actually, up to homotopy all the structures can be obtained using normalized cacti, see below 5.*

For a direct proof see [K3].

4.7. **Comments on an action on the loop space.** Given a manifold M we can consider its loop space LM. Using the configuration we have maps

$$Arc(n) \times LM^n \overset{Loop \times id}{\longleftarrow} Config(n) \times LM^n \overset{i}{\longleftarrow} L^{Config(n))}M \xrightarrow{e} LM \tag{4.8}$$

where $L^{Config(n))}M$ are continuous maps of the images L of the configurations into M, i.e., such a map takes a configuration $p : \coprod_i S^1_{m_i} \to L$ and produces a continuous $f : L \to M$; the maps i, e are given by $i(f) = (p : \coprod_i S^1_{m_i} \to L, f(p(S^1_{m_1}), \dots, f(p(S^1_{m_n})))$ and $e(f) = f(p(S^1_{m_0}))$.

One would like to follow the Pontrjagin–Thom construction of [CJ, V] so that the maps i and e would in turn induce maps on the level of homology

$$\begin{aligned} \text{“}H_*(\mathcal{ARC}(n)) \otimes H_*(LM^n) \simeq H_*(\mathcal{D}\mathcal{A}rc(n)) \otimes H_*(LM^n) \overset{Loop_*}{\to} \\ H_*(Config(n)) \otimes H_*(LM^n) \overset{i^!}{\to} H_*(L^{Config(n)}M) \overset{e_*}{\to} H_*(LM)\text{”} \end{aligned} \tag{4.9}$$

where $i^!$ is the “Umkehr” map, *but the map $i^!$ is only well defined on the subspace of cacti.*

If we restrict ourselves to this subspace we obtain from the above following [CS,V,CJ]:

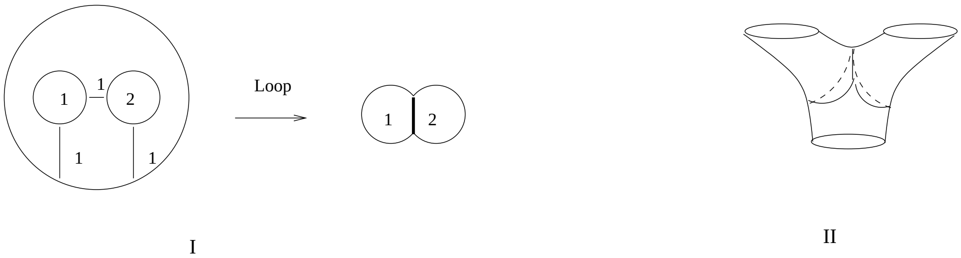


FIGURE 24. I. The Loop of a symmetric pair of pants; II. A closed string field theory picture of a pair of pants

Proposition 4.7.1. *The homology of the loop space of a manifold is an algebra over the suboperad of cyclic Chinese trees.*

4.7.1. **Remarks.**

1) It is clear that one desideratum is the extension of this result to all of $\mathcal{ARC}$.
2) The first example of an operation of composing loops which are not cacti would be given by the *Loop* of the pair of pants with three arcs as depicted in figure 24. This kind of composition first appeared in the considerations of closed string field theory.
3) If the image of Loop is not connected, then the information is partially lost. This can be refined however by using a prop version of our operad.
4) Factoring the operation through Loop has the effect that the internal topological structure is forgotten; thus, the torus with two boundary components has the same effect as the cylinder, for instance.

5. A cell decomposition for spineless cacti

One motivation for studying spineless cacti is that they give a well adapted chain model for the little discs operad. In fact, we will show below how the chains of spineless cacti operate naturally on the Hochschild complex of an associative algebra using the chain decomposition of this section.

5.1. Cact(i) and the (framed) little discs operad.

Theorem 5.1.1. *(Voronov)* [V] *The operad $\mathcal{C}acti$ is homotopy equivalent to the framed little discs operad.*

In [K3], we introduced the notion of semi–direct and bi–crossed products for operads which are the suitable generalization of the same notions for groups. We also proved in [K3]:

Theorem 5.1.2 ([K3])**.** *The operad of cacti is the bi–crossed product of the operad of spineless cacti with the operad $\mathcal{S}^1$ based on S^1 and furthermore this bi–crossed product is homotopic to the semi–direct product of the operad of cacti without spines with the circle group S^1 which is homotopy equivalent to the semi–direct product.*

$$Cacti \simeq Cact \bowtie \mathcal{S}^1 \sim Cact \rtimes \mathcal{S}^1 \tag{5.1}$$

From this and the fact that the famed little discs are the semi–direct product of the little discs with S^1 [SW] one easily obtains.

Corollary 5.1.3. *The operad $\mathcal{C}act$ is homotopy equivalent to the little discs operad.*

5.2. **A cell decomposition for spineless cacti.** Recall that the spaces $\mathcal{C}act^1(n)$ are the subspaces of $\mathcal{C}act(n)$ with the restriction that all the radii of the lobes are fixed to one.

This space inherits the obvious action by $\mathbb{S}_n$ of permuting the labels.

Definition 5.2.1. *The topological type of a spineless normalized cactus in $\mathcal{C}act^1(n)$ is defined to be the tree $\tau \in \mathcal{WLBPT}ree(n)$ which is its b/w graph together with the labeling induced from the labels of the cactus.*

Definition 5.2.2. *We define $\mathcal{WLBPT}ree(n)^k$ to be the elements of $\mathcal{WLBPT}ree(n)$ with $|E_w| = k$.*

Let Δ^n denote the standard n–simplex.

Definition 5.2.3. *For $\tau \in \mathcal{WLBPT}ree$ we define*

$$\Delta(\tau) := \times_{v \in V_w(\tau)} \Delta^{|v|} \tag{5.2}$$

Notice that $\dim(|\Delta(\tau)|) = |E_w(\tau)|$.

Theorem 5.2.4. *The space $\mathcal{C}act^1(n)$ is a CW complex whose k-cells are indexed by $\tau \in \mathcal{WLBPT}ree(n)^k$ with the cell $C(\tau) \sim |\Delta(\tau)|$. Moreover the map $\tau \mapsto C(\tau)$ is a map of differential operads and it identifies $\mathcal{WLBPT}ree(n)^k$ with $CC_k(\mathcal{C}act^1(n))$, where CC_k are the dimension k cellular chains.*

Proof. Given an element in $\mathcal{C}act^1$, we can view it as given by its topological type and a labeling of the arcs of its underlying arc family with the condition that the sum of all labels at each boundary is one. The number of incident arcs at each boundary i is $|v_i|$ and the condition of the weights summing to one translates to the weights being in $|\Delta^{|v_i|}|$. Vice versa given an element on the right hand side, the summand determines the topological type and it is obvious that any tree in $\mathcal{WLBPT}ree$ can be realized. Then the barycentric coordinates in the standard orientation define weights to the arcs in their fixed orientation of incidence. We orient the cells $|\Delta(\tau)|$ in the natural orientation induced from the linear order on the white edges. This makes the glueing well defined which can, for instance, be seen from the definition of the arc complex. $\square$

Definition 5.2.5. *We define the following operations*

$$\circ_i : Cact^1(n) \times Cact^1(m) \to Cact^1(n+m-1) \tag{5.3}$$

by the following procedure: given two normalized cacti without spines we reparameterize the i–th component circle of the first cactus to have length m and glue in the second cactus by identifying the outside circle of the second cactus with the i–th circle of the first cactus.

Lemma 5.2.6. *The glueings make the spaces $Cact^1(n)$ into a quasi–operad which is homotopy associative.*

The homotopies are actually controlled by the following analysis which shows that $\mathcal{C}act$ is a semi–direct product of $\mathcal{C}act^1$ with a scaling operad based on the spaces $\mathbb{R}^n_{>0}$.

5.2.1. **The scaling operad.** We define the scaling operad $\mathcal{R}_{>0}$ to be given by the spaces $\mathcal{R}_{>0}(n) := \mathbb{R}^n_{>0}$ with the permutation action by $\mathbb{S}_n$ and the following products

$$(r_1,\dots,r_n)\circ_i(r'_1,\dots,r'_m) = (r_1,\dots r_{i-1},\frac{r_i}{R}r'_1,\dots,\frac{r_i}{R}r'_m,r_{i+1},\dots r_n)$$

where $R=\sum_{k=1}^m r'_k$. It is easy to check that this indeed defines an operad.

5.2.2. **The action of the scaling operad on cacti.** Fix an element $\vec{r} := (r_1,\dots,r_n)\in\mathbb{R}^n_{>0}$ and set $R=\sum_i r_i$. Then $\vec{r}$ defines a map $\rho: S^1\to S^1$ via

$$\rho: S^1 \xrightarrow{rep^1_R} S^1_R \xrightarrow{cont_{\vec{r}}} S^1_n \xrightarrow{rep^n_1} S^1 \tag{5.4}$$

Where $cont_{\vec{r}}$ acts on S^1_R in the following way. Decompose S^1_R into a linear sequence of arcs of length r_i starting at zero. Now contract the i–th arc homogeneously with a scaling factor $\frac{1}{r_i}$ to length one. The maps rep^r_R are the natural reparameterizations mapping an S^1_r of radius r onto an S^1_R of radius R by using the angle as a parameter.

Using this map on the i–lobe of a normalized (spineless) cactus which we think of as an S^1 with marked points, where the marked points are the intersection points, the local zero and if applicable the global zero.

$$\begin{aligned}\rho_i: Cact^1(n)\times\mathcal{R}_{>0}(m) &\to Cact^1(n)\\ \rho_i: Cacti^1(n)\times\mathcal{R}_{>0}(m) &\to Cacti^1(n)\end{aligned}\tag{5.5}$$

Notice we would like to take a right action.

5.3. **The perturbed multiplication.** We define the perturbed multiplication via

$$\begin{aligned}\circ_i^{\mathcal{R}_{>0}}: Cacti^1(n)\times\mathcal{R}_{>0}(m)\times Cacti^1(m)&\\ \xrightarrow{\rho_i\times id} Cacti^1(n)\times Cacti^1(m) \xrightarrow{\circ_i} Cacti^1(n+m-1)&\end{aligned}\tag{5.6}$$

We also use the analogous perturbed action for $Cact^1$.

Theorem 5.3.1. *Cacti (without spines) are homotopic to normalized cacti without spines. Their operad structure is given by a semi–direct product of their normalized version with the scaling operad that is homotopic to the direct product.*

$$\begin{aligned}\rho_i: Cact^1(n)\times\mathbb{R}^n_{>0} &\simeq Cact(n)\\ \rho_i: Cacti^1(n)\times\mathbb{R}^n_{>0} &\simeq Cacti(n)\end{aligned}\tag{5.7}$$

and furthermore

$$\begin{aligned}Cact &\simeq \mathcal{R}_{>0}\ltimes Cact^1 \sim Cact^1\times\mathcal{R}_{>0}\\ Cacti &\simeq \mathcal{R}_{>0}\ltimes Cacti^1 \sim Cacti^1\times\mathcal{R}_{>0}\end{aligned}\tag{5.8}$$

as operads, where the operadic compositions are given by

$$(\vec{r},C)\circ_i(\vec{r}',C') = (\vec{r}\circ_i\vec{r'},C\circ_i^{\vec{r}'}C',)\tag{5.9}$$

Keeping track of the homotopies, one can see that $CC_*(\mathcal{C}act^1(n))$ is indeed a chain operad.

From our previous analysis about the structure of $\mathcal{C}act$ as a semi–direct product see §5.3.1 we obtain:

Theorem 5.3.2. *The glueings induced from the gluings of spineless normalized cacti make the spaces $CC_*(Cact^1(n))$ into a chain operad. And since $\mathcal{C}act, \mathcal{C}act^1$ and D_2 are all homotopy equivalent $CC_*(cact^1)$, is a model for the chains of the little discs operad.*

Remark 5.3.3. *If one would like a topological operad in the background, one can choose any chain model $Chain(\mathcal{R}_{>0})$ for the scaling operad, then use the mixed chains for $\mathcal{C}act$ i.e. $CC_*(\mathcal{C}act^1) \otimes Chain(\mathcal{R}_{>0})$. It follows that the inclusion of the cellular chains of $\mathcal{C}act^1$ into the mixed chains is an inclusion of operads up to homotopy.*

Finally, given an operation of the $CC_*(\mathcal{C}act^1)$ we can let the mixed chains of $\mathcal{C}act$ act by letting the mixed chains of bidegree $(n, 0)$ act as the component of $\mathcal{C}act^1$ and sending all the others to zero.

5.4. **Orientations of Chains.** To fix the generators and thereby the signs for the chain operad we have several choices, each of which is natural and has appeared in the literature.

To fix a generator $g(\tau)$ of $CC_*(Cact^1)$ corresponding to the cell indexed by $\tau \in \mathcal{WLBPT}ree(n)$ we need to specify an orientation for the arc family it represents, i.e. a parameterization or equivalently an order of the white edges of the tree.

The first orientation which we call Nat is the orientation given by the natural orientation of the arc family or equivalently the natural orientation for a planar planted tree.

We will also consider the orientation Op which is the enumeration of the white edges which is obtained by starting with the incoming edges of the white vertex labeled one, in the natural orientation of that vertex, then continuing with the incoming white edges into the vertex two, etc. until the last label is reached.

Lastly, for top–dimensional cells, we will consider the orientation of the edges induced by the labels, which we call Lab. It is obtained from Nat as follows: for $\tau \in \mathcal{WPT}ree$ let $\sigma \in \mathbb{S}_n$ be the permutation which permutes the vertices $v_1, \dots, v_n$ to their induced orientation $\prec^{(\tau)}$. Then let the enumeration of E_w be $\sigma(Nat)$, where the action of σ on E_w is given by the correspondence $v2e$ and the correspondence between black and white edges via $(v, N(v)) \mapsto (N(v), N^2(v))$ for top dimensional cells.

To compare with the literature it is also useful to introduce the orientations $\overline{Nat}$ and $\overline{Op}$ which are the reversed orientation of Nat and Op, i.e. reading them from right to left.

5.5. **Sign conventions for the operations in the Chain operad.** For the operation of the chain operad there are two possibilities.

The first is the operation induced by the topological operad. We denote this operation by $\tilde{\circ}$.

This corresponds to fixing the orientation of $g(\tau) \circ_i g(\tau')$ to be given by $E_w(\tau) \sqcup E_w(\tau')$. This makes $\tilde{\circ}$ natural w.r.t. the orientation Op.

The second convention is that the signs of the permutation obtained by inserting into the i–th vertex are already incorporated. This means that the orientation of $g(\tau) \circ_i g(\tau')$ is fixed to be be given by the orientation of $\tau \circ_i \tau'$. So if $\sigma : E_w(\tau) \sqcup E_w(\tau') \to E_w(\tau \circ_i \tau')$ is the permutation shuffling the white edges of τ' into the right position then $\circ_i := (-1)^{|\sigma|} \tilde{\circ}$. This makes $\circ$ natural w.r.t. the orientation Nat and Lab.

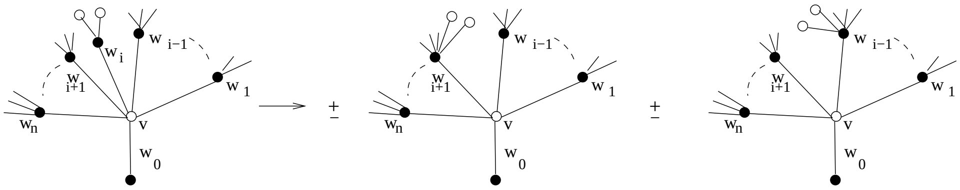


FIGURE 25. The trees τ_{e+} and τ_{e-}

5.5.1. **The differential on $\mathcal{WLBPT}ree$.** There is a natural differential on $\mathcal{WLBPT}ree$ which it inherits from its interpretation as $CC_*(\mathcal{C}act)$.

Definition 5.5.1. *We define the operator ∂ on the space $\mathcal{WLBPT}ree$ to be given by the following formulas*

$$\partial(\tau) := \sum_{v \in V_w(\tau)} \pm \partial_v(\tau)$$

where $\partial_v(\tau) := \sum_{e \in E_v(\tau)} \pm\tau_{e+} \pm \tau_{e-}$.

Here if $e = \{v, w\}$ then τ_{e+} is the tree with the vertex w deleted and the edges of w grafted to the vertex of the next edge keeping the cyclic order and τ_{e-} is the tree with the vertex w deleted and the edges of w grafted to the vertex of the previous edge also keeping the cyclic order. Here we use all edges with vertex v,i.e. the incoming as well as the outgoing edge and the sign $\pm$ is determined by a choice of generator for the cell corresponding to τ in $CC_*(\mathcal{C}act^1)$ see §5.4.

If we denote by $\partial : \mathcal{WLBPT}ree(n)^k$ the elements of $\mathcal{WLBPT}ree(n)$ with k white edges, then from the interpretation in terms of chains of $\mathcal{C}act^1$ we obtain:

Proposition 5.5.2. *The map $\partial : \mathcal{WLBPT}ree(n)^k \to \mathcal{WLBPT}ree(n)^{k-1}$ is a differential for $\mathcal{WLBPT}ree$ and turns $\mathcal{WLBPT}ree$ into a differential operad.*

5.6. **Operadic action of $\mathcal{WLBPT}ree$.** A natural way to let $\mathcal{WLBPT}ree$ act on a complex $(\mathcal{O}, \delta)$ is given by building a mixed complex by identifying the white edges of a tree with elements from $\mathcal{C}act^1$ and the vertices with elements from $\mathcal{O}$.

First we notice that if we are dealing with planted planar trees, we have the total linear order $\prec$ on the set of vertices and edges. For an action of the operad $\mathcal{WLBPT}ree$ on a graded space $\mathcal{O} = \sum O(n)$, we will consider maps

$$\rho : \mathcal{WLBPT}ree(k) \otimes O(n_1) \otimes \cdots \otimes O(n_k) \quad \to \quad O(m) \tag{5.10}$$

$$\tau \otimes f_1 \otimes \cdots \otimes f_k \quad \mapsto \quad \tau(f_1 \otimes \cdots \otimes f_k) \tag{5.11}$$

Actually, $\tau(f_1 \otimes \cdots \otimes f_k)$ will be zero unless $|v_i| = n_i$ and $m = \sum n_i - k$.

In the graded case, we have to fix the order of the tensor product on the l.h.s. of the expression (5.11). We do this by using $\prec^\tau$ to give tensor product the natural operadic order (i.e. $O(i)$ inserted into the vertex v_i. Let $N := |\{V(\tau) \amalg E_w(\tau)\}|$ and let $num : |\{V(\tau) \amalg E(\tau)\}| \to \{1, \dots, N\}$ be the bijection which is induced by $\prec^\tau$. We fix L_1 to be a line i.e. a free generated by an element of degree one. Now set

$$W_i := \begin{cases} O(n_j) & \text{if } num^{-1}(i) = v_j \\ L_1 & \text{if } num^{-1}(i) \text{ is a white edge} \end{cases} \tag{5.12}$$

We then define the order on tensor product on the l.h.s. of the expression (5.11) to be given by

$$W := W_1 \otimes W_N$$

Another way would be to include the sign which is necessary to permute the l.h.s. of 5.11 into W into the operation ρ.

5.6.1. **The action of the symmetric group.** The action of the symmetric group is induced by permuting the labels and permuting the elements of $\mathcal{O}$ respectively. This induces a sign by permutation on W.

5.7. **The action of $Chain(\mathcal{ARC})$ on itself and string topology.** A good example of this type of action is given in [KLP]. For the homotopy Gerstenhaber structure we need an action of $CC_*(\mathcal{C}act^1)$ on any choice of chain model for $\mathcal{ARC}$ or any of the suboperads which are stable under the linear trees suboperad. The action ρ is just given by the glueing in $\mathcal{ARC}$.

We get agreement with the signs of the operation on $\mathcal{ARC}$ which agree with those of string topology [CS1], if we denote the action of τ_1 as $*^{op}$ and τ_2^b as $\cdot$, see §3.5 for the operations and figure 27 for the definitions of the trees.

For the homotopy GBV structure we should consider the chains $CC_*(\mathcal{C}acti^1)$ and again any choice of chain model for $\mathcal{ARC}$ or any of the suboperads which are stable under the action of the trees suboperad.

6. Structures on Operads and Meta–Operads

Before going into the statement and proof of Deligne's conjecture, we would like to digress once more on operads. This helps to explain some choices of signs and explains the naturality of the construction of insertion operads which gives a special role to spineless cacti as their topological incarnation as well as to $\mathcal{ARC}$ as a natural generalization.

This analysis also enables us to relate spineless cacti to the renormalization Hopf algebra of Connes and Kreimer [CK].

Given any operad there are certain universal operations, i.e. maps of the operad to itself. We will first ignore possible signs and comment on them later on.

6.1. **The universal concatenations.** Given any operad, we have the structure maps

$$\circ_i : O(m) \otimes O(n) \to O(m+n-1)$$

and the concatenations of these, which can be described by their flow charts, which are given by $\tau \in \mathcal{WPTT}ree$. More precisely given k elements $op_k \in O(n_k)$, we can concatenate them with the $\circ_i$ to produce a tree flow chart where the inputs are the leaves and tails and the inner vertices are labeled by the operations op_k, where the a vertex v_k labeled by op_k necessarily has valence n_k. The number of leaves and the degrees $n-i$ of the op_i satisfy the condition

$$wt(\tau) = \sum_v |v| = \sum n_i = \#\text{leaves} + \#\text{inner vertices} = \#\text{leaves} + k \tag{6.1}$$

Notice, we might have white leaves, which allows one to consider operads also with a 0 component such as CH^* see below.

So let $\tau \in \mathcal{WPTT}ree(k)$ and let $n_i : i \in 1, \dots k := |v_i|$ then there is an operation

$$\circ(\tau)(O(n_1) \otimes \cdots \otimes O(n_k)) \to O(m) \tag{6.2}$$

by labeling the vertex v_i by $op_{n_i} \in \mathcal{O}(n_i)$.

Notice that, we used the linear order on a planted planar tree in order to associate the functions to the non–leaf vertices.

In general lifting the restriction on the n_i, we define the operations $\circ(\tau)$ to be zero of $|v_i| \neq n_i$.

If the category of the operad has direct sums (e.g. disjoint union for topological spaces, direct product for groups, $\oplus$ for vector spaces, etc.) then this gives rise to a partial non–Σ operad operation of non–labeled $\mathcal{WPTT}ree$ which can be made into an operation of the operad $\mathcal{WPTT}ree$ by using $\mathbb{S}_n$ equivariance. The partial concatenations $\circ_i$ insert a tree with k–tails into the vertex v_i if $|v_i| = k$, by connecting the incoming edges of v_i to the tail vertices in the linear order at v_i and contracting the tail edges.

So if we call $\mathcal{O} := \sum_{i\in\mathbb{N}} O(n)$ by abuse of notation, there is a map of partial operads

$$\mathcal{WPTT}ree \to Hom(\mathcal{O},\mathcal{O})$$

In this sense one can say that $\mathcal{WPTT}ree$ is the universal concatenation partial–operad.

6.2. **The pre–Lie Structure of an Operad.** In the category $\mathcal{C}hain$, one can go one step further to define the $\circ$ product of [G] and the iterated brace operation (cf. [Ge3, GV]), see below.

If the operad $\mathcal{O}$ is in $\mathcal{C}hain$ we can take the direct sum of the components, which by abuse of notation also call $\mathcal{O} := \sum_{i\in\mathbb{N}} O(n)$. The possibility of forming direct sums allows to put together all the operations $\circ_i$

Definition 6.2.1. *(Gerstenhaber) Given any operad $\mathcal{O}$ in $\mathcal{C}hain$, we can define the following map*

$$O(m)\otimes O(n) \to O(m+n-1) \tag{6.3}$$

$$op_m \otimes op_n \mapsto \sum_{i=1}^{m}(-1)^{(i-1)(n+1)} op_m \circ_i op_n \tag{6.4}$$

This extends to a map

$$\circ : \mathcal{O}\otimes\mathcal{O} \to \mathcal{O} \tag{6.5}$$

which we call the $\circ$ product.

Following Gerstenhaber's calculation (essentially using associativity) [G, MSS], we immediately have the following proposition

Proposition 6.2.2. *The product $\circ$ defines on $O := \bigoplus_{i\in\mathbb{N}} O(n)$ the structure of a graded pre–Lie algebra. Omitting the sign $(-1)^{(i-1)(n+1)}$ in the sum yields the structure of a non–graded pre–Lie algebra.*

We do not rewrite the proof here, but in graphical notation the proof follows from figure 26 below.

Without signs this notation is related to the one that can be found in [CL], for the case with signs see 7.5 below.

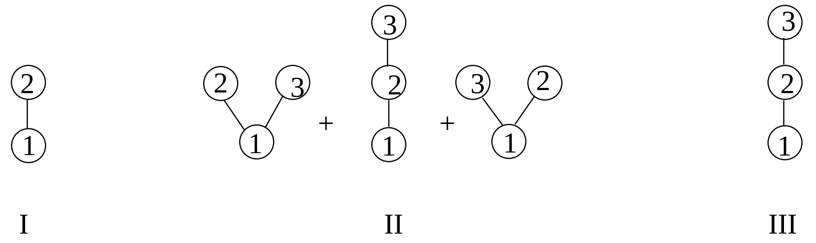


FIGURE 26. I. $f_1 \circ f_2$ II. $(f_1 \circ f_2) \circ f_3$ and III. $f_1 \circ (f_2 \circ f_3)$

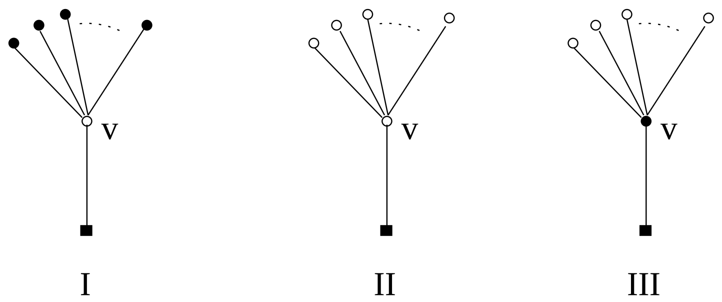


FIGURE 27. I The n–tail tree l_n, II. The white n–leaf tree τ_n, III. The black n–leaf tree τ_n^b

6.3. **The insertion operad.** The interesting property of the operation $\circ$ is that it effectively removes the dependence on the number of inputs of the factors.

Given an operad in $\mathcal{C}hain$ we can also define other operations similar to $\circ$ which are in natural correspondence with $\mathcal{PPT}ree$. In fact these operations all appear in the iterations of $\circ$. They are given by inserting the operations into each other according to the scheme of the tree and then distributing tails so that the equation (6.1) is satisfied. Examples of this are given in figure 26. Here the first tree yields the operation $f_1 \circ f_2$, i.e the insertion (at every place) of f_2 into f_1. Iterating this insertion we obtain expression II which shows that inserting f_3 into $f_1 \circ f_2$ gives rise to three topological types: inserting f_3 in front of f_2, into f_2 and behind f_2. In the opposite iteration one just inserts $f_2 \circ f_3$ into f_1 which gives a linear insertion of f_2 into f_1 and f_3 into f_2. From the figure (up to signs) one can read off the symmetry in the entries 2 and 3 of the associator.

We will not care about signs at the moment, but include them later, see 7.5.

6.3.1. **Notation.** There are some standard trees, which are essential in our study, these are the n–tail tree l_n, the white n–leaf tree τ_n, and the black n–leaf tree τ_n^b, as shown in figure 27

Essentially, if we would not like to *a priori* specify the number of leaves, i.e. inputs and degrees of the operations, we have to consider trees with all possible decorations by leaves. For this we need foliation operators in the botanical sense. To avoid confusion with the mathematical term "foliation", we choose to abuse the English language and call these operations "foliage" operators.

Definition 6.3.1. *Let l_n be the tree in $\mathcal{WLBPTT}ree$ with one white vertex labeled by v, n tails as depicted in figure 27. The foliage operator $F : \mathcal{WLPT}ree \rightarrow$*

$\mathcal{WLPTT}ree$ is defined by the following equation

$$F(\tau) := \sum_{n \in \mathbb{N}} l_n \circ_v \tau$$

Notice that the right hand side is infinite, but since $\mathcal{WLPTT}ree$ is graded by say the number of leaves, and $F(\tau)$ is finite for a fixed number of black leaves the definition does not pose any problems. Furthermore, one could let F take values in $\mathcal{WLPTT}ree[[t]]$ where t keeps track of the number of tails which would make the grading explicit.

Also notice that $F : \mathcal{WLBPT}ree \to \mathcal{WLBPTT}ree$ and $F : \mathcal{WPT}ree \to \mathcal{WPTT}ree$.

Recall that there is an operation of $\mathcal{WPTT}ree$ on homogeneous elements of $\mathcal{O}$ of the right degree. We extend this operation to all of $\mathcal{O}$ by extending linearly and setting to zero expressions which do not satisfy degree count, i.e. satisfy the equation (6.1).

Given τ in $\mathcal{WPT}ree(n)$, we can then define the operation

$$\circ_\tau(f_1 \otimes \cdots \otimes f_n) := F(\tau)(f_1 \otimes \cdots \otimes f_n)$$

Notice that, although $F(\tau)$ is an infinite linear combination, for given $f_1, \dots, f_n$ the expression on the right hand side is finite.

The following is almost automatic.

Theorem 6.3.2. *Any chain operad which affords a direct sum is an algebra over the operad $\mathcal{PPT}ree = \mathcal{WPT}ree$ with the insertion product.*

In fact, following Remark 6.3.3, we are forced to look at the insertion product.

Remark 6.3.3. *Thinking about F as a formal power–series, e.g. in $\mathcal{WPTT}ree[[t]]$, we can define a product $*$ by the formula*

$$F(\tau_1) \circ F(\tau_2) := F(\tau_1 * \tau_2) \tag{6.6}$$

Now, if the $*$ is thought of as operadic, i.e. $\tau_1 * \tau_2 = \gamma(*, \tau_1, \tau_2)$, then by linearity and associativity, we know how to define $\gamma(\tau, t_1, \dots, t_n)$ for any $\tau \in \mathcal{RT}ree \subset \mathcal{PPT}ree$, fixing the operations of the pre–lie operad. This is essentially the construction of [CL].

6.4. **The pre–Lie operad.** The pre–Lie operad is the suboperad of $\mathcal{PPT}ree$ which gives rise to the iterations of the operations $\circ_i$.

Proposition 6.4.1. *The map ppin gives an isomorphism of $\mathcal{P}reLie$ with $\mathcal{PPT}ree^{\mathbb{S}}$ where $\mathcal{PPT}ree^{\mathbb{S}}$ are the symmetric trees. This isomorphism is compatible with the operation on operads.*

6.5. **The Hopf algebra of an operad.** We have seen in 6.2 that any chain operad gives rise to a pre–Lie algebra. Now the defining property for a pre–Lie algebra is that the commutator of its product gives a Lie or in the graded case an odd Lie algebra.

We say that an $\mathcal{O}$ affords a direct sum, if it has a natural representation in $\mathcal{V}ect_k$, i.e. the n–th component corresponds to the n–th graded piece of a vector space, extending the operad structure. For this a functor from the given category to $\mathcal{V}ect_k$ suffices. This is the case for all linear, chain and combinatorial operads.

For $\mathcal{O}$ in $\mathcal{V}ect_k$ this is given as $\bigoplus_n O(n)$, if $\mathcal{O}$ is in $\mathcal{S}et$, we first take its linear representation and then the direct sum $\bigoplus_n k[O(n)]$, and finally if $\mathcal{O}$ is in $\mathcal{C}hain$, we first take extension of scalars and then the direct sum $\bigoplus_n O(n) \otimes k$.

Definition 6.5.1. *Given an operad $\mathcal{O}$, which affords a direct sum, we define its pre–Lie algebra $PL(\mathcal{O})$ to be the pre–Lie algebra $(\mathcal{O}, \circ)$, its Lie algebra $L(\mathcal{O})$ to be the Lie algebra (odd Lie) $(\mathcal{O}, \{\ \})$ where $\{\ \}$ is defined in equation (2.5). Lastly the Hopf algebra of an operad $H(\mathcal{O})$ is defined to be the universal enveloping algebra $U^*(L(\mathcal{O}))$ of its Lie algebra.*

In our notation we can rephrase the results of [CK, CL] as

Corollary 6.5.2. [CK, CL] *The renormalization Hopf algebra of Connes and Kreimer is the Hopf algebra $H(\mathcal{P}reLie)$.*

For the reader unfamiliar with this Hopf algebra this can also be a definition.

6.6. **Spineless cacti, the pre–Lie operad and Connes–Kreimer.** We denote the top–dimensional cells of $CC_n(\mathcal{C}act(n))$ by $CC_n^{top}(n)$. These cells again form an operad and they are indexed by trees with only white vertices (plus one input valanced black vertices), i.e. the subset $\mathcal{PPT}ree \subset \mathcal{WLBPT}ree$ using the map *bwbp*. Furthermore, the symmetric combinations of these cells again form an operad and hence

Theorem 6.6.1. *The concatenation*

$$\mathcal{RT}ree \stackrel{ppin}{\to} \mathcal{PPT}ree \stackrel{bw}{\to} \mathcal{WLBPT}ree \simeq CC_*(\mathcal{C}act^1)$$

induces an operad isomorphism of $\mathcal{P}reLie \simeq \mathcal{RT}ree$ onto $\mathcal{WLBPT}ree^{\mathbb{S}} \simeq CC_*^{top}(\mathcal{C}act^1)^{\mathbb{S}}$, if we use the operatic compositions $\circ_i$ and the orientations Lab of 5.4 for $CC_*^{top}(\mathcal{C}act^1)$.

We have used the notation $\mathcal{WLBPT}ree^{\mathbb{S}}$ for the orbits of τ under the group action of $\mathbb{S}(\tau)$.

So we see that we indeed have a cell and thus a topological interpretation of the pre–Lie operad in the $\mathcal{ARC}$ operad and since [CL, CK] H_{CK} is the Hopf algebra associated to the combinatorial operad $\mathcal{P}reLie$ it is also the Hopf algebra of the chain operad $CC_*^{top}(\mathcal{C}act)^{\mathbb{S}}$.

It is interesting to note that also the G and BV structures are inside the symmetric (graded symmetric) combinations, see below.

Remark 6.6.2. *We expect to obtain other interesting examples of such Hopf algebras by inserting other tree operads.*

7. Spineless cacti as a natural solution to Deligne's conjecture

7.1. **The Hochschild complex, its Gerstenhaber structure and Deligne's conjecture.** Let A be an associative algebra over a field k. We define $CH^*(A, A) := \bigoplus_{q \geq 0} \mathrm{Hom}(A^{\otimes q}, A)$

There are two natural operations

$$\circ_i : CH^m(A, A) \otimes CH^n(A, A) \to CH^{m+n-1}(A, A) \tag{7.1}$$

$$\cup : CH^n(A, A) \otimes CH^m(A, A) \to CH^{m+n}(A, A) \tag{7.2}$$

where the first morphism is as in 2.2.1 , i.e. for $f \in CH^p(A, A)$ and $g \in CH^q(A, A)$.

$$f \circ_i g(x_1, \dots, x_{p+q-1}) = f(x_1, \dots, x_{i-1}, g(x_i, \dots, x_{i+q-1}), x_{i+q}, \dots, x_{p+q-1})$$

and the second is given by the multiplication

$$f(a_1 \dots, a_m) \cup g(b_1, \dots, b_n) = f(a_1 \dots, a_m) g(b_1, \dots, b_n)$$

7.1.1. **The differential on CH^*.** The Hochschild complex also has a differential which is also derived from the algebra structure.

Given $f \in CH^n(A, A)$ then

$$\begin{aligned}\partial(f)(a_1, \dots, a_{n+1}) &:= a_1 f(a_2, \dots, a_{n+1}) - f(a_1 a_2, \dots, a_{n+1}) + \\ &\dots + (-1)^{n+1} f(a_1, \dots, a_n a_{n+1}) + (-1)^{n+2} f(a_1, \dots, a_n) a_{n+1}\end{aligned}$$

Definition 7.1.1. *The Hochschild complex is the complex (CH^*, ∂), its cohomology is called the Hochschild cohomology and denoted by $HH^*(A, A)$.*

7.1.2. **The Gerstenhaber structure.** Gerstenhaber [G] introduced the $\circ$ operations: for $f \in CH^p(A, A)$ and $g \in CH^q(A, A)$

$$f \circ g := \sum_{i=1}^{p} (-1)^{(i-1)(q+1)} f \circ_i g$$

and defined the bracket

$$\{f \bullet g\} := f \circ g - (-1)^{(p+1)(q+1)} g \circ f$$

and showed that this is indeed induces what is now called a Gerstenhaber bracket, i.e. odd Poisson for $\cup$, on $HH^*(A, A)$.

7.2. **Deligne's Conjecture.** Since $HH^*(A, A)$ has the structure of a Gerstenhaber algebra one knows from general theory that thereby $HH^*(A, A)$ is an algebra over the homology operad of the little discs operad.

The question of Deligne was: *Can one lift the action of the homology of the little discs operad to the chain respectively cochain level? Or in other words: is there a chain model for the little discs operad that operated on the Hochschild cochains which reduces to the usual action on the homology/cohomology level?*

This question has an affirmative answer in many ways by picking a suitable chain model for the little discs operad [Ko, T, MS, V2], see also [MSS] for a review of these constructions. We will provide a new and in a sense natural and minimal positive answer to this question, by giving an operation of $CC_*(\mathcal{C}act^1)$ on the Hochschild cochains.

There is a certain minimal set of operations necessary for the proof of such a statement which is given by iterations of the operations $\cup$ and $\circ_i$. These are, as we argue below in bijective correspondence with trees in $\mathcal{WLBPT}ree$, our model for the chains of the little discs operad $CC_*(\mathcal{C}act^1)$, has chains which are exactly indexed by these trees. Furthermore, the top dimensional cells which control the bracket are the universal concatenation operad. And lastly we will show that the differential of deleting arcs can be seen as a topological version of the Hochschild differential. This makes our new solution natural and minimal.

7.3. **The Operation of $CC_*(\mathcal{C}act^1)$ on $\mathcal{H}om_{CH}$.** For $\mathcal{O} = \mathcal{H}om_{CH}$, we define the map ρ of ed. 5.11, to be given by the operadic extension of the maps which send the tree τ_n to the non–intersecting brace operations i.e. for homogeneous f, g_i of degrees $|f|$ and $|g_i|$, $N = |f| + \sum_i |g_i| - n$pp

$$f\{g_1,\dots,g_n\}(x_1,\dots,x_N) := \sum_{\substack{1\leq i_1\leq\dots\leq i_n\leq |f|:\\ i_j+|g_j|\leq i_{j+1}}} f(x_1,\dots,x_{i_1-1},g_1(x_{i_1},\dots,x_{i_1+|g_1|}),\dots,x_{i_n-1},g_n(x_{i_n},\dots,x_{i_n+|g_n|}),\dots,x_N) \tag{7.3}$$

Brace operations first appeared in [Ge3].

In order to make signs match with [G], we will have to consider the opposite orientation for W i.e. $\overline{W} := W_N \otimes \cdots \otimes W_1$. To implement this change of sign we define $sign^W(\tau)$ to be the sign obtained by passing from W to $\overline{W}$. This basically means that in the orientation of W one would regard the operation $\circ^{op}, \cup^{op}$ on the Hochschild complex, where $f \circ^{op} g = (-1)^{pq+p+1} g \circ f$ and $f \cup^{op} g := (-1)^{pq} g \cup f$.

$$f \otimes L_{e_1} \otimes g_1 \otimes \cdots \otimes L_{e_n} \otimes g_n \mapsto (-1)^{sign^W(\tau)} f\{g_1,\dots,g_n\} \tag{7.4}$$

We let τ_0 act as the identity. and τ_n^b as

$$L_{e_1} \otimes g_1 \otimes \cdots \otimes L_{e_n} \otimes g_n \mapsto (-1) sign^W(\tau) g_1 \cup \cdots \cup g_n$$

The operadic extension means that we read the tree as a flow chart at each black vertex $|v|$ the operation $\tau^b_{|v|}$ is performed and at each white vertex the operation $\tau^w_{|v|}$ is performed. The $\mathbb{S}_n$ action is given by permutations and indeed induces the right signs on the Hochschild complex as seen by straightforward calculation.

For the operadic composition in $CC_*(\mathcal{C}act^1)$ we choose $\circ$ with the orientation Nat.

The following is straightforward:

Proposition 7.3.1. *The above procedure makes $CH^*(A,A)$ into an algebra over $CC_*(\mathcal{C}act^1)$.*

7.3.1. **The differential.** If we denote the differential on $CC_*(\mathcal{C}act^1)$ as ∂ and the differential of CH as δ then the action of $CC_*(\mathcal{C}act^1)$ on $\mathcal{H}om_{CH}$ commutes with the differential. On the space W there is a natural differential $\partial_W := \delta + \partial$. The calculations of [KLP] and the straightforward generalization to action of τ_n yield the following proposition.

Proposition 7.3.2.

$$\rho \circ (\partial_W) = \delta \circ \rho \tag{7.5}$$

7.4. **A second approach to the operation of $CC_*(\mathcal{C}act^1)$.** Another way to make $CC_*(\mathcal{C}act^1)$ or $\mathcal{WLBPT}ree$ act is by using the foliage operator. This approach was taken in [KS] and also stresses the fact that a function $f \in CH^q(A,A)$ is naturally depicted by τ_n. Notice for instance the compatibility of the differentials.

7.4.1. **Natural operations on CH^* and their tree depiction.** Given elements of the Hochschild cochain complex there are two types of natural operations which are defined for them. Suppose f_i is a homogeneous element, then it is given by a function $f: A^{\otimes n} \to A$. So treating the cochains as function, we have the operation of insertion, as in 2.2.1. The second type of operation comes from the fact that A is an associative algebra; therefore, for each collection $f_1,\dots,f_n \in CH^*(A,A)$ we have the n! ways of multiplying them together.

We will encode the concatenation of these operations into a black and white bipartite tree as follows: Suppose that we would like to build a cochain by using insertion and multiplication on the homogeneous cochains $f_1, \dots, f_n$. First we represent each function f_i as a white vertex with $|f_i|$ inputs and one output with the cyclic order according to the inputs $1, \dots, |f_i|$ of the function. For each insertion of a function into a function we put a black vertex of valence having as input edge the output of the function to be inserted and as an output edge the input of the function into which the insertion is being made. For a multiplication of $k \geq 2$ functions we put a black vertex whose inputs are the functions which are to be multiplied in the order of their multiplication. Finally we add tails to the tree by putting a black vertex at each input edge which has not yet been given a black vertex, and we decorate the tails by variables $a_1, \dots a_s$ according to their order in the total order of the vertices of the rooted planted planar tree. It is clear that this determines a black and white bipartite tree.

A rooted planted planar bipartite black and white tree whose tails are all black and decorated by variables $a_1, \dots a_s$ and whose white vertices are labeled by homogeneous elements $f_v \in CH^{|v|}(A, A)$ determines an element in $CH^s(A, A)$ by using the tree as a "flow chart", i.e. inserting for each black vertex of valence one and multiplying for each black vertex of higher valence. Notice that, since the algebra is associative, given an ordered set of elements there is a unique multiplication.

Remark 7.4.1. *The possible ways to compose k homogeneous elements of $CH^*(A, A)$ using insertion and cup product are bijectively enumerated by black and white bipartite planar rooted planted trees with tails and k white vertices labeled by k functions whose degree is equal to the valence of the vertex.*

We will fix A and use the short hand notation $CH := CH^*(A, A)$. For an element $f \in CH$, we write $f^{(d)}$ for its homogeneous component of degree d.

If we would like to consider non–homogeneous elements, then given a tree we can only use the homogeneous components of the elements of CH with the right degree. This leads to:

Definition 7.4.2. *For $\tau \in \mathcal{WLBPTT}ree(n)$ and $f_1, \dots, f_n \in CH$ we let $\tau(f_1, \dots, f_n)$ be the operation obtained in the above fashion by decorating the vertex v_i with label i with the homogeneous component of $f_i^{(|v_i|)}$. Notice that the result is zero if any of the homogeneous components $f_i^{(|v_i|)}$ vanish.*

Remark 7.4.3. *Up to the signs which are discussed below this gives an operation of $CC_*(\mathcal{C}act)$ on the Hochschild complex.*

7.5. **The operation of $\mathcal{WLBPT}ree$.**

Definition 7.5.1. *For a tree $\tau \in \mathcal{WLBPT}ree(n)$ with n white vertices we define a map $op(\tau) \in \mathrm{Hom}(CH^{\otimes n}, CH) = \mathcal{H}om_{CH}(n)$ by*

$$op(\tau)(f_1, \dots, f_n) := \pm op(ins(F(\tau), (f_1, \dots, f_n))$$

here ins inserts the function f_i into the label i and the signs are discussed below §7.5.

Proposition 7.5.2. *The Hochschild cochains are an algebra over $\mathcal{WLBPT}ree$.*

7.6. **Signs induced from $\mathcal{WLPT}ree$.** A good way to get to the correct grading is by using the calculus of [KS], i.e. to tensor with one dimensional spaces L_1 and L_2 of degrees -1 and -2 and their duals L_1^* and L_2^*. We can chose any pre-image $\mathcal{WLPT}ree$ under st_∞.

In order to make the correspondence, we have to choose the operation $\tilde{\circ}$ and the orientation Op. The differential of [KS] is our differential in $\mathcal{C}act$ acting from the right instead of from the left. I.e. with the sign change in passing from Op to $\overline{Op}$.

Notice that in the tree formalism a black edge is inserted upon differentiating, while in $\mathcal{ARC}$ an arc is erased which in $\mathcal{C}act$ corresponds to contracting an arc. Both these operations reduce the number of parameters by one. Of course omission of a parameter and insertion of a degree one space amounts to the same signs.

The action of $\mathcal{WLBPT}ree$ is then given by $(f_1, \dots f_n) \mapsto F(\tau)(f_1, \dots f_n)$ where the Hochschild complex is parameterized as in [KS].

Also using the action of $\mathcal{WLPT}ree$ of [KS], we could set $(f_1, \dots f_n) \mapsto F(st_\infty^{-1}(\tau))(f_1, \dots f_n)$ where $st_\infty^{-1}(\tau)$ is any lift and $m_3 = m_4 = \dots = 0$.

In this notation the operation of the cycle of 12 is the Gerstenhaber bracket up to the sign $(-1)^{pq+p}$ for homogeneous elements f of degree p and g of degree q.

7.6.1. **The differential.** As shown in [KS] or as a straightforward calculation shows, that if we define the operator $\partial(F) := \sum_n \partial(\tau_n)\circ_v$, then

$$\partial(\tau(f_1, \dots, f_k)) = \sum_n (\pm(\tau_n) \circ_v \tau)(f_1, \dots, f_k) := \partial(F(\tau)(\tau(f_1, \dots, f_k)))$$

7.7. **A solution of Deligne's conjecture from spineless cacti.**

7.7.1. **The action.** Using the cell decomposition 5.2.4 and the interpretation of $CC_*(\mathcal{C}act^1)$ as $\mathcal{WLBPT}ree$, $CC_*(\mathcal{C}act^1)$ acts naturally on the Hochschild complex, with the signs being fixed by one of the schemes above.

7.7.2. **Deligne's conjecture.** Notice that we have proven that $\mathcal{C}act$ is homotopy equivalent to the little discs operad in [K3] as well as to $\mathcal{C}act^1$ (see Corollary 5.1.3) and hence the cellular chains of $\mathcal{C}act^1$ give us a model of $D(2)$.

Although it is by now a corollary we formulate it as a theorem.

Theorem 7.7.1. *Deligne's conjecture is true for the chain model $CC_*(\mathcal{C}act)$ and moreover $CH^*(A, A)$ is even a dg–algebra over $CC_*(\mathcal{C}act)$.*

Remark 7.7.2. *As mentioned in §5.3.3 choosing a chain model $Chain(\mathcal{C}act)$ of $\mathcal{C}act$ by fixing a chain model for the scaling operad, we can let $Chain(\mathcal{C}act)$ by sending all cells of $Chain(\mathcal{C}act)$ which are not the product of a cell of $CC_*(\mathcal{C}act^1)$ and a zero dimensional cell of $\mathbb{R}^n_{>0}$ to zero and letting the cells of $CC_*(\mathcal{C}act^1)$ times a zero dimensional cell of $\mathbb{R}^n_{>0}$ i.e. cells of the type $\Delta(\tau) \times pt$ act via τ.*

Remark 7.7.3. *As noticed before, the fact that the chain operads of $\mathcal{C}act$ and $\mathcal{C}act^1$ possess the structure of a Gerstenhaber algebra up to homotopy, means that this structure exists also on $\mathcal{H}om_{CH^*(A,A)}$ up to homotopy and on $\mathcal{H}om_A$ on the nose – the latter being Gerstenhaber's original theorem* [G].

8. Outlook and speculations

8.1. **The A_∞ case.**

8.1.1. **Operation on the Hochschild complex of an A_∞ algebra.** In [KS] a natural action of $\mathcal{WLPT}ree$ called $\mathcal{M}$ in [KS] on $CH^*(A,A)$ was constructed in order to solve Deligne's conjecture, basically as above, only now the black vertices stand for the higher multiplications μ_n of the A_∞ structure. The solution was then established by constructing a quasi–isomorphism of the free operad of $\mathcal{M}$ onto the operad of the Fulton–MacPherson compactification of the configurations of $\mathbb{R}^2$. This beautiful construction is however indirect, as one has to invert the quasi–isomorphism and furthermore the map to the Fulton–McPherson compactification is rather involved and has considerably many choices. We would hope to find a direct interpretation of $\mathcal{WLPT}ree$.

8.1.2. **A putative cell decomposition.** In our situation an A_∞ version of Deligne's could be established, if we had a cellular decomposition of a suitable version of $Cact$ (e.g. $Cact^{\leq}$ where this is the space of cacti whose lobes have radius not greater than $2^{ht(v)}$) such that the cells are indexed by $\mathcal{WLPT}ree$ and are given by

$$cell(\tau) := \prod_{v\in V_W} C_{|v|} \times \prod_{v\in V_B} K_{|v|} \tag{8.1}$$

where $C_{|v|}$ is the $|v|$–dimensional cyclohedron and $K_{|v|}$ is the $|v|$–dimensional Stasheff polytope or associahedron [MSS].

Let $Cell(\mathcal{WLPT}ree)$ be the CW complex glued from the cells $cell(\tau)$ using the natural differential δ for the cyclohedra and associahedra and let ∂ of [KS]. It is then straightforward to show that:

Proposition 8.1.1. *The differential ∂ and δ agree on $CC_*(Cell(\mathcal{WLPT}ree))$*

This leads us to the conjecture

Conjecture 8.1.2. *There is a suitable suboperad of $Cact$ which is quasi–isomorphic to $Cact$ and whose cell decomposition is given by eq. (8.1).*

8.1.3. **Truncation of Simplices and Stasheff Polytopes.** A good candidate for the space for which the cell decomposition should be possible is the space $Cact^{\leq}$ of restricted cacti.

It is easy to see that this space is indeed a suboperad.

To find the cell decomposition, we should have a special realization of cyclohedra and associahedra. In particular, we need a realization of an n associahedron in the hyperplane $\sum_i x_i = 2^{n-1} \subset \mathbb{R}^{n-1}$ and the n cyclohedron in $\mathbb{R}^{2n+1}$.

The corners for the Associahedron K_n should be given by n tuples indexed by a binary tree and should be $2^{n-ht(v)}$ which are the weights obtained by using the homotopy associative operation $\cdot$ on $\mathcal{DA}rc$ using the bracketing given by a binary tree and reading off the total weights on the boundaries $\partial_i(F)$.

For $n = 2$ this is the point $(1,1)$ for $n = 3$ one can take the interval on the line through $(1,1,2)$ and $(2,1,1)$. For $n = 4$ the corners of the pentagon should be $(1,1,2,4)$, $(2,1,1,4)$, $(2,2,2,2)$, $(4,1,1,2)$ and $(4,2,1,1)$.

The cyclohedra should come from blowing up the n dimensional simplex Δ^n which corresponds to the τ_n, so that the tree boundary for the cyclohedra coincides with the geometric boundary of the operad.

Conjecture 8.1.3. *We conjecture that there is a truncation of simplices with the above corners that yields the above realizations of associahedra and cyclohedra.*

This has been checked for low dimensions, but we currently lack a general scheme. One step in this direction would be the construction of an explicit map of the compactification of the configuration space of n points on S^1 with one point fixed at the origin to the n–th cyclohedron.

8.2. **Relations to the Fulton–MacPherson compactification.** Finally, we expect a relation of spineless Cacti or the $\mathcal{ARC}$ operad to the Fulton–MacPherson compactification of the configuration space of points in $\mathbb{R}^2$. The idea is to use a variant of polar coordinates and then to keep track of the collision speeds in the width of the bands. A thorough analysis of this fact should result in a positive answer to a conjecture made in [KS]. This will be elaborated on elsewhere.

8.3. **Action of $\mathcal{ARC}$.** We also expect that $\mathcal{ARC}$ operates on $CH^*(A, A)$, for an algebra A together with a non–degenerate pairing as a cyclic operad. This has two steps, first the non–trees and second higher genus.

Conjecture 8.3.1. *We conjecture that all of $\mathcal{ARC}$ acts on CH^*.*

For the action of loop spaces there are two conjectures, one a little less bold:

Conjecture 8.3.2. *We conjecture that the Arc operad will act on LM by a map that factors through Loop.*

and a bolder conjecture

Conjecture 8.3.3. *We conjecture that the Arc operad will act on LM by a combinatorial Gromov–Witten type setup.*

This we understand as follows. If we wish to concatenate loops in the free loop space LM that do not intersect, we have to be able to move the loops into the a position in which they do. Regarding the image of a surfaces with arcs in M whose boundaries $1, \dots, n$ are exactly the loops to be multiplied will give us a way to move and multiply them into the loop 0. The versions of the discussed Props yield straightforward generalizations.

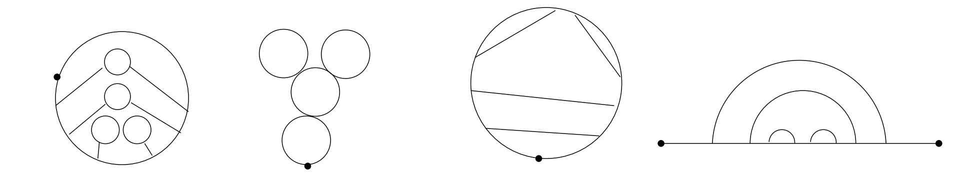
With a suitable version of mapping spaces and (virtual) fundamental classes one could hope to construct combinatorial invariants, pulling back families of loops and integrating over the moduli space.

8.4. **Rankin–Cohen Brackets.** Recently, in [CM] it was discovered that the Rankin Cohen brackets can be realized inside the foliation Hopf Algebra introduced by Connes and Moscovici. Since these brackets have a conjectured form in terms of naturally grown trees and due to the relationship of $\mathcal{ARC}$ to moduli spaces we dare to formulate:

Conjecture 8.4.1. *We conjecture that Rankin Cohen brackets are also realizable on $\mathcal{C}act$.*

8.5. **Open questions and future.** It still remains to find out the exact relationship of cacti to chord diagrams, the dihedral algebra of [Go] and polylogarithms and higher zeta values. For the latter the clarification of the relationship to configuration spaces and cacti should be key as well as the relationship to moduli spaces.

We wish to conclude by remarking that the new feature of the Chinese tree operad is that the chord diagrams of [K3] are no longer planar. Cutting at zero, we



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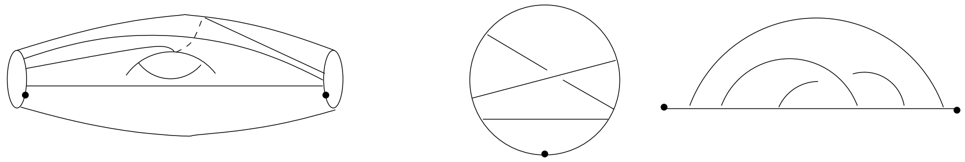

II

FIGURE 28. I. A genus one arc family, its cactus, cord diagram and rainbow diagram, II A genus one diagram giving a braid move

obtain not only rainbow diagrams as for genus zero, but (for) high enough genus any trivalent diagram of the types depicted in figure 28 which are known from high energy physics and knot theory [BN] and are making their appearance in biology in the form of folding problems for RNA.

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E-mail address: `ralphk@mpim-bonn.mpg.de`

Oklahoma State University, Stillwater OK, USA and Max–Planck Institut für Mathematik, Bonn, Germany